EL2320 Lab2

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1 Part 1

Particle Filters:

1. Particles are samples drawn from a distribution and hence are a representation of a probability density function. In the localization problem, the distribution that is sampled from is an approximation of the belief of the state $p(x_t|z_{1:t}, u_{1:t})$.

2. The target distribution is the distribution we would like to sample from. In the case of localization, the target distribution is the updated belief of the state at time t, $bel(x_t) = p(x_t|z_{1:t}, u_{1:t})$. The proposal distribution is the distribution that is actually sampled from. In the case of localization, it is the prior belief at time t, $\overline{bel}(x_t) = p(x_t|z_{1:t-1}, u_{1:t})$. The unnormalized importance weight for each particle m is the probability of the measurement given the sample state $x^{[m]}$, hence we have

$$\tilde{w}^{[m]} = p(z_t|x_t^{[m]}). \tag{1}$$

The unnormalized importance factor is the ratio of the target distribution to proposal distribution, hence we have

$$\tilde{w}^{[m]} = \frac{bel(x_t)}{\overline{bel}(x_t)} = p(z_t|x_t^{[m]}). \tag{2}$$

3. The cause of the particle deprivation is resampling when no new information has been received and hence causing the loss of valuable particles. This will lead to the density of the particles to become thin, where there is a significant probability for the actual posterior distribution.

4. Resampling forces the particles back to the approximate posterior $bel(x_t)$. If resampling is not done, more particles end up in regions with low probability and hence the predict stage will not be able to populate the likely regions with dense samples. In the end we will require more particles to accurately represent the posterior since we lose samples with high probability as time evolves.

5. When the posterior distribution is multi-modal, the average is not a good representation of the particle set. An example would be if we had four clusters at each corner of a rectangle room. The average would probably be somewhere in the middle, but this does not reflect the true distribution. A solution here is to use stratified sampling.

6. The solution here is to extract the density function from the particles. This can be done by simply fitting a Gaussian if the distribution is uni-modal. If it is multi-modal, a histogram can be

fitted to the distribution of the particles. Another option to represent multi-modal distributions is to use Kernel function at the place, where the samples are.

7. Sample variance can cause false estimates through repeated resampling, possibly culminating in particle deprivation. One remedy is to delay resampling until we are sure that new information is added. By reducing the resampling frequency, we decrease the loss of diversity in the particle population. Another remedy would be to use low variance sampling when resampling.

8.If the pose uncertainty of the robot is high, i.e. more spread of the true posteriori, then more particles are needed to accurately represent the posteriori distribution.

2 Part 2

1. Equation (6) has one dimension lower and hence the computational effort of the particle filter (which is exponential in the number of dimensions) is lower than in Equation (8). However, in Equation (6) the motion model assumes that the heading angle θ_0 is fixed and not inflicted with process noise, i.e. the input uncertainty of applying the heading angle is not modeled but rather only additive uncertainty in x- and y-direction, which is not accurate. Equation (8) models the uncertainty in the heading angle by introducing a new state which is subject to noise at each time step and is more realistic.

2. Since the heading angle ω_0 and the velocity v_0 are fixed, the motion model results in "exact" circular motions with constant radius in one direction (assuming zero noise). Hence, the two variables ω_0 and v_0 should be known in advance and fixed. The main limitation is that the mass can only drive with constant velocity with constant heading angle.

3. The constant part in Equation (10) normalizes the likelihood function to ensure it integrates to 1 over the domain of z and hence being a probability density function.

4. The multi-nomial resampling method requires M random samples for the resampling step, with M being the number of particles in the particle set. However, the systematic resampling method requires only one random number.

5. First we consider the case $w = \frac{1}{M} + \epsilon$, where $\epsilon > 0$. Let S denote the event of surviving the resampling step and S^C the counter event of not surviving. In the multi-nomial resampling case this leads to

$$P(S^C) = \left(1 - \frac{1}{M} + \epsilon\right)^M. \tag{3}$$

Hence, the probability of survival is

$$P(S) = 1 - P(S^C) = 1 - \left(1 - \frac{1}{M} + \epsilon\right)^M. \tag{4}$$

In the systematic resampling case, the probability of survival in this case is $P(S) = 1$. Since w.l.o.g we can assume $w_1 = w$ and hence for CDF(1) $> r_0$. This means in this case, there is a positive probability for the particle not to survive no matter how significant the probability is, and hence particle deprivation could occur. This is prevented in the systematic resampling case, since the probability to survive is 1. Secondly, for the case $0 \leq w < 1/M$, the probability of survival in the multi-nomial resampling case is

$$P(S) = 1 - P(S^C) = 1 - (1-w)^M, \tag{5}$$

while for the systematic resampling case the probability of survival is proportional to w, i.e. $P(s) = M \cdot w$. This means in this case, the probability of survival is proportional to w, no matter how small it is , which is not the case in the multi-nomial resampling case.

