Introductory Mathematics II (MTS 101) Trigonometry

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1 Introduction

1.1 Definition

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

1.2 Revision

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- Circular Measures
- Triangles
- Pythagoras' Theorem

2 Trigonometric Ratios

From the right-angled triangle shown in Figure 1, we have the following trigonometric ratios:

- $\sin\theta = \frac{a}{c}$
- $\cos\theta = \frac{b}{c}$
- $\tan\theta = \frac{a}{b}$
- $\csc\theta = \frac{1}{\sin\theta} = \frac{c}{a}$
- $\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$
- $\cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}$

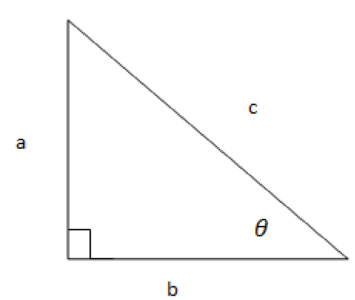

Figure 1: A Right-Angled Triangle

3 Trigonometric Identities

3.1 Basic Identities

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b} = \tan\theta \tag{3.1}$$

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{c}{a}}{\frac{c}{b}} = \frac{c}{a} \times \frac{b}{c} = \frac{b}{a} = \cot\theta \tag{3.2}$$

With reference to Figure 1, using Pythagoras' Theorem,

$$a^2 + b^2 = c^2 \tag{3.3}$$

Dividing (3.3) through by c^2, we have

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{3.4}$$

Dividing (3.3) through by b^2, we have

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{3.5}$$

Dividing (3.3) through by a^2, we have

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

$$1 + \cot^2\theta = \csc^2\theta \tag{3.6}$$

3.2 Examples

3.2.1 Example 1

To show that $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$,

$$\begin{aligned}
\sin^2\theta + (1+\cos\theta)^2 &= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta \\
&= 1 + 1 + 2\cos\theta \quad \text{(Recall that } \sin^2\theta + \cos^2\theta = 1\text{)} \\
&= 2 + 2\cos\theta \\
&= 2(1+\cos\theta)
\end{aligned}$$

$\therefore \ \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

3.2.2 Example 2

To show that $\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$,

$$\begin{aligned}
\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} &= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\frac{1}{\cos\theta}}{1+\frac{1}{\sin\theta}} \\
&= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+1}{\sin\theta}} \\
&= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \div \frac{\sin\theta+1}{\sin\theta} \\
&= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \cdot \frac{\sin\theta}{\sin\theta+1} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta
\end{aligned}$$

$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

3.2.3 Example 3

To show that $\frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{\sec\theta-1}{\sec\theta+1} = 1$,

$$\begin{aligned}
\frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{\sec\theta-1}{\sec\theta+1} &= \frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{\frac{1}{\cos\theta}-1}{\frac{1}{\cos\theta}+1} \\
&= \frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{1+\cos\theta}{\cos\theta}} \\
&= \frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{1-\cos\theta}{\cos\theta} \div \frac{1+\cos\theta}{\cos\theta} \\
&= \frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{1-\cos\theta}{\cos\theta} \cdot \frac{\cos\theta}{1+\cos\theta} \\
&= \frac{\cos\theta}{\cos\theta} \\
&= 1
\end{aligned}$$

$\therefore \frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{\sec\theta-1}{\sec\theta+1} = 1$

3.2.4 Example 4

To show that $(x^j)^2 + (y^j)^2 = x^2 + y^2$ if $x^j = x\cos\theta + y\sin\theta$ and $y^j = x\sin\theta - y\cos\theta$,

$$\begin{aligned}
(x^j)^2 + (y^j)^2 &= (x\cos\theta + y\sin\theta)^2 + (x\sin\theta - y\cos\theta)^2 \\
&= x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\
&= x^2\sin^2\theta + x^2\cos^2\theta + y^2\sin^2\theta + y^2\cos^2\theta \\
&= x^2(\sin^2\theta + \cos^2\theta) + y^2(\sin^2\theta + \cos^2\theta) \\
&= x^2(1) + y^2(1) \quad \text{(Recall that } \sin^2\theta + \cos^2\theta = 1\text{)} \\
&= x^2 + y^2
\end{aligned}$$

