Introduction to Quantum Computing

Abstract Space: determined by specific rules and properties and can be changed to linear vector space by matrices.

* $|a\rangle \Rightarrow$ vector $\quad \langle a| \Rightarrow$ dual vector

Hilbert Space: complete complex vector space which can represent every valid state of a "quantum system".

Inner Product:

$f(v) = \langle u|v\rangle$ inner product of v with u.

$= (|u\rangle)^*(|v\rangle)$

$\langle a| = (|a\rangle)^*$

$\underset{\text{row vector}}{\langle u|}\underset{\text{column vector}}{v\rangle} \in \mathbb{C}$ always

$||\vec{a}|| = \sqrt{\langle a|a\rangle}$

$\langle v|w\rangle = (\langle w|u\rangle)^*$

State vectors:

$|\psi\rangle$ is an independent state in the Hilbert space which can be represented by choosing basis such as position, momentum & energy.

$\langle v|w\rangle = \int v^*(x)w(x)t(x)dx$

$t =$ weight factor

* Inner product is very "similar" to dot product. In fact dot product is a special case of inner product.

→ Am not mentioning properties like positive definiteness, conjugate symmetry etc etc.

Outer Product:

$P = |v\rangle\langle w|$ is outer product of $|v\rangle$ with $|w\rangle$
column vector, row

hence P is a matrix

$|v\rangle$ be $n\times1$
$|w\rangle$ be $m\times1$ $\Rightarrow P$ is $n\times m$

→ Projection Operator

let $|v\rangle$ be vector

then $P = |v\rangle\langle v|$ is P.O.

let $P|w\rangle = |v\rangle\langle v||w\rangle$

$P|w\rangle = \langle v|w\rangle|v\rangle$

is the projection of $|w\rangle$ on $|v\rangle$ given that $|v\rangle$ is unit vector.

Operators:

$\hat{A}: V \to V$

This implies these operators are nothing but $n\times n$ matrices in linear alg.

let $H = \mathbb{C}^n$

$A|v\rangle = |w\rangle$

→ $\langle \phi|A\psi\rangle = \langle A^+\phi|\psi\rangle$

by basics of inner product.

★ given $A = A^*$ (A is hermitian)

$\langle A^*\psi|\psi\rangle = \langle\psi|A|\psi\rangle$

also $\langle\psi|A^*\psi\rangle = \langle A^*\psi|\psi\rangle^*$

$= \langle\psi|A\psi\rangle^*$

$(A=A^*)$ $\langle\psi|A\psi\rangle = \langle\psi|A\psi\rangle^*$

this also implies

$\langle\psi|A\psi\rangle$ must be real.

$\langle\psi|A\psi\rangle = \langle A\psi|\psi\rangle = \langle\psi|A\psi\rangle^*$

but what's the conclusion that we draw from here!

Orthonormalization.

- orthogonal v/s orthonormal

  ⇓

  $\langle v_i|v_j\rangle = \delta_{ij}$

  $\delta_{ij} = \begin{cases} 1; & i=j \\ 0; & \text{otherwise} \end{cases}$

- Gram Schmidt Process

Eigen values of operators:

$A|v\rangle = a|v\rangle$; $a \in \mathbb{C}$

a is eigen value of operator A and corresponding is eigen vector $|v\rangle$

1. $\langle v|A|v\rangle = a\langle v|v\rangle = \langle v|Av\rangle$
2. $\langle Av|v\rangle = a^*\langle v|v\rangle = \langle A^*v|v\rangle$
3. $\langle v|Av\rangle = a^*\langle v|v\rangle$

   (since A is hermitian)

   ⇒ real eigen values

   real possible states

In quantum mechanics all observables are represented by hermitian operators.

