Notes: MATH273 (F25) Basic Theory of Ordinary Differential Equations

Lecturer: Jiajie Chen
Notes by: Aden Chen

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Contents

1 Motivation, Preview of Application

Example 1.1 ((Stochastic) gradient descent). We are interested in solving

$$\min_{x \in D} g(x)$$

where g represents a cost, and $D \subset \mathbb{R}^n$. The FOC is $\nabla g(x) = 0$. If g is nonlinear and $n \gg 1$, this is a very hard problem. We can however always consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\nabla g(x(t)),$$

where g is given and $x(t)$ is unknown. If $\mathbb{R}^{n \times n} \ni \nabla^2 g > 0$ (is positive definite) and $x(t_0) = x_0$, then $x(t) \to x_*$, where $x_* \coloneqq \arg\min_{x \in D} g(x)$.

2 Basic Definitions and Examples

Definition 2.1 (Differential Equation)**.** A **differential equation** is an equation that relates a function y and its derivatives. A general representation is

$$F\left[x, y, \partial_i y, \partial_i^2 y, \ldots, \partial_i^{(n)} y\right] = 0.$$

Problem 2.2 (Heat Equation). Consider $u(t, x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

$$\partial_t u(t, x) = \Delta u(t, x) = \sum_{i=1}^{n} \partial_{x_i}^2 u(t, x).$$

This is a second order differential equation.

Problem 2.3.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u + \frac{\mathrm{d}}{\mathrm{d}t} u = u.$$

This is a second order ODE.

- **ODEs** contain only derivatives on one variable.
- **PDEs** can contain multiple partial derivatives.

Definition 2.4 (Order of a Differential Equation)**.** The **order** of a differential equation is the order of the highest order derivative that appears in the equation.

Definition 2.5 (Linear and Nonlinear ODEs)**.** We say an ODE is **linear** if

$$F\left[x, y(x), y'(x), \ldots, y^{(n)}(x)\right]$$

depends on $y, y', \ldots, y^{(n)}$ linearly. Note that we allow F to depend on x nonlinearly.
An **nonlinear** ODE is one that is not linear.

Note that we may always represent a linear ODE as

$$a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = b(x),$$

where each $a_i(x)$ can be nonlienar in x.

In general, linear ODEs are fully solvable by hand. For nonlinear ODEs, we may only be able to solve some special cases.

Problem 2.6.

$$\frac{\mathrm{d}}{\mathrm{d}t} y = c,$$

where c is a constant. Integrating, we get $y = ct + b$, where b is an arbitrary constant.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y = 0.$$

Integrating twice, we get $y = at + b$, where a, b are arbitrary constants.

These (non unique) are called **general solutions**. We thus sometimes prescribe also an initial condition (IC) to further determine the solution. An example of an IC for the second order ODE above is

$$y(0) = y_0, \quad y'(0) = v_0,$$

In these examples, note in particular that we have uniqueness results given the ICs.

2.1 First Order Linear ODEs

We may represent a first order linear ODE as

$$a_1(x)y'(x) + a_0(x)y(x) = b(t).$$

The general method is to rewrite the ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)] = f(t)$$

and integrate both sides.

When $a_1 \neq 0$, we can rewrite the ODE as

$$y'(x) + p(x)y(x) = g(x),$$

with p, g given. The particular case $b = 0$ ($g = 0$) is considered first:

2.1.1 The case b = 0

Problem 2.7. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = a(t)y(t). \tag{1}$$

Assuming $y(t) \neq 0$, we may rewrite this as

$$a(t) = \frac{1}{y(t)}\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\log|y(t)|].$$

Integrating, we get

$$\log|y(t)| = \int a(t)\,\mathrm{d}t + C,$$

and so

$$y(t) = \pm e^C \exp\left(\int a(t)\,\mathrm{d}t\right) = C' \exp\left(\int a(t)\,\mathrm{d}t\right),$$

where C' is an arbitrary constant (the case $C' = 0$ is attained when $y = 0$).

2.1.2 The Integrating Factor

Problem 2.8. Consider

$$y'(x) + p(x)y(x) = g(x). \tag{2}$$

Observe that for each $\mu(t) \neq 0$, the ODE is equivalent to

$$\mu y' + \mu p y = \mu g.$$

Let's guess that the left hand side can be written as $\frac{\mathrm{d}}{\mathrm{d}t}[a(t)y(t)]$ for some a. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}[a(t)y(t)] = a'(t)y(t) + a(t)y'(t) = \mu y' + \mu p y \implies \begin{cases} a = \mu, \\ \mu' = \mu p. \end{cases}$$

The function μ is called the **integrating factor**. It suffices to find one μ such that $\mu' = \mu p$. A μ is given by the previous case:

$$\mu(t) = \exp\left(\int_{t_0}^{t} p(s)\,\mathrm{d}s\right).$$

We now reduced the ODE to

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu(t)y(t)] = \mu(t)g(t),$$

which can be solved by integrating and dividing by μ:

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\,\mathrm{d}s + C\right).$$

Example 2.9.

$$y' + y = e^t.$$

We seek a μ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu y] = \mu y' + \mu y = \mu e^t.$$

This gives

$$\begin{cases} \mu' = \mu, \\ \mu = e^t. \end{cases}$$

Using this choice of μ we rewrite the ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^t y] = e^t e^t = e^{2t}$$

$$e^t y = \frac{1}{2}e^{2t} + C,$$

from which $y = \frac{1}{2}e^t + ce^{-t}$.

Determining C: Suppose we are given

$$y(t_0) = y_0.$$

Then

$$\mu y(t) = \int_{t_0}^{t} \mu g(s)\,\mathrm{d}s + c.$$

Taking $t = t_0$, we get

$$\mu y(t_0) = c$$

Since (given our choice of μ)

$$\mu(t_0) = 1,$$

we have $C = y(t_0) = y_0$, which we can plug back in the general solutions obtained above.

3 Separation of Variables

Recall that first order ODEs can be represented as

$$y'(t) + p(t)y(t) = g(t).$$

Using the implicit function theorem, we can in principle rewrite this as

$$y'(x) = f(x, y)$$

and then

$$M(x, y) + N(x, y)y' = 0.$$

Question: for which M, N can we solve this ODE?

Recall that last lecture we solved

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)] = g(t)$$

with y unknown.

A first special case (separation of variables) is when $M = M(x)$ and $N = N(y)$:

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Proof *(Formal Derivation)***.** If we formally treat $\mathrm{d}x$ and $\mathrm{d}y$ as differentials, we can rewrite the above as

$$N(y)\mathrm{d}y = -M(x)\mathrm{d}x.$$

In this view the variables x and y are separated. Integrating both sides, we obtain

$$\int N(y)\mathrm{d}y = -\int M(x)\mathrm{d}x + C.$$

If we can find n and m such that $n' = N$ and $m' = M$, then we have

$$n(y) = -m(x) + C,$$

from which we can solve for y. □

Proof *(Rigorous Derivation)***.** We integrate over x to get

$$\int M(x)\ \mathrm{d}x + \int N(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = 0.$$

With a change of variables we have

$$\int M(x)\ \mathrm{d}x + \int N(y)\ \mathrm{d}y = C.$$

□

3.0.1 Examples

Example 3.1.

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

We have $M = 1$ and $N =$ y. Integrating, we get

$$\frac{x^2}{2} + C + \int y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = 0$$

and so

$$\frac{x^2}{2} + \frac{y^2}{2} = C.$$

With additional initial conditions we can determine $y(x)$.

Example 3.2.

$$y + e^x\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Dividing by ye^x (assuming $y \neq 0$), we get

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = -e^{-x}.$$

and then

$$-e^{-x} + C + \log|y| = 0.$$

More generally, suppose the dependence of M and N on (x, y) can be separated in the following sense:

$$M_1(x)M_2(y) + N_1(x)N_2(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Again dividing both sides, we get

$$\frac{M_1}{N_1} + \frac{N_2}{M_2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Example 3.3.

$$e^{x+y} + xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Use above.

Warning: this method does not work for the following ODE:

$$M_1(x)M_2(y) + Z_1(x)Z_2(y) + N_1(x)N_2(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

3.1 Generalization

The method of integrating both sides cannot be pushed much further beyond the following case:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\varphi(x, y(x))\right] = g(x).$$

Integration gives

$$\varphi(x, y(x)) = \int g(x)\,\mathrm{d}x + c.$$

In principle we can solve for y by the implicit function theorem.

The ODE above can be written equivalently as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tilde{\varphi}(x, y(x))\right] \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\left[\varphi(x, y(x)) - \int g(x)\,\mathrm{d}x\right] = 0.$$

Thus we can without loss of generality suppose $g = 0$. We turn next thus to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x, y(x)) = 0.$$

For which M, N can we convert $M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ to the above form?

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x, y(x)) = \partial_1\varphi + \partial_2\varphi\frac{\mathrm{d}y}{\mathrm{d}x}.$$

This implies

$$\begin{cases} M(x, y) = \partial_1\varphi(x, y), \\ N(x, y) = \partial_2\varphi(x, y). \end{cases}$$

Definition 3.4. We say $M + Ny' = 0$ is an **exact equation** if there exists φ such that

$$M = \partial_1\varphi, \quad N = \partial_2\varphi.$$

What is the minimum requirement for M, N to be an exact equation? If φ is sufficiently smooth (C^2), then we would expect

$$\partial_2 M = \partial_2\partial_1\varphi = \partial_1\partial_2\varphi = \partial_1 N.$$

This in fact is also sufficient:

Theorem 3.5. *Suppose $M, N, \partial_2 M, \partial_1 N$ are continuous in the box $B = [a, b] \times [c, d]$ and $(x, y) \in B$. Then the equation $M(x, y) + N(x, y)y' = 0$ is exact if and only if*

$$\partial_2 M(x, y) = \partial_1 N(x, y).$$

That is, there exists φ such that

$$M = \partial_1\varphi, \quad N = \partial_2\varphi.$$

Example 3.6. The equation $M(x)\ \mathrm{d}x + N(y)\ \mathrm{d}y = 0$ is exact, with $\partial_2 M = 0 = \partial_1 N$. But observe also that

$$\partial_2[M(x) + y] = 1 = \partial_1[N(y) + x].$$

So we can solve the ODE

$$(M(x) + y) + (N(y) + x)\, y' = 0,$$

which is not separable.

Proof. That exactness implies $\partial_2 M = \partial_1 N$ is easy and shown above.

