The numbers are.

$a+d$, a and $a-d$.

Where $a=5$ and $d=+2$ or -2

We have. $5+2, 5, 5-2$.

$= 7, 5, 3$ or.

$5-2, 5, 5+2$.

$= 3, 5, 7$

(5) Find Six arithmetic means between 2 and 51. Also Compute three geometric means between 2 and 32.

Solution

At first

Six arithmetic means btw 2 and 51

Let the Six arithmetic means be

b, c, d, e, f, g.

$\therefore\ 2, b, c, d, e, f, g, 51$

first term $(a) = 2$.

8th term $= 51$

from $a+(n-1)d$.

$T_8 = 2+(8-1)d = 51$

$2+7d = 51$

$7d = 51-2$.

$7d = 49$

$d = \frac{49}{7}$

$d = 7$

Therefore.

$b = 2+7 = 9$.

$c = 9+7 = 16$

$d = 16+7 = 23$

$e = 23+7 = 30$

$f = 30+7 = 37$

$g = 37+7 = 44$

Three geometric means btw 2 and 32

Let the three mean $= h, i, J$

$\therefore\ 2, h, i, J, 32$

first term $(a) = 2$.

5th term $= 32$.

from ar^{n-1}

$T_5 = 2(r)^{5-1} = 32$

$2r^4 = 32$

$r^4 = \frac{32}{2}$

$r^4 = 16$

$r^4 = 2^4$.

$r = 2$

$h = 2\times 2 = 4$

$i = 4\times 2 = 8$

$J = 8\times 2 = 16$

(6) Find the three numbers in arithmetic Series whose Sum is 6 and whose product is -64.

Solution

Let the three numbers be.

$a-d$, a, and $a+d$.

Sum of the nos. $=$ ~~3a~~ 6

$(a+d)+(a)+(a-d) =$

$a+a+a+\cancel{d}-\cancel{d} = 3a$

$\therefore 3a = 6$

$a = \frac{6}{3} = 2$

product of the numbers $= -64$

$(a+d) \times (a) \times (a-d)$

$(a^2 + ad)(a-d)$

$a^3 - \cancel{a^2d} + \cancel{a^2d} - ad^2$

$a^3 - ad^2$

$\therefore a^3 - ad^2 = -64$

Since $a = 2$.

We have.

$a^3 - ad^2 = -64$

$(2)^3 - (2)d^2 = -64$

$8 - 2d^2 = -64$

$-2d^2 = -64 - 8$

$-2d^2 = -72$

$d^2 = \frac{\cancel{-}72}{\cancel{-}2}$

$d^2 = 36$

$d = \sqrt{36}$

$d = +6$ or $+6$.

the nos are

$a+d, a, a-d$.

$2+6, 2, 2-6$.

$= 8, 2, -4$

7) Evaluate

(i) $\sum_{k=1}^{6} (3k-2)$ (ii) $\sum_{k=1}^{m} 6k$.

(i) $\sum_{k=1}^{6} (3k-2)$

When

$k=1 \quad 3(1)-2 = 1$

$k=2 \quad 3(2)-2 = 4$

$k=3 \quad 3(3)-2 = 7$

$k=4 \quad 3(4)-2 = 10$

$k=5 \quad 3(5)-2 = 13$

$k=6 \quad 3(6)-2 = 16$

$\therefore \sum_{k=}^{6} 3k-2 = 1+4+7+10+13+16$

$= 51$

(ii) $\sum_{k=1}^{m} 6k$

When $k = 1 = 6(1) = 6$

$k=2 \quad 6(2) = 12$

$k=3 \quad ((3) = 18$

$k=m \quad 6(m) = 6m$

$\therefore \sum_{k=1}^{m} 6k = 6 + 12 + 18 + \dots + 6m$.

8) find the Sum of the first nine terms of the series $\frac{1}{5} + \frac{1}{6} + \frac{5}{36} + \dots$

Solution

$\frac{1}{5} + \frac{1}{6} + \frac{5}{36} + \dots$

$a = \frac{1}{5}$

$r = \frac{1}{6} \div \frac{1}{5} = \frac{1}{6} \times \frac{5}{1} = \frac{5}{6}$

$r < 1$

$$\frac{6r^2-15r+6}{3}=\frac{0}{3}.$$

$$\frac{6r^2}{3}-\frac{15r}{3}+\frac{6}{3}=0$$

$$2r^2-5r+2=0$$

$$2r^2-4r-r+2=0$$

$$2r(r-2)-1(r-2)=0$$

$$(2r-1)(r-2)=0$$

$$2r-1=0 \quad , \quad r-2=0$$

$$2r=1 \qquad r=2.$$

$$r=\tfrac{1}{2} \quad \text{or} \quad r=2.$$

the three nos are:

$\frac{3}{2}$, 3 and 6:

or 6, 3, $\frac{3}{2}$.

(10) Incorrect Question

(11) Find the sum to infinity of the following sequences.

(i) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

(ii) $\frac{1}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^4+\cdots$

(iii) $1-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots(-1<x<1)$

Solution

(i) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

This is a geometric progression

$\therefore\ a=\frac{1}{3} \quad r=\frac{1}{9}\div\frac{1}{3}.$

$$=\frac{1}{9}\times\frac{3}{1}.$$

$$r=\frac{3}{9}=\frac{1}{3}$$

$$S_\infty=\frac{a}{1-r}.$$

$$\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{\frac{1}{3}}{\frac{2}{3}}.$$

$$\frac{1}{3}\div\frac{2}{3}.$$

$$\frac{1}{\cancel{3}_1}\times\frac{\cancel{3}^1}{2}=\frac{1}{2}$$

(ii) $\frac{1}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^4+\cdots$

$a=\frac{1}{2} \qquad r=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2\div\frac{1}{2}$

$\frac{1}{18}\div\frac{1}{2}.. \qquad \left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{3}\right)^2\times\frac{\cancel{2}^1}{1}$

$\frac{2}{18} \qquad r=\left(\frac{1}{3}\right)^2=\frac{1}{3^2}$

$$r=\frac{1}{9}$$

$$S_\infty=\frac{a}{1-r}$$

$$=\frac{\frac{1}{2}}{1-\frac{1}{9}}=\frac{\frac{1}{2}}{\frac{8}{9}}.$$

$$\frac{1}{2}\div\frac{8}{9}.$$

$$\frac{1}{2}\times\frac{9}{8}=\frac{9}{16}$$

(iii) $1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \dots (-1 < x < 1$

$a = 1$ $\quad r = -\frac{1}{x} \div 1$

$r = -\frac{1}{x}$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{x}\right)}$$

$$= \frac{1}{1+\frac{1}{x}}$$

$$= \frac{1}{\frac{x+1}{x}} = 1 \times \frac{x}{x+1}$$

$$= \frac{x}{x+1} \text{ //}$$

(12) Express 0.272727 as a recurring fraction

$S_n = 0.27 + 0.0027 + 0.000027 + \dots + \infty$

$$S_n = \frac{27}{100} + \frac{27}{10000} + \frac{27}{1000000} + \dots \infty$$

$$= 0.272727.$$

$$a + ar + ar^2 + \dots + \infty = \frac{a}{1-r}.$$

This is a geometric series to infinity.

$a(\text{first term}) = \frac{27}{100}$

$r = $ Common ratio

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} \dots = \frac{T_\infty}{T_{\infty-1}}$$

(25)

$\frac{18}{20}$ $\frac{56}{60}$

$T_2 = \frac{27}{10000}$ and $T_1 = \frac{27}{100}$.

$r = \frac{T_2}{T_1} = \frac{27}{10000} \div \frac{27}{100}$

$\therefore \frac{\cancel{27}}{10000} \times \frac{100}{\cancel{27}} = \frac{100}{10000}$

$= \frac{1}{100}$

Therefore $r = \frac{1}{100}$

Sum to infinity. $S_\infty =$

$S_\infty = \frac{a}{1-r}$ Since $r < 1$

$\frac{\frac{27}{100}}{1-\frac{1}{100}} = \frac{\frac{27}{100}}{\frac{100-1}{100}}$

$\therefore \frac{\frac{27}{100}}{\frac{99}{100}} = \frac{27}{100} \div \frac{99}{100}$

$\frac{27}{\cancel{100}} \times \frac{\cancel{100}}{27} = \frac{27}{99}$

0.272727 recurring expressed as fraction gives $\frac{27}{99} = \frac{3}{11}$ //

(13) The sum of the first n terms of series is $3n^2+n$. find the nth term of the series and show that the series is an arithmetic progression. State the value of the first term and the common difference.

$3n^2+n$.

The nth term $= U_n$.

$U_n = S_n - S_{n-1}$

where S_n is sum of n^{th} terms.

$S_n = 3n^2+n$.

$S_{n-1} = 3(n-1)^2+n$.

$= 3(n^2-2n+1)+n$

$= 3n^2-6n+3+n$. $= 3n^2-5n+3$.

$\therefore S_n - S_{n-1}$

$= 3n^2+n - (3n^2-5n+3)$.

$\cancel{3n^2}+n-\cancel{3n^2}+5n-3$.

$= n+5n-3$ $\quad 3n^2+n-3n^2+5n-3$

$= 6n-3$

To show the series is an arithmetic progression using 6n

When $n=1$ $\quad 6-3=3$.

When $n=2$ $\quad 2(6)-3 = 12-3 = 9$

When $n=3$ $\quad 6(3)-3 = 18-3 = 15$

When $n=4$ $\quad 6(4)-3 = 24-3 = 21$

first term $= 3$.

Common difference

$= T_2 - T_1 = 9-3 = 6$ //

(14) The xth, yth, zth terms of a sequence are X, Y, Z respectively. Show that

(i) If the sequence is arithmetic
$$X(y-z) + Y(z-x) + Z(x-y) = 0$$

(ii) If the sequence is geometrical,
$$(y-z)\log X + (z-x)\log Y + (x-y)\log Z = 0$$

(i)
$$X(y-z) + Y(z-x) + Z(x-y) = 0.$$

for arithmetic series
$$T_n = a + (n-1)d.$$

$$T_x = a + (x-1)d = X \quad --- (1)$$
$$T_y = a + (y-1)d = Y \quad --- (2)$$
$$T_z = a + (z-1)d = Z \quad --- (3)$$

from (1)
$$X = a + (x-1)d$$
$$X = a + xd - d$$

from (2) $Y = a + (y-1)d$
$$Y = a + yd - d$$

from (3) $Z = a + (z-1)d$
$$Z = a + zd - d.$$

Putting $X = a + xd - d$, $Y = a + yd - d$ and $Z = a + zd - d$ into
$$X(y-z) + Y(z-x) + Z(x-y) = 0$$
$$= a + xd - d(y-z) + a + yd - d(z-x)$$
$$+ a + zd - d(x-y) = 0$$
$$= \cancel{ay} - \cancel{az} + x\cancel{d}y - x\cancel{d}z - \cancel{dy} + \cancel{dz} + \cancel{az}$$
$$- \cancel{ax} + y\cancel{d}z - y\cancel{d}x - \cancel{dz} + \cancel{dx} + \cancel{ax}.$$
$$- \cancel{ay} + z\cancel{d}x - z\cancel{d}y - \cancel{dx} + \cancel{dy} = 0$$

One cancels out the other leaving none. $0 = 0$

$$\therefore X(y-z) + Y(z-x) + Z(x-y) = 0$$

(ii) If the sequence is geometric
$$(y-z)\log X + (z-x)\log Y + (x-y)\log Z = 0$$

Since the nth term of a G.P
$$= ar^{n-1}$$

$$X\text{th term} = ar^{x-1}$$
$$Y\text{th term} = ar^{y-1}$$
$$Z\text{th term} = ar^{z-1}$$

Substitute equation (1), (2) and (3) into

$$(y-z)\log X + (z-x)\log Y + (x-y)\log Z = 0$$
$$(y-z)\log(ar^{x-1}) + (z-x)\log(ar^{y-1}) +$$
$$(x-y)\log(ar^{z-1}) = 0.$$

$\therefore$ Applying the law of logarithm $[\log a^n = n\log a]$.

$$\therefore [(y-z)(x-1)]\log ar + [(z-x)(y-1)]\log ar$$
$$+ [(x-y)(z-1)]\log ar = 0.$$

Open the brackets.

$$\therefore [yx - y - zx + z]\log ar + [zy - z - xy + x]$$
$$\log ar + [xz - x - yz + y]\log ar = 0$$

Expanding.

$$\therefore yx\log ar - y\log ar - zx\log ar$$
$$+ z\log ar + zy\log ar - z\log ar -$$
$$xy\log ar + x\log ar + xz\log ar -$$
$$x\log ar - yz\log ar + y\log ar = 0$$

Collect like terms.