★ let $A|v_1\rangle = a|v_1\rangle$

$A|v_2\rangle = b|v_2\rangle$ and $a \neq b$

$\langle v_2|Av_1\rangle = a\langle v_2|v_1\rangle$

$\langle v_1|Av_2\rangle = b\langle v_1|v_2\rangle$

(since b is real) $\langle Av_2|v_1\rangle = b\langle v_2|v_1\rangle$

$\langle v_2|Av_1\rangle = b\langle v_2|v_1\rangle$

this implies either $a=b$ or

$\langle v_2|v_1\rangle = 0$

Adjoint of an Operator:

$((A)^*)^* = A$ $(AB)^* = B^*A^*$

$(\alpha A+\beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$

Hermitian Operator:

$A = A^*$ & properties already written.

State vector:

let $|\psi\rangle = \sum c_i|v_i\rangle$

$\sum|c_i|^2 = 1$ (if $|\psi\rangle$ is a physically possible state and $|v_i\rangle$ are eigen states)

Ex: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \equiv \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$|\alpha|^2 + |\beta|^2 = 1$

$\langle 0|1\rangle = 0$

$\langle 1|1\rangle = 1$ (beforementioned)

$\langle 0|0\rangle = 1$

Dual vector space:

$V^* = \{ f : V \to \mathbb{C} \mid f \text{ is linear} \}$

$|v\rangle \in V$ then its dual vector or bra $\in V^*$

$\langle v|w\rangle \in \mathbb{C}$

$\langle v|w|v|w\rangle = \langle v|w\rangle$ (how??)

Tensor Product:

$V \otimes W$ (v tensor w) represents mn dimensional vector space.

$$|v\rangle \otimes |w\rangle = \begin{pmatrix} v_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ v_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}_{4\times 1}$$

still normalizable!

let $|v\rangle = v_1|0\rangle + v_2|1\rangle$

$|w\rangle = w_1|0\rangle + w_2|1\rangle$ $\left\{ |0\rangle \otimes |0\rangle = |00\rangle \right\}$

$|v\rangle \times |w\rangle = v_1 w_1 |00\rangle + v_1 w_2 |01\rangle + v_2 w_1 |10\rangle + v_2 w_2 |11\rangle$

→ that's what the matrix represented

new bases are $|00\rangle$ $|01\rangle$ ....

** $k_{mn} = V_m \otimes W_n$

↪ $|v\rangle \otimes |w\rangle$

XX $|k\rangle = |v\rangle \otimes |w\rangle$ Blunder!!

$|k\rangle = \sum_{ij} c_{ij} |v_i\rangle \otimes |w_j\rangle$

$|k_{ij}\rangle$

$|v\rangle = \sum a_i |v_i\rangle$

$|w\rangle = \sum b_j |w_j\rangle$ $\quad |v\rangle \otimes |w\rangle$

$= \sum_{ij} a_i b_j |v_i\rangle \otimes |w_j\rangle$

Example: $|v\rangle = \alpha|0\rangle + \beta|1\rangle$

$|w\rangle = \gamma|0\rangle + \delta|1\rangle$

$|k\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

would get $\alpha = \beta = \gamma = \delta = 0$

* for no $\alpha, \beta, \gamma, \delta$ $|v\rangle \otimes |w\rangle = |k\rangle$

→ $A_{n\times p} \otimes B_{k\times m} = C_{nk \times pm}$

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1p}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & \cdots & \cdots & a_{np}B \end{bmatrix}$$

→ $\underset{mn\times mn}{(\underset{n\times n}{A} \otimes \underset{m\times m}{B})}\,\underset{mn\times 1}{(\underset{n\times 1}{|v\rangle} \otimes \underset{m\times 1}{|w\rangle})} = \underset{mn\times 1}{\underset{n\times 1}{A|v\rangle} \otimes \underset{m\times 1}{B|w\rangle}}$

Properties:

$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

$(|v\rangle \otimes |w\rangle)^\dagger = (|v\rangle)^\dagger \otimes (|w\rangle)^\dagger$