It remains to prove that $\partial_2 M = \partial_1 N$ implies exactness. To that end we construct φ as follows:
Step 1: construct φ so that $\partial_1 \varphi = M(x, y)$. We set

$$\varphi(x, y) = \int_{x_0}^{x} M(s, y)\ \mathrm{d}s + h(y),$$

where h is to be determined.
Step 2: determine h so that $\partial_2 \varphi = N(x, y)$. Note that

$$\begin{aligned}
\partial_2 \varphi(x, y) &= \int_{x_0}^{x} \partial_2 M(s, y)\ \mathrm{d}s + h'(y) \\
&= \int_{x_0}^{x} \partial_1 N(s, y)\ \mathrm{d}s + h'(y) \\
&= N(x, y) - N(x_0, y) + h'(y),
\end{aligned}$$

from which we can specify h by

$$h(y) = \int_{y_0}^{y} N(x_0, s)\ \mathrm{d}s + C.$$

In sum, φ is given by

$$\varphi(x, y) = \int_{x_0}^{x} M(s, y)\ \mathrm{d}s + \int_{y_0}^{y} N(x_0, s)\ \mathrm{d}s + C.$$

This completes the proof. □

Example 3.7.

$$y^2 x + (1 + x^2 y) y' = 0.$$

We have

$$\partial_2 M = 2xy = \partial_1 N,$$

and may thus set

$$\varphi(x, y) = \int y^2 x\ \mathrm{d}x = \frac{x^2 y^2}{2} + h(y).$$

$$\partial_2 \varphi = x^2 y + h'(y) = 1 + x^2 y \implies h(y) = y + C.$$

Finally, we can rewrite the original ODE as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2y^2}{2}+y\right]=0 \implies \frac{x^2y^2}{2}+y=C,$$

with

$$\varphi(x,y)=\frac{x^2y^2}{2}+y+C'.$$

Suppose we have the IC $y(0)=1$. Then

$$C=\frac{0^2 1^2}{2}+1=1.$$

3.2

We consider further ODEs of the term

$$M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}=0.$$

We can equivalently write this as

$$\mu M+\mu N\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

for some $\mu \neq 0$. This is exact when

$$\partial_2(\mu M)=\partial_1(\mu N).$$

The goal, thus, is to find μ such that the above is true, when the original ODE might not be exact. If $\mu(x,y)=\mu(x)$ or $\mu(x,y)=\mu(y)$, then we need not deal with mixed partials.

Let's begin with $\mu=\mu(x)$: We would like to solve

$$\partial_2(\mu M)=\mu\partial_2 M=\mu' N+\mu\partial_1 N=\partial_1(\mu N),$$

or equivalently

$$\frac{\mu'}{\mu}=\frac{\partial_2 M-\partial_1 N}{N}.$$

This approach works when the right hand side is a function of x only.

A similar condition can be derived for $\mu=\mu(y)$.

4 Second Order Linear ODEs

A second order linear ODE can be written as

$$F(t, y, y', y'') = 0,$$

and by the inverse function theorem, as

$$y'' = f(t, y, y').$$

Definition 4.1. We say this ODE is **linear** if F depends on y, y', y'' linearly (note again that we do not require linearity in t). Thus a second order linear ODE can be written as

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

In the case that $P(t) \neq 0$, we can rewrite this as

$$y'' + p(t)y' + q(t)y = g(t),$$

Example 4.2. $y'' = 0$. The general solution is $y(t) = c_1 t + c_2$. We need 2 ICs to determine c_1, c_2, for examele

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = v_0 \end{cases}.$$

In general, for an n^{th} order ODE we need n ICs to determine a unique solution.

Definition 4.3. We say the ODE is **homogeneous** if $G = 0$ and **nonhomogeneous** otherwise.

Remark 4.4 (Property of homogeneous ODEs). If y solves $y'' + p(t)y' + q(t)y = 0$, then ay solves the same ODE for any $a \in \mathbb{Z}$.

We start with the homogeneous case.

4.1 Homogeneous Second Order Linear ODEs with Constant Coefficients

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}.$$

4.1.1 The Ansatz of Polynomials

We assume first that $y(t) = \sum_{j=0}^{n} a_j t_j$. Plugging this into the ODE, we get terms involving t^n which cannot be canceled.

4.1.2 Recall

If $a \equiv 0$, then this reduces to $by' + cy = 0$. This can be written as one of the following:

$$y' + \frac{c}{b}y = 0, \quad b\frac{y'}{y} + c = 0.$$

And in either case we will get $y(t) = e^{-\frac{c}{b}t} \cdot c_0$.

4.1.3 The Ansatz of Exponentials

Inspired by the first order case, we now try the ansatz $y(t) = c_0 e^{\lambda t}$. Plugging into the ODE, we get

$$c_0 \left[a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t}\right] = 0,$$

which reduces the original ODE to the following:

$$a\lambda^2 + b\lambda + c = 0.$$

4.1.4 The Operator L

Define the operator L as

$$(Ly)(t) = P(t)y'' + Q(t)y' + R(t)y.$$

Example 4.5. For any constants c_1, c_2 and functions y_1, y_2. Note that

$$\begin{aligned} L\left[c_1 y_1 + c_2 y_2\right] &= P(t)\left[c_1 y_1 + c_2 y_2\right]'' + Q(t)\left[c_1 y_1 + c_2 y_2\right]' + R(t)\left[c_1 y_1 + c_2 y_2\right] \\ &= c_1 L[y_1] + c_2 L[y_2], \end{aligned}$$

and so the operator L is linear.

A solution y to the ODE $P(t)y'' + Q(t)y' + R(t)y = 0$ then can equivalently be written as $Ly = 0$. Now note that by linearity, we have if $Ly_1 = Ly_2 = 0$ for two "different solutions" y_1 and y_2, then since

$$L[c_1 y_1 + c_2 y_2] = c_1 L y_1 + c_2 L y_2 = 0,$$

the general solution can be obtained as

$$y = c_1 y_1 + c_2 y_2.$$

This technique of obtaining the general solution is called **linear superposition**. It turns out that the correct notion of solutions being "different" is linear independence.

Example 4.6.

$$y'' - 5y' + 6y = 0.$$

We can solve $\lambda^2 - 5\lambda + 6 = 0$ to get

$$\lambda_1 = 2, \quad \lambda_2 = 3.$$

Thus the first solution is $y_1 = e^{2t}$ and the second solution is $y_2 = e^{3t}$.

4.1.5 Three Cases of Obtaining the General Solution

The solution of the characteristic polynomial can be classified into three cases:

(i) Two real roots. In this case the general solution is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

(ii) Two different complex roots that are complex conjugates $\lambda \pm \mathrm{i}\mu$ (since all coefficients are real). Recall that

$$e^z := \sum_{k\geq 0} \frac{z^k}{k!}, \quad z \in \mathbb{C}, \qquad e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}.$$

Thus noting that

$$\begin{aligned} e^{(\lambda+\mathrm{i}\mu)t} &= e^{\lambda t}\left[\cos\mu t + \mathrm{i}\sin\mu t\right] \\ e^{(\lambda-\mathrm{i}\mu)t} &= e^{\lambda t}\left[\cos\mu t - \mathrm{i}\sin\mu t\right], \end{aligned}$$

we see two ways to obtain a real solution:

- Choose $c_1 = c_2 \in \mathbb{R}$ to get a multiple of $y(t) = e^{\lambda t}\cos\mu t$.
- Choose $c_1 = -c_2 \in \mathrm{i}\mathbb{R}$ to get a multiple of $y(t) = e^{\lambda t}\sin\mu t$.

The general solution is then a linear combination of the above two:

$$y(t) = e^{\lambda t}\left[c_1\cos\mu t + c_2\sin\mu t\right], \quad c_1, c_2 \in \mathbb{R}.$$

Example 4.7.

$$y'' + y = 0.$$

Solving the characteristic polynomial gives $\lambda_1 = \mathrm{i}$ and $\lambda_2 = -\mathrm{i}$. Two real solutions are $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

(iii) One real root with a multiplicity two. For the characteristic polynomial $a\lambda^2 + b\lambda + c = 0$, we have $\lambda = \lambda_1 = \lambda_2 = -b/2a$ and $4ac = b^2$. This gives a solution $y(t) = e^\lambda t$. We seek another solution y_2 using the so called **reduction of order** method. We try the ansatz $y_2(t) = \mu(t)y_1(t)$.

Claim 4.8. μ solves a first order ODE.

Proof. Suppose y_1 solves $Py_1'' + Qy' + Ry = 0$. If $y_2 = \mu y_1$ satisfies the same ODE, then

$$P(\mu y_1)'' + Q(\mu y_1)' + R(\mu y_1) = 0.$$

In general after expanding the left hand side, we get $\sum_{i=0}^2 a_i(t)\mu^{(i)}(t) = 0$. We will show $a_0 = 0$. Expanding, we get

$$P\left[\mu'' y_1 + 2\mu' y_1' + \mu y_1''\right] + Q\left[\mu' y_1 + \mu y_1'\right] + R\mu y_1 = 0.$$

Note that the μ-terms sum to $\mu\left[Py_1'' + Qy_1' + Ry_1\right] = 0$. Thus μ solves the ODE involving μ' and μ''

$$\mu'' P y_1 + \mu' \left[2Py_1' + Qy_1\right] = 0.$$

This can be solved by separation of variables:

$$\frac{\mu''}{\mu'} = -\frac{2Py_1' + Qy_1}{Py_1}.$$

□

Example 4.9. Suppose $P \equiv a$, $Q \equiv b$, and $R \equiv c$. We have $y_1 = e^{\lambda t}$ where $\lambda := -b/2a$. We have that μ defined above solved

$$\mu'' a e^{\lambda t} + \mu' \left[2a(e^{\lambda t})' + be^{\lambda t}\right] = 0.$$

The term in the bracket evaluates to 0 by $2a\lambda + b = 0$. Thus $\mu'' = 0$ and so $\mu(t) = t + C$. Thus the general solution is

$$y(t) = (c_1 t + c_2)e^{\lambda t}.$$

4.2 Series Solution to Homogeneous Second Order Linear ODEs

Consider the ODE

$$Py'' + Qy' + Ry = 0.$$

We will use the ansatz $y(x) = \sum_{n\geq 0}^{\infty} a_n (x - x_0)^n$.

Remark 4.10. Recall the following facts about power series $\sum_{n\geq 0} a_n(x - x_0)^n$:

- The root test for convergence: Let $\mu := \limsup_{n\to\infty} |a_n|^{1/n}$. If $|x - x_0| < \mu^{-1}$, then the power series converges absolutely.
- Within the radius of convergence, we can differentiate and integrate the power series term by term.

  $$y'(x) = \sum_{n\geq 1} n a_n (x - x_0)^{n-1}.$$

  Similarly we can compute the k^{th} derivative:

  $$y^{(k)}(x) = \sum_{n\geq k} n(n-1)\dots(n-k+1) a_n (x - x_0)^{n-k}.$$

  The resulting power series have the same radius of convergence.
- We say y is analytic in $B(x_0, \mu^{-1})$.

