Mathematics of AI

Deep Learning for Applied Mathematicians

Higham & Higham (2019)

This section

Deep learning and related topics

- Week 6: Deep Learning for Applied Mathematicians

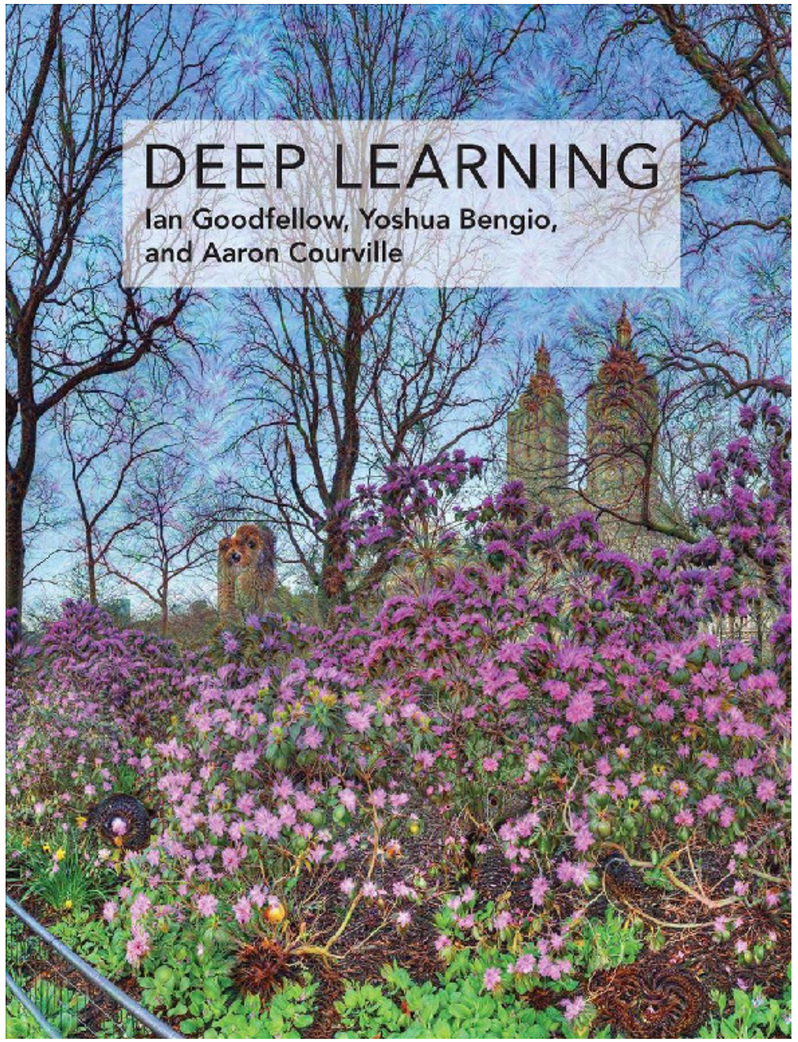
- Week 7: Further topics in optimisation
  - SGD and variants
  - Batch and minibatch algorithms
  - Practical issues (+ PyTorch implementation)

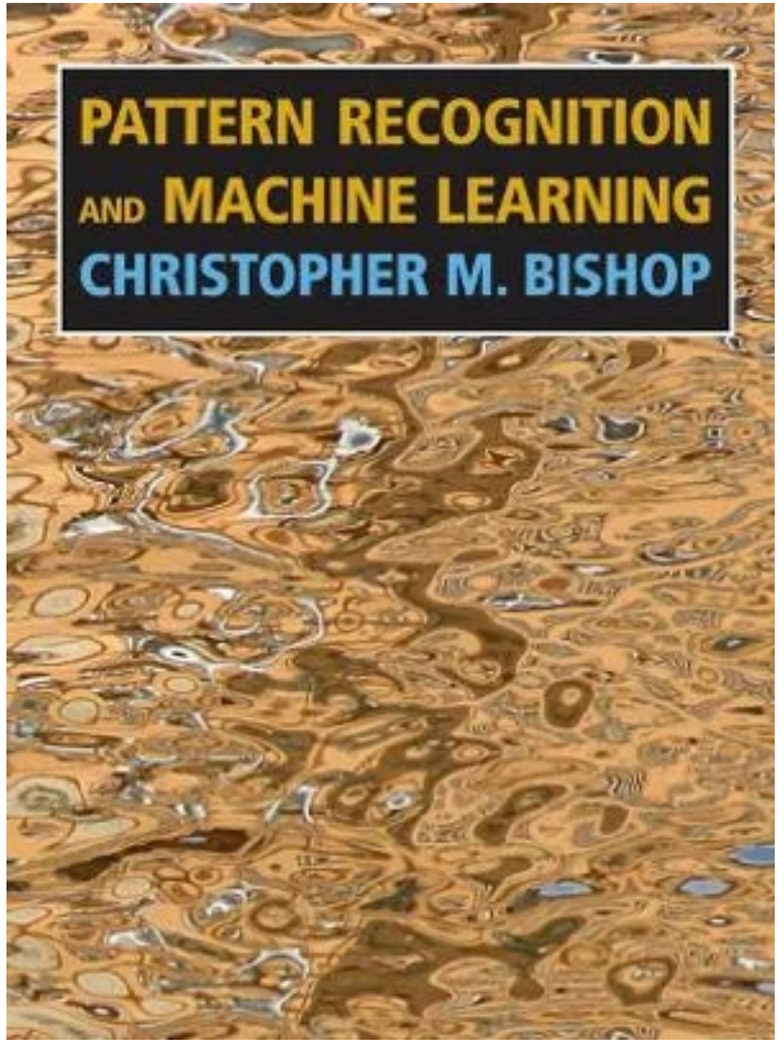
- Week 8: Regularisation
  - In “traditional” learning
  - In deep learning

