COURSE SYNOPSIS:-

CHAPTER ONE: INTRODUCTION TO DIFFERENTIAL EQUATION

CHAPTER TWO: METHODS OF SOLVING DIFFERENTIAL EQUATION

CHAPTER THREE: HIGHER DERIVATIVE

CHAPTER FOUR: SOLUTION OF DIFFERENTIAL EQUATION BY OPERATOR **D**

CHAPTER FIVE: LAPLACE TRANSFORMATION

REFERENCES

CHAPTER ONE

1.1 DEFINITION

An equation which involves differential co-efficient is called a differential equation.

Example:

$$(1)\frac{dy}{dx}=\frac{1+x^3}{1-y^3} \quad (2)\ \frac{d^2y}{dx^2}+4\frac{dy}{dx}-8y=0$$

Differential equation can be divided into (2)

1) Ordinary Differential Equation
2) Partial Differential Equation

Example:

- Ordinary differential equation are as follows:

$$(1)\ \frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=0 \quad (2)\ \frac{d^5y}{dx^5}+\frac{d^2y}{dx^2}+3y=x$$

- Partial differential equation are as follows;

$$(1)\ p\frac{\partial^2y}{\partial p^2}+\frac{\partial y}{\partial x}+p=0 \quad (2)\ \frac{\partial^5y}{\partial x^5}+\frac{\partial^2y}{\partial x^2}\ 4y=8$$

1.2 ORDER AND DEGREE OF A DIFFERENTIAL OF A DIFFERENTIAL EQUATION

- ORDER: The order of a differential equation is the order of the highest differential co-efficient present in the equation. Consider

$$(1)\ L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=Esin\omega t \qquad it's\ of\ order\ 2$$

$$(2)\ \left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2 \qquad it's\ of\ order\ 2$$

$$(3)\ \frac{d^5y}{dx^5}+\frac{d^2y}{dx^2}+4y=8x \qquad it's\ of\ order\ 5$$

$$(4)\ \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0 \qquad it's\ of\ order\ 3$$

DEGREE: The degree of the differential equation is the highest derivative in a differential equation.

Example:

$$(1)\ \left(\frac{\partial y}{\partial p}\right)^2 + P\frac{\partial^2 y}{\partial p^2} + \frac{\partial^2 y}{\partial r^2} = 0$$

Ans. Order 2, Degree 1

$$(2)\ \frac{\partial^5 y}{\partial x^5} + \frac{\partial^2 y}{\partial x^2} + 4y = 8$$

Ans. Order 5, Degree 1

$$(3)\ \left[1 + \left(\frac{d^3y}{dx^3}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)$$

Ans. Order 3, Degree 6

EXERCISE 1.1

Classify each of the following differential equations as an ordinary Differential Equation (O.D.E.) and Partial Differential Equation (P.D.E), give the **Order** and **Degree** of each of them.

PDE, order 2, degree 1

1). $\left(\frac{\partial T}{\partial t}\right)^3 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + KT$

ODE order 2, degree 1

2). $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + 3y = x^2$

3). $4\left(\frac{d^3W}{dV^3}\right) + \left(\frac{d^5W}{dV^5}\right)^3 + W = VW$

4). $\frac{d^3P}{dq^3} = \sqrt{P + \left(\frac{dy}{dx}\right)}$

5). $x^2\left(\frac{d^2y}{dx^2}\right)^3 = \sqrt{\frac{dy}{dx} + x}$

6). $\frac{\partial^2 y}{\partial x^2} + a^2x = 0$

Answer to above exercises:

(1) P.D.E, Order 2, Degree 1 (2) O.D.E, Order 2, Degree 1
(3) O.D.E, Order 5,Degree 3 (4) O.D.E, Order 3, Degree 2
(5) O.D.E, Order 2, Degree 6 (6) P.D.E, Order 2, Degree 1

CHAPTER TWO

METHOD OF SOLVING DIFFERENTIAL EQUATION

We will discuss the standard methods of solving the differential equation of the following types;

1) Variable Separable
2) Exact Equation
3) Non Exact Equation
4) Linear Equation
5) Bernoulli Equation
6) Homogeneous Equation

NOTE: the general equation of differential equation is given as

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . a_{n-1}\frac{dy}{dx}+a_ny = P(x)$$

Where $a_0, a_1, a_{2,\dots\dots\dots\dots} a_n$ are functions of x which are the Independent Variable, If $P(x) \neq 0$. It is called Non Homogeneous.

VARIABLE SEPARABLE

If a differential equation can be given in the form

$$F(y)dy = \emptyset(x)dx$$

We say that variable is separable, y on left hand side and x on the right hand side. We get the solution by integrating both sides.

$$dy = \frac{1}{x(x-1)}dx$$

Step 2: Integrate both sides.

$$\int dy = \int \frac{1}{x(x-1)}dx$$

$$y = \int \frac{1}{x(x-1)}dx ------------ (*)$$

We make use of partial fraction

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x+1}$$

From the right hand side (LCM)

$$\frac{1}{\cancel{x(x-1)}} = \frac{A(x-1) + Bx}{\cancel{x(x-1)}}$$

$$put\ x = 0,\ A = -1$$

$$x = 1,\ B = 1$$

Recall;

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x+1}$$

Replace in with 1 and -1

$$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x+1}$$

Recall from equation (*)

$$y = \frac{dx}{x(x+1)}$$

$$y = -\int \frac{1}{x}dx + \int \frac{dx}{(x-1)}$$

$$y = -\ln x + \ln(x-1)$$

$$y = \ln x - \ln(x-1)$$

$$y = \ln\left(\frac{x-1}{x}\right) + c$$

Example 3:

$$y(1+x^2)\frac{dy}{dx} + 1 = y^2$$

Solution;

Step 1: multiply through by dx.

$$y(1+x^2)dy = (y^2-1)dx$$

Step 2: Divide through by $y(1+x^2)(y^2-1)$

$$\int \frac{y}{(y^2-1)} dy = \int \frac{\cancel{y}}{(1+x^2)} dx$$

Introduce the constant $^2/_2$ on LHS

$$^2/_2 \int \frac{y}{(y^2-1)} dy = \int \frac{\cancel{y}}{(1+x^2)} dx$$

$$^1/_2 \int \frac{2y}{y^2-1} dy = \int \frac{dx}{1+x^2}$$

$$^1/_2 \ln(y^2-1) = \tan^{-1} x + K$$

Let $K = \ln P$. Such that P is an arbitrary constant

$$^1/_2 \ln(y^2-1) = \tan^{-1} x + \ln P$$

Multiply through by 2

$$\ln(y^2-1) = 2\tan^{-1} x + 2\ln P$$

$$\ln(y^2-1) = 2\tan^{-1} x + \ln P^2$$

$$\ln(y^2-1) - \ln P^2 = 2\tan^{-1} x$$

$$\ln\left(\frac{y^2-1}{P^2}\right) = 2\tan^{-1} x$$

EXERCISE 1.2

(1) $(1+x)\frac{dy}{dx} = (1-y)$

(2) $(xy^2 + x)dx + (yx^2 + y)dy = 0$ {Hint:$x(y^2 + 1)dx + y(x^2 + 1)dy$}
then solve.

(3) $(1 + x^2)dy - xydx = 0$

(4) $(4x + y)^2 \frac{dy}{dx} = 1$

(5) $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

EXACT DIFFERENTIAL EQUATION

An exact differential equation is formed by direct differentiating its Primitive (solution) without any other process. If the given equation of a differential equation is $M(x, y)dx + N(x, y)dy = 0 \dots\dots\dots\dots\dots\dots (1)$ is said to be an exact differential equation if it's satisfies the following condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Where $\frac{\partial M}{\partial y}$, denotes the differential co-efficient of M with respect to y keeping x constant and $\frac{\partial N}{\partial x}$, the differential co-efficient of N with respect to x keeping y

Method for solving Exact Differential Equation

Step 1: integrate M w.r.t x keep y constant

Step 2: integrate w.r.t y, only those terms of N which do not contain x.

Step 3: result of 1 + result of 2 = constant.

