Introductory Mathematics I (MTS 101)
Sequences and Series

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1. Objective

After studying the topics, student should:

1. be able to recognize geometric and arithmetic sequences
2. understand Σ notation for sum of series
3. be familiar with standard formula for Σr , Σr^2 and Σr^3.

2. Introduction

Suppose you go on a sponsored walk. In the first hour you walk 3 miles, in the second hour 2 miles and in each succeeding hour $\frac{2}{3}$ of the distance the previous hour. Hour far would you walk in 10 hours? How far would you go if you kept on like this forever?

Arithmetic Sequence

Arithmetic Sequence deals with common difference between consecutive terms .
In an experiment to measure the descent of a trolley rolling down a slope, a ticker tape timer is used to measure the distance traveled in each second.
The sequence 3,5,7,9,11,13,15 are example of an arithmetic sequence. The sequence starts with S and thereafter each term is 2 times more than the previous one. The difference of 2 is known as the common difference.

It is also useful to find the total distance traveled in the first 7 seconds by adding the numbers together. A quick numerical trick for doing this is to imagine writing the numbers out twice, once forward once backward as shown below:

$$\begin{array}{ccccccc} 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 15 & 13 & 11 & 9 & 7 & 5 & 3 \end{array}$$

Each pair of vertical numbers add up to 18. So adding the sequences, you have 7×18 between them. Hence, the sum of the Original series is
$\frac{1}{2}(7 \times 18) = 63$
The sum of terms of an arithmetic sequence is called arithmetic series or progression, often called AP for short.
Using the above example of a trolley rolling down a slope to answer the following:

...ample: Consider the arithmetic sequence 6, 10, 14, 18,... , find expressions

- for U_n (the nth term)
- for S_n.

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this A.P, the fist term is 6 and the common difference is 4

$U_1 = 6$, $U_2 = 6 + 4$, $U_3 = 6 + 2(4)$ and $U_4 = 6 + 3(4)$

is obtained by adding on the common difference(n-1) times

$U_n = 6 + 4(n-1)$

$= 2 + 4n$,

find S_n, follow the procedure explained previously

6, 10,..., 4n-2 , 4n+2

4n+2 , 4n-2 , 10 , 6

Each pair adds up to 4n+8. There are n pairs so that

$$2S_n = n(4n+8)$$
$$2S_n = 4n(n+2)$$
$$S_n = 2n(n+2).$$

1 Exercise

1. Obtain the formulas for U_n and S_n in the sequences
   - 1,4,7,10,
   - 12,21,30,39,
   - 60,56,50,45,
   - 1, $2\frac{1}{2}$, 4, $5\frac{1}{2}$.
2. Use the 'numerical...

6 (

$U_1 =$
$U_2 =$
$U_3 =$
$U_4 =$
$\boxed{U_n =}$
U_n
=

6 14 18 4n+2, 4n+2
-2 10 14 10 6
4n(n
n(4n+8)

1 4 7
+3 +3 +3
$1 + 3(n-1)$
$n = 0 \sim \infty$
12
3x4 9

- $3+4+11+\ldots+27$,
- $52+46+40+\ldots+4$,
- the sum of all the numbers on a traditional clock face,
- the sum of all the odd numbers between 1 and 99.

3. A model railway manufacturer makes pieces of tracks of length 8cm, 10cm, 12cm, e.t.c. up to and including 38cm. An enthusiast buys 5 pieces of each length. What total length of track can be made?

The general arithmetic sequence is often denoted by:

$$a,\ a+d,\ a+2d,\ a+3d,\ \text{e.t.c.}$$

$a, a+d$

To sum the series of the first n terms of the sequence,

$$S_n = a + (a+d) + (a+2d) + \ldots + [a + (n-1)d].$$

Note that the order can be reversed to give

$$S_n = [a + (n-1)d] + \ldots + (a+2d) + (a+d) + a.$$

Adding the two sequences for S_n gives

$$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + \ldots + [2a + (n-1)d]$$
$$= n[2a + (n-1)d].$$

So

$$S_n = \tfrac{n}{2}[2a + (n-1)d].$$

An alternative form for S_n is given in terms of nth first and last term, a and l, where

$$l = a + (n-1)d.$$

Since the nth term of the sequence is given by

$$U_n = a + (n-1)d.$$

Thus ,

$$S_n = \frac{n}{2}(a+l)$$

Example

- Sum the series 2 + 6 + 10 + 14 + 18 + ... to the 20th term.

Solution:

This is an arithmetic sequence with first term 2 and common difference 4; so

$$S_{20} = \frac{20}{2}[2 \times 2 + (19 \times 4)]$$
$$= 800$$

- The sum of the series 1 + 6 + 11 +16 +21 + ... is 100. Find the number of term in the series.
- How many term does the series 1 + 8+ 15 + ... =396 contains?.

