Geometric structure of thermal cones

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QUIT Physics - Thermoroundtable
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Outline

I. Setting the scene

II. Statement of the problem

III. Results

IV. Outlook

In collaboration with:


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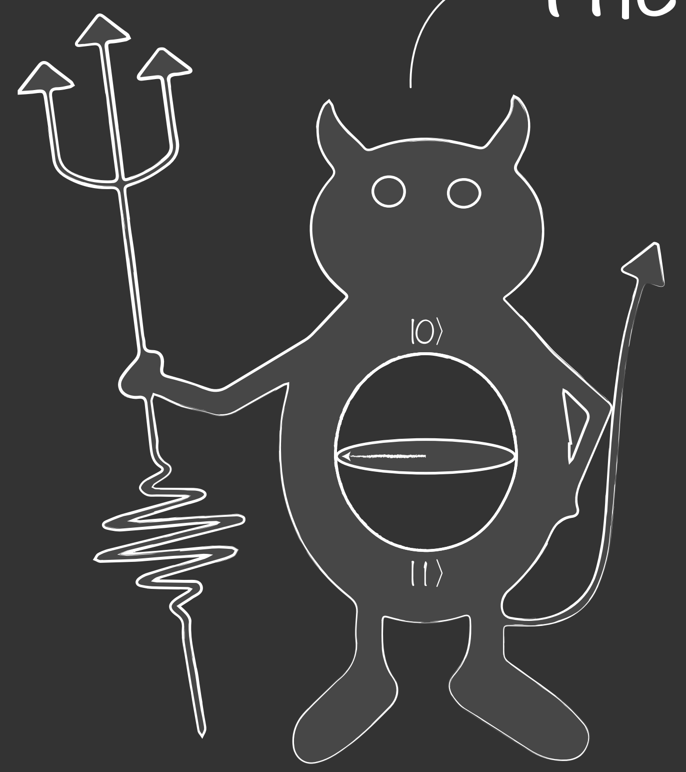
Kamil Korzekwa


Karol Życzkowski

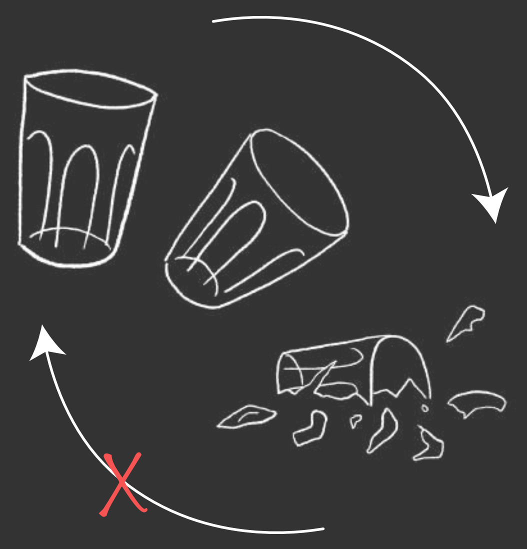
Jagiellonian University, Krakow

Based on:
arXiv. 2207.02237 - Framerwork & Applications

Thermodynamic arrow of time



The 2nd law → fundamental asymetry:



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A. Eddington, 1927

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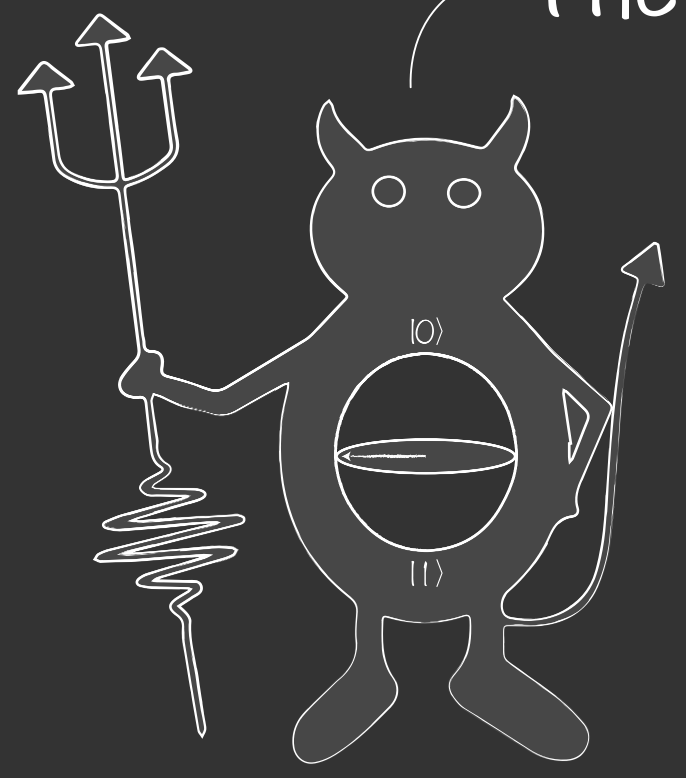

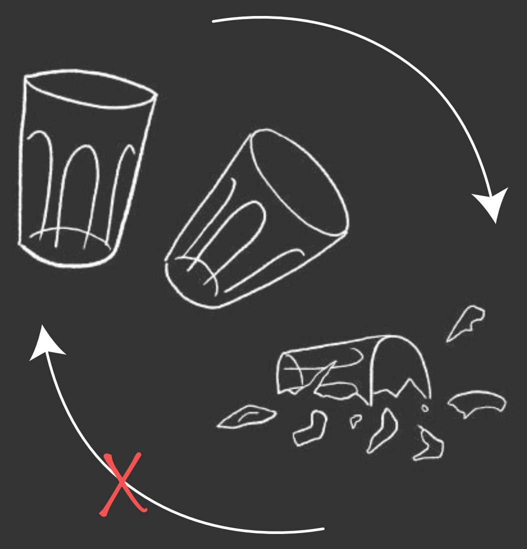

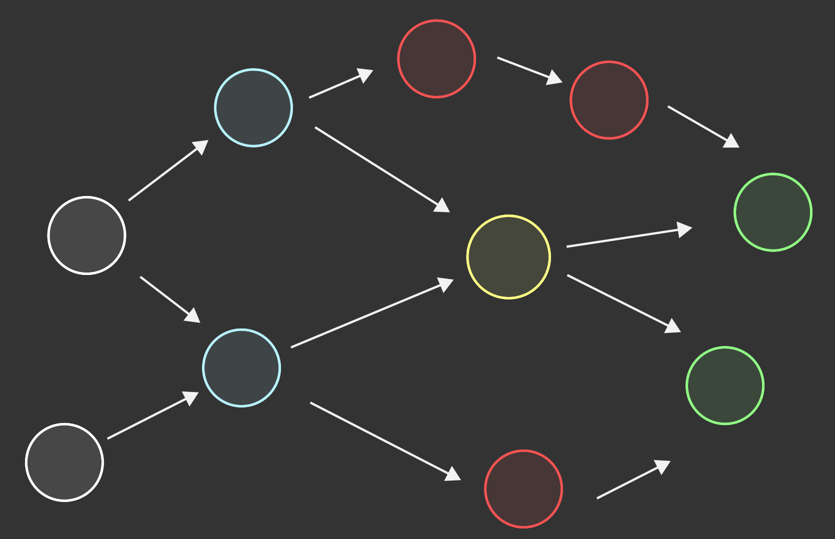

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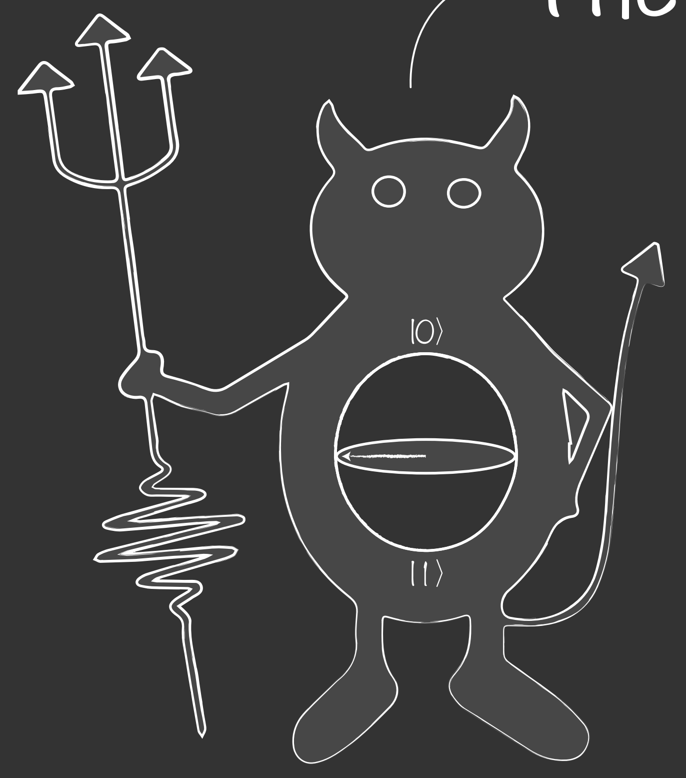

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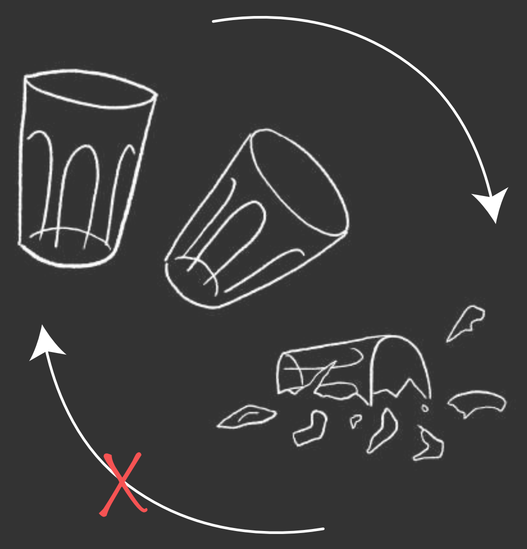
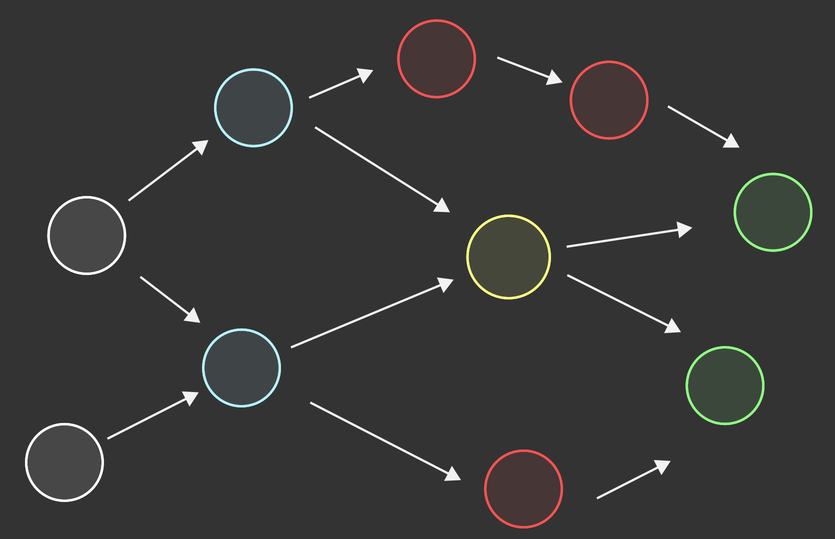


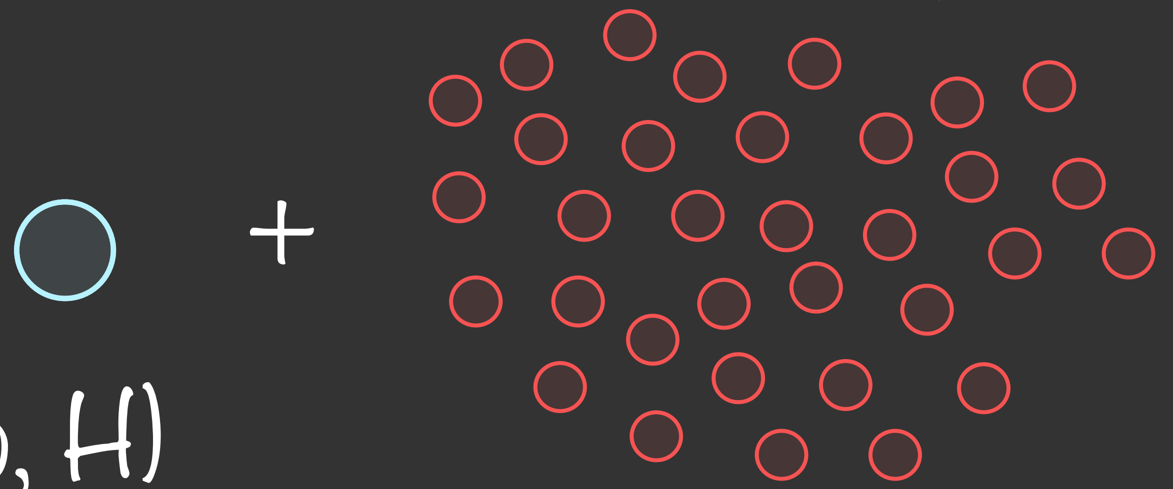


Given a quantum system what can we say about its past, incomparable and future?

Setting the scene

Setting the scene

$\beta = 1/\kappa_B T$



(ρ, H)

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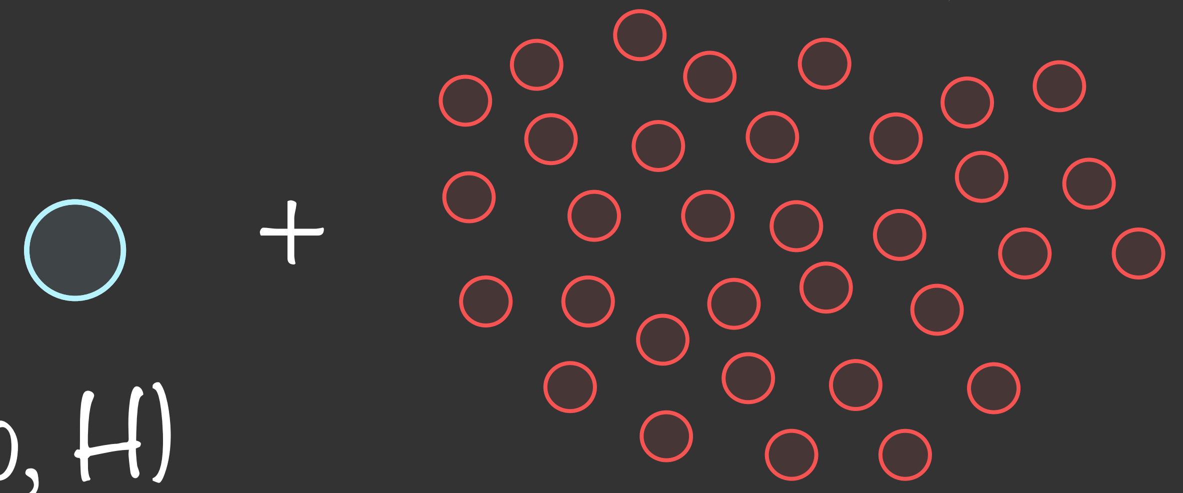

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$$\xrightarrow[\text{operations}]{\text{thermal}}$$

$$\xi(\rho) = \mathrm{tr}_E\left[U\left(\rho \otimes \gamma_E\right) U^\dagger\right] \quad \text{with} \quad \left[U, H \otimes \mathbb{1}_E + \mathbb{1} \otimes H_E\right] = 0$$

D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth,
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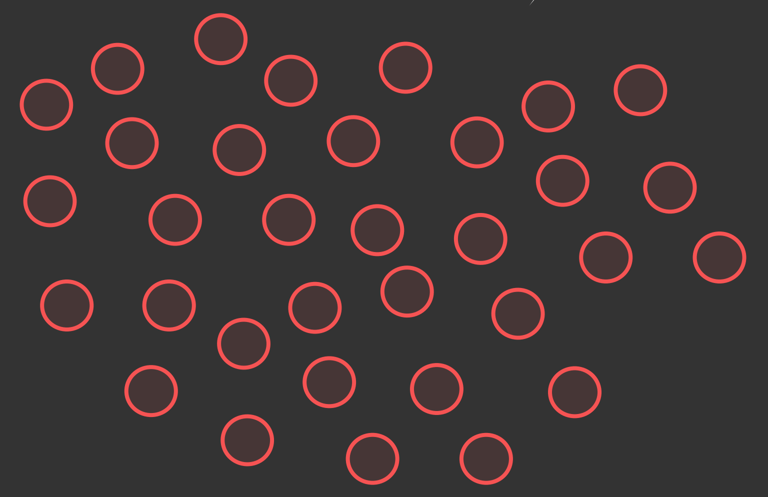
 + 

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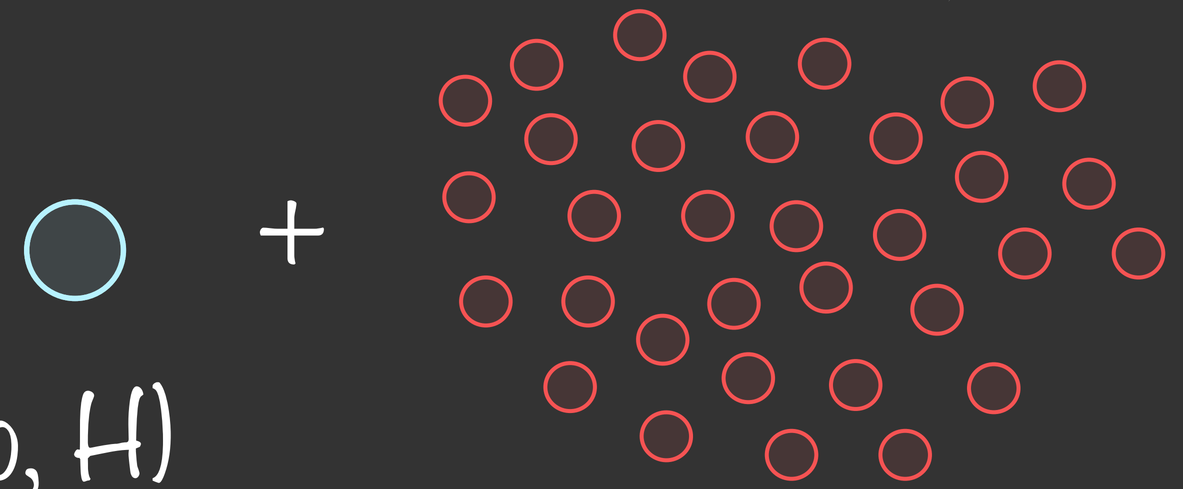

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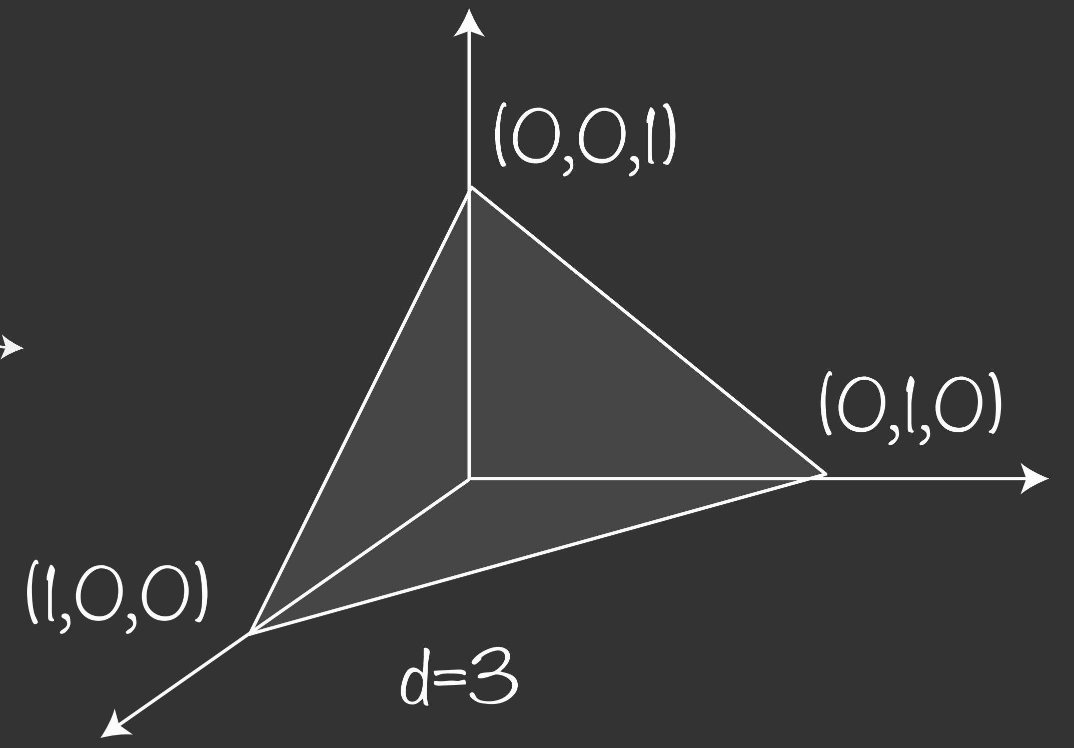

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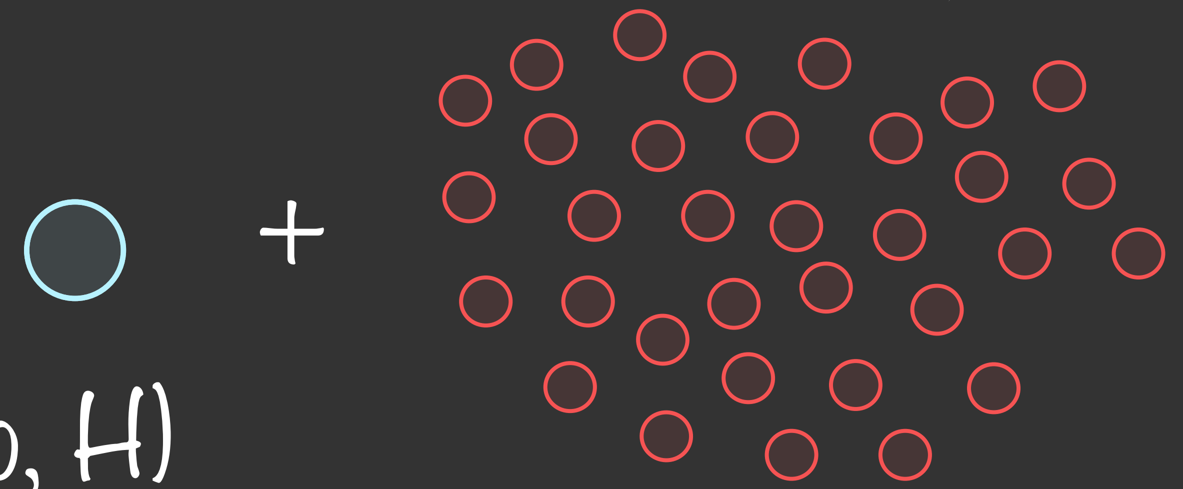

