The Canonical Approach to Quantum Gravity: General Ideas and Geometrodynamics*

Domenico Giulini[1] and Claus Kiefer[2]

[1] Physikalisches Institut, Universität Freiburg
Hermann-Herder-Straße 3, D-79104 Freiburg, Germany
[2] Institut für Theoretische Physik, Universität zu Köln
Zülpicher Straße 77, D-50937 Köln, Germany

Summary. We give an introduction to the canonical formalism of Einstein's theory of general relativity. This then serves as the starting point for one approach to quantum gravity called quantum geometrodynamics. The main features and applications of this approach are briefly summarized.

1 Introduction

The really novel feature of General Relativity (henceforth abbreviated GR), as compared to other field theories in physics, is that spacetime is not a fixed background arena that merely stages physical processes. Rather, spacetime is itself a dynamical entity, meaning that its properties depend in parts on its specific matter content. Hence, contrary to the Newtonian picture, in which spacetime acts (via its inertial structure) but is not acted upon by matter, the interaction between matter and spacetime now goes both ways.

Saying that the spacetime is 'dynamic' does not mean that it 'changes' with respect to any given external time. Time is clearly within, not external to spacetime. Accordingly, solutions to Einstein's equations, which are whole spacetimes, do not as such describe anything evolving. In order to take such an evolutionary form, which is, for example, necessary to formulate an initial value problem, we have to re-introduce a notion of 'time' with reference to which we may speak of 'evolution'. This is done by introducing a structure that somehow allows to split spacetime into space and time.

Let us explain this in more detail: suppose we are give a spacetime, that is, a four dimensional differentiable manifold M with Lorentzian metric g. We assume that M can be foliated by a family $\{\Sigma_t \mid t \in \mathbb{R}\}$ of spacelike leaves. That is, for each number t there is an embedding of a fixed 3-dimensional manifold Σ into M,

$$\mathcal{E}_t : \Sigma \to M \,, \tag{1}$$

whose image $\mathcal{E}_t(\Sigma) \subset M$ is just Σ_t, which is a spacelike submanifold of M; see Fig. 1. It receives a Riemannian metric by restricting the Lorentzian metric

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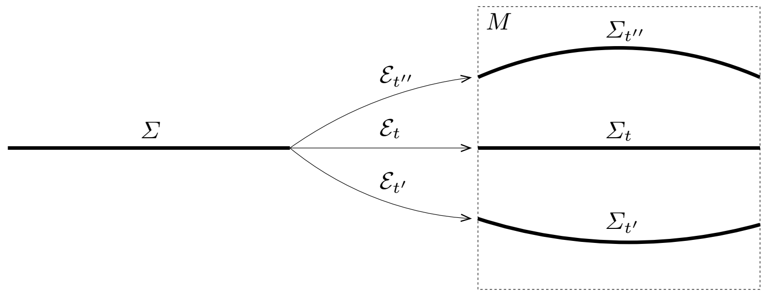

Fig. 1. Foliation of spacetime M by a one-parameter family of embeddings $\mathcal{E}_t$ of the 3-manifold Σ into M. Σ_t is the image in M of Σ under $\mathcal{E}_t$. Here the leaf $\Sigma_{t'}$ is drawn to lie to the past and $\Sigma_{t''}$ to the future of Σ_t.

g of M to the tangent vectors of Σ_t. This can be expressed in terms of the 3-manifold Σ. If we endow Σ with the Riemannian metric

$$h_t := \mathcal{E}_t^* g\,, \tag{2}$$

then (Σ, h_t) is isometric to the submanifold Σ_t with the induced metric.

Each three dimensional leaf Σ_t now corresponds to an instant of time t, where t is so far only a topological time: it faithfully labels instants in a continuous fashion, but no implication is made as to its relation to actual clock readings. The statement of such relations can eventually only be made on the basis of dynamical models for clocks coupled to the gravitational field.

By means of the foliation we now recover a notion of time: we view spacetime, (M, g), as the one-parameter family of spaces, $t \mapsto (\Sigma, h_t)$. Spacetime then becomes nothing but a 'trajectory of spaces'. In this way we obtain a dynamical system whose configuration variable is the Riemannian metric on a 3-manifold Σ. It is to make this point precise that we carefully distinguish between the manifold Σ and its images Σ_t in M. In the dynamical formulation given now, there simply is no spacetime to start with and hence no possibility to embed Σ into something. Only *after* solving the dynamical equations can we construct spacetime and interpret the time dependence of the metric of Σ as being brought about by 'wafting' Σ through M via a one-parameter family of embeddings $\mathcal{E}_t$. But initially there is only a 3-manifold Σ of some topological type[3] and the equations of motion together with some suitable

[3] It can be shown that the Einstein equations do not pose any obstruction to the topology of Σ, that is, solutions exist for *any* topology. However, one often imposes additional requirements on the solution. For example, one may require that there exists a moment of time symmetry, which will make the corresponding instant Σ_t a totally geodesic submanifold of M, like e.g. in recollapsing cosmological models at the moment of maximal expansion. In this case the topology of Σ will be severely restricted. In fact, most topologies Σ will only support geometries that always expand or contract somewhere.

initial data. For a fuller discussion we refer to the comprehensive work by Isham and Kuchař [13, 14].

2 The Initial-Value Formulation of GR

Whereas a specified motion of Σ through M, characterized by the family of embeddings (1), gives rise to a one-parameter family of metrics h_t, the converse is not true. That is to say, it is not true that *any* one-parameter family of metrics h_t of Σ can be obtained by finding a spacetime (M, g) and a one parameter family of embeddings $\mathcal{E}_t$, such that (2) holds.

Moreover, there is clearly a huge redundancy in creating (M, g) from the family $\{(\Sigma, h_t) \mid t \in \mathbb{R}\}$, since there are obviously many different motions of Σ through the same M, which give rise to apparently different solution curves h_t. This redundancy can be locally parameterized by four functions, on Σ: a scalar field α and a vector field β. In the embedding picture, they describe the components of the velocity vector field

$$\frac{\partial}{\partial t} := \frac{d}{dt}\mathcal{E}_t \tag{3}$$

normal and tangential to the leaves Σ_t respectively. We write

$$\frac{\partial}{\partial t} = \alpha n + \beta, \tag{4}$$

where n is the normal to Σ_t. The tangential component, β, just generates intrinsic diffeomorphisms on each Σ_t, whereas the normal component, α, really advances one leaf Σ_t to the next one; see Fig. 2.