This is an arithmetic sequence with first term 1 and common difference 5. Let the number of terms in the sequence be n

S_n
$= \frac{n}{2}[2 + 5(\text{n-1})] = 100$
$n[2 + 5(n-1)] = 200$
$n(5n-3) = 200$
$5n^2 - 3n - 200 = 0.$

Exercise

1. Ten brothers receives 100 shekels between them. Each brother receives a constant amount more than the next oldest. The seventh oldest receives 7 shekels. How much does each brothers receives?

2. The last three terms of an arithmetic sequence with 18 terms is as follows; ..., 67, 72, 77. Find the first term and sum of the series.

3. Find the nth term and the sum of the series 67,72, 77....

4. How many terms are there if:

- $52 + 49 + 46 + ... = 385$

- $0.35 + 0.52 + 0.69 + ... = 35.72.$

5. The first term of an arithmetic series is 16 and the last is 60. The sum of the arithmetic series is 342. Find the common difference.

Geometric Sequence

The series of number 1,2,4,8,16,... is an example of a geometric sequence, sometimes called a geometric progression. Each term in the progression is found multiplying by the previous number by 2.

For example: If you invested £2000 in an account with fixed interest rate of 8 percent per annum, then the amount of money in the account in 1 year, 2 year, 3 year, e.t.c would be as shown below. The first number in the sequence is 2000 and each successive number is fixed by multiplying by 1.08 each time

No. of year	Money in account
0	2000
1	2160
2	2332.80
3	2159.42
4	2720.98

The general form of a geometric sequence with nth term is

$$a, ar, ar^2, ..., ar^{n-1}$$

The ratio r of consecutive terms is known as the common ratio.
Notice that the nth term of the sequence is ar^{n-1}
The sum of a geometric sequence with n term gives

$$S_n = a + ar + ar^2 + ... + ar^{n-1} \quad ....(1)$$

Multiplying both sides by r,

$$rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n \quad ....(2)$$

Note that the expression for S_n and rS_n are identical, with the exception of the terms a and ar^n. Subtracting equation (1) from (2)

$$\begin{aligned} rS_n - S_n &= ar^n - a \\ (r-1)S_n &= a(r^n - 1) \\ S_n &= \frac{a(r^n-1)}{r-1}, if r \neq 1 \end{aligned}$$

Sometimes it is useful to write

$$S_n = \frac{a(1-r^n)}{1-r} \text{ instead of } S_n = \frac{a(r^n-1)}{r-1}$$

Why are these formulas identical? When might it be more convenient to use the alternate form?

Example: Find.

1. $4 + 6 + 9 + ... + 4 \times (1.5)^{10}$
2. $8 + 6 + 4.5 + ... + 8 \times (0.75)^{25}$ 4%

Example: A plant grows 1.67cm in its first week. Each week it grows by 4 percent more than it did the week before. By how much does it grow in nine weeks, including the first week

Solution

The growth in the first nine weeks are as follows:

$$1.67, 1.67 \times 1.04, 1.67 \times 1.04^2, 1.67 \times 1.04^3, ...$$

Total growth in the first nine weeks is $S_9 = \frac{a(r^n-1)}{r-1} = \frac{1.67(1.04^9-1)}{1.04-1} =$

$$S_9 = \frac{1.67(1.04^9-1)}{1.04-1} = 17.67\text{cm to 4 significance figure.f}$$

Example: After how many complete years will a starting capital of £5000 first exceed £10000 if it grows at 6 percent per annum?

Exercise

1. Why is this a geometric sequence?

   $7.2\% \times 500$

   $\frac{7.2}{100} \times \frac{500}{1}$

   1, -2, 4, -8, 16,....

   What is its common ratio? What is its nth term ? What is its S_n?

2. Dave invest £500 in a building society account at the start of each year, the interest rate is 7.2 percent per annum. Immediately after he invest his $12th$ installment, he calculates how much money the account should contain. Show this calculation as the sum of a GP and as the formula for S_n to evaluate it.

Limit of a Sequence

2.2 Convergence and Divergence

We say the sequence a_n converges if $lim\ a_n$ is a real number. If $lim_{n\longrightarrow\infty} a_n$ is infinite or does not exist, the sequence diverges.

Example: Does the sequence $\{n^2\}$ converge or diverge?

Example: Does the sequence $\{\frac{n^2+1}{3n^2+4n+2}\}$ converge or diverge?

Example: Consider the sequence -1, 1, -1, 1, ...

Example: Consider the sequence $lim_{x\longrightarrow\infty} \sin x$

Exercise:

1. Find a formula for the general term a_n of the sequence assuming the pattern of the first few term continues

   $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \cdots$

- $\frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \ldots$
- $3, 2\frac{1}{2}, 2, 1\frac{1}{2}, 1, \ldots$
- $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$
- $10, 50, 250, 1250, \ldots$
- $\frac{1}{1}, \frac{2^2}{3}, \frac{3^3}{5}, \frac{4^4}{7}, \ldots$

2. Determine whether the given sequence converges or diverges. If it converges, find the limit.

$a_n = \left\{\frac{n^2-1}{n^2+1}\right\}_{n=1}^{\infty}$
$a_n = \left\{\frac{1+3n}{n+2}\right\}_{n=1}^{\infty}$
$a_n = \left\{\frac{n}{1+e^{-n}}\right\}_{n=1}^{\infty}$

3. The Infinite Geometric Series

Recall that the common geometric series is; $a + ar + ar^2 + \ldots$.
Then,
$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}r^n$

Now if , r^n decreases as n increases, we say that the limiting value of r^n is zero. Thus, as n increases , S_n approaches the limiting value.

