1

Quantum Gravity and Precision Tests

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Abstract

This article provides a cartoon of the quantization of General Relativity using the ideas of effective field theory. These ideas underpin the use of General Relativity as a theory from which precise predictions are possible, since they show why quantum corrections to standard classical calculations are small. Quantum corrections can be computed controllably provided they are made for the weakly-curved geometries associated with precision tests of General Relativity, such as within the solar system or for binary pulsars. They also bring gravity back into the mainstream of physics, by showing that its quantization (at low energies) exactly parallels the quantization of other, better understood, non-renormalizable field theories which arise elsewhere in physics. Of course effective field theory techniques do *not* solve the fundamental problems of quantum gravity discussed elsewhere in these pages, but they do helpfully show that these problems are specific to applications on very small distance scales. They also show why we may safely reject any proposals to modify gravity at *long* distances if these involve low-energy problems (like ghosts or instabilities), since such problems are unlikely to be removed by the details of the ultimate understanding of gravity at microscopic scales.

1.1 Introduction

Any of us who has used the Global Positioning System (GPS) in one of the gadgets of everyday life has also relied on the accuracy of the predictions of Einstein's theory of gravity, General Relativity (GR). GPS systems accurately provide your position relative to satellites positioned thousands of kilometres from the earth, and their ability to do so requires being able to understand time and position measurements to better than 1 part in 10^{10}. Such an accuracy is comparable to the predicted relativistic effects for such measurements in the Earth's gravitational field, which are of order $GM_\oplus/R_\oplus c^2 \sim 10^{-10}$, where G is Newton's constant, $M_\oplus$ and $R_\oplus$ are the Earth's mass and mean radius, and c is the speed of light. GR also does well when compared with other precise measurements within the solar system, as well as in some extra-solar settings [1].

So we live in an age when engineers must know about General Relativity in order to understand why some their instruments work so accurately. And yet we also are often told there is a crisis in reconciling GR with quantum mechanics, with the size of quantum effects being said to be infinite (or — what is the same — to be unpredictable) for gravitating systems. But since precision agreement with experiment implies agreement within both theoretical and observational errors, and since uncomputable quantum corrections fall into the broad category of (large) theoretical error, how can uncontrolled quantum errors be consistent with the fantastic success of classical GR as a precision description of gravity?

This chapter aims to explain how this puzzle is resolved, by showing why quantum effects in fact *are* calculable within GR, at least for systems which are sufficiently weakly curved (in a sense explained quantitatively below). Since all of the extant measurements are performed within such weakly-curved environments, quantum corrections to them can be computed and are predicted to be fantastically small. In this sense we quantitatively understand *why* the classical approximation to GR works so well within the solar system, and so why in practical situations quantum corrections to gravity need not be included as an uncontrolled part of the budget of overall theoretical error.

More precisely, the belief that quantum effects are incalculable within GR arises because GR is what is called a non-renormalizable theory. Non-renormalizability means that the short-wavelength divergences — which are ubiquitous within quantum field theory — cannot be absorbed

into the definitions of a finite number of parameters (like masses and charges), as they are in renormalizable theories like Quantum Electrodynamics (QED) or the Standard Model (SM) of the strong and electroweak interactions. Although this does preclude making quantum predictions *of arbitrary accuracy*, it does not preclude making predictions to any finite order in an appropriate low-energy expansion, and this is what allows the predictivity on which precise comparison with experiment relies. In fact gravity is not at all special in this regard, as we know of other non-renormalizable theories which describe nature — such as the chiral perturbation theory which describes the low-energy interactions of pions and kaons, or the Fermi theory of the weak interactions, or a wide variety of condensed matter models. In many of these other systems quantum corrections are not only computable, they can be measured, with results which agree remarkably well with observations.

One thing this chapter is *not* intended to do is to argue that it is silly to think about the problems of quantum gravity, or that there are no interesting fundamental issues remaining to be addressed (such as many of those described elsewhere in these pages). What is intended is instead to identify more precisely where these more fundamental issues become important (at very short distances), and why they do not hopelessly pollute the detailed comparison of GR with observations. My presentation here follows that of my longer review of ref. [2], in which the arguments given here are provided in more detail.

1.2 Nonrenormalizability and the low-energy approximation

Since the perceived difficulties with calculating quantum corrections in weak gravitational fields revolve around the problem of calculating with non-renormalizable theories, the first step is to describe the modern point of view as to how this should be done. It is convenient to do so first with a simpler toy model, before returning to GR in all of its complicated glory.

1.2.1 A toy model

Consider therefore the theory of a complex scalar field, ϕ, described by the Lagrangian density

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi)\,, \qquad (1.1)$$

with the following scalar potential

$$V = \frac{\lambda^2}{4} \left(\phi^* \phi - v^2 \right)^2 . \tag{1.2}$$

This theory is renormalizable, so we can compute its quantum implications in some detail.