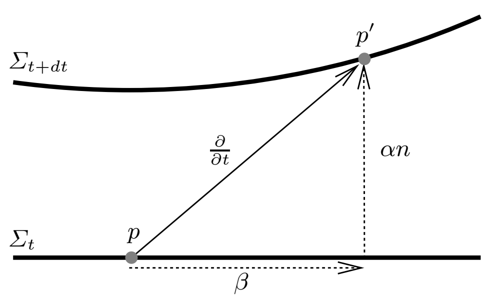

Fig. 2. Infinitesimally nearby leaves Σ_t and Σ_{t+dt}. For some point $q \in \Sigma$, the image points $p = \mathcal{E}_t(q)$ and $p' = \mathcal{E}_{t+dt}(q)$ are connected by the vector $\partial/\partial t|_p$, whose components tangential and normal to Σ_t are β and αn, respectively. n is the normal to Σ_t in M, β is called the 'shift vector-field' and α the 'lapse function' on Σ_t.

For the initial-value problem it is the derivative along the normal n of the 3-metric h, denoted by K, that gives the essential information. Hence we write

$$\frac{\partial h_t}{\partial t} = \alpha K_t + L_\beta h_t \,. \tag{5}$$

In the embedding picture K_t is the extrinsic curvature of Σ_t in M.

The first order evolution equations that result from Einstein's field equations are then of the general form

$$\begin{aligned} \frac{\partial h_t}{\partial t} &= F_1(h_t, K_t; \alpha, \beta) \;, && (6)\\ \frac{\partial K_t}{\partial t} &= F_2(h_t, K_t; \alpha, \beta; \text{matter}) \;, && (7) \end{aligned}$$

where F_1 in (6) is given by the right-hand side of (5). F_2 is a more complicated function whose precise structure need not interest us now and which also depends on matter variables; see e.g. [8].

3 Why constraints

As we have seen, the initial data for the gravitational variables consist of a differentiable 3-manifold Σ, a Riemannian metric h – the configuration variable, and another symmetric second rank tensor field K on Σ – the velocity variable. However, the pair (h, K) cannot be chosen arbitrarily. This is because there is a large redundancy in describing a fixed spacetime M by a foliation (1). On the infinitesimal level this gauge freedom is just the freedom of choosing α and β. The gauge transformations generated by β are just the spatial diffeomorphisms of Σ. β may be an arbitrary function of t, which corresponds to the fact that we may arbitrarily permute the points in each leaf Σ_t separately (only restricted by some differentiability conditions). The gauge transformations generated by α correspond to pointwise changes in the velocities with which the leaves Σ_t push through M. These too may vary arbitrarily within the leaves as well as with coordinate time t.

Whenever there is gauge freedom in a dynamical theory, there are so-called *constraints*, that is, conditions which restrict the initial data; see e.g. [10]. For each gauge freedom parameterized by an arbitrary function, there is one functional combination of the initial data which has to vanish. In our case there are four gauge functions, α, and the three components of β. Accordingly there are four constraints, which group into one *scalar or Hamiltonian constraint*, $H[h, K] = 0$, and three combined in the *vector or diffeomorphism constraint*, $D[h, K] = 0$. Their explicit expressions are:[4]

$$\begin{aligned} H[h, K] &= (2\kappa)^{-1}\, G^{ab\,cd} K_{ab} K_{cd} - (2\kappa)^{-1} \sqrt{h}\big({}^{(3)}R - 2\Lambda\big) + \sqrt{h}\rho \,, && (8)\\ D^a[h, K] &= -\,\kappa^{-1}\, G^{ab\,cd} \nabla_b K_{cd} + \sqrt{h} j^a \,. && (9) \end{aligned}$$

[4] Here and below we shall write $\sqrt{h} := \sqrt{\det\{h_{ab}\}}$ and use the abbreviation $\kappa = 8\pi G/c^4$. Hence κ has the physical dimension of $s^2 \cdot m^{-1} \cdot Kg^{-1}$. We shall set $c = 1$ throughout.

Here ρ and j^a are the energy- and momentum densities of the matter, ∇ and ${}^{(3)}R$ are the Levi-Civita connection and its associated scalar curvature of (Σ, h). Finally $G^{ab\,cd}$ is the so called DeWitt metric, which at each point of Σ defines an h-dependent Lorentzian metric on the $1+5$ – dimensional space of symmetric second-rank tensors at that point.[5] Its explicit form is given by

$$G^{ab\,cd} = \tfrac{\sqrt{h}}{2}\big(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}\big) \tag{10}$$

Note that the linear space of symmetric second-rank tensors is viewed here as the tangent space ('velocity space') of the space $\mathrm{Riem}(\Sigma)$ of Riemannian metrics on Σ. From (10) one sees that it is the trace part of the 'velocities', corresponding to changes of the scale (conformal part) of the Riemannian metric, that span the negative-norm velocity directions.

4 Comparison with conventional form of Einstein's equations

The presence of constraints and their relation to the evolution equations is the key structure in canonical GR. It is therefore instructive to point out how this structure arises from the conventional, four dimensional form of Einstein's equations. Before doing this, it is useful to first remind ourselves on the analogous situation in electrodynamics.

So let us first consider electrodynamics in Minkowski space. As usual, we write the field tensor F as exterior differential of a vector potential A, that is $F = dA$. In components this reads $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here $E_i = F_{0i}$ are the components of the electric, $B_i = -F_{jk}$ of the magnetic field, where ijk is a cyclic permutation of 123. The homogeneous Maxwell equations now simply read $dF = 0$, whereas the inhomogeneous Maxwell equations are given by (in components):

$$M^\mu := \partial_\nu F^{\mu\nu} + \tfrac{4\pi}{c} j^\mu = 0\,, \tag{11}$$

where here j^μ is the electric four-current. Due to its antisymmetry, the field tensor obeys the identity

$$\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0\,. \tag{12}$$

Taking the divergence of (11) and using (12) leads to

$$\partial_\mu M^\mu \equiv \tfrac{4\pi}{c}\partial_\mu j^\mu = 0\,, \tag{13}$$

showing the well known fact that Maxwell's equations imply charge conservation as integrability condition.

Let us now interpret the role of charge conservation in the initial-value problem. Decomposing (12) into space and time derivatives yields

[5] The Lorentzian signature of the DeWitt Metric has nothing to do with the Lorentzian signature of the space-time metric: it persists in Euclidean gravity .

$$\partial_0 \partial_\nu F^{0\nu} \equiv -\partial_a \partial_\nu F^{a\nu} \,. \tag{14}$$

Even though the right-hand side contains third derivatives in the field A_μ, time derivatives appear at most in second order (since ∂_a is spatial). Hence, since it is an identity, $\partial_\nu F^{0\nu}$ contains time derivatives only up to first order. But the initial data for the second order equation (11) consist of the field A_μ and its first time derivative. Hence the time component M^0 of Maxwell's equations gives a relation amongst initial data, in other words, it is a *constraint*. Clearly this is just the Gauß constraint $\boldsymbol{\nabla} \cdot \boldsymbol{E} - 4\pi\rho = 0$ (here ρ is the electric charge density). Only the three spatial components of (11) contain second time derivatives and hence propagate the fields. They provide the evolutionary part of Maxwell's equations.