$= \langle v| \otimes \langle w|$

$(|v\rangle \otimes |w\rangle)^\dagger (|v\rangle \otimes |w\rangle)$

$= \langle v|v\rangle \langle w|w\rangle \in \mathbb{C}$

Important Notes:

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hadamard gate

$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$|\psi\rangle = \sum c_i v_i\rangle$

$\langle\psi|\psi\rangle = \sum_i c_i c_i^* = \sum_i |c_i|^2 = 1$

expectation value or Norm

* let $|\psi\rangle \rightarrow e^{i\theta}|\psi\rangle$

$\langle\psi|\psi\rangle = \langle\psi|e^{-i\theta}|e^{i\theta}|\psi\rangle$

remains same!!

Unitary Matrix:

$UU^* = I$

let $|\psi'\rangle = A|\psi\rangle$ (A is unitary)

evolution defined by unitary matrix.

$\langle\psi'|\psi'\rangle = \langle\psi|A^* A|\psi\rangle$

$= \langle\psi|\psi\rangle = 1$

also if $|\psi'\rangle = A|\psi\rangle$

$A^*|\psi'\rangle = |\psi\rangle$

we get the state back.

Kronecker Product:

Commutator $[A,B] = AB - BA$

Anticommutator $\{A,B\} = AB + BA$

Pauli Matrices

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

* The system doesn't move in space but rotates in Hilbert space.

Time Evolution

$|\psi\rangle \rightarrow |\psi'\rangle$

Hilbert space $H \rightarrow$ n dimensional

$|\psi\rangle = \sum_i c_i |v_i\rangle$

$|\psi'\rangle = \sum_j c_j |w_j\rangle$

$|\psi'\rangle = U|\psi\rangle$

$U = \sum_i |w_i\rangle\langle v_i|$

$U|v_i\rangle = w_i$

(Obvious from here)

(c_{ij} kyu nahi hai!)

$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \underbrace{H}_{\text{hamiltonian}} |\psi(t)\rangle$

$|\psi(0)\rangle \rightarrow |\psi(t)\rangle$

$|\psi(t)\rangle = U(t)|\psi(0)\rangle$

$i\hbar\left(\frac{d}{dt} U(t)\right)|\psi(0)\rangle = H(U(t)\psi(0))$

$i\hbar \frac{dU(t)}{dt} = H(U(t))$

($\psi(0) = f(x)$ then consider kaise kiya?)

$U(t) = e^{\frac{-iHt}{\hbar}}$

Operator Function & Spectral Decomposition

For an operator A

$A = \sum_i \lambda_i |i\rangle\langle i|$ λ_i's are eigen values

$e^A = \sum_i e^{\lambda_i} |i\rangle\langle i\rangle$ (But how??)

$f(A) = \sum_i f(\lambda_i) |i\rangle\langle i|$

returning.....

$e^{At} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

$\frac{-iH}{\hbar} = A$ $U(t) = e^{At}$

$\frac{d}{dt} e^{At} = \frac{-iH}{\hbar} e^{At}$

$\star$ $\frac{d}{dt} e^{At} = Ae^{At}$

(wasn't it obvious?) NO!

and what's the meaning e to the power some matrix!!

$U(t) = e^{\frac{-iHt}{\hbar}}$

we got the same thing

$U^*(t) = e^{iHt/\hbar}$

$U \cdot U^* = I$

Ex: A hermitian matrix

$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

eigen values

3	-1
↓	↓
$\|0\rangle$	$\|1\rangle$

eigen vectors

$A = 3|0\rangle\langle 0\rangle| + -1|1\rangle\langle 1\rangle|$

$3\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$

$3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ verified ✓✓

Matrix representation of an operator

$A|v\rangle = $ — ?

$|v\rangle = \sum c_i |v_i\rangle$

$A|v_i\rangle = \sum_j A_{ji} |w_j\rangle$

$A|v\rangle = |w\rangle$

$m\times n \quad n\times 1 \qquad m\times 1$

$\langle w_j | A v_i\rangle = A_{ji}$ (directly)

* Any hermitian matrix

$\alpha = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix}$ can be written as sum of (linear comb) Pauli matrices.

