Mixing of passive scalars advected by incompressible fields

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Mixing

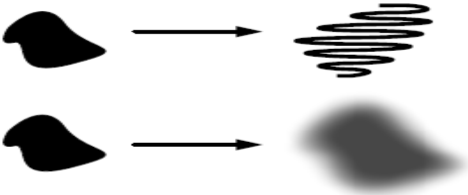


Figure: The transformation at the top shows mixing by homogenization and the bottom shows mixing by filamentation.

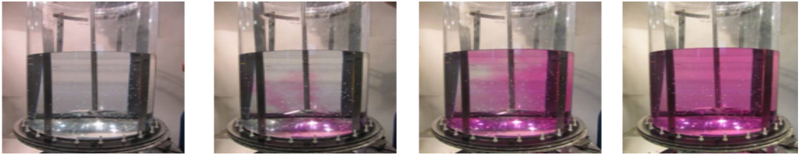


Figure: Mixing by homogenization.

Aim & Problem Statement

The questions that begs for answers are:

- How well can passive scalars be mixed given constraints on the advecting velocity?
- What is the optimal incompressible flow?

Mixing of a passive scalar $\Psi(x,t)$ by a smooth incompressible flow field $u(x,t)$ is described by the advection-diffusion equation below:

$$\frac{\partial \Psi}{\partial t} + u \cdot \nabla \Psi = k \Delta \Psi, \quad \Psi(x,0) = \Psi_0(x) \tag{0.1}$$

$$\nabla \cdot u = 0. \tag{0.2}$$

Given the following constraints on the velocity field

- Energy constraint: $\|u(x,t)\|_{L^2}^2 = U^2 L^d$.
- Enstrophy constraint: $\|\nabla u\|_{L^2}^2 = F$.

Variance as a measure of mixing

- $$\text{var }\Psi = ||\Psi||^2_{\mathrm{L}^2} - \langle\Psi\rangle^2 \tag{0.3}$$

  where $||\Psi||^2 = \frac{1}{|\Omega|}\int_\Omega \Psi^2 \, \mathrm{d}\Omega$, $\langle\Psi\rangle = \frac{1}{|\Omega|}\int_\Omega \Psi \, \mathrm{d}\Omega$.
- Advection-diffusion equation

  $$\frac{\partial\Psi}{\partial t} + u.\nabla\Psi = k\Delta\Psi. \tag{0.4}$$

- Multiply (0.11) by Ψ and then integrate by parts and using the incompressibility condition, we have: $\frac{d}{dt}\langle\Psi\rangle = 0$,
  $\frac{d}{dt}||\Psi||^2_{L^2} = -2k||\nabla\Psi||^2_{L^2}$
  $\Longrightarrow$

  $$\frac{d}{dt}\text{var }\Psi = -2k||\nabla\Psi||^2_{L^2}. \tag{0.5}$$

Better Measure: H^{-a} norm

Zin Lin et al [5] introduced the mix-norm $||\Psi(.,t)||_{H^{-a}}$, $a > 0$.

Definition: Negative Sobolev Norm

The negative Sobolev norm of Ψ is given as

$$\|\Psi(.,t)\|^2_{H^{-a}} = \||\nabla|^{-a}\Psi(.,t)\|^2_{L^2} = \sum_{\mathbf{k}\neq 0} k^{-2a}|\hat{\Psi}_{\mathbf{k}}(t)|^2.$$

$k = |\mathbf{k}|$ is the wave number and $|\hat{\Psi}_{\mathbf{k}}(t)|$ is the Fourier amplitude where $\hat{\Psi}_{\mathbf{k}}(t) = \frac{1}{L^{\frac{d}{2}}}\int_{[0,L]^d} e^{-i\mathbf{k}x}\Psi(x,t)\,\mathrm{d}x$.