$\therefore (x^j)^2 + (y^j)^2 = x^2 + y^2$ if $x^j = x\cos\theta + y\sin\theta$ and $y^j = x\sin\theta - y\cos\theta$

3.2.5 Example 5

To show that $x^2 + y^2 + z^2 = c^2$ if $x = c\sin\theta\cos\varphi$, $y = c\sin\theta\sin\varphi$ and $z = c\cos\theta$,

$$\begin{aligned}
x^2 + y^2 + z^2 &= (c\sin\theta\cos\varphi)^2 + (c\sin\theta\sin\varphi)^2 + (c\cos\theta)^2 \\
&= c^2\sin^2\theta\cos^2\varphi + c^2\sin^2\theta\sin^2\varphi + c^2\cos^2\theta \\
&= c^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + c^2\cos^2\theta \\
&= c^2\sin^2\theta(1) + c^2\cos^2\theta \\
&= c^2\sin^2\theta + c^2\cos^2\theta \\
&= c^2(\sin^2\theta + \cos^2\theta) \\
&= c^2(1) \\
&= c^2
\end{aligned}$$

$\therefore\ x^2 + y^2 + z^2 = c^2$ if $x = c\sin\theta\cos\varphi$, $y = c\sin\theta\sin\varphi$ and $z = c\cos\theta$

4 The Addition Formulae

4.1 Basic Formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{4.1}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{4.2}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{4.3}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{4.4}$$

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \\
&= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}
\end{aligned}$$

$$\therefore\ \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \tag{4.5}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \tag{4.6}$$

4.2 Examples

4.2.1 Example 1

To calculate $\sin 15^\circ$ given that $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$, $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$,

$$\begin{aligned}
\sin 15^\circ &= \sin(45-30)^\circ \\
&= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ \\
&= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}\sin 15^\circ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

4.2.2 Example 2

To show that $\tan(45^\circ + A) = \frac{1+\tan A}{1-\tan A}$,

Recall that $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, so

$$\begin{aligned}\tan(45^\circ + A) &= \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A}\\ &= \frac{1+\tan A}{1-(1)\tan A} \quad \text{(Recall that } \tan 45^\circ = 1)\\ \therefore\ \tan(45^\circ + A) &= \frac{1+\tan A}{1-\tan A}\end{aligned}$$

4.2.3 Example 3

To show that $(\cos\theta + \cos\varphi)^2 + (\sin\theta + \sin\varphi)^2 = 2 + 2\cos(\theta - \varphi)$,

$$\begin{aligned}(\cos\theta + \cos\varphi)^2 + (\sin\theta + \sin\varphi)^2 &= \cos^2\theta + 2\cos\theta\cos\varphi + \cos^2\varphi + \sin^2\theta + 2\sin\theta\sin\varphi + \sin^2\varphi\\ &= (\cos^2\theta + \sin^2\theta) + (\cos^2\varphi + \sin^2\varphi) + (2\cos\theta\cos\varphi + 2\sin\theta\sin\varphi)\\ &= 1 + 1 + 2(\cos\theta\cos\varphi + \sin\theta\sin\varphi)\\ &= 2 + 2\cos(\theta-\varphi) \quad [\text{Recall that } \cos(A-B) = \cos A\cos B + \sin A\sin B]\end{aligned}$$

