Vector spaces

A Note on Representation

One of the things I used to find confusing about a vector space like $\mathbb{R}^n$ is that the objects that make up the space, the “vectors”, and the basis for the space, and the representation of the objects in terms of the basis are all just arrays of numbers.

The examples I find a more informative example are function spaces. The space L_2, for example, is the space of *square-integrable* function (equipped with the L_2 inner product and norm). There are some technical details of such a space, but we can do all the usual vector space stuff, *e.g.,* add functions or multiply them by scalar.

If we constrain the function space we are working with to only look at periodic functions, then we have as a basis the sinusoids, $\sin{(nx)}$ and $\cos{(nx)}$. Thus we can write a representation for the functions in these spaces as

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \sin{(nx)} + \sum_{n=1}^{\infty} b_n \cos{(nx)}$,

or we sometimes use complex numbers instead of two sets of coefficients, *e.g.,*

$f(x) = \sum_n z_n \exp{(inx)}$.

Then we can represent the function as $(z_0, z_1, z_2, \ldots)$. A one level this just the Fourier series of the function. At another we have just created a representation with a countable number of elements for something that looked uncountable, which seems a neat trick, but also makes the two “things” more distinct.

The other part of this that helps (me) understand is that the coefficients here, the a_n for instance, can be obtained using the inner products (the same way you do in $\mathbb{R}^2$), *e.g.,* using the L_2 inner product modified for the periodic space with period 2π we get

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos{(nx)} dx$.

PS The Fourier series is not the Fourier transform, which we will look at later.

So a vector space is a pretty general concept, and once you start think of the generalisations it actually becomes easier to think about than when we were just stuck in $\mathbb{R}^n$.

A Weird Vector Space

Let’s start this bit with an experiment. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}.$$

Pick a exactly number from each row/column and add them. That is pick five numbers no two of which share a row or a column. I will psychically guess you answer.

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How did I know, well, everyone should have the same answer because it doesn't matter which numbers you chose (as long as you followed the rules).

Call a set of locations (with unique row and column) in a matrix a *transveral*, and a matrix with the property that its transversals are all the same a *constant transversal matrix*.

Then here's an idea. Let's assume that we are working in a space called V_n of $n \times n$ constant transversal matrices. Then the zero matrix is in V_n, and if I take two matrices $A, B \in V_n$ then $A + B \in V_n$ and $\alpha A \in V_n$ so the space V_n is closed under linear combinations. So V_n is a vector space.

If V_n is a vector space, we should be able to find a basis for it. So let's go. PIck a number n – let's go with 3 to make it easier – and try to find a basis.

Questions:

- What is the maximum number of basis "vectors"?
- Do they all have completely different values like the example?
- What is the simplest (non-zero) matrix that has this property?
- How many vectors are there in the basis for V_n?
- Can you come up with a inner product and/or norm for this space?

Links

- https://cns.gatech.edu/~predrag/courses/PHYS-6124-12/StGoChap2.pdf
- https://mathworld.wolfram.com/FourierSeries.html
- A Tricky Linear Algebra Example, David Sprows, The College Mathematics Journal, Vol.39, No.1, 2008, pp.54-56, https://www.maa.org/sites/default/files/Sprows.pdf