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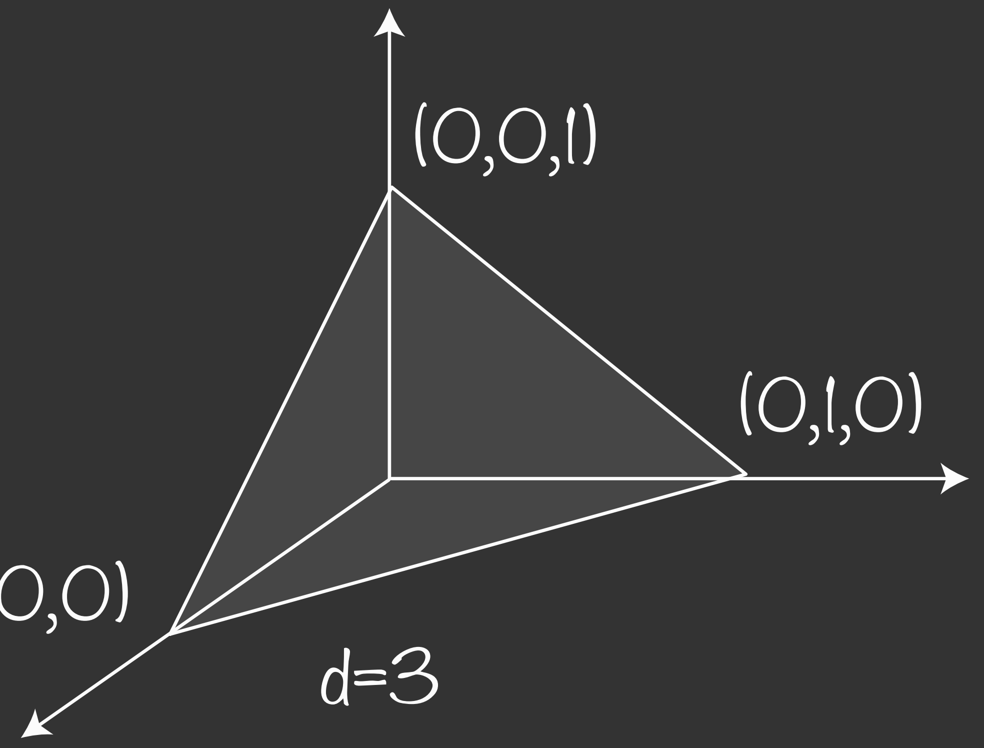
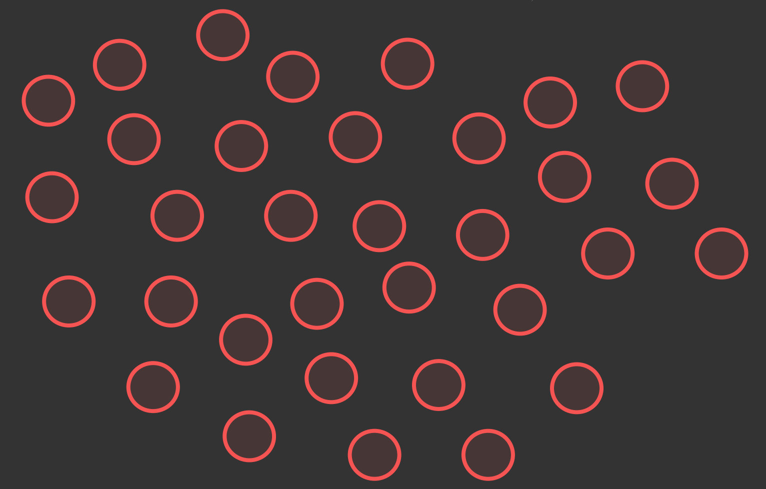


$$\xrightarrow[\text{conditions}]{\text{transformation}}$$

$\left(\Lambda_{ij} \geq 0 \;\&\; \sum_i \Lambda_{ij} = 1\right)$

$$\boxed{\exists \xi : \xi(\rho) = \sigma, \text{ if and only if, } \Lambda\gamma = \gamma \;\&\; \underbrace{\Lambda \mathbf{p} = \mathbf{q}}_{\text{eigenvalues of } \rho \text{ and } \sigma}}$$

Setting the scene



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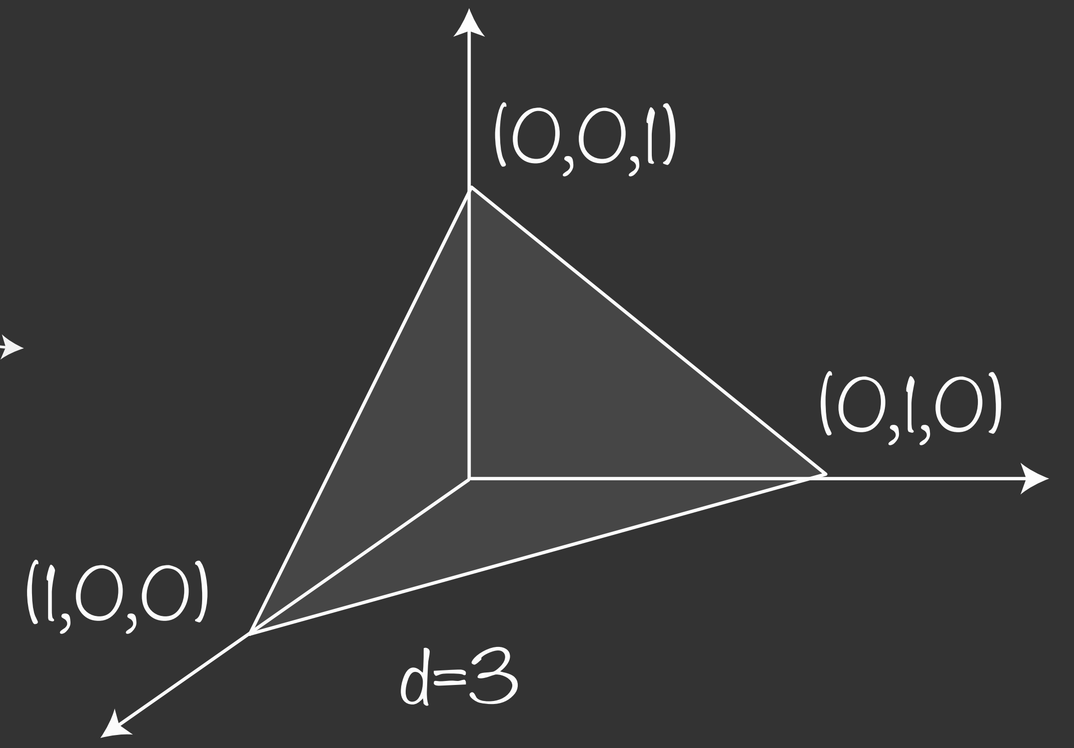


$\xrightarrow[\text{conditions}]{\text{transformation}}$

$$\boxed{\exists\, \Lambda: \Lambda\mathbf{p} = \mathbf{q} \text{ if and only if, } \mathbf{p} \succ_\beta \mathbf{q}}$$

M. Horodecki & Jonathan Oppenheim,
Nature Communication (2013)

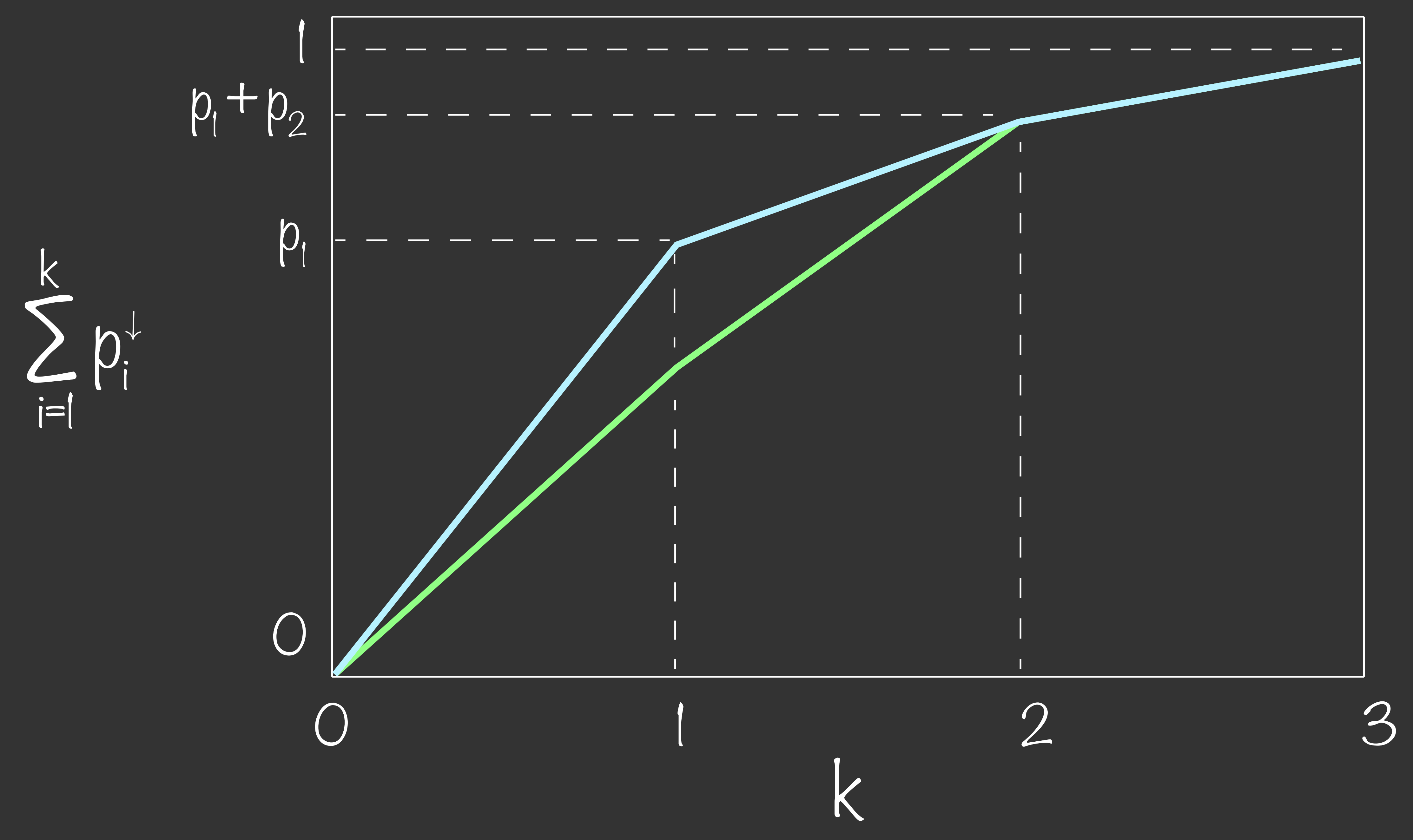
Partial-order relation interlude

Majorisation

A given vector $\mathbf{p}$ is said to majorise $\mathbf{q}$ if and only if

$$\sum_{i=1}^{k} p_i^{\downarrow} \geq \sum_{i=1}^{k} q_i^{\downarrow} \quad \text{for all } k \in \{1, \ldots, d\}$$

$\mathbf{p} \succ \mathbf{q}$



Majorisation curve

Partial-order relation interlude

! Non-increasing ordering $\longrightarrow$ β-ordering

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Example

$$\mathbf{p} = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right), \quad \mathbf{q} = \left(\frac{1}{2}, \frac{3}{8}, \frac{1}{8}\right) \text{ and } \boldsymbol{\gamma} = \left(\frac{3}{6}, \frac{2}{6}, \frac{1}{6}\right)$$

$$\pi^{\beta}_{\mathbf{p}} = (3,2,1) \qquad \pi^{\beta}_{\mathbf{q}} = (2,1,3)$$

$$\downarrow \qquad\qquad \downarrow$$

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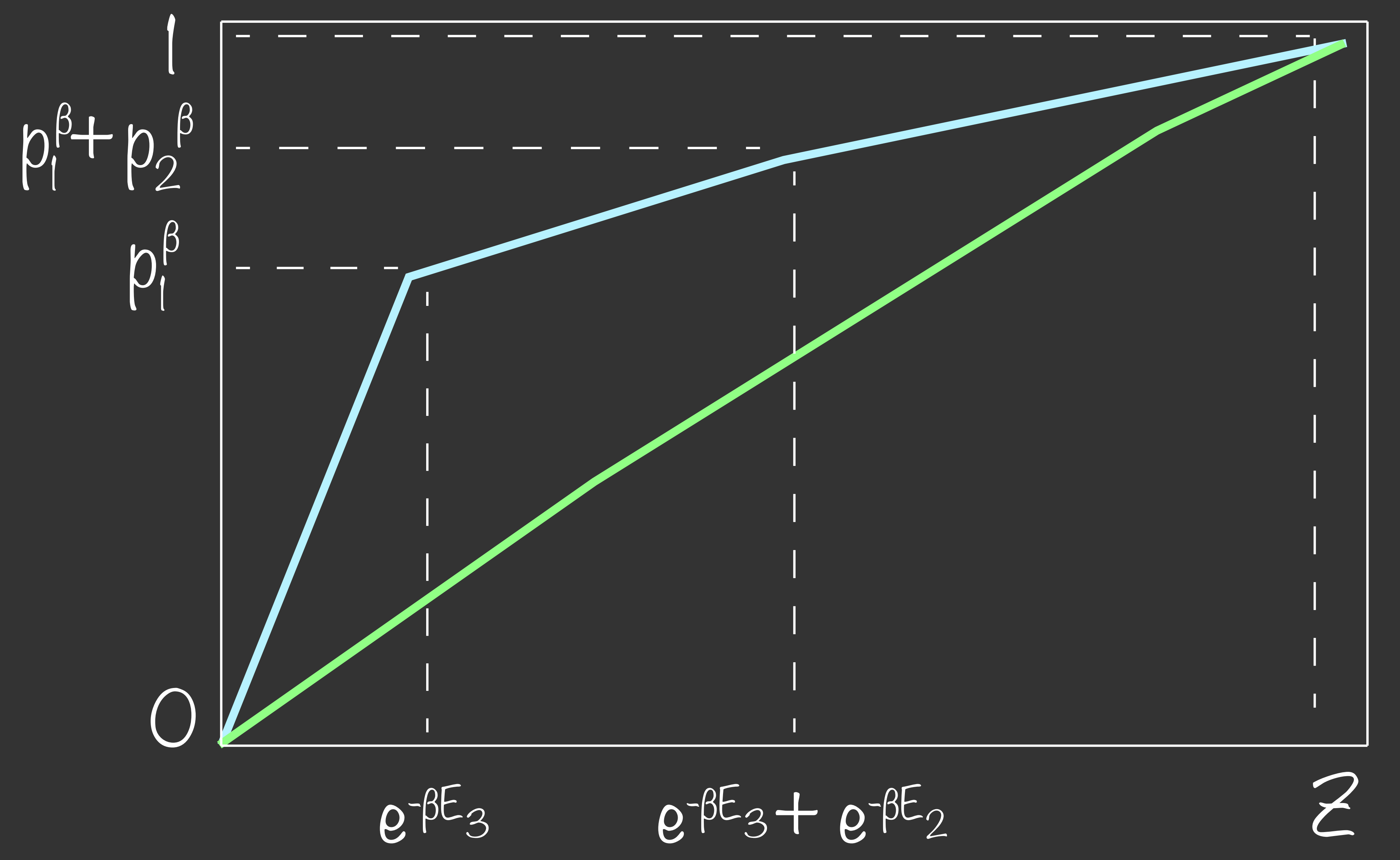
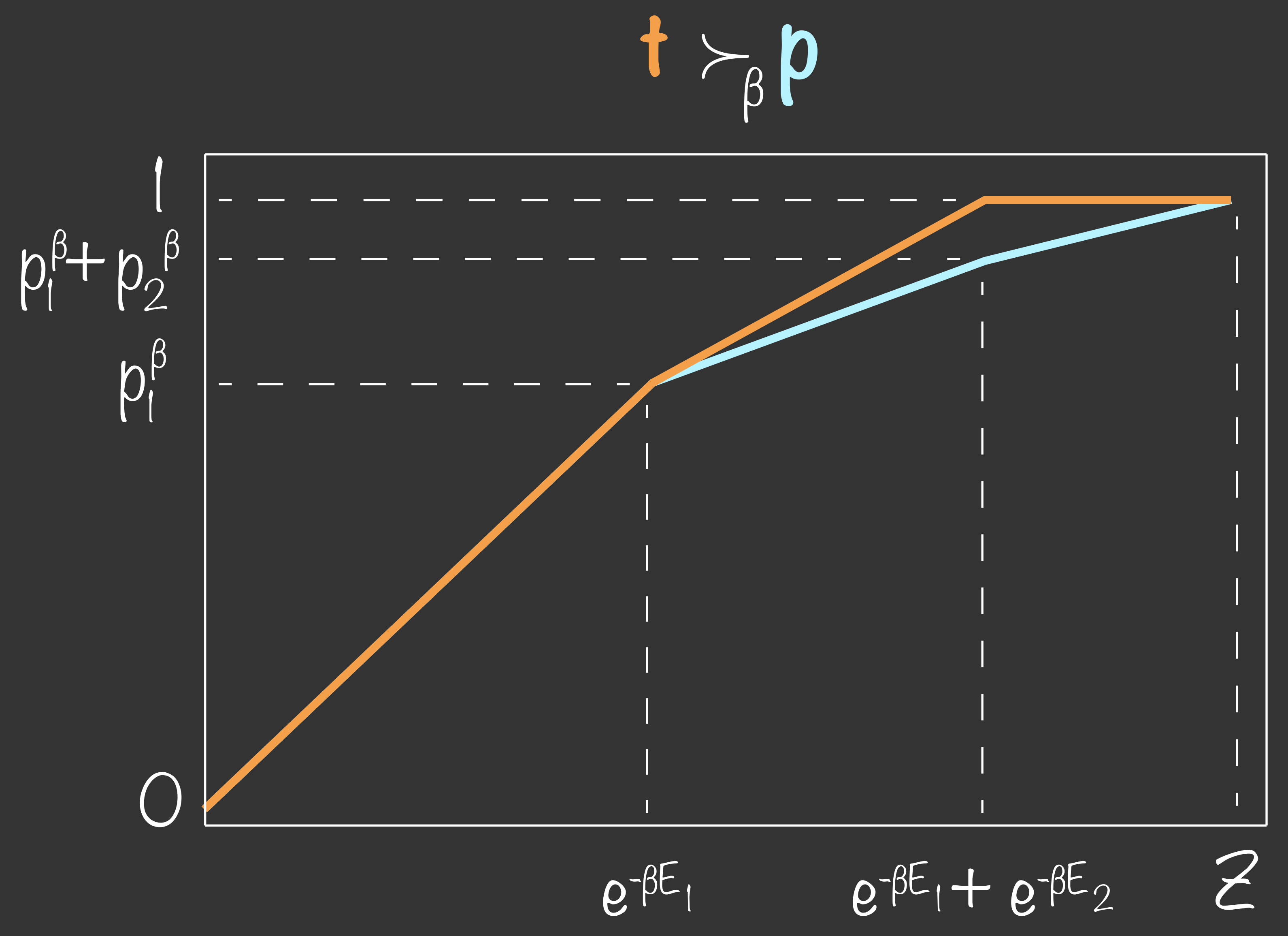
$\mathbf{p} \succ_{\beta} \mathbf{q}$



Partial-order relation interlude

Given **p** there exist three different cases:



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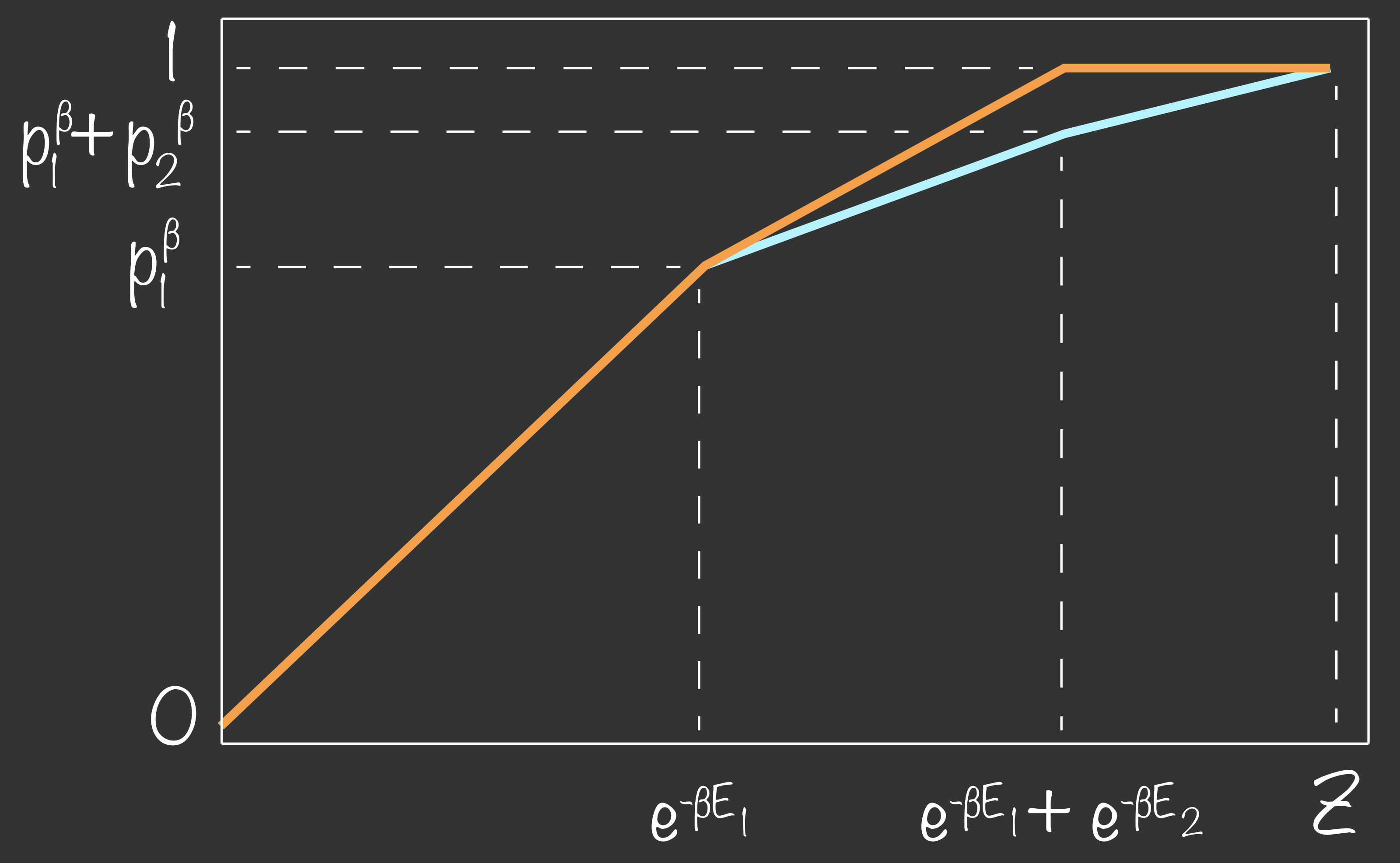
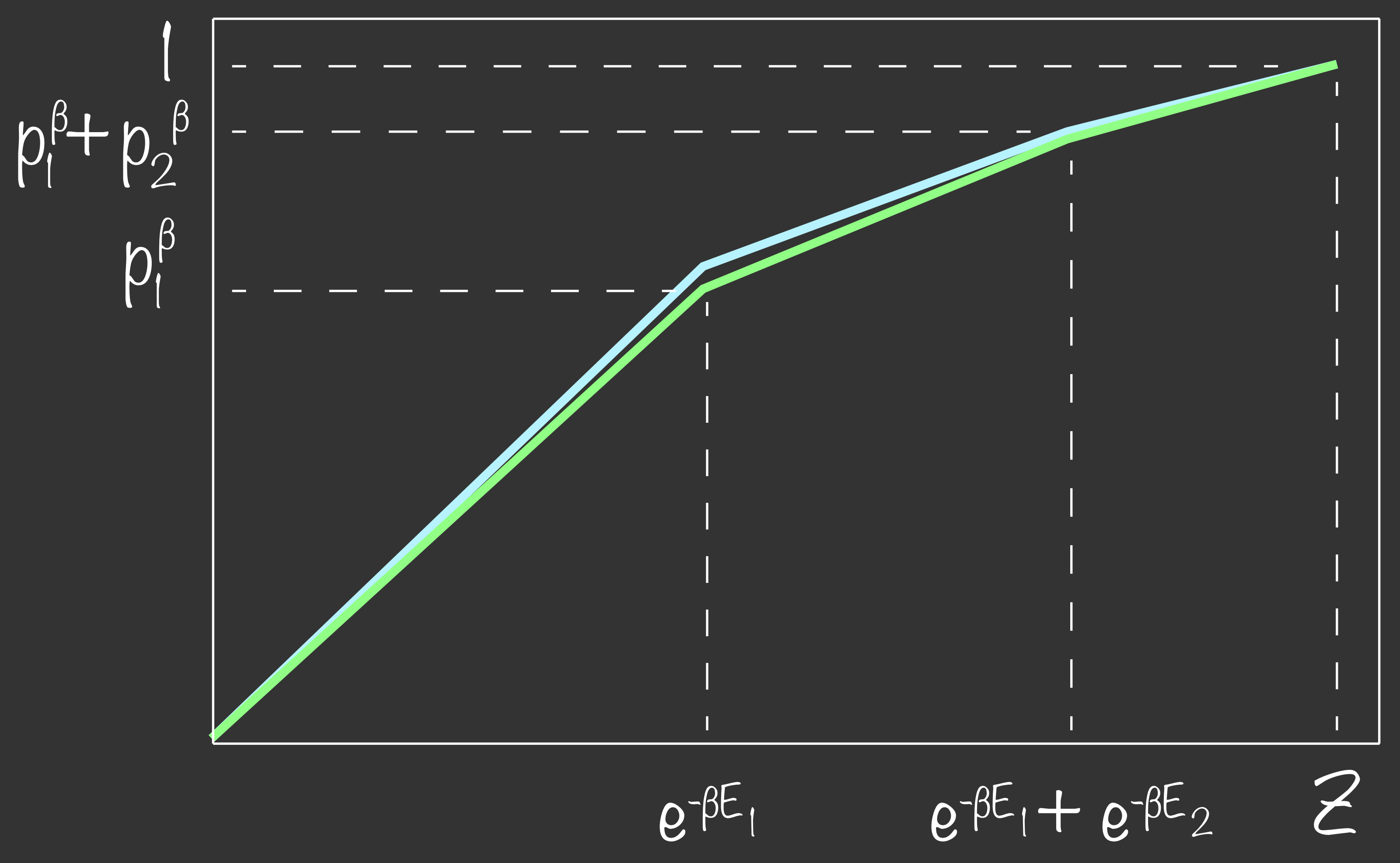
$\mathbf{t} \succ_\beta \mathbf{p}$