$yx\log ar - xy\log ar - y\log ar +$
$y\log ar - zx\log ar + xz\log ar$
$+ z\log ar - z\log ar + zy\log ar$
$-yz\log ar + x\log ar - x\log ar = 0$.
$0 = 0$.

∴ Therefore

$(y-z)\log X + (z-x)\log Y + (x-y)\log Z = 0$

Proved.

(15) If the fifth term of both arithmetic series and geometric series are 11 and 243 respectively. Let the common difference be 2. Find the sum of the first ten terms of both arithmetic and geometric series.

for arithmetic series.

$T_n = a + (n-1)d$.

$\therefore T_5 = a + (5-1)d = 11$

$a + 4d = 11$ --- (1)

for geometric series

$T_n = ar^{n-1}$

$T_5 = ar^{5-1} = 243$

$ar^4 = 243$ --- (2)

If the common difference $d = 2$.

from (1)

$\therefore a + 4(2) = 11$

$a + 8 = 11$

$a = 11 - 8 = 3$.

$\therefore a = 3$.

Subtitute ~~a into~~ $a = 3$ into equ (2)

$ar^4 = 243$

$3r^4 = 243$

Divide both sides by 3

$\frac{3r^4}{3} = \frac{243}{3}$

$r^4 = 81$

$\sqrt[4]{r^4} = \sqrt[4]{81}$

$r = \sqrt[4]{81}$

$r = 3$//

therefore the sum of the ~~10th~~ 10 terms in the A.P series

is $\frac{n}{2}(2a + (n-1)d) = S_n$

where $n = 10$ $a = 3$ $d = 2$.

$S_{10} = \frac{10}{2}(2(3) + (10-1)2)$.

$S_{10} = 5(6 + 9(2))$.

$S_{10} = 5(6 + 18)$.

$S_{10} = 5(24)$

$S_{10} = 120$.

the sum of the first 10 terms in the GP series.

$S_n = \frac{a(r^n - 1)}{r - 1}$

where $a = 3$ $r = 3$. $n = 10$

$S_{10} = \frac{3(3^{10} - 1)}{3 - 1}$

$S_{10} = \frac{3(3^{10} - 1)}{2}$.

$= \frac{3}{2}(59048)$

$S_{10} = 3(29524)$

$S_{10} = 88572$

(16) p, q, r are three consecutive terms of an A.P, whose sum is 18. The ratio of $p : r = 7 : -1$. Find p, q, r.

Since p, q and r are consecutive let assume p, q, r to be

$a-d$, a and $a+d$.

$\therefore$ Sum of p, q and r

$$(a-d)+(a)+(a+d) = 18$$
$$= a-d+a+a+d = 18$$
$$= 3a = 18$$
$$3a = 18 \qquad \frac{p}{r} = \frac{7}{-1}$$
$$a = \frac{18}{3} = 6$$

The ratio of ~~p and r = 7 : -1~~ $a-d$

$$\therefore \frac{a-d}{a+d} = \frac{7}{-1} \qquad \overset{a-d}{p} \; \overset{a}{q} \; \overset{a+d}{r}$$

Cross multiply.

$$a-d(-1) = a+d(7)$$

Recall that $a = 6$.

$$6-d(-1) = 6+d(7).$$
$$-6+d = 42+7d.$$
$$-6-42 = 7d-d.$$
$$-48 = 6d$$
$$d = \frac{-48}{6}$$
$$d = -8$$

Therefore p, q and r is.

$$a-d, \; a, \; a+d.$$
$$6-(-8), \; 6, \; 6+(-8)$$
$$6+8, \; 6, \; 6-8$$
$$14, \; 6, \; -2$$
$$p = 14 \quad q = 6 \text{ and } r = -2$$

(17) The 3pth term of an A.P is 56 more than the pth term, and the (p+1) term is 60. Find the first term.

$$T_{3p} = a+(3p-1)d.$$
$$T_p = a+(p-1)d.$$

$$T_{3p} = T_p + 56$$
$$a+(3p-1)d = a+(p-1)d+56$$
$$a+3pd-d = a+pd-d+56$$

Collect like terms.

$$\cancel{a}-\cancel{a}+3pd-pd\cancel{-d}+\cancel{d} = 56$$
$$2pd = 56$$
$$pd = \frac{56}{2}$$
$$pd = 28$$

$$T_{p+1} = a+((p+1)-1)d$$
$$a+(p+1-1)d$$
$$T_{p+1} = a+pd = 60 \quad \text{---} (1)$$

Substitute $pd = 28$ into equ (1)

$$\therefore a+(28) = 60$$
$$a = 60-28$$
$$a = 32$$

First term = 32.

(19) By expanding $(\sqrt{a}-\sqrt{b})^2 > 0$, show that the arithmetic mean of two unequal numbers is greater than their geometric mean.

$(\sqrt{a}-\sqrt{b})^2 > 0$

By expansion

$(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{b})$

$= a - 2\sqrt{ab} + b \geq 0$

$a+b-2\sqrt{ab} \geq 0$

$(\sqrt{a}-\sqrt{b})^2 = a+b \geq 2\sqrt{ab}$.

Divide through by 2

$(\sqrt{a}-\sqrt{b})^2 =$

$\frac{a+b}{2} = \frac{\cancel{2}\sqrt{ab}}{\cancel{2}}$

$\frac{a+b}{2} = \sqrt{ab}$.

$\frac{a+b}{2} = \sqrt{ab}$ when $a = b$

$\frac{a+b}{2} \geq \sqrt{ab}$ when $a > b$ or $b > a$

Recall $\frac{a+b}{2}$ Arithmetic mean

$\sqrt{ab}$ Geometric mean

Proved

(20) If $S_n = an^3 + bn^2$ and $T_1 = -1$ and $T_5 = 15$, find the values of a and b, and the sum of the first 12 terms from the eight term.

$S_n = an^3 + bn^2$.

$S_1 = T_1 = a(1)^3 + b(1)^2 = -1$

$a + b = -1$ --- (i)

$S_5 - S_4 = T_5$.

$S_5 = a(5)^3 + b(5)^2 = 125a + 25b$

$S_4 = a(4)^3 + b(4)^2 = 64a + 16b$

$S_5 - S_4 = 125a + 25b - (64a + 16b) = 15$

$125a - 64a + 25b - 16b = 15$

$61a + 9b = 15$ -- (ii)

Solving equation (i) and (ii) using simultaneous method

$a+b=-1$ ...

$61a+9b=15$ --- (ii)

Multiply equ (i) by 9 and (ii) by 1

$9\times \quad a+b=-1$

$1\times \quad 61a+9b=15$

$9a+9b=-9$ --- (iii)

$61a+9b=15$ --- (iv)

Subtract equation (iii) from (iv)

$52a=24$

$a=\frac{24}{52}=\frac{6}{13}$

Since $a=\frac{6}{13}$

Sub $a=\frac{6}{13}$ into equ (i)

$a+b=-1$

$\frac{6}{13}+b=-1$

$b=-1-\frac{6}{13} \quad b=\frac{-19}{13}$

If $a=\frac{6}{13}$ and $b=\frac{-19}{13}$

$S_n=\frac{6}{13}n^3-\frac{19}{13}n^2$

$\therefore$ Sum of 12 terms from the eighth term is

$S_{12}-S_7$.

$S_{12}=\frac{6}{13}(12)^3-\frac{19}{13}(12)^2$

$\frac{10368}{13}-\frac{2736}{13}$

$S_{12}=\frac{7632}{13}$

$S_7=\frac{6}{13}(7)^3-\frac{19}{13}(7)^2$

$\frac{\quad}{13} \quad \frac{\quad}{13}$

$S_7 \quad \frac{1127}{13}$

$\therefore S_{12}-S_7$

$\frac{7632}{13}-\frac{1127}{13}$

$\frac{6505}{13}=500.38$

21) A Sequence of numbers $U_1, U_2, U_3, \ldots$ Satisfies the relation $U_{n+1}=5+\frac{1}{5}U_n$ for $n\geq 1$. If $U_1=6$, find an expression for U_n interms of n.

Since $n\geq 1$

When $n=1$

$U_{1+1}=5+\frac{1}{5}U_1$

$U_2=5+\frac{1}{5}U_1$

Where $U_1=6$.

$U_2=5+\frac{1}{5}(6)$

$U_2=5+\frac{6}{5} \qquad U_2=\frac{31}{5}=6.2$

When $n=2$.

$U_{2+1}=5+\frac{1}{5}U_2$.

$U_3=5+\frac{1}{5}(6.2) \qquad U_3=\frac{156}{25}=6.$

$\frac{5}{1}\frac{6}{5}$ $\frac{25+6}{5}=\frac{31}{5}$

When $n=3$

$U_{3+1}=5+\frac{1}{5}U_3$

$U_4=5+\frac{1}{5}(6.24)$

$U_4=\frac{781}{125}=6.248$

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∴ Therefore

$U_1 = 6 \quad U_2 = 6.2 \quad U_3 = 6.24$

$U_4 = 6.248$

(22) The nth term of a series is given as $U_n = a\left[\frac{1}{4}\right]^n + b\left[\frac{1}{3}\right]^n$. If $U_1 = 1$ and $U_2 = \frac{1}{6}$, find the values of a and b, and obtain a formula for the first n terms of the series.

$$U_n = a\left[\frac{1}{4}\right]^n + b\left[\frac{1}{3}\right]^n$$

when $n=1$

$$U_1 = a\left[\frac{1}{4}\right]^1 + b\left[\frac{1}{3}\right]^1 = 1$$

$$\frac{a}{4} + \frac{b}{3} = 1$$

$$3a + 4b = 12 \quad \text{--- (i)}$$

when $n=2$

$$U_2 = a\left[\frac{1}{4}\right]^2 + b\left[\frac{1}{3}\right]^2 = \frac{1}{6}$$

$$\frac{a}{16} + \frac{b}{9} = \frac{1}{6}$$

(32)

$$9a + 16b = 24 \quad \text{--- (ii)}$$

Solving equ (i) and (ii) simultaneously

$$9a + 16b = 24$$
$$3a + 4b = 12$$

Multiply equation (ii) by 3 and (i) by 1

$$9a + 16b = 24 \quad \times 1$$
$$= \quad 3a + 4b = 12 \quad \times 3$$

$$9a + 16b = 24 \quad \text{--- (iii)}$$
$$= \quad 9a + 12b = 36 \quad \text{--- (iv)}$$

Subtract (iv) from (iii)

$$4b = -12$$
$$b = -3$$

Sub $b=-3$ into equ (i)

$$3a + 4b = 12$$
$$3a + 4(-3) = 12$$
$$3a - 12 = 12$$
$$3a = 12 + 12$$
$$3a = 24$$
$$a = 8$$

Substitute $a=8$ and $b=-3$ into

$$U_n = a\left[\frac{1}{4}\right]^n + b\left[\frac{1}{3}\right]^n$$

$$8\left[\frac{1}{4}\right]^n + (-3)\left[\frac{1}{3}\right]^n$$

$$\frac{8}{4^n} - \frac{3}{3^n}$$

(23) $T_{n+1} = a + \left[\frac{1}{b}\right] T_n$ for a Sequence $T_1, T_2, T_3 \cdots$, Show that

$T_n = \frac{\;}{1} + \frac{1}{b^{n-1}}\left(T_1 - \frac{ab}{b-1}\right]$ and $T_{n+1} - T_n = \frac{1}{b^{n-1}}\left[a - \left[\frac{b-1}{b}\right] T_1\right]$

$T_r = a + \left[\frac{1}{b}\right] T_n$

$T_n = \frac{ab}{b-1} + \frac{1}{b^{n-1}}\left(T_1 - \frac{ab}{b-1}\right)$

$T_{n+1} - T_n = \frac{1}{b^{n-1}}\left(a - \left[\frac{b-1}{b}\right] T_1\right]$.