Density Matrix ??

Polar Decomposition:

Linear operator $A = UP$

$A^{+} = P^{+}U^{+}$

$A^{+}A = P^{+}P$

$P = \sqrt{A^{+}A}$

ye kaise aa gaya!

U is unique if A is invertible

P is hermitian and positive semi definite

$P^2 = A^{+}A$

Quantum Measurement Theorem:

$\psi(x) \xrightarrow{x=x_0} \delta(x-x_0)$

delta function!

* $H \rightarrow$ Energy

$|\psi\rangle = \alpha|E_1\rangle + \beta|E_2\rangle$ $\quad E_1$ & E_2 are eigen values

↓ after measurement

$|\psi'\rangle = |E_1\rangle$

$A|v\rangle = \lambda|v\rangle \rightarrow \{|v_1\rangle, |v_2\rangle, \ldots |v_n\rangle\}$

n-eigen vectors

n-fold generate.

$A \rightarrow \lambda_i \rightarrow |\lambda_i, j\rangle$

$j \in \{1,2,3, \ldots n\}$

λ_i is n-fold degenerate.

$|1,1\rangle \quad |1,2\rangle \ldots \quad |1,n\rangle$

$|\psi\rangle = \sum_i \lambda_i |v_i\rangle$

$|\psi\rangle = \sum_{i,j} \lambda_i |\lambda_i, j\rangle$

let λ_3: 2 fold degenerate

$\lambda_3|\lambda_3, 1\rangle + \lambda_3|\lambda_3, 2\rangle$

* let a state

$|\psi\rangle = \sum_j c_{1j} \underline{|\lambda_1, j\rangle} + \sum_j c_{2j} \underline{|\lambda_2, j\rangle}$

$P_{\lambda_1} = \sum_j |\lambda_1, j\rangle\langle\lambda_1, j|$

probability of having λ_1 ✗

projector ✓

$P_{\lambda_1}|\psi\rangle = \sum_j (c_{1j})^2 \Rightarrow$ probability

when $|\lambda_1, 1\rangle, |\lambda_2, 2\rangle, \ldots$

ye sab orthonormal kaise?

$$|\psi'\rangle = \sum_j b_{ij} |\lambda_i, j\rangle$$ superimposition

this is meant by collapsing

$$b_{ij} = \frac{c_{ij}}{\sqrt{\sum_j c_{ij}^2}}$$ ⇒ conserving probability

Quantum Measurement Theory

$\{M_m\}$, $m \to$ measurement outcome.

$$p(m) = \langle\psi|M^{+}M|\psi\rangle$$

$$|\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M^{+}|M|\psi\rangle}}$$

(For probability to be 1, we divided by this factor)

$$|\psi\rangle \longrightarrow M_m|\psi\rangle \longrightarrow |\psi'\rangle$$

initial   intermediate   final

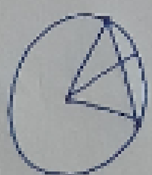
* $\sum_m p(m) = 1$ & $\sum_m M_m^{+}M_m = 1$

M_m represents part of state associated with m.

- Interpretation:

$$|\psi'\rangle = M_m|\psi\rangle$$

$$p(m) = \langle\psi'|\psi'\rangle = \langle\psi|M^{+}M|\psi\rangle$$

Ex:  what is explained here!?