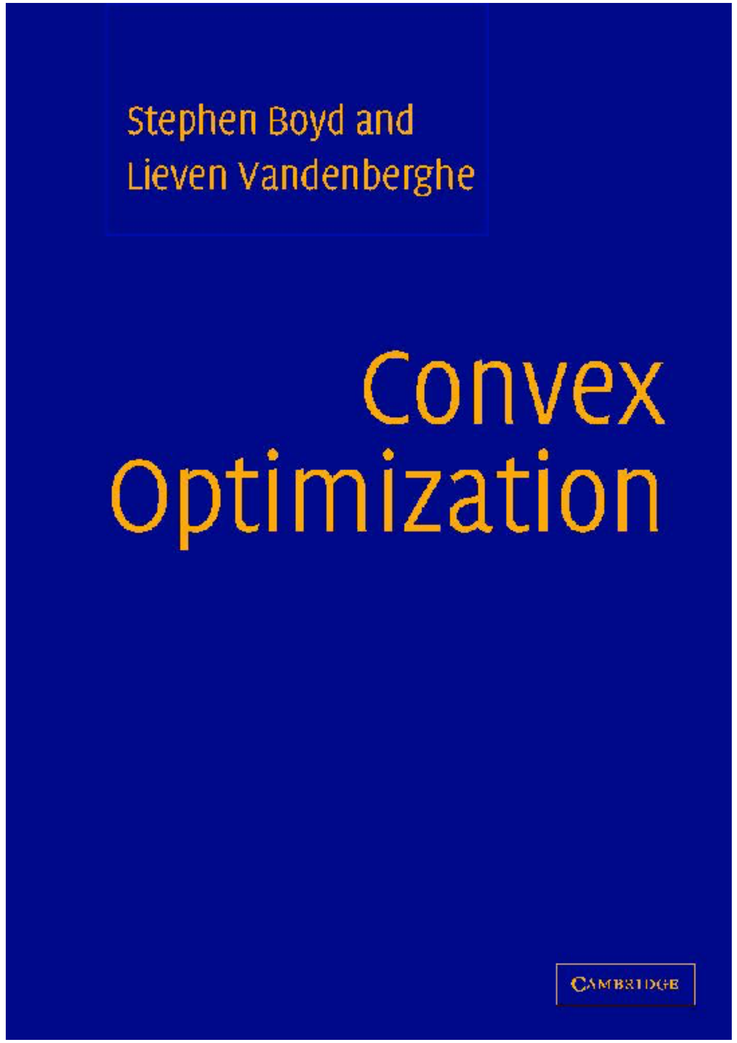
- Week 9: Compressive sensing redux
  - Regularisation + compression!

Resources

My favourite books for this section (all in Course Readings)







Mathematics > History and Overview

[Submitted on 17 Jan 2018]

Deep Learning: An Introduction for Applied Mathematicians

Catherine F. Higham, Desmond J. Higham

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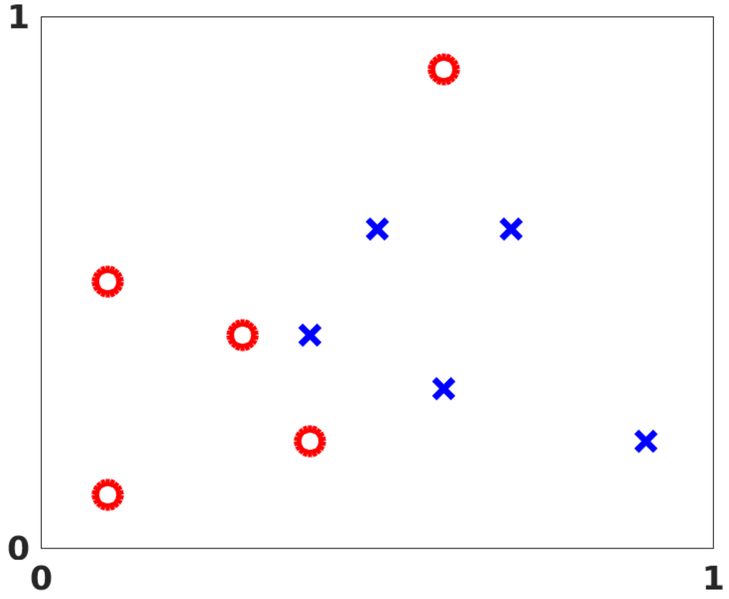

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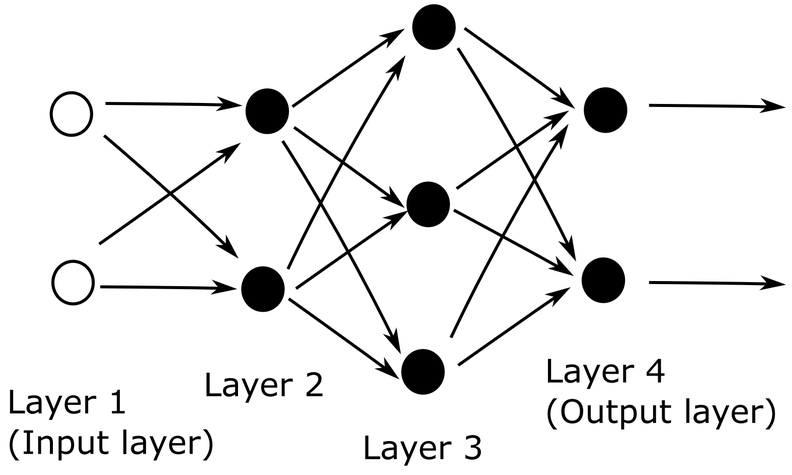