4.2.4 Example 4

To show that $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan A\tan C - \tan B\tan C}$,

$$\begin{aligned}\tan(A+B+C) &= \frac{\sin(A+B+C)}{\cos(A+B+C)}\\ &= \frac{\sin[(A+B)+C]}{\cos[(A+B)+C]}\\ &= \frac{\sin(A+B)\cos C + \cos(A+B)\sin C}{\cos(A+B)\cos C - \sin(A+B)\sin C}\\ &= \frac{(\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C}{(\cos A\cos B - \sin A\sin B)\cos C - (\sin A\cos B + \cos A\sin B)\sin C}\\ &= \frac{\sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C}{\cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C}\\ &= \frac{\frac{\sin A\cos B\cos C}{\cos A\cos B\cos C} + \frac{\cos A\sin B\cos C}{\cos A\cos B\cos C} + \frac{\cos A\cos B\sin C}{\cos A\cos B\cos C} - \frac{\sin A\sin B\sin C}{\cos A\cos B\cos C}}{\frac{\cos A\cos B\cos C}{\cos A\cos B\cos C} - \frac{\sin A\sin B\cos C}{\cos A\cos B\cos C} - \frac{\sin A\cos B\sin C}{\cos A\cos B\cos C} - \frac{\cos A\sin B\sin C}{\cos A\cos B\cos C}}\\ &= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan A\tan C - \tan B\tan C}\end{aligned}$$

$\therefore\ \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$

4.2.5 Example 5

To show that $P + Q + R = 45^\circ$ if $\tan P = \frac{1}{2}$, $\tan Q = \frac{1}{5}$ and $\tan R = \frac{1}{8}$,

$$\begin{aligned}
\tan(P+Q+R) &= \frac{\tan P + \tan Q + \tan R - \tan P \tan Q \tan R}{1 - \tan P \tan Q - \tan P \tan R - \tan Q \tan R} \\
&= \frac{\frac{1}{2} + \frac{1}{5} + \frac{1}{8} - \frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{8}}{1 - \frac{1}{2} \cdot \frac{1}{5} - \frac{1}{2} \cdot \frac{1}{8} - \frac{1}{5} \cdot \frac{1}{8}} \\
&= \frac{\frac{40+16+10-1}{80}}{\frac{80-8-5-2}{80}} \\
&= \frac{\frac{65}{80}}{\frac{65}{80}} \\
&= \frac{65}{80} \div \frac{65}{80} \\
&= \frac{65}{80} \cdot \frac{80}{65} \\
&= 1 \\
P + Q + R &= \tan^{-1} 1 \\
&= 45^\circ
\end{aligned}$$

$\therefore\ P + Q + R = 45^\circ$ if $\tan P = \frac{1}{2}$, $\tan Q = \frac{1}{5}$ and $\tan R = \frac{1}{8}$

5 Multiple Angles

5.1 Double Angles

$$\begin{aligned}
\sin 2A &= \sin(A + A) \\
&= \sin A \cos A + \cos A \sin A \\
&= 2 \sin A \cos A
\end{aligned}$$

$$\begin{aligned}
\cos 2A &= \cos(A + A) \\
&= \cos A \cos A - \sin A \sin A \\
&= \cos^2 A - \sin^2 A \\
&= \cos^2 A - (1 - \cos^2 A) \\
&= \cos^2 A - 1 + \cos^2 A \\
&= 2\cos^2 A - 1
\end{aligned}$$

$$\begin{aligned}
\tan 2A &= \tan(A + A) \\
&= \frac{\tan A + \tan A}{1 - \tan A \tan A} \\
&= \frac{2 \tan A}{1 - \tan^2 A}
\end{aligned}$$