Example 4.11 (Airy's Equation). We seek the solution to the ODE $y'' - xy = 0$ near $x_0 = 0$.

$$\begin{aligned}
y &= \sum_{n\geq 0} a_n x^n, \\
y'' &= \sum_{n\geq 2} a_n n(n-1)x^{n-2} = \sum_{n\geq 0} a_{n+2}(n+2)(n+1)x^n, \\
xy &= \sum_{n\geq 0} a_n x^{n+1} = \sum_{n\geq 1} a_{n-1}x^n.
\end{aligned}$$

We collect terms to get

$$\begin{aligned}
x^0 : &\qquad a_2 \cdot 2 \cdot 1 = 0 \\
x_n, n \geq 1 : &\qquad a_{n+2}(n+2)(n+1) - a_{n-1} = 0.
\end{aligned}$$

This gives $a_2 = 0$ and

$$a_{m+3} = \frac{a_m}{(m+3)(m+2)}, \quad m := n-1 \geq 0$$

Equivalently, for $i = 0, 1, 2$, with $m + 3 = 3k + i$ we have

$$a_{3k+i} = \frac{a_{3k+i-3}}{(3k+i)(3k+i-1)}.$$

This can be solved using iterative substitution (See remark below). With

$$b_k = a_{3k+i}, \quad c_k = \frac{1}{(3k+i)(3k+i-1)},$$

we get

$$a_{3k+i} = a_i \prod_{j=1}^{k} \frac{1}{(3j+i)(3j+i-1)}$$

with $a_2 = 0$ and a_0, a_1 free. Thus the general solution is

$$y(x) = a_0 \sum_{i\geq 0} A_k x^{3k} + a_1 \sum_{k\geq 0} B_k x^{3k+1},$$

where

$$A_k = \prod_{j=1}^{k} \frac{1}{(3j)(3j-1)}, \quad B_k = \prod_{j=1}^{k} \frac{1}{(3j+1)(3j)}.$$

Note in particular that $|A_k|, |B_k| \leq 1$ for each k, and thus $\mu := \limsup[\cdot]^{1/n} \leq 1$. Thus the solution is analytic on a neighborhood of 0 with radius of convergence at least 1.

Remark 4.12. Suppose $b_k = c_k b_{k-1}$ for $k \geq 1$ and c_k and b_0 are given. Then we have the following solution by iterative substitution:

$$\begin{aligned} b_k &= c_k\left[c_{k-1}b_{k-2}\right] = c_k c_{k-1}\left[c_{k-2}b_{k-3}\right] = \cdots \\ &= c_k c_{k-1}\ldots c_1 b_0 = b_0 \prod_{i=1}^{k} c_i. \end{aligned}$$

Example 4.13 (Airy's Equation, around $x_0 = 1$). We seek the solution to the ODE $y'' - xy = 0$ near $x_0 = 1$.

$$\begin{aligned} y &= \sum_{n\geq 0} a_n (x-1)^n, \\ y'' &= \sum_{n\geq 2} a_n n(n-1)(x-1)^{n-2} = \sum_{n\geq 0} a_{n+2}(n+2)(n+1)(x-1)^n, \\ xy &= (x-1+1)y \\ &= \sum_{n\geq 0} a_n (x-1)^{n+1} + \sum_{n\geq 0} a_n (x-1)^n = \sum_{n\geq 1} a_{n-1}(x-1)^n + \sum_{n\geq 0} a_n (x-1)^n. \end{aligned}$$

We collect terms:

$$\begin{aligned} (x-1)^0 :& \qquad a_2\cdot 2\cdot 1 - a_0 = 0 \\ (x-1)^n, n\geq 1 :& \qquad a_{n+2}(n+2)(n+1) - a_{n-1} - a_n = 0. \end{aligned}$$

This gives $a_2 = a_0/2$ and

$$a_3 = \frac{a_0 + a_2}{3\cdot 2}, \quad a_4 = \frac{a_2 + a_1}{4\cdot 3} = \frac{a_0}{2\cdot 3\cdot 4} + \frac{a_1}{3\cdot 4}, \quad \ldots$$

In this case it is hard to obtain a closed form for a_n. We note that the general solution is determined by a_0, a_1.

Remark 4.14. This power series is useful for numerical approximation of the solution. We can truncate the series at some N and use the first N terms to approximate the solution.

Theorem 4.15 (5.3.1, BDM 9th edition)**.** *Consider $P(x)y'' + Q(x)y' + R(x)y = 0$, where we assume P, Q, R are analytic near x_0 with convergence radius R and write*

$$P(x) = \sum P_j (x-x_0)^j, \forall |x-x_0| < R$$

and similarly for Q and R. If $P(x_0) \neq 0$ in the ball $B(x_0, R)$, we can consider

$$y'' + \frac{Q}{P}y' + \frac{R}{P}y = 0.$$

Then there exists a power series solution y to the ODE of the form

$$y(x) = \sum_{n\geq 0} a_n (x-x_0)^n = a_0 y_1 + a_1 y_2$$

with convergence radius at least R.

Proof. Omitted. $\square$

Example 4.16. Examples of analytic functions:

- Polynomials. $R = +\infty$.
- e^x. $R = +\infty$.
- $\log(1 + x)$.
- $\sin x$, $\cos x$.

Examples of non-analytic functions:

- Any non-smooth function. E.g., $|x|^{1/2}$.

4.3 Non-Homogeneous Second Order Linear ODEs

$$Py'' + Qy' + Ry = G.$$

Observation: if φ and ψ are two solutions to the ODE above, then $\varphi - \psi$ solves the corresponding homogeneous ODE $Py'' + Qy' + Ry = 0$.

Proposition 4.17. *Suppose that y_0 solves $Py'' + Qy' + Ry = G$ and y_1, y_2 are two different solutions to $Py'' + Qy' + Ry = 0$. Then the general solution to the non-homogeneous ODE is*

$$y = y_0 + c_1 y_1 + c_2 y_2, \quad \forall c_1, c_2 \in \mathbb{R}.$$

That is, the general solution is a particular solution plus the general solution to the corresponding homogeneous ODE.

With this in mind, we see that we need only find one solution y_0 to the non-homogeneous ODE.

4.3.1 Variation of Parameters / Constants

Assume that y_1, y_2 are two different solutions to the corresponding homogeneous ODE $y''+py'+qy = 0$. Recall that the goal is a particular solution y_0 to the non-homogeneous ODE $y'' + py' + qy = g$. We try the ansatz

$$y_0(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t),$$

where functions μ_1, μ_2 are to be determined. Plugging into the $y'' + py' + qy$, we will get in general terms involving μ_i, μ_i', μ_i''. We will select μ_i in a way that the μ_i and μ_i'' terms vanish.

Note that

$$y' = \mu_1' y_1 + \mu_1 y_1' + \mu_2' y_2 + \mu_2 y_2' = \mu_1 y_1' + \mu_2 y_2',$$

where the last equality results after we *impose the restriction* $\mu_1' y_1 + \mu_2' y_2 = 0$. Now,

$$y'' = \mu_1' y_1' + \mu_2 y_2' + \mu_1 y_1'' + \mu_2 y_2''$$

and so

$$y'' + py' + qy = \mu_1' y_1' + \mu_2' y_2' + \mu_1(y_1'' + py_1' + qy_1) + \mu_2(y_2'' + py_2' + qy_2),$$

where y_1 and y_2 solve the homogeneous ODE, and so the last two terms vanish. If we set

$$\begin{cases} \mu_1' y_1 + \mu_2' y_2 = 0, \\ \mu_1' y_1' + \mu_2' y_2' = g, \end{cases} \iff \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

we get a 2×2 linear system for μ_1', μ_2'. This is solvable if and only the matrix if invertible:

$$\begin{pmatrix} \mu_2 \\ \mu_2 \end{pmatrix} (x) = \int_{x_0}^{x} A^{-1}(t) \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \, \mathrm{d}t.$$

5 First order ODE system

5.1 Motivation and Setup

The ODE

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = g(t)$$

can be rewritten as a first order system of ODEs. Write for each j, $x_j(t) := y^{(j-1)}(t)$ so that we have

$$y^{(j)} = \frac{\mathrm{d}}{\mathrm{d}t} x_j.$$

Now the original ODE can be rewritten as

$$a_n x_n'(t) + a_{n-1} x_n(t) + \cdots + a_1 x_2(t) + a_0 x_1(t) = g(t).$$

This gives the system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = \frac{1}{a_n}\left[g(t) - a_{n-1}x_n - \cdots - a_1 x_2 - a_0 x_1\right] \end{cases}.$$

This can be summarized as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-1}}{a_n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{g(t)}{a_n} \end{pmatrix}.$$

Let's for now abstract away from the above construction and consider the general first order system. With $F_1, \dots, F_n$ given and $x_1(t), \dots, x_n(t)$ unknown, we consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1(t, x_1, \dots, x_n) \\ F_2(t, x_1, \dots, x_n) \\ \vdots \\ F_n(t, x_1, \dots, x_n) \end{pmatrix},$$

which can be summarized as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{F}(t, \mathbf{x}).$$

This is a linear system if we can write

$$\mathbf{F}(t, \mathbf{x}) = A(t)\mathbf{x} + \mathbf{b}(t)$$

for $A(t) \in \mathbb{R}^{n \times n}$ and $\mathbf{b}(t) \in \mathbb{R}^{n \times 1}$.

Example 5.1. The first order ODE system that arises from the n^{th} order ODE is linear.

Definition 5.2. We say the system $\mathbf{F}(t, \mathbf{x}) = A(t)\mathbf{x} + \mathbf{b}(t)$ is **homogeneous** if $\mathbf{b} \equiv 0$ and **non-homogeneous** otherwise.

Example 5.3.

$$y'' + py' + qy = g.$$

Let $x_1 := y$ and $x_2 := y'$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Initial conditions for the first order system can be written as

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^{n\times 1}.$$

Theorem 5.4 (Existence and Uniqueness)**.** *Suppose F_j is continuous and $|\partial_{x_i} F_j| \leq M$ (Lipschitz) for $t \in (a_0, b_0) =: I_0$ and $x_i \in (a_i, b_i) =: I_i$. Then for each $t_0 \in I_0$, $x_{0,i} \in I_i$ there exists $\delta > 0$ and a unique solution $\mathbf{x}(t)$ to the ODE system on $(t_0 - \delta, t_0 + \delta)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| < \delta$.*

Note that for the special case

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A(t)\mathbf{x} + \mathbf{b}, \quad \mathbf{x}, \mathbf{b} \in \mathbb{R}^{n\times 1}, \quad A \in \mathbb{R}^{n\times n},$$

it is important to restrict A and $\mathbf{b}$ to be real, since complex numbers can roughly be identified as two real numbers, and in those cases solutions may not be unique.

Example 5.5. Consider $F = A(\mathbf{x})\mathbf{x} + \mathbf{b}$. If $|A| \leq C$, then there exists unique solution in a small neighborhood.

