UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2020/2021

DEPARTMENT OF MATHEMATICS

MATH 121: ALGEBRA AND TRIGONOMETRY (3 credits)

INSTRUCTION:

ANSWER ANY 3 OUT OF THE FOLLOWING 5 QUESTIONS

TIME ALLOWED:

TWO HOURS (2 hours)

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1. (a) The statement p is given by

$$p: \text{The number of elements in } \{0\} \text{ is } 0.$$

What is the truth value of p? Justify your answer. **[10 marks]**

(b) List all the elements in the following sets:

i. $M \times N$, where $M = \{1, 2\}$ and $N = \{a, b\}$ **[8 marks]**

ii. $T = \{x \in P \mid x \text{ is divisible by 3 or 5}\}$ where P is the set given by $P = \{x \in \mathbb{Z} \mid x \text{ is a perfect square less than or equal to 49}\}$. **[8 marks]**

(c) Find $I \cap J$, given that I and J are the intervals given by $I = (0, 1]$ and $J = (1, 3]$. **[9 marks]**

(d) Let A and B be any two sets and $\emptyset$ the empty set. Show that

$$\text{if } A \times B = \emptyset \text{ then } A = \emptyset \text{ or } B = \emptyset.$$

[*Hint* : Consider a proof by contradiction.] **[15 marks]**

2. (a) Let n be an odd integer. Show that $n^2 + n$ is an even integer. **[20 marks]**

(b) By using the principle of mathematical induction, prove that for any integer $n \geq 1$,

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$$

[30 marks]

3. (a) (a) State the truth value of the following statement and write down its negation.

$$\text{For all } x \in \mathbb{Q}, \quad x^2 - 4 \neq 0.$$

[8 marks]

(b) Write down the converse and contrapositive of the following statement.

"A day that is sunny is a good day for walking on the beach"

[10 marks]

(c) By the use of truth table, show that the statement

$$[(p \vee q) \wedge (r \vee \sim q)] \Rightarrow (p \vee r)$$

is a tautology. **[16 marks]**

(d) Determine whether or not the following statements are logically equivalent.

$$\sim [(p \vee q) \wedge r] \quad \text{and} \quad [(\sim q \vee \sim r) \wedge (\sim q \vee \sim r)]$$

[16 marks]

4. (a) The function h is defined by

$$h(x) = \frac{x}{x^2 - 1}, \; x > 0.$$

i. State the domain of h. **[2 marks]**

ii. Find the range of h. **[8 marks]**

iii. Show that h is injective. **[10 marks]**

(b) Let

$$f(x) = \frac{1}{2}x + \pi \quad \text{and} \quad g(x) = \alpha x + \beta$$

where $\alpha \neq 1$.
If $(f \circ g)(x) = (g \circ f)(x)$, find the constant k such that

$$\beta = k(1 - \alpha).$$

If α and β also satisfy

$$\beta = \alpha^2 + 2\pi,$$

find the values of α and β. **[15 marks]**

(c) Find the values of x satisfying

$$\sqrt{3x + 4} - 3 = \sqrt{x - 3}.$$

[15 marks]

5. (a) Let $a, b, c \in \mathbb{Z}$. Show that if a divides b and b divides c, then a divides c. **[20 marks]**

(b) Provide a counterexample to disprove the following statement:

For all $a, b, c \in \mathbb{Z}$, if a divides $b + c$ then a divides b or a divides c.

[15 marks]

(c) Let U, A and B be given by

$$U = \{0, 1, 2, 3, 4, a, b, c, d\}, \quad A = \{1, a, 2, b\} \text{ and } B = \{2, b, 3, c\}$$

respectively, where U is the universal set. Find the following:

i. The difference of A and B **[5 marks]**

ii. The symmetric difference of A and B **[5 marks]**

iii. The complement of A. **[5 marks]**