5.2 Triple Angles

$$
\begin{aligned}
\sin 3A &= \sin(A + 2A) \\
&= \sin A \cos 2A + \cos A \sin 2A \\
&= \sin A \cos(A + A) + \cos A \sin(A + A) \\
&= \sin A(\cos A \cos A - \sin A \sin A) + \cos A(\sin A \cos A + \cos A \sin A) \\
&= \sin A(\cos^2 A - \sin^2 A) + \cos A(2 \sin A \cos A) \\
&= \sin A \cos^2 A - \sin^3 A + 2 \sin A \cos^2 A \\
&= 3 \sin A \cos^2 A - \sin^3 A \\
&= 3 \sin A(1 - \sin^2 A) - \sin^3 A \\
&= 3 \sin A - 3 \sin^3 A - \sin^3 A \\
&= 3 \sin A - 4 \sin^3 A
\end{aligned}
$$

$$
\begin{aligned}
\cos 3A &= \cos(A + 2A) \\
&= \cos A \cos 2A - \sin A \sin 2A \\
&= \cos A \cos(A + A) - \sin A \sin(A + A) \\
&= \cos A(\cos A \cos A - \sin A \sin A) - \sin A(\sin A \cos A + \cos A \sin A) \\
&= \cos A(\cos^2 A - \sin^2 A) - \sin A(2 \sin A \cos A) \\
&= \cos^3 A - \sin^2 A \cos A - 2 \sin^2 A \cos A \\
&= \cos^3 A - 3 \sin^2 A \cos A \\
&= \cos^3 A - 3(1 - \cos^2 A) \cos A \\
&= \cos^3 A - 3 \cos A + 3 \cos^3 A \\
&= 4 \cos^3 A - 3 \cos A
\end{aligned}
$$

$$
\begin{aligned}
\tan 3A &= \tan(A + 2A) \\
&= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A} \\
&= \frac{\tan A + \tan(A + A)}{1 - \tan A \tan(A + A)} \\
&= \frac{\tan A + \left(\frac{\tan A + \tan A}{1 - \tan A \tan A}\right)}{1 - \tan A\left(\frac{\tan A + \tan A}{1 - \tan A \tan A}\right)} \\
&= \frac{\tan A + \frac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A\left(\frac{2 \tan A}{1 - \tan^2 A}\right)} \\
&= \frac{\tan A(1 - \tan^2 A) + 2 \tan A}{(1 - \tan^2 A) - 2 \tan^2 A} \qquad \text{(Multiplying through by } 1 - \tan^2 A\text{)} \\
&= \frac{\tan A - \tan^3 A + 2 \tan A}{1 - \tan^2 A - 2 \tan^2 A} \\
&= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}
\end{aligned}
$$

6 Submultiple Angles

$$\begin{aligned}\sin A &= \sin\left(\frac{A}{2}+\frac{A}{2}\right)\\ &= \sin\frac{A}{2}\cos\frac{A}{2}+\cos\frac{A}{2}\sin\frac{A}{2}\\ &= 2\sin\frac{A}{2}\cos\frac{A}{2}\end{aligned}$$

$$\begin{aligned}\cos A &= \cos\left(\frac{A}{2}+\frac{A}{2}\right)\\ &= \cos\frac{A}{2}\cos\frac{A}{2}-\sin\frac{A}{2}\sin\frac{A}{2}\\ &= \cos^2\frac{A}{2}-\sin^2\frac{A}{2}\\ &= \cos^2\frac{A}{2}-\left(1-\cos^2\frac{A}{2}\right)\\ &= \cos^2\frac{A}{2}-1+\cos^2\frac{A}{2}\\ &= 2\cos^2\frac{A}{2}-1\end{aligned}$$

$$\begin{aligned}\tan A &= \tan\left(\frac{A}{2}+\frac{A}{2}\right)\\ &= \frac{\tan\frac{A}{2}+\tan\frac{A}{2}}{1-\tan\frac{A}{2}\tan\frac{A}{2}}\\ &= \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}\end{aligned}$$

7 Product and Sum Formulae

7.1 Derivation of the Formulae

Recall the following:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \tag{7.1}$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B \tag{7.2}$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B \tag{7.3}$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B \tag{7.4}$$