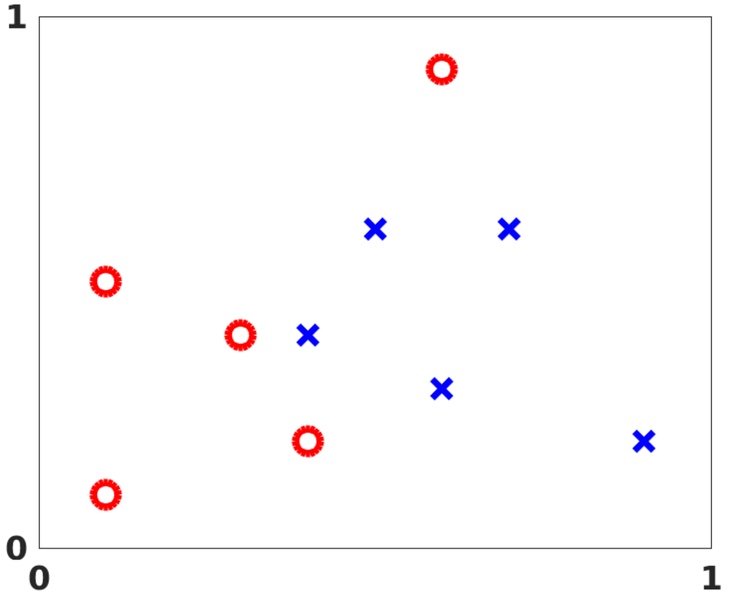

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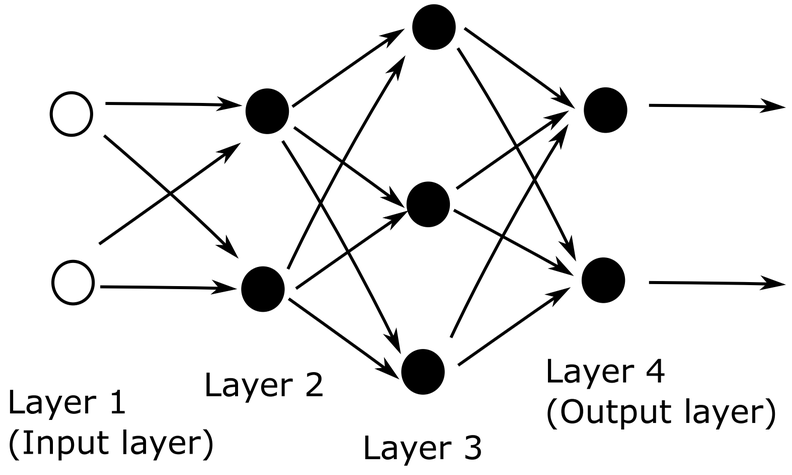
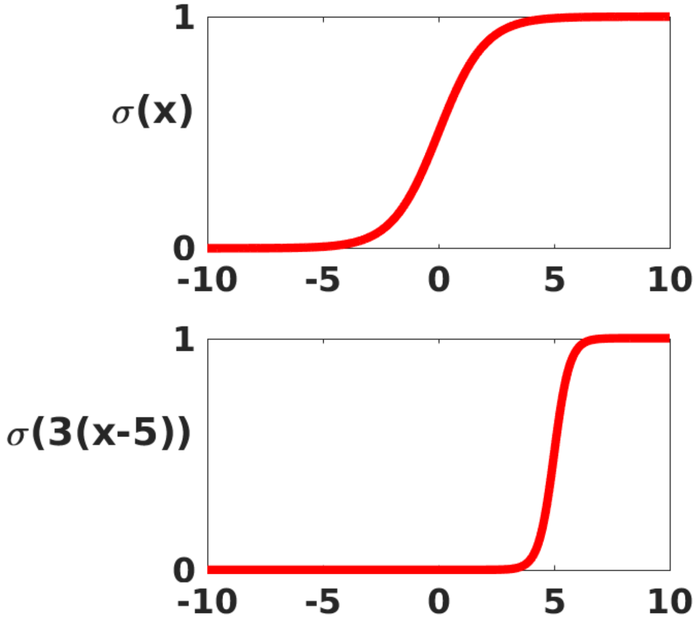


$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



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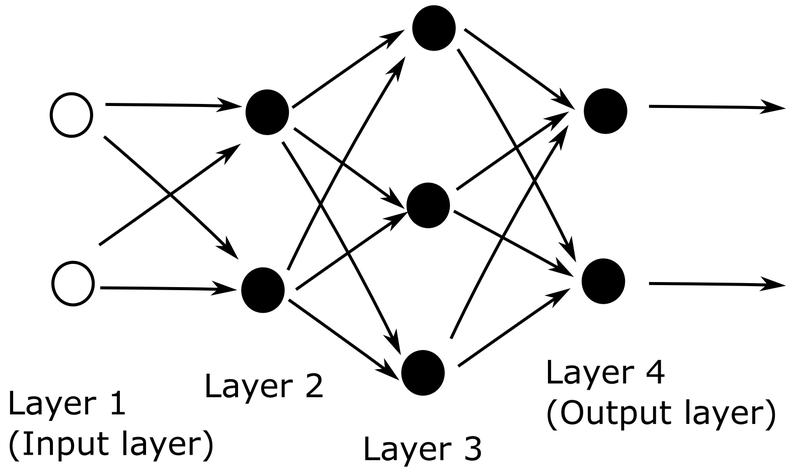
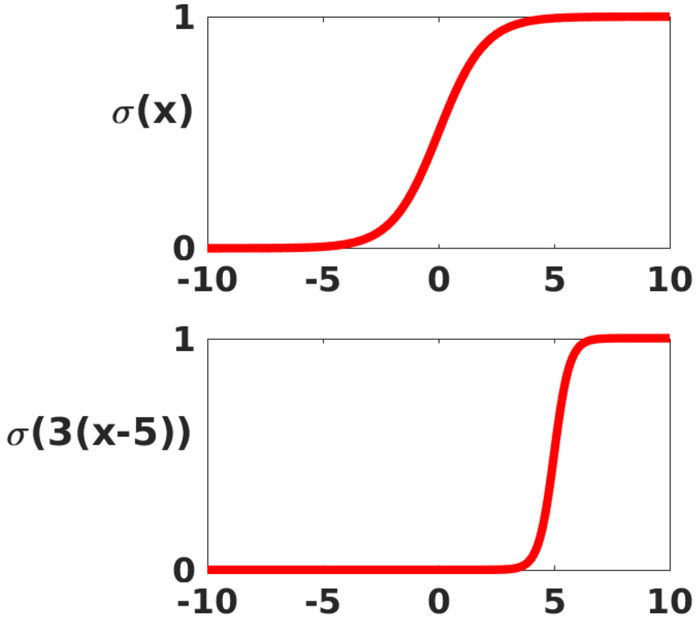