Example: Test for exactness of the following

1) $(12x + 5y - 9)dx + (5x + 2y - 4)dy = 0$

Solution;

$M = 12x + 5y - 9$

$\frac{\partial M}{\partial y} = 5$

$N = 5x + 2y - 4$

$$\frac{\partial N}{\partial x} = 5$$

Recall$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Since$5 = 5$, then the equation is exact.

2) $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y + 3x^2y^2 - 5y^4)dy = 0$

Solution;

$$M = 5x^4 + 3x^2y^2 - 2xy^3$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

$$N = 2x^3y + 3x^2y^2 - 5y^4$$

$$\frac{\partial N}{\partial y} = 6x^2y - 6xy^2$$

Recall $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$ since$6x^2y - 6xy^2 = 6x^2y - 6xy^2$, then the equation is exact.

3) $(e^y + 1)cosxdx + e^y sinxdy = 0$

Solution;

$$M = (e^y + 1)cosxdx$$

$$\frac{\partial M}{\partial y} = e^y cosx$$

$$N = e^y sinxdy$$

$$\frac{\partial N}{\partial x} = e^y cosx$$

Recall, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$e^y cosx = e^y cosx$, then, the equation is exact.

NOTE: When the differential equation is exact, then it can be solve.

Example:

4) $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$

Solution;

Step 1:

$$\frac{\partial M}{\partial y} = 4x$$

$$\frac{\partial N}{\partial y} = 4x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

$4x = 4x$, the equation is exact. Then it can be solved.

Step 2: integrate $M(x, y)dx$

$$F(x, y) = \int M(x, y)dx + \emptyset y$$

$$= \int (3x^2 + 4xy)dx + \emptyset y$$

$$= \frac{3x^3}{3} + \frac{4x^2y}{2} + \emptyset y$$

$$= x^3 + 2x^2y + \emptyset y$$

Hence differentiate w.r.t y.

$$\frac{\partial F}{\partial y} = 2x^2 + \emptyset' y$$

Recall $\frac{\partial F}{\partial y} = 2x^2 + 2y$

$$\cancel{2x^2} + 2y = \cancel{2x^2} + \emptyset' y$$

$$\emptyset' y = 2y$$

Integrate both sides,

$$\int \emptyset' y = \int 2y$$

$$\emptyset y = 2y$$

$$\therefore F(x, y) = x^3 + 2x^2y + y^2 + c$$

Test

$$\frac{\partial F}{\partial x} = 3x^2 + 4xy$$

$$\frac{\partial F}{\partial y} = 2x^2 + 2y$$

Example 5

$$(12x + 5y - 9)dx + (5x + 2y - 4)dy = 0$$

Solution;

Step 1:

$$\frac{\partial M}{\partial y} = 5$$

$$\frac{\partial N}{\partial x} = 5$$

Hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The equation is exact.

Step 2: integrate any of $M(x,y)dx \; or \; N(x,y)dy$

$$F(x,y) = \int M(x,y)dx + \emptyset y$$

$$F(x,y) = \int(12x + 5y - 9)dx + \emptyset y$$

$$F(x,y) = \frac{12x^2}{2} + 5xy - 9x + \emptyset y$$

$$F(x,y) = 6x^2 + 5xy - 9x + \emptyset y --------(*)$$

Differentiate w.r.t y

$$\frac{\partial F}{\partial y} = 5x + \emptyset' y$$

Recall $\frac{\partial F}{\partial y} = 5x + 2y - 4$

Then $\cancel{5x} + 2y - 4 = \cancel{5x} + \emptyset' y$

$$\emptyset' y = 2y - 4$$

Integrate both sides,

$$\int \emptyset' y = \int (2y - 4)dy$$

$$\emptyset' y = y^2 - 4y + c \quad put\ this\ in\ equation\ (*)$$

$$F(x, y) = 6x^2 + 5xy - 9x + y^2 - 4y + c$$

Test :

$$\frac{\partial F}{\partial x} = 12x + 5y - 9$$

$$\frac{\partial F}{\partial y} = 5x + 2y - 4$$

TEST & EXAM QUESTIONS

Determine the constant Ksuch that the equation is exact and solve.

$$(x^2 + 3xy)dx + (Kx^2 + 4y)dy = 0$$

$$M = x^2 + 3xy$$

$$\frac{\partial M}{\partial y} = 3x$$

$$N = Kx^2 + 4y$$

$$\frac{\partial N}{\partial y} = 2Kx$$

Recall, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$3\cancel{x} = 2K\cancel{x}$$

$$K = {}^3/_2$$

Hence, $(x^2 + 3xy)dx + \left({}^3/_2 x^2 + 4y\right)dy = 0$

Test for the exactness of the equation.

$$M = x^2 + 3xy, \qquad \frac{\partial M}{\partial y} = 3x$$

$$N = {}^3/_2\, x^2 + 4y, \qquad \frac{\partial N}{\partial x} = 3x$$

Recall; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ that is $3x = 3x$

Thus, the equation is exact ant it can then be solved

$$M(x,y)dx + \emptyset y \; or \; N(x,y)dy + \emptyset x$$

$$F(x,y) = \int M(x,y)dx + \emptyset \cancel{x} y$$

$$= \int (x^2 + 3xy)dx + \emptyset y$$

$$= \frac{x^3}{3} + \frac{3x^2 y}{2} + \emptyset y$$

Differentiate w.r.t y.

$$\frac{\partial F}{\partial y} = \frac{3x^2}{2} + \emptyset' y - - - - - - - - - (*)$$

Recall $\frac{\partial F}{\partial y} = \frac{3}{2}x^2 + 4y$, then equate it with equation (*)

$$\cancel{\frac{3}{2}x^2} + 4y = \cancel{\frac{3}{2}x^2} + \emptyset' y$$

$$\emptyset' y = 4y$$

Integrate both sides,

$$\int \emptyset' y = \int 4y$$

$$\emptyset' y = \int 4y dy$$

$$\emptyset' y = 2y^2$$

Recall, $F(x,y) \quad = \quad \frac{x^3}{3} + \frac{3x^2 y}{2} + \emptyset' y$

$$F(x,y) \quad = \quad \frac{x^3}{3} + \frac{3x^2 y}{2} + 2y^2$$

Check

$$\frac{\partial F}{\partial x} = \frac{3x^2}{3} + \frac{6xy}{2}$$

$$= x^2 + 3xy$$

$$\frac{\partial F}{\partial y} = \frac{3}{2}x^2 + 4y$$

NON EXACT EQUATION

Suppose the differential equation $M(x,y)dx + N(x,y)dy = 0$ does not exact. That is $\left|\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right|$, then in order to make it an exact equation, we need to simplify the equation by an Integrating Factor (I.F).

Rule 1: if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = F(x)$ is a function of x alone, say $F(x)$, then

$$I.F = e^{\int F(x)dx}$$

Rule 2: if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = F(y)$ is a function of y alone, say $F(y)$ then

$$I.F = e^{\int F(y)dy}$$

Example 7: solve the equation $(x^2 + y^2 + x)dx + xydy = 0 - - - - - - (*)$

Solution;

<u>Step 1</u> $\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = y$

$$\frac{\partial M}{\partial y} = \frac{\partial M}{\partial y}, \quad 2y \neq y$$

The equation is not exact.

<u>Step 2</u> $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = F(x)$

$$\frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x}$$

$$F(x) = \frac{1}{x}$$

$$I.F = e^{\int F(x)dx}$$

$$I.F = e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$I.F = x$$

Multiply equation (*) by $I.F = (x)$.

$$(x^2 + y^2 + x)xdx + (xy)xdy = 0$$

$$(x^3 + xy^2 + x^2)dx + (x^2y)dy = 0$$

Now, $\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = 2xy$

Then, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$2xy = 2xy$$

The equation is now exact and it can be solved.