(7.1) + (7.2) gives

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B \tag{7.5}$$

(7.1) - (7.2) gives

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B \tag{7.6}$$

(7.3) + (7.4) gives

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B \tag{7.7}$$

(7.3) - (7.4) gives

$$\begin{aligned} \cos(A+B) - \cos(A-B) &= -2\sin A\sin B && (7.8)\\ \cos(A-B) - \cos(A+B) &= 2\sin A\sin B && (7.9) \end{aligned}$$

Given that

$$A + B = C \tag{7.10}$$

$$A - B = D \tag{7.11}$$

We have

$$A = \frac{C+D}{2} \tag{7.12}$$

$$B = \frac{C-D}{2} \tag{7.13}$$

Substituting (7.10)-(7.13) into (7.5)-(7.9), we have

$$\begin{aligned} \sin C + \sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) && (7.14)\\ \sin C - \sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) && (7.15)\\ \cos C + \cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) && (7.16)\\ \cos D - \cos C &= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) && (7.17) \end{aligned}$$

7.2 Examples

7.2.1 Example 1

To express $\sin 5x + \sin x$ as a product of trigonometric ratios,

Comparing

$$\sin 5x + \sin x$$

with

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

It is clear that $C = 5x$ and $D = x$, so

$$\begin{aligned} \sin 5x + \sin x &= 2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right)\\ &= 2\sin\left(\frac{6x}{2}\right)\cos\left(\frac{4x}{2}\right)\\ &= 2\sin 3x\cos 2x \end{aligned}$$

7.2.2 Example 2

To express 2 sin 3*x* cos 2*x* as a sum of trigonometric ratios,

Comparing

$$2\sin 3x\cos 2x$$

with

$$2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) = \sin C + \sin D$$

We see that,

$$\frac{C+D}{2} = 3x$$

$$C + D = 6x \tag{7.18}$$

Also,

$$\frac{C-D}{2} = 2x$$

$$C - D = 4x \tag{7.19}$$

(7.18)+(7.19) gives

$$2C = 10x$$

$$C = 5x$$

(7.18)-(7.19) gives

$$2D = 2x$$

$$D = x$$

$\therefore\ 2\sin 3x\cos 2x = \sin 5x + \sin x$

7.2.3 Example 3

To show that sin 2*A* cos 4*A* + sin 3*A* cos 9*A* = $\frac{1}{2}$ (sin 12*A* − sin 2*A*),

From

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

We have

$$\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) = \frac{1}{2}(\sin C + \sin D)$$

For sin 2*A* cos 4*A*,

$$\frac{C+D}{2} = 2A$$

$$C + D = 4A \tag{7.20}$$

$$\frac{C-D}{2} = 4A$$

$$C - D = 8A \tag{7.21}$$

(7.20)+(7.21) gives

$$2C = 12A$$
$$C = 6A$$

(7.20)-(7.21) gives

$$2D = -4A$$
$$D = -2A$$

$\therefore\ \sin 2A \cos 4A = \frac{1}{2}(\sin 6A + \sin(-2A)) = \frac{1}{2}(\sin 6A - \sin 2A)$

For $\sin 3A \cos 9A$,

$$\frac{C+D}{2} = 3A$$
$$C + D = 6A \tag{7.22}$$

$$\frac{C-D}{2} = 9A$$
$$C - D = 18A \tag{7.23}$$

(7.22)+(7.23) gives

$$2C = 24A$$
$$C = 12A$$

(7.22)-(7.23) gives

$$2D = -12A$$
$$D = -6A$$

$\therefore\ \sin 3A \cos 9A = \frac{1}{2}(\sin 12A + \sin(-6A)) = \frac{1}{2}(\sin 12A - \sin 6A)$