$\mathbf{p} \succ_\beta \mathbf{q}$



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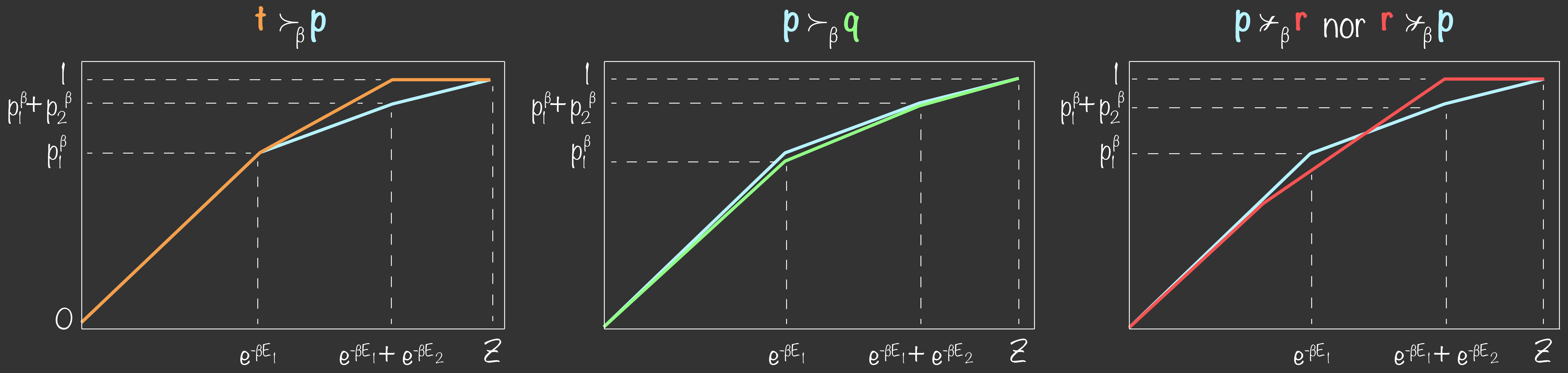
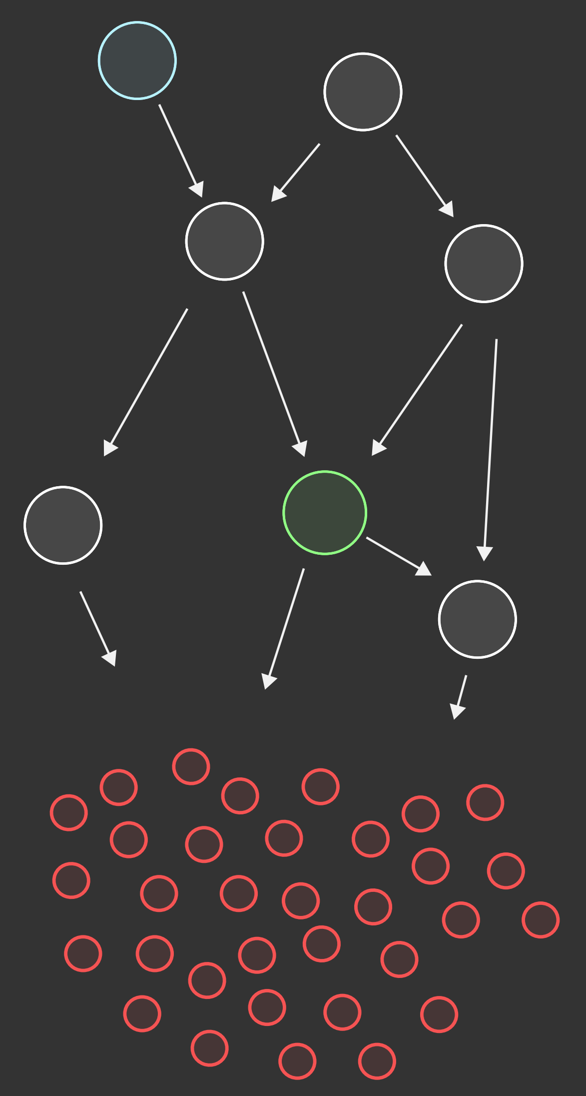


Summarising

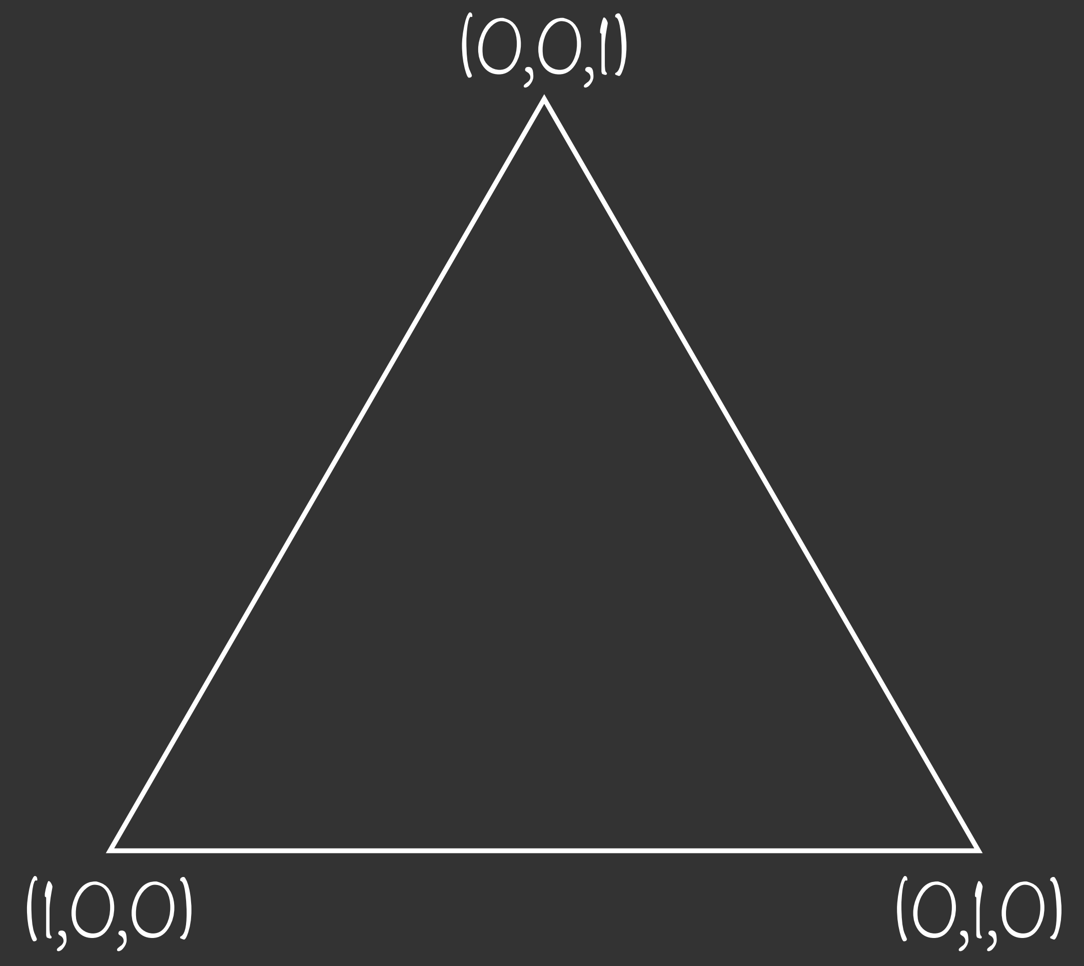
out of equilibrium



equilibrium

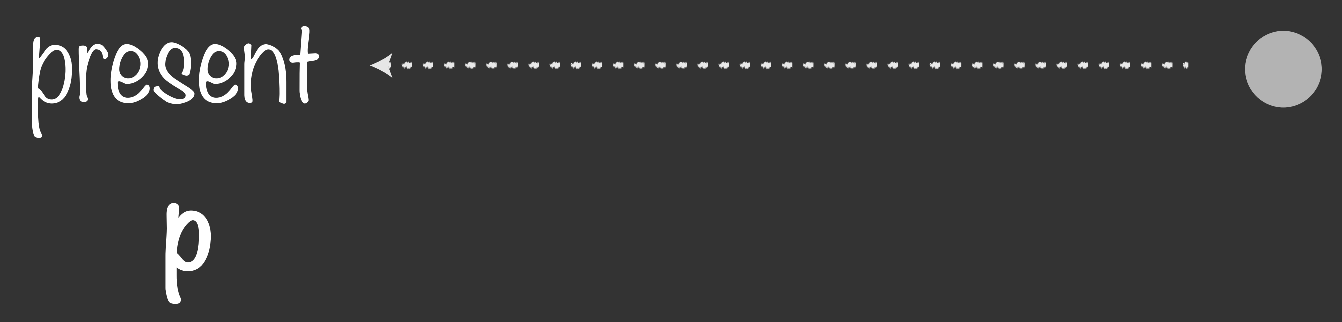
- Energy-incoherent states: $(\rho, H) = \left(\sum_{i=1}^{d} p_i |E_i\rangle\langle E_i| , \sum_{i=1}^{d} E_i |E_i\rangle\langle E_i| \right) \longrightarrow \mathbf{p} = (p_1, \ldots, p_d)$

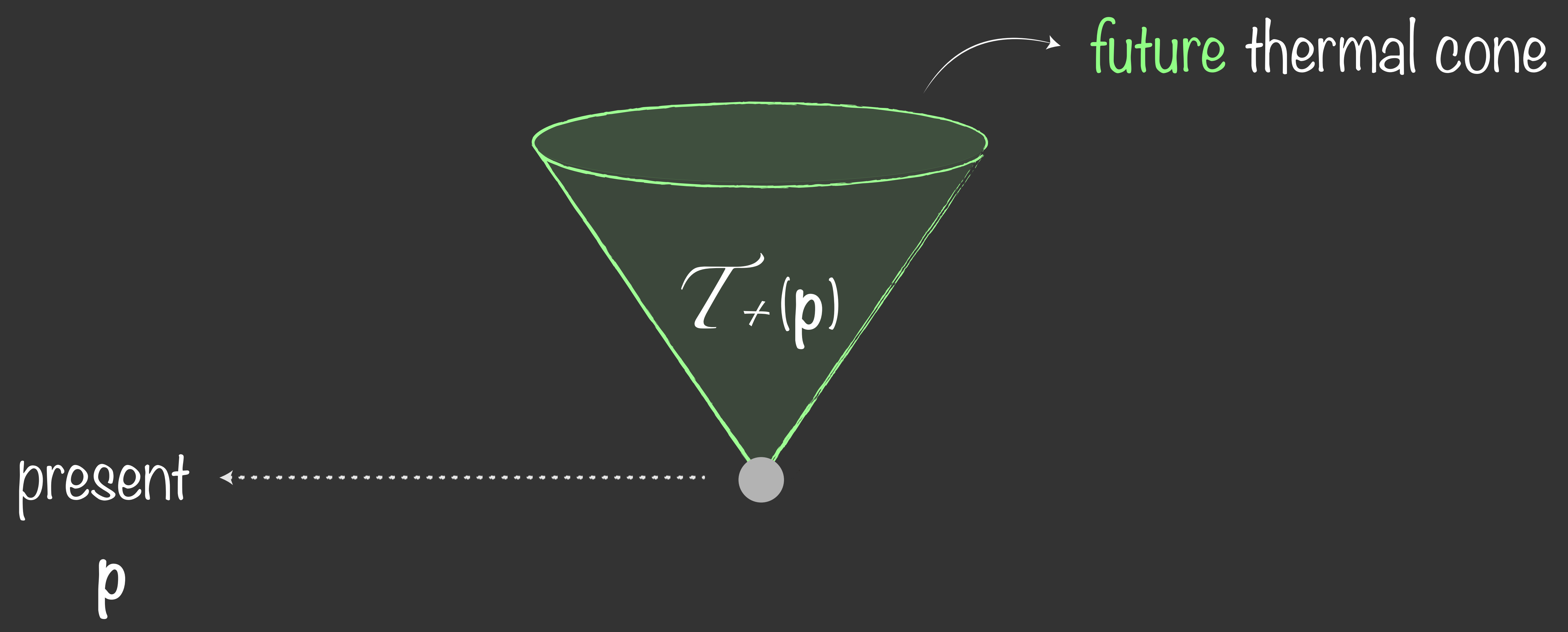
- Probability simplex:



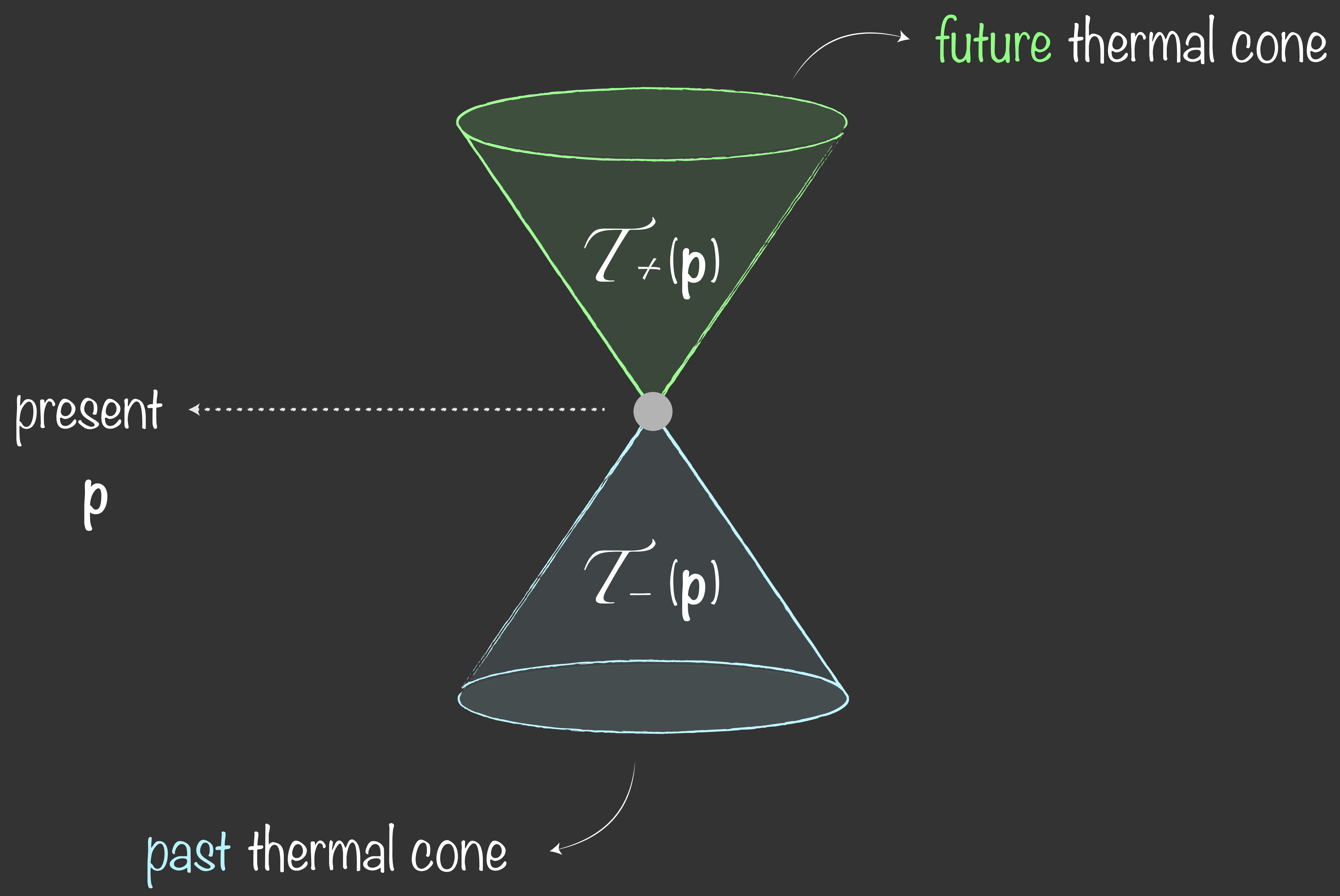
- States transformations: $\mathbf{p} \succ_\beta \mathbf{q}$

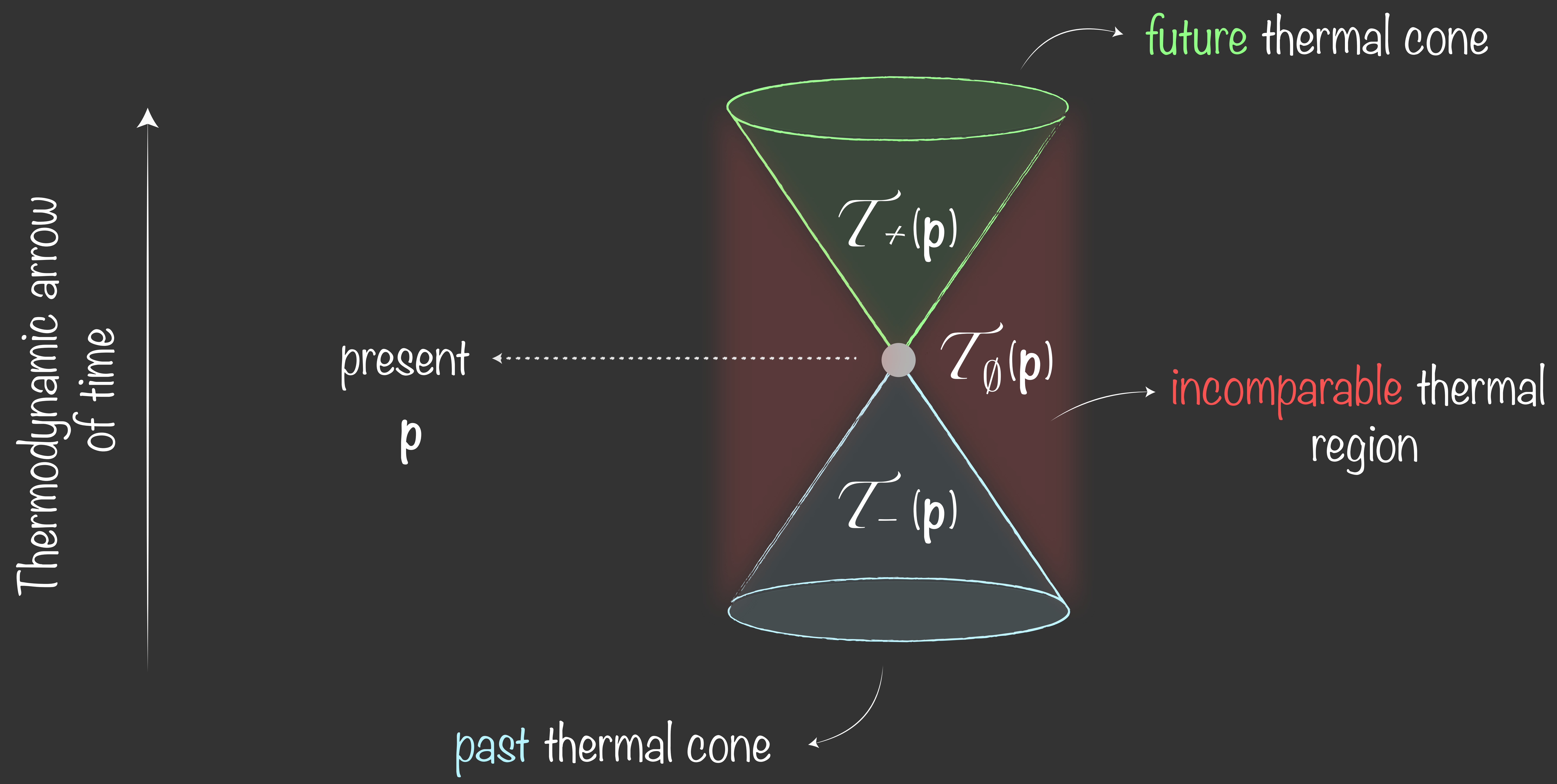
Statement of the problem


present
p


future thermal cone
$\mathcal{T}_+(\mathbf{p})$
present
$\mathbf{p}$

GEOMETRIC STRUCTURE OF THERMAL CONES





How one can characterise the thermal cones?

Results

Majorisation cones

Infinite temperature limit $T \to \infty / \beta = 0$: $\gamma = \eta := (1/d, ..., 1/d)$

Majorisation cones

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B: The set of n x n bistochastic matrices is a convex set whose extreme points are permutation matrices

G. Birkhoff, Univ. Nac. Tucumán. Revista A. (1946)

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HLP: There exists a bistochastic matrix mapping $\mathbf{p}$ into $\mathbf{q}$ if and only if $\mathbf{p}$ majorises $\mathbf{q}$: $\Lambda\mathbf{p} = \mathbf{q}$, $\Lambda\boldsymbol{\eta} = \boldsymbol{\eta}$

G. Hardy, J. Littlewood, and G. Polya, Inequalities, (1952)

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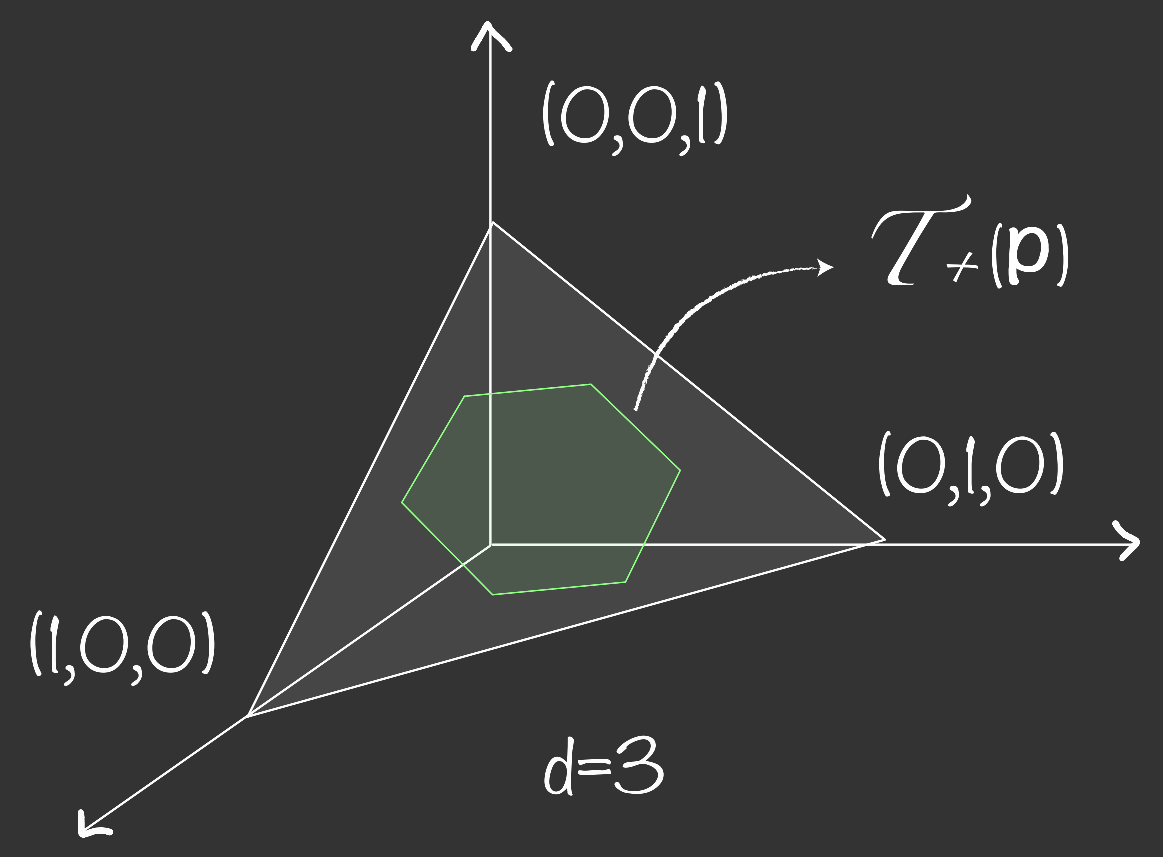
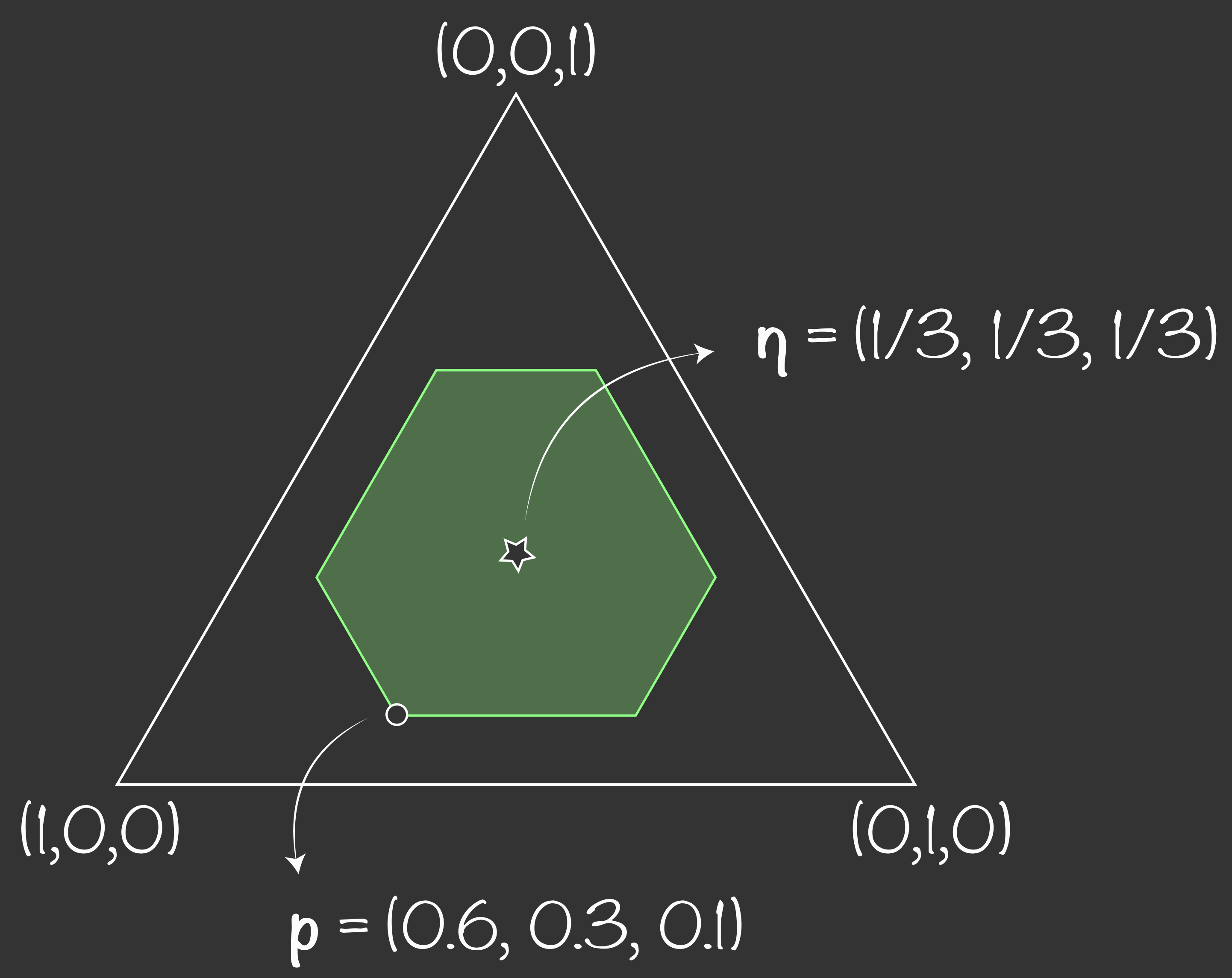