To Show that $T_n = \frac{ab}{b-1} + \frac{1}{b^{n-1}}\left[T_1 - \frac{ab}{b-1}\right]$

When $n=1$ i.e $T_n = T_1$

$T_1 = \frac{ab}{b-1} + \frac{1}{b^{n-1}}\left[T_1 - \frac{ab}{b-1}\right]$

$= T_1 = \frac{c}{b} + \frac{1}{b^{1-1}}\left(T_1 - \frac{ab}{b-1}\right] \implies T_1 = \frac{ab}{b-1} + \frac{1}{b^0}\left(T_1 - \frac{ab}{b-1}\right]$

$= T_1 = \frac{a}{\;} + \frac{1}{1}\left(T_1 - \frac{ab}{b-1}\right] \rightarrow T_1 = \frac{ab}{b-1} + T_1 - \frac{ab}{b-1}$

Collect like terms.

$T_1 = T_1 + \frac{a\cancel{b}}{\cancel{b}-1} - \frac{a\cancel{b}}{\cancel{b}-1}$

$T_1 = T_1$

Thus Shows that $T_n = \frac{ab}{b-1} + \frac{1}{b^{n-1}}\left(T_1 - \frac{ab}{b-1}\right)$ is valid

To show that

$T_{n+1} - T_n = \frac{1}{b^{n-1}}\left[a - \frac{(b-1)}{b}\right] T_1$.

Recall that $T_{n+1} = a + \left[\frac{1}{b}\right] T_n$.

$\left[\frac{1}{b}\right] T_n - \left[\frac{ab}{b-1} + \frac{1}{b^{n-1}}\left(T_1 - \frac{ab}{b-1}\right)\right] = \frac{1}{b^{n-1}}\left[a - \left[\frac{b-1}{b}\right] T_1\right]$

$a \cdot \frac{n}{b} - \left[\frac{ab}{b-1} - \frac{T_1}{b^{n-1}} - \frac{ab}{b^{n-1}(b-1)}\right] = \frac{a}{b^{n-1}} - \left[\frac{T_1(b-1)}{b^{n-1}(b)}\right]$

$$a + \frac{T_n}{b} - \frac{ab}{b-1} - \frac{T_1}{b^{n-1}} + \frac{ab}{b^{n-1}(b-1)} = \frac{a}{b^{n-1}} - \left[\frac{T_1 b - T_1}{b^{n-1}(b)}\right]$$

$$a + \frac{T_n}{b} - \frac{ab}{b-1} - \frac{T_1}{b^{n-1}} + \frac{ab}{b^{n-1}(b-1)} = \frac{a}{b^{n-1}} - \frac{T_1 \cancel{b}}{b^{n-1}(\cancel{b})} + \frac{T_1}{b^{n-1}(b)}$$

$$a + \frac{T_n}{b} - \frac{T_1}{b^{n-1}} - \frac{ab}{b-1} + \frac{ab}{b^{n-1}(b-1)} = \frac{a}{b^{n-1}} - \frac{T_1}{b^{n-1}} + \frac{T_1}{b^{n-1}(b)}$$

$$a + \frac{T_n}{b} - \frac{T_1}{b^{n-1}} - \frac{ab}{b-1}\left[1 - \frac{1}{b^{n-1}}\right] = \frac{a}{b^{n-1}} - \frac{T_1}{b^{n-1}} + \frac{T_1}{b^{n-1}(b)}$$

Let $n = 1$

$$a + \frac{T_1}{b} - \frac{T_1}{b^{1-1}} - \frac{ab}{b-1}\left[1 - \frac{1}{b^{1-1}}\right] = \frac{a}{b^{1-1}} - \frac{T_1}{b^{1-1}} + \frac{T_1}{b^{1-1}(b)}$$

$$a + \frac{T_1}{b} - \frac{T_1}{b^0} - \frac{ab}{b-1}\left[1 - \frac{1}{b^0}\right] = \frac{a}{b^0} - \frac{T_1}{b^0} + \frac{T_1}{b^0(b)}$$

N.B $B^0 = \underline{1}$ (Law of indices).

$$a + \frac{T_1}{b} - T_1 - \frac{ab}{b-1}[1-1] = a - T_1 + \frac{T_1}{b}$$

$$a + \frac{T_1}{b} - T_1 - \frac{a\cancel{b}}{\cancel{b}-1}[0] = a - T_1\left[1 - \frac{1}{b}\right]$$

$$a + \frac{T_1}{b} - T_1 - 0 = a - T_1 + \frac{T_1}{b}$$

$$a + \frac{T_1}{b} - T_1 = a - T_1 + \frac{T_1}{b} \Rightarrow a + \frac{T_1}{b} - T_1 = a + \frac{T_1}{b} - T_1$$

$$T_{n+1} - T_n = \frac{1}{b^{n-1}}\left[a - \left(\frac{b-1}{b}\right)T_1\right]$$

$$\therefore \quad T_{n+1} - T_n = \frac{1}{b^{n-1}}\left[a - \left(\frac{b-1}{b}\right)T_1\right]$$

PROVED.

(24) " Sum to infinity of G.P is th... first term, find the Common terms. If the Sum to infinity of the series $x + x^3 + x^5 + \cdots$ is $\frac{1}{8}$, find the value of x.

Sum of Infinity $S_\infty = \frac{a}{1-r}$

...rm (a).

Simultaneously = 3× first term

$= 3 \times a = \frac{1}{8}$

$= 3a = \frac{1}{8}$

$= \frac{1}{3} = a = \frac{1}{24}$.

Substitution $a = \frac{1}{24}$ into $\frac{a}{1-r}$

$= \frac{1}{8}$

$= \quad \frac{1}{} = 1-r$

$= \quad 1-r \quad = r = \frac{2}{3}$.

Com... (r) =

$\frac{x^5}{x^3} = x^2$.

and $r = x^2$.

$= \frac{2}{3}$.

$x = \sqrt{\frac{2}{3}}$ //

(25) Check Your Manual

(26) Refer to Your Manual

(27) Open Your Manual

(28) Solution in the Manual

(29) Already in the Manual.

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TUTORIAL SHEET 4

1) If the equation $x^2-3x+1 = p(x-3)$ has equal roots. Find the possible values p

Solution

$x^2-3x+1 = p(x-3)$

$x^2-3x+1 = px-3p$

$x^2-3x-px+1+3p=0$

Comparing with ax^2+bx+c

$a=1,\ b=(-3-p),\ c=1+3p$

For equal root $b^2-4ac=0$

$(-3-p)^2-4(1)(1+3p)=0$

$9+6p+p^2-4-12p=0$

Factorize the arranging

$p^2+6p-12p+9-4=0$

$p^2-6p+5=0$

$p^2-p-5p+5=0$ $-5\ -1$

$p(p-1)-5(p-1)=0$ $-1p,\ 5p$ $-5p-1p$

$(p-1)\ (p-5)=0$

$p-1=0$ OR $p-5=0$

$p=1$ OR $p=5$

2) Show that $a^2x^2-ax+1=0$ can never have real roots.

Solution

Comparing with

ax^2+bx+c

$a=a^2,\ b=-a,\ c=1$

If the expression does not have real root, it has complete root.

Hence for complete root, $b^2-4ac<0$

Substituting for $b^2-4ac<0$

$a-4a^2<0$

Hence the equation can never have real root.

5) Discuss the nature of the roots of the following equations

(i) $3x^2-2x+4=0$

Comparing with

ax^2+bx+c

$a=3,\ b=-2,\ c=1$

Using b^2-4ac

$(-2)^2-4(3)(1)$

$4-12=-10$

Hence since b^2-4ac is less than zero, then it has complex root

(ii) $4x^2-28x+49=0$

Comparing with ax^2+bx+c

$a=4,\ b=-28\ \ c=49$

Using b^2-4ac

$(-28)^2-4(4)\times 49$

$784-784=0$

Since $b^2-4ac=0$

It has equal root

Complex

$b^2-4ac<0$

$b^2=4ac$ or b^2-c ...

$b^2-4ac>0$

(iii) $x^2=x-5$

Collecting the equation together

$x^2-x+5=0$

Comparing with ax^2+bx+c

$a=1,\ b=-1,\ c=5$

Using b^2-4ac

$(-1)^2-4(1)(5)$

$1-20=-19$

Since the b^2-4ac is less than zero, hence the equation has complex root

(6) If α and β are the roots of the equation $5x^2-3x-1=0$, form the equations with integral coefficients which have roots

(i) $1/\alpha^2$ and $1/\beta^2$ (ii) $\frac{\alpha^2}{\beta}$ and β^2/α

Solution

$5x^2-3x-1=0$

$a=5,\ b=-3,\ c=-1$

Quadratic equation =

x^2-(sum of the root)x +(Product of the root)

The sum of the root $=\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$

But $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\therefore \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$\alpha+\beta=-b/a=-\frac{(-3)}{5}=\frac{3}{5}$

$\alpha\beta=\frac{c}{a}=\frac{-1}{5}$

Substituting it

$=\frac{\left(\frac{3}{5}\right)^2-2\left(-\frac{1}{5}\right)}{\left(-\frac{1}{5}\right)^2}$

$\frac{\frac{9}{25}+\frac{2}{5}}{\frac{1}{25}}=\frac{\frac{9+10}{25}}{\frac{1}{25}}$

(i,c) $=\frac{19}{25}\div\frac{1}{25}$

$=\frac{19}{25}\times\frac{25}{1}$

$=19$

Hence Sum of the root = 19

For the product of the roots

$\frac{1}{\alpha^2}\times\frac{1}{\beta^2}=\frac{1}{\alpha^2\beta^2}=\frac{1}{(\alpha\beta)^2}$

$=\frac{1}{\left(-\frac{1}{5}\right)^2}$

$=\frac{1}{\frac{1}{25}}$

$=25$

Hence Product of the root is 25

From x^2-(sum of the root)x +(Product of the ro

$=x^2-(19)x+(25)$

$=x^2-19x+25$

(ii) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

sum of the root $=\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

$\frac{\alpha^3+\beta^3}{\alpha\beta}$

Recall that

$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$=\frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{\alpha\beta}$

$\therefore \alpha+\beta=3/5\quad \alpha\beta=-\frac{1}{5}$

$$\frac{\left(\frac{3}{5}\right)(\alpha^2+\beta^2-\alpha\beta)}{\alpha\beta}$$

Since $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$\frac{\left(\frac{3}{5}\right)\left((\alpha+\beta)^2-2\alpha\beta-\alpha\beta\right)}{\alpha\beta}$$

$$\frac{\left(\frac{3}{5}\right)\left((\alpha+\beta)^2-3\alpha\beta\right)}{\alpha\beta}$$

$$=\frac{\left(\frac{3}{5}\right)\left(\left(\frac{3}{5}\right)^2-3\left(-\frac{1}{5}\right)\right)}{-1/5}$$

$$=\frac{\frac{3}{5}\left(\left(\frac{9}{25}\right)+\frac{3}{5}\right)}{-\frac{1}{5}}$$

$$=\frac{\frac{3}{5}\left(\frac{9}{25}+\frac{3}{5}\right)}{-\frac{1}{5}} = \frac{\frac{3}{5}\left(\frac{9+15}{25}\right)}{-1/5}$$

$$\frac{3}{5}\left(\frac{24}{25}\right)\times\frac{5}{-1}$$

$$=-\frac{72}{25}$$

Product of the root is given by

$$\frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha} = \frac{(\alpha\beta)^2}{\alpha\beta} = \frac{\alpha\beta\times\alpha\beta}{\alpha\beta}$$

$$=\alpha\beta = -\frac{1}{5}$$

from $x^2-(\text{Sum of root})x+(\text{product of the root})$

$$= x^2-\left(-\frac{72}{25}\right)x+\left(-\frac{1}{5}\right)$$

$$= x^2+\frac{72}{25}x-\frac{1}{5}$$

multiply through by 25

$$= 25x^2+72x-5=0$$

The equation of the roots is

$$25x^2+72x-5=0$$

(7) If α and β are the roots of the equation ax^2+bx+c, obtain in terms of a, b and c the following.
(i) $\alpha-\beta$ (ii) $\frac{1}{\alpha}+\frac{1}{\beta}$ (iii) $\alpha^4+\beta^4$