Now,

$$\begin{aligned}
\sin 2A \cos 4A + \sin 3A \cos 9A &= \frac{1}{2}(\sin 6A - \sin 2A) + \frac{1}{2}(\sin 12A - \sin 6A) \\
&= \frac{1}{2}[(\sin 6A - \sin 2A) + (\sin 12A - \sin 6A)] \\
&= \frac{1}{2}(\sin 6A - \sin 2A + \sin 12A - \sin 6A) \\
&= \frac{1}{2}(\sin 12A - \sin 2A)
\end{aligned}$$

$\therefore\ \sin 2A \cos 4A + \sin 3A \cos 9A = \frac{1}{2}(\sin 12A - \sin 2A)$

7.2.4 Example 4

To show that $\sin 7x + \sin x - 2\sin 2x \cos 3x = 4\cos^2 3x \sin x$,

Since $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$, we have

$$\begin{aligned}
\sin 7x + \sin x - 2\sin 2x\cos 3x &= 2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) - 2\sin 2x\cos 3x \\
&= 2\sin\left(\frac{8x}{2}\right)\cos\left(\frac{6x}{2}\right) - 2\sin 2x\cos 3x \\
&= 2\sin 4x\cos 3x - 2\sin 2x\cos 3x \\
&= 2\cos 3x(\sin 4x - \sin 2x)
\end{aligned}$$

Since $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$, we have

$$\begin{aligned}
\sin 7x + \sin x - 2\sin 2x\cos 3x &= 2\cos 3x\left(2\cos\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)\right) \\
&= 2\cos 3x\left(2\cos\left(\frac{6x}{2}\right)\sin\left(\frac{2x}{2}\right)\right) \\
&= 2\cos 3x(2\cos 3x\sin x) \\
&= 4\cos^2 3x\sin x
\end{aligned}$$

$\therefore\ \sin 7x + \sin x - 2\sin 2x\cos 3x = 4\cos^2 3x\sin x$

7.2.5 Example 5

If $\sin\theta + \sin\varphi = a$ and $\cos\theta + \cos\varphi = b$, to show that $\cos^2\frac{1}{2}(\theta - \varphi) = \frac{1}{4}(a^2 + b^2)$,

Since $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$,

$$\begin{aligned}
\sin\theta + \sin\varphi &= a \\
2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) &= a \qquad (7.24)
\end{aligned}$$

Since $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$,

$$\begin{aligned}
\cos\theta + \cos\varphi &= b \\
2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) &= b \qquad (7.25)
\end{aligned}$$

Squaring (7.24) and (7.25),

$$\begin{aligned}
4\sin^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) &= a^2 \qquad (7.26) \\
4\cos^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) &= b^2 \qquad (7.27)
\end{aligned}$$

Adding (7.26) and (7.27),

$$
\begin{aligned}
4\sin^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right)+4\cos^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) &= a^2+b^2\\
4\cos^2\left(\frac{\theta-\varphi}{2}\right)\left(\sin^2\left(\frac{\theta+\varphi}{2}\right)+\cos^2\left(\frac{\theta+\varphi}{2}\right)\right) &= a^2+b^2\\
4\cos^2\left(\frac{\theta-\varphi}{2}\right)(1) &= a^2+b^2\\
\cos^2\left(\frac{\theta-\varphi}{2}\right) &= \frac{1}{4}(a^2+b^2)\\
\cos^2\frac{1}{2}(\theta-\varphi) &= \frac{1}{4}(a^2+b^2)
\end{aligned}
$$

$\therefore\ \cos^2\frac{1}{2}(\theta-\varphi)=\frac{1}{4}(a^2+b^2)$ given that $\sin\theta+\sin\varphi=a$ and $\cos\theta+\cos\varphi=b$

8 A Revisit to Sub-Multiple Angles

8.1 A Transformation of the Sub-Multiple Angles

Recall that

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

If $\tan\frac{x}{2}=t$, then

$$\tan x=\frac{2t}{1-t^2} \tag{8.1}$$

For $\sin x$, we proceed as follows

$$
\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\
&= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}\\
&= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\\
&= \frac{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}\\
&= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\\
\therefore\ \sin x &= \frac{2t}{1+t^2}
\end{aligned}
\tag{8.2}
$$