Future cone

$$\mathcal{T}_+(\mathbf{p}) = \mathrm{conv}[\{\Pi\mathbf{p} \,,\, S_d \ni \boldsymbol{\pi} \mapsto \Pi\}]$$



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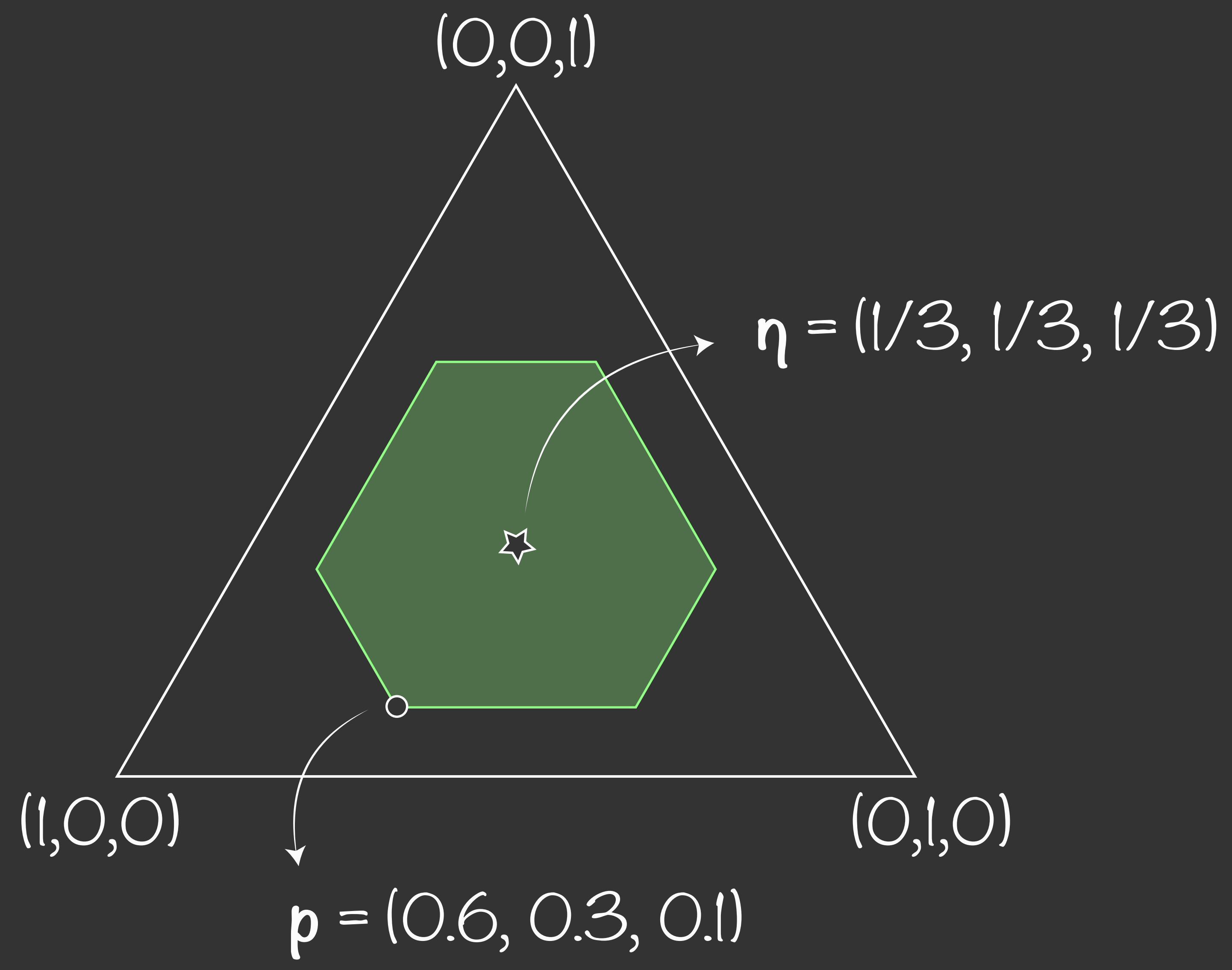
Q. Given a present state **p**, how to characterise its incomparable region and past cone?



Majorisation cones

Lemma: For a d-dimensional energy incohent state $\mathbf{p} = (p_1, \ldots, p_d)$, consider the quasi-distributions $\mathbf{t}^{(n)}$ constructed for each $n \in \{1, \ldots, d\}$,

$$\mathbf{t}^{(n)} = (\mathbf{t}_1^{(n)}, p_n^{\downarrow}, \ldots, p_n^{\downarrow}, \mathbf{t}_d^{(n)}) \quad \text{with} \quad \mathbf{t}_1^{(n)} = \sum_{i=1}^{n-1} p_i^{\downarrow} - (n-2)\, p_n^{\downarrow} \quad \text{and} \quad \mathbf{t}_d^{(n)} = 1 - \mathbf{t}_1^{(n)} - (d-2)\, p_n^{\downarrow},$$

and define the following set

$$T := \bigcup_{j=1}^{d-1} \mathrm{conv}\left[\mathcal{T}_{\neq}\left(\mathbf{t}^{(j)}\right) \cup \mathcal{T}_{\neq}\left(\mathbf{t}^{(j+1)}\right)\right].$$

Then, the incomparable cone of $\mathbf{p}$ is given by

$$\mathcal{T}_{\emptyset}(\mathbf{p}) = [\mathrm{int}(T) \setminus \mathcal{T}_{\neq}(\mathbf{p})] \cap \Delta_d$$

Majorisation cones

Theorem: The past cone of **p** is given by

$$\mathcal{T}_-(\mathbf{p}) = \Delta_d \setminus \mathrm{int}(T)$$

Majorisation cones

Example. $\mathbf{p} = (0.6, 0.3, 0.1) \longrightarrow \begin{cases} \mathbf{t}^{(1)} = (0.6, 0.6, -0.2) \\ \mathbf{t}^{(2)} = (0.6, 0.3, 0.1) \\ \mathbf{t}^{(3)} = (0.8, 0.1, 0.1) \end{cases}$

Incomparable cone

$$T = \bigcup_{j=1}^{2} \mathrm{conv}\left[\mathcal{T}_{\neq}\left(\mathbf{t}^{(j)}\right) \cup \mathcal{T}_{\neq}\left(\mathbf{t}^{(j+1)}\right)\right]$$

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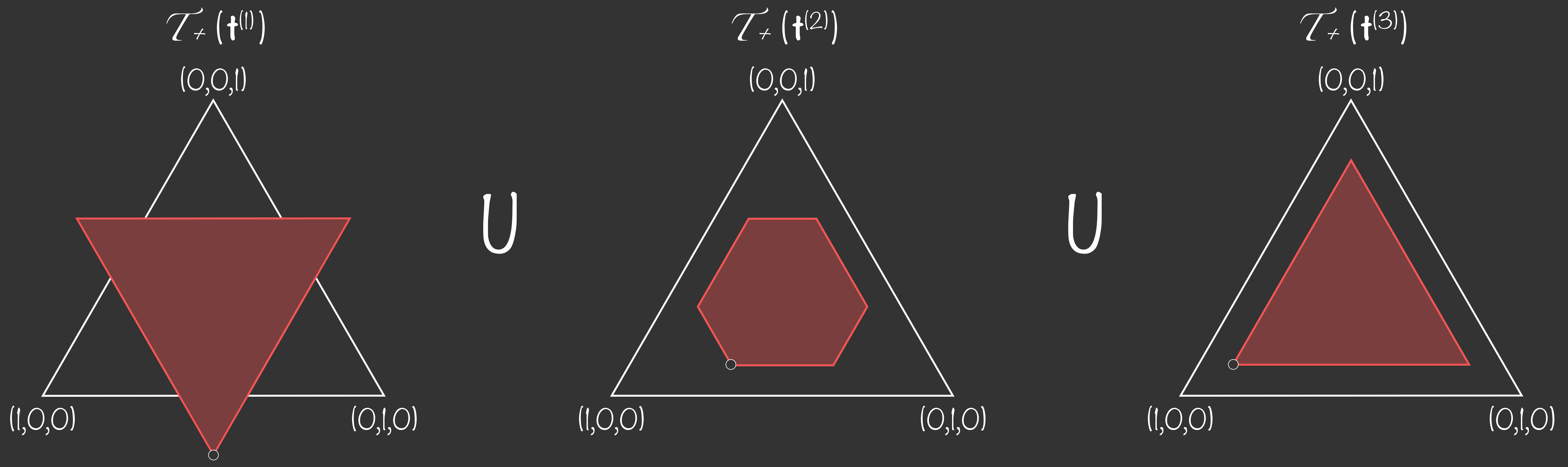

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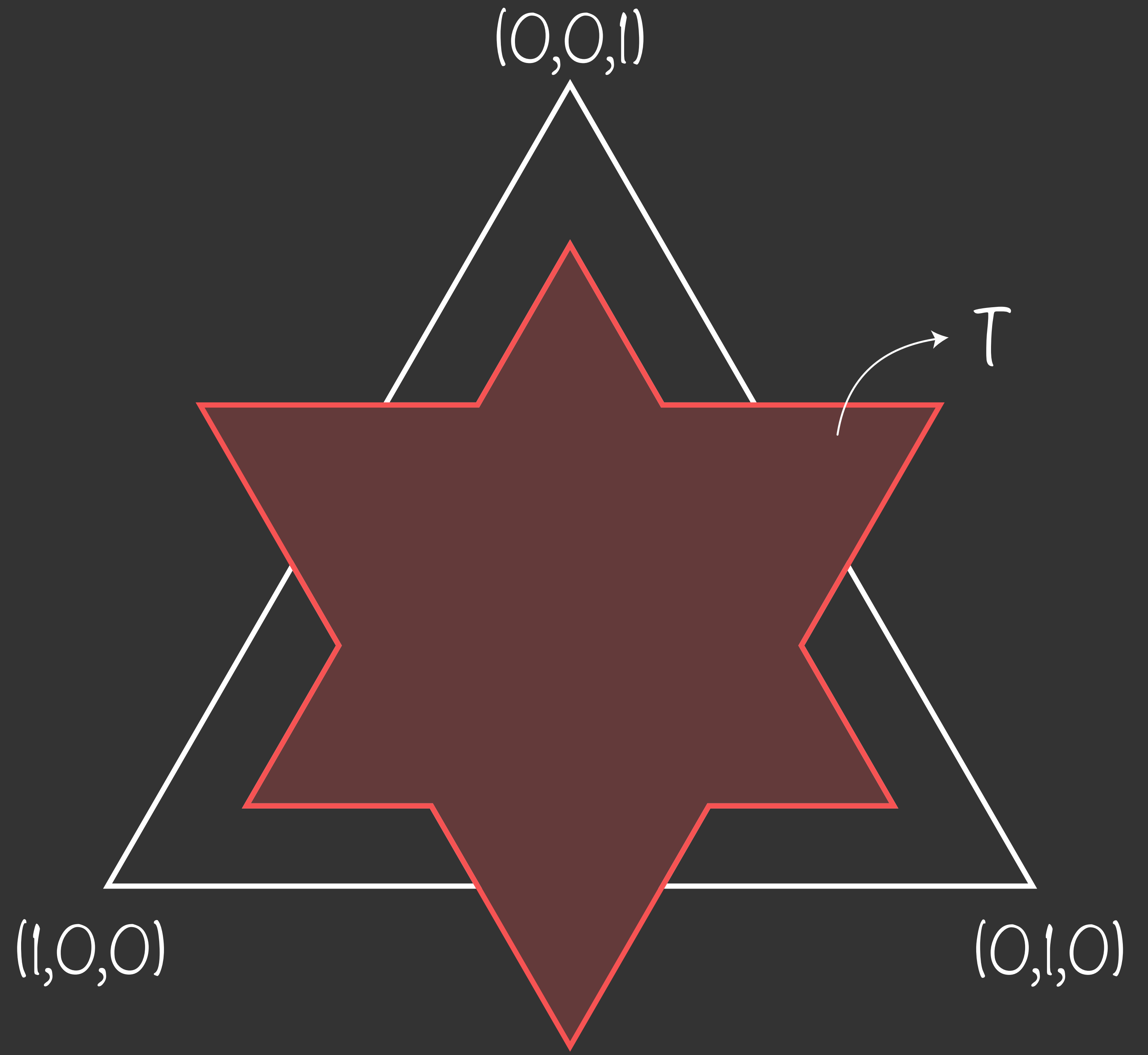

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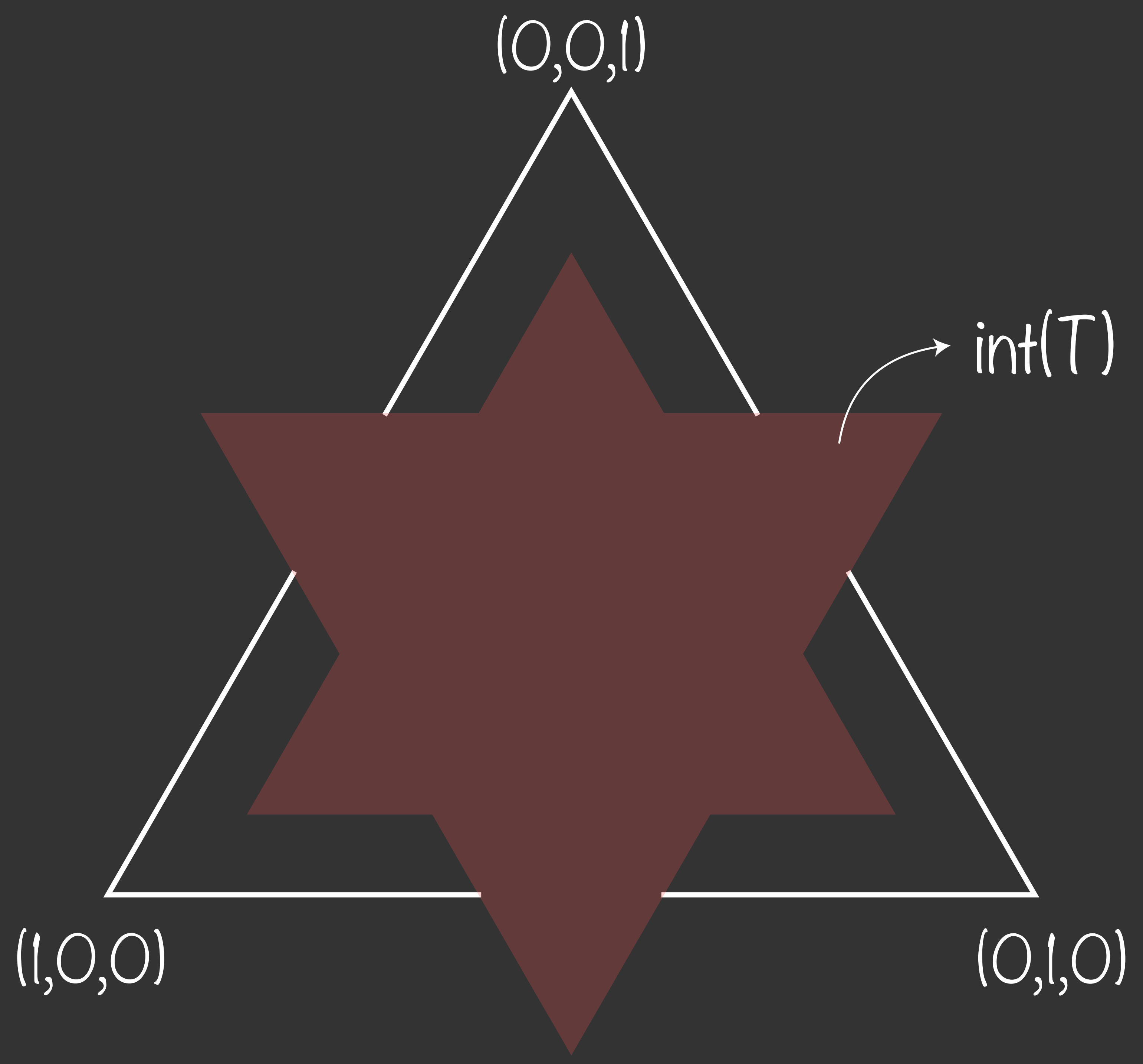

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Majorisation cones

Example. $\mathbf{p} = (0.6, 0.3, 0.1) \longrightarrow \begin{cases} \mathbf{t}^{(1)} = (0.6, 0.6, -0.2) \\ \mathbf{t}^{(2)} = (0.6, 0.3, 0.1) \\ \mathbf{t}^{(3)} = (0.8, 0.1, 0.1) \end{cases}$

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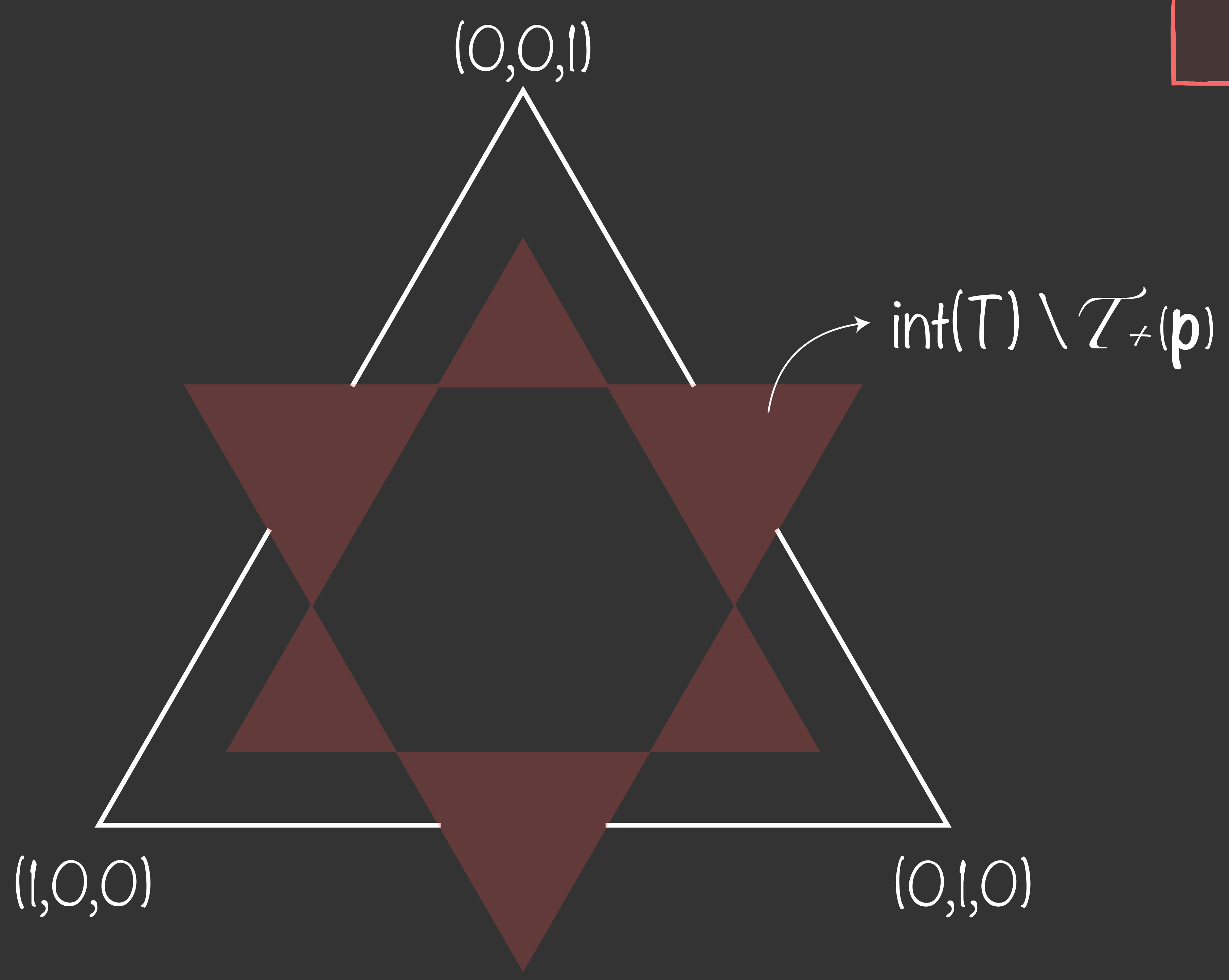