5.2 Solving the Homogeneous First Order Linear ODE System

Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A(t)\mathbf{x}(t)$$

Suppose $\mathbf{x}^{(1)} = \left(x_1^{(1)}, \ldots, x_n^{(1)}\right)', \ldots, \mathbf{x}^{(j)} = \left(x_1^{(2)}, \ldots, x_n^{(2)}\right)'$ are j solutions to the ODE. Then by linearity, so are $\sum_j c_j \mathbf{x}^{(j)}$.

How do we differentiate different solutions?

Definition 5.6. We say $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linear independent if the **Wronski matrix** is not singular, i.e., if the **Wronskian**

$$W(t) := \det\left(x^1, \ldots, x^n\right)$$

is not identically zero.

Note that if $W(t_0) = 0$, then $x^{(j_0)}(t_0) = \sum_{j \neq j_0} c_j x^j(t_0)$.

Theorem 5.7.
$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = \operatorname{tr}(A(t))\, W(t).$$

Proof. Write

$$M(t) = \begin{pmatrix} x^1(t) & x^2(t) & \dots & x^n(t) \end{pmatrix} = \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \dots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \dots & x_2^{(n)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \dots & x_n^{(n)}(t) \end{pmatrix} =: \begin{pmatrix} \mathbf{b}_1(t) \\ \mathbf{b}_2(t) \\ \vdots \\ \mathbf{b}_n(t) \end{pmatrix}.$$

We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix} = \det \begin{pmatrix} b_1' \\ b_2 \\ \dots \\ b_n \end{pmatrix} + \det \begin{pmatrix} b_1 \\ b_2' \\ \dots \\ b_n \end{pmatrix} + \dots + \det \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n' \end{pmatrix}.$$

To see this we recall

$$\frac{\mathrm{d}}{\mathrm{d}t}(C_1 \dots C_n) = \sum C_1 \dots C_{k-1} C_k' C_{k+1} \dots C_n$$

and

$$\det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \sum a_{1i_1} a_{2i_2} \dots a_{ni_n} \operatorname{sgn}(\sigma).$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ b_k' \\ \dots \\ b_n \end{pmatrix}.$$

Recalling $\frac{\mathrm{d}}{\mathrm{d}t} x^i = Ax^i$ for each i and fixing k, we have $\frac{\mathrm{d}}{\mathrm{d}t} x_k^i = [Ax^i]_k = \sum_j A_{kj} x_j^i$ for each i. Thus by stacking we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b_k = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} X_k^{(1)} & \dots & X_k^{(n)} \end{pmatrix} = \sum_j A_{kj} \begin{pmatrix} X_j^{(1)} & \dots & X_j^{(n)} \end{pmatrix} = \sum_j A_{kj} b_j.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ b_k' \\ \dots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ \sum_j A_{kj} b_j \\ \dots \\ b_n \end{pmatrix}$$

$$= \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ A_{kk} b_k \\ \dots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} A_{kk} \det \begin{pmatrix} b_1 \\ \dots \\ b_k \\ \dots \\ b_n \end{pmatrix} = \sum_{k} A_{kk} W(t).$$

□

Corollary 5.8.

- *$W(t_0) = 0$ for some t if and only if $W(t) = 0$ for each $t \in I$.*
- *$W(t_0) \neq 0$ for some $t_0 \in I$ if and only if $W(t) \neq 0$ for each $t \in I$.*

In particular, it suffices to check $W(t_0)$ at any t_0.

Proof. Note that

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\, \mathrm{d}s \right),$$

where the last term is never zero. Write $\mu \coloneqq \operatorname{tr} \circ A$. We have if $\mu(t) \neq 0$ for each t, then $W(t_0) = 0$ if and only if $W(t) = 0$ for each $t \in I$.

$W(t_0) \neq 0$ if and only if $W(t) \not\equiv 0$. □

Theorem 5.9. *Suppose that $x^1, \dots, x^n$ are n linearly independent* real *solutions to the homogeneous ODE system $\frac{\mathrm{d}}{\mathrm{d}t}x = A(t)x$. Then any solution x to the ODE can be written uniquely as*

$$x(t) = \sum_{j=1}^{n} c_j x^{(j)}(t), \quad c_j \in \mathbb{R}.$$

In particular, this tells us that the space of solutions to the homogeneous ODE system (or any n^{th} order ODE) is an n-dimensional vector space.

Proof. For each c write

$$y_c(t) = x(t) - \sum_i c_i x^i(t).$$

By previous results, y_c solves the homogeneous ODE.

Now fix t_0, we seek c_i such that

$$\begin{pmatrix} x^1(t_0) & x^2(t_0) & \dots & x^n(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = x(t_0).$$

Since $x^1, \ldots, x^n$ are linearly independent, the Wronski matrix

$$M(t_0) := \begin{pmatrix} x^1(t_0) & x^2(t_0) & \ldots & x^n(t_0) \end{pmatrix}$$

is invertible, and we can obtain $\mathbf{c}$ as $M^{-1}(t_0)x(t_0)$.

In particular we have $y_c(t_0) = 0$ and y_c solves the homogeneous ODE. Since 0 is also a solution to the homogeneous ODE, by uniqueness we have $y_c(t) \equiv 0$, which gives $x(t) \equiv \sum c_i x^i$. □

Theorem 5.10. *Suppose that $X^{(i)}$ solves $\frac{\mathrm{d}}{\mathrm{d}t}X^{(i)} = A(t)X^{(i)}$ with the IC $X^{(i)}(t_0) = \mathbf{e}_i$ for $1 \leq i \leq n$. Then $X^{(i)}$ are linearly independent solutions.*

Proof. The existence and uniqueness theorem guarantees that $X^{(i)}$ exist and are unique. To check that they are independent, we need to check that

$$W(t) = \det \begin{pmatrix} X^{(1)}(t) & X^{(2)}(t) & \ldots & X^{(n)}(t) \end{pmatrix}$$

is not identically zero. Recall that to do this it suffices to check $W(t_0) \neq 0$:

$$W(t_0) = \det I = 1 \neq 0.$$

But note of course that we can pick any n linearly independent initial conditions in $\mathbb{R}^n$. □

Remark 5.11. The preceding two theorems together implies that there are exactly n linearly independent solutions to the homogeneous ODE system.

Note that the theorem above also gives a method to find n linearly independent solutions.

Example 5.12.

$$y'' + py' + qy = 0.$$

This is a second order ODE which can be rewritten as a 2×2 first order ODE system. The results above imply that there are exactly two linearly independent solutions to the ODE. The linear independence turns out to be equivalent to $y_1 \neq cy_2$ in this case.

5.3 Finding Solutions Explicitly

Recall that for $n = 1$, we have that the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = a(t)x(t), \quad \in \mathbb{R}$$

has solution

$$x(t) = x(t_0) \exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right).$$

In the special case that $A(t)$ is diagonal with $n \geq 2$, we may easily generalize the result above: $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = A\mathbf{x}$ has solution

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i = A_{ii}X_i, \quad 1 \leq i \leq n.$$

We say that the solution is **decoupled**.

For more general $A(t)$, we can only restrict to the special case $A(t) \equiv A$ is constant. To see why further generalization is hard, consider the case $n = 2$ and

$$y'' + py' + qy = 0.$$

Recall that we can only solve this explicitly if p, q are constant.

5.3.1 The case A is constant

Recall that for $n = 1$, we know that the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t), \quad a \in \mathbb{R}$$

has solution $x(t) = ce^t$.

We seek to generalized this to the case

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A\mathbf{x}(t), \quad \mathbf{a} \in \mathbb{R}^{n\times n}.$$

The examples above for $n = 1$ motivates the ansatz

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^{n\times 1}, \lambda \in \mathbb{R}.$$

Note that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\mathbf{v}\right) = \lambda e^{\lambda t}\mathbf{v} \\ &A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v}. \end{aligned}$$

Thus $\mathbf{x}(t)$ solves the ODE system if and only if $\lambda\mathbf{v} = A\mathbf{v}$, or if and only if $(\lambda, \mathbf{v})$ is an eigenvalue-eigenvector pair of A.

Remark 5.13.

- Special linear combinations of $x_i(t)$ are solutions to a corresponding 1×1 ODE.
- After projecting in the direction of the eigenvectors, the ODE system decouples.

☕

We have the following cases:

(i) A has n distinct real eigenvalues.

(ii) A has complex eigenvectors, but all eigenvalues are distinct.

(iii) Repeated eigenvalues.

(iv) n linearly independent real eigenvectors.

The first three cases are disjoint, while the last case can overlap with the first three.

5.3.2 Case (i): A has n distinct real eigenvalues

Recall that when eigenvalues are real, so are the eigenvectors. Then we have the following n solutions:

$$\mathbf{x}^{(i)} = e^{\lambda_i t}\mathbf{v}_i, \quad 1 \le i \le n.$$

It can be shown that if $\lambda_1, \dots, \lambda_n$ are distinct, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent. This gives $W(0) = \det\left(\mathbf{v}_1, \dots, \mathbf{v}_n\right) \neq 0$ and so the solutions above are linearly independent.

5.3.3 Case (iv): A has n linearly independent real eigenvectors

Note that case (iv) includes case (i). This is a strict subset:

Example 5.14.

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}.$$

Since $\mathbf{v}_i$ are real, so are λ_i. We have

$$A\begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \iff AP = P\Lambda.$$

We have $\mathbf{v}_i$ are linearly independent if and only if P is invertible. In such case we have $A = P\Lambda P^{-1}$, or A is diagonalizable.

Recall the following:

Theorem 5.15 (Symmetric Matrix)**.** *If $A = A^\intercal \in \mathbb{R}^{n\times n}$, then $A = Q\Lambda Q^\intercal$, where $Q \in \mathbb{R}^{n\times n}$, $QQ^\intercal = Q^\intercal Q = I_n$, $Q^{-1} = Q^\intercal$, and Λ is diagonal with real eigenvalues.*

Thus if A is symmetric, we can find n linearly independent solutions to the ODE system.