For cos x, we have

$$\begin{aligned}\cos x &= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}\\ &= \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1}\\ &= \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}\\ &= \frac{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}\\ &= \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\\ \therefore\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned} \tag{8.3}$$

8.2 Examples

8.2.1 Example 1

To calculate $\tan\frac{1}{2}\theta$, if $\tan\theta = \frac{24}{7}$ and θ is acute,

$$\begin{aligned}\tan\theta &= \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}\\ \tan\theta &= \frac{2t}{1-t^2} \quad (\text{Since } \tan\frac{1}{2}\theta = t)\\ \frac{24}{7} &= \frac{2t}{1-t^2}\\ 7(2t) &= 24(1-t^2)\\ 14t &= 24-24t^2\\ 24t^2+14t-24 &= 0\\ 12t^2+7t-12 &= 0\\ 12t^2+16t-9t-12 &= 0\\ (12t^2+16t)-(9t+12) &= 0\\ 4t(3t+4)-3(3t+4) &= 0\\ (3t+4)(4t-3) &= 0\\ t &= -\frac{4}{3} \text{ or } \frac{3}{4}\end{aligned}$$

8.2.2 Example 2

Given that $\sec\theta - \tan\theta = x$. Show that $t = \frac{1-x}{1+x}$ where $t = \tan(\frac{\theta}{2})$.
(This is left as an exercise for students).

9 Trigonometric Equations

9.1 Overview

Trigonometric equations are equations which involve the six trigonometric ratios.

In solving trigonometric equations, we get the values of the angles in the four quadrants from $0^\circ - 360^\circ$ that satisfy the equations.

We recall the following about the four quadrants:

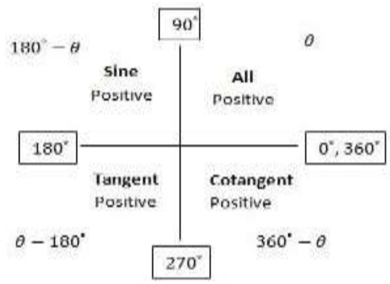

Figure 2: The Four Quadrants

In the first quadrant,

$$\sin\theta = \sin\theta$$
$$\cos\theta = \cos\theta$$
$$\tan\theta = \tan\theta$$

In the second quadrant,

$$\sin\theta = \sin(180 - \theta)^\circ$$
$$\cos\theta = -\cos(180 - \theta)^\circ$$
$$\tan\theta = -\tan(180 - \theta)^\circ$$

In the third quadrant,

$$\sin\theta = -\sin(\theta - 180)^\circ$$
$$\cos\theta = -\cos(\theta - 180)^\circ$$
$$\tan\theta = \tan(\theta - 180)^\circ$$

In the fourth quadrant,

$$\sin\theta = -\sin(360 - \theta)^\circ$$
$$\cos\theta = \cos(360 - \theta)^\circ$$
$$\tan\theta = -\tan(360 - \theta)^\circ$$

9.2 Examples

9.2.1 Example 1

To find the angles less than 360° which satisfy $6\sin\theta = \tan\theta$,

$$
\begin{aligned}
6\sin\theta &= \tan\theta \\
6\sin\theta &= \frac{\sin\theta}{\cos\theta} \\
6\sin\theta\cos\theta &= \sin\theta \\
6\cos\theta &= 1 \\
\cos\theta &= \frac{1}{6} \\
\theta &= \cos^{-1}\frac{1}{6} \\
\theta &= \cos^{-1}(0.1667) \\
\theta &= 80.4^\circ,\ 279.6^\circ
\end{aligned}
$$