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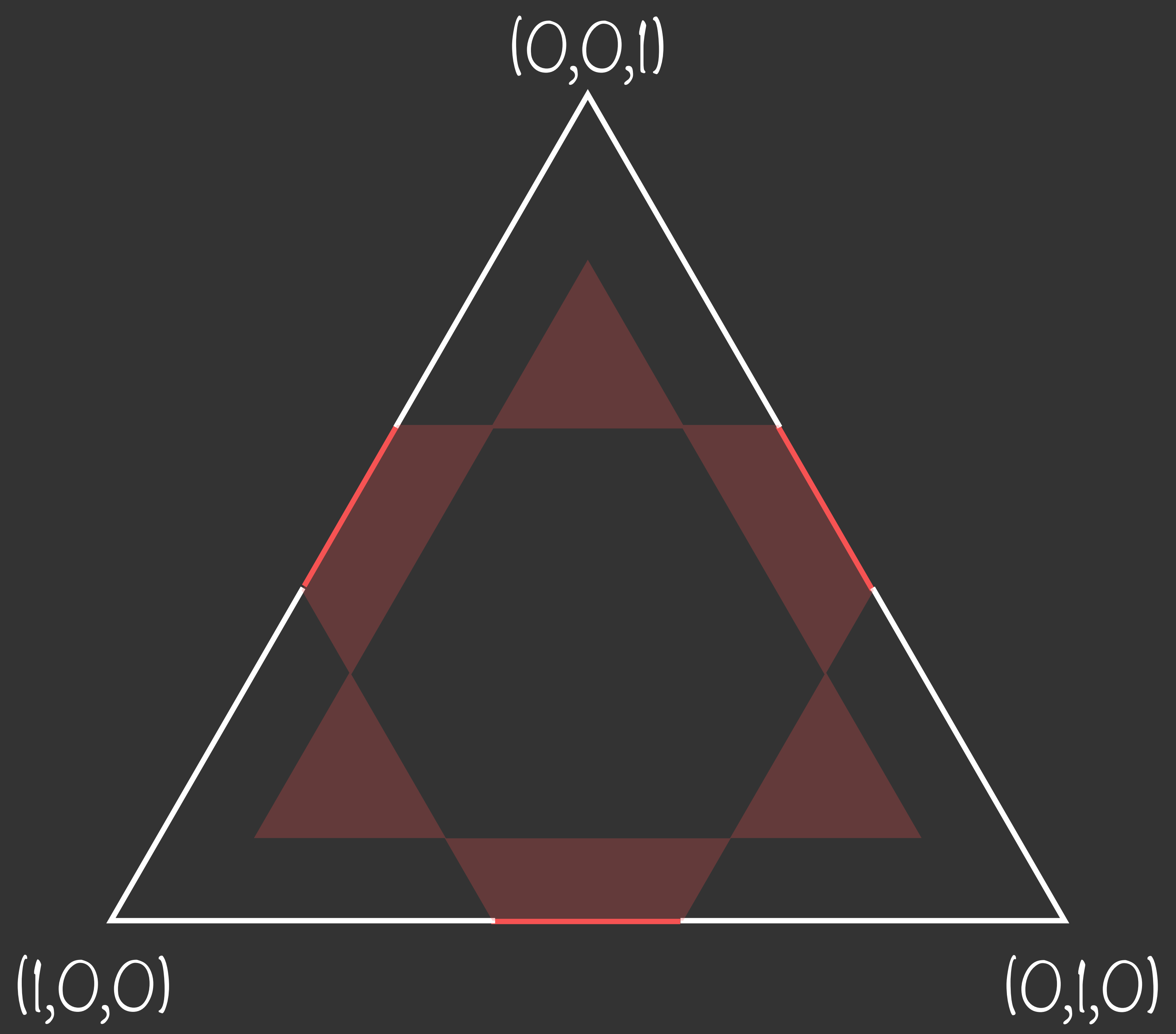

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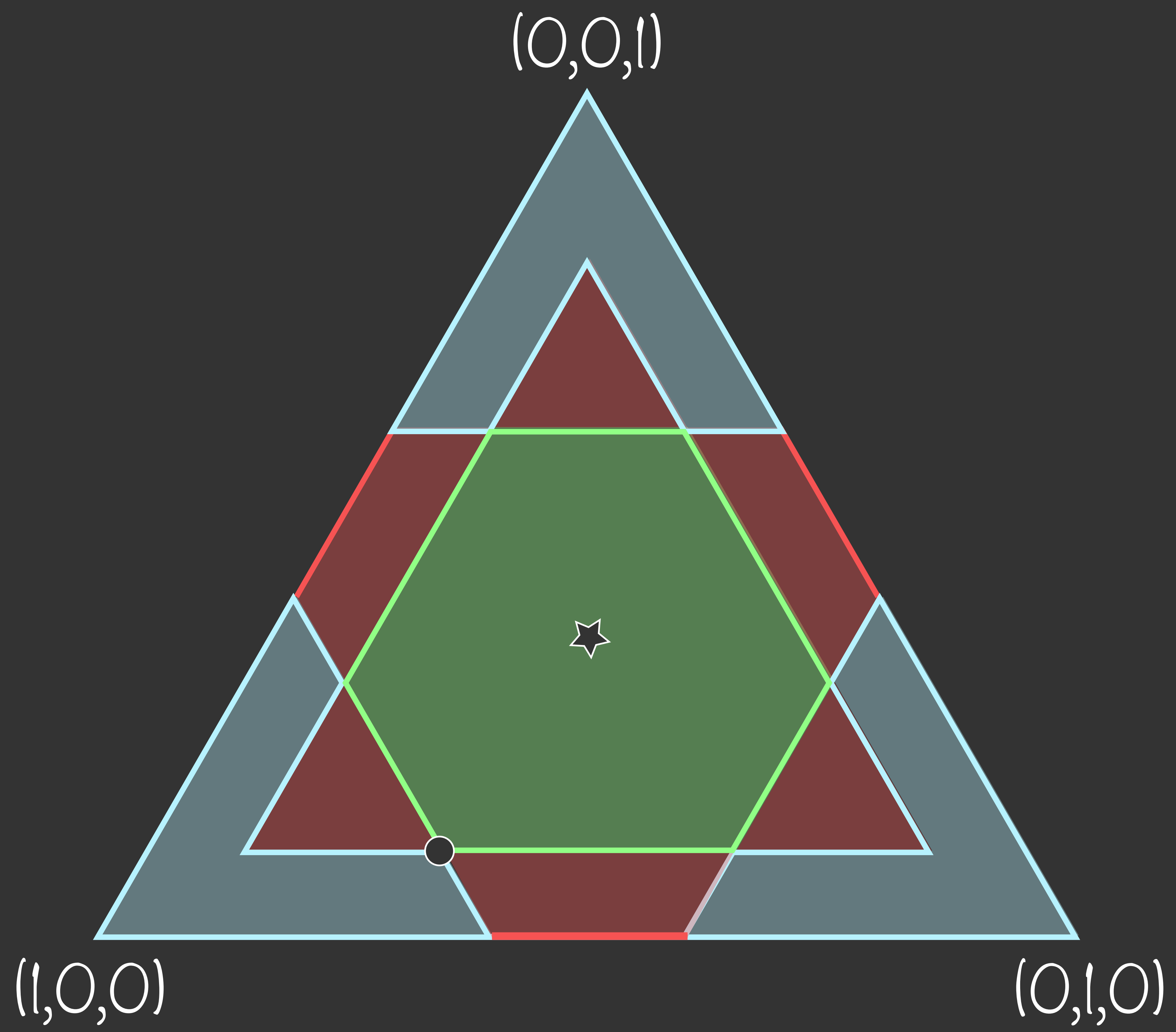

Future cone

$$\mathcal{T}_{+}(\mathbf{p}) = \mathrm{conv}[\{\Pi\mathbf{p}\,,\ S_d \ni \boldsymbol{\pi} \mapsto \Pi\}]$$

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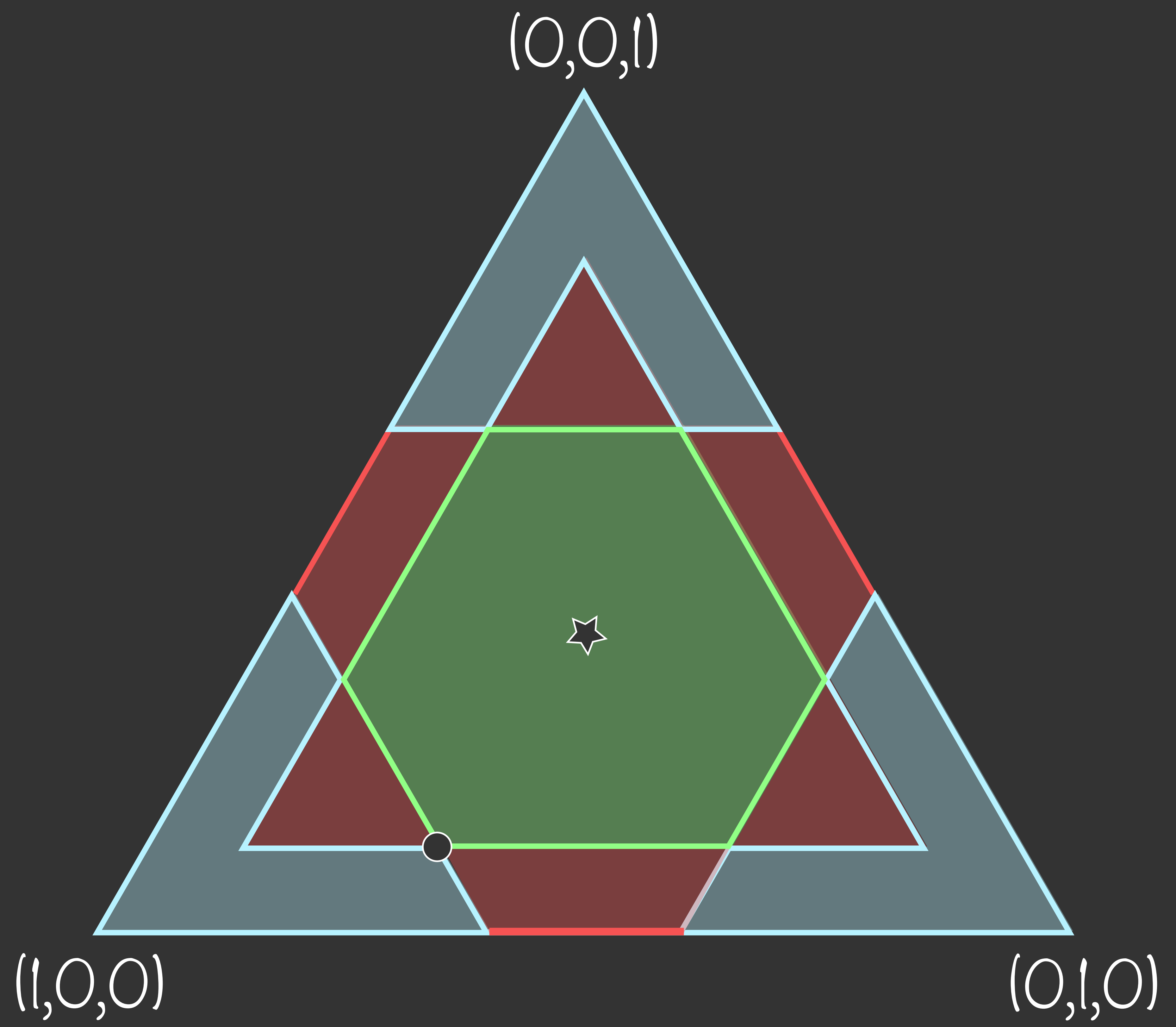

- Only the future is convex.

Majorisation cones

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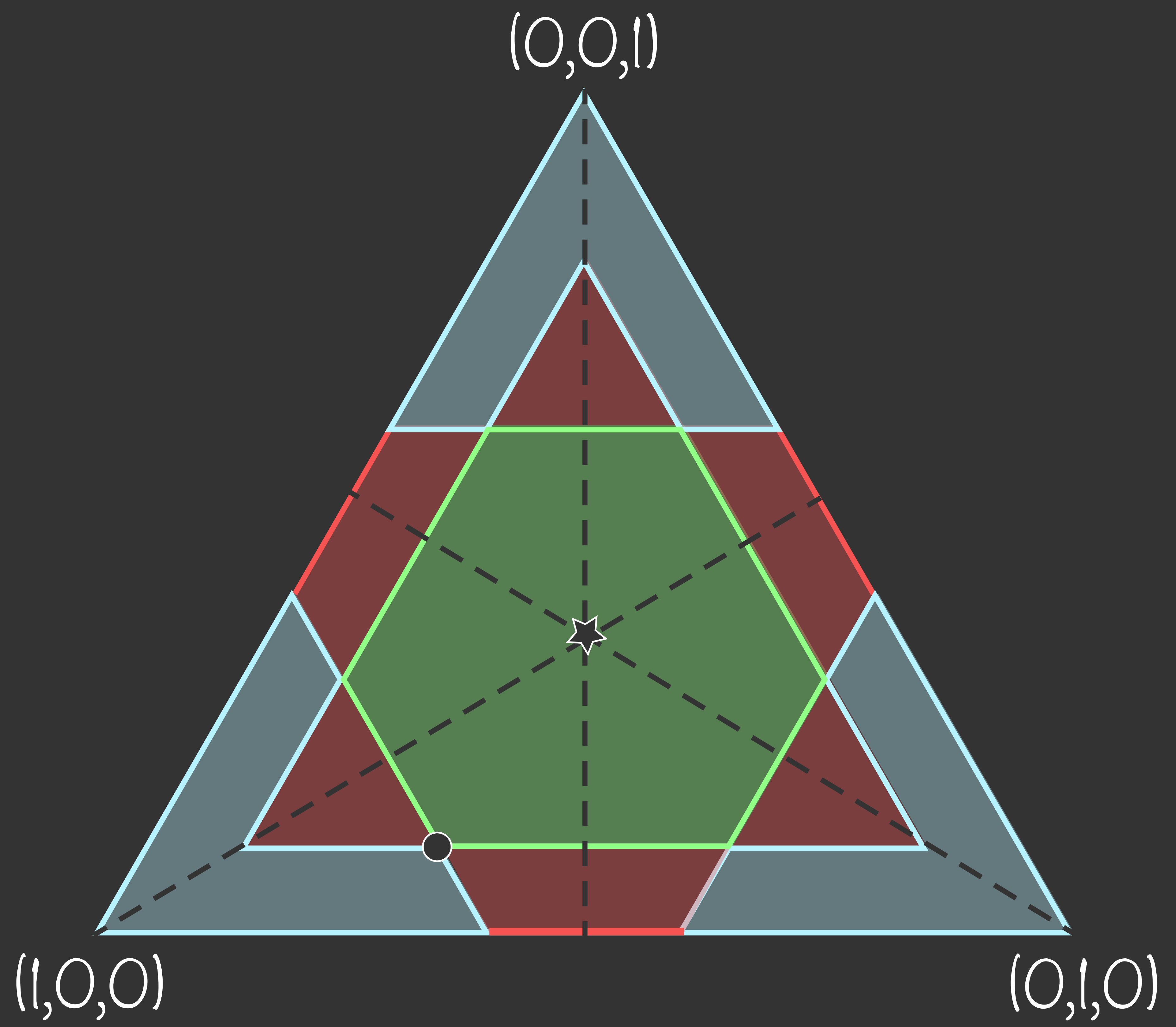

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- The past is the union of d! convex pieces.

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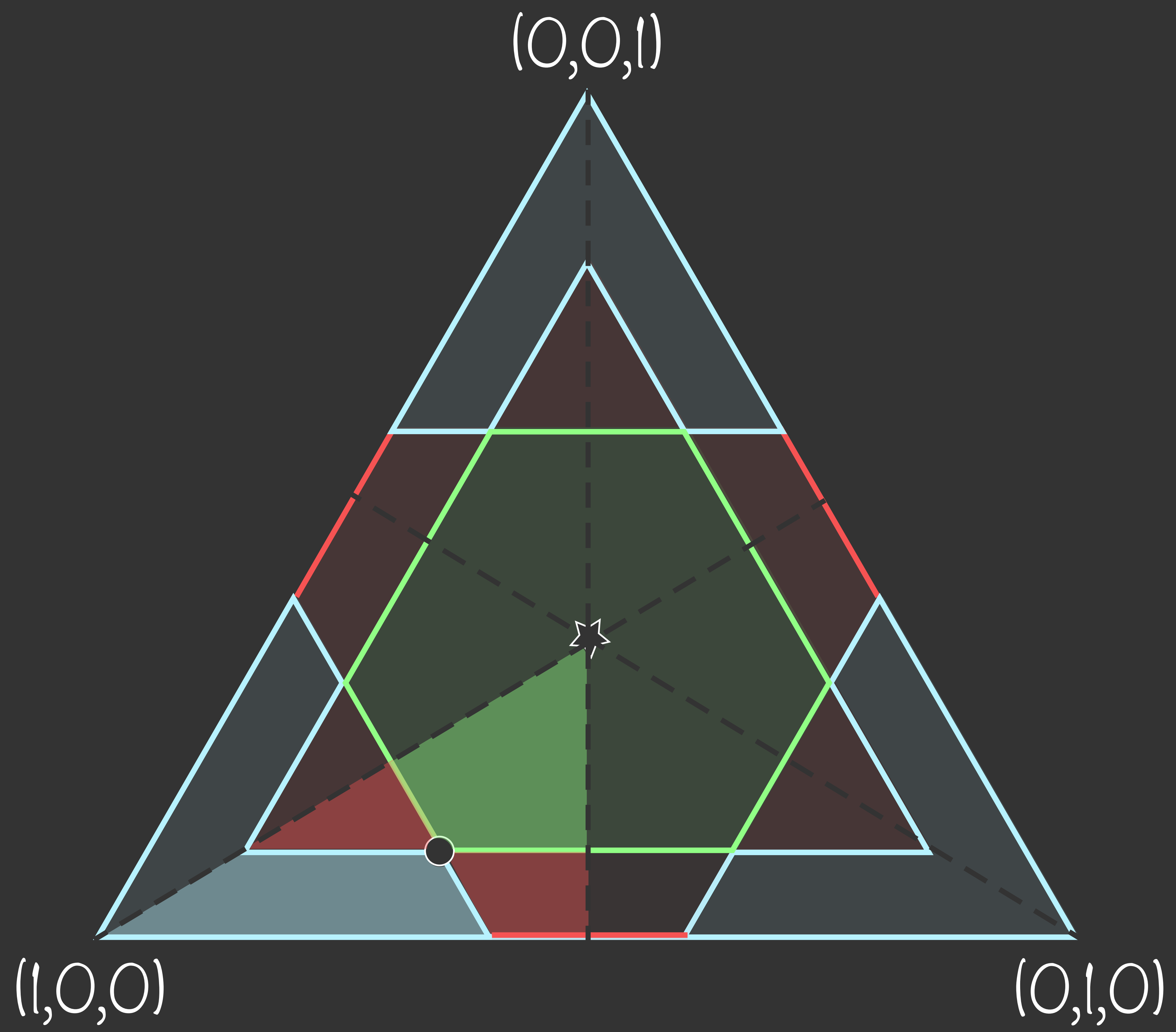

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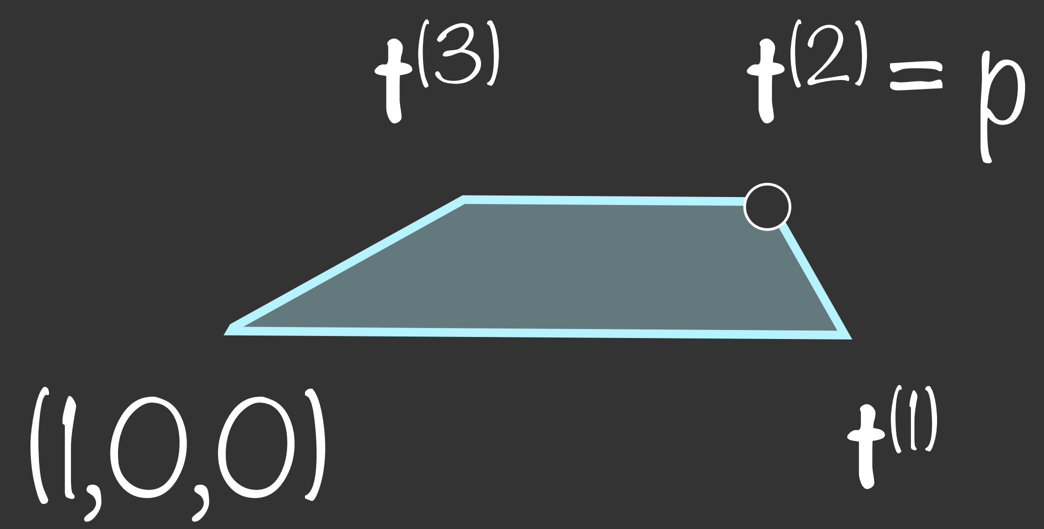
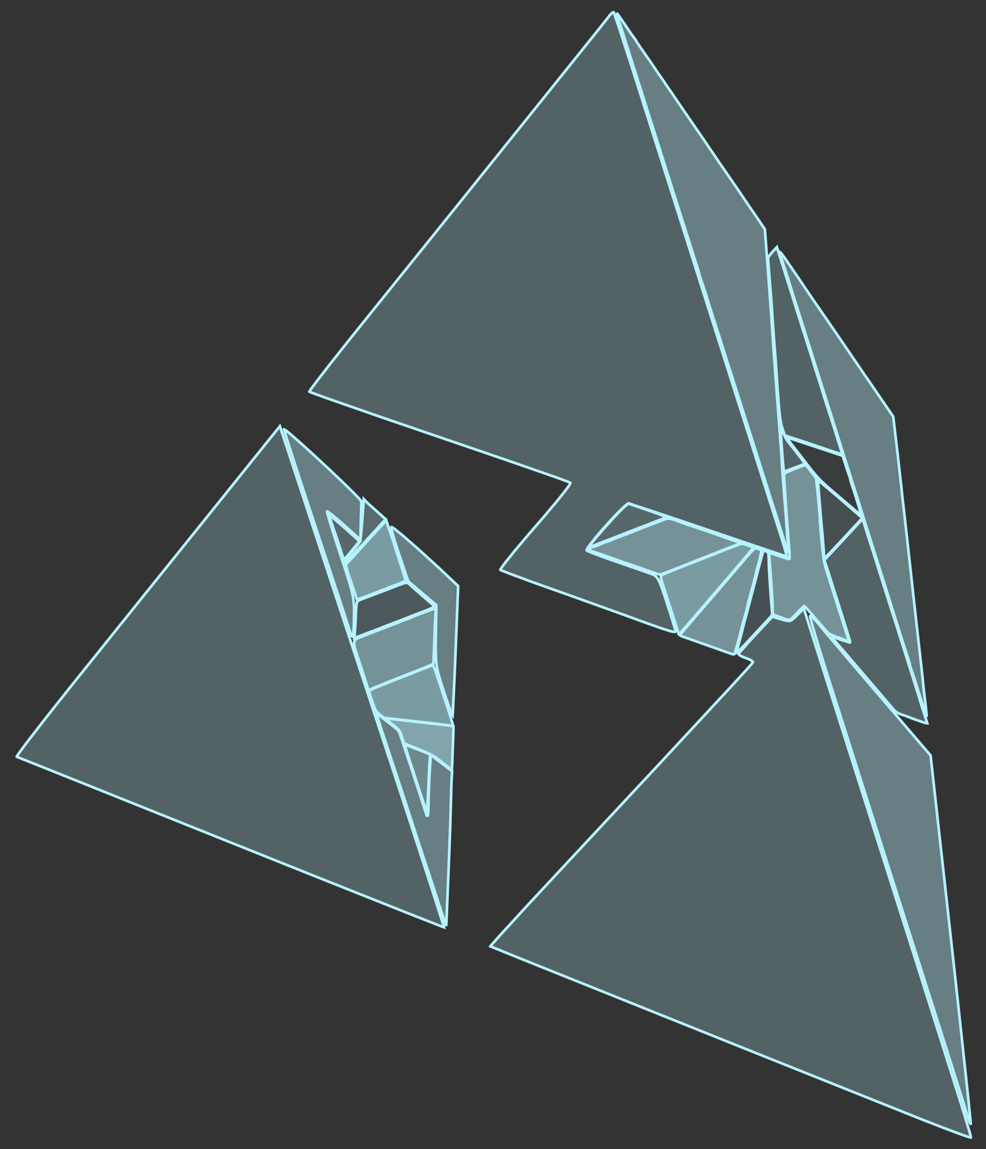
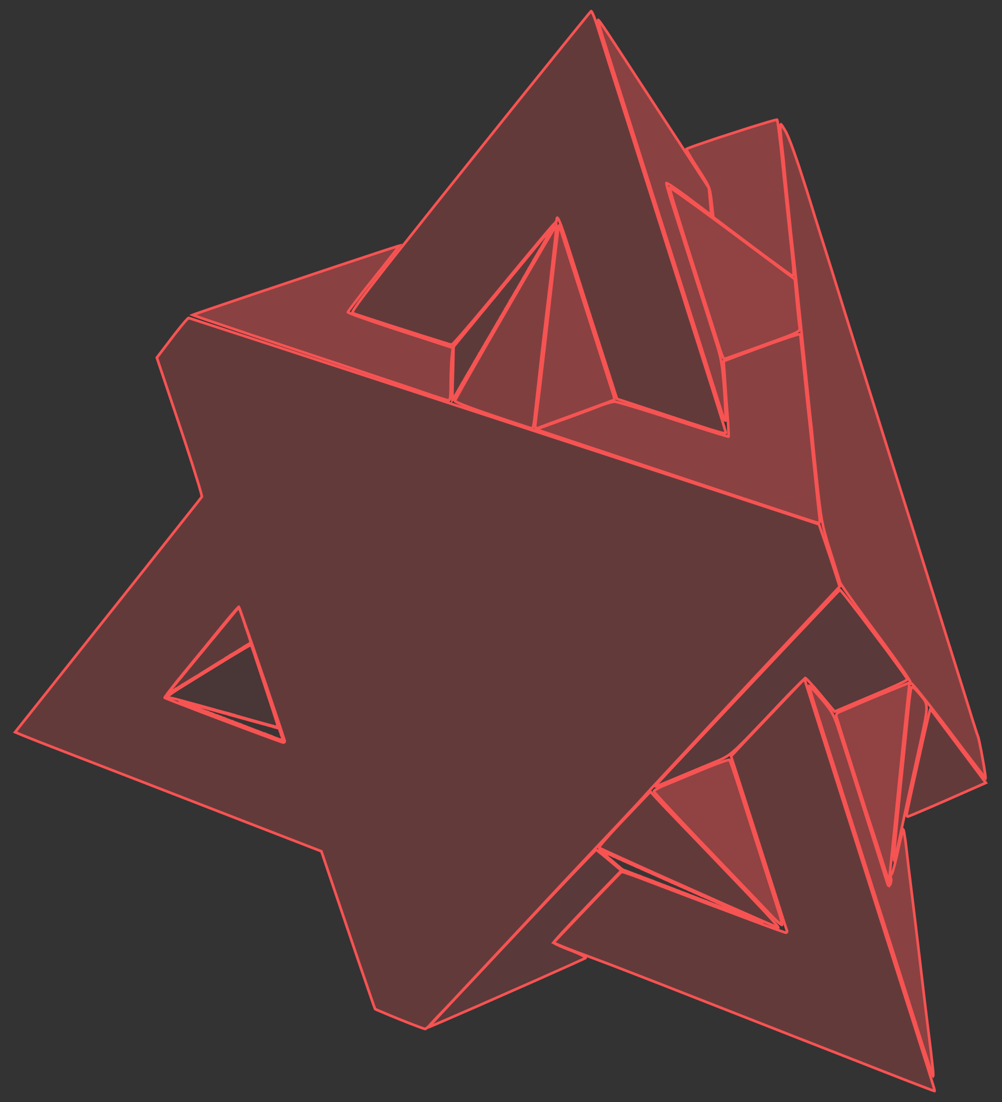