Overall, we can say that with systematic resampling we can guarantee that particles with significant importance factor are guaranteed to survive and in the case of uniform weights $w = 1/M$, no particles are lost.

6. The variable `Sigma_Q` models the measurement noise, while the variable `Sigma_R` models the process noise.

7. This is exactly the situation described in the book and literature. Since the process noise is zero, the motion model will result in the same previous state. After weighting with the importance factor, most of the particles will be lost. No matter the initial distribution, particle deprivation will occur, and eventually the whole particle set will be M copies of the same particle with high certainty, which is misleading. This is seen in Figure 1.

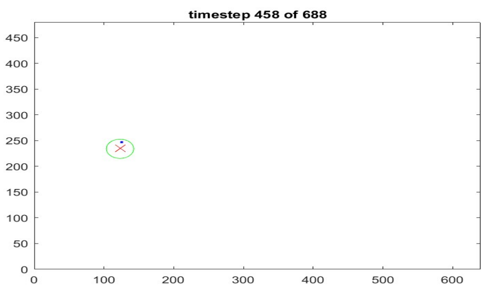

Figure 1: Experiment with fixed target and zero process noise, where all 10000 particles are reduced to a multiple copies of the same particle.

8. If resampling is not done, then the particles will remain scattered over the state space. Furthermore, since the target is not moving, at each iteration, the particles are only changed by the process noise. Hence, in the end, the particles do not converge to the true state distribution. The scattered particles are shown at the end of the simulation in Figure 2.
9. For very large measurement noise covariance, the particle set still converges to the true state distribution, however the particles are spread out around the actual location of the fixed target due to the high measurement noise. This is seen in Figure 3. The mean absolute error will hence depend on the measurement noise. On the other hand, for very small measurement noise covariance, the particles do not converge to the true state distribution and instead form clusters arbitrarily in the state space. This situation is illustrated in Figure **??**. Since, the covariance is small the unnormalized weights are small and hence are disregarded as outliers. This has been remedied in the code by applying uniform weights across all particles. Since the probability to select weights are uniform, it is probable to select the same particle more than once in each iteration. As each cluster grows, the probability of the next particle being drawn from the same cluster is bigger. The mean absolute error in this case is larger, since the particles do not converge to the true state distribution.

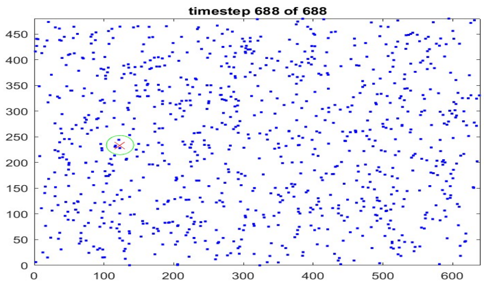

Figure 2: Experiment with fixed target no resampling step. Particles remain scattered over the state space at the end of the simulation and no convergence to the true state belief distribution occurs.

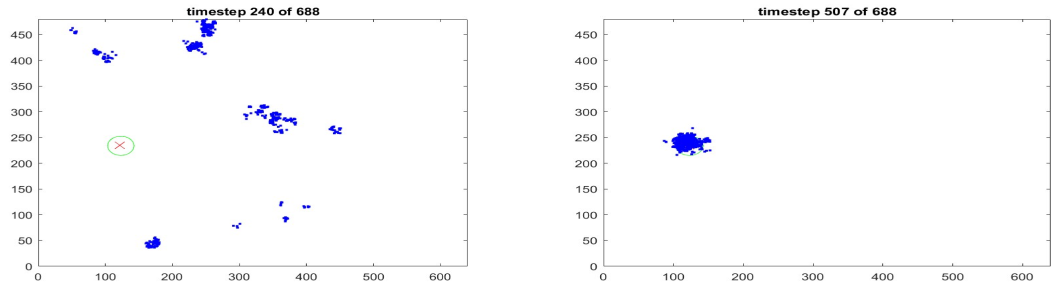

Figure 3: Experiment with fixed target and resampling step. (Left) Measurement covariance matrix is set to $Q = 0.0001 \cdot I_2$, (Right) Measurement covariance matrix is set to $Q = 10000 \cdot I_2$.