Now, assume we are given initial data satisfying the constraint $M^0 = 0$, which we evolve according to $M^a = 0$. How can we be sure that the evolved data again satisfy the constraint? To see when this is the case, we use the identity (13) and solve it for the time derivative of M^0:

$$\partial_0 M^0 \equiv -\partial_a M^a + \tfrac{4\pi}{c} \partial_\mu j^\mu \,. \tag{15}$$

This shows: if initially $M^a = 0$ (and hence $\partial_a M^a = 0$), then the constraint $M^0 = 0$ is preserved in time if and only if $\partial_\mu j^\mu = 0$. Charge conservation is thus recognized as the necessary and sufficient condition for the compatibility between the constraint part and the evolutionary part of Maxwell's equations.

Finally we wish to make another remark concerning the interplay between constraints and evolution equations. It is clear that a solution $F^{\mu\nu}$ to (11) satisfies the constraint on *any* simultaneity hypersurface of an inertial observer (i.e. spacelike plane). If the normal to the hypersurface is n_μ, this just states that $M^\mu = 0$ implies $M^\mu n_\mu = 0$. But the converse is obviously also true: if $M^\mu n_\mu = 0$ for all timelike n_μ, then $M^\mu = 0$. In words: given an electromagnetic field that satisfies the constraint (for given external current j^μ) on *any* spacelike plane in Minkowski space, then this field must necessarily satisfy Maxwell's equations. In this sense, Maxwell's equations are the *unique* propagation law that is compatible with Gauß' constraint.

After this digression we return to GR, where we can perform an entirely analogous reasoning. We start with Einstein's equations, in which the spacetime metric $g_{\mu\nu}$ is the analog of A_μ and the Einstein tensor $G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is the analog of $\partial_\nu F^{\mu\nu}$. They read

$$E^{\mu\nu} := G^{\mu\nu} - \Lambda - \kappa T^{\mu\nu} = 0 \,. \tag{16}$$

Due to four dimensional diffeomorphism invariance, we have the identity (twice contracted second Bianchi-Identity):

$$\nabla_\mu G^{\mu\nu} \equiv 0 \,, \tag{17}$$

which is the analog of (12). Taking the covariant divergence of (16) and using (17) yields

$$\nabla_\mu E^{\mu\nu} = -\kappa \nabla_\mu T^{\mu\nu} = 0\,, \tag{18}$$

which is the analog of (13). Hence the vanishing covariant divergence of $T^{\mu\nu}$ is an integrability condition of Einstein's equations, just as the divergencelessness of the electric four-current was an integrability condition of Maxwell's equations.[6]

In order to talk about 'evolution', we consider the foliation (1) of M and locally use coordinates $\{x^0, x^a\}$ such that $\partial/\partial x^0$ is the normal n to the leaves and all $\partial/\partial x^a$ are tangential. Expanding (17) in terms of partial derivatives gives:

$$\partial_0 G^{0\nu} = -\partial_a G^{a\nu} - \Gamma^\mu_{\mu\lambda} G^{\lambda\nu} - \Gamma^\nu_{\mu\lambda} G^{\mu\lambda}\,, \tag{19}$$

which is the analog of (14). Now, since the $G^{\mu\nu}$ contain at most second and the $\Gamma^\lambda_{\mu\nu}$ at most first derivatives of the metric $g_{\mu\nu}$, this identity immediately shows that the four components $G^{0\nu}$ ($\nu = 0,1,2,3$) contain at most first time derivatives $\partial/\partial x^0$. But Einstein's equations are of second order, hence the four equations $E^{0\nu} = 0$ are relations amongst the initial data, rather than being evolution equations. In fact, up to a factor of -2 they are just the constraints (8–9):

$$\begin{aligned} H &= -2E^{00} = -2(G^{00} - \Lambda - \kappa T^{00})\,, && (20)\\ D^a &= -2E^{0a} = -2(G^{0a} - \Lambda - \kappa T^{0a})\,. && (21)\end{aligned}$$

Moreover, the remaining purely spatial components of Einstein's equations are equivalent to the twelve first-order evolution equations (6-7).

The interplay between constraints and evolution equations can now be followed along the very same lines as for the electrodynamic analogy. Expanding the left equality of (18) in terms of partial derivatives gives

$$\partial_0 E^{0\nu} = -\partial_a E^{a\nu} - \Gamma^\mu_{\mu\lambda} E^{\lambda\nu} - \Gamma^\nu_{\mu\lambda} E^{\mu\lambda} - \kappa \nabla_\mu T^{\mu\nu}\,, \tag{22}$$

which is the analog of (15). It shows that the constraints are preserved by the evolution if and only if the energy-momentum tensor of the matter has vanishing covariant divergence.

Let us now turn to the last analogy: the uniqueness of the evolution preserving constraints. Clearly Einstein's equations $E^{\mu\nu}$ imply $E^{\mu\nu} n_\mu = 0$ for any timelike vector field n_μ. Hence the constraints are satisfied on any spacelike slice through spacetime. Again the converse is also true: given a gravitational field such that $E^{\mu\nu} n_\mu = 0$ for any timelike n_μ (and given external $T^{\mu\nu}$), then this field must necessarily satisfy Einstein's equations. In this sense Einstein's equations follow uniquely from the condition of constraint preservation.

[6] There is, however, a notable difference in the physical interpretation of divergencelessness of a tensor field on one hand, and a vector field on the other: $\nabla_\mu T^{\mu\nu} = 0$ does not as such imply a conservation law. Only in presence of a spacetime symmetry, i.e. a Killing vector field K_ν, the current $J^\mu = T^{\mu\nu} K_\nu$ is conserved, $\nabla_\mu J^\mu = 0$, and hence gives rise to a conserved quantity.

This property will be crucial for the interpretation of the quantum theory discussed below. We know from quantum mechanics that the classical trajectories have completely disappeared at the fundamental level. As we have discussed above, the analogue to a trajectory is in GR provided by a spacetime given as a set of three dimensional geometries. In quantum gravity, the spacetime will therefore disappear like the classical trajectory in quantum mechanics. It is therefore not surprising that the evolution equations (6) and (7) will be absent in quantum gravity. All the information will be contained in the quantized form of the constraints (8) and (9).