Majorisation cones

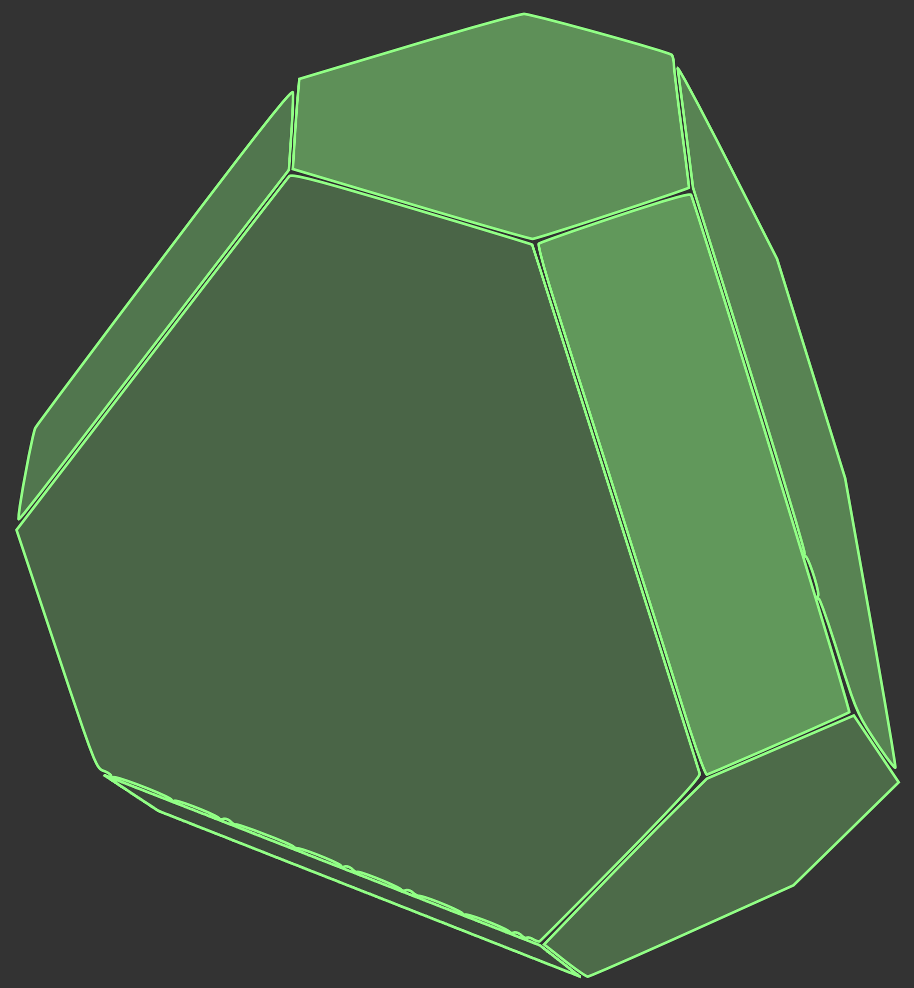
Example. $\mathbf{p} = (0.37, 0.31, 0.23, 0.09)$



Past cone



Incomparable cone



Future cone

Thermal cones

Finite limit temperature $\beta > 0$:

B: The set of bistochastic matrices is a convex set whose extreme points are permutation matrices ✗

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G. Birkhoff, Univ. Nac. Tucumán. Revista A. (1946)

Thm: The set of GP matrices is a convex set whose extreme points are β-permutation matrices ✓

M. Lostaglio, ÁM. Alhambra, C. Perry, Quantum (2018)

P. Mazurek & M. Horodecki New Journal of Physics (2018)

Thermal cones

Finite limit temperature $\beta > 0$:

Thm: For a d-dimensional energy inconherent state **p** with Hamiltonian H, its future thermal cone is given by

$$\mathcal{T}_{+}(\mathbf{p}) = \mathrm{conv}[\{\Pi_i^{\beta}\, \mathbf{p}\ ,\ i \in (1, \ldots d!)\ \}]$$

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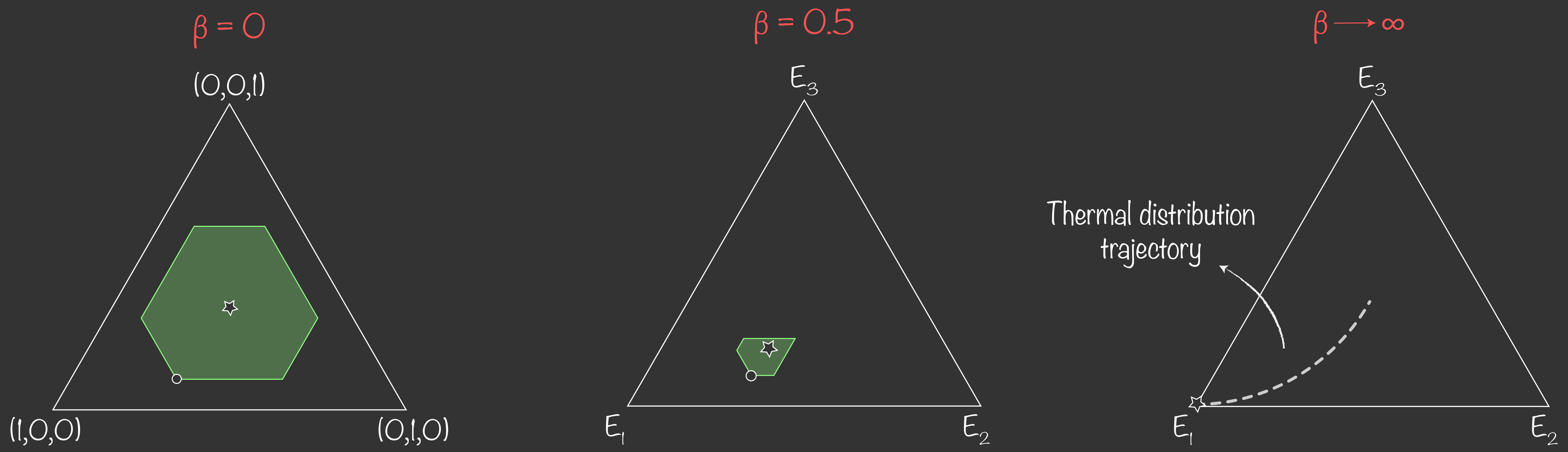

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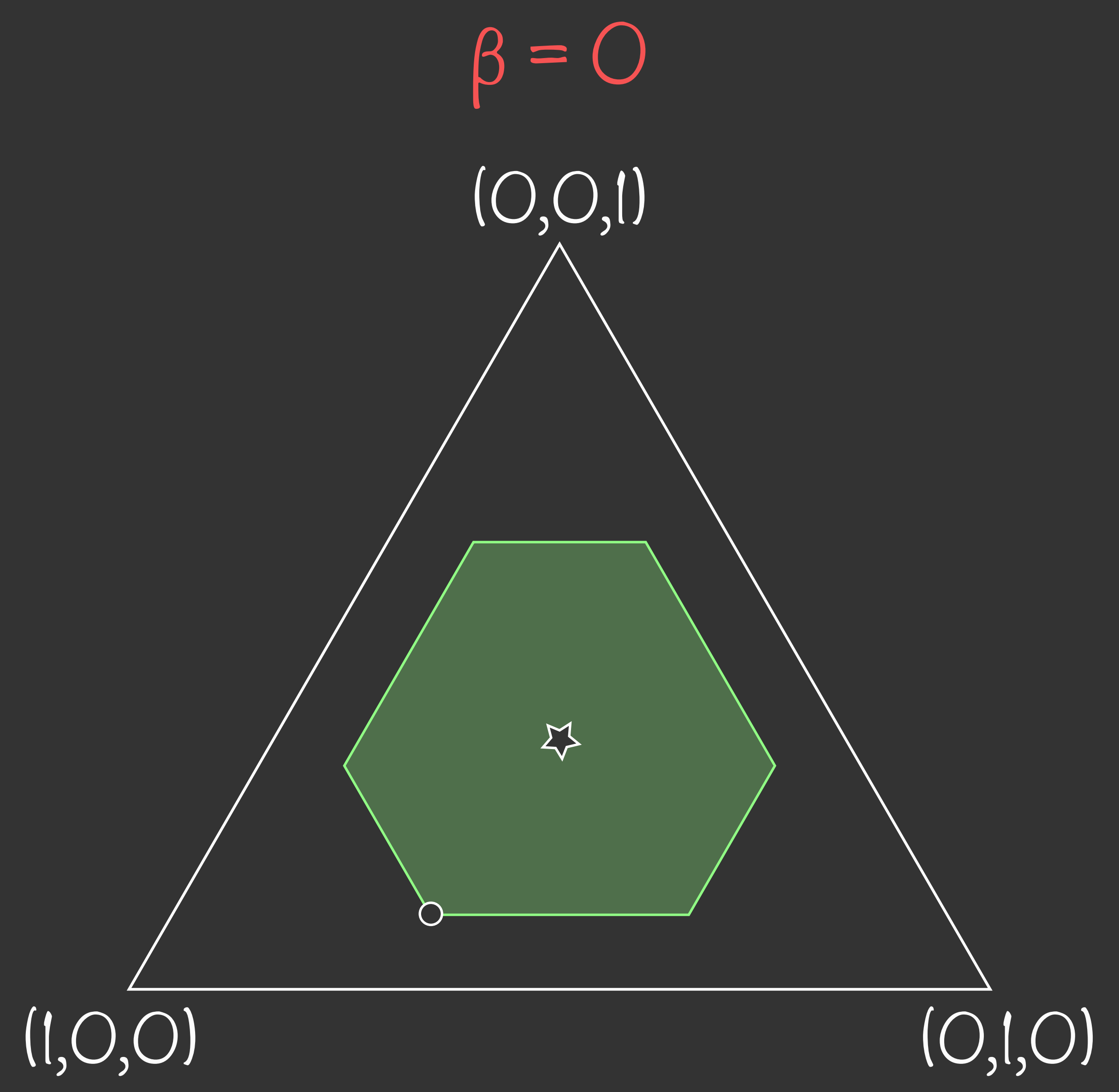
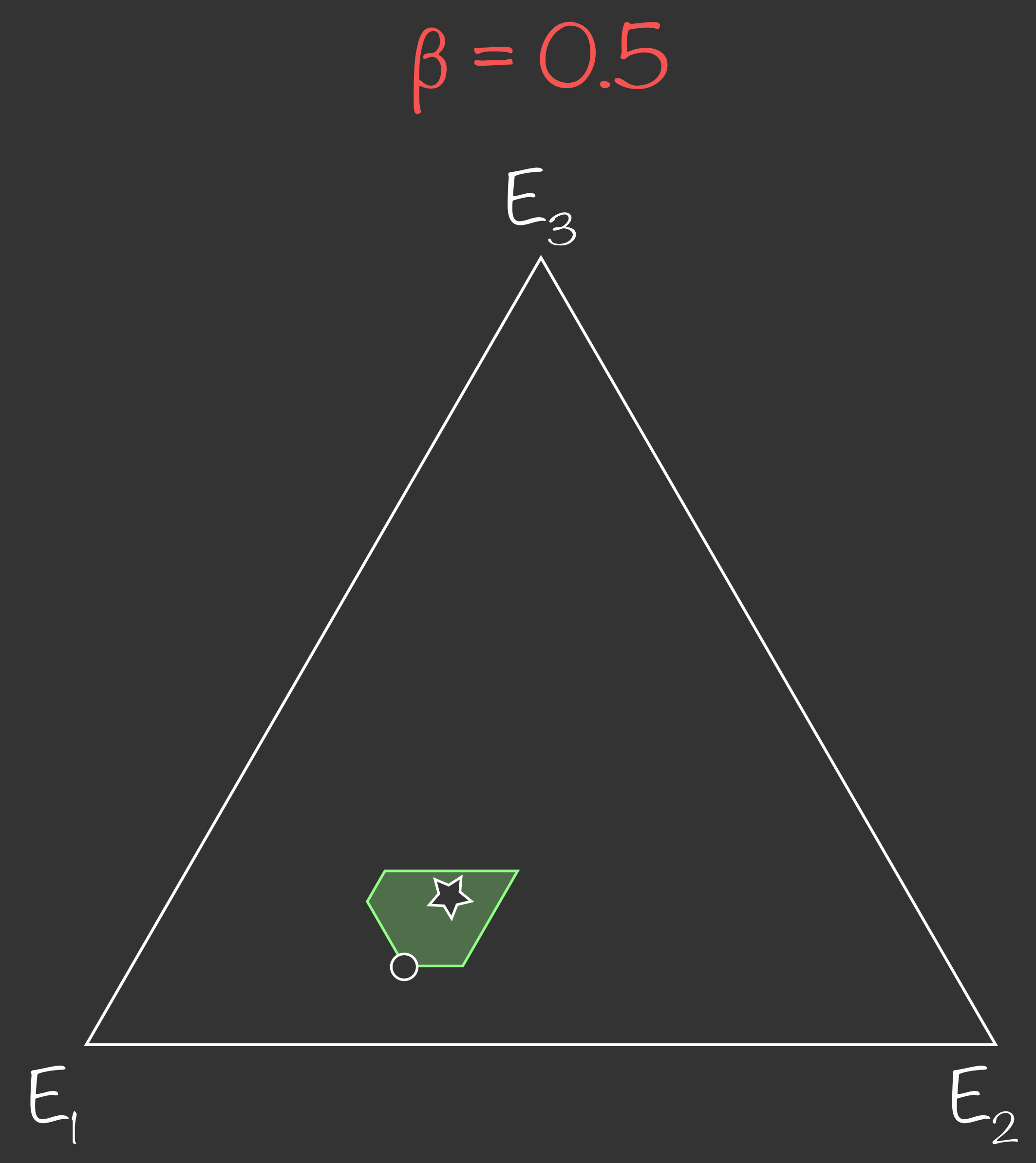




Q. How to characterise the incomparable region and past thermal cone of **p**?

Thermal cones

Lemma: Given an energy-incoherent state $\mathbf{p}$ and a thermal state $\boldsymbol{\gamma}$ consider the distribution $\mathbf{t}^{(n,\pi)}$ in their β-ordered form, constructed for each permutation permutation $\pi \in S_d$,

$$[\mathbf{t}^{(n,\pi)}]^\beta = \left(t_{\pi(1)}^{(n,\pi)},\ \frac{p^\beta_n \gamma_{\pi(2)}}{\gamma^\beta_n}, \dots, \frac{p^\beta_n \gamma_{\pi(d-1)}}{\gamma^\beta_n},\ t_{\pi(d)}^{(n,\pi)} \right)$$

with

$$t_{\pi(1)}^{(n,\pi)} = p^\beta_i - \frac{p^\beta_n}{\gamma^\beta_n}\left(\sum_{i=1}^{n} \gamma^\beta_i - \gamma_{\pi(i)} \right) \quad \text{and} \quad t_{\pi(d)}^{(n,\pi)} = 1 - t_{\pi(1)}^{(n,\pi)} - \frac{p^\beta_n}{\gamma^\beta_n}\left(\sum_{i=2}^{d-1} \gamma_{\pi(i)} \right).$$

Define the set $T^\beta = \bigcup_{\pi \in S_d} \bigcup_{j=1}^{d-1} \mathrm{conv}[\mathcal{T}_{\neq}^{\beta}(\mathbf{t}^{(j,\pi)}), \mathcal{T}_{\neq}^{\beta}(\mathbf{t}^{(j+1,\pi)})]$, the incomparable region of $\mathbf{p}$ is given by

$$\mathcal{T}_{\emptyset}^{\beta}(\mathbf{p}) = [\mathrm{int}(T^\beta) \setminus \mathcal{T}_{\neq}^{\beta}(\mathbf{p})] \cap \Delta_d$$

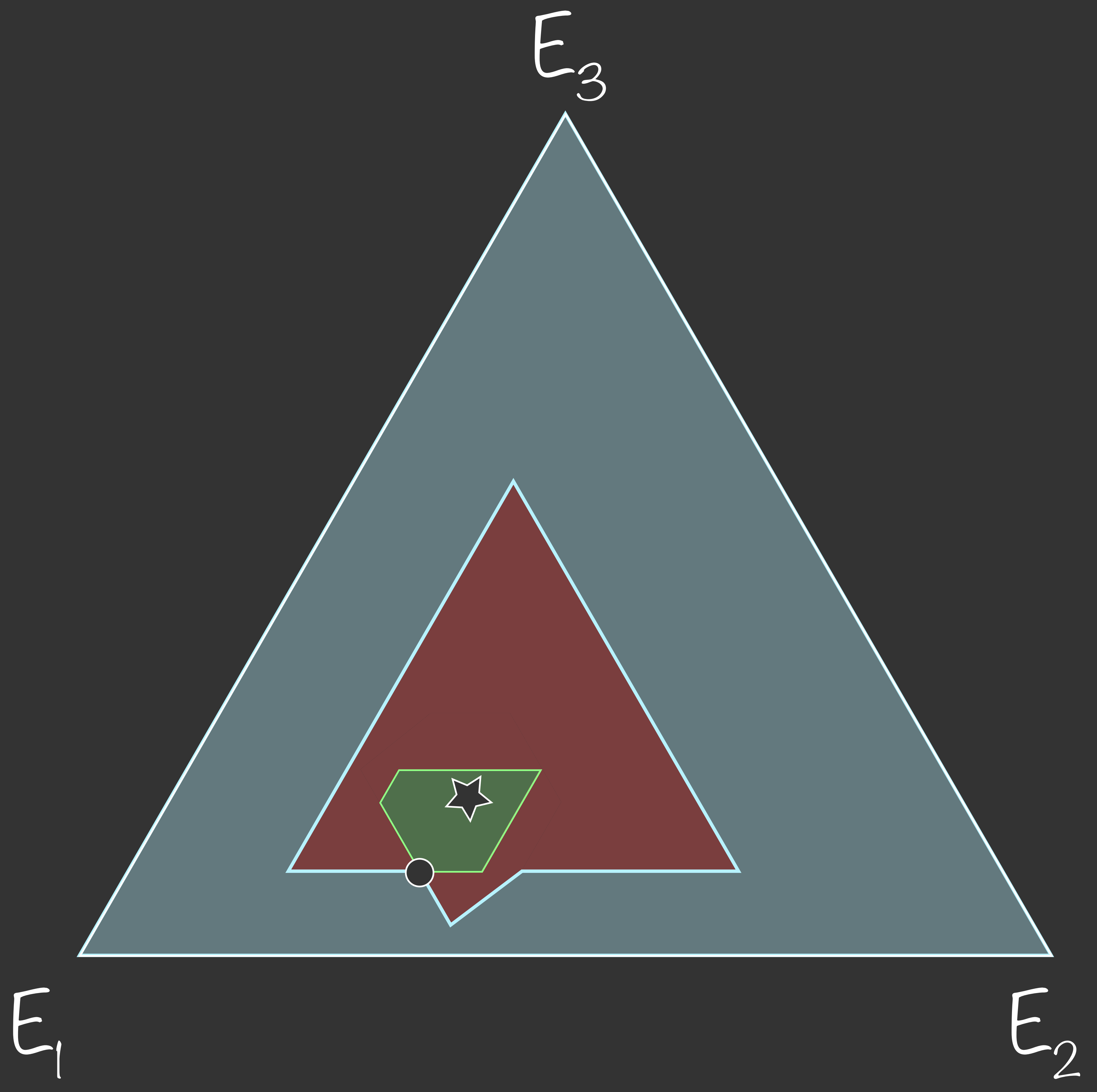
Thermal cones

Theorem: The past thermal cone of $\mathbf{p}$ is given by

$$\mathcal{T}_{-}^{\beta}(\mathbf{p}) = \Delta_d \setminus \mathrm{int}(T^{\beta})$$

Thermal cones

Example. $\mathbf{p} = (0.6, 0.3, 0.1)$ and $\beta = 0.5$



Future thermal cone

$$\mathcal{T}_{+}(\mathbf{p}) = \mathrm{conv}[\{\Pi_i^{\beta}\,\mathbf{p}\,,\ i\in(1,\ldots d!)\}]$$

