UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2021/2022

DEPARTMENT OF MATHEMATICS

MATH 123: VECTORS AND GEOMETRY (3 credits)

INSTRUCTION:

ANSWER ALL QUESTIONS

TIME ALLOWED:

TWO HOURS (2 hours)

Multiple-Choice: [30 Marks]

- Answer all questions (1 - 15) on page 1 of the answer booklet provided.
- For questions followed by four options lettered (a), (b), (c), (d), write **ONLY** the letter of the correct answer in the answer booklet.
- Each multiple-choice question is worth **2 marks.**
- You are allowed to use **pages 3 and above** in the answer booklet for your rough work .
- **Notation**: Vector $\boldsymbol{a}$, denoted in boldface, is equivalent to the notations $\underline{a}$ or $\overrightarrow{a}$.

1. Consider the triangle ABC. The points $D(\boldsymbol{d})$ and $E(\boldsymbol{e})$ are midpoints of the sides AC and AB respectively. The point $F(\boldsymbol{f})$ divides the line segment CB externally in the ratio $3:1$. Given that the position vectors of the points A, B and C are $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ respectively, which of the following is true?

   (a) $\boldsymbol{e} = \frac{2\boldsymbol{a}+\boldsymbol{b}}{2}$

   (b) $\boldsymbol{d} = \frac{\boldsymbol{a}-\boldsymbol{c}}{2}$

   (c) $\boldsymbol{b} = \frac{2\boldsymbol{f}+\boldsymbol{c}}{3}$

   (d) $\boldsymbol{f} = \frac{3\boldsymbol{b}+\boldsymbol{c}}{2}$

2. Compute the scalar product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that $|\boldsymbol{a}| = 2, |\boldsymbol{b}| = 5$ and the angle between them is $\frac{\pi}{5}$.

   (a) 5

   (b) 3.333

   (c) 8.660

   (d) $\frac{10\pi}{3}$

3. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two orthornomal vectors. Which of the following equations is true?

   (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

   (b) $\boldsymbol{a} \cdot \boldsymbol{b} = 1$

   (c) $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$

   (d) $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$

4. Find the magnitude of the centroid of the vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{k}$ and $\boldsymbol{c} = 3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$.

   (a) 2

   (b) $\sqrt{6}$

   (c) $2\sqrt{3}$

   (d) 4

5. Given that

$$\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} \quad \text{and} \quad \boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}.$$

   Find the coordinates of the point Q if $\overrightarrow{PQ} = \boldsymbol{a} + 3\boldsymbol{b}$ and the coordinates of the point P are $(1, 2, 3)$.

   (a) (8,10,1)

   (b) (6,8, 1)

   (c) (6,10,2)

   (d) (8, 8, 2)

6. Given that the points $A(-4, -6, 1), B(-1, -3, 2)$ and $C(5, 3, 4)$ are collinear, find the ratio in which C divides $\overrightarrow{AB}$ externally.

   (a) 2 : 1

(b) $1:2$

(c) $1:1$

(d) $3:2$

7. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be vectors and λ a scalar. Which of the following equations is not true?

   (a) $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$

   (b) $\lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (\lambda \boldsymbol{b})$

   (c) $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

   (d) $\boldsymbol{a} + (\boldsymbol{b} \cdot \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

Consider the line L that passes through the point with position vector $2\boldsymbol{i}+3\boldsymbol{j}$ and is parallel to the vector $5\boldsymbol{i}-7\boldsymbol{j}$. Use this information to answer questions 8 and 9.

8. Find a vector equation of the line L.

   (a) $\boldsymbol{r} = (2-5\lambda)\boldsymbol{i} + (-3+7\lambda)\boldsymbol{j}$

   (b) $\boldsymbol{r} = (2+5\lambda)\boldsymbol{i} + (3+7\lambda)\boldsymbol{j}$

   (c) $\boldsymbol{r} = (2+5\lambda)\boldsymbol{i} + (3-7\lambda)\boldsymbol{j}$

   (d) $\boldsymbol{r} = (2-5\lambda)\boldsymbol{i} + (3-7\lambda)\boldsymbol{j}$

9. Compute the unique value of λ for which the point M with position vector $17\boldsymbol{i}-18\boldsymbol{j}$ lies on the line L.

   (a) $\lambda = 3$

   (b) $\lambda = 4$

   (c) $\lambda = -3$

   (d) $\lambda = 5$

10. If $A(3,-5)$ and $B(-7,1)$ are any two points on the $x-y$ plane, then the equation of the line in the plane that passes through these two points is