Since we return to it below, it is worth elaborating briefly on the criterion for renormalizability. To this end we follow standard practice and define the 'engineering' dimension of a coupling as p, where the coupling is written as $(\text{mass})^p$ in units where $\hbar = c = 1$ (which are used throughout).† For instance the coupling λ^2 which pre-multiplies $(\phi^* \phi)^2$ above is dimensionless in these units, and so has $p = 0$, while the coupling $\lambda^2 v^2$ pre-multiplying $\phi^* \phi$ has $p = 2$.

A theory is renormalizable if $p \geq 0$ for all of its couplings, and if for any given dimension all possible couplings have been included consistent with the symmetries of the theory. Both of these are clearly true for the Lagrangian of eqs. (1.1) and (1.2), since all possible terms are written consistent with $p \geq 0$ and the $U(1)$ symmetry $\phi \to e^{i\omega} \phi$.

1.2.1.1 Spectrum and scattering

We next analyze the spectrum and interactions, within the semiclassical approximation which applies in the limit $\lambda \ll 1$. In this case the field takes a nonzero expectation value, $\langle \phi \rangle = v$, in the vacuum. The particle spectrum about this vacuum consists of two weakly-interacting particle types, one of which – φ_0 – is massless and the other – φ_m – has mass $m = \lambda v$. These particles interact with one another through an interaction potential of the form

$$V = \frac{1}{2} \left[m\varphi_m + \frac{\lambda}{2\sqrt{2}} \left(\varphi_m^2 + \varphi_0^2 \right) \right]^2 , \tag{1.3}$$

as may be seen by writing $\phi = v + (\varphi_m + i\varphi_0)/\sqrt{2}$. For instance, these interactions imply the following invariant scattering amplitude for the scattering process $\varphi_0(p)\varphi_0(q) \to \varphi_0(p')\varphi_0(q')$

$$\begin{aligned}
\mathcal{A} = {} & -\frac{3\lambda^2}{2} + \left(\frac{\lambda m}{\sqrt{2}} \right)^2 \left[\frac{1}{(p+q)^2 + m^2 - i\epsilon} \right. \\
& \left. + \frac{1}{(p-p')^2 + m^2 - i\epsilon} + \frac{1}{(p-q')^2 + m^2 - i\epsilon} \right] \tag{1.4}
\end{aligned}$$

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† It is implicit in this statement that the relevant fields are canonically normalized, and so have dimensionless kinetic terms.

This amplitude has an interesting property in the limit that the center-of-mass scattering energy, E, is much smaller than the mass m. As may be explored by expanding $\mathcal{A}$ in powers of external four-momenta, in this limit the $O(\lambda^2)$ and $O(\lambda^2 E^2/m^2)$ terms both vanish, leaving a result $\mathcal{A} = O(\lambda^2 E^4/m^4)$. Clearly the massless particles interact more weakly than would be expected given a cursory inspection of the scalar potential, eq. (1.3).

The weakness of the scattering of φ_0 particles at low energy is a consequence of their being Nambu-Goldstone bosons [3, 4] for the theory's $U(1)$ symmetry: $\phi \to e^{i\omega}\phi$. This can be seen more explicitly by changing variables to polar coordinates in field space, $\phi = \chi\, e^{i\theta}$, rather than the variables φ_0 and φ_m. In terms of θ and χ the action of the $U(1)$ symmetry is simply $\theta \to \theta + \omega$, and the model's Lagrangian becomes:

$$\mathcal{L} = -\partial_\mu \chi \partial^\mu \chi - \chi^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda^2}{4}(\chi^2 - v^2)^2 \,, \qquad (1.5)$$

and semiclassical calculations can be performed as before by expanding in terms of canonically-normalized fluctuations: $\chi = v + \hat{\varphi}_m/\sqrt{2}$ and $\theta = \hat{\varphi}_0/v\sqrt{2}$, revealing that $\hat{\varphi}_m$ describes the massive particle while $\hat{\varphi}_0$ describes the massless one. Because $\hat{\varphi}_0$ appears in $\mathcal{L}$ only explicitly differentiated (as it must because of the symmetry $\hat{\varphi}_0 \to \hat{\varphi}_0 + \omega v\sqrt{2}$), its scattering is suppressed by powers of E/m at low energies.