5.3.4 Case (ii): A has complex eigenvalues, but all eigenvalues are distinct

Note that if $(\lambda, \mathbf{v})$ is an eigenvalue-eigenvector pair, then so is $(\overline{\lambda}, \overline{\mathbf{v}})$, since $A\mathbf{v} = \lambda\mathbf{v} \iff A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$ (Note that this relies on A being real). We seek two real solutions from $e^{\lambda t}v$ and $e^{\overline{\lambda}t}\overline{v}$. Suppose then that $\lambda = \alpha + \mathrm{i}\beta$ and $v = a + \mathrm{i}b$ for $\alpha, \beta \in \mathbb{R}$, $a, b \in \mathbb{R}^{n\times 1}$. Then

$$\begin{aligned} e^{\lambda t}v &= e^{(\alpha+\mathrm{i}\beta)t}(a + \mathrm{i}b) \\ &= e^{\alpha t}(\cos(\beta t) + \mathrm{i}\sin(\beta t))(a + \mathrm{i}b) \\ &= e^{\alpha t}\left[(\cos(\beta t)a - \sin(\beta t)b) + \mathrm{i}(\sin(\beta t)a + \cos(\beta t)b)\right] \end{aligned}$$

and similarly,

$$e^{\overline{\lambda}t}\overline{v} = e^{\alpha t}\left[(\cos(\beta t)a - \sin(\beta t)b) - \mathrm{i}(\sin(\beta t)a + \cos(\beta t)b)\right].$$

From this we see that

$$
\begin{aligned}
\operatorname{Re} e^{\lambda t} v &= \frac{e^{\lambda t} v + e^{\overline{\lambda} t}\overline{v}}{2} = e^{\alpha t}(\cos(\beta t) a - \sin(\beta t) b) \\
\operatorname{Im} e^{\lambda t} v &= \frac{e^{\lambda t} v - e^{\overline{\lambda} t}\overline{v}}{2\mathrm{i}} = e^{\alpha t}(\sin(\beta t) a + \cos(\beta t) b)
\end{aligned}
$$

are two real solutions to the ODE system.

Thus each complex eigenvalue-eigenvector pair gives two real solutions to the ODE system.

5.3.5 Case (iii): A has Repeated eigenvalues

Example 5.16.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}\mathbf{x}.
$$

The eigenvalue is λ with multiplicity 3. This matrix is not diagonalizable, since

$$
(A - \lambda I)\mathbf{v} = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff v_2 = v_3 = 0,
$$

and so each eigenvector is of the form $\mathbf{v} = \begin{pmatrix} v_1 & 0 & 0 \end{pmatrix}^\top$.

The methods discussed above thus does not work. Note, however, that x_3 can be solved easily: $x_3(t) = c_3 e^{\lambda t}$. Then the restriction on x_2 becomes $x_2' = \lambda x_2 + c_3 e^{\lambda t}$. Using the integrating factor $e^{-\lambda t}$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda t} x_2\right) = c_3,
$$

which gives $x_2(t) = e^{\lambda t}(c_2 + c_3 t)$. Finally, the restriction on x_1 becomes $x_1' = \lambda x_1 + e^{\lambda t}(c_2 + c_3 t)$. Using the integrating factor $e^{-\lambda t}$ again, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda t} x_1\right) = c_2 + c_3 t,
$$

which gives $x_1(t) = e^{\lambda t}\left(c_1 + c_2 t + \frac{c_3 t^2}{2}\right)$. Thus the general solution to the ODE system is

$$
\mathbf{x}(t) = e^{\lambda t}\begin{pmatrix} c_1 + c_2 t + \frac{c_3 t^2}{2} \\ c_2 + c_3 t \\ c_3 \end{pmatrix} = c_1 e^{\lambda t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t}\begin{pmatrix} \frac{t^2}{2} \\ t \\ 1 \end{pmatrix}.
$$

5.4 Matrix Exponential

Consider the $n = 1$ case $x' = ax$, $a \in \mathbb{R}$. We have solution $x(t) = e^{at}x_0$. We seek to generalize this to the case $n \geq 2$ where

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

But of course we need first to make sense of e^{At} for $A \in \mathbb{R}^{n\times n}$. The hope is that a definition will be consistent with $(e^{At})' = Ae^{At}$.

Definition 5.17.

$$\exp[A] \coloneqq \sum_{k=0}^{\infty} \frac{A^k}{k!} \in \mathbb{R}^{n\times n},$$

if the series converges.

Example 5.18.

- If $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, we have

$$\exp[A] = \operatorname{diag}\left(e^{\lambda_1}, \ldots, e^{\lambda_n}\right).$$

- $A = \begin{pmatrix} & \beta \\ -\beta & \end{pmatrix} = \beta J$, where $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. We have $A^k = \beta^k J^k$. Note that $J^2 = -I$, $J^3 = -J$, $J^4 = I$. Thus we have

$$A^{4k+i} = \beta^{4k+i}J^i, \quad i = 0, 1, 2, 3,$$

  and then

$$\exp[A] = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

Remark 5.19. In general $AB \neq BA$. If however $A = P^a$ and $B = P^b$, then $AB = BA$.

For $\exp[A]$ to be well-defined, we require $\sum_{k\geq 0}(A^k)ij/k!$ to converge for each $1 \leq i, j \leq n$. Let's suppose $\max_{i,j}|A_{ij}| \leq a$.

Proposition 5.20.

$$\left|(A^k)_{ij}\right| \leq (na)^{k-1}a.$$

Proof. We use induction. The case $k = 1$ is clear. Now suppose the condition holds for $k \geq 1$. From $(A^{k+1})_{ij} = (A^kA)_{ij} = \sum_{l=1}^{n}(A^k)_{il}A_{lj}$ we have

$$|(A^{k+1})_{ij}| \leq \sum_{l=1}^{n}|(A^k)_{il}||A_{lj}| \leq \sum_{l=1}^{n}(na)^{k-1}a^2 = (na)^k a.$$

□

In light of the proposition above, we have for each N.

$$\sum_{k=0}^{N} \frac{|(A^k)_{ij}|}{k!} \le \sum_{k=0}^{\infty} \frac{(na)^{k-1}a}{k!} = \sum_{k=0}^{\infty} \frac{1}{n} \frac{(na)^k}{k!} = \frac{e^{na}}{n} < \infty$$

Thus $\exp[A]$ is well-defined for all $A \in \mathbb{R}^{n\times n}$.

Lemma 5.21.

(i) *If* $B = T^{-1}AT$*, then* $\exp[B] = T^{-1}\exp[A]T$.

(ii) *If* $AB = BA$*, then* $\exp[A+B] = \exp[A]\cdot\exp[B]$.

(iii) $(\exp[A])^{-1} = \exp[-A]$.

Proof.

(i) Note only that $B^k = T^{-1}A^kT$.

(ii) Note that

$$\begin{aligned}\exp[A+B] &= \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k\ge 0} \frac{1}{k!}\left(\sum_j \binom{k}{j} A^j B^{k-j}\right)\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{A^j B^{k-j}}{j!(k-j)!} = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \frac{A^j}{j!}\frac{B^{k-j}}{(k-j)!} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{A^j}{j!}\frac{B^k}{k!}\\ &= \exp[A]\exp[B].\end{aligned}$$

(iii) Applying property (ii) with $B = -A$ gives $\exp[A]\exp[-A] = \exp[0] = I$. Similarly, $\exp[-A]\exp[A] = I$.

□

Proposition 5.22. *Suppose* $Av = \lambda v$*. Then* $\exp[A]v = e^{\lambda}v$.

Proof. Use the fact that $A^k v = \lambda^k v$. □

Proposition 5.23.

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp[tA] = A\exp[tA] = \exp[tA]A.$$

Proof. A previous calculation shows that if $\max|(A)_{ij}| \le a$, then $\sum(\dots) \le e^{nat}/n < \infty$. So $t \mapsto \exp[tA]$ behaves like a power series with radius of convergence ∞.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty}\frac{(tA)^k}{k!}\right) &= \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{(tA)^k}{k!} = \sum_{k=1}^{\infty}\frac{A^k}{k!}kt^{k-1} = \sum_{k\ge 1}\frac{A^k}{(k-1)!}t^{k-1}\\ &= A\sum_{k=1}^{\infty}\frac{(tA)^{k-1}}{(k-1)!} = A\exp[tA].\end{aligned}$$

But in the last line we may as well place A at the end to obtain $\frac{\mathrm{d}}{\mathrm{d}t} = \exp[tA]A$. □

Theorem 5.24. *Suppose that $A \in \mathbb{R}^{n\times n}$. Then the solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax, \quad x(t_0) = x_0 \in \mathbb{R}^{n\times 1}$$

is given by

$$x(t) = e^{A(t-t_0)}x_0.$$

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A(t-t_0)}x_0\right) = Ae^{A(t-t_0)}x_0 = Ax(t).$$

At $t = t_0$ we have

$$x(t_0) = e^{A\cdot 0}x_0 = Ix_0 = x_0.$$

By the existence and uniqueness theorem, this is the unique solution. □

Remark 5.25.

(i) e^{At} is called the **Fundamental matrix** of the ODE system.

(ii) If $x_0 = v$, $Av = \lambda v$, then $x(t) = e^{At}v = e^{\lambda t}v$. In this connection we see that the eigenvalue-eigenvector method is a special case of the matrix exponential method. More generally, see next point:

(iii) Suppose now A has n linearly independent eigenvectors $v_1, \dots, v_n \in \mathbb{C}^n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ (where note that we allow $\mathbb{C}$). Then by writing $P = (v_1, \dots, v_n)$, we have $AP = P\Lambda$, and so A is diagonalizable with $A = P\Lambda P^{-1}$. In particular,

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\mathrm{diag}\left(e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right)P^{-1}.$$

Thus

$$e^{At}Pe_i = e^{\lambda_i t}v_i.$$

☕

Theorem 5.26 (Jordan Normal Form)**.** *Suppose that $A \in \mathbb{C}^{n\times n}$. Then there exists $U, J \in \mathbb{C}^{n\times n}$ such that*

$$A = UJU^{-1},$$

where J is given by

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}, \quad J_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} \in \mathbb{C}^{m_j\times m_j}, \quad \sum m_j = n.$$

The columns of U are called generalized eigenvectors *and satisfy*

$$(A - \lambda I)^k u_i = 0, \quad k \le n.$$

Given a Jordan normal form decomposition $A = UJU^{-1}$, we have

$$e^{At} = U \exp[Jt] U^{-1},$$

Here,

$$J^k = \begin{pmatrix} J_1^k & & \\ & \ddots & \\ & & J_m^k \end{pmatrix}, \quad J_j^k = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}^k.$$

If $J_i = \alpha_i I + N$, where $N \in \mathbb{R}^{l \times l}$ is the nilpotent matrix with 1 on the superdiagonal and 0 elsewhere, then

$$e^{J_i t} = e^{\alpha_i t} e^{Nt} = \exp[\alpha_i t I] \exp(Nt),$$

since $IN = NI$. We have $\exp(\alpha_i t I) = e^{\alpha_i t} I$ and N^k is the matrix with 1 on the k^{th} superdiagonal and 0 elsewhere (Ex.). Thus

$$\exp[Nt] = I + Nt + \frac{(Nt)^2}{2!} + \cdots + \frac{(Nt)^{l-1}}{(l-1)!} \qquad = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{l-1}}{(l-1)!} \\ 0 & 1 & t & \dots & \frac{t^{l-2}}{(l-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Example 5.27.