$S_n = \frac{a}{1-r}$, we say that the series converges to sum of $\frac{a}{1-r}$ the sum to infinity of the series.

Thus, if

the sum to infinity of the geometric series

$$a + ar + ar^2 + ar^3 + \ldots \text{ is}$$
$$lim_{n\longrightarrow\infty}S_n = S = \frac{a}{1-r}$$

3.1 Example

1. To what sum does the following series converges $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$?
2. Express the recurring decimal 0.272727^{-} as a fraction in its lowest term

3.2 Exercise

1. An Uncle places a sum of money in a savings account for a nephew when he his born. On each succeeding birthday, the uncle deposits two times more than the

previous birthday, the total sum of the first 11 deposit is 20,450.00. How larg was the first deposit? What is the sum for eight deposits.

2. Find x, y such that the sequence 2, x, y, 9 will have the property that the first three are terms of $A.S$ and the last three belongs to $G.S$.

3. If the sum of the first n terms of an $A.S$ is $2n$ and the sum of the first $2n$ terms is n , find the sum of the first $4n$ terms.

4. The sum of an $A.P$ whose first and last term are respectively 4 and 26 is 180. How many terms are there in the sequence?

5. Find the first three terms of the $G.P$ whose sum and product 14 and 64 respectively.

6. The third and the sixth term of a $G.P$ are 36 and $\frac{243}{2}$ respectively. Find the $G.P$

7. Express the recurring decimal 0.31^{-} as a fraction in its lowest term.

8. The xth , yth , zth term of a sequence are X, Y, Z respectively
Show that;

- if the sequence is arithmetic
$X(y-z) + Y(z-x) + Z(x-y) = 0$
- If the sequence is geometric
$(y-z)logX + (z-x)logY + (x-y)logZ = 0$

9. The sum of the first n terms of a geometric series is 145 and sum of the reciprocal is $\frac{145}{33}$. The first term is 1, find n and the common ratio.

10. The $3p^{th}$ term of an $A.P$ is 56 more than p^{th} term toward the $(p+1)$ term is 60. Find the first term.

11. Find the three number in arithmetic progression whose sum is 6 and whose product is 64.

12. Find the sum to infinity of the sequence 0.252525^{-}.

13. Find the sum to infinity of the sequence $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$

14. The first term term of an arithmetic series is 16 and the last term is 60. The sum of the series is 342. Find the common difference.

15. The first two terms of an arithmetic series are -2 and 3. How many terms are needed for the sum to equal 306?

16. The sum of three numbers in arithmetic progression is 18 and the sum of their squares is 158. Find the numbers.

17. The sum of the [illegible] terms of a series is [illegible], the sum of their reciprocal is $\frac{[illegible]}{[illegible]}$ and the first term is 1. Find n and the common ratio.

18. Find three numbers of arithmetic progression whose sum is 3 and their product [illegible]

1. The first term of an arithmetic series is 3, the common difference is 4 and the sum of all term is 820. Find the number of terms and the last term.

2. Find the sum to which the series converges $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots$

3. What is the ninth term of the geometric sequence? If the third term and the seventh term are -1 and -81?

4. What is the second term of the geometric sequence if the three numbers in geometric sequence whose their sum is 13 and their product is 64.

5. Evaluate the tenth term of the series $3+9+27+81+\ldots$

6. Find the three numbers in arithmetic progression whose sum is 3 and whose product is -15.

7. If $U_1=-1, U_2=-5$ and $U_r=a+br$, find a and b.

8. To what sum does the following series converges $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots$

9. The first term and the last term of a geometric series are 3 and 768, if the sum of the terms is 1533. Find the common ratio.

10. Find the first term and common ratio of the geometric series if the third term and the seventh term are 81 and 16 respectively.

11. The sum of the first twenty terms of an arithmetic progression is 45, and the sum of the first forty term is 290. Find the first term and the common difference.

12. Find the formula for $12,6,3,\frac{3}{2},\frac{3}{4},\ldots$

13. What limit does the sequence $\left\{\frac{n^2+1}{3n^2+4n+2}\right\}_{n=1}^{\infty}$ converges into?

14. Find the S_n for the arithmetic sequence $8,12,16,20,\ldots$

15. Find the sum of all the even numbers between 2 and 100.

16. A model railway manufacturer makes pieces of track of lengths 8cm, 10cm, 12cm etc up to and including 38cm. An enthusiast buys 5 pieces of each length. What is total length of track can be made?

17. The general formula for the reciprocal of geometric sequences is...

18. The sums $1, 1+2, 1+2+3,\ldots$ referred to as