1.2.1.2 The low-energy effective theory

Properties such as this which arise (sometimes unexpectedly) when observables are expanded at low energies in powers of E/m are explored most easily by 'integrating out' the heavy particle to construct the *effective field theory* describing the low-energy dynamics of the massless particle alone. One way to do so in the case under consideration here would be to define 'light' degrees of freedom to be those modes (in momentum space) of $\hat{\varphi}_0$ which satisfy $p^2 < \Lambda^2$ (in Euclidean signature), for some cutoff Λ satisfying $E \ll \Lambda \ll m$. All other modes are, by definition, 'heavy'. Denoting the heavy and light modes schematically by h and ℓ, then the effective theory governing the light fields may be defined by

$$\exp\left[i \int d^4x\, \mathcal{L}_{\text{eff}}(\ell, \Lambda)\right] = \int \mathcal{D}h_\Lambda\, \exp\left[i \int d^4x\, \mathcal{L}(\ell, h)\right] \,, \qquad (1.6)$$

where the functional integral is performed over all of the heavy modes (including the large-momentum components of $\hat{\varphi}_0$).

$\mathcal{L}_{\text{eff}}$ defined this way necessarily depends on Λ, but it does so in just

the way required in order to have Λ cancel with the explicit Λ's which cut off the loop integrals for the functional integration over the light fields, ℓ. All Λ's must cancel in observables because Λ is just a bookmark which we use to organize the calculation. Because of this cancellation the detailed form of the regularization is largely immaterial and can be chosen for convenience of calculation.

For this reason it is actually preferable instead to define $\mathcal{L}_{\text{eff}}$ using dimensional regularization rather than a cutoff. Paradoxically, this is possible even though one keeps both short- and long-wavelength modes of the light fields in the low-energy theory when dimensionally regularizing, which seems to contradict the spirit of what a low-energy effective theory is. In practice it is possible because the difference between the cutoff- and dimensionally-regularized low-energy theory can itself be parameterized by an appropriate choice for the effective couplings within the low-energy theory. This is the choice we shall make below when discussing quantum effects within the effective theory.

With this definition, physical observables at low energies are now computed by performing the remaining path integral over the light degrees of freedom only, weighted by the low-energy effective Lagrangian: $\exp\left[i \int d^4x \, \mathcal{L}_{\text{eff}}(\ell)\right]$. The effects of virtual contributions of heavy states appear within this low-energy theory through the contributions of new effective interactions. When applied to the toy model to leading order in λ this leads to the following result for $\mathcal{L}_{\text{eff}}$:

$$\mathcal{L}_{\text{eff}} \;\; = \;\; v^2 \left[-\partial_\mu\theta\,\partial^\mu\theta + \frac{1}{4m^2}(\partial_\mu\theta\,\partial^\mu\theta)^2 - \frac{1}{4m^4}\,(\partial_\mu\theta\,\partial^\mu\theta)^3 \right. \qquad (1.7)$$
$$\left. + \frac{1}{4m^4}\,(\partial_\mu\theta\,\partial^\mu\theta)\partial_\lambda\partial^\lambda(\partial_\nu\theta\,\partial^\nu\theta) + \cdots \right] ,$$

where the ellipses in $\mathcal{L}$ represent terms which are suppressed by more than four inverse powers of m. The inverse powers of m which premultiply all of the interactions in this Lagrangian are a consequence of the virtual $\hat{\varphi}_m$ exchanges which are required in order to produce them within the full theory. The explicit numerical factors in each term are an artifact of leading order perturbation theory, and receive corrections order by order in λ. Computing 2-particle scattering using this effective theory gives a result for which the low-energy suppression by powers of E/m are explicit due to the derivative form of the interactions.

What is interesting about the lagrangian, eq. (1.7), for the present purposes is that the successive effective couplings involve successively more powers of $1/m^2$. In particular, this keeps them from having non-

negative engineering dimension and so makes the effective theory manifestly non-renormalizable. If someone were to hand us this theory we might therefore throw up our hands and conclude that we cannot predictively compute quantum corrections. However in this case we know this theory simply expresses the low-energy limit of a full theory which *is* renormalizable, and so for which quantum corrections can be explicitly computed. Why can't these corrections also be expressed using the effective theory?

The answer is that they can, and this is by far the most efficient way to compute these corrections to observables in the low-energy limit where $E \ll m$. The key to computing these corrections is to systematically exploit the low-energy expansion in powers of E/m which underlies using the action, eq. (1.7) in the first place.