5 Canonical gravity

We have seen above that Einstein's equations can be written as a dynamical system (6–7) with constraints (8–9). Here we wish to give its canonical formulation. Basically this means to introduce momenta for the velocities and write the first-order equations of motions as Hamilton equations. This means to identify the Poisson structure and the Hamiltonian. The result is this: As before, the configuration variable is the Riemannian metric h_{ab} on Σ. Its canonical momentum is now given by

$$\pi^{ab} = (2\kappa)^{-1}\, G^{ab\,cd} K_{cd} = (2\kappa)^{-1}\sqrt{h}(K^{ab} - h^{ab}K^c_c)\,, \tag{23}$$

so that the Poisson brackets are

$$\{h_{ab}(x), \pi^{cd}(y)\} = \tfrac{1}{2}(\delta^c_a\delta^d_b + \delta^d_a\delta^c_b)\delta^{(3)}(x,y)\,, \tag{24}$$

where $\delta^{(3)}(x,y)$ is the Dirac distribution on Σ.

Elimination of K_{ab} in favour of π^{ab} in the constraints leads to their canonical form:

$$H[h,\pi] \;=\; 2\kappa\, G_{ab\,cd}\pi^{ab}\pi^{cd} \;-\; (2\kappa)^{-1}\sqrt{h}({}^{(3)}R - 2\Lambda) \;+\; \sqrt{h}\rho\,, \tag{25}$$

$$D^a[h,\pi] \;=\; -2\nabla_b\pi^{ab} \;+\; \sqrt{h}j^a\,, \tag{26}$$

where now[7]

$$G_{ab\,cd} = \frac{1}{2\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd})\,. \tag{27}$$

Likewise, rewriting (6–7) in terms of the canonical variables shows that they are just the flow equations for the following Hamiltonian:

$$\mathcal{H}[h,\pi] = \int_\Sigma d^3x \big\{\alpha(x)H[h,\pi_{ab}](x) + \beta^a(x)D_a[h,\pi](x)\big\} + \text{boundary terms}\,. \tag{28}$$

[7] Note the difference in the factor of two in the last term, as compared to (10). $G_{ab\,cd}$ is the inverse to $G^{ab\,cd}$, i.e. $G^{ab\,nm}G_{nm\,cd} = \frac{1}{2}(\delta^a_c\delta^b_d + \delta^a_d\delta^b_c)$, and *not* obtained by simply lowering the indices using h_{ab}.

The crucial observation to be made here is, that, up to boundary terms, the total Hamiltonian is a combination of pure constraints. The boundary terms generally appear if Σ is non-compact, as it will be the case for the description of isolated systems, like stars or black holes. In this case the boundary terms are taken over closed surfaces at spatial infinity and represent conserved Poincaré charges, like energy, linear- and angular momentum, and the quantity associated with asymptotic boost transformations. If, however, Σ is closed (i.e. compact without boundary) all of the evolution will be generated by constraints, that is, pure gauge transformations! In that case, evolution, as described here, is not an observable change. For that to be the case we would need an extrinsic clock, with respect to which 'change' can be defined. But a closed universe already contains – by definition – everything physical, so that no external clock exists. Accordingly, there is no external time parameter. Rather, all physical time parameters are to be constructed from within our system, that is, as functional of the canonical variables. A priori there is no preferred choice of such an intrinsic time parameter. The absence of an extrinsic time and the non-preference of an intrinsic one is commonly known as the *problem of time* in Hamiltonian (quantum-) cosmology.

Finally we turn to the commutation relation between the various constraints. For this it is convenient to integrate the local constraints (25–26) over lapse and shift functions. Hence we set (suppressing the phase-space argument $[h, \pi]$)

$$\mathcal{H}(\alpha) = \int_\Sigma d^3x \, H(x) \, \alpha(x) \,, \tag{29}$$

$$\mathcal{D}(\beta) = \int_\Sigma d^3x \, D^a(x) \, \beta_a(x) \,. \tag{30}$$

A straightforward but slightly tedious computation gives

$$\{\mathcal{D}(\beta), \mathcal{D}(\beta')\} = \mathcal{D}([\beta, \beta']) \,, \tag{31}$$

$$\{\mathcal{D}(\beta), \mathcal{H}(\alpha)\} = \mathcal{H}(\beta(\alpha)) \,, \tag{32}$$

$$\{\mathcal{H}(\alpha), \mathcal{H}(\alpha')\} = \mathcal{D}(\alpha\nabla\alpha' - \alpha'\nabla\alpha) \,. \tag{33}$$

There are three remarks we wish to make concerning these relations. First, (31) shows that the diffeomorphism generators form a Lie subalgebra. Second, (32) shows that this Lie subalgebra is not a Lie ideal. This means that the flow of the Hamiltonian constraint does not leave invariant the constraint hypersurface of the diffeomorphism constraint. Finally, the term $\alpha\nabla\alpha' - \alpha'\nabla\alpha$ in (33) contains the canonical variable h, which is used implicitly to raise the index in the differential in order to get the gradient ∇. This means that the relations above do not make the set of all $\mathcal{H}(\alpha)$ and all $\mathcal{D}(\beta)$ into a Lie algebra.[8]

[8] Sometimes this is expressed by saying that this is an 'algebra with structure functions'.

6 The general kinematics of hypersurface deformations

In this section we wish to point out that the relations (31–33) follow a general pattern, namely to represent the 'algebra' of hypersurface deformations, or in other words, infinitesimal changes of embeddings $\mathcal{E}:\Sigma\rightarrow M$. To make this explicit, we introduce local coordinates x^a on Σ and y^μ on M. An embedding is then locally given by four functions $y^\mu(x)$, such that the 3×4 matrix $y^\mu_{,a}$ has its maximum rank 3 (we write $y^\mu_{,a}:=\partial_a y^\mu$). The components of the normal to the image $\mathcal{E}(\Sigma)\subset M$ are denoted by n^μ, which should be considered as functional of $y^\mu(x)$. The generators of normal and tangential deformations of $\mathcal{E}$ with respect to the lapse function α and shift vector field β are then given by

$$N_\alpha \;=\; \int_\Sigma d^3x\;\alpha(x)\,n^\mu[y(x)]\,\frac{\delta}{\delta y^\mu(x)}\,, \tag{34}$$

$$T_\beta \;=\; \int_\Sigma d^3x\;\beta^a(x)\,y^\mu_{,a}(x)\,\frac{\delta}{\delta y^\mu(x)}\,, \tag{35}$$

which may be understood as tangent vectors to the space of embeddings of Σ into M. A calculation[9] then leads to the following commutation relations

$$[T_\beta\,,\,T_{\beta'}] \;=\; -\,T_{[\beta,\beta']}\,, \tag{36}$$

$$[T_\beta\,,\,N_\alpha] \;=\; -\,N_{\beta(\alpha)}\,, \tag{37}$$

$$[N_\alpha\,,\,N_{\alpha'}] \;=\; -\,T_{\alpha\nabla\alpha'-\alpha'\nabla\alpha}\,. \tag{38}$$

Up to the minus signs this is just (31-33). The minus signs are just the usual ones that one always picks up in going from the action of vector fields to the Poisson action of the corresponding phase-space functions. (In technical terms, the mapping from vector fields to phase-space functions is a Lie-*anti*-homomorphism.)

