Assignment 0:

This assignment contains revision questions based on what you should have learned from past work and the reading for week 1. Do not hand this in – self-assess your work.

1. Prove that the intersection of two convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ is also a convex set.

2. Show whether $f(x) = |x|$ is a convex function or not.

3. (a) What is the definition of linear independence?

   (b) If you have a set of $n+1$ vectors in R^n, can you conclude in general whether those vectors linearly dependent or linearly independent and why?

4. Show that MM^T is a symmetric matrix, for **all** matrices M.

5. Write the following system in augmented matrix form.

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 2 \\ x & & & + & z & = & 0 \\ 2x & + & y & + & 3z & = & 2 \end{array}$$

6. Which of the following are linear equations in x_1, x_2 and x_3?
   (a) $x_1 + 4x_2x_3 + 5x_3 = 1$, (b) $3x_1 - x_2 + x_3 = 0$, (c) $x_1 \leq x_2$, (d) $x_1 - x_2 + \sqrt{x_3} = 6$

7. (a) What does it mean when two systems of linear equations are equivalent?

   (b) Find the solution sets for the following two systems. Are they are equivalent?

   (i) $$\begin{array}{rcl} 2x + 2y & = & 12 \\ x - z & = & 1 \\ y & = & 2 \end{array}$$

   (ii) $$\begin{array}{rcr} x + 3z & = & 13 \\ -x - 2z & = & -10 \\ x + y - z & = & 3 \end{array}$$

8. In each case, find a matrix whose product with A_1 transforms the matrix into A_2.

   (a)

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 3 & 7 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

   (b)

$$A_1 = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 3 & 9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 2 & 2 \end{bmatrix}.$$

9. What is the rank of the following matrices.

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 9/2 & 0 \\ 0 & 1 & 0 & -17/4 & 0 \\ 0 & 0 & 1 & -7/12 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{(b)} \begin{bmatrix} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 10 \end{bmatrix} \text{(c)} \begin{bmatrix} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & 11 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix} \text{(d)} \begin{bmatrix} 1 & 0 & 5 & 8 & 5 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. Consider the following system of equations $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_2 - 8x_3 &= 20 \\ -3x_1 + 12x_2 - 3x_3 &= -36 \\ 2x_1 - 8x_2 - 6x_3 &= 14 \end{aligned}$$

(a) Use Gauss-Jordan elimination on the augmented matrix $[A\, I\, \mathbf{b}]$.

(b) Writing the final matrix as $[A'\, W\, \mathbf{b}']$ (where A reduces to A', and so on), what is A'?

(c) Verify that $WA = A'$ and $W\mathbf{b} = \mathbf{b}'$. What is W?

(d) Solve the system of equations.

11. Sketch the following sets in $\mathbf{R}^2$, showing all the vertices.

(a) $\{(x_1, x_2) \mid 2x_1 + 3x_2 \leq 9,\ 2x_1 - x_2 \geq 2,\ x_1 \geq 0,\ x_2 \geq 0\}$

(b) $\{(x_1, x_2) \mid 2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \leq 4,\ x_1 \geq 0,\ x_2 \geq 0\}$

12. Let A be an $n \times n$ matrix.

(a) Define what it means for $\mathbf{x} \in \mathbb{R}$ to be an eigenvector of A with eigenvalue λ.

(b) Define the characteristic polynomial of A.

(c) Find all of the eigenvalues and eigenvectors of the following matrices.

$$\text{(i)} \quad A_1 = \begin{bmatrix} 0 & 3 \\ 6 & -3 \end{bmatrix} \qquad \text{(ii)} \quad A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

You can use computer code to do your calculations.

13. Let A be an $n \times n$ matrix of real or complex numbers. Which of the following statements are equivalent to: "the matrix A is invertible"?

(a) The columns of A are linearly independent.

(b) The columns of A span R^n.

(c) The rows of A are linearly independent.

(d) The kernel of A is 0.

(e) The only solution of the homogeneous equations $Ax = 0$ is $x = 0$.

(f) The linear transformation $T_A : R^n ßR^n$ defined by A is 1-1.

(g) The linear transformation $T_A : R^n ßR^n$ defined by A is onto.

(h) The rank of A is n

(i) $\det A \neq 0$

14. Is every real upper triangular $n \times n$ matrix A with $A_{ii} = 1$, for $i = 1, 2, \ldots, n$, invertible? Provide a proof or counterexample.

15. Let A, B, and C be any three $n \times n$ matrices.

(a) Show that trace(ABC) = trace(CAB) = trace(BCA).

(b) Is trace(ABC) = trace(BAC). Provide a proof or counterexample.

16. Use linear algebra to derive that the nth Fibonacci number F_n is given by

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$