1.2.2 Computing loops

To explore quantum effects consider evaluating loop graphs using the toy-model effective lagrangian, which we may write in the general form

$$\mathcal{L}_{\text{eff}} = v^2 m^2 \sum_{id} \frac{c_{id}}{m^d}\, \mathcal{O}_{id}, \qquad (1.8)$$

where the sum is over interactions, $\mathcal{O}_{id}$, involving i powers of the dimensionless field θ and d derivatives. The power of m pre-multiplying each term is chosen to ensure that the coefficient c_{id} is dimensionless, and we have seen that these coefficients are $O(1)$ at leading order in λ^2. To be completely explicit, in the case of the interaction $\mathcal{O} = (\partial_\mu \theta\, \partial^\mu \theta)^2$ we have $i = d = 4$ and we found earlier that $c_{44} = \frac{1}{4} + O(\lambda^2)$ for this term. Notice that Lorentz invariance requires d must be even, and the $U(1)$ symmetry implies every factor of θ is differentiated at least once, and so $d \geq i$. We may ignore all terms with $i = 1$ since these are linear in $\partial_\mu \theta$ and so must be a total derivative.† Furthermore, the only term with $i = 2$ is the kinetic term, which we take as the unperturbed Lagrangian, and so for the interactions we may restrict the sum to $i \geq 3$.

With these definitions it is straightforward to track the powers of v and m that interactions of the form (1.8) contribute to an L-loop contribution to a scattering amplitude, $\mathcal{A}(E)$, at centre-of-mass energy E. (The steps presented here closely follow the discussion of refs. [2, 4].) Imagine using this lagrangian to compute a contribution to the scattering

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† Terms like total derivatives, which do not contribute to the observables of interest, are called *redundant* and may be omitted when writing the effective Lagrangian.

amplitude, $\mathcal{A}(E)$, coming from a Feynman graph involving $\mathcal{E}$ external lines; I internal lines and V_{ik} vertices. (The subscript i here counts the number of lines which converge at the vertex, while k counts the power of momentum which appears.) These constants are not all independent, since they are related by the identity $2I + E = \sum_{ik} i\, V_{ik}$. It is also convenient to trade the number of internal lines, I, for the number of loops, L, defined by $L = 1 + I - \sum_{ik} V_{ik}$.

We now use dimensional analysis to estimate the result of performing the integration over the internal momenta, using dimensional regularization to regulate the ultraviolet divergences. If all external momenta and energies are of order E then the size of a dimensionally-regularized integral is given on dimensional grounds by the appropriate power of E, we find

$$
\begin{aligned}
\mathcal{A}(E) \quad &\sim \quad v^2 m^2 \left(\frac{1}{v}\right)^{\mathcal{E}} \left(\frac{m}{4\pi v}\right)^{2L} \left(\frac{E}{m}\right)^{P} \\
&\sim \quad v^2 E^2 \left(\frac{1}{v}\right)^{\mathcal{E}} \left(\frac{E}{4\pi v}\right)^{2L} \prod_i \prod_{d>2} \left(\frac{E}{m}\right)^{(d-2)V_{id}} .
\end{aligned}
\tag{1.9}
$$

where $P = 2 + 2L + \sum_{id}(d-2)V_{id}$. This is the main result, since it shows which graphs contribute to any order in E/m using a nonrenormalizable theory.†

To see how eqs. (1.9) are used, consider the first few powers of E in the toy model. For any $\mathcal{E}$ the leading contributions for small E come from *tree* graphs, *i.e.* those having $L = 0$. The tree graphs that dominate are those for which $\sum'_{id}(d-2)V_{id}$ takes the smallest possible value. For example, for 2-particle scattering $\mathcal{E} = 4$ and so precisely one tree graph is possible for which $\sum'_{id}(d-2)V_{id} = 2$, corresponding to $V_{44} = 1$ and all other $V_{id} = 0$. This identifies the single graph which dominates the 4-point function at low energies, and shows that the resulting leading energy dependence in this case is $\mathcal{A}(E) \sim E^4/(v^2\, m^2)$, as was also found earlier in the full theory. The numerical coefficient can be obtained in terms of the effective couplings by more explicit evaluation of the appropriate Feynman graph.

The next-to-leading behavior is also easily computed using the same arguments. Order E^6 contributions are achieved if and only if either: (*i*) $L = 1$ and $V_{i4} = 1$, with all others zero; or (*ii*) $L = 0$ and $\sum_i \left(4V_{i6} + \right.$

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† It is here that the convenience of dimensional regularization is clear, since it avoids keeping track of powers of a cutoff like Λ, which drops out of the final answer for an observable in any case.

$2V_{i4}\Big) = 4$. Since there are no $d = 2$ interactions, no one-loop graphs having 4 external lines can be built using precisely one $d = 4$ vertex and so only tree graphs can contribute. Of these, the only two choices allowed by $\mathcal{E} = 4$ at order E^6 are therefore the choices: $V_{46} = 1$, or $V_{34} = 2$. Both of these contribute a result of order $\mathcal{A}(E) \sim E^6/(v^2\, m^4)$.