Solution

(i) $\alpha-\beta$

from $(\alpha-\beta)^2 = (\alpha-\beta)(\alpha-\beta)$

open the bracket

$$(\alpha-\beta)^2 = \alpha^2-\alpha\beta-\alpha\beta+\beta^2$$

$$(\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta \quad (*)$$

Recall that $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

substituting it in eq *

$$(\alpha-\beta)^2 = (\alpha+\beta)^2-2\alpha\beta-2\alpha\beta$$

$$(\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta$$

Square root both sides

$$\sqrt{(\alpha-\beta)^2} = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$(\alpha-\beta) = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

But ax^2+bx+c

$$\alpha+\beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

$$(\alpha-\beta) = \sqrt{\left(\frac{-b}{a}\right)^2-4\left(\frac{c}{a}\right)}$$

$$(\alpha-\beta) = \sqrt{\frac{b^2}{a^2}-\frac{4c}{a}}$$

Finding the LCM we have

$(\alpha-\beta) = \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}$

$(\alpha-\beta) = \sqrt{\frac{b^2-4ac}{a^2}}$

$\alpha-\beta = \frac{\sqrt{b^2-4ac}}{a}$

(11) $\frac{1}{\alpha}+\frac{1}{\beta}$

Finding the LCM

$\frac{\alpha+\beta}{\alpha\beta}$

Since $\alpha+\beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

$= \frac{-\frac{b}{a}}{c/a}$

$= -\frac{b}{a} \times \frac{a}{c}$

$= -\frac{b}{c}$

(13) $\alpha^4+\beta^4$

$= (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$

But $(\alpha^2+\beta^2) = (\alpha+\beta)^2 - 2\alpha\beta$

$((\alpha+\beta)^2 - 2\alpha\beta)^2 - 2(\alpha\beta)^2$

But $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = c/a$

$\left[\left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right]^2 - 2\left[c/a\right]^2$

$\left[\frac{b^2}{a^2} - \frac{2c}{a}\right]^2 - 2\left[\frac{c^2}{a^2}\right]$

(8) Find the value of p for which the equation $(x-2)(x-3)=p$ has roots which differ by 2

Solution

Let the roots be $\alpha+2$ & α

$(x-2)(x-3) = p$

Open the bracket

$x^2-3x-2x+6 = p$

$x^2-5x+6 = p$

$x^2-5x+6-p=0$

$a=1$, $b=-5$, $c=6-p$

$\alpha+\beta = -\frac{b}{a} = -\left(\frac{-5}{1}\right) = 5$

$\alpha\beta = \frac{c}{a} = 6-p$

$(\alpha-\beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

Roots which differ by 2

$2^2 = (5^2) - 4(6-p)$

$4 = 25 - 24 - 4p$

$4p = 4 - 25 + 24$

$4p = 3$

$p = 3/4$

(9) let the roots be α and 2α

$\alpha + 2\alpha = -\frac{q}{p}$

$3\alpha = -\frac{q}{p}$

$\alpha = -\frac{q}{3p}$

$\alpha \times 2\alpha = 2\alpha^2 = 2\left(\frac{-q}{3p}\right)^2 = \frac{r}{p}$

$\frac{2q^2}{9p^2} = \frac{r}{p}$

$= 2q^2 p = 0$

$= 2q^2 = 9l$

$= 2q^2 - 9p \cdot 0$

(10) Form the quadratic equation for which the sum of the roots is 5 and the sum of the square of the roots is 53.

Solution

$\alpha + \beta = 5$ —— (i)

$\alpha^2 + \beta^2 = 53$ —— (ii)

Recall

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$(\alpha + \beta)^2 - 2\alpha\beta = 53$

$5^2 - 2\alpha\beta = 53$

$25 - 2\alpha\beta = 53$

$-2\alpha\beta = 53 - 25$

$-2\alpha\beta = 28$

$\alpha\beta = -14$

$x^2 - (\alpha + \beta)x + (\alpha\beta)$

$x^2 - (5)x + (-14) = 0$

$x^2 - 5x - 14 = 0$

(11) Given the roots of the equation $x^2 - x - 1 = 0$ are α and β, find, in its simplest form, the equation with numerical coefficient whose roots are $\frac{1+\alpha}{2-\alpha}$ and $\frac{1+\beta}{2-\beta}$

Solution

Given $x^2 - x - 1 = 0$

$a = 1, \; b = -1, \; c = -1$

$\alpha + \beta = \frac{-b}{a} = -\frac{(-1)}{1} = 1$

$\alpha\beta = \frac{c}{a} = \frac{-1}{1} = -1$

The equation will be given by

$x^2 - (\text{sum of the root})x + (\text{product of root})$

Sum of the root:

$\frac{1+\alpha}{2-\alpha} + \frac{1+\beta}{2-\beta}$

Finding the LCM expand

$\frac{(2-\beta)(1+\alpha) + (2-\alpha)(1+\beta)}{(2-\alpha)(2-\beta)}$

$= \frac{(2 + 2\alpha - \beta - \alpha\beta) + (2 + 2\beta - \alpha - \alpha\beta)}{4 - 2\beta - 2\alpha + \alpha\beta}$

$= \frac{2 + 2\alpha - \beta - \alpha\beta + 2 + 2\beta - \alpha - \alpha\beta}{4 - 2\beta - 2\alpha + \alpha\beta}$

$= \frac{4 + \alpha + \beta - 2\alpha\beta}{4 - 2(\beta + \alpha) + \alpha\beta}$

Substitute for $\alpha + \beta$ and $\alpha\beta$

$= \frac{4 + (1) - 2(-1)}{4 - 2(1) + (-1)}$

$= \frac{4 + 1 + 2}{4 - 2 - 1} = \frac{7}{1}$

$= 7$

The sum of the root $= 7$

For the product of the roots

$\frac{1+\alpha}{2-\alpha} \times \frac{1+\beta}{2-\beta} = \frac{(1+\alpha)(1+\beta)}{(2-\alpha)(2-\beta)}$

Expanding the above

$\frac{1 + \alpha + \beta + \alpha\beta}{4 - 2\alpha - 2\beta + \alpha\beta}$ —— (i)

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Substituting the value of $\alpha+\beta$ and $\alpha\beta$ into eqn (i)

$$= \frac{1+(1)+(-1)}{4-2(1)+(1)} = \frac{1+1-1}{4-2+1}$$

$$= \frac{1}{4-3} = \frac{1}{1} = 1$$

Hence the equation is given by x^2 - (sum of the root) x + (product of the root) = 0

$$x^2 - 7x + 1 = 0$$

(14) If α and β are the roots of the equation $2\gamma^2x^2 + 2\gamma px + p^2 - 3\gamma^2 = 0$, show that $\alpha^2+\beta^2$ is Independent of γ and p

Solution

$$2\gamma^2x^2 + 2\gamma px + p^2 - 3\gamma^2 = 0$$

$$\alpha+\beta = \frac{-b}{a} = \frac{-2\gamma p}{2\gamma^2} = \frac{-p}{\gamma}$$

$$\alpha\beta = \frac{c}{a} = \frac{p^2-3\gamma^2}{2\gamma^2}$$

where $a = 2\gamma^2$

$b = 2\gamma p$

$c = p^2 - 3\gamma^2$

Recall

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

by substituting, the values of $\alpha+\beta$ and $\alpha\beta$ in the above equation we have

$$\alpha^2+\beta^2 = \left(\frac{-p}{\gamma}\right)^2 - \cancel{2}\left(\frac{p^2-3\gamma^2}{\cancel{2}\gamma^2}\right)$$

$$\alpha^2+\beta^2 = \frac{p^2}{\gamma^2} - \frac{p^2+3\gamma^2}{\gamma^2}$$

Simplifying

$$\alpha^2+\beta^2 = \frac{p^2-p^2+3\gamma^2}{\gamma^2}$$

$$\alpha^2+\beta^2 = \frac{3\cancel{\gamma^2}}{\cancel{\gamma^2}}$$

$$\alpha^2+\beta^2 = 3$$

Since $\alpha^2+\beta^2$ has No variable, then It is Independent of γ & p

TUTORIAL SHEET FIVE.

(1) Solved in the Manual

(2) Use the binomial theorem to Simplify $(3+2x)^4$

Using pascal triangle the Coefficients are 1 4 6 4 1

$= 1\cdot(3)^4(2x)^0 + 4\cdot(3)^3(2x) + 6\cdot(3)^2(2x)^2 + 4\cdot(3)^1(2x)^3 + 1\cdot(3)^0(2x)^4$

$= 3^4 + 4\cdot27\cdot(2x) + 6\cdot9\cdot4x^2 + 4\cdot3\cdot8x^3 + 16x^4$

$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$ //

(3) Solved in the manual

(4) Also Solved in the manual

(5) Solved in the manual

(6) Use the binomial theorem and multiplication to expand $(1-x^2)(2-3x)^3$

Expanding $(2-3x)^3$ with binomial theorem using pascal triangle, the Coefficients are 1 3 3 1

$= 1\cdot(2)^3(-3x)^0 + 3\cdot(2)^2(-3x)^1 + 3(2)^1(-3x)^2 + 1\cdot(2)^0\cdot(-3x)^3$

$= 2^3 + 3\cdot4\cdot(-3x) + 3\cdot2\cdot9x^2 + 1\cdot1\cdot(-27x^3)$

$= 8 - 36x + 54x^2 - 27x^3$

$\therefore (2-3x)^3 = 8-36x+54x^2-27x^3$

$\therefore (1-x^2)(8-36x+54x^3-27x^3)$

$= 8-36x+54x^2-27x^3-8x^2+36x^3-54x^4+27x^5$

Collect like terms

$= 8-36x+54x^2-8x^2-27x^3+36x^3-54x^4-27x^5$

$= 8-36x+46x^2+9x^3-54x^4-27x^5$ //

(7) Without using tables, find $(2+\sqrt{3})^6 + (2-\sqrt{3})^6$

Expanding using pascal triangle the Coefficients are 1 6 15 20 15 6 1

$(2+\sqrt{3})^6 = 1\cdot(2)^6(\sqrt{3})^0 + 6\cdot(2)^5(\sqrt{3})^1 + 15\cdot(2)^4(\sqrt{3})^2 + 20\cdot(2)^3\cdot(\sqrt{3})^3 + 15\cdot(2)^2\cdot(\sqrt{3})^4 + 6\cdot(2)^1(\sqrt{3})^5 + 1\cdot(2)^0(\sqrt{3})^6$

$(2+\sqrt{3})^6 = 2^6 + 6\cdot32\cdot\sqrt{3} + 15\cdot16\cdot3 + 20\cdot8\cdot(\sqrt{3})^3 + 15\cdot4\cdot(\sqrt{3})^4 + 6\cdot2\cdot(\sqrt{3})^5 + (\sqrt{3})^6$

$(2-\sqrt{3}$ ... $(-\sqrt{3})^0 + 6 \cdot 2^5 \cdot (-\sqrt{3})^1$

$+$ ... $)^2 + 20 \cdot (2)^3 (-\sqrt{3})^3 +$

15 ... $+ 6 \cdot 2 \cdot (-\sqrt{3})^5 +$

1 ... $)^6$

$(\ \ 2^6 - 6 \cdot 32 \cdot \sqrt{3} +$

1 ... $20 \cdot 8 \cdot (\sqrt{3})^3 +$

15 ... $6 \cdot 2 \cdot (\sqrt{3})^5 + (\sqrt{3})^6$

$\therefore$ ... $- (2-\sqrt{3})^6$

$= [$ ... $-\sqrt{3}) + (15 \cdot 16 \cdot 3) +$

$20 \cdot$ ... $15 \cdot 4 \cdot (\sqrt{3})^4 + 6 \cdot 2 \cdot (\sqrt{3})^5 +$

$(\sqrt{3})$ ... $2^6 - (6 \cdot 32 \cdot \sqrt{3}) + (15 \cdot 16 \cdot 3)$

$-$ ... $15 \cdot 4 (\sqrt{3})^4 - 6 \cdot 2 \cdot (\sqrt{3})^5$

$+($ ...

brackets and

... terms.