Mixing Without Diffusion

Zin Lin et al [5] calculated the absolute limits on mixing as follows.

$$\partial_t \Psi + u \cdot \nabla \Psi = 0 \tag{0.6}$$

Multiply Eq (0.6) with $-\Delta^{-1}\Psi$ and integrate over the domain and use integration by parts, with the incompressibility condition, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}||\Psi||^2_{H^{-1}} = -2\int \Psi u \cdot \nabla(\Delta^{-1}\Psi)\ \mathrm{d}x = -2\int \Psi u \cdot \nabla^{-1}\Psi\ \mathrm{d}x.$$

With **energy constraint** $\|u(x,t)\|^2_{L^2} = U^2 L^d$ and using Cauchy-Schwartz inequality, we have

$$\|\Psi(.,t)\|_{H^{-1}} \geqslant \|\Psi(.,0)\|_{H^{-1}} - UL^{\frac{d}{2}}\|\Psi_0\|_{L^\infty} t. \tag{0.7}$$

The lower bound implies that perfect mixing is possible in finite time.

Mixing Without Diffusion

Gautam Iyer al [2] calculated the absolute limits on mixing as follows. The main idea is to relate the notion of "mixed to scale δ" to the H^{-1} norm, and use Crippa and DeLellis Theorem.

Definition: δ-mixed

Let $k \in (0, \frac{1}{2})$ be fixed. For $\delta > 0$, we say a set $A \subset \mathbb{T}^d$ is δ-semi-mixed if

$$\frac{m(A \cap B(x,\delta))}{m(B(x,\delta))} \leqslant 1 - k \quad \forall x \in \mathbb{T}^d \tag{0.8}$$

Also, A^c is also δ-semi-mixed, then we say A is $\underline{\delta - mixed}$.

Lemma

Let $\lambda \in (0,1]$ and $\Psi \in L^\infty(\mathbb{T}^d)$. If for any integer $n > 0$, $k \in \left(0, \frac{\lambda}{1+\lambda}\right)$ there exists an explicit constant $c_0 = c_0(d,k,\lambda,n)$ such that $\|\Psi\|_{H^{-n}} \leqslant \frac{\|\Psi\|_{L^\infty}\delta^{n+\frac{d}{2}}}{c_0}$ implies A_λ is δ-semi-mixed where $A_\lambda = \{\Psi > \lambda\|\Psi\|_{L^\infty}\}$.

Mixing Without Diffusion

Crippa and DeLellis Theorem

Let H to be the left half of the torus, and Γ be the flow generated by an incompressible vector field u. If after time T, the image of H under the flow Γ is δ-mixed, then there exists a constant C_p such that

$$\int_0^T ||\nabla u(., t)||_{L^p} \mathrm{d}t \geqslant \frac{|ln\ \delta|}{C_p}. \tag{0.9}$$

After the relation, we have

$$\|\Psi(t)\|_{H^{-1}} \geqslant \varepsilon_0 r_0^{\frac{d}{2}+1} \|\Psi_0\|_{L^\infty} exp\left(\frac{-c}{m(\Psi_0)^{\frac{1}{2}}} \int_0^t \|\nabla u(s)\|_{L^2} \mathrm{d}s \right). \tag{0.10}$$

With **enstrophy constraint** $\|\nabla u(x, t)\|_{L^2}^2 = F$, the lower bound implies that perfect mixing is not possible in finite time but an exponential decay of the mix-norm.

Optimal Stirring Velocity

Zin Lin et al [5] introduced the local-in-time optimal velocity using the method of steepest descent. Multiply Eq(0.6) with $-\Delta^{-1}\Psi$ and integrate over the domain and use integration by parts, with the incompressibility condition, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi\|_{H^{-1}}^2 = -2\int \Psi u\cdot\nabla(\Delta^{-1}\Psi)\ \mathrm{d}x = -2\int u\cdot\mathbb{P}(\Psi\nabla(\phi))\ \mathrm{d}x$$

Where $\phi = \Delta^{-1}\Psi$ and $\mathbb{P}$ is the Leray projection defined by $\mathbb{P}(v) = v - \nabla\Delta^{-1}(\nabla\cdot v)$.