Incomparable thermal region

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Past thermal cone

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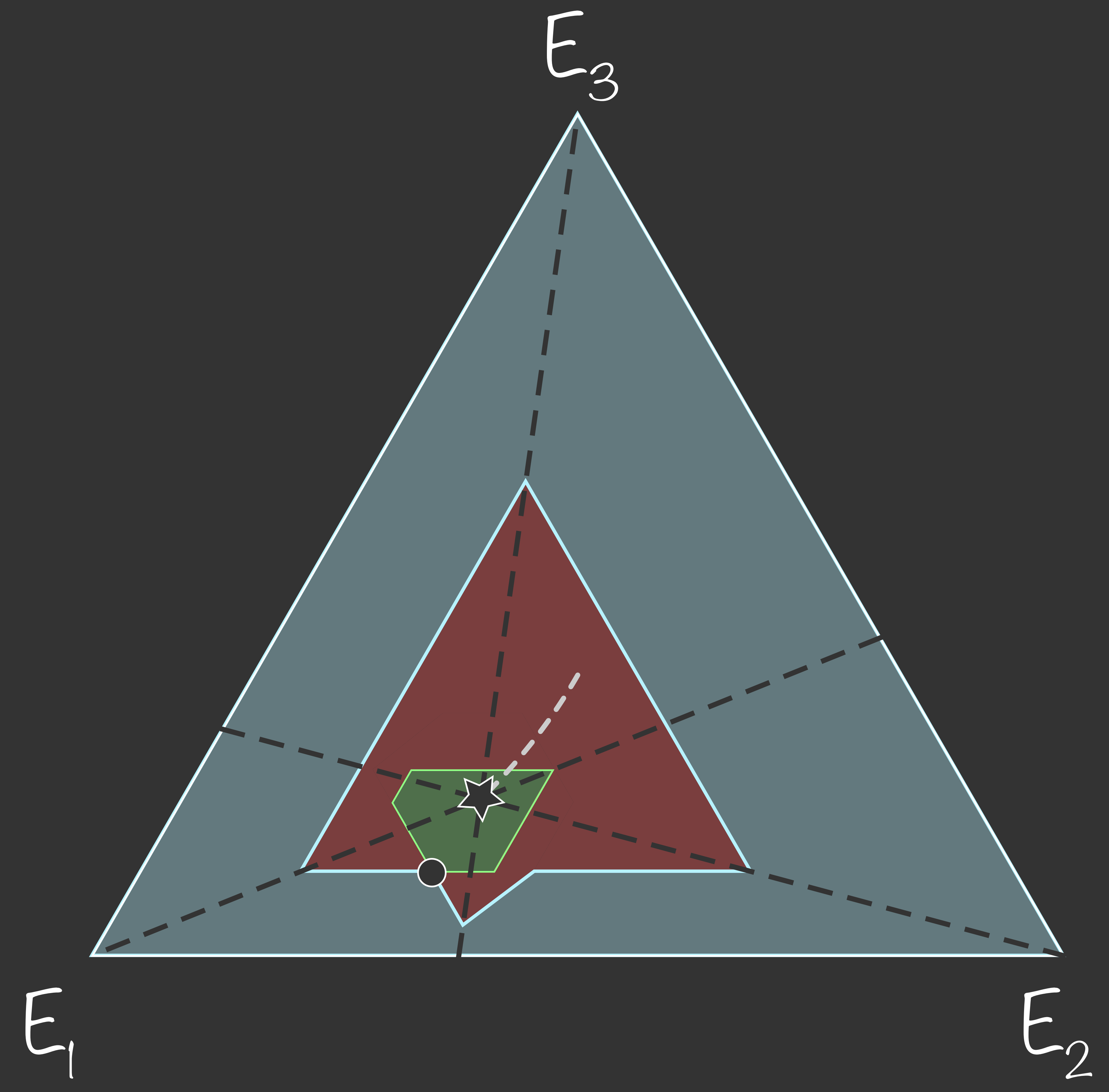
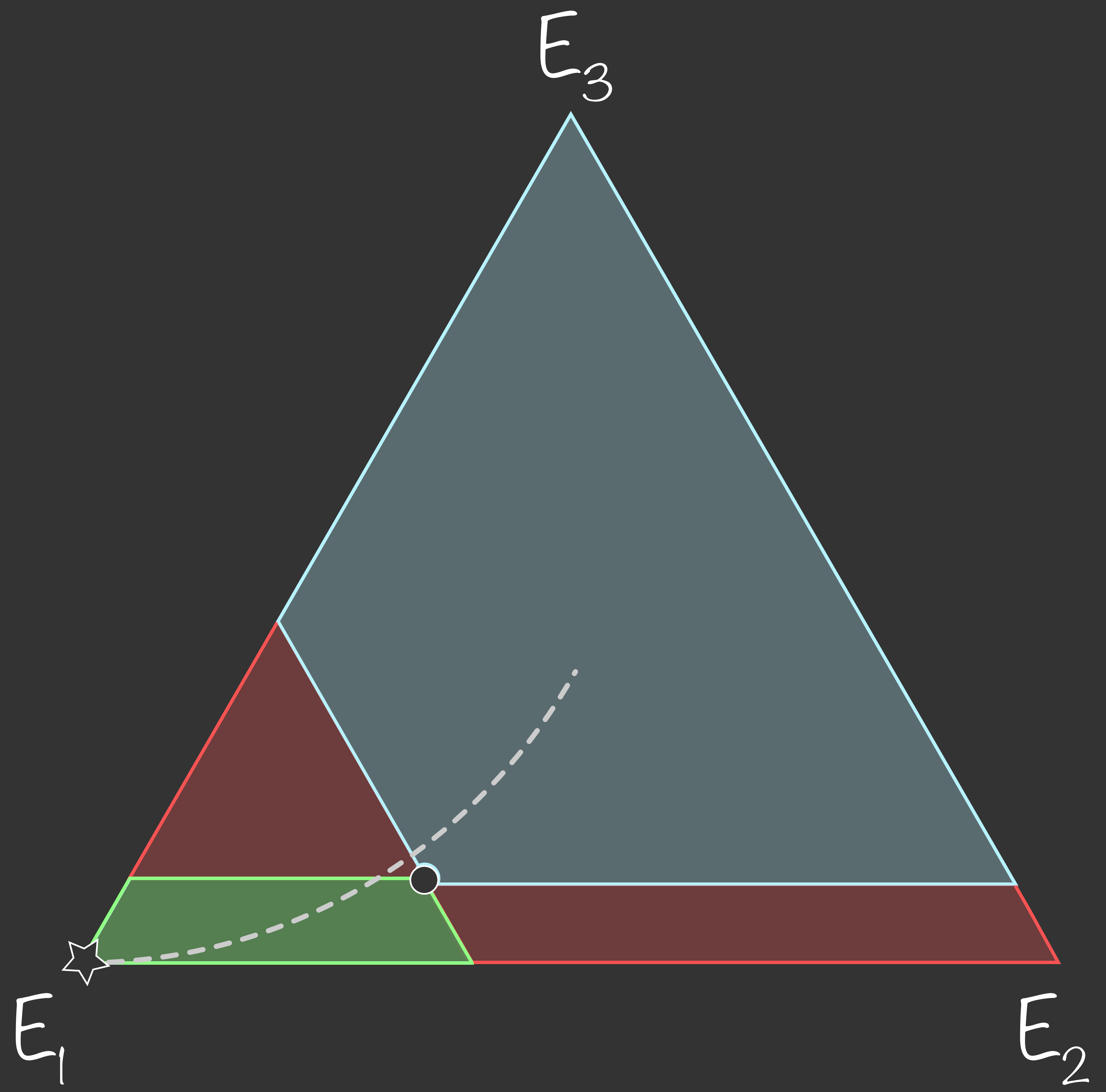


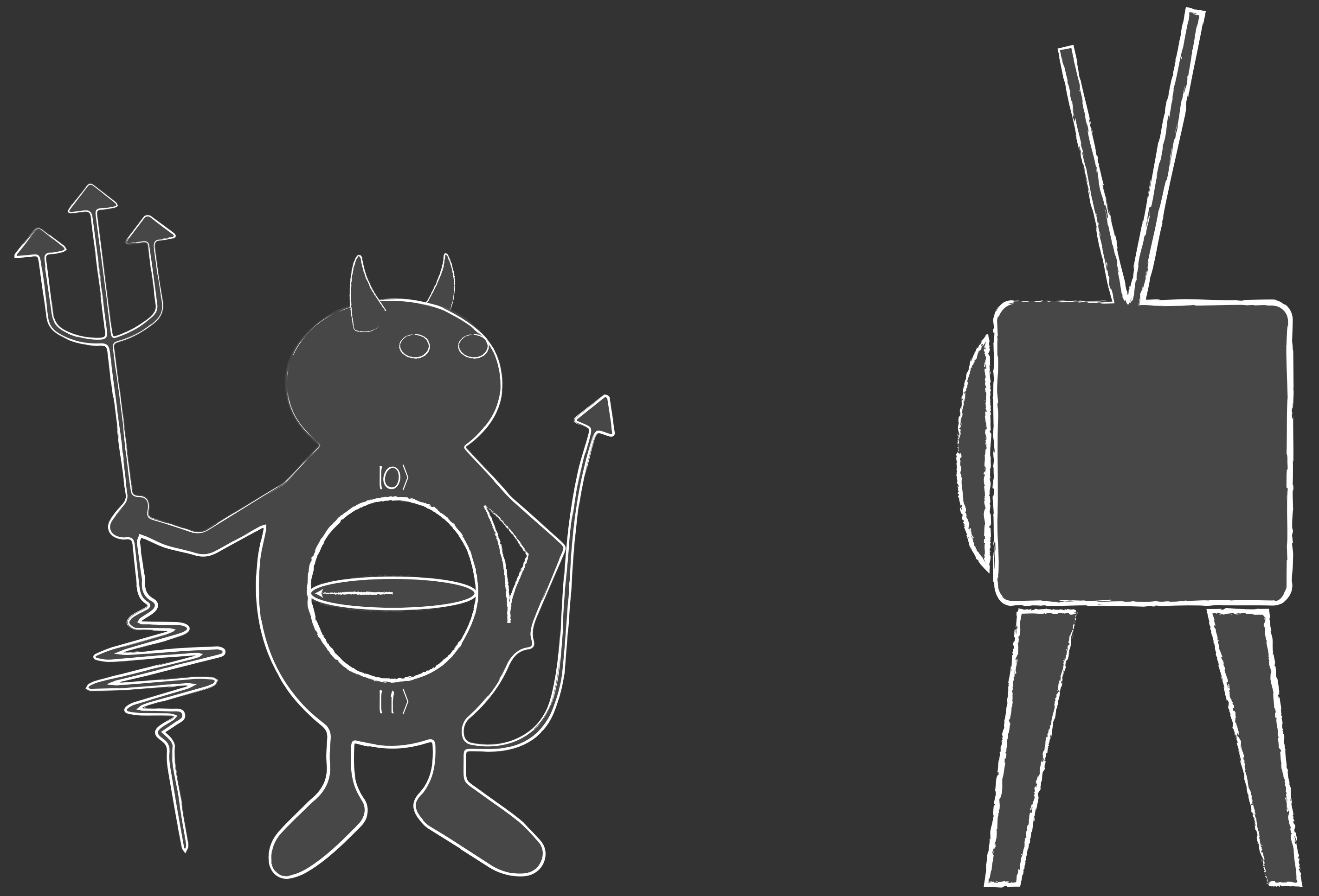
- Only the future is convex, but...

Thermal cones

Example. $\mathbf{p} = (0.6, 0.3, 0.1)$ and $\beta \longrightarrow \infty$



- The past becomes convex!


$|0\rangle$
$|1\rangle$

Outlook

What I didn't about:

1. Our results also apply to entanglement and coherence!
2. Probabilistic transformations

Open problems

1. Markovian thermal cones
2. Coherent thermal cones

See more about thermal cones:

github.com/AdeOliveiraJunior/Thermal-Cones

$\mathbf{p} = (0.7, 0.2, 0.1)$, $E = (0, 1, 2)$

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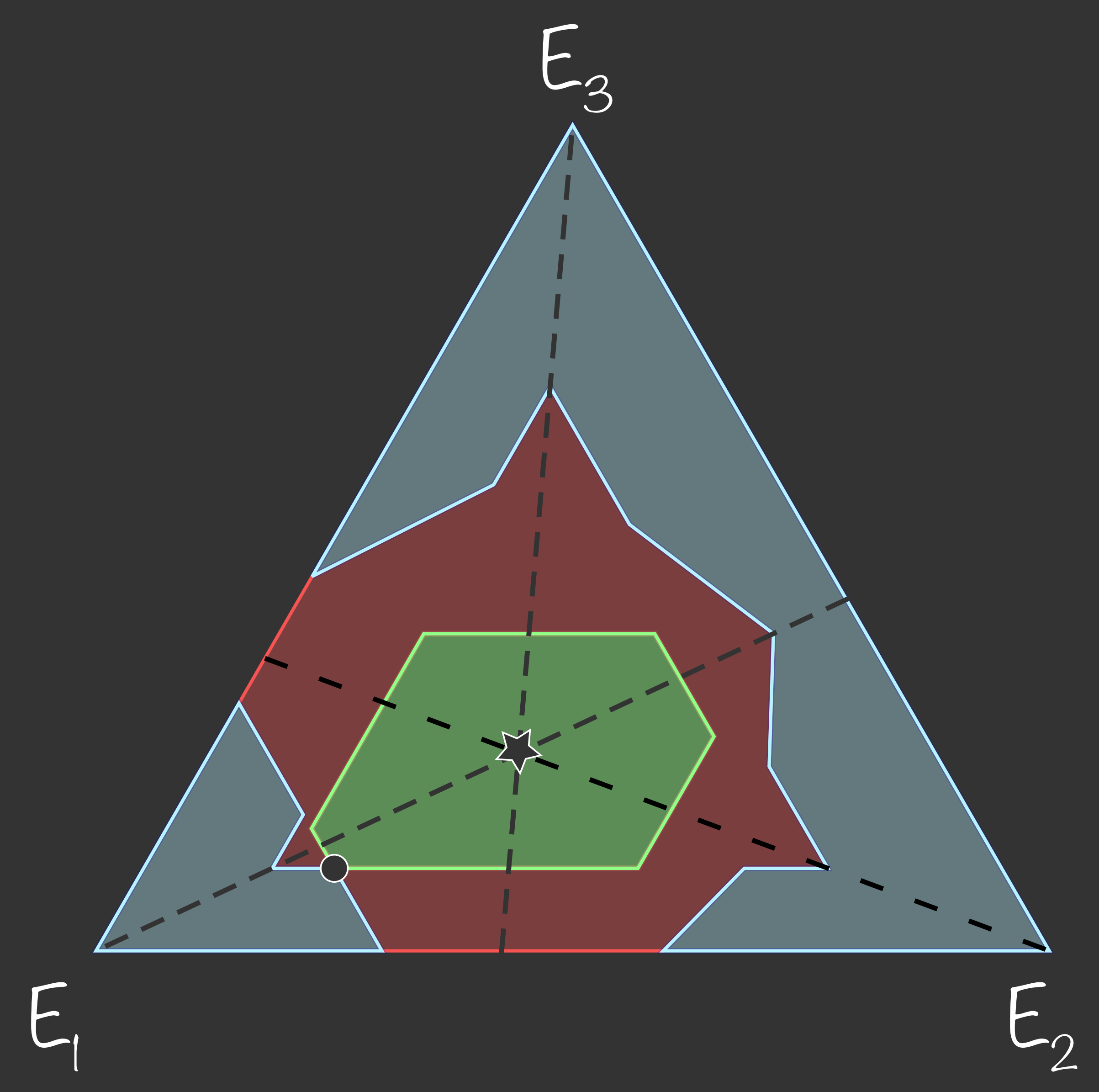

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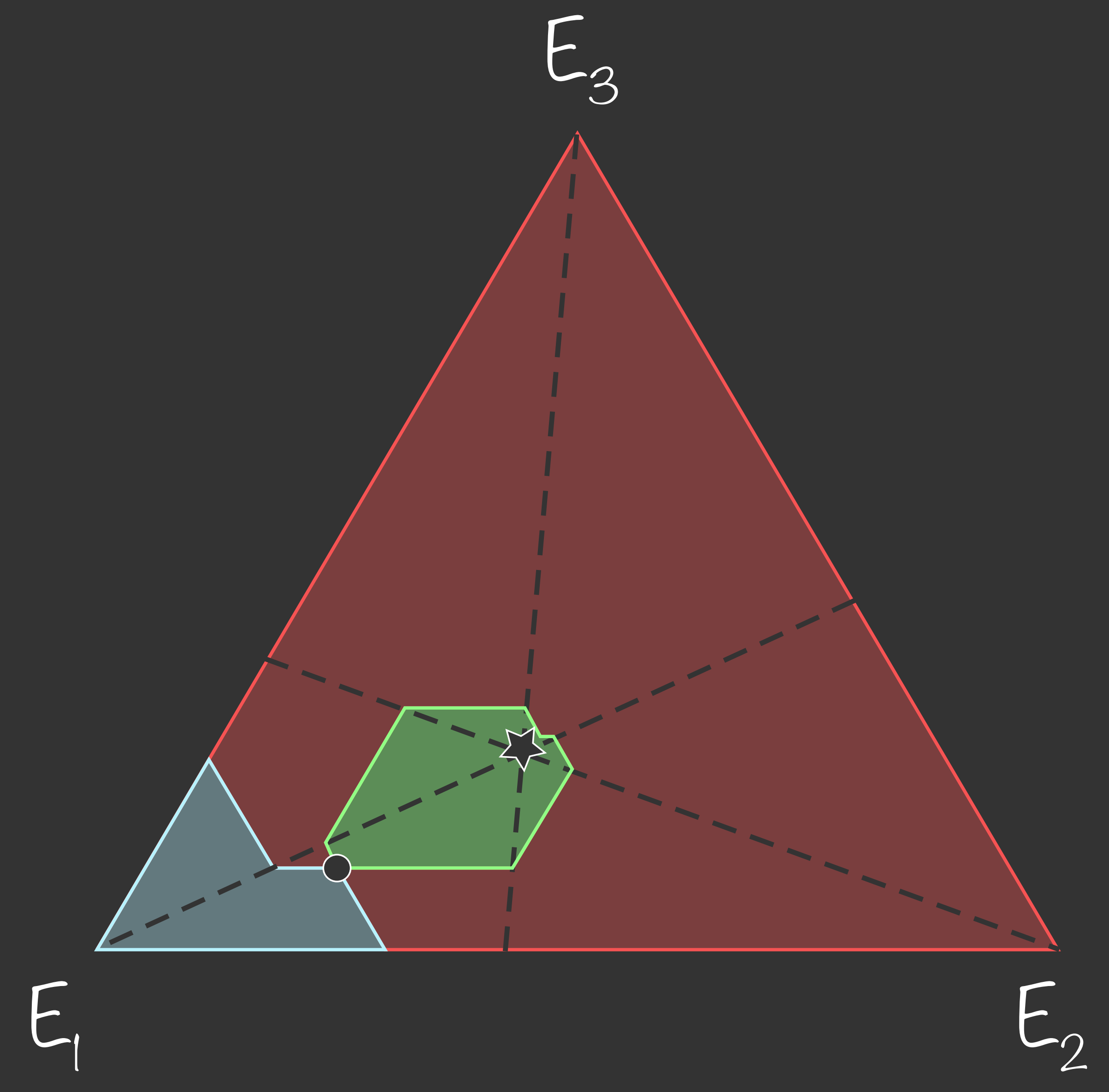

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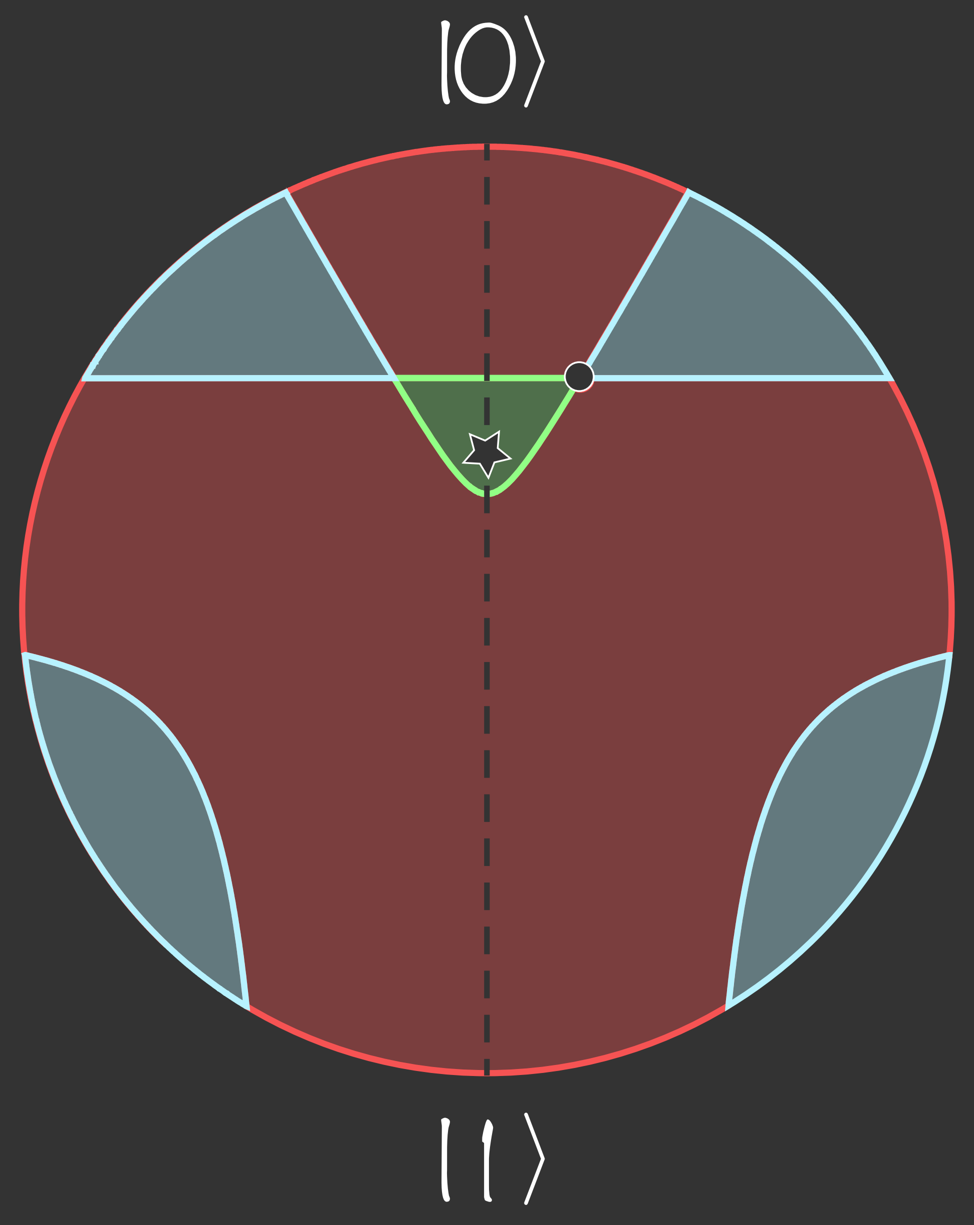
$$\mathbf{r}_\rho = (0.2, 0, 0.5), \ \mathbf{r}_\gamma = \left(0, 0, \frac{1}{3}\right)$$



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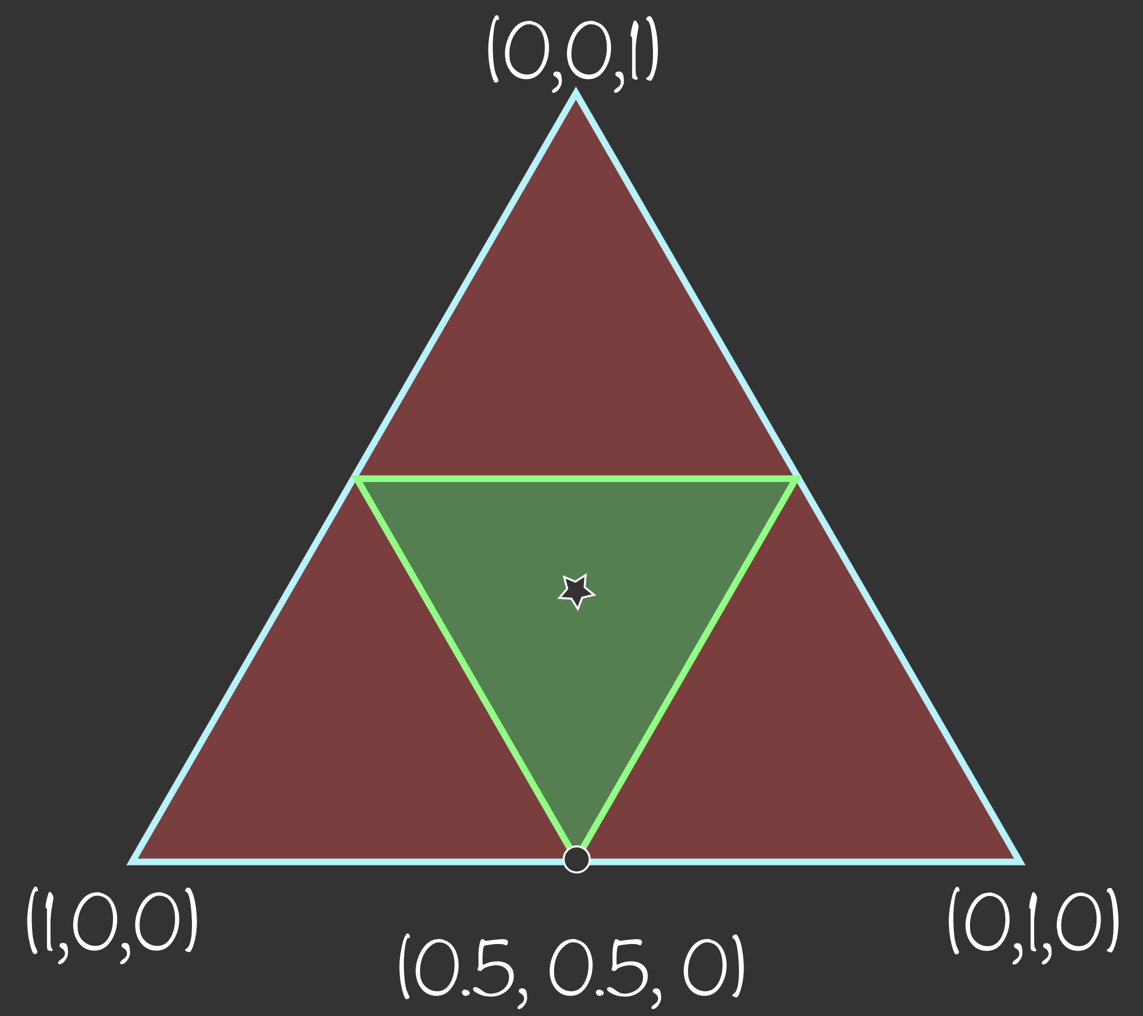