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} x$$

has solution

$$e^{At} = e^{\lambda t} e^{Nt} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

5.5 Nonhomogeneous ODE Systems

Consider the nonhomogeneous ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t} x = A(t)x + G(t), \quad x(t_0) = x_0.$$

Recall that in the $n = 1$ case ($y'' + py' + qy = g(t)$), we first find solutions y_1 and y_2 to the corresponding homogeneous ODE, and then use the ansatz $y = \mu_1 y_1 + \mu_2 y_2$ with some clever restrictions on μ_1, μ_2 (variation of parameters/constants).

To generalize to nonhomogeneous ODE systems, we first recall the case $n = 1$

$$\frac{\mathrm{d}}{\mathrm{d}t} x = ax + g(t)$$

with a constant. Using the integrating factor e^{-at}, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-at}x\right) = e^{-at}g(t)$$

and so

$$e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^{t} e^{-as}g(s)\,\mathrm{d}s,$$

giving

$$x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^{t} e^{a(t-s)}g(s)\,\mathrm{d}s$$

or equivalently

$$x(t) = e^{a(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-a(s-t_0)}g(s)\,\mathrm{d}s\right].$$

Note that the first term $e^{a(t-t_0)}x_0$ solves the homogeneous ODE with the given IC, and for fixed s, the term $e^{a(t-s)}g(s)$ solves the homogeneous ODE with IC $g(s)$ at $t = s$. We may think of them as solutions to the following two ODEs:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_1 = ax_1, & x_1(t_0) = x_0 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 = g(t), & x_2(t_0) = 0 \end{cases}$$

Theorem 5.28. *Fur $n \geq 2$, the solution to $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + G(t)$, $x(t_0) = x_0$ is given by*

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}G(s)\,\mathrm{d}s \\ &= e^{A(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-A(s-t_0)}G(s)\,\mathrm{d}s\right]. \end{aligned}$$

Proof. Define

$$c(t) \coloneqq x_0 + \int_{t_0}^{t} e^{-A(s-t_0)}G(s)\,\mathrm{d}s.$$

Then the proposed solution is $x(t) = e^{A(t-t_0)}c(t)$. In like of existence and uniqueness, we just need to verify that this solves the ODE with the IC. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A(t-t_0)}c(t)\right] = Ae^{A(t-t_0)}c(t) + e^{A(t-t_0)}\dot{c}(t).$$

Now, using the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t_0}^{t} b(s)\,\mathrm{d}s\right) = b(t)$, we have

$$\dot{c}(t) = e^{-A(s-t_0)}G(t)\Big|_{s=t} = e^{-A(t-t_0)}G(t),$$

Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A(t-t_0)}c(t)\right] &= Ae^{A(t-t_0)}c(t) + e^{A(t-t_0)}\dot{c}(t) \\ &= Ax + G(t). \end{aligned}$$

$\square$

Remark 5.29. From this result we have **Duhamel's formula**: The differential equation

$$\partial_t f = Lf + g(t,x), \qquad f|_{t=0} = f_0$$

where L is a t-independent linear operator (e.g., ∂_x), has solution

$$f = e^{Lt} f_0 + \int_0^t e^{L(t-s)} g(s,x) \, \mathrm{d}s.$$

☕

6 The Theory of Existence and Uniqueness

We focus on the case $n = 1$, but the proof generalizes to $n \geq 2$ easily.

Theorem 6.1 (Existence and Uniqueness)**.** *Consider the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} y = f(t, y(t)), \qquad y(t_0) = y_0$$

in the region $R : |t - t_0| \leq a$, $|y - y_0| \leq b$.

We assume that f is continuous in R and Lipschitz in y with Lipschitz constant L. Then, there exists $h > 0$ such that the ODE admits a unique C^1 solution for $|t - t_0| \leq h$.

Proof. The idea is to construct a sequence $\varphi_0, \varphi_1, \ldots$ so that $\varphi_n \to \varphi$. One idea is to define $\varphi_0 = y_0$, $\varphi_1(t) = y_0 + f(t_0, y_0)(t - t_0)$, and so on. But this requires f to be differentiable.

Alternatively, we may integrate:

$$\varphi(t) := t_0 + \int_{t_0}^{t} f(s, \varphi(s)) \, \mathrm{d}s.$$

Note that differencing both sides gives $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = f(t, \varphi(t))$. Again we set $\varphi_0 = y_0$. For $n \geq 0$, we define

$$\varphi_{n+1}(t) := y_0 + \int_{t_0}^{t} f(s, \varphi_n(s)) \, \mathrm{d}s$$

and show that φ_n converges. This method is called **Picaro iteration**.[1]

We will show first that $|\varphi_n(t) - y_0| \leq b$ in a small neighborhood of t_0. From f being continuous, we know that $|f| \leq M$ in R. Thus

$$|\varphi_{n+1}(t) - y_0| \leq \int_{t_0}^{t} |f(s, \varphi_n(s))| \, \mathrm{d}s \leq M|t - t_0|.$$

And so by setting $h \leq b/M$, we have $|\varphi_n(t) - y_0| \leq b$ for all n and $|t - t_0| \leq h$.

Next, we show that $|\varphi_{n+1} - \varphi_n| \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} |\varphi_{n+1}(t) - \varphi_n(t)| &= \left| \int_{t_0}^{t} f(s, \varphi_n(s)) - f(s, \varphi_{n-1}(s)) \, \mathrm{d}s \right| \\ &\leq L \int_{t_0}^{t} |\varphi_n(s) - \varphi_{n-1}(s)| \, \mathrm{d}s. \end{aligned}$$

Note recall that we have the uniform in time bound

$$|\varphi_1 - \varphi_0| = |\varphi_1 - y_0| \leq Mh.$$

Iterating the inequality above gives

$$|\varphi_{n+1}(t) - \varphi_n(t)| \leq (Lh)^n Mh$$

[1] Another way is to use a fixed point theorem on a suitable function space.

which converges if we choose h such that $Lh < 1$. We define thus

$$\varphi := \lim_{n\to\infty} ved\varphi_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} (\varphi_{i+1} - \varphi_i) + \varphi_0.$$

Note that the series on the right converges uniformly in light of the bound above. In particular,

$$|\varphi - \varphi_n| \leq \sum_{i\geq n} |\varphi_{i+1} - \varphi_i| \leq Mh \frac{(Lh)^n}{1 - Lh}.$$

To see φ solves the ODE, note that from (exercise)

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = f(t, \varphi(t)) \implies \varphi \in C^1.$$

It remains to show uniqueness. Suppose that ψ is another solution to the ODE with the same IC. Note that

$$\varphi(t) = t_0 + \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s, \quad \psi(t) = t_0 + \int_{t_0}^{t} f(s, \psi(s))\, \mathrm{d}s.$$

Thus

$$d(t) := \varphi(t) - \psi(t) = \int_{t_0}^{t} [f(s, \varphi(s)) - f(s, \psi(s))]\, \mathrm{d}s.$$

The Lipschitz condition gives

$$|d(t)| \leq L \int_{t_0}^{t} |d(s)|\, \mathrm{d}s \implies |d|'(t) \leq LD'(t), \quad \text{where } D(t) := \int_{t_0}^{t} |d(s)|\, \mathrm{d}s.$$

Using the integrating factor e^{-Lt} we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-Lt} D(t)\right) \leq 0, \quad D(t) \geq 0$$

which gives

$$e^{-Lt} D(t) \leq e^{-Lt_0} D(t_0) = 0$$

and in turn

$$D(t) = 0, d = 0, \varphi = \psi.$$

$\square$

Remark 6.2. The assumption $f \in C^0$ already gives existence (Lipschitz is not required). The Lipschitz assumption gives uniqueness (no continuity required). ☕

7 Quantitative Estimates

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(t,x), \quad x(t_0) = x_0.$$

Recall that

- $f \in C^0 \implies$ Existence
- f is Lipschitz in $x \implies$ Uniqueness

Recall that with

$$|\varphi(t) - \psi(t)| \leq L \int_{t_0}^{t} |\varphi(s) - \psi(s)| \,\mathrm{d}s,$$

we have

$$d(t) := |\varphi(t) - \psi(t)| \leq L \int_{t_0}^{t} d(s) \,\mathrm{d}s.$$

We generalize this result:

Lemma 7.1 (Gronwall's Inequality)**.** *For given α, β, assume $\beta(t) \geq 0$, $\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s) \,\mathrm{d}s$ and define*

$$B(s) := \exp\left(\int_0^s \beta(t) \,\mathrm{d}t\right).$$

Then,

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\frac{B(t)}{B(s)} \,\mathrm{d}s.$$

If $\alpha \in C^1$, the right hand side may be written by integration by parts as

$$\alpha(0)B(t) + \int_0^t \alpha'(s)\frac{B(t)}{B(s)} \,\mathrm{d}s.$$

Proof. We write

$$A(t) := \int_0^t \beta(s)\psi(s) \,\mathrm{d}s$$

and seek an ODE inequality for A. The idea is to solve an ODE inequality for A of the form $\frac{\mathrm{d}}{\mathrm{d}t}[\text{unknown}] \leq \text{known}$ (note that this works only if the RHS is known). Note that we have $A'(t) = \beta(t)\psi(t)$. Using the assumptions that $\psi(t) \leq \alpha(t) + A(t)$ and $\beta \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \psi\beta \leq \alpha\beta + A\beta.$$

This is a first order linear "ODE inequality" in A. Using the integrating factor $1/B(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A(t)}{B(t)}\right) \leq \alpha(t)\beta(t)\frac{1}{B(t)},$$

where we note that

$$B^{-1} = \exp\left(-\int_0^t \beta \, \mathrm{d}t\right), \quad \dot{B}^{-1} = -\beta B^{-1}.$$

Integrating both sides from 0 to t gives

$$A(t)B^{-1}(t) \leq \int_0^t \alpha(s)\beta(s)B^{-1}(s) \, \mathrm{d}s + A(0)B^{-1}(0),$$

where the last term is zero since $A(0) = 0$ by definition. Now we have

$$\psi(t) \leq \alpha(t) + A(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\frac{B(t)}{B(s)} \, \mathrm{d}s.$$

□

Remark 7.2. Applications:

- $\alpha = 0, \beta = L, \psi = d$ and we recover the motivating example.
- $\alpha = 0$ implies $\psi \leq 0$.
- $\alpha(t) = \alpha_0 + \alpha_1 t, \beta = L$. Then

  $$B(s) = \exp\left(\int_0^s L \, \mathrm{d}s\right) = e^{Ls}$$

  and so

  $$\psi(t) \leq \alpha_0 e^{Lt} + \alpha_1 \int_0^t e^{L(t-s)} \, \mathrm{d}s.$$

  In some sense this is similar to the Duhamel's formula for the ODE system

  $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x = Lx + \alpha_1 \\ x(0) = \alpha_0 \end{cases} \implies x(t) = e^{Lt}\alpha_0 + \alpha_1 \int_0^t e^{L(t-s)} \, \mathrm{d}s.$$

☕

Consider now a small perturbation in x_0 or f in the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(t,x), \quad x(t_0) = x_0$$

so that it becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}y = g(t,y), \quad y(t_0) = y_0.$$

How do these errors affect x? The hope is that x does not change too much. Otherwise the model is not robust.