This shows that (31–33) just mean that we have a Hamiltonian realization of hypersurface deformations. In particular, (31–33) is neither characteristic of the action nor the field content: *Any* four dimensional diffeomorphism invariant theory will gives rise to this very same 'algebra'. It can be shown that under certain general locality assumptions the expressions (25) and (26) give the unique 2-parameter (here κ and Λ) family of realizations for N and T satisfying (36–38) on the phase space parameterized by $(h_{ab}\,,\,\pi^{ab})$; see [11] and also [18].

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[9] Equation (36) is immediate. To verify (37–38) one needs to compute $\delta n^\mu[y(x)]/\delta y^\nu(x')$. This can be done in a straightforward way by varying

$$g(y(x))_{\mu\nu}n^\mu[y(x)]n^\nu[y(x)]=-1 \quad\text{and}\quad g_{\mu\nu}(y(x))\,y^\mu_{,a}(x)\,n^\nu[y(x)]=0$$

with respect to $y(x)$.

7 Topological issues

As we have just discussed, Einstein's equations take the form of a constrained Hamiltonian system if put into canonical form. The unconstrained configuration space is the space of all Riemannian metrics on some chosen 3-manifold Σ. This space is denoted by $\mathrm{Riem}(\Sigma)$. Any two Riemannian metrics that differ by an action of the diffeomorphism constraint are gauge equivalent and hence to be considered as physically indistinguishable. Let us briefly mention that the question of whether and when the diffeomorphism constraint actually generates all diffeomorphisms of Σ is rather subtle. Certainly, what is generated lies only in the identity component of the latter, but even on that it may not be onto. This occurs, for example, in the case where Σ contains asymptotically flat ends with non-vanishing Poincaré charges associated. Asymptotic Poincaré transformations are then not interpreted as gauge transformations (otherwise the Poincaré charges were necessarily zero), but as proper physical symmetries (i.e. changes of state that are observable in principle).

Leaving aside the possible difference between what is generated by the constraints and the full group $\mathrm{Diff}(\Sigma)$ of diffeomorphisms of Σ, we may consider the quotient space $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$ of Riemannian *geometries*. This space is called *superspace* in the relativity community (this has nothing to do with supersymmetry), which we denote by $\mathcal{S}(\Sigma)$. Now from a topological viewpoint $\mathrm{Riem}(\Sigma)$ is rather trivial. It is a cone[10] in the (infinite dimensional) vector space of all symmetric second-rank tensor fields. But upon factoring out $\mathrm{Diff}(\Sigma)$ the quotient space $\mathcal{S}(\Sigma)$ inherits some of the topological information concerning Σ, basically because $\mathrm{Diff}(\Sigma)$ contains that information [6]. This is schematically drawn in Fig. 3.

In a certain generalized sense, GR is a dynamical system on the phase space (i.e. cotangent bundle) built over superspace. The topology of superspace is characteristic for the topology of Σ, though in a rather involved way. Note that, by construction, the Hamiltonian evolution is that of a varying embedding of Σ into spacetime. Hence the images Σ_t are all of the same topological type. This is why canonical gravity in the formulation given here cannot describe transitions of topology.

Note, however, that this is not at all an implication by Einstein's equations. Rather, it is a consequence of our restriction to spacetimes that admit a global spacelike foliation. There are many solutions to Einstein's equations that do not admit such foliations globally. This means, that these spacetimes cannot be constructed by integrating the equations of motions (6–7) successively from some initial data. Should we rule out all other solutions? The general feeling seems to be, that at least in quantum gravity, topology changing classical solutions should not be ruled out as possible contributors in the sum over histories (path integral). Fig. 4 shows two such histories. Whereas

[10] Any real positive multiple λh of $h \in \mathrm{Riem}(\Sigma)$ is again an element of $\mathrm{Riem}(\Sigma)$.

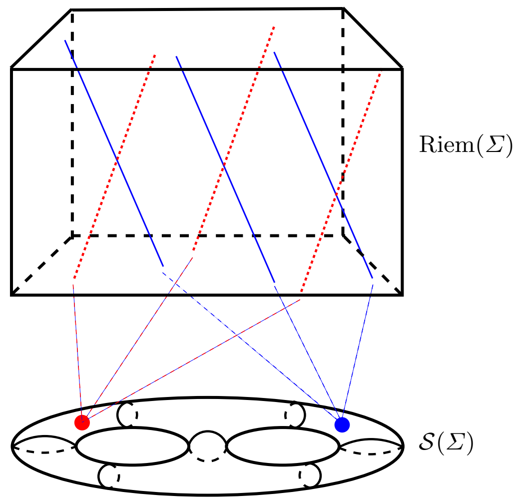


Fig. 3. The topologically trivial space Riem(Σ), here drawn as the box above, is fibered by the action of the diffeomorphism group. The fibers are the straight lines in the box, where the sets consisting of three dashed and three solid lines, respectively, form one fiber each. In the quotient space $\mathcal{S}(\Sigma)$ each fiber is represented by one point only. By taking the quotient, $\mathcal{S}(\Sigma)$ receives the non-trivial topology from Diff(Σ). To indicate this, $\mathcal{S}(\Sigma)$ is represented as a double torus.

in the left picture the universe simply 'grows a nose', it bifurcates in the right example to become disconnected.

One may ask whether there are topological restrictions to such transitions. First of all, it is true (though not at all obvious) that for any given two 3-manifolds $\Sigma_{\rm i}$, $\Sigma_{\rm f}$ (neither needs to be connected) there is a 4-manifold M whose boundary is just $\Sigma_{\rm i} \cup \Sigma_{\rm f}$. In fact, there are infinitely many such M. Amongst them, one can always find some which can be endowed with a globally regular Lorentz metric g, such that $\Sigma_{\rm i}$ and $\Sigma_{\rm f}$ are spacelike. However, if topology changes, (M, g) necessarily contains closed timelike curves [2]. This fact has sometimes been taken as rationale for ruling out topology change in (classical) GR. But it should be stressed that closed timelike curves do not necessarily ruin conventional concepts of predictability. In any case, let us accept this slight pathology and ask what other structures we wish to define on M. For example, in order to define fermionic matter fields on M we certainly wish to endow M with a $SL(2,\mathbb{C})$ spin structure. This is where now the first real obstructions for topological transitions appear [3].[11] It is then possible to translate them into selection rules for transitions between all known 3-manifolds [4].