Integrate any of $M(x,y)dx + \emptyset y$ or $N(x,y)dy + \emptyset x$

$$F(x,y) = \int M(x,y)dx + \emptyset y$$

$$= \int (x^3 + xy^2 + x^2)dx + \emptyset y$$

$$= \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + \emptyset y$$

Differentiate w.r.t y

$$\frac{\partial F(x,y)}{\partial y} = x^2y + \emptyset y$$

Recall $\frac{\partial F}{\partial y} = x^2y$

$$\cancel{x^2y} = \cancel{x^2y} + \emptyset y$$

$$\emptyset' y = 0$$

Integrate both sides,

$$\int \emptyset' y = \int 0 \, dy$$

$$\emptyset' y = c$$

$$\therefore \quad F(x,y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + c$$

<u>Check</u>

$$\frac{\partial F}{\partial x} = x^3 + xy^2 + x^2$$

$$\frac{\partial F}{\partial y} = x^2 y$$

Exercise 1.3

1) $(3x^2y + 2)dx - (x^3 - y)dy = 0$
2) $(\theta^2 + 1)\cos r dr + 2\theta \sin r d\theta = 0$
3) $(2y \sin x \cos x + y^2 \sin x)dx + (\sin^2 x - 2y \cos x)dy = 0 \; at \; y(0) = 3$

<u>Hint:</u> $\frac{\partial M}{\partial y} = 2 \sin x \cos x + 2y \sin x$

$$\frac{\partial N}{\partial x} = 2 \sin x \cos x + 2y \sin x$$

Then solve answer should be 9

4) Determine the most general function M or N. such that the equation is exact.
   (a) $(M(x,y)dx + (2x^2y^3 + x^4y)dy = 0$
   (b) $(x^2y^{-2} + xy^{-3})dx + N(x,y)dy = 0$
5) $(e^y + 2)\sin x \, dx - e^y \cos x \, dy = 0$
6) $(y^2 + 2yx^2)dx + (2x^3 - xy)dy = 0$

<u>LINEAR EQUATION</u>

The general form of the linear differential equation is given as $\frac{dy}{dx} + P(x)y = q(x)$ where P and q is function of x only.

Solution Method

Let $R(x)$ be a function define as $R(x) = e^{\int P(x)dx}$ then the solution of the equation is given as $R(x)y = \int R(x)q(x) + c$

Example 1: $\frac{dy}{dx} + ay = xe^{ax}$ ------------------------------ (1)

Note: $P(x) = a, q(x) = xe^{ax}$

Step 1: Find the integrating factor (I.F)

$$I.F = R(x)$$

$$R(x) = e^{\int F(x)dx}$$

$$R(x) = e^{\int adx} = e^{ax}$$

Step 2: Multiply the $I.F = R(x)$ with eqn. (1)

$$e^{ax}\frac{dy}{dx} + ae^{ax}y \quad = \quad xe^{2ax}$$

$$\frac{d}{dx}(e^{ax}y) \quad = \quad xe^{2ax}$$

$$d(e^{ax}y) \quad = \quad xe^{2ax}dx$$

Integrate both sides

$$\int d(e^{ax}y) \quad = \quad \int xe^{2ax}dx$$

$$e^{ax}y \; = \; \int xe^{2ax}dx + c - - - - - - - - - (*)$$

Using integration by part,

Hence, $\int xe^{2ax}dx$

Recall $\int udv = uv - \int vdu$

$$u = x, \qquad dv = e^{2ax}$$

We get v from dv, you integrate both sides

$$\int dv = \int e^{2ax}$$

$$v = \int e^{2ax}$$

$$v = {}^{1}/_{2}\, ae^{2ax}$$

$$\int xe^{2ax} = {}^{1}/_{2}\, axe^{2ax} - {}^{1}/_{2}\, a \int e^{2ax} dx$$

$$= {}^{1}/_{2}\, axe^{2ax} - {}^{1}/_{2}\, a * {}^{1}/_{2}\, ae^{2ax} + D$$

Recall from equation (*)

$$e^{ax} y = \int xe^{2ax} dx + c$$

$$e^{ax} y = {}^{1}/_{2}\, axe^{2ax} - {}^{1}/_{4}\, a^2 e^{2ax} + E$$

Divide through by e^{ax}

$$y = e^{-ax}({}^{1}/_{2}\, axe^{2ax} - {}^{1}/_{4}\, a^2 e^{2ax} + E)$$

$$y = {}^{1}/_{2}\, axe^{2ax} - {}^{1}/_{4}\, a^2 e^{2ax} + E$$

Example 2

$$\frac{dy}{dx} + (2x + 1)y = e^{-2x} \quad - - - - - - - - (*)$$

Solution;

$$P(x) = (2x + 1), \quad q(x) = e^{-2x}$$

Step 1: find the $I.F = R(x)$

$$R(x) = e^{\int P(x) dx}$$

$$R(x) = e^{\int (2x+1) dx}$$

$$R(x) = e^{x^2 + x}$$

Multiply $R(x)$ with equation (*).

$$e^{x^2+x} \frac{dy}{dx} + (2x + 1)e^{x^2+x} y = e^{x^2+x} * e^{-2x}$$

$$\frac{d}{dx}\left(e^{x^2+x}y\right) = e^{x^2-x}$$

$$d\left(e^{x^2+x}y\right) = e^{x^2-x}dx$$

Integrate both sides,

$$\int d\left(e^{x^2+x}y\right) = \int e^{x^2-x}dx$$

$$e^{x^2+x}y = \int e^{x^2-x}dx + c$$

$$e^{x^2+x}y \quad = \quad \frac{1}{2x-1}e^{x^2-x}$$

$$y \quad = \quad \frac{\frac{1}{2x-1}e^{x^2-x}}{e^{x^2-x}}$$

<u>Example 3</u> $\frac{dy}{dx} + (\cot x)y \quad = \quad \cos x$

Solution,

$$P(x) = \cot x, \qquad q(x) = \cos x$$

Find the $I.F = R(x)$

$$R(x) \quad = \quad e^{\int P(x)dx}$$

$$R(x) \quad = \quad e^{\int \cot x dx}$$

$$R(x) \quad = \quad e^{\int \frac{\cos x}{\sin x}dx}$$

$$R(x) \quad = \quad e^{\ln \sin x}$$

$$R(x) = \sin x$$

Multiply through by $\sin x$ i.e. $I.F$

$$\sin x\frac{dy}{dx} + (\sin x)(\cot x)y \quad = \quad \sin x \cos x$$

$$\frac{d}{dx}(\sin x)y \quad = \quad \sin x \cos x$$

$$d(\sin x)y \quad = \quad \sin x \cos x \, dx$$

Integrate both sides,

$$\int d(\sin x)y \quad = \quad \int \sin x \cos x \, dx$$

$$(\sin x)y \quad = \quad \int \sin x \cos x \, dx$$

NOTE: from the R.H.S.

$$u = \sin x, \qquad \frac{du}{dx} = \cos x$$

$$du = \cos x \, dx, \qquad dx = \frac{du}{\cos x}$$

$$\int u * \cancel{\cos x} * \frac{du}{\cancel{\cos x}}$$

$$\int u du \quad = \quad \frac{u^2}{2} + D$$

Recall, $(\sin x)y = \dfrac{(\sin x)^2}{2} + E$

Divide through by $\sin x$.

$$y \; = \; \frac{\frac{(\sin x)^2}{2} + E}{\sin x}$$

$$y \; = \; \frac{\cancel{(\sin x)^2}}{2} * \frac{1}{\cancel{\sin x}} + \frac{E}{\sin x}$$

$$y \; = \; \frac{\sin x}{2} + E \sin^{-1} x$$

EXERCISE 1.4

(1) $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$

(2) $x\dfrac{dy}{dx} + y = x \sin x$

(3) $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2$

Bernoulli equation

These are an improvement of linear equation, and it is given in this form.

$$\frac{dy}{dx} + p(x)y = q(x)y^n \text{-------------------}(*)$$

SOLUTION METHOD

Convert the equation to linear equation.