$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\sigma\left(\sum_j w_{ij}a_j + b_i\right)$$



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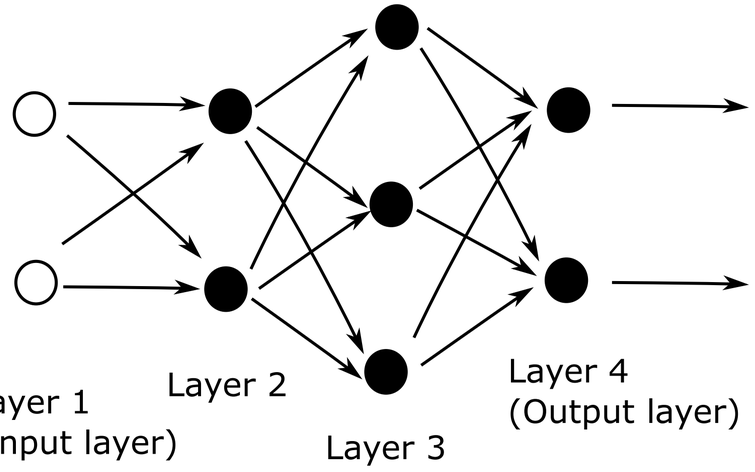

$$F(x) = \sigma\left(W^{[4]}\sigma\left(W^{[3]}\sigma(W^{[2]}x + b^{[2]}) + b^{[3]}\right) + b^{[4]}\right) \in \mathbb{R}^2.$$

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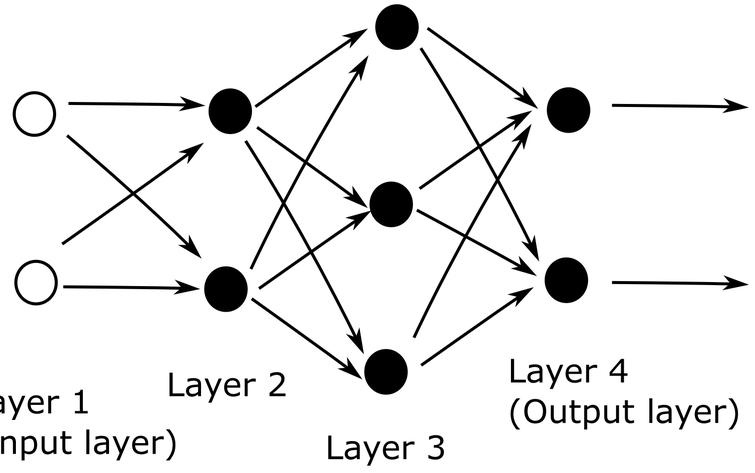

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$$\text{Cost}\left(W^{[2]}, W^{[3]}, W^{[4]}, b^{[2]}, b^{[3]}, b^{[4]}\right) = \frac{1}{10}\sum_{i=1}^{10} \frac{1}{2}\|y(x^{\{i\}}) - F(x^{\{i\}})\|_2^2.$$

https://arxiv.org/abs/1801.05894

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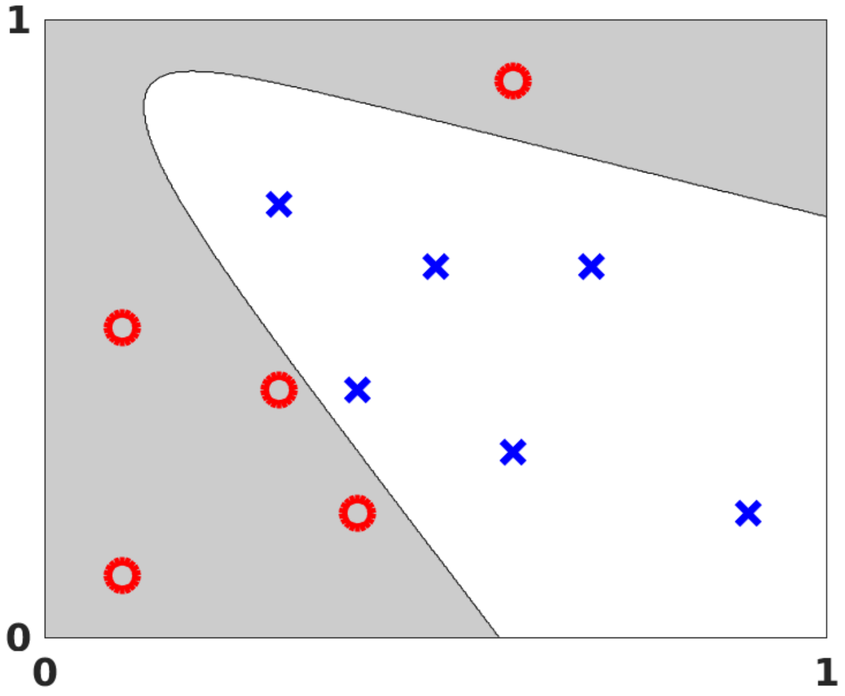

