Eigenvalues and vectors

Intro

The definition of an eigenvalue and vector of a matrix A is simple: it is a pair λ and $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda v.$$

But that isn't very intuitive. And the name doesn't help.

All the theory should be at your fingertips, so I won't do that again.

We can talk about why they are important, but that will come up naturally when we start talking about SVD and PCA. Google's pagerank algorithm is another good example that links them up to Markov chains. They are a huge part of linear systems theory, with applications all over the place as well.

What I thought I would try to do is come up with a strange take on them that gives a different view.

1. There is a kind of algebra of eigenvalues: if A has the eigenvalues $\lambda_1, \ldots, \lambda_n$ then
    - A^n has eigenvalues λ_i^n
    - kA has eigenvalues $k\lambda_i$
    - $A + kI$ has eigenvalues $\lambda + k$
    - But unfortunately $A + B$ doesn't work like this (in general).
2. Non-negative matrices (or positive matrices, or stochastic matrices) have an associated theorem called the Perron-Frobenius theorem that guarantees a real, largest eigenvalue called the *spectral radius* and often denoted $\rho(A)$, and this has an associated eigenvector that is non-negative. That is really important, because when the matrix is stochastic (and irreducible) the spectral radius is 1, and the eigenvector can be interpreted as a stationary probability. That's the basis for half the stochastic process stuff out there. There's a lot more to PF that I won't cover here.

3. Certain types of matrices have easier to calculate eigenvalues:

    - The eigenvalues of an upper (or lower) triangular matrix are the diagonal elements.

    - Toeplitz matrices are matrices with constant diagonal, and they come up ion connection with convolutions (much more on these later), *e.g.,*

    $$T = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}.$$

    Toeplitz matrices form a subspace of the general matrix vector space, and lots of operations on them are simpler (numerically).

4. Eigenvalues of certain ensembles of random matrices have interesting properties of their own.

    - For instance: take an $n \times n$ matrix (with big n) Gaussian IID random variables with mean zero and standard deviation $1/\sqrt{n}$. Look at the complex eigenvalues.

- Another example has upper triangular elements that that are Gaussian IID random variables with mean zero and standard deviation 1 (except for the diagonals that get standard deviation $\sqrt{2}$). Then make the matrix symmetric (this is called the Gaussian Orthogonal Ensemble). Look at the gaps between their magnitudes.

Let's do some code for that.

Links

- https://towardsdatascience.com/eigenvectors-and-eigenvalues-all-you-need-to-know-df92780c591f
- https://www.adelaide.edu.au/mathslearning/ua/media/120/evalue-magic-tricks-handout.pdf
- https://www.intechopen.com/chapters/50675
- https://nhigham.com/2021/07/13/what-is-the-perron-frobenius-theorem/
- https://people.maths.ox.ac.uk/trefethen/dec11.pdf