$=$ ... $32 \cdot \sqrt{3}) + 15 \cdot \cancel{16} \cdot 3 +$

$15 \cdot \cancel{4} (\sqrt{3})^4 + 6 \cdot 2 (\sqrt{3})^5$

$6 \cdot 32 \cdot \sqrt{3} - 15 \cdot \cancel{16} \cdot 3$

$+$ ... $- 15 \cdot \cancel{4} (\sqrt{3})^4 +$

$+$ ... $(\sqrt{3})^6$

... the odds

$+ 20 \cdot 8 (\sqrt{3})^3 + 6 \cdot 2 (\sqrt{3})^5$

$=$ ... $+ 20 \cdot 8 (\sqrt{3})^3 + 6 \cdot 2 (\sqrt{3})^5$

$+$

$=$ ... $\sqrt{3}) + 2[20 \cdot 8 (\sqrt{3})^3]$

... $(\sqrt{3})^5]$

$= 2(332.55) + 2(831.3)$
$+ 2(187.06)$

$= 665.1 + 1662.8 + 374.12$

$= 2701.82$ //

$(2+x)^8 = (5)^8$

(8) Use the binomial theorem to evaluate $(1.004)^8$ Correct to 5 dp

$(1+x)^8$

if $1.004 = 1 + x$.

$x = 1.004 - 1$

$x = 0.004$.

$\therefore (1 + 0.004)^8$

Using pascal triangle the Coefficients are 1, 8, 28, 56, 70, 56, 28, and 8 and 1

$1.004^8 = 1 + 8(1)^7(0.004)^1 + 28(1)^6(0.004)^2 + 56(1)^5(0.004)^3 + 70(1)^4(0.004)^4 + 56(1)^3(0.004)^5 + 28(1)^2(0.004)^6 + 8(1)(0.004)^7 + (0.004)^8$

$1.004^8 = 1 + 0.032 + 4.48 \times 10^{-4} + 3.584 \times 10^{-6} + 1.792 \times 10^{-8} + 5.735 \times 10^{-11} + 1.14688 \times 10^{-13} + 1.3107 \times 10^{-16} + 6.5536 \times 10^{-20}$

$= 1.032451602$

(10) Find the coefficient of α^6 in the expansion $\left(\frac{1}{\alpha^2}-\alpha\right)^{18}$

The $(r+1)$th term in the expansion of $\left(\frac{1}{\alpha^2}-\alpha\right)^{18}$ is ${}^{18}C\left(\frac{1}{\alpha^2}\right)^{18}(-\alpha)$

$$= \binom{18}{r}\left(\frac{1}{\alpha^2}\right)^{18-r}(-\alpha)^r$$

$+ {}^{17}C\left(\frac{1}{\alpha^2}\right)^{17}(-\alpha)$

$$= \binom{18}{r}(\alpha^{-2})^{18-r}(-\alpha^r)$$

$+ {}^{16}C_2\left(\frac{1}{\alpha^2}\right)^{16}(-\alpha)^2$

$$= -\binom{18}{r}\alpha^{-36+2r}(\alpha^r)$$

$$= -\binom{18}{r}\alpha^{-36+2r+r}$$

$$= -\binom{18}{r}\alpha^{-36+3r}$$

Since $\alpha^{-36+3r} = \alpha^6$

$-36+3r = 6$

$3r = 6+36$

$3r = 42$

$r = 14$

$$\therefore -\binom{18}{14}\alpha^{-36+3(14)}$$

$$= -\frac{18\times17\times16\times15\times14!}{(18-14)!\ 14!}\alpha^6$$

$$= -\frac{73440}{4\times3\times2}\alpha^6$$

$$= -3060\,\alpha^6$$

α^6 Coefficient is

$= -3060$

(11) Show that $\frac{(1+x)^{3/2}-(1+\frac{x}{2})^3}{\sqrt{1-x}} \approx -\frac{3}{8}x^2$ as a polynomial of degree two.

By expansion using binomial theorem

Recall that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\cdot2}x^2+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3+\dots+$$

$$\therefore (1+x)^{3/2} = 1+\tfrac{3}{2}(x)+\frac{\frac{3}{2}(\frac{3}{2}-1)}{2}x^2$$

$$(1+x)^{3/2} = 1+\tfrac{3}{2}x+\tfrac{3}{8}x^2$$

$$\left(1+\frac{x}{2}\right)^3 = 1+3\left(\frac{x}{2}\right)+\frac{3(3-1)}{2}\left(\frac{x}{2}\right)^2$$

$$\left(1+\frac{x}{2}\right)^3 = 1+\frac{3x}{2}+\frac{3x^2}{4}$$

$$(1-x)^{1/2} = 1+\frac{1}{2}(-x)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(-x)^2$$

$$(1-x)^{1/2} = 1-\frac{x}{2}-\tfrac{1}{4}x^2$$

$$\therefore \frac{(1+x)^{3/2}-(1+\frac{x}{2})^3}{\sqrt{1-x}} =$$

$$= \frac{1+\frac{3}{2}x+\frac{3}{8}x^2-\left(1+\frac{3}{2}x+\frac{3x^2}{4}\right)}{1-\frac{x}{2}-\frac{1}{4}x^2}$$

to 5 d.p

$= 1.03245$

(9). Express $f(x) = (1 - x + x^2)^6$ in ascending power of x as far as the term in x^5. If $f(x) = 0.91^6$, find x and determine the approximate value of 0.91^6 using the expansion above, to nearest 2 decimal places.

$$= 1 + \frac{9}{4}x^2 + \frac{189x^4}{32} + \frac{6237}{384}x^6 + \dots$$

(14) $(2-x^2)^{1/3}$

$$(2-x^2)^{1/3} = \left[2\left(1-\frac{x^2}{2}\right)\right]^{1/3}$$

$$2^{1/3}\left(1-\frac{x^2}{2}\right)^{1/3}$$

$$= 2^{1/3}\left[1 + \left(\frac{1}{3}\right)\left(\frac{-x^2}{2}\right) + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{1\cdot 2}\left(\frac{-x^2}{2}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{1\cdot 2\cdot 3}\left(\frac{-x^2}{2}\right)^3 + \dots\right]$$

$$= 2^{1/3}\left[1 - \frac{x^2}{6} - \frac{2x^4}{36} - \frac{10}{216}x^6 + \dots\right]$$

$$= 2^{1/3}\left[1 - \frac{x^2}{6} - \frac{x^4}{18} - \frac{5}{108}x^6 + \dots\right]$$

(15) $(1-3x^2)^{-3/4}$

$$= \left[1 + \left(-\tfrac{3}{4}\right)(-3x^2) + \frac{\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{1\cdot 2}(-3x^2)^2 + \frac{\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)}{1\cdot 2\cdot 3}(-3x^2)^3 + \dots\right]$$

(16) Find the fourth terms in the expansion of $\frac{3}{(1+x^2)(1-2x)}$

$$\frac{3}{(1+x^2)(1-2x)} = 3\left[(1+x^2)^{-1}(1-2x)^{-1}\right]$$

$$(1+x^2)^{-1} = 1 - x^2 + 2x^4 + \dots$$

$$(1-2x)^{-1} = \left[1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3!}(-2x)^3 + \dots\right.$$

$$(1-2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + \dots$$

$$(1+x^2)^{-1}(1-2x)^{-1} = (1-x^2+2x^4)(1+2x+4x^2+8x^3)$$

$$= 1 + 2x + 4x^2 + 8x^3 - x^2 - 2x^3 - 4x^4 - 8x^5 + 2x^4 + 4x^5 + 8x^6 + 16x^7$$

$$= 1 + 2x + 3x^2 + 6x^3 - 2x^4 - 4x^5 + 8x^6 + 16x^7$$

$$\therefore \frac{3}{(1+x^2)(1-2x)} = 3\left[(1+x^2)^{-1}(1-2x)^{-1}\right]$$

$$3\left[1 + 2x + 3x^2 + 6x^3 - 2x^4 - 4x^5 + 8x^6 + 16x^7\right]$$

$$= 3 + 6x + 9x^2 + 18x^3 - 6x^4 - 12x^5 + 24x^6 + 48x^7 + \dots$$

$$\frac{\cancel{1} + \cancel{\frac{3}{2}x} + \frac{3}{8}x^2 \cancel{-1} - \cancel{\frac{3}{2}x} - \frac{3x^2}{4}}{1 - \frac{x}{2} - \frac{1}{4}x^2}$$

$$= \frac{\frac{3}{8}x^2 - \frac{3x^2}{4}}{1 - \frac{x}{2} - \frac{1}{4}x^2}$$

$$= \frac{-\frac{3x^2}{8}}{1 - \frac{x}{2} - \frac{1}{4}x^2}.$$

Since the value of x is so small (something of 0.000000001) Therefore the denominator is 1

$$= \frac{-\frac{3x^2}{8}}{1}$$

$$\frac{(1+x)^{3/2} - (1+\frac{x}{2})^3}{\sqrt{1-x}} \simeq -\frac{3x^2}{8}.$$

(12) $(3+2x)^{-2}$.

$$(3+2x)^{-2} = \frac{1}{(3+2x)^2}.$$

$$= \frac{1}{[3(1+\frac{2}{3}x)]^2} = \frac{1}{3^2}(1+\frac{2}{3}x)^{-2}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot2}x^2 + \frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3 + \cdots +$$

$\therefore (3+2x)^{-2} =$

$$= \frac{1}{3^2}\left[1 + (-2)(\tfrac{2}{3}x) + \frac{(-2)(-3)}{2}(\tfrac{2}{3}x)^2 + \frac{(-2)(-3)(-4)}{1\cdot2\cdot3}(\tfrac{2}{3}x)^3 + \cdots +\right]$$

$$= \frac{1}{9}\left[1 - \frac{4}{3}x + \frac{24}{18}x^2 - \frac{192}{162}x^3 + \cdots\right]$$

$$= \frac{1}{9} - \frac{4}{27}x + \frac{24}{162}x^2 - \frac{192}{1458}x^3 + \cdots$$

(13) $(4+2x^2)^{-1/2}$

$$(4+2x^2)^{-1/2} = \frac{1}{(4+2x^2)^{1/2}}.$$

$$= \frac{1}{4(1+\frac{2x^2}{4})^{1/2}} = \frac{1}{[4(1+\frac{x^2}{2})]^{1/2}}$$

$$= \frac{1}{4^{1/2}(1+\frac{x^2}{2})^{1/2}} = \frac{1}{2}(1+\frac{x^2}{2})^{-1/2}.$$

$$= \frac{1}{2}\left[1 + (-\tfrac{1}{2})(\tfrac{x^2}{2}) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\tfrac{x^2}{2})^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\cdot2\cdot3}(\tfrac{x^2}{2})^3 + \cdots +\right]$$

$$= \frac{1}{2}\left[1 - \frac{x^2}{4} + \frac{3}{32}x^2 - \frac{15}{384}x^3 + \cdots\right]$$

$$= \frac{1}{2} - \frac{x^2}{8} + \frac{3x^2}{64} - \frac{15x^3}{768} + \cdots$$

The fourth term is $18x^3$.
And the general terms is
$3+6x+9x^2+18x^3-6x^4-12x^5+$
$24x^6+48x^7+\dots+$ //