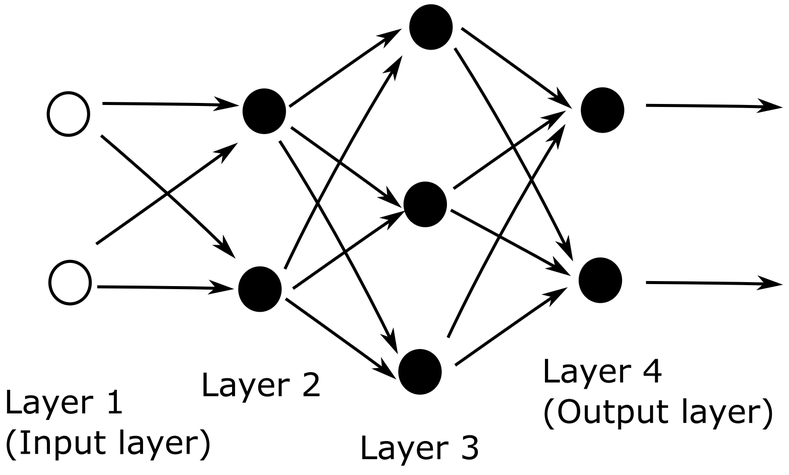

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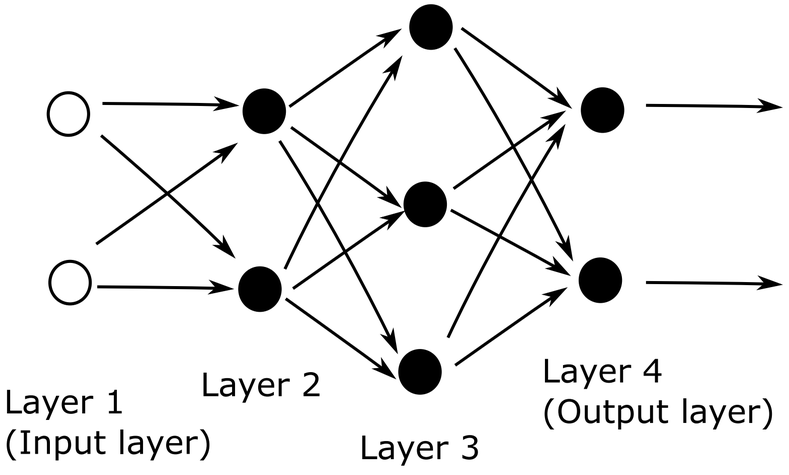
How many parameters?



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Deep Feedforward Networks

Lecture slides for Chapter 6 of *Deep Learning*
www.deeplearningbook.org
Ian Goodfellow
Last updated 2016-10-04

Roadmap

- Example: Learning XOR
- Gradient-Based Learning
- Hidden Units
- Architecture Design
- Back-Propagation