9.2.2 Example 2

To find the angles less than 360° which satisfy $3\cos^2\theta + 5\sin^2\theta = 4$,

$$
\begin{aligned}
3\cos^2\theta + 5\sin^2\theta &= 4 \\
3\cos^2\theta + 5(1-\cos^2\theta) &= 4 \\
3\cos^2\theta + 5 - 5\cos^2\theta &= 4 \\
3\cos^2\theta - 5\cos^2\theta &= 4-5 \\
-2\cos^2\theta &= -1 \\
2\cos^2\theta &= 1 \\
\cos^2\theta &= \frac{1}{2} \\
\cos\theta &= \pm\sqrt{\frac{1}{2}} \\
\cos\theta &= \pm\frac{1}{\sqrt{2}} \\
\cos\theta = \frac{1}{\sqrt{2}} \text{ or } \cos\theta &= -\frac{1}{\sqrt{2}} \\
\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \text{ or } \theta &= \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) \\
\theta = 45^\circ, 315^\circ \text{ or } \theta &= 135^\circ, 225^\circ \\
\therefore\ \theta &= 45^\circ, 315^\circ, 135^\circ, 225^\circ
\end{aligned}
$$

9.2.3 Example 3

To solve $5\tan^2\theta - \sec^2\theta = 11$ for $0^\circ \le \theta \le 360^\circ$,

$$
\begin{aligned}
5\tan^2\theta - \sec^2\theta &= 11 \\
5\tan^2\theta - (1+\tan^2\theta) &= 11 \\
5\tan^2\theta - 1 - \tan^2\theta &= 11 \\
5\tan^2\theta - \tan^2\theta &= 11+1 \\
4\tan^2\theta &= 12
\end{aligned}
$$

$$\tan^2\theta = 3$$
$$\tan\theta = \pm\sqrt{3}$$
$$\theta = \tan^{-1}\sqrt{3} \text{ or } \theta = \tan^{-1}(-\sqrt{3})$$
$$\theta = 60^\circ, 240^\circ \text{ or } \theta = 120^\circ, 300^\circ$$
$$\theta = 60^\circ, 240^\circ, 120^\circ, 300^\circ$$

9.2.4 Example 4

To solve $6\cos^2\theta + \sin\theta - 5 = 0$ for $0^\circ \leq \theta \leq 360^\circ$,

$$6\cos^2\theta + \sin\theta - 5 = 0$$
$$6(1-\sin^2\theta) + \sin\theta - 5 = 0$$
$$6 - 6\sin^2\theta + \sin\theta - 5 = 0$$
$$-6\sin^2\theta + \sin\theta + 1 = 0$$
$$6\sin^2\theta - \sin\theta - 1 = 0$$
$$(\text{Let } \sin\theta = x) \quad 6x^2 - x - 1 = 0$$
$$6x^2 - 3x + 2x - 1 = 0$$
$$(6x^2 - 3x) + (2x - 1) = 0$$
$$3x(2x-1) + 1(2x-1) = 0$$
$$2x - 1 = 0 \text{ or } 3x + 1 = 0$$
$$x = \frac{1}{2} \text{ or } x = -\frac{1}{3}$$
$$(\text{Recall that } x = \sin\theta) \quad \sin\theta = \frac{1}{2} \text{ or } \sin\theta = -\frac{1}{3}$$
$$\theta = \sin^{-1}\left(\frac{1}{2}\right) \text{ or } \theta = \sin^{-1}\left(-\frac{1}{3}\right)$$
$$\theta = 30^\circ,\ 150^\circ \text{ or } \theta = 340.5^\circ,\ 119.5^\circ$$

9.3 Exercise

Solve the following for $0^\circ \leq \theta \leq 360^\circ$:

1. $4\cos 2\theta - 3 = 0$
2. $3\sin 2\theta = 18$
3. $4\cos\theta = 3\tan\theta$
4. $2\tan^2\theta - 3\tan\theta + 1 = 0$
5. $\sin\theta + \sin^2\theta = 0$

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