7) Determine the greatest Coefficient
in the binomial expansion of $(3x+1)^8$.

$$(3x+1)^8 = \sum_{k=0}^{8} \binom{8}{k} (3x)^k . 1^{8-k}.$$

for the kth term.

$$\binom{8}{k} (3x)^k . 1^{8-k}$$

for the $(k+1)$th term.

$$\binom{8}{k+1}(3x)^{k+1} . 1^{8-(k+1)}$$

$\therefore$ The ratio of $(k+1)$th to kth
term is.

$$= \frac{\binom{8}{k+1}(3x)^{k+1} . 1^{8-k-1}}{\binom{8}{k}(3x)^k . 1^{8-k}}$$

$$= \frac{\binom{8}{k+1}}{\binom{8}{k}} \cdot \frac{3^{k+1} . x^{k+1} . 1^{7-k}}{3^k . x^k . 1^{8-k}}$$

$$= \frac{\binom{8}{k+1}}{\binom{8}{k}} 3^{k+1-k} . x^{k+1-k} . 1^{\cancel{7-k}-8\cancel{+k}}$$

We ignore x since we need
coefficient

$$\frac{\binom{8}{k+1}}{\binom{8}{k}} \cdot 3$$

Solving for $\frac{\binom{8}{k+1}}{\binom{8}{k}}$

$$= \frac{\frac{8!}{(k+1)!\,(8-(k+1))!}}{\frac{8!}{(8-k)!\,k!}}$$

$$= \frac{\cancel{8!}}{(k+1)!\,[(8-(k+1)]!} \times \frac{(8-k)(8-k-1)!\ k!}{\cancel{8!}}$$

$$= \frac{(8-k)(8-k-1)!\ k!}{(k+1)!\ (8-k-1)!}$$

$$= \frac{(8-k)\cancel{k!}}{(k+1)\ \cancel{(k)!}} = \frac{8-k}{k+1}$$

$$\therefore \left[\frac{8-k}{k+1}\right] \cdot \frac{3}{1} = \frac{24-3k}{k+1}$$

$$\therefore \frac{24-3k}{k+1} > 1$$

$24-3k > k+1$
$= -4k > -23$
$k > \frac{23}{4}$
$\therefore k > 5.75$
$k = 6$

$$(3x+1)^8 = \sum_{k=6}^{8} \binom{8}{6} 3^6 . 1^{8-6}$$

$$\binom{8}{6} \cdot 3^6 = 729 \times 28$$
$$= 20412$$ //

(48)

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(22) Use binomial expansion to evaluate $\sqrt{\frac{1}{17}}$

$$\sqrt{\frac{1}{17}} = \frac{1}{\sqrt{17}} = \frac{1}{\sqrt{16+1}}$$

$$= \frac{1}{(16+1)^{\frac{1}{2}}} = \frac{1}{\left[16\left(1+\frac{1}{16}\right)\right]^{\frac{1}{2}}}$$

$$= \frac{1}{16^{\frac{1}{2}}\left(1+\frac{1}{16}\right)^{\frac{1}{2}}} = \frac{1}{4}\left(1+\frac{1}{16}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{4}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{16}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{16}\right)^2}{1\cdot 2} + \cdots\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{32} + \frac{3}{2048} + \cdots\right]$$

$$\simeq \frac{1}{4}\left[1 - 0.03215 + 0.0014648\right]$$

$$\sqrt{\frac{1}{17}} \simeq \frac{1}{4}\left[0.9702148\right]$$

$$\sqrt{\frac{1}{17}} \simeq 0.2425537$$

(23) Use binomial expansion to evaluate $\sqrt[3]{\frac{1}{28}}$

$$\sqrt[3]{\frac{1}{28}} = \frac{\sqrt[3]{1}}{\sqrt[3]{28}} = \frac{1}{\sqrt[3]{27+1}}$$

$$= \frac{1}{(27+1)^{\frac{1}{3}}} = \frac{1}{(27+1)^{\frac{1}{3}}}$$

$$\frac{1}{(27+1)^{\frac{1}{3}}} = \frac{1}{27^{\frac{1}{3}}\left(1+\frac{1}{27}\right)^{\frac{1}{3}}}$$

$$\frac{1}{3}\left(1+\frac{1}{27}\right)^{-\frac{1}{3}}$$

$$= \frac{1}{3}\left[1 + \left(-\frac{1}{3}\right)\left(\frac{1}{27}\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(\frac{1}{27}\right)^2}{2} + \cdots\right]$$

$$\simeq \frac{1}{3}\left[1 - \frac{1}{81} + \frac{4}{13122}\right]$$

$$\sqrt{\frac{1}{17}} = \frac{1}{3}\left[1 - 0.01234 + 0.0003048\right]$$

$$\sqrt{\frac{1}{17}} = \frac{1}{3}\left[0.9879\right]$$

$$\sqrt{\frac{1}{17}} \simeq 0.3293$$

(24) Use binomial expansion to evaluate $\left(\frac{1}{80}\right)^{\frac{1}{4}}$

$$\left(\frac{1}{80}\right)^{\frac{1}{4}} = \frac{1^{\frac{1}{4}}}{(80)^{\frac{1}{4}}} = \frac{1}{(81-1)^{\frac{1}{4}}}$$

$$\frac{1}{(81-1)^{\frac{1}{4}}} = \frac{1}{\left[81\left(1-\frac{1}{81}\right)\right]^{\frac{1}{4}}}$$

$$= \frac{1}{81^{\frac{1}{4}}\left(1-\frac{1}{81}\right)^{\frac{1}{4}}} = \frac{1}{3\left(1-\frac{1}{81}\right)^{\frac{1}{4}}}$$

$$\frac{1}{3}\left(1-\frac{1}{81}\right)^{-\frac{1}{4}}$$

$$= \frac{1}{3}\left(1 + \left(-\frac{1}{4}\right)\left(-\frac{1}{81}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{1}{81}\right)^2}{2}\right]$$

$$= \frac{1}{3}\left[1 + \frac{1}{324} - \frac{5}{209952}\right]$$

$$\frac{1}{3}\left[1 + 0.003086 - 0.000238\right]$$

$$\left(\frac{1}{81}\right)^{\frac{1}{4}} \simeq \frac{1.00306}{3} = 0.3343$$

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(20) Find the Largest term in the expansion of $(3x+2y)^8$

Using pascal triangle the Coefficients are

1 8 28 56 70 56 28 8 1

$\therefore (3x+2y)^8 = (3x)^8 + 8\cdot(3x)^7(2y)$
$+ 28(3x)^6(2y)^2 + 56(3x)^5(2y)^3$
$+ 70(3x)^4(2y)^4 + 56(3x)^3(2y)^5$
$+ 28(3x)^2(2y)^6 + 8(3x)(2y)^7$
$+ (2y)^8$.

$(3x+2y)^8 = 6561x^8 + 34992x^7y$
$+ 326592x^6y^2 + 108862x^5y^3$
$+ 90720x^4y^4 + 48384x^3y^5$
$+ 16128x^2y^6 + 3072xy^7$
$+ 256y^8$.

Therefore the ~~8~~ largest term in the expansion of $(3x+2y)^8$ is,

(21) Use binormal expansion to evaluate $\sqrt{\frac{1}{26}}$

$\sqrt{\frac{1}{26}} = \frac{\sqrt{1}}{\sqrt{26}} = \frac{1}{\sqrt{26}}$

$\frac{1}{\sqrt{26}} = \frac{1}{\sqrt{25+1}} = \frac{1}{(25+1)^{\frac{1}{2}}}$

$= \frac{1}{\left[25\left(1+\frac{1}{25}\right)\right]^{\frac{1}{2}}} = \frac{1}{25^{\frac{1}{2}}\left(1+\frac{1}{25}\right)^{\frac{1}{2}}}$

$= \frac{1}{25^{\frac{1}{2}}} \cdot \left(1+\frac{1}{25}\right)^{-\frac{1}{2}}$.

$= \frac{1}{5} \cdot \left(1+\frac{1}{25}\right)^{-\frac{1}{2}}$.

Expanding $\left(1+\frac{1}{25}\right)^{-\frac{1}{2}}$.

$= \left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{25}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{25}\right)^2}{1.2} + \dots\right]$

$= \frac{1}{5}\left[1 - \frac{1}{50} + \frac{3}{5000} + \dots\right]$

$\sqrt{\frac{1}{26}} \simeq \frac{1}{5}\left[1 - 0.02 + 0.0006 + \dots\right]$

$\simeq \frac{1}{5}[0.9806$

$\sqrt{\frac{1}{26}} \simeq 0.19612$

25 Use binomial expansion to evaluate $\sqrt{626}$.

$\sqrt{626} = \sqrt{625+1}$

$(625+1)^{1/2} = 625^{1/2}\left(1+\frac{1}{625}\right)^{1/2}$

$= 25\left(1+\frac{1}{625}\right)^{1/2} =$

$= 25\left(1 + \frac{1}{2}\left(\frac{1}{625}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{625}\right)\right)$

$= 25\left(1 + \frac{1}{1250} - \frac{1}{2500}\right)$

$= 25(1 + 0.0008 - 0.0004)$

$= 25.01$

26
27
28
29
30

All Solved in the MANUAL.

TUTORIAL SHEET SIX

...estion 1 - 3

...eady Solved Inside manual.

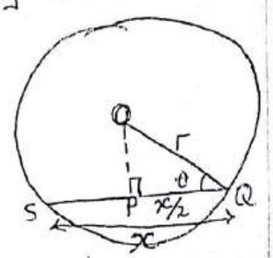
Question 4

Chord of a circle of radius r cm ...ubstends an angle θ on the cir...mference of the circle. If the ...hord is x cm long. Show that

$$\tan\theta = x(4r^2 - x^2)^{-\frac{1}{2}}$$

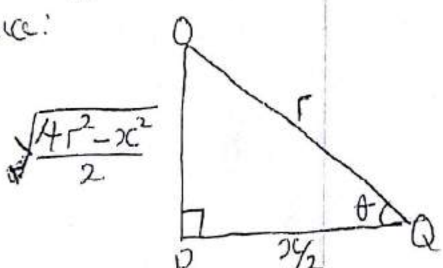
Solution

...rawing the circle we have:



...et the angle substended by the radius ...m of the circle be θ

Bringing out the triangle OPQ we have:



...he length of the chord x cm has been ...ivided into two Hence $PQ = \left(\frac{x}{2}\right)$ cm

To get the side OP using Pythagoras theorem $r^2 = \left(\frac{x}{2}\right)^2 + (OP)^2$

$$\Rightarrow OP^2 = r^2 - \frac{x^2}{4}$$

$$OP = \sqrt{r^2 - \frac{x^2}{4}} = \sqrt{\frac{4r^2 - x^2}{4}}$$

$$OP = \frac{\sqrt{4r^2 - x^2}}{2} \text{ cm}$$

where $\tan\theta = \frac{OPP}{ADJ}$

i.e $\tan\theta = \dfrac{\frac{\sqrt{4r^2 - x^2}}{2}}{x/2}$

$$\tan\theta = \frac{\sqrt{4r^2 - x^2}}{2} \div \frac{x}{2}$$

$$= \frac{\sqrt{4r^2 - x^2}}{\cancel{2}} \times \frac{\cancel{2}}{x}$$

$$\tan\theta = \frac{\sqrt{4r^2 - x^2}}{x}$$

making the inverse of the R.H.S

$$\tan\theta = x(4r^2 - x^2)^{-\frac{1}{2}}$$

Shown

Question 5

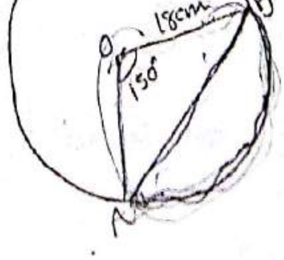
A chord of circle of radius 18 cm, substends an angle of 150° at the centre O of the circle. Calculate, correct to one decimal place.