    (a) $\boldsymbol{r} = (3-5\lambda)\boldsymbol{i} - (5-3\lambda)\boldsymbol{j}$

    (b) $\boldsymbol{r} = (3-5\lambda)\boldsymbol{i} - (5+3\lambda)\boldsymbol{j}$

    (c) $\boldsymbol{r} = (3+5\lambda)\boldsymbol{i} - (5-3\lambda)\boldsymbol{j}$

    (d) $\boldsymbol{r} = (3+5\lambda)\boldsymbol{i} - (5+3\lambda)\boldsymbol{j}$

11. Find the vector equation of the line with gradient $-\frac{1}{3}$ that passes through the point $A(2,-3)$.

    (a) $\boldsymbol{r} = (2-3\lambda)\boldsymbol{i} - (3-\lambda)\boldsymbol{j}$

    (b) $\boldsymbol{r} = (2-3\lambda)\boldsymbol{i} + (3-\lambda)\boldsymbol{j}$

    (c) $\boldsymbol{r} = (2-3\lambda)\boldsymbol{i} - (3+\lambda)\boldsymbol{j}$

    (d) $\boldsymbol{r} = (2+3\lambda)\boldsymbol{i} - (3-\lambda)\boldsymbol{j}$

12. Find the acute angle θ between the straight lines defined by the equations

$$\begin{aligned} \boldsymbol{r} &= (1+3t)\boldsymbol{i} + 2(1+2t)\boldsymbol{j}, \\ \boldsymbol{r} &= (1+2s)\boldsymbol{i} + (1-s)\boldsymbol{j}. \end{aligned}$$

    (a) $\theta = \cos^{-1}\left(\frac{2\sqrt{5}}{25}\right)$

(b) $\theta = \cos^{-1}\left(-\frac{2\sqrt{5}}{5}\right)$

(c) $\theta = \cos^{-1}\left(\frac{2\sqrt{5}}{5}\right)$

(d) $\theta = \sin^{-1}\left(\frac{2\sqrt{5}}{25}\right)$

13. Find, in normal form, the equation of the line that passes through the point $A(-8, 3)$ and is perpendicular to the vector $\boldsymbol{n} = 6\boldsymbol{i} - 5\boldsymbol{j}$

(a) $\boldsymbol{r} \cdot (6\boldsymbol{i} - 5\boldsymbol{j}) + 63 = 0$

(b) $\boldsymbol{r} \cdot (6\boldsymbol{i} - \boldsymbol{j}) + 63 = 0$

(c) $\boldsymbol{r} \cdot (6\boldsymbol{i} + 5\boldsymbol{j}) + 63 = 0$

(d) $\boldsymbol{r} \cdot (6\boldsymbol{i} - 5\boldsymbol{j}) - 63 = 0$

14. If the lines $\boldsymbol{r} \cdot (\boldsymbol{i} - 3\boldsymbol{j}) = -3$, $\boldsymbol{r} \cdot (\boldsymbol{i} + 5\boldsymbol{j}) = 7$ and $\boldsymbol{r} \cdot (2\boldsymbol{i} - 2\boldsymbol{j}) = \lambda$ are concurrent, find the value of λ.

(a) $\lambda = -\frac{1}{2}$

(b) $\lambda = -1$

(c) $\lambda = -2$

(d) $\lambda = 2$

15. Find the equation of the line which has gradient $-\frac{2}{3}$ and passes through the point of intersection of the lines

$$\begin{aligned} \boldsymbol{r} \cdot (\boldsymbol{i} - 4\boldsymbol{j}) - 7 &= 0, \\ \boldsymbol{r} \cdot (2\boldsymbol{i} + \boldsymbol{j}) - 1 &= 0. \end{aligned}$$

(a) $\boldsymbol{r} \cdot (-8\boldsymbol{i} + 27\boldsymbol{j}) + 17 = 0$

(b) $\boldsymbol{r} \cdot (-18\boldsymbol{i} + 2\boldsymbol{j}) + 17 = 0$

(c) $\boldsymbol{r} \cdot (-18\boldsymbol{i} + 27\boldsymbol{j}) + 17 = 0$

(d) $\boldsymbol{r} \cdot (-18\boldsymbol{i} + 27\boldsymbol{j}) - 17 = 0$

Short-Answer Questions: [20 Marks]

- Answer all questions (16 - 20) on **page 2** of the answer booklet provided.
- For each short-answer question, write **ONLY** the correct answer in the answer booklet.
- Each short-answer question is worth **4 marks.**
- You are allowed to use **pages 3 and above** in the answer booklet for your rough work .

16. Let $\boldsymbol{a} = \boldsymbol{i} + m\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{k}$ be two vectors. Find the value of m if $\boldsymbol{a} \times \boldsymbol{b} = -2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$.

17. Given the two vector combinations

$$\boldsymbol{a} + 4\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k} \quad \text{and} \quad \boldsymbol{a} - 2\boldsymbol{b} = \boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k}.$$

Find the magnitude of the vector $\boldsymbol{b}$.

18. Simplify the expression $(3\boldsymbol{a}+4\boldsymbol{b})\cdot(3\boldsymbol{a}-2\boldsymbol{b})$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors, parallel to each other and moving in the same direction.

19. The position vectors of the points A and B relative to O are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. P is the midpoint of OA and Q lies on OB with $OQ = \frac{1}{3}OB$. If QP meets BA at X, find the position vector of X. Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$

20. Find the equations of the diagonals of the parallelogram whose sides have the equations;

$L1: \ 3x + y = 1, \quad L2: \ 3y - 5x = 3, \quad L3: \ 3x + y = 15$ and $L4: \ 3y - 5x = -11.$