XOR is not linearly separable

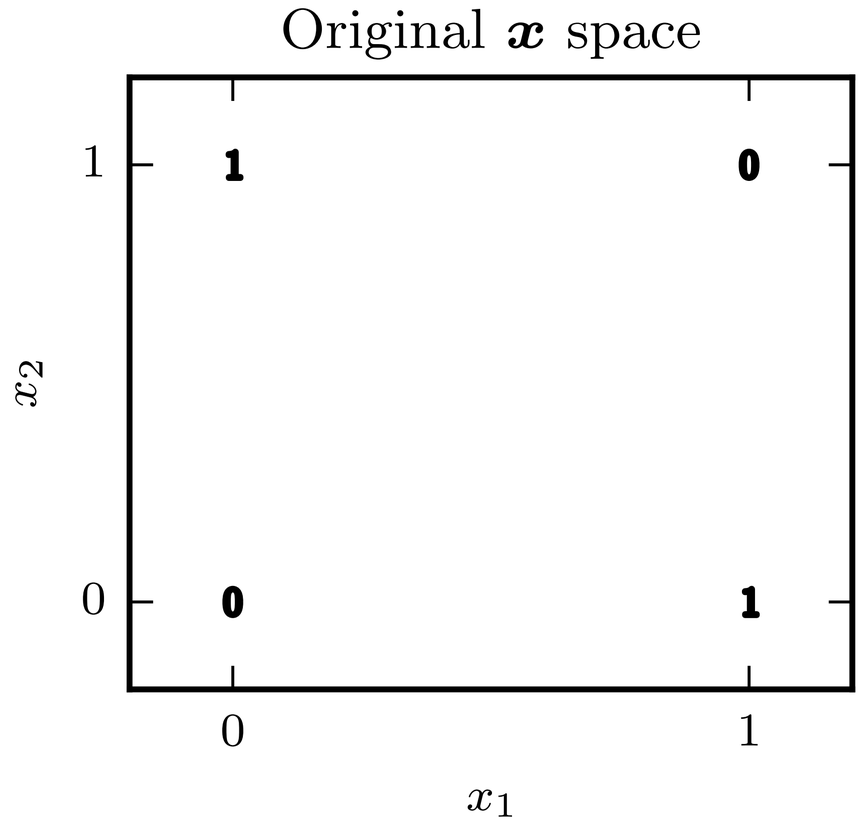

Figure 6.1, left

Rectified Linear Activation

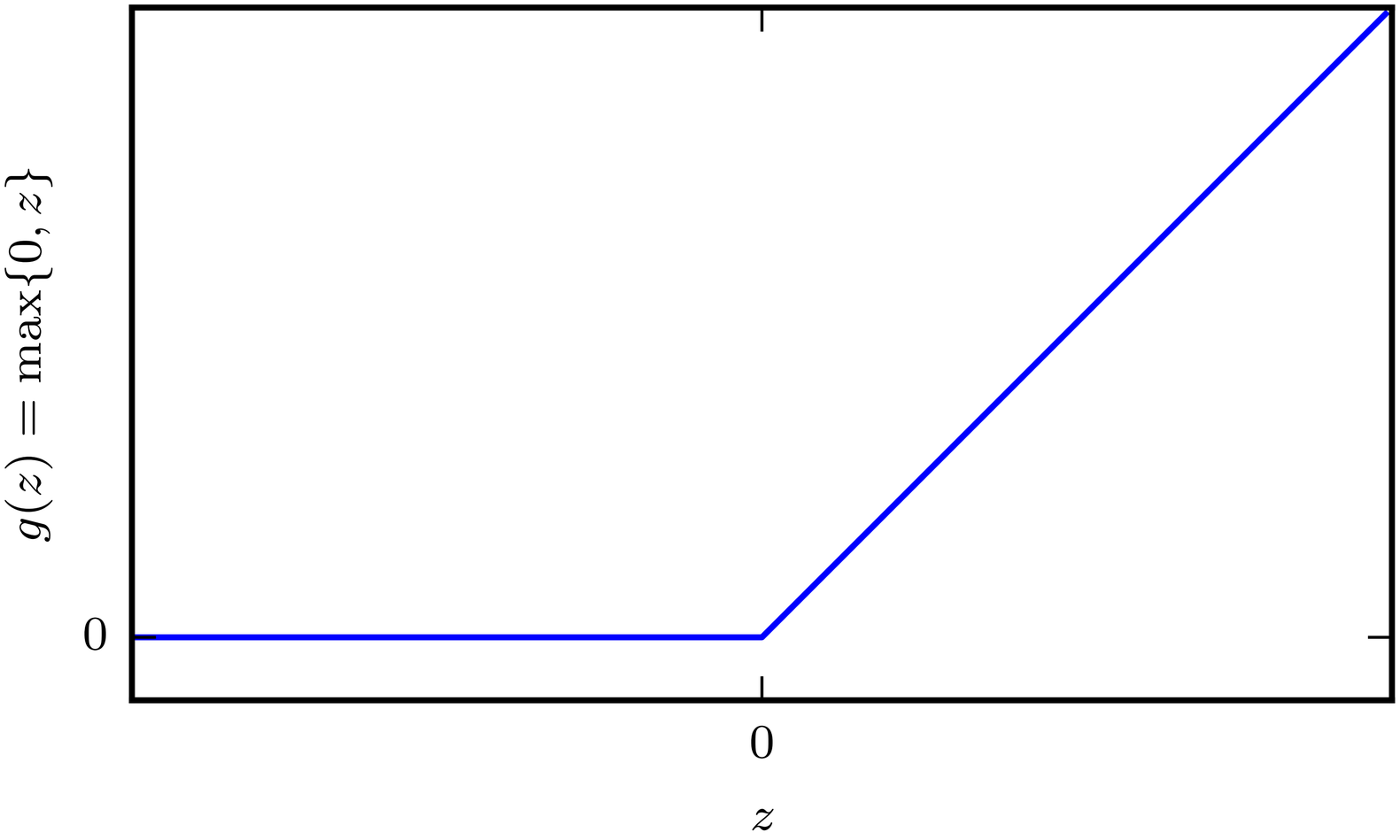

Figure 6.3

Network Diagrams

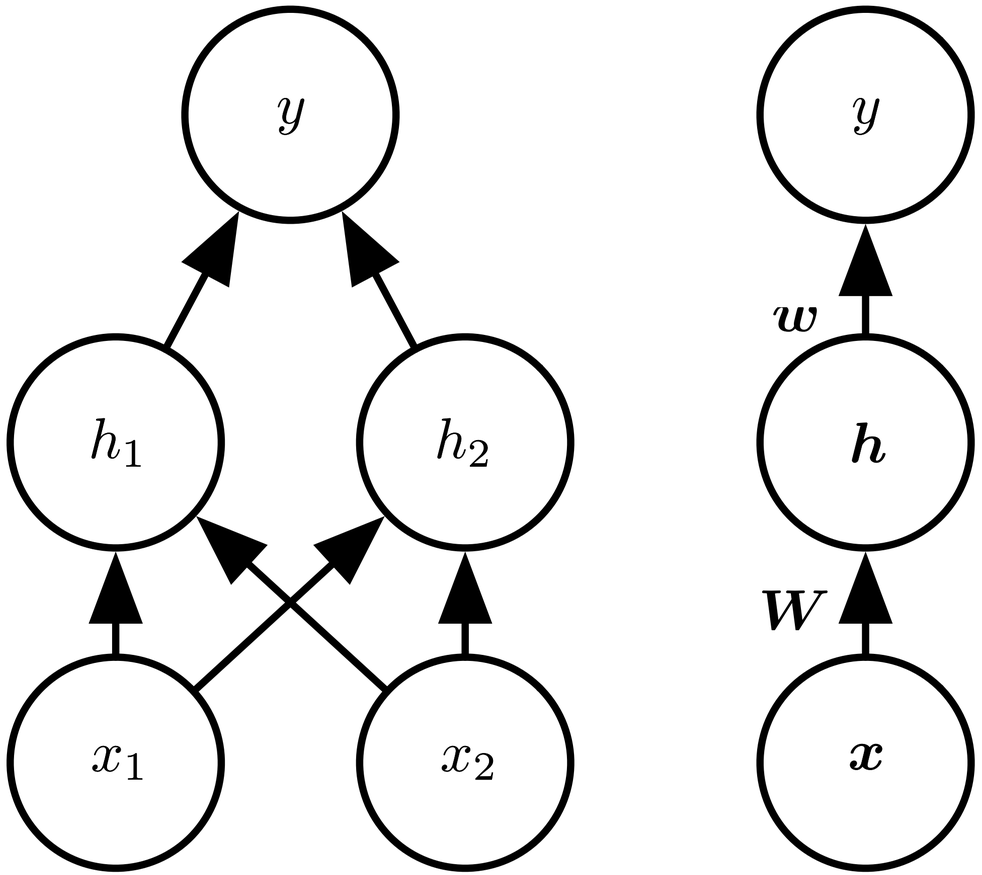

Figure 6.2

Solving XOR

$$f(\boldsymbol{x}; \boldsymbol{W}, \boldsymbol{c}, \boldsymbol{w}, b) = \boldsymbol{w}^\top \max\{0, \boldsymbol{W}^\top \boldsymbol{x} + \boldsymbol{c}\} + b. \tag{6.3}$$

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{6.4}$$

$$\boldsymbol{c} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{6.5}$$