Besides showing how to use the effective theory to compute to any order in E/m, eq. (1.9) also shows the domain of approximation of the effective-theory calculation. The validity of perturbation theory within the effective theory relies only on the assumptions $E \ll 4\pi v$ and $E \ll m$. In particular, it does *not* rely on the ratio $m/4\pi v = \lambda/4\pi$ being small, even though there is a factor of this order appearing for each loop. This factor does not count loops in the effective theory because it is partially cancelled by another factor, E/m, which also comes with every loop. $\lambda/4\pi$ *does* count loops within the full theory, of course. This calculation simply shows that the small-λ approximation is only relevant for predicting the values of the effective couplings, but are irrelevant to the problem of computing the energetics of scattering amplitudes given these couplings.

1.2.3 The effective lagrangian logic

These calculations show how to calculate predictively — *including loops* — using a non-renormalizable effective theory.

Step I: Choose the accuracy desired in the answer (*e.g.* a 1% accuracy might be desired.)

Step II: Determine how many powers of E/m are required in order to achieve the desired accuracy.

Step III: Use a calculation like the one above to identify which effective couplings in $\mathcal{L}_{\text{eff}}$ can contribute to the observable of interest to the desired order in E/m. This always requires only a finite number (say: N) of terms in $\mathcal{L}_{\text{eff}}$ to any finite accuracy.

There are two alternative versions of the fourth and final step, depending on whether or not the underlying microscopic theory — like the ϕ theory in the toy model — is known.

Step IV-A: If the underlying theory is known and calculable, then compute the required coefficients of the N required effective interactions to the accuracy required. Alternatively,

Step IV-B: If the underlying theory is unknown, or is too complicated to

permit the calculation of $\mathcal{L}_{\text{eff}}$, then leave the N required coefficients as free parameters. The procedure is nevertheless predictive if more than N observables can be identified whose predictions depend only on these parameters.

The effective lagrangian is in this way seen to be predictive even though it is not renormalizable in the usual sense. Renormalizable theories are simply the special case of Step *IV-B* where one stops at zeroeth order in E/m, and so are the ones which dominate in the limit that the light and heavy scales are very widely separated. In fact, this is *why* renormalizable interactions are so important when describing Nature.

The success of the above approach is well-established in many areas outside of gravitational physics, with non-renormalizability being the signal that one is seeing the virtual effects due to some sort of heavier physics. Historically, one of earliest examples known was the non-renormalizable interactions of chiral perturbation theory which describe well the low-energy scattering of pions, kaons and nucleons. It is noteworthy that this success requires the inclusion of the loop corrections within this effective theory. The heavier physics in this case is the confining physics of the quarks and gluons from which these particles are built, and whose complicated dynamics has so far precluded calculating the effective couplings from first principles. The effective theory works so long as one restricts to center-of-mass energies smaller than roughly 1 GeV.

The Fermi (or $V - A$) theory of the weak interactions is a similar example, which describes the effects of virtual W-boson exchange at energies well below the W-boson mass, $M_W = 80$ GeV. This theory provides an efficient description of the low-energy experiments, with an effective coupling, $G_F/\sqrt{2} = g^2/8M_W^2$ which in this case is calculable in terms of the mass and coupling, g, of the W boson. In this case agreement with the precision of the measurements again requires the inclusion of loops within the effective theory.

1.3 Gravity as an effective theory

Given the previous discussion of of the toy model, it is time to return to the real application of interest for this chapter: General Relativity. The goal is to be able to describe quantitatively quantum processes in GR, and to be able to compute the size of quantum corrections to the classical processes on which the tests of GR are founded.

Historically, the main obstacle to this program has been that GR is

not renormalizable, as might be expected given that its coupling (Newton's constant), $G = (8\pi M_p^2)^{-1}$, has engineering dimension (mass)$^{-2}$ in units where $\hbar = c = 1$. But we have seen that non-renormalizable theories can be predictive in much the same way as are renormalizable ones, provided that they are interpreted as being the low-energy limit of some more fundamental microscopic theory. For gravity, this more microscopic theory is as yet unknown, although these pages contain several proposals for what it might be. Happily, as we have seen for the toy model, their effective use at low energies does not require knowledge of whatever this microscopic theory might be. In this section the goal is to identify more thoroughly what the precise form of the low-energy theory really is for gravity, as well as to identify what the scales are above which the effective theory should not be applied.

1.3.1 The effective action

For GR the low-energy fields consist of the metric itself, $g_{\mu\nu}$. Furthermore, since we do not know what the underlying, more microscopic theory is, we cannot hope to compute the effective theory from first principles. Experience with the toy model of the previous section instead suggests we should construct the most general effective Lagrangian which is built from the metric and organize it into a derivative expansion, with the terms with the fewest derivatives being expected to dominate at low energies. Furthermore we must keep only those effective interactions which are consistent with the symmetries of the problem, which for gravity we can take to be general covariance.