[11] Their result is the following: Let $\Sigma = \Sigma_{\rm i} \cup \Sigma_{\rm f}$ be the spacelike boundary of the Lorentz manifold M, then $\dim\big(H^0(\Sigma, \mathbf{Z}_2)\big) + \dim\big(H^1(\Sigma, \mathbf{Z}_2)\big)$ has to be even for M to admit an $SL(2,\mathbb{C})$ spin structure.

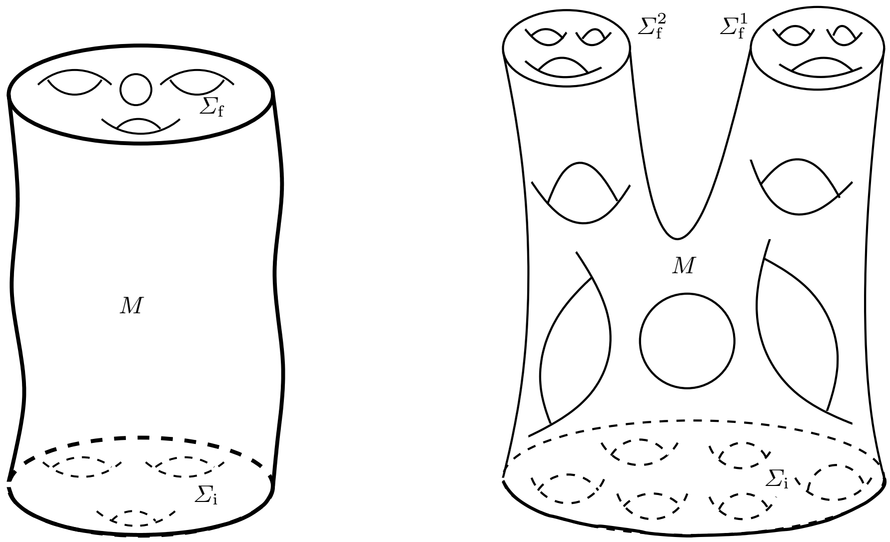


Fig. 4. Spacetimes in which spatial sections change topology. In the left picture the initial universe Σ_i has three, the final Σ_f four topological features ('holes') – it 'grows a nose' while staying connected. In the right picture the initial universe Σ_i splits into two copies $\Sigma_\mathrm{f}^{1,2}$, so that $\Sigma_f = \Sigma_\mathrm{f}^1 \cup \Sigma_\mathrm{f}^2$. In both cases, the interpolating spacetime M can be chosen to carry a Lorentzian metric with respect to which initial and final hypersurfaces are spacelike, possibly at the price of making M topologically complicated, like indicated in the right picture.

So far the considerations were purely kinematical. What additional obstructions arise if the spacetime (M, g) is required to satisfy the field equations? Here the situation becomes worse. It is, for example, known that any topology-changing spacetime that satisfies Einstein's equations with matter that satisfies the weak-energy condition $T_{\mu\nu}l^\mu l^\nu \geq 0$ for all lightlike l^μ must necessarily be singular.[12] Hence it seems that we need to consider degenerate metrics already on the classical level if topology change is to occur. Can we relax the notion of 'solution to Einstein's equations' so as to contain these degenerate cases as well? The answer is 'yes' if instead of taking the metric as basic variable we rewrite the equations in terms of vierbeine and connections (first oder formalism). It turns out that the kind of singularities one has to cope with are very mild indeed: the vierbeine become degenerate on sets of measure zero but, somewhat surprisingly, the curvature stays bounded everywhere. In fact, there is a very general method to generate an abundance of such solutions [12].

It is a much debated question whether topology changing amplitudes are suppressed or, to the contrary, needed in quantum gravity. On one hand,

[12] In fact, this result can be considerably strengthened: Instead of invoking Einstein's equations we only need to require $R_{\mu\nu}l^\mu l^\nu \geq 0$ for all lightlike l^μ.

it has been shown in the context of specific lower dimensional models that matter fields on topology-changing backgrounds may give rise to singularities corresponding to infinite densities of particle production [1]. On the other hand, leaving out topology changing amplitudes in the sum-over-histories approach is heuristically argued to be in conflict with expected properties of localized pseudo-particle-like excitations in gravity (so called geons), like, for example, the usual spin-statistic relation [19]. Here there still seems to be much room for speculations.

8 Geometric issues

Just in the same way as any Lagrangian theory endows the configuration space with the kinetic-energy metric, $\mathrm{Riem}(\Sigma)$ inherits a metric structure from the 'kinetic-energy' part of (8). Tangent vectors at $h \in \mathrm{Riem}(\Sigma)$ are symmetric second-rank tensor fields on Σ and their inner product is given by the so-called *Wheeler–DeWitt metric*:

$$\mathcal{G}_h(V,V') = \int_\Sigma d^3x\, G^{ab\,cd} V_{ab} V'_{cd}\,. \tag{39}$$

Due to the pointwise Lorentzian signature (1+5) of $G^{ab\,cd}$ it is of a hyper-Lorentzian structure with infinitely many negative, null, and positive directions each. However, not all directions in the tangent space $T_h(\mathrm{Riem}(\Sigma))$ correspond to physical changes. Those generated by diffeomorphism, which are of the form $V_{ab} = \nabla_a\beta_b + \nabla_b\beta_a$ for some vector field β on Σ are pure gauge. We call them *vertical*. The diffeomorphism constraint (26) for $j^a = 0$ – a case to which we now restrict for simplicity – now simply says that V must be $\mathcal{G}$–orthogonal to such vertical directions. We call such orthogonal directions *horizontal*. Moreover, it is easily seen that the inner product (39) is invariant under $\mathrm{Diff}(\Sigma)$. All this suggests how to endow superspace, $\mathcal{S}(\Sigma)$, with a natural metric: take two tangent vectors at a point $[h]$ in $\mathcal{S}(\Sigma)$, lift them to horizontal vectors at h in $\mathrm{Riem}(\Sigma)$ and there take the inner product according to (39).

However, this procedure only works if the horizontal subspace of $T_h(\mathrm{Riem}(\Sigma))$ is truly complementary to the vertical space of gauge directions. However, this is not guaranteed due to $\mathcal{G}$ not being positive definite: whenever there are vertical directions of zero $\mathcal{G}$–norm, there will be non-trivial intersections of horizontal and vertical spaces. Sufficient conditions on h for this *not* to happen can be derived, like, for example, a strictly negative Ricci tensor [7]. The emerging picture is that there are open sets in $\mathcal{S}(\Sigma)$ in which well defined hyper-Lorentzian geometries exist, which are separated by closed transition regions in which the signature of these metrics change. The transition regions precisely consist of those geometries $[h]$ which possess vertical directions of zero $\mathcal{G}$–norm; see Fig. 5.