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \tag{6.6}$$

$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise: transform **X** by hand.

Solving XOR

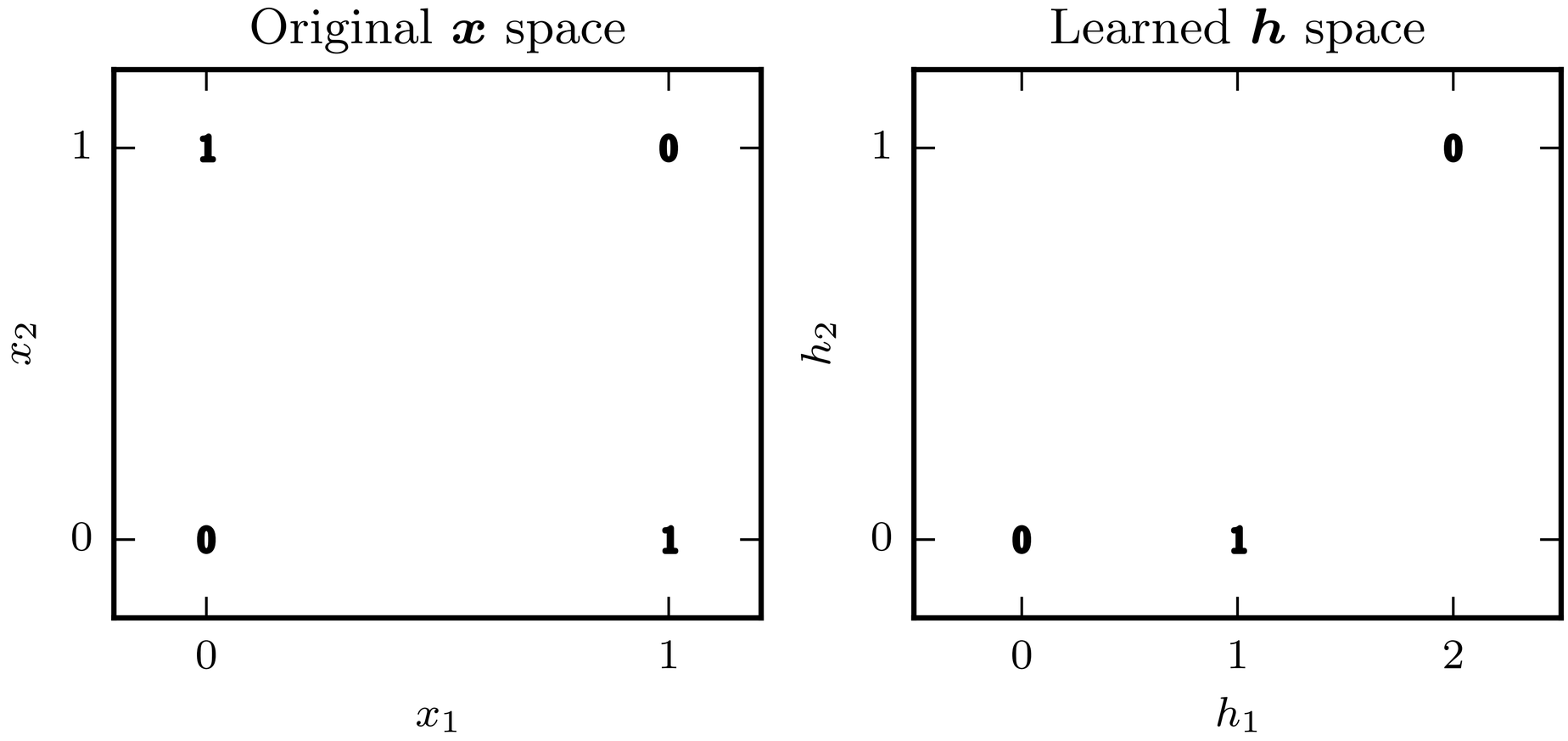

Figure 6.1

Comments/questions

- Note the transformation: stretched into a “larger” space, but low-dimensional. (This is a theme!)
- This is a global minimum: why?
- Are other transformations possible? How many? What does the cost function surface look like?
- Train a neural network to do this: MLWM workshop 1 (in PyTorch!)