These considerations lead us to expect that the Einstein-Hilbert action of GR should be considered to be just one term in an expansion of the action in terms of derivatives of the metric tensor. General covariance requires this to be written in terms of powers of the curvature tensor and its covariant derivatives,

$$-\frac{\mathcal{L}_{\text{eff}}}{\sqrt{-g}} = \lambda + \frac{M_p^2}{2} R + a_1 R_{\mu\nu} R^{\mu\nu} + a_2 R^2 + a_3 R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + a_4 \Box R$$

$$+ \frac{b_1}{m^2} R^3 + \frac{b_2}{m^2} R R_{\mu\nu} R^{\mu\nu} + \frac{b_3}{m^2} R_{\mu\nu} R^{\nu\lambda} R_\lambda{}^\mu + \cdots \quad (1.10)$$

where $R^\mu{}_{\nu\lambda\rho}$ is the metric's Riemann tensor, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ is its Ricci tensor, and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, each of which involves precisely two derivatives of the metric.

The first term in eq. (1.10) is the cosmological constant, which is

dropped in what follows since observations imply λ is (for some reason) extremely small. Once this is done the leading term in the derivative expansion is the Einstein-Hilbert action whose coefficient, $M_p \sim 10^{18}$ GeV, has dimensions of squared mass, whose value defines Newton's constant. This is followed by curvature-squared terms having dimensionless effective couplings, a_i, and curvature-cubed terms with couplings inversely proportional to a mass, b_i/m^2, (not all of which are written in eq. (1.10)). Although the numerical value of M_p is known, the mass scale m appearing in the curvature-cubed (and higher) terms is not. But since it appears in the denominator it is the lowest mass scale which has been integrated out which should be expected to dominate. For this reason m is unlikely to be M_p, and one might reasonably use the electron mass, $m_e = 5 \times 10^{-4}$ GeV, or neutrino masses, $m_\nu \gtrsim 10^{-11}$ GeV, when considering applications over the distances relevant in astrophysics.

Experience with the toy model shows that not all of the interactions in the lagrangian (1.10) need contribute independently (or at all) to physical observables. For instance, for most applications we may drop total derivatives (like $\Box R$), as well as those terms which can be eliminated by performing judicious field redefinitions [2]. Since the existence of these terms does not affect the arguments about to be made, we do not bother to identify and drop these terms explicitly here.

1.3.2 Power counting

Of all of the terms in the effective action, only the Einstein-Hilbert term is familiar from applications of classical GR. Although we expect naively that this should dominate at low energies (since it involves the fewest derivatives), we now make this more precise by identifying which interactions contribute to which order in a low-energy expansion. We do so by considering the low-energy scattering of weak gravitational waves about flat space, and by repeating the power-counting exercise performed above for the toy model to keep track of how different effective couplings contribute. In this way we can see how the scales M_p and m enter into observables.

In order to perform this power counting we expand the above action flat space, trading the full metric for a canonically normalized fluctuation: $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}/M_p$. For the present purposes what is important is that the expansion of the curvature tensor (and its Ricci contractions) produces terms involving all possible powers of $h_{\mu\nu}$, with each term involving precisely two derivatives. Proceeding as before gives an

estimate for the leading energy-dependence of an L-loop contribution to the a scattering amplitude, $\mathcal{A}$, which involves $\mathcal{E}$ external lines and V_{id} vertices involving d derivatives and i attached graviton lines. (The main difference from the previous section's analysis is the appearance here of interactions involving two derivatives, coming from the Einstein-Hilbert term.)

This leads to the estimate:

$$\mathcal{A}(E) \sim m^2 M_p^2 \left(\frac{1}{M_p}\right)^{\mathcal{E}} \left(\frac{m}{4\pi M_p}\right)^{2L} \left(\frac{m^2}{M_p^2}\right)^{Z} \left(\frac{E}{m}\right)^{P} \qquad (1.11)$$

where $Z = \sum'_{id} V_{id}$ and $P = 2 + 2L + \sum'_{id}(d-2)V_{id}$. The prime on both of these sums indicates the omission of the case $d = 2$ from the sum over d. Grouping instead the terms involving powers of L and V_{ik}, eq. (1.11) becomes

$$\mathcal{A}_{\mathcal{E}}(E) \sim E^2 M_p^2 \left(\frac{1}{M_p}\right)^{\mathcal{E}} \left(\frac{E}{4\pi M_p}\right)^{2L} \prod_i \prod_{d>2} \left[\frac{E^2}{M_p^2}\left(\frac{E}{m}\right)^{(d-4)}\right]^{V_{id}} .$$
$$(1.12)$$

Notice that no negative powers of E appear here because d is even and because of the condition $d > 2$ in the product.