10. For small process noise, the situation is similar to the case in Question 7. Since the process noise is small and target is stationary, the particles do not move much from one iteration to the other. Eventually, some particles are weighted more resulting in higher probability for those particles and less probability for the others. Every time we resample this occurs and we end up with particle deprivation again and no guarantee for convergence towards the true state distribution. That can be seen in Figure 4. For high process noise, the particles end up moving arbitrarily around its previous location, those particles close to the actual state will end up being weighted more and will survive after resampling. What is observed is that each time resampling is done, most of the particles end up far from their previous values and hence lost while only those close to the actual state will survive. If no state is near the actual target, then the measurement is regarded as outlier and all particles are given equal weight. This is illustrated in Figure 4.

11. The process noise model compensates for motion model inaccuracies. Hence, if the motion model is accurate and gives the actual state precisely, the process noise can be kept small. If the model is inaccurate, the process noise should increase accounting for the uncertainty from the unmodeled dynamics.

12. If the motion model is accurate, then the precision of the results increase and less particles will be needed for the filter. However, if the motion model is not accurate, then motion model must compensate that with increased uncertainty and hence we need more particles to cover the

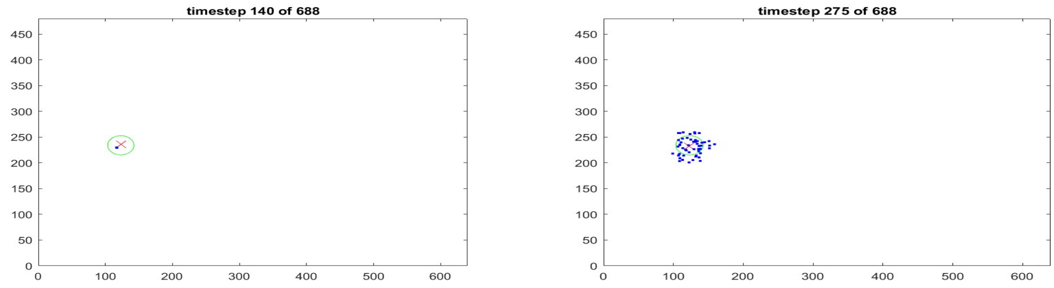

Figure 4: Experiment with fixed target and resampling step. (Left) Measurement covariance matrix is set to $R = 0.0001 \cdot I_2$, (Right) Measurement covariance matrix is set to $R = 10000 \cdot I_2$.

possible state distribution. This will result in less precise results than with an accurate motion model.

13. One possible solution for this is already implemented in the code. For the third dataset 50 % of the measurements are outliers. Hence, at a given stage we can compute the unnormalized importance factors, i.e. the probability of the measurements given the set of particles. This corresponds to

$$\tilde{w}_k^{(i)} = p(z|x_k^{(i)}). \tag{6}$$

If the average over the whole particle set is below a threshold value, we can disregard this measurement as an outlier.

14. The best parameters along with their corresponding mean absolute error are shown in Table 1. For the fixed motion case, the filter is very sensitive to the parameters Q and R. The sensitivity on the tuning parameters decreases in the case of linear motion and more so for the circular motion.

Table 1: Mean absolute error and the tuning parameters for the fixed, linear and circular motion.

	Fixed	Linear	Circular
Q	diag(20,20)	diag(30,30)	diag(5,5)
R	diag(1,1,0.1)	diag(50,50,0.1)	diag(100,100,1)
MAE	0.5 ± 0.3	1.1 ± 0.5	1.2 ± 0.5

15. The threshold on the average likelihood of the particles λ_Ψ affects the outlier detection. If the threshold is increased, then more measurements are regarded as outliers since the average of the likelihood measurements will have to exceed the increased threshold. For the case it is decreased, the inverse applies. Furthermore, the measurement noise affects the outlier detection as well, since it affects the measurement likelihood. If the uncertainty is high, then the likelihood is more forgiving towards imprecise measurements and hence less outliers are detected. For the case that Q goes towards zero, then the measurements should be very precise in order to not be assigned a low likelihood. This will lead to a low average likelihood for all the particles and hence more outliers being deteced.

16. If outliers are not detected, then the measurement likelihood will give low probability to states that are close to the true state and will give high probability to states that are far from the true

state (those explaining the measurements). This will result in the particle filter giving false estimates since the particle set does not approximate the posteriori distribution of the state.