Divide through by y^n

$$y^{-n}\frac{dy}{dx}+p(x)y^{1-n}=q(x)\text{--------------------(**)}$$

Let $Z=y^{1-n}$

Differentiate Z with respect to x

$$\frac{dZ}{dx}=(1-n)y^{-n}\frac{dy}{dx}$$

$$y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dZ}{dx}$$

$$\frac{1}{1-n}\frac{dZ}{dx}+p(x)Z=q(x)$$

$$\frac{dZ}{dx}+(1-n)p(x)Z=q(x)$$

$$\frac{dZ}{dx}+p(x)Z=q(x)$$

Where $p(x)=(1-n)p(x)$

$$q(x)=(1-n)q(x)$$

Question,

$$(1)\ \frac{dy}{dx}\ +\ y\ =\ xy^3$$

$$Z\ =\ y^{1-n}$$

$$Z\ =\ y^{-2}$$

Step 1: divide through by y^3

$$y^{-3}\ \frac{dy}{dx}\ +\ y^{-2}\ =\ x\text{-------------------------(*)}$$

$$Z\ =\ y^{1-n}$$

$$Z\ =\ y^{-2}$$

$$\frac{dZ}{dx}\ =\ -2y^{-3}\ \frac{dy}{dx}$$

Divide through by -2

$$y^{-3}\ \frac{dy}{dx} \quad = \quad -\frac{1}{2}\ \frac{dZ}{dx}$$ (Put this to equation *)

$$-\frac{1}{2}\ \frac{dZ}{dx} \ + \ Z \ = \ x$$

Multiply through by -2

$$\frac{dZ}{dx} \ - \ 2Z \ = \ -2x$$

$$p(x) = -2, \ \ q(x) = -2x$$

Step 2:

Look for I.F

$$R(x) \quad = \quad e^{\int P(x)dx}$$

$$= \quad e^{\int -2dx}$$

$$R(x) \quad = \quad e^{-2x}$$

Multiply by R(x)

$$e^{-2}\frac{dZ}{dx} - 2e^{-2x}Z \quad = \quad 2xe^{-2x}$$

$$\frac{d}{dx}[e^{-2x}Z] \ = \ -2xe^{-2x}$$

$$d[e^{-2x}Z] \ = \ -2xe^{-2x}dx$$

Integrate both sides.

$$e^{-2x}Z \ = \ \int -2xe^{-2x}dx$$

$$= -2\int xe^{-2x}dx$$

From the right hand side. using integration by part.

$$\int udv \ = \ uv - \int vdu$$

$$u = x, \ \ dv \ = \ e^{-2x}, \ \ v = -\frac{1}{2}e^{-2x}$$

$$\frac{-xe^{-2x}}{2} \ + \ \frac{1}{2}\int e^{-2x}dx$$

$$\frac{-xe^{-2x}}{2} + \frac{1}{4}e^{-2x} + D$$

Recall,

$$e^{-2x}Z = \int -2xe^{-2x}\,dx$$

$$e^{-2x}Z = -xe^{-2x} + \frac{1}{2}e^{-2x} + E$$

$$Z = \frac{-xe^{-2x}}{e^{-2x}} + \frac{\frac{1}{2}e^{-2x}}{e^{-2x}} + \frac{E}{e^{-2x}}$$

$$Z = -x + \frac{1}{2} + e^{2x}E$$

Recall $Z = y^{-2}$

$$y^{-2} = x + \frac{1}{2} + E(e^{2x})$$

Question 2:

$$x^2y - x^3\frac{dy}{dx} = y^4\cos x$$

Solution,

Divide through by $-x^3$

$$\frac{-1}{x}y + \frac{dy}{dx} = \frac{y^4\cos x}{-x^3}$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{y^4\cos x}{-x^3}$$

Divide through by y^4

$$y^{-4}\frac{dy}{dx} - \frac{1}{x}y^{-3} = -\frac{\cos x}{x^3}$$

Recall $Z = y^{-3}$ (differentiate)

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$

Divide through by -3

$$-\frac{1}{3}\frac{dz}{dx} - \frac{1}{x}Z = -\frac{\cos x}{x^3}$$

Multiply through by -3

$$\frac{dz}{dx} + \frac{3}{x}Z = \frac{\cos x}{x^3} \quad \text{---------------------------(*)}$$

$$f(x) = \frac{3}{x}, \quad q(x) = \frac{3\cos x}{x^3}$$

Step 2: integrating factor,

CHAPTER THREE

APPLICATION OF FIRST ORDER DIFFERENTIAL EQUATION

Under the application of first order differential equation we talk about

(i) Electrical circuit
(ii) Mechanics
(iii) Heat conduction

MECHANICS

Example.

A particle falls in a vertical line under gravity (supposed constant) and the force of air resistance to its motion is proportional to its velocity. Show that its velocity cannot exceed a particular limit.

Solution.

Let V be the velocity when the particle has fallen a distance S n time t from rest. If the resistance is KV, then the equation of motion is

$$\frac{dV}{dt} = g - KV$$

$$\frac{dV}{g-KV} = dt$$

Integrate both sides,

$$\int \frac{dV}{g-KV} = \int dt$$

$$-\frac{1}{K}\log(g - KV) = t + C \text{ -------------------(*)}$$

Initial conditions, V=0, t=0

$$C = -\frac{1}{K}\log g$$

Put C in to equation (*)

Then it becomes,

$$-\frac{1}{K}\log(g - KV) = t - \frac{1}{K}\log g$$

$$t = -\frac{1}{K}\log\frac{g-KV}{g} \quad or \quad \frac{g-KV}{g} = e^{-Kt}$$

$$V = \frac{g}{K}(1 - e^{-Kt})$$

t is always positive :- $1 > 1 - e^{-Kt}$

Limiting Velocity or Maximum Velocity

$$V = \frac{g}{K}$$

Example.

A moving body is opposed by a force per unit mass of value Cx and resistance per unit mass of value bV^2 where x and V are the displacement and velocity of the particle at that instant. Find the velocity of the particle in term of$x$, if it starts from rest.

Solution.

By Newton Second law of motion, the equation of the body is,

$$V\frac{dV}{dx} = -Cx - bV^2$$

$$V\frac{dV}{dx} + bV^2 = -Cx$$

Putting $V^2 = z, 2V\frac{dV}{dx} = \frac{dZ}{dx}$

$$\frac{1}{2}\frac{dZ}{dx} + bZ = -cx$$

$$\frac{dZ}{dx} + 2bz = -2Cx$$

$$I.F = e^{\int 2bdx} = e^{2bx}$$

$$z.e^{2bx} = \int -2cx\, e^{2bx} dx + C$$

$$= -2c\left[\frac{xe^{2bx}}{2b} - \int \frac{e^{2bx}}{2b} dx\right] + C' = -\frac{c}{b}xe^{2bx} + \frac{c}{b}\frac{e^{2bx}}{2b} + C'$$

$$= -\frac{c}{b}xe^{2bx} + \frac{c}{2b^2}e^{2bx} + C'$$

$$Z = -\frac{c}{b}x + \frac{c}{2b^2} + C'e^{-2bx}$$

Initially $V = 0$ where $x = 0$

$$0 = \frac{c}{2b^2} + C' \;\; or\; C' = -\frac{c}{2b^2}$$

$$V^2 = -\frac{cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2}e^{-2bx}$$

In mechanics, Newton Law of motion which state that a time rate of change of linear momentum of a body is proportional to the impressed force. i.e.

$$\frac{d}{dt}MV \;\; \alpha \;\; F$$

$\frac{d}{dt}MV = KF$. Where M is the mass of the body V is the velocity F is the force K is the constant of proportionality

$$M\frac{d}{dt}V = KF$$

$$F = \frac{M}{K}\frac{dV}{dt}$$

If K=1

$$F = M\frac{dV}{dt} = Ma \quad \text{-------------------- (1)}$$

By chain rule,

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

Where x is the displacement.

$$\frac{dV}{dt} = V\frac{dV}{dx} \text{ -------------------------}*$$

Make V the subject,

$$V = \frac{dV}{dt} * \frac{dx}{dV}$$

$$V = \frac{dx}{dt}$$

From equation * and 1, it can be written as

$$F = M \, . \, V\frac{dV}{dt} \text{ -------------------- (2)}$$

By definition of acceleration A.

$$a = \frac{dV}{dt} = V\frac{dV}{dx}$$

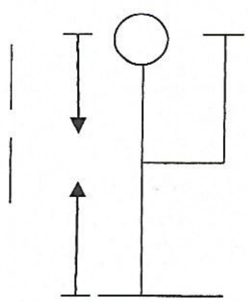
$$a = \frac{dV}{dt} = \frac{d^2x}{dt^2}$$

$$F = M\,\frac{d^2x}{dt^2} \text{ -------------------------- (3)}$$

Example 3.