Projective Measurements:

described by set of orthogonal projections $\{P_m\}$

- observable $\hat{A}$ have eigen value m which is n-fold degenerate

then $P_m = \sum^{n} |m_i\rangle\langle m_i|$

let's verify

$$P_{\lambda_1} = \sum_j |\lambda_1, j\rangle\langle\lambda_1, j| = \alpha$$

$$P_{\lambda_1}|\psi\rangle = \sum_{j,k} c_{ij} |\lambda_1, k\rangle\langle\lambda_1, k|\lambda_1, j\rangle$$

$$+ \sum_{j,k} c_{ij} |\lambda_2, k\rangle\langle\lambda_1, k|\lambda_2, j\rangle$$

$$= \sum_j c_{ij} |\lambda_1, j\rangle$$

* $P_m^{+} = P_m = P_m^2$

but $M_m = P_m$ in before formulas.

as seen $\hat{A} = \sum i |i\rangle\langle i|$

$$\hat{A} = \sum m P_m$$

POVM measurements:

we care most about probability

$\{M_m\} \to$ no interest

$M_m^{+}M_m$ or $E_m = M^{+}(M)$

positive operator

$$p(m) = \langle\psi|E_m|\psi\rangle$$

$$p(m) = \langle\psi|M^{+}M|\psi\rangle = \langle\psi'|\psi'\rangle$$

$$\Rightarrow p(m) \geq 0$$

Phase:

global phase and local phase

→ no change in expectation value   change is there

$$e^{i\theta}|\psi\rangle \sim |\psi\rangle$$ global phase

$$\underbrace{e^{i\theta_1}}_{\text{global}}\left(|\psi_1\rangle + \underbrace{e^{i(\theta_2-\theta_1)}}_{\text{local}}|\psi_2\rangle\right)$$

* Superdense Coding: we can send 2 classical bits using 1 qubit.

At a basic level,

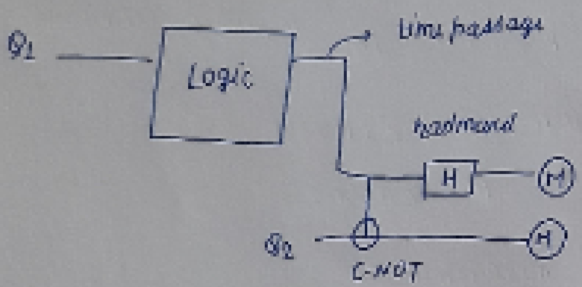
$|\psi\rangle$ — quantum channel $=$ 2 bits

when you measure qubit collapse

Shared entanglement:

$$\downarrow$$

Alice $\downarrow$ Qubit1 $\qquad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad$ Bob $\downarrow$ Qubit2



Logic:

$00 : I \otimes I \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$01 : X \otimes I \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$10 : Z \otimes I$

$11 : ZX \otimes I \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

C-NOT: first bit 0 $\Rightarrow$ nothing
first bit 1 $\Rightarrow$ flip second

$$|00\rangle + |11\rangle \xrightarrow{\text{C-NOT}} |00\rangle + |10\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Example: let Alice had to transfer 10

$$(Z \otimes I)\frac{(|00\rangle + |11\rangle)}{\sqrt{2}} = Z \otimes I |\psi\rangle$$

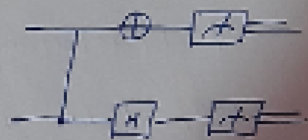
$$= (Z \otimes I)\frac{(|0\rangle \otimes |0\rangle)}{\sqrt{2}} + (Z \otimes I)\frac{(|1\rangle \otimes |1\rangle)}{\sqrt{2}}$$

$$= \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{Z|0\rangle \otimes |0\rangle + Z|1\rangle \otimes |1\rangle}{\sqrt{2}}$$

$\downarrow$ C-NOT $\qquad \left\{ \begin{array}{l} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{array} \right\}$

$$= \frac{|0,0\rangle - |1,0\rangle}{\sqrt{2}}$$

Bob's end



hence Bob can distinguish!

Quantum teleportation:
allows sending / teleporting from one location to other!

* doing $A \otimes I$ allows to act upon
$\langle 1 \otimes \langle 0 | \qquad |1\rangle \otimes |0\rangle$

then $A|1\rangle \otimes |0\rangle$

gate A would act on $|1\rangle$.

But how exactly Bob distinguish??