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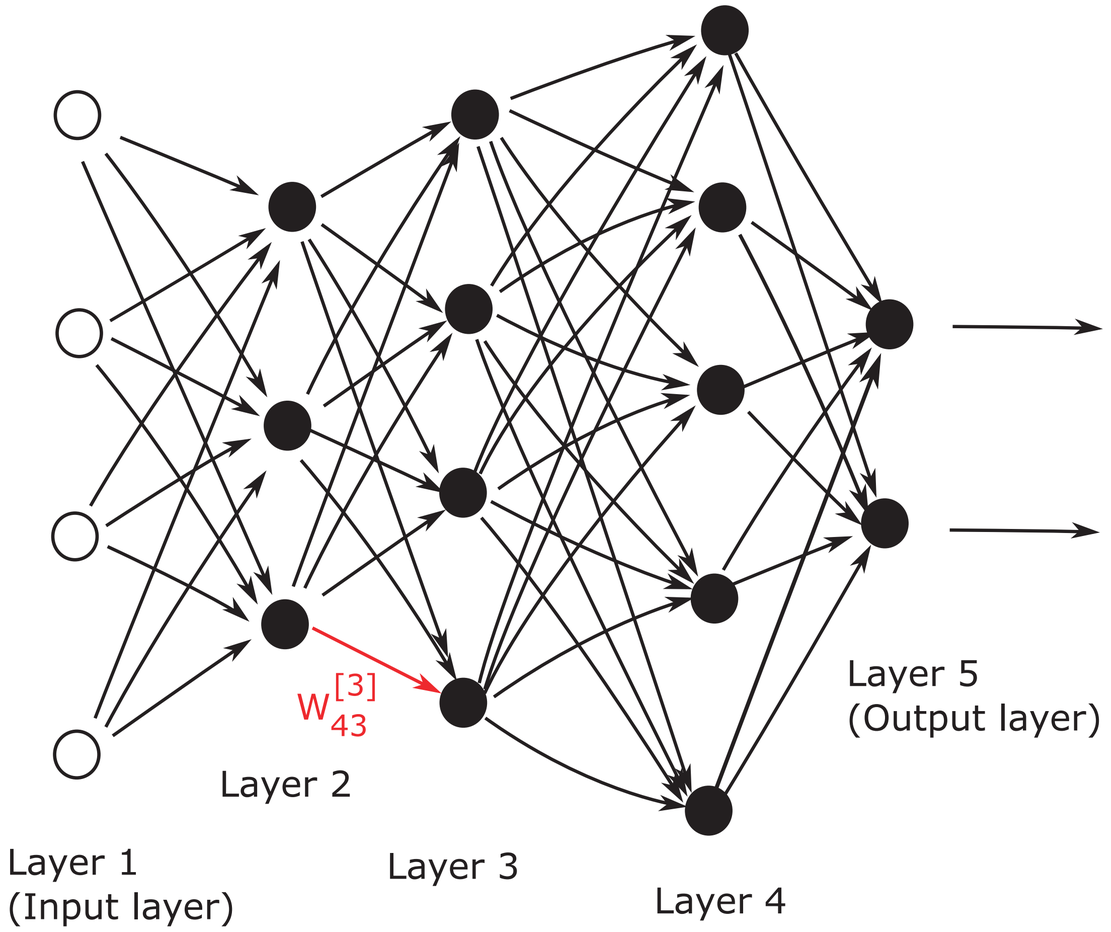

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Stochastic Gradient Descent: 2 approaches

$$p \to p - \eta \nabla \mathrm{Cost}(p). \qquad \text{where} \qquad \nabla \mathrm{Cost}(p) = \frac{1}{N} \sum_{i=1}^{N} \nabla C_{x\{i\}}(p). \qquad \text{Too expensive!}$$

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Approach 1: (epochs)

1. Shuffle the integers $\{1, 2, 3, \ldots, N\}$ into a new order, $\{k_1, k_2, k_3, \ldots, k_N\}$.
2. For $i = 1$ upto N, update

$$(4.7) \qquad p \to p - \eta \nabla C_{x\{k_i\}}(p).$$

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2. For $i = 1$ upto N, update

$$p \to p - \eta \nabla C_{x\{k_i\}}(p). \tag{4.7}$$

Approach 2: (minibatch)

1. Choose m integers, $k_1, k_2, \ldots, k_m$, uniformly at random from $\{1, 2, 3, \ldots, N\}$.
2. Update

$$p \to p - \eta \frac{1}{m} \sum_{i=1}^{m} \nabla C_{x\{k_i\}}(p). \tag{4.8}$$

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Calculating the gradient: Back propagation

The important definitions:

$$C = \tfrac{1}{2}\|y - a^{[L]}\|_2^2. \tag{5.1}$$

$$z^{[l]} = W^{[l]}a^{[l-1]} + b^{[l]} \in \mathbb{R}^{n_l} \qquad \text{for } l = 2, 3, \ldots, L. \tag{5.2}$$

$$a^{[l]} = \sigma\left(z^{[l]}\right) \qquad \text{for } l = 2, 3, \ldots, L. \tag{5.3}$$

$$\delta_j^{[l]} = \frac{\partial C}{\partial z_j^{[l]}} \qquad \text{for } 1 \le j \le n_l \quad \text{and} \quad 2 \le l \le L. \tag{5.4}$$

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LEMMA 5.1. *We have*

$$\delta^{[L]} = \sigma'(z^{[L]}) \circ (a^L - y), \tag{5.5}$$

$$\delta^{[l]} = \sigma'(z^{[l]}) \circ (W^{[l+1]})^T \delta^{[l+1]} \tag{5.6}$$

$$\frac{\partial C}{\partial b_j^{[l]}} = \delta_j^{[l]} \tag{5.7}$$

$$\frac{\partial C}{\partial w_{jk}^{[l]}} = \delta_j^{[l]} a_k^{[l-1]} \tag{5.8}$$

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Deep Mysteries of Deep Learning

- Why does it converge at all?
- Why does it work well?
  - non-convex optimisation
  - many local minima
  - generalises to unseen data
- How does regularisation happen?
- Why do high dimensions seem to help?
  - segue into project…