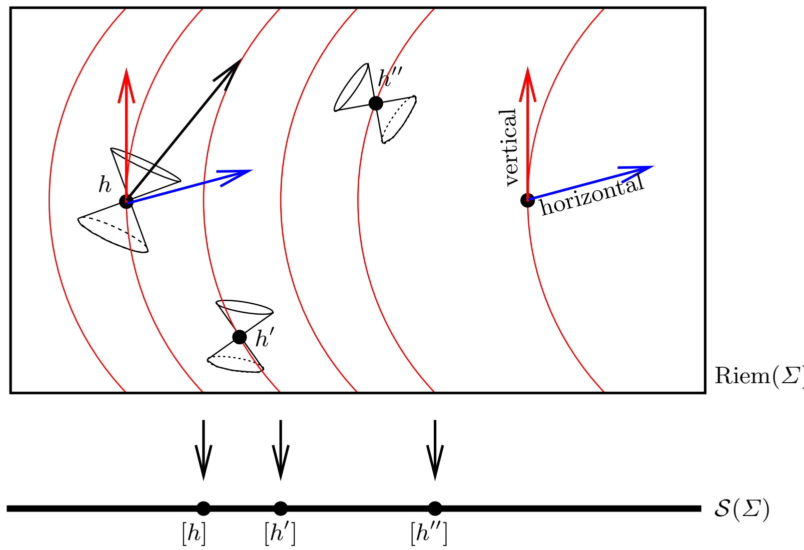


Fig. 5. The space Riem(Σ), fibered by the orbits of Diff(Σ) (curved vertical lines). Tangent directions to these orbits are called 'vertical', the $\mathcal{G}$-orthogonal directions 'horizontal'. Horizontal and vertical directions intersect whenever the 'hyper-light-cone' touches the vertical directions, as in point h'. At h, h', and h'' the vertical direction is depicted as time-, light-, and spacelike respectively. Hence $[h']$ corresponds to a transition point where the signature of the metric in superspace changes.

9 Quantum geometrodynamics

Einstein's theory of GR has now been brought into a form where it can be subject to the procedure of canonical quantization. As we have argued above, all the information that is needed is encoded in the constraints (25) and (26). However, quantizing them is far from trivial [16]. One might first attempt to solve the constraints on the classical level and then to quantize only the reduced, physical, degrees of freedom. This is already impossible in quantum electrodynamics (except the case of freely propagating fields), and it is illusory to achieve in GR. One therefore usually follows the procedure proposed by Dirac and tries to implement the constraints as conditions on physically allowed wave functionals. The constraints (25) and (26) the become the quantum conditions

$$\hat{H}\Psi = 0 \ , \tag{40}$$

$$\hat{D}^a\Psi = 0 \ , \tag{41}$$

where the 'hat' is a symbolic indication for the fact that the classical expressions have been turned into operators. This procedure also applies if other variables instead of the three-metric and its momentum are used; for example, such quantum constraints also play the role in loop quantum gravity, cf. the contributions of Nicolai and Peeters as well as Thiemann to this volume. In the present case the resulting formalism is called quantum geometrodynamics.

Quantum geometrodynamics is defined by the transformation of $h_{ab}(x)$ into a multiplication operator and π^{cd} into a functional derivative operator, $\pi^{cd} \to -\mathrm{i}\hbar\delta/\delta h_{cd}(x)$. The constraints (25) and (26) then assume the form, restricting here to the vacuum case for simplicity,

$$\hat{H}\Psi \equiv \left(-2\kappa\hbar^2 G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - (2\kappa)^{-1}\sqrt{h}\big({}^{(3)}R - 2\Lambda\big)\right)\Psi = 0 \ , \quad (42)$$

$$\hat{D}^a\Psi \equiv -2\nabla_b \frac{\hbar}{\mathrm{i}}\frac{\delta\Psi}{\delta h_{ab}} = 0 \ . \quad (43)$$

Equation (42) is called the *Wheeler–DeWitt equation* in honour of the work by Bryce DeWitt and John Wheeler; see e.g. [16] for details and references. In fact, these are again infinitely many equations (one equation per space point). The constraints (43) are called the *quantum diffeomorphism (or momentum) constraints*. Occasionally, both (42) and (43) are referred to as Wheeler–DeWitt equations. In the presence of non-gravitational fields, these equations are augmented by the corresponding terms.

The argument of the wave functional Ψ is the three-metric $h_{ab}(x)$ (plus non-gravitational fields). However, because of (43), Ψ is invariant under coordinate transformations on three dimensional space (it may acquire a phase with respect to 'large diffeomorphisms' that are not connected with the identity). A most remarkable feature of the quantum constraint equations is their 'timeless' nature – the external parameter t has completely disappeared.[13] Instead of an external time one may consider an 'intrinsic time' that is distinguished by the kinetic term of (42). As can be recognized from the signature of the DeWitt metric (10), the Wheeler–DeWitt equation is locally hyperbolic, that is, it assumes the form of a local wave equation. The intrinsic timelike direction is related to the conformal part of the three-metric. With respect to the discussion in the last section one may ask whether there are regions in superspace where the Wheeler–DeWitt metric exists and has precisely one negative direction. In that case the Wheeler–DeWitt equation would be strictly hyperbolic (rather than ultrahyperbolic) in a neighbourhood of that point. It has been shown that such regions indeed exist and that they include neighbourhoods of the standard round three-sphere geometry [7]. This implies that the full Wheeler–DeWitt equation that describes fluctuations around the positive curvature Friedmann universe is strictly hyperbolic. In this case the scale factor of the Friedmann universe could serve as an intrinsic

[13] In the case of asymptotic spaces such a parameter may be present in connection with Poincaré transformations at spatial infinity. We do not consider this case here.

time. The indefinite nature of the kinetic term reflects the fact that gravity is attractive [5].

There are many problems associated with the quantum constraints (42) and (43). An obvious problem is the 'factor-ordering problem': the precise form of the kinetic term is open – there could be additional terms proportional to $\hbar$ containing at most first derivatives in the metric. Since second functional derivatives at the same space point usually lead to undefined expressions such as $\delta(0)$, a regularization (and perhaps renormalization) scheme has to be employed. Connected with this is the potential presence of anomalies, cf. the contribution by Nicolai and Peeters. Another central problem is what choice of Hilbert space one has to make, if any, for the interpretation of the wave functionals. No final answer to this problem is available in this approach [16].