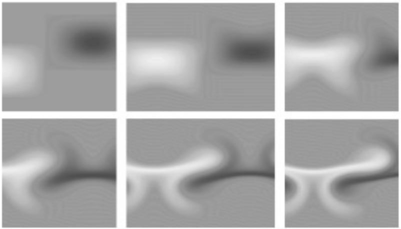
Given energy constraint, we have the optimization problem

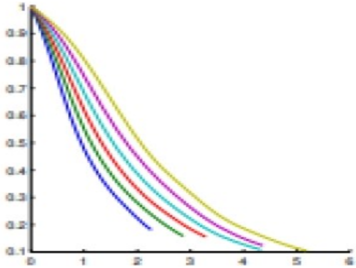
$$\max\int u\cdot\mathbb{P}(\Psi\nabla(\phi))\ \mathrm{d}x \ \text{s.t}\ \|u(x,t)\|^2 = U^2L^d.$$

The local-in-time optimal velocity can be calculated as $u_e = \frac{UL^{d/2}\|\mathbb{P}(\Psi\nabla(\phi))\|}{\|\mathbb{P}(\Psi\nabla(\phi))\|}$. Also with enstrophy constraint, the optimal mixer is $u_p = \frac{1}{\tau}\frac{-\Delta^{-1}\mathbb{P}(\Psi\nabla\phi)}{\langle|\nabla^{-1}\mathbb{P}(\Psi\nabla\phi)|^2\rangle^{\frac{1}{2}}}$ provided the norm in the denominator does not vanish.

Numerical Results

Following the work of Gautam Iyer et al [2] and varying the value of the initial data, the local-in-time optimal velocity is implemented numerically for the enstrophy constraint case. The numerical simulation agrees with the analysis (0.10).





Mixing With Diffusion

$$\partial_t \Psi + u \cdot \nabla \Psi = k \Delta \Psi \qquad k \neq 0 \tag{0.11}$$

- **Enstrophy constrained** $||\nabla u||_{L^\infty} = 1$:

$$||\nabla^{-1}\Psi||_{L^2} \geqslant ||\nabla^{-1}\Psi_0||_{L^2} \exp\left[-t - \frac{k}{2}\frac{||\nabla\Psi_0||_{L^2}^2}{||\Psi_0||_{L^2}}(e^{2t} - 1)\right].$$

  This suggests that perfect mixing in finite time is not possible for L^∞ enstrophy constrained flow.

- **Energy constrained** $||u||_{L^\infty} = 1$:

$$||\nabla^{-1}\Psi||_{L^2} \geqslant \frac{||\Psi_0||_{L^2}^2}{||\nabla\Psi_0||_{L^2}} \exp\left[\frac{-t}{2k} - k^2\frac{||\nabla\Psi_0||_{L^2}^2}{||\Psi_0||_{L^2}^2}(e^{\frac{t}{k}} - 1)\right].$$

  This also suggests that perfect mixing in finite time is not possible for L^∞ energy constrained flow.

REFERENCES

- Theodore D.Drivas, Tarek M.Elgindi, Gautam Iyer, In-Jee jeong (2020)
  Anomalous Dissipation in Passive Scalar Transport
- Gautam Iyer, Alexander Kiselev, Xiaoqian Xu (2014)
  Lower bounds on the mix norm of passive scalers advected by incompressible enstrophy-constrained flows
- C. J. Miles and C. R. Doering.
  Diffusion-limited mixing by incompressible flows.
  Nonlinearity, 31(5):2346, 2018.
- www.math.cmu.edu/gautam/research/201208-mix-bounds
- Z. Lin, J.-L. Thiffeault, and C. R. Doering.
  Optimal stirring strategies for passive scalar mixing. J. Fluid Mech., 2011
- Prof. Suzanne Kresta
  "What does well mixed mean?", University of Alberta, San Francisco 2013

Thank You