(i) The area of ΔAOB

(ii) The area of the minor segment cut off by the cord $\overline{AB}$ and the arc AB (Take $\pi = 3.142$)

Solution

(i)



... the triangle is given by

$A = \frac{1}{2}ab\sin\theta = \frac{1}{2}\times 18\times 18\times \sin 150$

$= 162 \times 0.5$

$= 81\,cm^2$.

(ii) Area of the minor segment is given by:

The area of the sector AOB − Area of the triangle

Area of the sector $= \frac{\theta}{360}\times \pi r^2$

$= \frac{150}{360}\times 3.142 \times 18^2$

$= 424.17$

$\simeq 424.2\,cm^2$

$\therefore$ Area of minor segment $= 424.2\,cm^2 - 81\,cm^2$

$= 343.2\,cm^2$

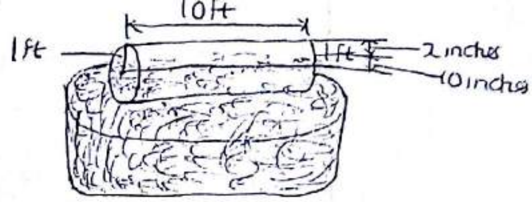
Question 6

Already solved inside manual

Question 7

A cylindrical log, radius of circular section 1ft, length 10ft, is floating in water with its axis horizontal and its highest point 2 inches above the level of the water. Find the volume not immersed.

Solution



1 ft = 12 inches

$\Rightarrow$ Radius r = 12 inches

Diameter D = 24 inches

$\Rightarrow$ 1 inch $= \frac{1}{12}$ ft

2 inches $= 2\times\frac{1}{12}$ ft $= \frac{1}{6}$ ft

Volume of a cylinder is given by

$V = \pi r^2 h$

$r = 1$ ft

$h = 10$ ft

$V_T = 3.142 \times 1^2 \times 10$

$= 31.42$ cubic ft

Total diameter of the cylinder = 24 inches

Hence the % of the cylinder not inserted in the water is

$\frac{2}{24}\times 100 = 8.33\%$ of the cylinder

$\therefore$ Hence 8.33% of the total volume is not inserted

$\Rightarrow$ 8.33% of 31.42 is not inserted

$= \frac{8.33}{100}\times 31.42 = 2.62$ cubic ft

OR

In foot total diameter = 2ft

Part of cylinder not inserted = 2 inches $\equiv \frac{1}{6}$ foot $= 0.167$ ft

Hence the % of cylinder not inserted in the water is $\frac{0.167}{2} \times 100 = 8.33\%$

also 8.33% of the total volume of cylinder is given by $\frac{8.33}{100} \times 31.42$

$= 2.62$ cubic ft

Question 8 - 10

Already solved in the manual

Question 11

If $x = a\cos\theta$, simplify (i) $a^2 - x^2$

(ii) $\left(1 - \frac{x^2}{a^2}\right)^{5/2}$ $a = \frac{x}{\cos\theta}$

Solution

$x = a\cos\theta \Rightarrow a = \frac{x}{\cos\theta}$

$\left(\frac{x}{\cos\theta}\right)^2 - (a\cos\theta)^2 = a^2 - x^2$

$\left(\frac{x}{\cos\theta}\right)^2 - \frac{x^2}{1} = \frac{x^2}{\cos^2\theta} - \frac{x^2}{1}$

$= \frac{x^2 - x^2\cos^2\theta}{\cos^2\theta} = \frac{x^2(1-\cos^2\theta)}{\cos^2\theta}$

recall $1 - \cos^2\theta = \sin^2\theta$

$\Rightarrow \frac{x^2\sin^2\theta}{\cos^2\theta} = x^2\tan^2\theta$

$= (x\tan\theta)^2$

OR

$a^2 - (a\cos\theta)^2 = a^2 - a^2\cos^2\theta$

$= a^2(1-\cos^2\theta) = a^2\sin^2\theta$

but $a = \frac{x}{\cos\theta}$

$a^2\sin^2\theta = \frac{x^2}{\cos^2\theta} \cdot \sin^2\theta = x^2\tan^2\theta$

$= (x\tan\theta)^2$

(ii) $\left(1 - \frac{x^2}{a^2}\right)^{5/2}$

$\left(1 - \frac{a^2\cos^2\theta}{a^2}\right)^{5/2}$

$\left(\frac{a^2 - a^2\cos^2\theta}{a^2}\right)^{5/2}$

$\left[\frac{\cancel{a^2}(1-\cos^2\theta)}{\cancel{a^2}}\right]^{5/2}$

$= (\sin^2\theta)^{5/2} = (\sin\theta)^{\cancel{2}\times 5/\cancel{2}}$

$= \sin^5\theta$

Question 12

Given that $\tan\lambda = \mu$, show that

$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \tan(\theta + \lambda)$

Solution

from R.H.S

$\tan(\theta + \lambda) = \frac{\tan\theta + \tan\lambda}{1 - \tan\theta\tan\lambda}$

If $\tan\lambda = \mu$ then

$= \frac{\tan\theta + \mu}{1 - \mu\tan\theta}$

$\cos\theta$

$$\Rightarrow \frac{\frac{\sin\theta}{\cos\theta}+\mu}{1-\mu\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{\frac{\sin\theta+\mu\cos\theta}{\cos\theta}}{\frac{\cos\theta-\mu\sin\theta}{\cancel{\cos\theta}}}$$

$$= \frac{\sin\theta+\mu\cos\theta}{\cos\theta-\mu\sin\theta} = \tan(\theta+\lambda)$$

Proved.

from L.H.S

$$\frac{\sin\theta+\mu\cos\theta}{\cos\theta-\mu\sin\theta}$$

divide through by $\cos\theta$

$$\frac{\frac{\sin\theta}{\cos\theta}+\frac{\mu\cancel{\cos\theta}}{\cancel{\cos\theta}}}{\frac{\cancel{\cos\theta}}{\cancel{\cos\theta}}-\frac{\mu\sin\theta}{\cos\theta}} = \frac{\tan\theta+\mu}{1-\mu\tan\theta}$$

Substitute $\mu = \tan\lambda$

then $= \dfrac{\tan\theta+\tan\lambda}{1-\tan\lambda\tan\theta} = \tan(\theta+\lambda)$

Proved

Question 13

Show that $\tan(A+B+C) =$

$$\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Solution

Simplify the L.H.S

$\tan(A+B+C) = \tan[(A+B)+C]$

where $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$ ----(1)

Hence $\tan[(A+B)+C]$

$$= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

by substituting (1)

$$\tan[(A+B)+C] = \frac{\frac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left[\frac{\tan A+\tan B}{1-\tan A\tan B}\right]\tan C}$$

find the L.C.M and simplify

$$\tan(A+B+C) = \frac{\frac{\tan A+\tan B+\tan C(1-\tan A\tan B)}{1-\tan A\tan B}}{1-\left(\frac{\tan A\tan C+\tan B\tan C}{1-\tan A\tan B}\right)}$$

$$= \frac{\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{\cancel{1-\tan A\tan B}}}{\frac{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}{\cancel{1-\tan A\tan B}}}$$

$\therefore \tan(A+B+C) =$

$$\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$$

Since L.H.S = R.H.S Shown

Question 14

If $x = a\tan\theta$, Simplify

(i) $\dfrac{1}{a^2+x^2}$ (ii) $\sqrt{\left(1+\frac{x^2}{a^2}\right)}$

Solution

$x = a\tan\theta$

(i) $\dfrac{1}{a^2+x^2} = \dfrac{1}{a^2+(a\tan\theta)^2}$

$$= \frac{1}{a^2+a^2\tan^2\theta} = \frac{1}{a^2(1+\tan^2\theta)}$$

$$= \frac{1}{a^2\sec^2\theta}$$

OR

where $a = x/\tan\theta$

$$\frac{1}{\left(\frac{x}{\tan\theta}\right)^2 \cdot \sec^2\theta} = \frac{1}{\frac{x^2}{\tan^2\theta} \cdot \sec^2\theta}$$

$$= \frac{\tan^2\theta}{x^2\sec^2\theta}$$

(ii) $\sqrt{\left(1+\frac{x^2}{a^2}\right)}$

$$= \sqrt{1+\frac{a\tan\theta}{a^2}} = \sqrt{\frac{a^2+a\tan\theta}{a^2}}$$

$$= \sqrt{\frac{a(a+\tan\theta)}{a^2}} = \sqrt{\frac{a+\tan\theta}{a}}$$

$$= \sqrt{\frac{a}{a}+\frac{\tan\theta}{a}} = \sqrt{1+\frac{\tan\theta}{a}}$$

Question 15

If $a\cos^2\theta + b\sin^2\theta = C$ Show that

$$\tan^2\theta = \frac{c-a}{b-C}$$

Solution

$$b\sin^2\theta = C - a\cos^2\theta$$

$$b(1-\cos^2\theta) = C - a\cos^2\theta$$

$$b - b\cos^2\theta = C - a\cos^2\theta$$

$$b - C = b\cos^2\theta - a\cos^2\theta$$

$$\Rightarrow b - C = (b-a)\cos^2\theta$$

$$\Rightarrow \cos^2\theta = \frac{b-c}{b-a} \quad \text{------(i)}$$

also

$$b\sin^2\theta = C - a\cos^2\theta$$

$$b\sin^2\theta = C - a(1-\sin^2\theta)$$

$$b\sin^2\theta = C - a + a\sin^2\theta$$

$$b\sin^2\theta - a\sin^2\theta = C - a$$

$$\sin^2\theta(b-a) = C-a$$

$$\sin^2\theta = \frac{C-a}{b-a} \quad \text{------(2)}$$

recall $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Hence $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{C-a}{b-a} \Big/ \frac{b-c}{b-a}$

$$= \frac{C-a}{\cancel{b-a}} \times \frac{\cancel{b-a}}{b-C}$$

$$\Rightarrow \tan^2\theta = \frac{C-a}{b-C}$$

Prooved

Question 16

If A, B and C are the angles of a triangle. Show that:

(i) $\cos A + \cos(B-C) = 2\sin B\sin C$

(ii) $\cos\frac{1}{2}C + \sin\frac{1}{2}(A-B) = 2\sin\frac{1}{2}A\cos\frac{1}{2}B$

Solution

(i) $\cos A + \cos(B-C) = 2\sin B\sin C$

Solving the L.H.S

$$\cos A + [\cos B\cos C + \sin B\sin C]$$

$$= \cos A + \cos B\cos C + \sin B\sin C$$

Since it is the angles of the triangle

then $A + B + C = 180$

$$\Rightarrow A = 180 - B - C$$

$$= 180 - (B+C)$$

Substitute into the equation

$$= \cos[180-(B+C)] + \cos B\cos C + \sin B\sin C$$

$$= \cos 180\cos(B+C) + \sin 180\sin(B+C) + \cos B\cos C + \sin B\sin C$$

[where $\cos(A-B) = \cos A\cos B + \sin A\sin B$]

$\cos 180 = -1$ and $\sin 180 = 0$

Also $\cos(B+C) = \cos B\cos C - \sin B\sin C$

$\Rightarrow (-1)(\cos B\cos C - \sin B\sin C) + 0(\sin(B+C)) +$
$\cos B\cos C + \sin B\sin C$

$= -\cancel{\cos B\cos C} + \sin B\sin C + \cancel{\cos B\cos C} + \sin B\sin C$

$= 2\sin B\sin C$

Hence $\cos A + \cos(B-C) = 2\sin B\sin C$

Proved

(ii) $\cos\frac{1}{2}C + \sin\frac{1}{2}(A-B) = 2\sin\frac{1}{2}A\cos\frac{1}{2}B$

where $\sin\frac{1}{2}(A-B) = \sin\frac{1}{2}A\cos\frac{1}{2}B - \cos\frac{1}{2}A\sin\frac{1}{2}B$