Theorem 7.3 (Continuous Dependence in IC, parameters, etc.)**.** *Suppose f is Lipschitz in the domain D of interest with Lipschitz constant L and denote*

$$M \coloneqq \max_{(t,x)\in D} |f(t,x) - g(t,x)|.$$

Then,

$$|x(t) - y(t)| \le e^{L|t-t_0|}|x_0 - y_0| + \frac{M}{L}\left(e^{L|t-t_0|} - 1\right),$$

where the first part is due to the IC error and the second part is due to the ODE error.

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - y(t)] = f(t, x(t)) - g(t, y(t)).$$

Integrating,

$$|x(t) - y(t)| \le |x_0 - y_0| + \int_{t_0}^{t} |f(s, x(s)) - g(s, y(s))| \,\mathrm{d}s.$$

The integrand may be bounded as

$$\begin{aligned} |f(s,x(s)) - g(s,y(s))| &\le |f(s,x(s)) - f(s,y(s))| + |f(s,y(s)) - g(s,y(s))| \\ &\le L|x(s) - y(s)| + M. \end{aligned}$$

Observe that

$$|x(t) - y(t)| \le |x_0 - y_0| + \int_{t_0}^{t} \left[L|x(s) - y(s)| + M\right] \,\mathrm{d}s.$$

Setting $\alpha \coloneqq |x_0 - y_0| + M(t - t_0)$ and $\psi(t) \coloneqq |x(t) - y(t)|$, we may apply Gronwall's inequality with $\beta = L$ to get

$$\begin{aligned} |x(t) - y(t)| &\le e^{L(t-t_0)}|x_0 - y_0| + \int_{t_0}^{t} MLe^{L(t-s)} \,\mathrm{d}s \\ &\le e^{L|t-t_0|}|x_0 - y_0| + \frac{M}{L}\left(e^{L|t-t_0|} - 1\right). \end{aligned}$$

□

Corollary 7.4.

- *If $f = g$, $x_0 = y_0$, then $x = y$.*
- *If $f = g$ then*

  $$|x(t) - y(t)| \le e^{L|t-t_0|}|x_0 - y_0|.$$

  That is, we have continuous (and in fact Lipschitz) dependence on the IC. In other words, if we define $\Phi(t, x_0)$ to be the solution of

  $$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = f(t, \Phi), \quad \Phi(t_0, x_0) = x_0,$$

  then Φ is Lipschitz in x_0.

- *If $x_0 = y_0$ and say $|f - g| \le \varepsilon$, then*

$$|x(t) - y(t)| \le \frac{\varepsilon}{L}\left(e^{L|t-t_0|} - 1\right).$$

In particular, if $f_\alpha(t,x) \coloneqq \alpha f_0 + (1-\alpha) f_1$, then

$$|f_\alpha - f_\beta| \le |\alpha - \beta| \left(|f_1| + |f_0|\right).$$

Example 7.5. We perturb the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \frac{x^2}{1 + t^2 + x^2}, \quad x(0) = 0$$

to get

$$\frac{\mathrm{d}}{\mathrm{d}t} y = \frac{y^2 + \varepsilon}{1 + t^2 + y^2} + \varepsilon, \quad y(0) = \varepsilon.$$

We have $|f - g| \le \varepsilon$ and $|x_0 - y_0| \le \varepsilon$, and we may check that $|\partial_x f(t,x)| \le L$ for any t, x. Thus

$$|x(t) - y(t)| \le e^{Lt}\varepsilon + \frac{\varepsilon}{L}\left(e^{Lt} - 1\right) = \varepsilon\left(e^{Lt} + \frac{e^{Lt} - 1}{L}\right).$$

Now consider

$$y' \le F(t, y(t)), \quad t \in [a, b].$$

Theorem 7.6 (Comparison)**.** *Suppose F is Lipschitz. Let $f, g \in C^1$ are such that*

$$f' \le F(t, f(t)), \quad g' = F(t, g(t)).$$

If $f(a) \le g(a)$, then

$$f(t) \le g(t), \quad t \in [a, b].$$

Proof. Suppose for contradiction that $f(t_1) > g(t_1)$. Let $\Omega \coloneqq \{t : f(t) \le f(t)\}$. We have $a \in \Omega$. Let $t_0 \coloneqq \max\{t \in [a, t_0] \cap \Omega\}$. We know $f(t_0) = g(t_0)$ and $f(t) \le g(t)$ for each $t \in \Omega$. Moreover, $f(t) > g(t)$ for each $t \in (t_0, t_1]$. But this and continuity implies that $f(t_0) \ge g(t_0)$. Thus $f(t_0) = g(t_0)$.

Now, from assumption we have

$$f'(t) - g'(t) \le F(t, f(t)) - F(t, g(t))$$

By integrating both sides,

$$\begin{aligned} f(t) - g(t) &\le f(t_0) - g(t_0) + \int_{t_0}^{t} F(s, f(s)) - F(s, g(s))\, \mathrm{d}s \\ &\le \int_{t_0}^{t} L(f(s) - g(s))\, \mathrm{d}s, \end{aligned}$$

where the second inequality is justified from $f(t) \ge g(t)$ on $(t_0, t_1]$. By Gronwall's inequality, we have

$$f(t) - g(t) \le 0, \quad t \in [t_0, t_1]$$

a contradiction. □

Now, if F is not Lipschitz, this no longer holds true:

Example 7.7.

$$\frac{\mathrm{d}}{\mathrm{d}t} y = y^{\frac{1}{3}}, \quad y(0) = 0$$

has solution $g(t) \equiv 0$. Now try the ansatz $f(t) = ct^{\alpha}$. We have

$$f'(t) = c\alpha t^{\alpha - 1}, \quad f^{\frac{1}{3}}(t) = c^{\frac{1}{3}} t^{\frac{\alpha}{3}},$$

giving

$$\alpha = \frac{3}{2}, \quad c = \pm \left(\frac{3}{2}\right)^{\frac{3}{2}}.$$

In particular, a solution is

$$f(t) = \begin{cases} ct^{\frac{3}{2}}, & t \geq 0 \\ -c(-t)^{\frac{3}{2}}, & t < 0 \end{cases},$$

which crosses g at $t = 0$ and is C^1 (since in particular both parts of f is C^1 at 0).

Remark 7.8. Another version of the comparison theorem is as follows: $f' < F(t, f)$ with strict inequality and $f(a) < g(a)$, f is continuous (Lipschitz not required??).

Theorem 7.9.

$$\frac{\mathrm{d}}{\mathrm{d}t} x = f(t, x), \quad \frac{\mathrm{d}}{\mathrm{d}t} y = g(t, y).$$

If $f(t, z) \leq g(t, z)$ for any t, z in the domain of interest D, and if f or g is Lipschitz in D, and that $x(a) \leq y(a)$, then $x(t) \leq y(t)$ for each $t \in [a, b]$.

Proof. Suppose g is Lipschitz (the other case is similar). We have

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(t, x) \leq g(t, x(t)), \quad \frac{\mathrm{d}}{\mathrm{d}t} y(t) = g(t, y).$$

By the comparison theorem, we have $x(t) \leq y(t)$ for each $t \in [a, b]$. □

Example 7.10. Consider

$$\frac{\mathrm{d}}{\mathrm{d}y} x_{\alpha}(t) = f(t, x_{\alpha}(t)) + \alpha, \quad x_{\alpha}(t_0) = x_0.$$

Then the solution satisfies

$$x_{\alpha}(t) \leq x_{\beta}(t), \quad t \in [a, b], \quad \text{if } \alpha \leq \beta.$$

Example 7.11.

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = P_n(x, t), \quad t \in [0, 1], \quad x(0) = x_0,$$

where P_n is a polynomial such that $P_n(x, t) \leq Ce^x$ for some constant $C > 0$. We know x is bounded above by the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} y = Ce^y, \quad y(0) = x_0.$$

Of course, this works for any upper bounding function.

7.1 Extension

In this section we will always assume that $f \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz in the second argument in any bounded region.

Example 7.12 (of Blowup).

$$\frac{\mathrm{d}}{\mathrm{d}t} y = y^2, \quad y(0) = 1.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} y^{-1} = -1 \implies y^{-1}(t) = 1 - t \implies y = \frac{1}{1-t}.$$

But observe that the solution exists up to 1^-: $\lim_{t \to 1^-} y(t) = +\infty$.

How can we detect/characterize blowup?

Lemma 7.13 (Gluing)**.** *Suppose $f \in C(\mathbb{R}, \mathbb{R})$ and $f(t,x)$ is Lipschitz in any bounded region:*

$$|f(t,x_1) - f(t,x_2)| \leq L_D |x_1 - x_2|, \quad (t,x_1),(t,x_2) \in D \subset \mathbb{R}^2.$$

Suppose $\varphi_i(t)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t} x = f(t,x), \quad x(t_0) = x_0$$

on $t \in I_i \coloneqq (a_i, b_i)$. If $t_0 \in I_1 \cap I_2 = (a,b)$ and $\varphi_1(t_0) = \varphi_2(t_0)$. Then $\varphi_1(t) = \varphi_2(t), t \in I_1 \cap I_2$ and

$$\varphi(t) \coloneqq \begin{cases} \varphi_1(t), & t \in I_1 \\ \varphi_2(t), & t \in I_2 \end{cases}$$

is a C^1 solution to the ODE on $I_1 \cup I_2$.

Proof. The hard part is to show $\varphi_1 = \varphi_2$ on $I_1 \cap I_2$.

Apply existence and uniqueness on to the ODE with IC at t_0 to get $\varphi_1 = \varphi_2$ on $t_0 \pm h$. Write

$$J_- \coloneqq \{p : \varphi_1(t) = \varphi_2(t), t \in [p, t_0 + h] \cap (a,b)\}.$$

Then we can prove that $a = \inf J_- =: A$. Suppose not, then $A > a$. There exists a sequence $P_n \to A$, each $P_n \in J$, so that $\varphi_1 = \varphi_2$ on each $[P_n, t_0 + h]$. By continuity, $\varphi_1(P_n) = \varphi_2(P_n)$. Again by continuity, $\varphi_1(A) = \varphi_2(A)$. But by existence and uniqueness, $\varphi_1 = \varphi_2$ on $[A - h', t_0 + h]$ for some $h' > 0$, contradicting the definition of A.