3 Experiments:

3.1 Dataset 4:

First, tracking is performed using the default parameters and a particle size of 1000. The evolution of the error and the covariance matrix is shown for the duration of the simulation in Figure 5.

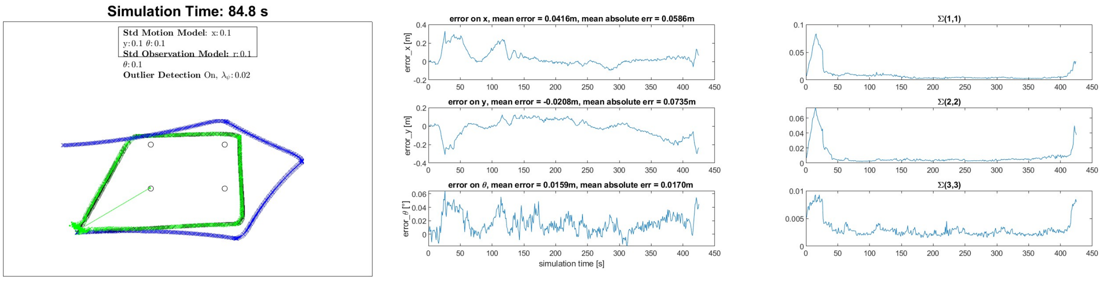

Figure 5: Particle Filter with a particle size of 1000 performing tracking. (Left) The simulation at the end time with the correct state estimate in green and the ground truth. (Middle) The evolution of the error over the simulation data. (Right) The evolution of the covariance matrix.

Then, globalization is performed with two particle sizes, once with 1000 particles and with 5000 particles, again using the default parameters. In the case of 1000 particles, the particles converge to an incorrect estimate. This is shown in the left plot in Figure 6, where the particles (in green) converge to the false state estimate. If the particle size is increased, the particles converge to the true estimate as seen in the right plot of Figure 6. The middle plot shows the multiple hypothesis of the particles right before converging to the true estimate.

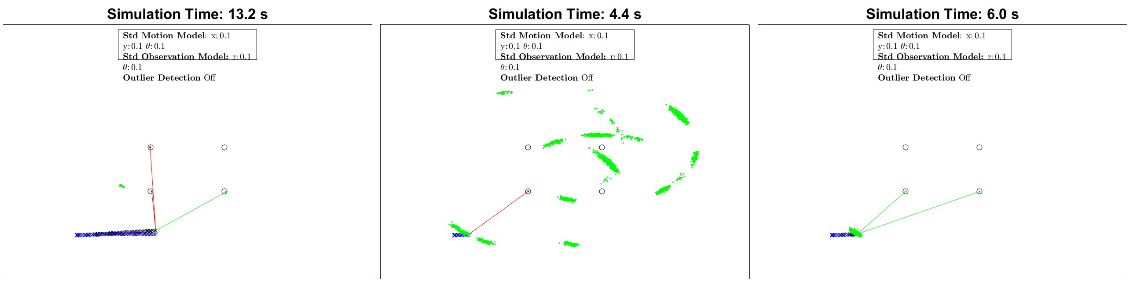

Figure 6: Particle Filter with particle sizes of 1000 and 5000 performing global localization. (Left) Particle size is 1000 and convergence failed. (Middle) Particle size is 5000 with multiple hypotheses before convergence. (Right) Particle size 5000 after convergence of the particles.

The evolution of the error and the covariance matrix for the case with 5000 particles is shown for the duration of the simulation in Figure 7.

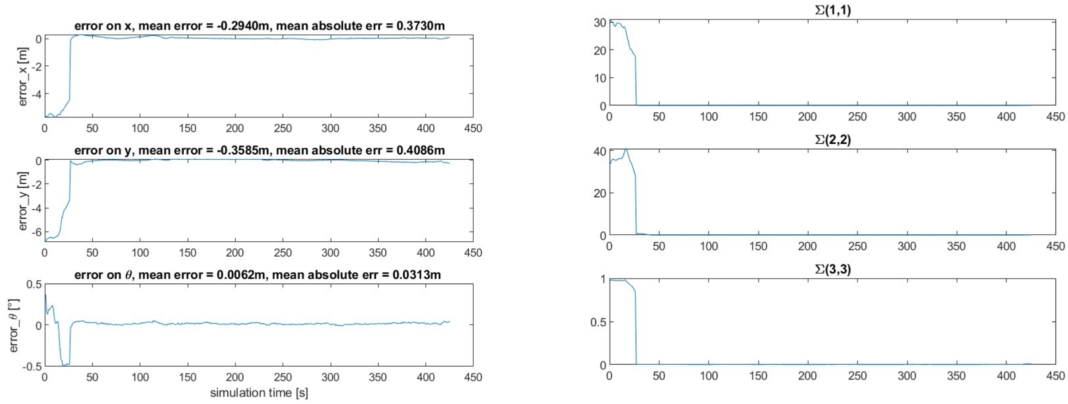

Figure 7: Particle Filter with a particle size of 5000 performing tracking. (Left) The evolution of the error over the simulation time. (Right) The evolution of the covariance matrix over simulation time.