What about the semiclassical approximation and the recovery of an appropriate external time parameter in some limit? For the full quantum constraints this can at least be achieved in a formal sense (i.e., treating functional derivatives as if they were ordinary derivatives and neglecting the problem of anomalies); see [16, 17]. The discussion is also connected to the question: Where does the imaginary unit i in the (functional) Schrödinger equation come from? The full Wheeler–DeWitt equation is real, and one would thus also expect real solutions for Ψ. An approximate solution is found through a Born–Oppenheimer-type of scheme, in analogy to molecular physics. The state then assumes the form

$$\Psi \approx \exp(\mathrm{i}S_0[h]/\hbar)\, \psi[h,\phi] \ , \tag{44}$$

where h is an abbreviation for the three-metric, and ϕ stands for non-gravitational fields. In short, one finds that

- S_0 obeys the Hamilton–Jacobi equation for the gravitational field and thereby defines a classical spacetime which is a solution to Einstein's equations (this order is formally similar to the recovery of geometrical optics from wave optics via the eikonal equation).
- ψ obeys an approximate (functional) Schrödinger equation,

$$\mathrm{i}\hbar \underbrace{\nabla S_0\, \nabla\psi}_{\frac{\partial\psi}{\partial t}} \approx H_{\mathrm{m}}\, \psi \ , \tag{45}$$

  where H_{m} denotes the Hamiltonian for the non-gravitational fields ϕ. Note that the expression on the left-hand side of (45) is a shorthand notation for an integral over space, in which ∇ stands for functional derivatives with respect to the three-metric. Semiclassical time t is thus defined in this limit from the dynamical variables.
- The next order of the Born-Oppenheimer scheme yields quantum gravitational correction terms proportional to the inverse Planck mass squared, $1/m_{\mathrm{P}}^2$. The presence of such terms may in principle lead to observable

effects, for example, in the anisotropy spectrum of the cosmic microwave background radiation.

The Born–Oppenheimer expansion scheme distinguishes a state of the form (44) from its complex conjugate. In fact, in a generic situation both states will decohere from each other, that is, they will become dynamically independent [15]. This is a type of symmetry breaking, in analogy to the occurrence of parity violating states in chiral molecules. It is through this mechanism that the i and the t in the Schrödinger equation emerge.

The recovery of the Schrödinger equation (45) raises an interesting issue. It is well known that the notion of Hilbert space is connected with the conservation of probability (unitarity) and thus with the presence of an external time (with respect to which the probability is conserved). The question then arises whether the concept of a Hilbert space is still required in the *full* theory where no external time is present. It could be that this concept makes sense only on the semiclassical level where (45) holds.

10 Applications

The major physical applications of quantum gravity concern cosmology and black holes. Although the above presented formalism exists, as yet, only on a formal level, one can study models that present no mathematical obstacles. Typically, such models are obtained by imposing symmetries on the equations [16]. Examples are spherical symmetry (useful for black holes) and homogeneity and isotropy (useful for cosmology).

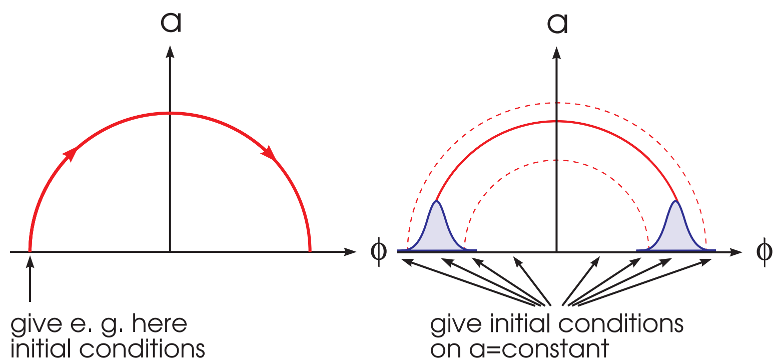
Quantum cosmology is the application of quantum theory to the universe as a whole. Let us consider a simple example: a Friedmann universe with scale factor $a \equiv \mathrm{e}^{\alpha}$ containing a massive scalar field ϕ. In this case, the diffeomorphism constraints (43) are identically fulfilled, and the Wheeler–DeWitt equation (42) reads

$$\hat{H}\psi \equiv \left(G\hbar^2 \frac{\partial^2}{\partial\alpha^2} - \hbar^2 \frac{\partial^2}{\partial\phi^2} + m^2\phi^2 \mathrm{e}^{6\alpha} - \frac{\mathrm{e}^{4\alpha}}{G} \right) \psi(\alpha,\phi) = 0 \ . \qquad (46)$$

This equation is simple enough to find solutions (at least numerically) and to study physical aspects such as the dynamics of wave packets and the semiclassical limit [16].

There is one interesting aspect in quantum cosmology that possesses far-reaching physical consequences. Because (42) does not contain an external time parameter t, the quantum theory exhibits a kind of determinism drastically different from the classical theory [20][16]. Consider a model with a two-dimensional configuration space spanned by the scale factor, a, and a homogeneous scalar field, ϕ, see Fig. 6. (Such a model is described, for example, by (46) with $m = 0$.) The classical model be such that there are solutions where the universe expands from an initial singularity, reaches a maximum, and recollapses to a final singularity. Classically, one would impose, in a Lagrangian formulation, $a, \dot{a}, \phi, \dot{\phi}$ (satisfying the constraint) at some t_0 (for example, at the left leg of the trajectory), and then the trajectory would be determined. This is indicated on the left-hand side of Fig. 6. In the quantum

Fig. 6. The classical and the quantum theory of gravity exhibit drastically different notions of determinism.



theory, on the other hand, there is no t. The hyperbolic nature of a minisuperspace equation such as (46) suggests to impose boundary conditions at $a =$ constant. In order to represent the classical trajectory by narrow wave packets, the 'returning part' of the packet must be present 'initially' (with respect to a). The determinism of the quantum theory then proceeds from small a to large a, not along a classical trajectory (which does not exist). This behaviour has consequences for the validity of the semiclassical approximation and the arrow of time. In fact, it may in principle be possible to understand the origin of irreversibility from quantum cosmology, by the very fact that the Wheeler–DeWitt equation is asymmetric with respect to the intrinsic time given by a. The framework of canonical quantum cosmology is also suitable to address the quantum-to-classical transition for cosmological variables such as the volume of the universe [15][16]. Using the approach of loop quantum gravity (see Thiemann's contribution) one arrives at a Wheeler–DeWitt equation in cosmology which is fundamentally a difference equation instead of a differential equation of the type (46). In the ensuing framework of loop quantum gravity it seems that the classical singularities of GR can be avoided.

Singularity avoidance for collapse situations can also be found from spherically symmetric models of quantum geometrodynamics. For example, in a model with a collapsing null dust cloud, an initially collapsing wave packet evolves into a superposition of collapsing and expanding packet [9]. This leads to destructive interference at the place where the singularity in the classical theory occurs. Other issues, such as the attempt to give a microscopic derivation of the Bekenstein–Hawking entropy (see the contribution by C. Kiefer to this volume), have been mainly addressed in loop quantum gravity. A final, clear-cut, derivation remains, however, elusive.

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