This last expression is the result we seek because it is what shows how to make systematic quantum predictions for graviton scattering. It does so by showing that the predictions of the full gravitational effective lagrangian (involving all powers of curvatures) can be organized into powers of E/M_p and E/m, and so we can hope to make sensible predictions provided that both of these two quantities are small. Furthermore, all of the corrections involve powers of $(E/M_p)^2$ and/or $(E/m)^2$, implying that they may be expected to be *extremely* small for any applications for which $E \ll m$. For instance, notice that even if $E/m \sim 1$ then $(E/M_p)^2 \sim 10^{-42}$ if m is taken to be the electron mass. (Notice that factors of the larger parameter E/m do not arise until curvature-cubed interactions are important, and this first occurs at subleading order in E/M_p.)

Furthermore, it shows in detail what we were in any case inclined to believe: that classical General Relativity governs the dominant low-energy dynamics of gravitational waves. This can be seen by asking which graphs are least suppressed by these small energy ratios, which turns out to be those for which $L = 0$ and $P = 2$. That is, arbitrary tree graphs constructed purely from the Einstein-Hilbert action — precisely the predictions of classical General Relativity. For instance, for

2-graviton scattering we have $\mathcal{E} = 4$, and so the above arguments predict the dominant energy-dependence to be $\mathcal{A}(E) \propto (E/M_p)^2 + \cdots$. This is borne out by explicit tree-level calculations [5] for graviton scattering, which give:

$$\mathcal{A}_{\text{tree}} = 8\pi i G \left(\frac{s^3}{tu} \right), \tag{1.13}$$

for an appropriate choice of graviton polarizations. Here $s = -(p_1 + p_2)^2$, $t = (p_1 - p_1')^2$ and $u = (p_1 - p_2')^2$ are the usual Lorentz-invariant Mandelstam variables built from the initial and final particle four momenta, all of which are proportional to E^2. This shows both that $\mathcal{A} \sim (E/M_p)^2$ to leading order, and that it is the physical, invariant centre-of-mass energy, E, which is the relevant energy for the power-counting analysis.

But the real beauty of a result like eq. (1.12) is that it also identifies which graphs give the subdominant corrections to classical GR. The leading such a correction arises in one of two ways: either (*i*) $L = 1$ and $V_{id} = 0$ for any $d \neq 2$; or (*ii*) $L = 0$, $\sum_i V_{i4} = 1$, V_{i2} is arbitrary, and all other V_{id} vanish. That is, compute the one-loop corrections using only Einstein gravity; or instead work to tree level and include precisely one vertex from one of the curvature-squared interactions in addition to any number of interactions from the Einstein-Hilbert term. Both are suppressed compared to the leading term by a factor of $(E/M_p)^2$, and the one-loop contribution carries an additional factor of $(1/4\pi)^2$. This (plus logarithmic complications due to infrared divergences) are also borne out by explicit one-loop calculations [6, 7, 8]. Although the use of curvature-squared terms potentially introduces additional effective couplings into the results,† useful predictions can nonetheless be made provided more observables are examined than there are free parameters.

Although conceptually instructive, calculating graviton scattering is at this point a purely academic exercise, and is likely to remain so until gravitational waves are eventually detected and their properties are measured in detail. In practice it is of more pressing interest to obtain these power-counting estimates for observables which are of more direct interest for precision measurements of GR, such as within the solar system. It happens that the extension to these kinds of observables is often not straightforward (and in some cases has not yet been done in a completely systematic way), because they involve non-relativistic sources

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† For graviton scattering in 4D with no matter no new couplings enter in this way because all of the curvature-squared interactions turn out to be redundant. By contrast, one new coupling turns out to arise describing a contact interaction when computing the sub-leading corrections to fields sourced by point masses.

(like planets and stars). Non-relativistic sources considerably complicate the above power-counting arguments because they introduce a new dimensionless small quantity, v^2/c^2, whose dependence is not properly captured by the simple dimensional arguments given above [9].

Nevertheless the leading corrections have been computed for some kinds of non-relativistic sources in asymptotically-flat spacetimes [10, 11]. These show that while relativistic corrections to the observables situated a distance r away from a gravitating mass M are of order GM/rc^2, the leading quantum corrections are suppressed by powers of the much smaller quantity $G\hbar/r^2c^3$. For instance, while on the surface of the Sun relativistic corrections are of order $GM_\odot/R_\odot c^2 \sim 10^{-6}$, quantum corrections are completely negligible, being of order $G\hbar/R_\odot^2 c^3 \sim 10^{-88}$. Clearly the classical approximation to GR is *extremely* good for solar-system applications.