With increased particle size, the particles can cover the state space well and therefore the global localization succeeds with increased particle sizes. Multinomial resampling performed worse than systematic resampling. This verifies the previous considerations, since valuable particles can still be lost due to sample variance, which could lead to particle deprivation. This phenomena is reduced when using systematic resampling.

If stronger model noises are used, then convergence is slowed down and the filter keeps track of multiple hypothesis for longer. This can be good for this dataset, since the landscape is symmetrical and quick convergence could lead to incorrect estimates. On the other hand, if weaker measurement noises, then convergence is fast and thus, the risk of incorrect estimates.

3.2 Dataset 5:

The filter is run with a particle size of 10000 and the default parameters. Figure **??** shows the particles before and after symmetry is broken and the convergence to the true state estimate.

The evolution of the error and the covariance matrix for the case with 10000 particles is shown for the duration of the simulation in Figure 9.

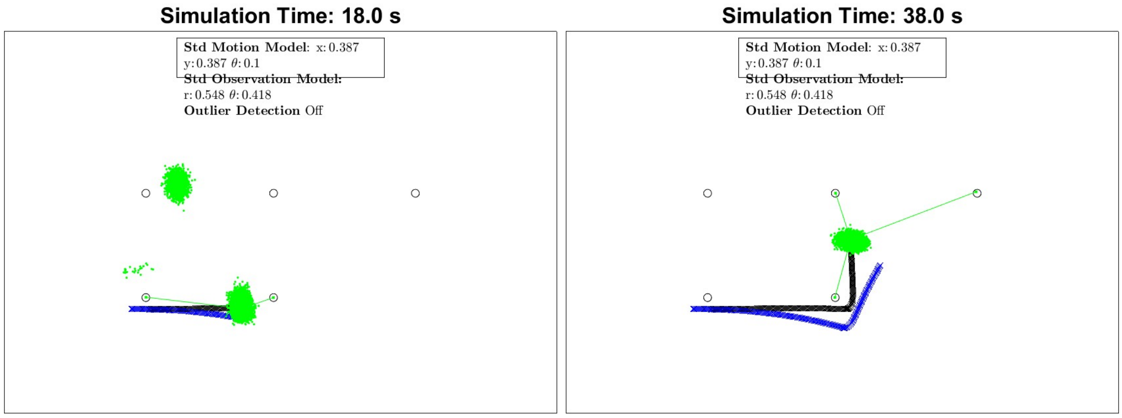

Figure 8: Particle Filter with particle size of 10000 performing global localization. (Left) Due to the symmetry in the map, there are two clusters (hypotheses). (Right)After observing the asymmetrical landscape the particles converge to the true state estimate.

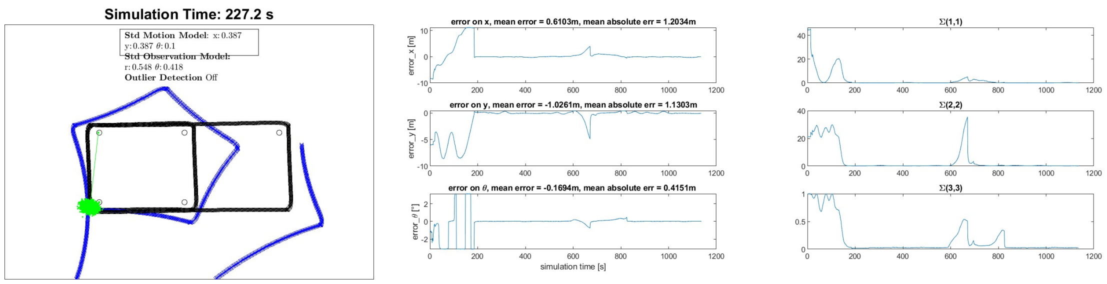

Figure 9: Particle Filter with particle size of 1000 performing global localization. (Left) State estimate at the end of simulation. (Middle) Evolution of error over simulation time. (Right) Evolution of covariance matrix over simulation time.