UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2020/2021

DEPARTMENT OF MATHEMATICS

MATH 123: VECTORS AND GEOMETRY (3 credits)

INSTRUCTION:

ANSWER ALL QUESTIONS IN SECTION A AND TWO QUESTIONS FROM SECTION B

TIME ALLOWED:

TWO HOURS (2 hours)

NOTE: BOLD TYPEFACE LETTERS DENOTE VECTORS.

Section A (Multiple Choice Questions): [100 Marks]

Answer all questions in this section in the answer booklet provided. Each question is followed by four options lettered (a), (b), (c), (d). Write the letter of the correct answer only for each question in the answer booklet. You are however allowed to do your rough work in the answer booklet. Page(s) in the answer booklet used for the rough work must be clearly marked as such.

1. Given that the points $A(-4, -6, 1)$, $B(-1, -3, 2)$, $C(5, 3, 4)$ are collinear, find the ratio in which B divides AC.

   (a) $2:1$

   (b) $1:2$

   (c) $1:1$

   (d) $3:2$

2. Three vertices of a parallelogram $ABCD$ are $A(3, -4, 7)$, $B(5, 3, -2)$, $C(1, 2, -3)$. Find the coordinates of the fourth vertex.

   (a) $(-1, -5, 6)$

   (b) $(2, 7, -9)$

   (c) $(-4, -1, -1)$

   (d) $(9, 4, -1)$

3. ABC is a triangle with vertices $A(3, -4, 6)$, $B(3, 4, 2)$, $C(9, -6, 4)$. What are the coordinates of the point of concurrency of the medians of the triangle?

   (a) $(6, -1, 3)$

   (b) $(3, 0, 4)$

   (c) $(5, -2, 4)$

   (d) $(2, -4, 6)$

4. $ABCD$ is a parallelogram and $(3, -4, 6)$ are the coordinates of the vertex A. If $\mathbf{AC} = -4\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$ and $\mathbf{BD} = -4\mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$, what is the position vector of the point D?

   (a) $-\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

   (b) $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

   (c) $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

   (d) $-\mathbf{i} + 2\mathbf{j}$

5. D, E, F are the midpoints of the sides BC, CA, AB respectively of a triangle ABC. Which of the following is true?

   (a) $\mathbf{AD} + \mathbf{BE} = \mathbf{FC}$

   (b) $\mathbf{AD} + \mathbf{BE} = \mathbf{CF}$

   (c) $\mathbf{DA} + \mathbf{BE} = \mathbf{FC}$

   (d) $\mathbf{AD} + \mathbf{EB} = \mathbf{FC}$

6. The vector $(a - 4)\mathbf{i} - b\mathbf{j}$ is parallel to $\mathbf{j}$ and the vector $(a - 8)\mathbf{i} + b\mathbf{j}$ is parallel to the vector $-\mathbf{i} - \mathbf{j}$. Find the values of a and b.

   (a) $a = 4$; $b = 0$

   (b) $a = -4$; $b = 4$

   (c) $a = 4$; $b = -4$

   (d) $a = 0$; $b = 4$

7. In triangle ABC, P divides BC in the ratio $-2 : 1$, Q divides CA in the ratio $3 : 4$ and R divides AB in the ratio $2 : 3$. In what ratio does P divide RQ?

   (a) $7 : 5$

   (b) $5 : 7$

   (c) $-5 : 7$

   (d) $-7 : 5$

8. If the foot of the perpendicular from the point $C(6, 1)$ to the line $\mathbf{r} \cdot (-\mathbf{i} + 3\mathbf{j}) + 1 = 0$ is $D(5, 4)$, what is the mirror image of C in the line?

   (a) $(4, 7)$

   (b) $(7, 4)$

   (c) $(7, -2)$

   (d) $(4, -7)$

9. The parametric equations of the line $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j}) = 5$ are

   (a) $x = 3 + \lambda$ and $y = 2 - \lambda$

   (b) $x = 3 + \lambda$ and $y = 2 + \lambda$

   (c) $x = 3 - \lambda$ and $y = 2 - \lambda$

   (d) $x = \lambda - 3$ and $y = 2 - \lambda$

10. Which of the following equations does not represent the line $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j}) + 5 = 0$?

   (a) $\mathbf{r} = 5\mathbf{i} + 5\mathbf{j} + \lambda (3\mathbf{i} + 2\mathbf{j})$

   (b) $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \lambda (3\mathbf{i} + 2\mathbf{j})$

   (c) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda (3\mathbf{i} + 2\mathbf{j})$

   (d) $\mathbf{r} = -4\mathbf{i} - \mathbf{j} + \lambda (3\mathbf{i} + 2\mathbf{j})$

11. What is the equation of the line that passes through the point with position vector $-\mathbf{i} + 2\mathbf{j}$ and is perpendicular to the line $\mathbf{r} \cdot (3\mathbf{i} + 4\mathbf{j}) + 7 = 0$?