A stone weight 4kg fall rest towards the earth from a great height as it fall it experiences air resistance which numerically in equal $\frac{1}{2}V(N)$, where V is the velocity in Km. find the velocity and the displacement at time t.

Solution.



We assume that the +ve direction is vertically downward along the part of the body and the origin at which the body fall, the force acting on the body consist is weight and air resistance which opposite the motion.

Resultant force = $F_2 + F_1$

$$F = 4 + (-\frac{1}{2}V)$$

$$F = 4 - \frac{1}{2}V$$

$$Ma = 4 - \frac{1}{2}V$$

$$M\frac{dV}{dt} = 4 - \frac{1}{2}V \text{ ---------------------- } (1)$$

Note, W = Mg, g=10m/s^2

$$M = \frac{w}{g} = \frac{4}{10}$$ Put it in equation (1).

$$\frac{4}{10}\frac{dV}{dt} = 4 - \frac{1}{2}V$$

$$4\frac{dV}{dt} = 40 - 5V$$

$$\frac{dV}{dt} = \frac{40-5V}{4}$$

$$\frac{dV}{40-5V} = \frac{1}{4}dt$$

Integrate both sides,

$$-\frac{1}{5}\int \frac{dV}{40-5V} = \int \frac{dt}{4}$$

$$-\frac{1}{5}\ln(40-5V) = \frac{t}{4} + C$$

$$\ln(40-5V) = -\frac{5}{4}t - 5C$$

$$\ln(40-5V) = -\frac{5}{4}t - D \qquad (D = -5C)$$

Taking the exponential of both sides.

$$e^{\ln(40-5V)} = e^{-\frac{5}{4}t} \,.\, e^{D}$$

Let $e^D = K$

$$40 - 5V = e^{-\frac{5t}{4}} \,.\, K$$

$$40 - 5V = Ke^{-\frac{5t}{4}}$$

$$5V = 40 - Ke^{-\frac{5t}{4}}$$

Let $P = \frac{K}{5}$

$V = 8 - Pe^{-\frac{5}{4}t}$ ---------------------------(*)

At V = 0, t = 0

$$0 = 8 - Pe^{-\frac{5(0)}{4}}$$

$$0 = 8 - Pe^{0}$$

$$0 = 8 - P$$

$P = 8$ Put it into equation (*).

$$V = 8(1 - e^{-\frac{5t}{4}}).$$

(ii) Let $x(t)$ be the displacement

$$V = \frac{dx(t)}{dt}$$

$$8\left(1 - e^{-\frac{5t}{4}}\right) = \frac{dx(t)}{dt}$$

$$8\left(1-e^{-\frac{5t}{4}}\right)dt = dx(t)$$

Integrate both sides.

$$\int dx(t) = \int 8\left(1-e^{-\frac{5t}{4}}\right)dt$$

$$x(t) = 8\left(t+\frac{4}{5}e^{-\frac{5t}{4}}\right)+R \text{ ---------------------- } (*)$$

For particular solution, at t = 0, x (t) = 0

X (0) = 0

$$0 = 8\left(0+\frac{4}{5}\right)+R$$

$$= \frac{32}{5}+R$$

$R = -\frac{32}{5}$ Put it into equation (*)

$$x(t) = 8\left(t+\frac{4}{5}e^{-\frac{5t}{4}}\right)-\frac{32}{5}$$

EXERCISES,

(1) Under certain conditions, cane sugar is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If out of 75grams of sugar at t=0, 8grams are converted during the first 3mins. Find the amount the converted in $1\frac{1}{2}\,hours.$

Hint: let M be the amount of cane sugar converted into dextrose

$$\frac{dm}{dt} = K(M-m) \;\; or \;\; \frac{dm}{dt}+Km = KM$$

(2) Water at temperature 100^{0}C cool in 10munites to 88^{0}C in a room of temperature 25^{0}C. Find the temperature of water after 20munites.

Hint: $\frac{dT}{dt} = -K(T-T_0).$

Where T is the temperature of the body at time t and T_0 the constant temperature of the medium.

SECOND ORDER OF DIFFERENTIAL EQUATION

Linear equation with constant coefficient, many practical problems engineering give rice to the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + Cy = f(x)$$

Where f(x) is a given function and a, b, c are constant coefficient.

Case 1: when f(x) = 0

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + Cy = 0$$

The type of solution we get depend on the root of the auxiliary equation. The auxiliary equation is given as, $am^2 + bm + C = 0$

CONDITION:

(1) REAL BUT DIFFERENT ROOT

$$y = Ae^{m_1x} + Be^{m_2x}$$

Where m_1m_2 are the roots of the auxiliary equation.

(2) REAL AND EQUAL ROOT

$$y = e^{mx}(A + Bx)$$

(3) COMPLEX ROOT

$$y = e^{ax}(Acosbx + Bsinbx)$$

$a + ib$ ------------------- Real and imaginary part.

Example 1,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Solution,

Case 2: form the auxiliary equation

$$m^2 - m - 2 = 0$$

Factorize the equation,

$$m^2 + m - 2m - 2 = 0$$

$$m(m + 1) - 2(m + 1) = 0$$

$$(m-2)(m+1)=0$$
$$m=2, or -1$$

Case 2: since it is real but different root.

$$y=Ae^{m_1x}+Be^{m_2x}$$
$$m_1=2, m_2=-1$$
$$y=Ae^{2x}+Be^{-x}$$

Example 2,

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=0$$

Solution,

Case 1: form an auxiliary equation

$$m^2-4m+3=0$$

Factorize the equation

$$m^2+m+3m+3=0$$
$$m(m+1)+3(m+1)=0$$
$$(m+3)(m+1)=0$$
$$m=-3 \ or -1$$

Case 2: since it is real but different root

$$y=Ae^{m_1x}+Be^{m_2x}$$
$$y=Ae^{-3x}+Be^{-x}$$

Example 3:

$$\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=0$$

Solution,

Case 1: form an auxiliary equation.

$$m^2-8m+16=0$$

Factorize the equation,

$$m^2+4m+4m+16=0$$

$$m(m+4)+4(m+4)=0$$

$$(m+4)(m+4)=0$$

$m = -4$ *twice which is equal root.*

$$y = e^{mx}(A + Bx)$$

$$y = e^{-4x}(A + Bx)$$

Example 4:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 9y = 0$$

Solution,

Case 1: form the auxiliary equation

$$m^2 - 4m + 9 = 0$$

NOTE: it can be factorize by using quadratic formular.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = 4, c = 9$$

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(9)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{-20}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{-5}}{2}$$

$$x = -2 \pm \sqrt{-5}$$

$$i^2 = -1$$

$$i = \sqrt{-1}$$

$$x = -2 \pm i\sqrt{-5}$$

-2 Is the real part while $i\sqrt{-5}$ is the imaginary part.

Case 2: since it is of a complex root

$$y = e^{ax}(Acosbx + Bsinbx)$$

$$y = e^{-2x}(Acos\sqrt{5}x + Bsin\sqrt{5}x)$$

EXERCISE

(1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

(2) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$

(3) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 4y = 0$

(4) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

Special cases

$$\frac{d^2y}{dx^2} + n^2y = 0$$

Form the auxiliary equation.