$\cos\frac{1}{2}(A-B) = \cos\frac{1}{2}A\cos\frac{1}{2}B + \sin\frac{1}{2}A\sin\frac{1}{2}B$

from L.H.S

$\cos\frac{1}{2}C + \sin\frac{1}{2}(A-B) = \cos\frac{1}{2}C + \sin\frac{1}{2}A\cos\frac{1}{2}B$
$- \cos\frac{1}{2}A\sin\frac{1}{2}B$

Since the angles make up a Δ then
$A+B+C = 180 \Rightarrow C = 180-(A+B)$

Then
$\cos\frac{1}{2}[180-(A+B)] + \sin\frac{1}{2}A\cos\frac{1}{2}B - \cos\frac{1}{2}A\sin\frac{1}{2}B$

Opening the brackets we have

$\cos\frac{1}{2}(180)\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}(180)\sin\frac{1}{2}(A+B)$
$+ \sin\frac{1}{2}A\cos\frac{1}{2}B - \cos\frac{1}{2}A\sin\frac{1}{2}B$

$= \cos 90\cos\frac{1}{2}(A+B) + \sin 90\sin\frac{1}{2}(A+B) +$
$\sin\frac{1}{2}A\cos\frac{1}{2}B - \cos\frac{1}{2}A\sin\frac{1}{2}B$

recall $\cos 90 = 0$ $\sin 90 = 1$ Substituting

$= 0[\cos\frac{1}{2}(A+B)] + 1[\sin\frac{1}{2}(A+B)] + \sin\frac{1}{2}A\cos\frac{1}{2}B$
$- \cos\frac{1}{2}A\sin\frac{1}{2}B$

$= \sin\frac{1}{2}(A+B) + \sin\frac{1}{2}A\cos\frac{1}{2}B - \cos\frac{1}{2}A\sin\frac{1}{2}B$

$= \sin\frac{1}{2}A\cos\frac{1}{2}B + \cancel{\cos\frac{1}{2}A\sin\frac{1}{2}B} + \sin\frac{1}{2}A\cos\frac{1}{2}B$
$- \cancel{\cos\frac{1}{2}A\sin\frac{1}{2}B}$

$= 2\sin\frac{1}{2}A\cos\frac{1}{2}B$

Proved.

Question 17

Show that

$\sin x + \sin\left(x + \frac{2}{3}\pi\right) + \sin\left(x + \frac{4}{3}\pi\right) = 0$

Solution

recall $\pi = 180$

$\therefore \sin x + \sin\left[x + \frac{2}{3}(180)\right] + \sin\left[x + \frac{4}{3}(180)\right]$

$= \sin x + \sin(x+120) + \sin(x+240)$

where $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\therefore \sin x + \sin x\cos 120 + \cos x\sin 120 +$
$\sin x\cos 240 + \cos x\sin 240$

$= \sin x + \sin x(-\frac{1}{2}) + \cos x(\frac{\sqrt{3}}{2}) + \sin x(-\frac{1}{2})$
$+ \cos x(-\frac{\sqrt{3}}{2})$

$= \sin x - \frac{1}{2}\sin x + \cancel{\frac{\sqrt{3}}{2}\cos x} - \frac{1}{2}\sin x - \cancel{\frac{\sqrt{3}}{2}\cos x}$

$= \sin x - \sin x = 0$

Hence $\sin x + \sin(x + \frac{2}{3}\pi) + \sin(x + \frac{4}{3}\pi) = 0$

Shown

Question 18

Show that $\tan(\frac{1}{4}\pi + x)\tan(\frac{1}{4}\pi - x) = 1$

Solution

from the L.H.S where $\pi = 180$

$\tan\left[\frac{1}{4}(180) + x\right]\tan\left[\frac{1}{4}(180) - x\right]$

$= \tan(45+x)\tan(45-x)$

where $\tan A \pm B = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$\therefore \dfrac{\tan 45 + \tan x}{1 - \tan 45\tan x} \times \dfrac{\tan 45 - \tan x}{1 + \tan 45\tan x}$

Where $\tan 45 = 1$

$\Rightarrow \frac{1+\tan x}{1-\tan x} \times \frac{1-\tan x}{1+\tan x} = 1$

Hence

$\tan(\frac{1}{4}\pi + x)\tan(\frac{1}{4}\pi - x) = 1$

Proved

Question 22a

If $\tan 2\emptyset - \sin 2\emptyset = x$ and $\tan 2\emptyset + \sin 2\emptyset = y$ Show that

i) $\frac{x}{y} = \tan^2\emptyset$ (ii) $(x^2-y^2)^2 = 16xy$

Solution

i) $\frac{x}{y} = \frac{\tan 2\emptyset - \sin 2\emptyset}{\tan 2\emptyset + \sin 2\emptyset}$

recall $\tan\emptyset = \frac{\sin\emptyset}{\cos\emptyset}$

$\Rightarrow \frac{x}{y} = \frac{\frac{\sin 2\emptyset}{\cos 2\emptyset} - \sin 2\emptyset}{\frac{\sin 2\emptyset}{\cos 2\emptyset} + \sin 2\emptyset}$

finding the L.C.M of both numerator and denominator

$\frac{x}{y} = \frac{\frac{\sin 2\emptyset - \sin 2\emptyset\cos 2\emptyset}{\cancel{\cos 2\emptyset}}}{\frac{\sin 2\emptyset + \sin 2\emptyset\cos 2\emptyset}{\cancel{\cos 2\emptyset}}}$

$\frac{x}{y} = \frac{\cancel{\sin 2\emptyset}(1-\cos 2\emptyset)}{\cancel{\sin 2\emptyset}(1+\cos 2\emptyset)}$

$\frac{x}{y} = \frac{1-\cos 2\emptyset}{1+\cos 2\emptyset}$

recall $\cos 2\emptyset = \cos^2\emptyset - \sin^2\emptyset$

$= 1 - 2\sin^2\emptyset$

$= 2\cos^2\emptyset - 1$

Substituting in the above we have,

$\frac{x}{y} = \frac{1-(1-2\sin^2\emptyset)}{1+(2\cos^2\emptyset - 1)}$

$= \frac{\cancel{1}-\cancel{1}+\cancel{2}\sin^2\emptyset}{\cancel{1}+\cancel{2}\cos^2\emptyset\cancel{-1}} = \tan^2\emptyset$

Hence $\frac{x}{y} = \tan^2\emptyset$

Proved

Question 24 a

Prove that $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

Solution

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

So $\sin(\alpha+\beta)\sin(\alpha-\beta) =$

$(\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= \sin^2\alpha\cos^2\beta - \sin\alpha\cancel{\cos\alpha}\sin\beta\cos\beta + \cancel{\cos\alpha\sin\alpha}$
$\cos\beta\sin\beta - \cos^2\alpha\sin^2\beta$

$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta$

recall $\sin^2\theta + \cos^2\theta = 1$

$= \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$

open brackets

$= \sin^2\alpha - \sin^2\cancel{\alpha}\sin^2\beta - \sin^2\beta + \sin^2\cancel{\alpha}\sin\beta$

$= \sin^2\alpha - \sin^2\beta$

$\therefore \sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

Proved

① Already Solved in the Manual

② If $\frac{1-\tan^2 67\frac{1}{2}}{1+\tan^2 67\frac{1}{2}} = \cos 135^\circ$, find $\tan 67\frac{1}{2}$ in surd form.

Solution

Since $\cos 135^\circ = -\cos 45^\circ$

$= -\frac{1}{\sqrt{2}}$

let $a = \tan^2 67\frac{1}{2}$

$\frac{1-a}{1+a} = \frac{-1}{\sqrt{2}}$

$\sqrt{2} - a\sqrt{2} = -1 - a$

$a\sqrt{2} - a = \sqrt{2}+1$

$a(\sqrt{2}-1) = \sqrt{2}+1$

$a = \frac{\sqrt{2}+1}{\sqrt{2}-1}$

Since $a = \tan^2 67\frac{1}{2}$

$\tan^2 67\frac{1}{2} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$

$\tan 67\frac{1}{2} = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}$

③ Find the values of $\frac{\tan 300^\circ + \tan 210^\circ}{1 - \tan 300^\circ \times \tan 210^\circ}$

Solution

Recall that $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\therefore \tan(300+210) = \tan 510$

④ Identify all values of α, if $0^\circ < \alpha < 360^\circ$ and satisfies the equation

$4\sin^2\alpha - \cos\alpha + 1 = 0$

Solution

Recall that $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2\alpha = 1 - \cos^2\alpha$

$4(1-\cos^2\alpha) - \cos\alpha + 1 = 0$

$4 - 4\cos^2\alpha - \cos\alpha + 1 = 0$

$4\cos^2\alpha + \cos\alpha - 5 = 0$

$(4\cos^2\alpha - 4\cos\alpha) + (5\cos\alpha - 5) = 0$

$4\cos\alpha(\cos\alpha - 1) + 5(\cos\alpha - 1) = 0$

$4\cos\alpha + 5 = 0$ OR $\cos\alpha - 1 = 0$

$4\cos\alpha = -5$ OR $\cos\alpha = 1$

$\cos\alpha = \frac{-5}{4}$ OR $\cos\alpha = 1$

$\alpha = \cos^{-1}\left(-\frac{5}{4}\right)$ OR $\alpha = \cos^{-1}(1)$

$=$ $= 0, 360$

(5) Given that $\tan\theta = \frac{2t}{1-t^2}$ and θ is an acute angle, express $\sin\theta$ and $\cos\theta$ in terms of t

Solution

$\sin x = 2\sin\frac{1}{2}x\cos\frac{1}{2}x$

Since $\sin x = \sin(\frac{1}{2}x + \frac{1}{2}x)$

$$\sin x = \frac{2\sin\frac{1}{2}x\cos\frac{1}{2}x}{\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x} \quad (\text{Since } \cos^2\tfrac{1}{2}x + \sin^2\tfrac{1}{2} = 1)$$

on dividing the numerator & denominator by $\cos^2\frac{1}{2}x$, we have

$$\sin x = \frac{\frac{2\sin\frac{1}{2}x}{\cos\frac{1}{2}x}}{1 + \frac{\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x}}$$

Recall $\frac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x} = \tan\frac{1}{2}x$

let $t = \tan\frac{1}{2}x$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x}$$

Since $\cos x = \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

dividing through by $\cos^2\frac{1}{2}x$

$$\cos x = \frac{1 - \frac{\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x}}{1 + \frac{\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x}}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

(6) Show that $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \operatorname{cosec}\theta$

Solution

From L.H.S

$(\sin\theta + \cos\theta)(\tan\theta + \cot\theta)$

Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

$$(\sin\theta + \cos\theta)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$(\sin\theta + \cos\theta)\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)$$

Recall that $\sin^2\theta + \cos^2\theta = 1$

$$(\sin\theta + \cos\theta)\left(\frac{1}{\cos\theta\sin\theta}\right)$$

To open the bracket

$$\frac{\cancel{\sin\theta}}{\cos\theta\cancel{\sin\theta}} + \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}\sin\theta}$$

$$\frac{1}{\cos\theta} + \frac{1}{\sin\theta}$$

$= \sec\theta + \operatorname{cosec}\theta$ PROVED

(7) Simplify $(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2$

Solution

$(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2$

by Expanding each bracket

$(\cos\theta - \sin\theta)(\cos\theta - \sin\theta)$

$\cos^2\theta - \cos\theta\sin\theta - \cos\theta\sin\theta + \sin^2\theta$

$\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta$ ---- (i)

$(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)$

$\cos^2\theta + \cos\theta\sin\theta + \cos\theta\sin\theta + \sin^2\theta$

$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$ ---- (ii)

from (i) $\cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta$

$1 - 2\cos\theta\sin\theta$

from (ii) $\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta$

$1 + 2\cos\theta\sin\theta$