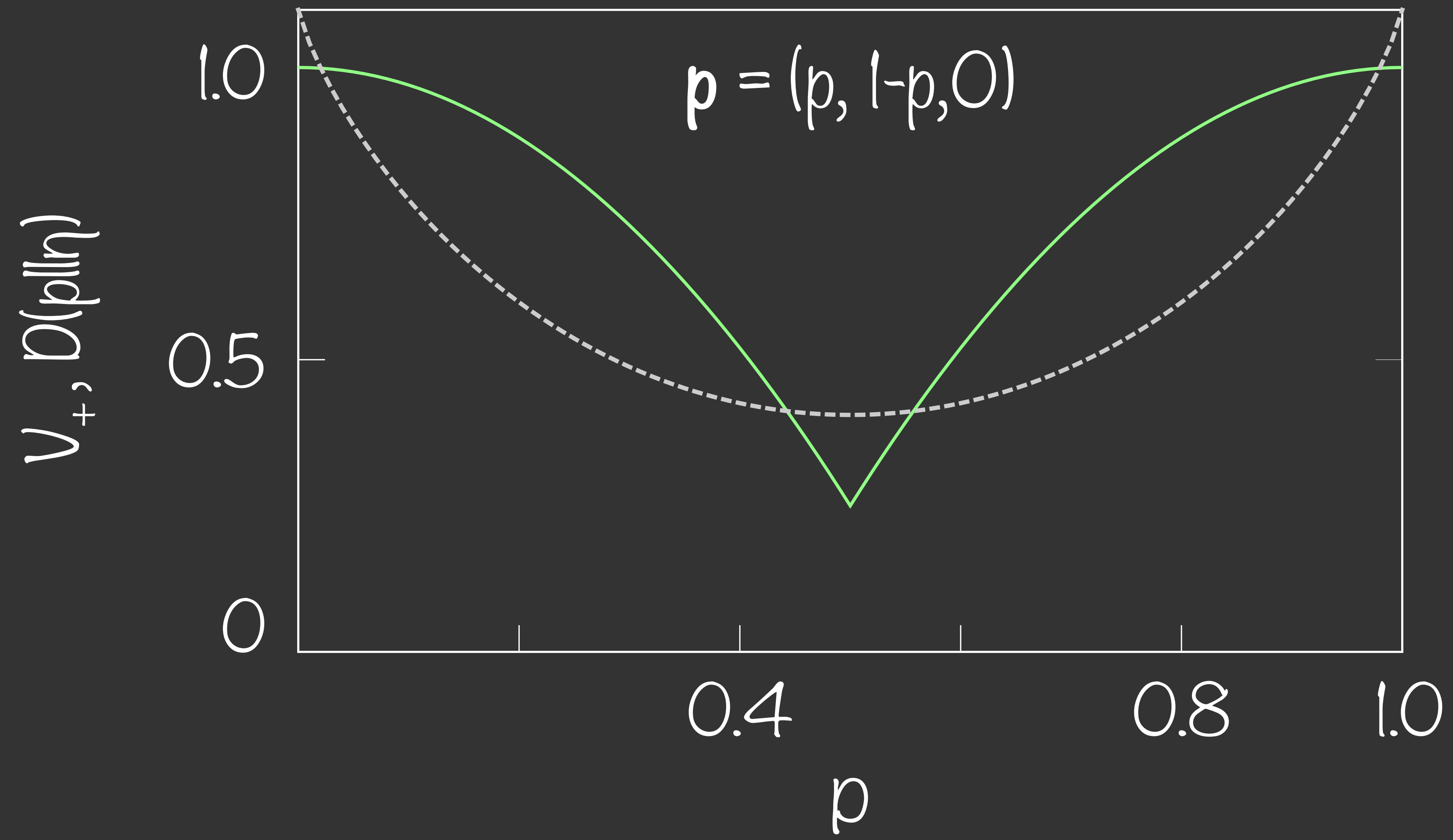

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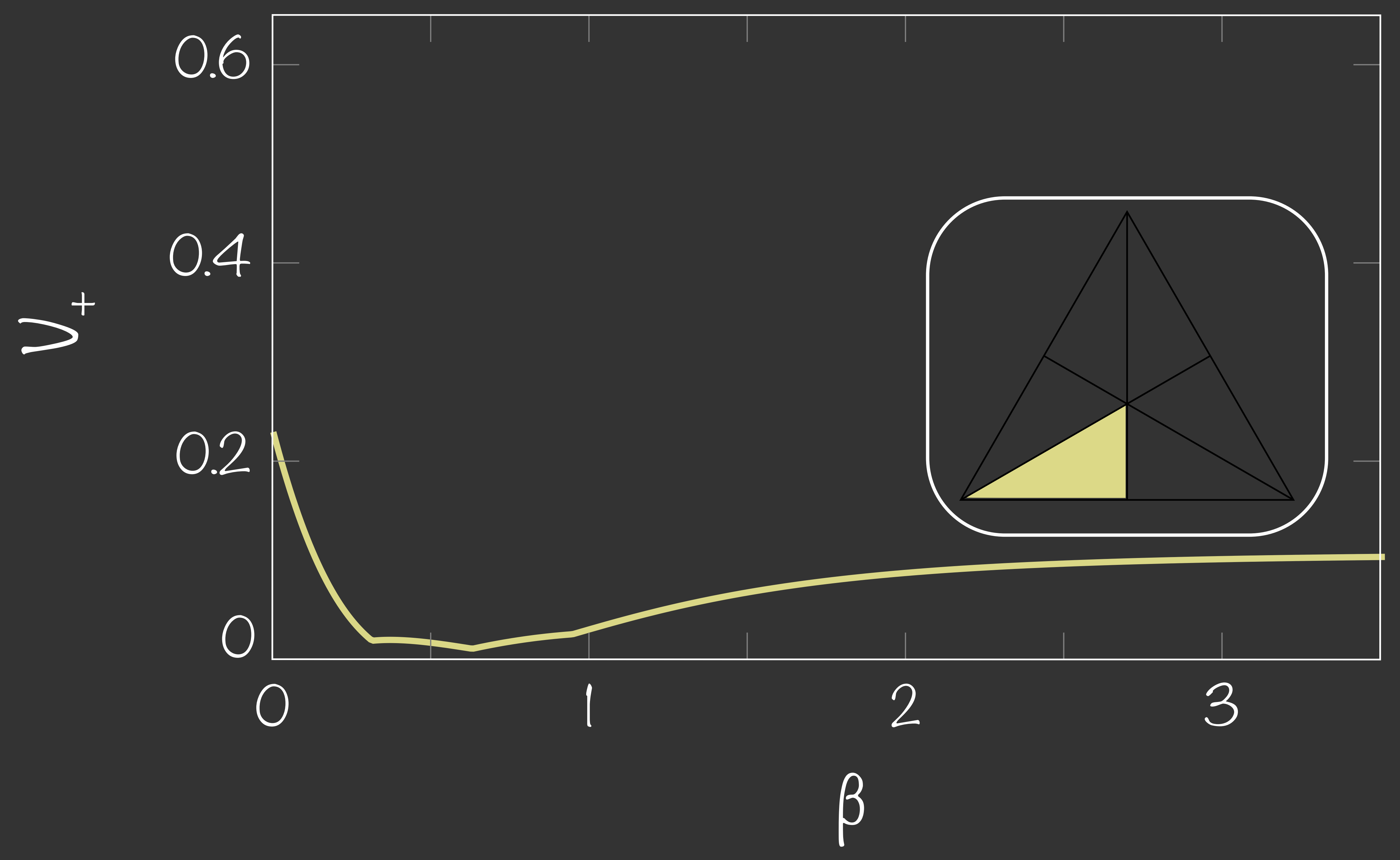

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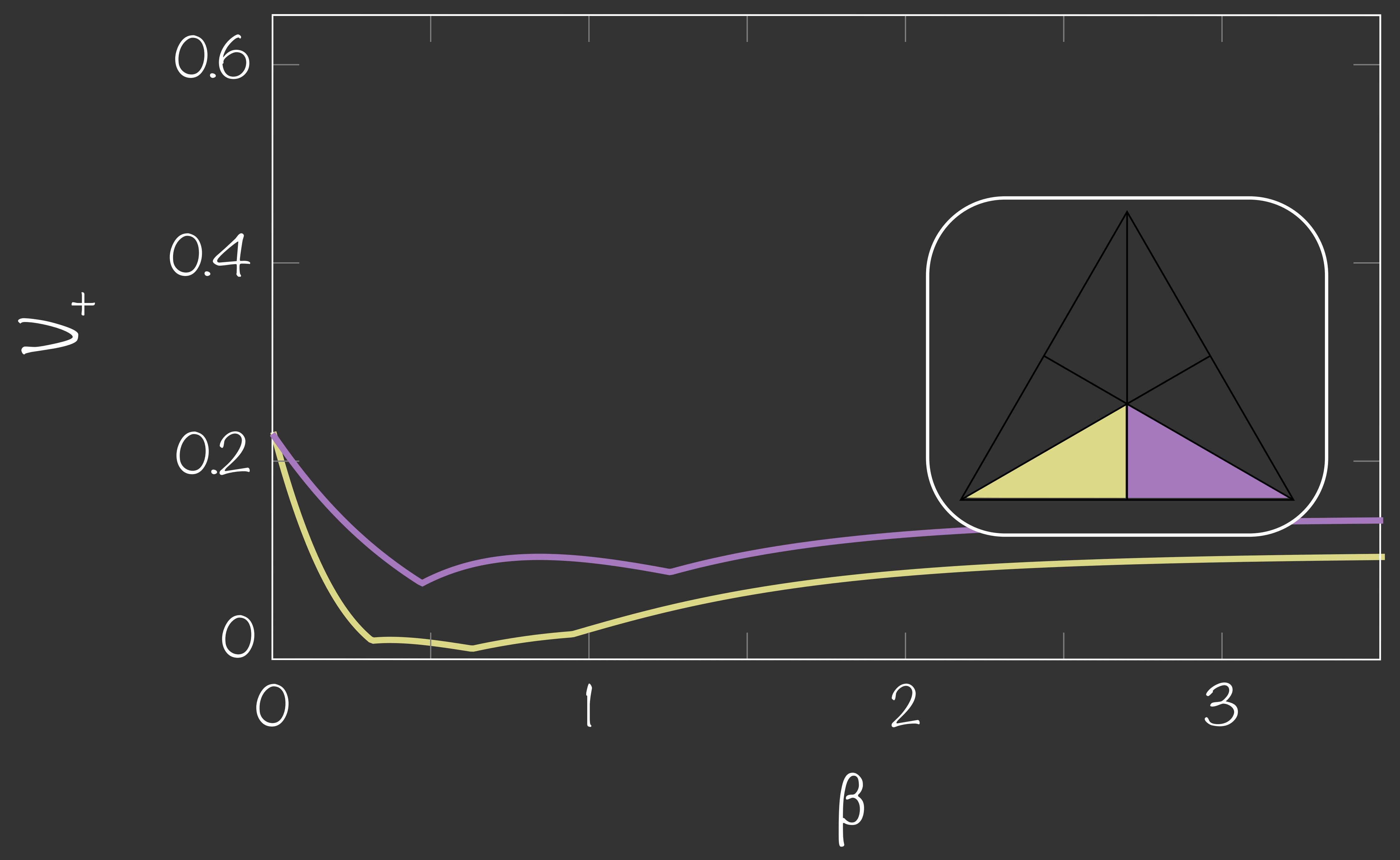

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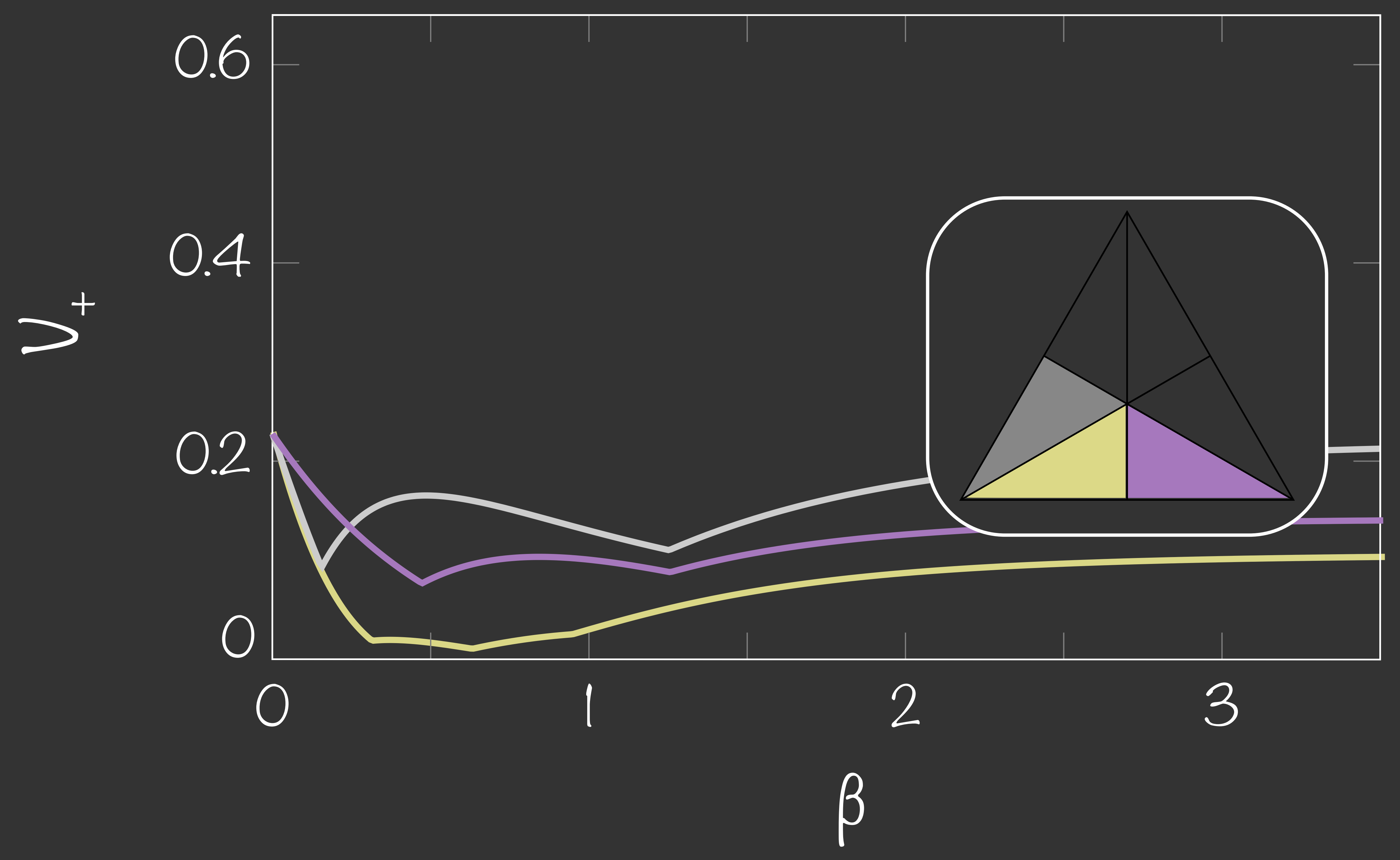

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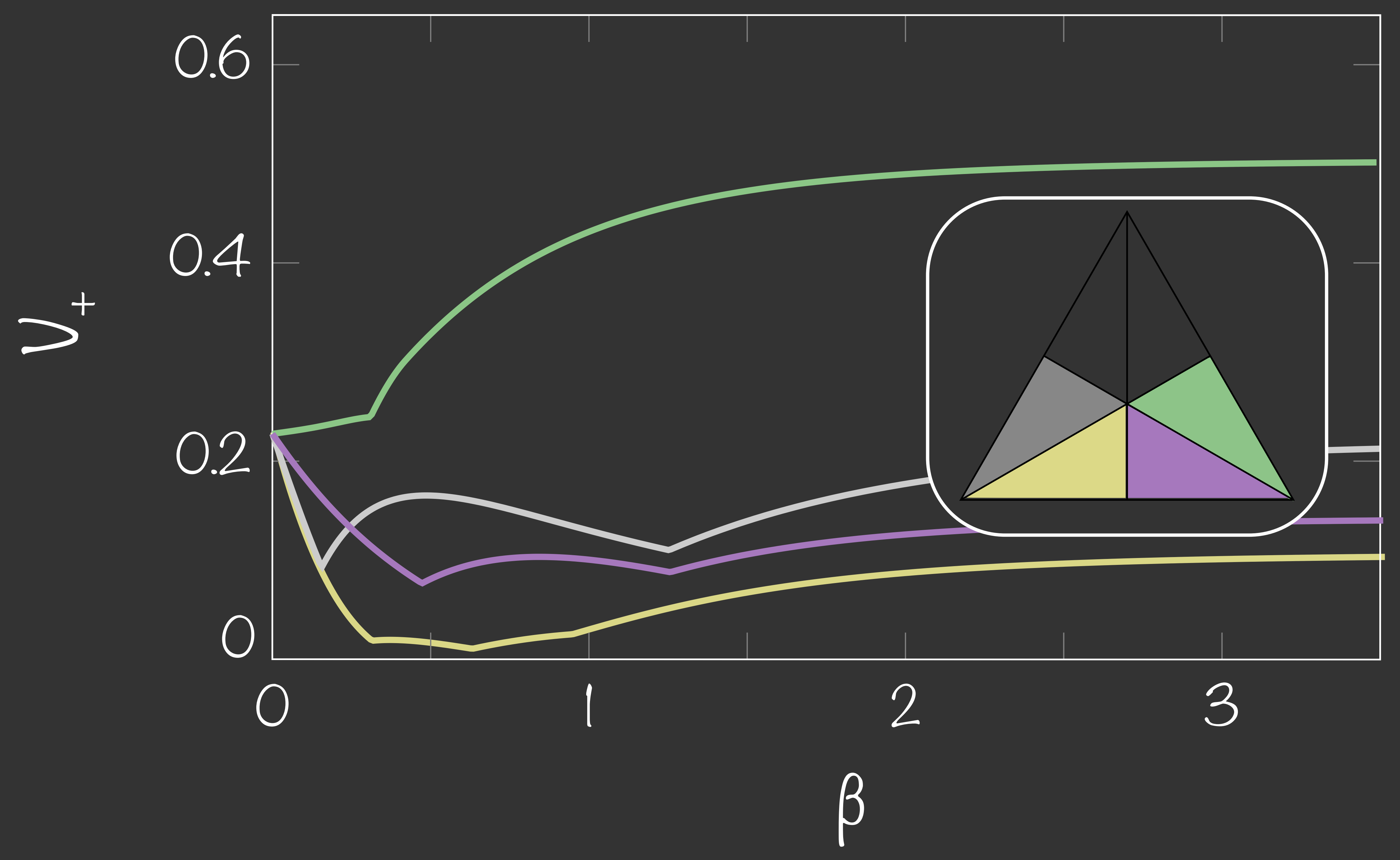

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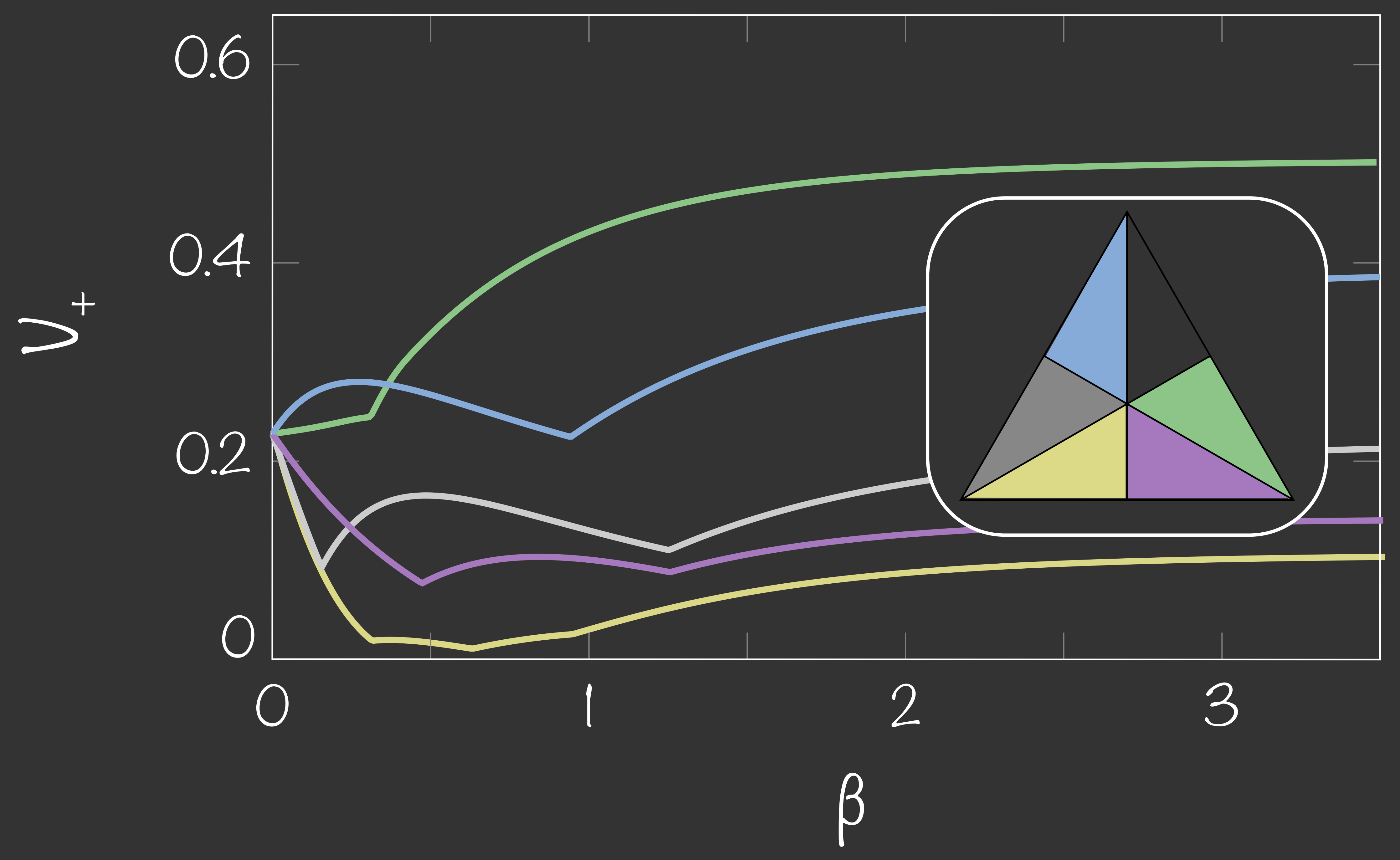

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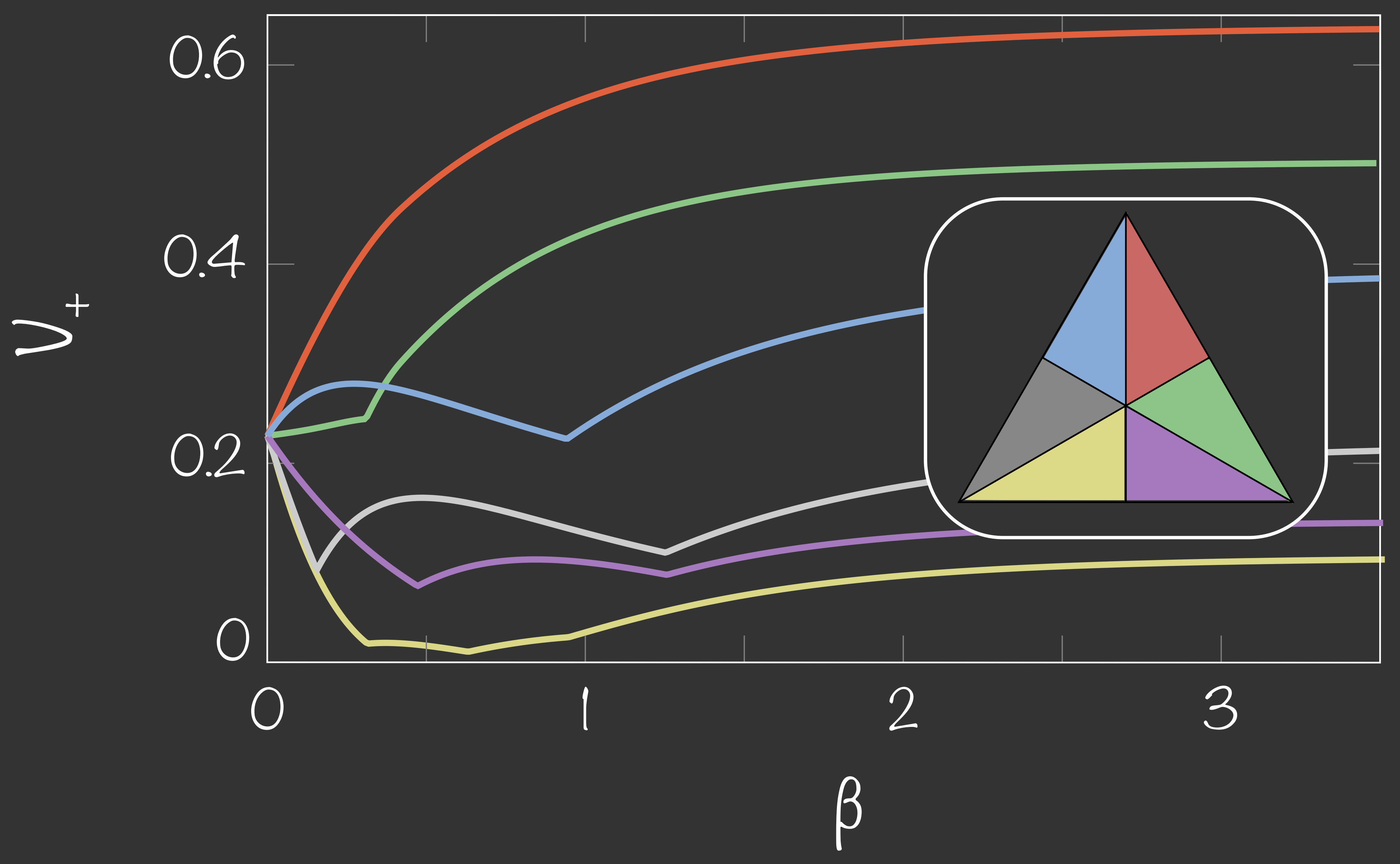

- The passive and maximally active states have minimum and maximum volumes among all permutions for a given $\mathbf{p}$.