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm\sqrt{-n^2}$$

$$m = \pm in$$

$$y = (Acosnx + Bsinnx)$$

Example,

$$\frac{d^2y}{dx^2} + 16y = 0$$

Solution,

$$m^2 + 16 = 0$$

$$m^2 = -16$$

$$m = \pm\sqrt{-16}$$

$$m = \pm i4$$

$$y = Acos4x + Bsin4x$$

$$\frac{d^2y}{dx^2} + n^2y = 0$$

Form the auxiliary equation,

$$m^2 - n^2 = 0$$

$$m^2 = n^2$$

$$m = n$$

$$y = Acoshnx + Bsinhnx$$

Case 2: NON HOMOGENEOUS

A linear ordinary differential equation of order 'n' in the dependent variable y and the independent variable x is an equation of the form.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots\dots\dots\dots\dots\dots a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

$$i.e\ a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + Cy = f(x) \text{ ------------------(*)}$$

Were our $a_0(x)$ is not equal to zero$[a_0(x) \neq 0]$. The right hand side member is called the non-homogenous. But if $f(x) = 0$ the equation becomes homogenous equation.

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + Cy = 0$. This is called homogenous equation.

When $f(x) \neq 0$ write equation (*) we used the method of undetermined co-efficient to solve the equation in the case $f(x)$ contain a polynomial or term of the form $sinKx, cosKx, e^{Kx}$e.t.c. where K is a constant, then the general solution is given as $\boldsymbol{y_g = y_c + y_p}$

Example 1,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10sinx$$

Solution,

$$f(x) = 2e^x - 10sinx$$

$$y_g = y_c + y_p$$

Step 1: calculate the y_c. Were y_c is called the y complement.

$$y_c = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y$$

Form the auxiliary equation,

$$m^2 - 2m - 3 = 0$$

Factorize the equation,

$$m^2 - 3m + m - 3 = 0$$

$$m(m-3) + 1(m-3) = 0$$

$$(m+1)(m-3) = 0$$

$$m = -1 \ or \ 3$$

Since it is real but different root.

$$y_c = Ae^{m_1x} + Be^{m_2x}$$

$$= Ae^{-x} + Be^{3x}$$

Step 2: calculate for y_p. y_p is called y particular.

For y particular, form the set $S = \{e^x, \sin x, \cos x\}$

$$y'_P = Pe^x + Qcosx - Rsinx$$

$$y''_P = Pe^x - Qsinx - Rcosx$$

Recall,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10sinx$$

$Pe^x - Qsinx - Rcosx - 2(Pe^x + Qcosx - Rsinx) - 3Pe^x - 3sinx -$
$3cosx$

$$\gg -4Pe^x + sinx(-4Q + 2R) + cosx(-4R - 2Q) = 2e^x - 10sinx$$

Hence, equate both sides.

$$-4Pe^x = 2e^x$$

$-2Pe^{x} = e^{x}$

$-2P = 1$

$P = -1/2$

$\cancel{\sin x}(-4Q + 2R) = -10\cancel{sinx}$

$-4Q + 2R = -10$ ----------------- (i)

$\text{Cos}\, x\,(-4R - 2Q) = 0$

Divide through by $\cos x$

$-4R - 2Q = 0$

$R = -\frac{1}{2}Q$ ------------------ (ii)

Put equation (ii) into equation (i),

$-4Q + 2\left(-\frac{1}{2}Q\ \right) = -10$

$-5Q = -10$

$Q = 2$

$P = -\frac{1}{2}, Q = 2, R = -1$

Put in y particular,

$$y_p = -\frac{1}{2}e^{-x} - 2\sin x - \cos x$$

Recall,

$y_g = y_c + y_p$

$y_p = Ae^{-x} + Be^{3x} - \frac{1}{2}e^{x} + 2\sin x - \cos x$

$= 2e^{x} - 10\sin x$

Example 2,

$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

Solution,

$$f(x) = x^2$$

$$y_g = y_c + y_p$$

Step 1: y_c

$$y_c = \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Form the auxiliary equation,

$$m^2 - 5m + 6 = 0$$

Factorize the auxiliary equation,

$$m^2 - 3m - 2m + 6 = 0$$

$$m(m-3) - 2(m-3)$$

$$(m-2)(m-3) = 0$$

Since they have different root,

$$y_c = Ae^{m_1x} + Be^{m_2x}$$

$$y_c = Ae^{3x} + Be^{2x}$$

For y particular, form a set of non-homogenous term.

$$S = \{x^2, x, 1\}$$

$$y_p = Dx^2 + Ex + F$$

$$y'_p = 2Dx + E$$

$$y''_p = 2D$$

Recall, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$$2D - 5(2Dx + E) + 6(Dx^2 + Ex + F) = x^2$$

$$2D - 10Dx - 5E + 6Dx^2 + 6Ex + 6F = x^2$$

Collect the like terms,

$$6Dx^2 - 10Dx - 5E + 6Ex + 6F + 2D = x^2$$

$$6Dx^2 = x^2$$

$$D = \frac{1}{6}$$

$$(6E - 10D)x - 5E + 6F + 2D = 0$$

$$6E - 10D = 0$$

$$6E = 10\text{D}$$

$$E = \frac{10D}{6} = \frac{10 \cdot \frac{1}{6}}{6} = \frac{10}{36}$$

$$-5E + 6F + 2D = 0$$

$$-5\left(\frac{10}{36}\right) + 6F + 2\left(\frac{1}{6}\right) = 0$$

$$-\frac{50}{36} + 6F + \frac{1}{3} = 0$$

$$6F = \frac{50}{36} - \frac{1}{3}$$

$$6F = \frac{38}{36}$$

$$F = \frac{38}{36} \times \frac{1}{6} = \frac{38}{216}$$

$$y_p = Dx^2 + Ex + F$$

$$= \frac{1}{6}x^2 + \frac{10}{36}x + \frac{38}{216}$$

$$y_g = Ae^{3x} + Be^{2x} + \frac{1}{6}x^2 + \frac{10}{36}x + \frac{38}{216}$$

Example 3,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$$

Solution,

Form the auxiliary equation.

$$m^2 + m - 2 = 0$$

Factorize the equation,

$$m^2 + 2m - m - 2 = 0$$

$$m(m - 2) - 1(m + 2) = 0$$

$$(m - 1)(m - 2) = 0$$

$$m = 1 \ or - 2$$

$$Ae^{mx} + Be^{mx}$$

$$Ae^{x} + Be^{-2x} = y_c$$

Step 2: y particular form the set from the non-homogenous,

$$S = \{e^x\}$$

NOTE: multiply x with the exponential function.

$$y_p = xe^x$$

Using product rule to differentiate,

$$y'_p = D[xe^x + e^x]$$

$$y''_p = D[xe^x + 2e^x]$$

Recall $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$

$$Dxe^x + 2De^x + Dxe^x + De^x - 2Dxe^x = e^x$$

Equating both sides,

$$3De^x = e^x$$

$$D = \frac{1}{3}$$

$$y_g = Ae^{-2x} + Be^x + \frac{1}{3}xe^x$$

EXERCISE

(1) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 8x^2 + 12e^{-x}$

(2) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$$

(3) $$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 2\sin 3x$$

(4) $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{2x} + 10\sin 3x$$

(5) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{2x} + \cos x$$

CHAPTER FOUR

SOLUTION OF DIFFERENTIAL EQUATION BY OPERATOR *D*

The general solution is $y = y_c + y_p$. The methods of solving differential equation are:

(1) Real but different root $y_c = Ae^{mx} + Be^{mx}$
(2) Equal root. $y_c = e^{mx}(A + Bx)$
(3) Complex root. $y_c = e^{ax}(Acosbx + Bsinbx)$
(4) Only one root. $y_c = Acoshnx + Bsinhnx$

But for y particular,

(i) $F(D)\ \{e^{ax}\} = e^{ax}f(a)$
(ii) $F(D)\ \{e^{ax}\} = e^{ax}F(D + a)\ \ \{V\}$
(iii) $F(D^2)\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} = F(-a^2)\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix}$

Example 1: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

Solution,

Step 1:

Form an auxiliary equation,

$m^2 + 4m + 3 = 0$

Factorize the equation,

$m^2 + 3m + m + 3 = 0$

$m_1 = -1\ \ and\ \ m_2 = -3$

$y_c = Ae^{mx} + Be^{mx}$

$y_c = Ae^{-x} + Bx^{-3x}$

Step 2: to find y_p

To find y_p,

Given $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

Where $D(y) = \frac{dy}{dx}$

$$D^2y + 4Dy + 3y = e^{2x}$$

$$y(D^2 + 4D + 3) = e^{2x}$$

Where $y = y_p$

$$y_p = \frac{1}{D^2+4D+3}e^{2x}$$

Where $F(D) = 2$

$$y_p = \frac{1}{2^2+4(2)+3}e^{2x}$$

$$y_p = \frac{1}{4+8+3}e^{2x}$$

$$y_p = \frac{e^{2x}}{15}$$

Given $y = y_c + y_p$

Where $y_c = Ae^{-x} + Be^{-3x}$ and $y_p = \frac{e^{2x}}{15}$

$$y = Ae^{-x} + Be^{-3x} + y_p = \frac{e^{2x}}{15}$$

Example 2:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 3\cos 4x$$

Step 1:

$$y = y_c + y_p$$

For the auxiliary equation,

$$m^2 + 10m + 25 = 0$$

$$m^2 + 5m + 5m + 25 = 0$$

$$(m + 5) = 0$$

$$m = -5 \ twice.$$

Using $y_c = e^{mx}(A + Bx)$

When $m = 5, then\ y_c = e^{-5x}(A + Bx)$

Step 2: find y particular,

$$\text{Given } \frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 3\cos 4x$$

Where $\frac{dy}{dx} = Dy$

$$D^2y + 10Dy + 25y = 3\cos 4x$$

$$y(D^2 + 10D + 25) = 3\cos 4x$$

Where $y = y_p$

$$y_p = \frac{3\cos 4x}{(D^2+10D+25)}$$

Given, $F(D^2)\ \{\cos 4x\} = F(-16)\ \{\cos 4x\}$

$$D^2 = -16$$

$$y_p = \frac{3\cos 4x}{-16+25+10D} = \frac{3\cos 4x}{10D+9}$$

By rationalizing denominator,

$$y_p = \frac{(10D-9)(3\cos 4x)}{(10D-9)(10D+9)}$$

$$y_p = \frac{(10D-9)(3\cos 4x)}{100D^2-81} \qquad (D^2 = -16)$$

$$y_p = \frac{(10D-9)(3\cos 4x)}{-1600-81}$$

$$y_p = \frac{(10D-9)(3\cos 4x)}{-1681}$$

$$y_p = \frac{-1}{1681}[10D(3\cos 4x) - 9(3\cos 4x)]$$

$$y_p = \frac{-1}{1681}[-10 \times 3 \times 4\sin 4x - 9(3\cos 4x)]$$

$$y_p = -\left(\frac{-1}{1681}\right)[120\sin 4x + 27\cos 4x]$$

$$y_p = \frac{1}{1681}(120\sin 4x + 27\cos 4x)$$

$$y_g = y_c + y_p$$

$$y_g = e^{-5x}(A + Bx) + \frac{1}{1681}(120\sin 4x + 27\cos 4x)$$

EXERCISE

Using operator D to solve the following question.

(1) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = e^{2x}\sin 3x$

(2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$

(3) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

(4) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

(5) $\frac{d^2y}{dx^2} + 9y = \cos x$

WRONSKIAN

By definition: if $y_1, y_2 ------ y_n$ are given function and we have $\propto_1, \propto_2 --- \propto_n$. Therefore $\propto_1 y_1 + \propto_2 y_2 + ---- + \propto_n y_n$ is called the linear combination of $y_1, y_2 ---- y_n$.

If $y_1, y_2 ------ y_n$ are solution of homogenous of differential equation.

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots\dots\dots\dots + a_ny = 0 ------ (*)$$

Then the linear combination $\propto_1 y_1 + \propto_2 y_2 + ---- + \propto_n y_n$ is also a solution of equation (*).

NOTE:

(1) $y_1, y_2, ----, y_n$ are called the fundamental solution of equation (*).

(2) The linear combination $\propto_1 y_1 + \propto_2 y_2 + ---- + \propto_n y_n$ is called the general solution of equation (*).

AC

LINEAR INDEPENDENT

A set of function $y_1, y_2, ----, y_n$ is said to be linearly independent in an interval, if there exist a set of constant.

$$\propto_1, \propto_2, ----, \propto_n \rightarrow \propto_1 y_1 + \propto_2 y_2 + ----- + \propto_n y_n = 0$$

If $\propto_1 = \propto_2 = \propto_n = 0$

Otherwise, they are linearly dependent.

Suppose we have set of n function $\exists$ they possesses at least $n-1$.

Derivative the set linearly independent on an interval of the determinant.

$$W(y_1, y_2, ----, y_n) = \begin{vmatrix} y_1 & y_2 --- & y_n \\ y'_1 & y'_2 --- & y_n \\ & - & \\ & - & \\ & - & \\ y_1^{n-1} & y_2^{n-1} -- & y_n^{n-1} \end{vmatrix} \neq 0$$

For wronskian if $\neq 0$ it is linearly independent but if $= 0$, they are linearly dependent.

The determinant W is called Wronskian if $y_1, y_2, ----, y_n$.

Example 1:

Show that the solution e^x, e^{-x}, e^{2x} of the differential equation

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

Are linearly independent of all interval.

Solution,

$$y_1 = e^x,\ y'_1 = e^x, y''_1 = e^x$$

$$y_2 = e^{-x}, y'_2 = -e^{-x}, y''_2 = e^{-x}$$

$$y_3 = e^{2x}, y_3 = 2e^{2x},\ y_3 = 4e^{2x}$$

Hence $W(e^{x}, e^{-x},\ e^{2x}) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{x} & e^{-x} & e^{2x} \\ e^{x} & -e^{-x} & 2e^{2x} \\ e^{x} & e^{-x} & 4e^{2x} \end{vmatrix}$$

Take the determinant.

$$e^{x}\begin{vmatrix} -e^{-x} & -e^{-x} \\ e^{x} & e^{-x} \end{vmatrix} - e^{-x}\begin{vmatrix} e^{x} & 2e^{2x} \\ e^{x} & 4e^{4x} \end{vmatrix} + e^{2x}\begin{vmatrix} e^{x} & -e^{-x} \\ e^{x} & e^{-x} \end{vmatrix}$$

$$e^{x}[-4e^{x} - 2e^{x}] - e^{-x}[4e^{3x} - 2e^{3x}] + e^{2x}[e^{0} + e^{0}]$$

Example:

If $\propto$ *and* β are number, $\beta \neq 0$, then $y_1 = e^{\propto x}\cos\beta x,\ y_1 = e^{\propto x}\sin\beta x.$ Determine whether it s linear independent or dependent.

CHAPTER FIVE

LAPLACE TRANSFORMATION

Laplace transformations help in solving the differential equation with boundary values without finding the general solution and the values of the arbitrary constants. By definition, let $f(t)$ be function defined for all positive values of t, then $f(s) = \int_0^{\infty} e^{-st} f(t)dt.$

Provided the integral exist, is called the Laplace transformation of $f(t)$. It is denoted as $L[f(t)] = f(s) = \int_0^{\infty} e^{-st} f(t)dt.$

Properties of Laplace Transformation.