Another important limitation to the discussion as given above is its restriction to perturbations about flat space. After all, quantum effects are also of interest for small fluctuations about other spacetimes. In particular, quantum fluctuations generated during a past epoch of cosmological inflationary expansion appear to provide a good description of the observed properties of the cosmic microwave background radiation. Similarly, phenomena like Hawking radiation rely on quantum effects near black holes, and the many foundational questions these raise have stimulated their extensive theoretical study, even though these studies may not lead in the near term to observational consequences. Both black holes and cosmology provide regimes for which detailed quantum gravitational predictions are of interest, but for which perturbations about flat space need not directly apply.

A proper power-counting of the size of quantum corrections is also possible for these kinds of spacetimes by examining perturbations about the relevant cosmological or black-hole geometry, although in these situations momentum-space techniques are often less useful. Position-space methods, like operator-product expansions, can then provide useful alternatives, although as of this writing comparatively few explicit power-counting calculations have been done using these. The interested reader is referred to the longer review, [2], for more discussion of this, as well as of related questions which arise concerning the use of effective field theories within time-dependent backgrounds and in the presence of event horizons.

1.4 Summary

General Relativity provides a detailed quantitative description of gravitational experiments in terms of a field theory which is not renormalizable. It is the purpose of the present article to underline the observation that gravity is not the only area of physics for which a non-renormalizable theory is found to provide a good description of experimental observations, and we should use this information to guide our understanding of what the limits to validity might be to its use.

The lesson from other areas of physics is clear: the success of a non-renormalizable theory points to the existence of a new short-distance scale whose physics is partially relevant to the observations of interest. What makes this problematic for understanding the theory's quantum predictions is that it is often the case that we do often not understand what the relevant new physics is, and so its effects must be parameterized in terms of numerous unknown effective couplings. How can predictions be made in such a situation?

What makes predictions possible is the observation that only comparatively few of these unknown couplings are important at low energies (or long distances), and so only a finite number of them enter into predictions at any fixed level of accuracy. Predictions remain possible so long as more observables are computed than there are parameters, but explicit progress relies on being able to identify which of the parameters enter into predictions to any given degree of precision.

In the previous pages it is shown how this identification can be made for the comparatively simple case of graviton scattering in flat space, for which case the size of the contribution from any given effective coupling can be explicitly estimated. The central tool is a power-counting estimate which tracks the power of energy which enters into any given Feynman graph, and which duplicates for GR the similar estimates which are made in other areas of physics. The result shows how General Relativity emerges is the leading contribution to an effective theory of some more fundamental picture, with its classical contributions being shown to be the dominant ones, but with computable corrections which can be explicitly evaluated in a systematic expansion to any given order in a low-energy expansion. This shows how a theory's non-renormalizability need not preclude its use for making sensible quantum predictions, provided these are made only for low energies and long distances.

This kind of picture is satisfying because it emphasizes the similarity between many of the problems which are encountered in GR and

in other areas of physics. It is also conceptually important because it provides control over the size of the theoretical errors which quantum effects would introduce into the classical calculations against which precision measurements of General Relativity are compared. These estimates show that the errors associated with ignoring quantum effects is negligible for the systems of practical interest.

There is a sense for which this success is mundane, in that it largely confirms our prejudices as to the expected size of quantum effects for macroscopic systems based purely on dimensional analysis performed by building dimensionless quantities out of the relevant parameters like G, $\hbar$, c, M and R. However the power-counting result is much more powerful: it identifies which Feynman graphs contribute at any given power of energy, and so permits the detailed calculation of observables as part of a systematic low-energy expansion.

It is certainly true the small size of quantum contributions in the solar system in no way reduces the fundamental mysteries described elsewhere in these pages that must be resolved in order to properly understand quantum gravity at fundamentally small distances. However it is important to understand that these problems are associated with small distance scales and not with large ones, since this focusses the discussion as to what is possible and what is not when entertaining modifications to GR. In particular, although it shows that we are comparatively free to modify gravity at short distances without ruining our understanding of gravitational physics within the solar system, it also shows that we are not similarly protected from *long*-distance modifications to GR.

This observation is consistent with long experience, which shows that it is notoriously difficult to modify GR at long distances in a way which does not introduce unacceptable problems such as various sorts of instabilities to the vacuum. Such vacuum-stability problems are often simply ignored in some circles on the grounds that 'quantum gravity' is not yet understood, in the hope that once it is it will somehow also fix the stability issues. However our ability to quantify the size of low-energy quantum effects in gravity shows that we need not await a more complete understanding of gravity at high energies in order to make accurate predictions at low energies. And since the vacuum is the lowest-energy state there is, we cannot expect unknown short-distance physics to be able to save us from long-distance sicknesses.

Calculability at low energies is a double-edged sword. It allows us to understand why precision comparison between GR and experiment is possible in the solar system, but it equally forces us to reject alterna-

tive theories which have low-energy problems (like instabilities) as being inadequate.

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