   (a) $\mathbf{r} \cdot (3\mathbf{i} + 4\mathbf{j}) - 5 = 0$

   (b) $\mathbf{r} \cdot (4\mathbf{i} - 3\mathbf{j}) + 10 = 0$

   (c) $\mathbf{r} \cdot (4\mathbf{i} + 3\mathbf{j}) - 2 = 0$

   (d) $\mathbf{r} \cdot (4\mathbf{i} - 3\mathbf{j}) - 10 = 0$

12. What are the coordinates of the point of intersection of the lines $\mathbf{r} \cdot (-\mathbf{i} + 2\mathbf{j}) - 6 = 0$ and $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j}) + 2 = 0$?

   (a) $(-2, 2)$

   (b) $\left(-\frac{7}{2}, \frac{5}{4}\right)$

   (c) $\left(\frac{2}{3}, \frac{10}{3}\right)$

   (d) $(2, 2)$

13. What is the distance between the parallel lines $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j}) + 7 = 0$ and $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j}) - 11 = 0$?

   (a) $\sqrt{2}$

   (b) $7\sqrt{2}$

   (c) $9\sqrt{2}$

   (d) $18\sqrt{2}$

14. In a triangle ABC, D divides BC in the ratio $1 : 3$, E divides AC in the ratio $1 : 3$. What is the value of the vector $\mathbf{ED}$?

   (a) $\frac{1}{4}\mathbf{AB}$

   (b) $\frac{1}{2}\mathbf{AB}$

   (c) $\frac{3}{4}\mathbf{AB}$

   (d) $\frac{1}{3}\mathbf{AB}$

15. What is the perpendicular distance of the point $(-5, 1, 2)$ from the line

$$\frac{x - 6}{-12} = \frac{y + 6}{9} \text{ and } z - 2 = 0?$$

   (a) $\sqrt{2}$

   (b) 1

   (c) 2

   (d) $\frac{3}{2}$

16. Find the mirror image of the point $(1, 6, 3)$ in the line $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

   (a) $(1, 3, 5)$

   (b) $(0, 1, 7)$

   (c) $(1, 0, -7)$

   (d) $(1, 0, 7)$

17. If there exist scalars α, μ, γ, λ and δ, not all zero, such that

$$\alpha\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{c} + \lambda\mathbf{d} + \delta\mathbf{e} = 0 \text{ and } \alpha + \mu + \gamma + \lambda + \delta = 0$$

then the points whose position vectors are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ lie

   (a) in a plane

   (b) on a line

   (c) in space

   (d) on a line in a plane

18. Let $\mathbf{r} = -6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$ be the vector equations of the lines L_1 and L_2 respectively in space. Which of the following best describes L_1 and L_2?

   (a) L_1 and L_2 intersect

   (b) L_1 and L_2 do not intersect

   (c) L_1 and L_2 are skew

   (d) L_1 and L_2 are coplanar

19. What are the Cartesian equations of the line $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{k})$?

   (a) $x + 2 = z - 2$, $y = 4$

   (b) $x - 2 = z + 2$, $y = 4$

   (c) $y - 2 = z + 2$, $x = 4$

   (d) $x - 2 = z - 2$, $y + 4 = 0$

20. The points $A(-2, -2, 1)$, $B(-1, -1, 3)$, $C(0, -2, 3)$ are the vertices of a triangle. Which of the following best describes this triangle?

   (a) scalene

   (b) right-angle isosceles

   (c) right-angle scalene

   (d) equilateral

Section B (Essay Questions): **[100 Marks]**

Answer any two questions out of the three questions in this section in your answer booklet.

1. (a) In a trapezium $ABCD$, $\mathbf{DC} = 3\mathbf{AB}$. If P is the midpoint of AD and Q is the midpoint of BC, prove that $\mathbf{PQ} = 2\mathbf{AB}$. **[12 marks]**

   (b) The position vectors of the points A, B, C and D are $\mathbf{a}$, $\mathbf{b}$, $3\mathbf{a} - \mathbf{b}$, $\mathbf{a} + \mathbf{b}$ respectively. Show that $\mathbf{CD}$ is parallel to $\mathbf{AB}$ and find the ratio $|\mathbf{AB}| : |\mathbf{CD}|$. **[10 marks]**

   (c) In a parallelogram $ABCD$, P and Q are the midpoints of the sides AB and BC respectively. Show that AC and DP meet in a common point of trisection and similarly, AC and DQ also meet in a common point of trisection. **[28 marks]**

2. (a) Given that the lines $\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 5\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 7\mathbf{j} - \mathbf{k})$ are skew, find a direction vector for the common perpendicular. Hence, determine the shortest distance between the two lines. **[28 marks]**

   (b) Show that the line through the points $A(3, 2, 1)$ and $B(6, 5, -1)$ intersects the line through the points $C(4, 5, 5)$ and $D(4, 2, -2)$. Find the position vector of their point of intersection.

   [22 marks]

3. (a) The position vectors of the points A and B are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j}$ respectively. Write down a vector equation of the line AB and find the value of α if the point C with position vector $\mathbf{c} = 7\mathbf{i} + \alpha\mathbf{j}$ lies on AB. **[10 marks]**

   (b) Find the reflection of the point $(2, 1)$ in the line L defined by the equation $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j}) + 2 = 0$. Hence find the equation of the reflection of the line $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) - 1 = 0$ in the line L. Determine the cosine of the acute angle between L and $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) - 1 = 0$. Leave your answer in surd form. **[40 marks]**