1) If ∞ and β are real constant and $f(t)$ and $g(t)$ are continuous function or piecewise continuous.

$$L\{\propto f(t) \pm \beta g(t)\} = \propto L\{f(t)\} \pm \beta L\{g(t)\}$$

Important formulae

Prove $L(k) = \frac{K}{s}$

Solution,

$$L(k) = L\{f(t)\}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

$$L(k) = \int_0^\infty e^{-st} dt$$

$$L(k) = k\int_0^\infty e^{-st} dt$$

$$= k\left[\frac{-e^{-st}}{s}\right]_0^\infty$$

$$= -\frac{k}{s}[e^\infty - e^0]$$

$$= -\frac{k}{s}[0-1] = \frac{k}{s} \qquad proved$$

2) $L(e^{at}) = \frac{1}{S-1}$

Solution,

$$L(f(t)) = \int_0^\infty e^{-st} f(t) dt = f(s)$$

$$L(e^{at}) = \int_0^\infty e^{-st}.e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{s-a}[e^{-\infty} - e^0]$$

$$= \frac{1}{s-a}$$

$$L(e^{at}) = \frac{1}{s-a} = f(s) \qquad proved$$

3) $L(t^n) = \int_0^\infty e^{-st}\, t^n dt$

Solution,

Let $p = st, t = \frac{p}{s}, \frac{dp}{dt} = s,\ dp = sdt.$

$$dt = \frac{dp}{s}$$

Recall $\int_0^\infty e^{-st}.t^n dt$

$$= \int_0^\infty e^{-p} \left(\frac{p}{s}\right)^n . \frac{dp}{s}$$

$$= \int_0^\infty e^{-p} \frac{p^n}{s^n} . \frac{dp}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}.p^n dp \text{ - - - - - - - - - - - - } (*)$$

FOR GAMMA FUNCTION

$$\lceil\alpha = \int_0^\infty u^{\alpha-1}.e^u du$$

Relate the equation (*) with the gamma function p=u.

$$= \int_0^\infty p^n.e^{-p} dp \text{ ------------------------------- } 1$$

$$n = \alpha - 1, \quad \alpha = n + 1$$

$$= \frac{1}{s^{n+1}} \int_0^\infty p^{(n+1)-1}.e^{-p} dp \text{ ----------------------- } 2$$

1 and 2 are the same.

$$= \frac{\lceil n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$L(t^n) = \frac{n!}{s^{n+1}} \quad proved.$$

4) $L(\sin at) = \frac{a}{s^2+a^2}$

Solution,

From hyperbolic function,

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\} = \frac{1}{2i} L\left(e^{iat} - e^{-iat}\right)$$

$$= \frac{1}{2i}[L\{e^{iat}\} - L\{e^{-iat}\}]$$

NOTE:

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ai-s+ai}{(s-ia)(s+ia)}\right] \qquad note\ i^2 = -1$$

$$= \frac{1}{2i}\left[\frac{2ai}{s^2+a^2}\right] = \frac{a}{s^2+a^2}$$

$$\sin at = \frac{a}{s^2+a^2} \qquad (proved)$$

5) $\cos at$: in the same way,

$$\cos at = \frac{s}{s^2+a^2}$$

6) $L(\cos hat) = \frac{s}{s^2-a^2}$

Proof $L(\cos hat) = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$$\left(\cosh at = \frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{1}{2}\left[\frac{2s}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+a^2} \qquad proved$$

$$L(\cosh at) = \frac{s}{s^2+a^2}$$

7) $\sinh at$: Solve it the same way as $\cosh at$. Try it.

$$L(\cosh at) = \frac{s}{s^2-a^2}$$

Examples,

(1) $L(1) = \frac{1}{s}$

(2) $L(e^{5t} + 3) = \frac{1}{s-5} + \frac{3}{s}$

(3) $L(8 \cos 3t) = 8\left(\frac{s}{s^2+9}\right)$

(4) $L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

SHIFT THEOREM

Theorem1: if f (t) is a piecewise continuous function and is of exponential order.

$$L\{f(t)\} = f(s) \text{ Then,}$$
$$L(e^{at}f(t)) = f(s-a) \text{--------} (1)$$

NOTE: find the Laplace transformation of the function first and subtract a from the S.

Example1, $L\{e^{at} \sin \propto t\} = \frac{\propto}{t^2+\propto^2}$

$$= \frac{\propto}{(t-a)^2+\propto^2}$$

Example2,

$L\{2e^{3t} \sin 3t\}$

Solution,

$L(2e^{3t}) = \frac{2}{s-3}$

$L(\sin 3t) = \frac{3}{s^2+9}$

Combining the two,

$L\{2e^{3t} \sin 3t\} = \frac{6}{(s-3)^2+9}$

Theorem 2:

MULTIPLYING BY t

Theorem 2: if $f(t)$ is a piecewise continuous function and of exponential order.

If $L\{f(t)\} = f(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n f(s)}{ds^n}$

Example,

$$L\{t^2 \cos 2t\}$$

Solution,

$$L\{t^2 \cos 2t\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

Using quotient rule,

$$\frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right] \quad \text{Re-differentiate}$$

$$\frac{d^2}{ds^2}\left[\frac{(s^2+4)^2(-2s)-(4-s^2)(s^2+4)(4s)}{(s^2+4)^4}\right]$$

$$(s^2+4)\left[\frac{-2s^3-8s-16s+4s^3}{(s^2+4)^2}\right]$$

$$\frac{2s^3-24s}{(s^2+4)^3} = \frac{2s(s^2-12)}{(s^2-4)^3}$$

Division Theorem,

If $f(t)$ is a piecewise continuous function and of exponential order,

If $f(t) = f(s)$ then,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} f(s)ds$$

Example,

$$L\left\{\frac{1-e^{-t}}{t}\right\}$$

Solution,

$$L\left\{\frac{1-e^{-t}}{t}\right\} = L\left\{\frac{1}{t}\right\} - L\left\{\frac{e^{-t}}{t}\right\}$$

$$= \int_s^{\infty} \left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= \int_s^{\infty} \frac{1}{s} - \int_s^{\infty} \frac{1}{s+1}$$

$$= [\ln s - \ln s + 1]_s^{\infty}$$

$$= \left[\ln \frac{s}{s+1}\right]_s^{\infty} \text{ Divide through by 3}$$

$$= \left[\ln \frac{1}{1+\frac{1}{s}}\right]_s^{\infty}$$

Put the limit value in the function.

$$= \ln 1 - \ln \frac{s}{s+1}$$

$$= 0 - [\ln s - \ln(s+1)]$$

$$= \ln(s+1) - \ln s$$

$$= \left(\frac{s+1}{s}\right)$$

Questions,

Solve the differential equation using Laplace transformation

$$y'' - 3y' + 2y = 2e^{-t} \text{ At,}$$

$$y(0) = 2, \quad y'(0) = -1$$

$$y'' = f''(t) = \ddot{y}, \; y' = f'(t) = \dot{y}$$

Applying Laplace Transformation,

$$L\{\ddot{y}\} = 3L\{\dot{y}\} + 2L\{y\} = 2L\{e^{-t}\}$$

$$[s^2L\{y\} - sy\{0\} - y'(0)] - 3[sL\{y\} - y(0)] + 2L\{y\} = \frac{2}{s+1}$$

Substitute the initial value condition,

$$s^2L\{y\} - 2s + 1 - 3[sL\{y\} - 2] + 2L\{y\} = \frac{2}{s+1}$$

$$s^2L\{y\} - 2s + 1 - 3sL\{y\} + 6 + 2L\{y\} = \frac{2}{s+1}$$

Collect the like terms,

$$\{s^2 - 3s + 2\}L\{y\} - 2s + 7 = \frac{2}{s+1}$$

$$\{s^2 - 3s + 2\}L\{y\} = \frac{2}{s+1} + \frac{2s-7}{1}$$

Find the L.C.M.

$$= \frac{2-(2s-7)(s+1)}{s+1}$$

$$= \frac{2+2s^2-5s-7}{s+1}$$

$$(s^2 - 3s + 2)L\{y\} = \frac{2s^2-5s-5}{s+1}$$

$$L\{y\} = \frac{2s^2-5s-5}{(s+1)(s^2-3s+2)}$$

$$L\{y\} = \frac{2s^2-5s-5}{(s+1)(s-1)(s-2)}$$

$$\frac{2s^2-5s-5}{(s+1)(s-1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$= A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$$

$$2s^2 - 5s - 5 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$$

When $s = 1$

$$2(1)^2 - 5 - 5 = A(0) + B(2)(-1) + C(2)(0)$$

$$2 - 10 = 0 - 2B + 0$$

$$-8 = -2B$$

$$B = 4$$

When

$$s = -1$$

$$2(-1)^2 - 5(-1) - 5 = A(-2)(-3) + B(0) + C(0)$$

$$2 + 5 - 5 = 6A$$