Alternatively:

Set $J \coloneqq \{t \in I_1 \cap I_2 : \varphi_1(t) = \varphi_2(t)\}$. The goal is to show $J = I_1 \cap I_2, J \neq \varnothing$. By continuity, J is relatively closed. But since it is also relatively open (by local existence and uniqueness), we have $J = I_1 \cap I_2$.

Yet another alternative: Using Gronwall's inequality, from

$$|\varphi_1(t) - \varphi_2(t)| \leq \int_{t_0}^{t} L |\varphi_1(s) - \varphi_2(s)| \,\mathrm{d}s,$$

we have $\varphi_1 = \varphi_2$ on $I_1 \cap I_2$.

We now know that φ is well-defined. It is C^1 since each φ_i is C^1 on I_i, and it solves the ODE since each φ_i does so on I_i. □

What happens if f is not Lipschitz? Well then φ_1 may not be identically equal to φ_2 on $I_1 \cap I_2$. Then there are multiple ways to choose φ, and the second statement is ambiguous.

Theorem 7.14 (Blowup Criterion)**.** *Suppose $f \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz in x in any bounded region. Suppose that φ is a C^1 solution to $\frac{\mathrm{d}}{\mathrm{d}t}x = f(t,x)$ on (t_-, t_+). The solution can be extended to $(t_-, t_+ + \varepsilon)$ for some $\varepsilon > 0$ if and only if one of the following equialent conditions hold:*

(i) There exists a sequence $\{t_n\}$ such that $t_n \longrightarrow (t_+)^-$ and $\varphi(t_n) \longrightarrow \varphi_0 \neq \pm\infty$. Note that this is true for oscillating behavior but not monotonic blowup.

(ii) $\limsup_{t \to (t_+)^-} |\varphi(t)| < +\infty$.

In particular, the solution cannot be extended beyond t_+ if $\lim_{t \to (t_+)^-} |\varphi(t)| = +\infty$.

Proof. Note that the forward direction is implied by the gluing lemma. For the converse, suppose $|\varphi(t_n) - \varphi_0| \leq 1$ for each $n \geq N$. The idea is to apply existence and uniqueness at $(t_n, \varphi(t_n))$ and hope that we get a solution on $(t_n - h, t_n + h)$ with $t_n + h > t_+$.

Write

$$S := [t_+ - 1, t_+ + 1] \times [\varphi_0 - 1, \varphi_0 + 1]$$
$$D := [t_+ - 2, t_+ + 2] \times [\varphi_0 - 2, \varphi_0 + 2]$$

We will control $(t_n, \varphi(t_n))$ to be in S, and φ to be in D. By f being continuous and Lipschitz in x in D, we have

$$\begin{cases} |f| \leq M \\ |f(t,x) - f(t,y)| \leq L|x - y| \end{cases}$$

in D for some $L, M > 0$. In particular, $|\varphi'| \leq M$ in D. Applying existence and uniqueness at $(t_n, \varphi(t_n))$ gives a unique solution ψ such that $\psi(t_n) = \varphi(t_n)$ and with domain $t_n - h, t_n + h$, where h can be chosen as

$$0 < h < \min\left\{1, \frac{1}{M}, \frac{1}{L}\right\}.$$

Since h is independent of n, we may choose n large enough so that $t_n + h > t_+$. □

Q: which ODEs admit global solutions?

Proposition 7.15. *Consider $x' = f(t,x)$ with $|f| \leq M$ for each (t,x), $f \in C(\mathbb{R}, \mathbb{R})$ and Lipschitz in x in any bounded region. Then the solution exists globally and is unique.*

Proof. Assume the solution exists on (t_-, t_+). Note that

$$|x(t) - x(t_0)| = \left| \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s \right| \leq M|t - t_0| < \infty.$$

By the blowup criterion, we know that x can be extended beyond t_+ and t_-. □

Remark 7.16. The idea of existence-uniqueness / blowup criterion applies also to some time evolution PDEs e.g., $\partial_t u = N(u, \partial u, \dots)$. We similarly have local existence-uniqueness, and we cannot extend beyond t_+ if and only if $\lim_{t \to (t_+)^-} \|u(t,x)\|_Y = \infty$.

7.2 Finite Time Blowup

Consider the Riccatti ODE $y' = y^p$, $y(0) = 1$.

- $p = 1$. The unique solution is $y(t) = e^t$.
- $p > 1$. By the comparison principle we have $y \geq e^t$ since $y(0) \geq 1$. By separation of variables, we have

  $$\frac{y'}{y^p} = 1 \implies y^{1-p}(t) = 1 + (1-p)t.$$

  Note that

  $$y^{p-1}(t) = \frac{1}{1-(p-1)t} \to +\infty \text{ as } t \to \left(\frac{1}{p-1}\right)^-.$$

  So we have blow up at $t_+ = 1/(p-1) < \infty$.

Example 7.17. Consider $y' = e^y \geq y^2$. By comparison with $y' = y^2$, we have finite time blowup. Consider $y' = y^n + \cdots + 1 + e^t$, $y(0) = 1$. By comparison with $y' = y^2$, we have finite time blowup.

Example 7.18. $y' = y^2$, $y(0) = y_0$.

- If $y_0 > 0$, we have finite time blowup.
- If $y_0 = 0$, we have $y \equiv 0$.
- If $y_0 < 0$, we have $y = y_0/(1 - ty_0)$. Note that $1 - ty_0 > 0$ for $t > 0$. Thus $y_0 \sim -1/t$ as $t \to +\infty$.

From this we see nonlinear stabilization: negative IC gives global solution decaying to 0, while positive IC gives finite time blowup.

Example 7.19. Consider $y' = y^2 + \varepsilon$ for $\varepsilon > 0$ and $y(0) = y_0$. Note that $y(t) \geq y_0 + \varepsilon t$ and so $y(t_0) > 0$ for t_0 large. Then, we have $\dot{y} > y^2$ and thus blowup.

Example 7.20. Consider $y' = y^2 - \varepsilon^2$. We have $y_1 \equiv \varepsilon$ and $y_2 \equiv -\varepsilon$. For $y_0 > \varepsilon$ we again have $y' \geq y^2$ and so blowup.

8 Autonomous ODE, Stability, and Phase Portrait/Plane

Recall that a general first order ODE is $x' = F(t, x)$, where $x, F \in \mathbb{R}^{n\times 1}$. An **autonomous** ODE is one where F does not depend on t:

$$x' = F(x).$$

A special solution is the constant solution$x(t) \equiv x_*$. We say x_* is in **equilibrium** or a **critical point** if $F(x_*) = 0$.

Example 8.1. $x' = x^2 - x + t$ has no constant solution.

We call $(t, x(t))$ the **trajectory** of the solution $x(t)$. For $n = 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix},$$

we call the x_1-x_2 plane the **phase plane**. We may think of x_2 as a function of x_1, $x_2 = x_2(x_1)$, to remove the dependency on t: By the chain rule,

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{\mathrm{d}x_2/\mathrm{d}t}{\mathrm{d}x_1/\mathrm{d}t} = \frac{f_1}{f_2}(x_1, x_2).$$

The representative set of trajectories is called the **phase portrait**.

Example 8.2. The ODE $x' = x^2 - x =: f$ has critical points at $x_* = 0, 1$.

Example 8.3. The system

$$\begin{cases} x' = x^2 - 2y \\ y' = x + y \end{cases}.$$

Note that $x^2 = 2y = -2x$ gives $x = 0$ or $x = -2$. This gives the critical points $(0, 0)$ and $(-2, 2)$.

8.1 Phase Portrait of 2 x 2 Systems

Consider the linear 2×2 system:

$$x' = Ax, \quad A \in \mathbb{R}^{2\times 2}.$$

The equilibrium is at $x_* = 0$ and the real eigen-vectors associated with the eigenvalue 0. But what happens as $t \to \infty$ for a general solution x?

Remark 8.4. Recall first that for

$$z(t) := e^{\alpha t}\left(\cos\beta t + \mathrm{i}\sin\beta t\right),$$

we have

(i) If $\alpha > 0$, then $|z(t)| \to \infty$ as $t \to \infty$.

(ii) If $\alpha = 0$, then $z(t)$ is bounded, periodic, and in particular does not converge.

(iii) If $\alpha < 0$, then $z(t) \to 0$ exponentially fast as $t \to \infty$.

We consider the following three cases:

Case 1:

$\lambda_1 < \lambda_2$ real. The general solution is

$$x(t) = \underbrace{e^{\lambda_t}}_{\text{amplitude}} \underbrace{\left(e^{(\lambda_1-\lambda_2)t}c_1v_1 + c_2v_2\right)}_{\text{direction}}.$$

(i) If $\lambda_1 < 0$, then any solution of the form

$$x(t) = c_1 e^{\lambda_1 t} v_1$$

will converge to 0 as $t \to \infty$, remaining on the line spanned by v_1 on the way.

(ii) (**Sink**) $\lambda_1 < \lambda_2 < 0$, then $x(t) \approx e^{\lambda_2 t} c_2 v_2$ for large t. Thus $x(t) \to 0$ as $t \to \infty$ (converging in the v_1 direction faster).

(iii) (**Saddle**) $\lambda_1 < 0 < \lambda_2$. We have $x \to 0$ only if $x \equiv 0$.

(iv) (**Source**) $0 < \lambda_1 < \lambda_2$, then $x(t) \to \infty$ (diverging in the v_2 direction faster) as $t \to \infty$ unless $x \equiv 0$.

These can be visualized in a phase portrait.

Case 2:

$\lambda = \alpha \pm \mathrm{i}\beta$, $v = a \pm \mathrm{i}b$, $b \neq 0$. The general solution is

$$\begin{aligned} x(t) &= c_1 \operatorname{Re}(e^{\lambda t} v) + c_2 \operatorname{Im}(e^{\lambda t} v) \\ &= e^{\alpha t}\left[c_1 (\cos\beta t a - \sin\beta t b) + c_2 (\sin\beta t a + \cos\beta t b)\right] \\ &= e^{\alpha t} V(t), \end{aligned}$$

where

$$V(t) \coloneqq c_1 (\cos\beta t a - \sin\beta t b) + c_2 (\sin\beta t a + \cos\beta t b)$$

is periodic in t.

(i) If $\alpha < 0$, $|x| \to 0$.

(ii) If $\alpha = 0$, then $x(t) = V(t)$ is bounded and periodic (but does not converge).

(iii) If $\alpha > 0$, then $|x| \to \infty$.

Case 3:

One real eigen-value with two linearly independent eigen-vectors. General solution

$$x(t) = e^{\lambda t} \underbrace{(c_1 v_1 + c_2(t v_1 + v_2))}_{\text{constant}}.$$

The trajectory is along a straight line.

Case 4:

One eigenvector v_1 and one generalized eigen-vector v_2 such that $(A-\lambda)^2 v_2 = 0$. General solution:

$$\begin{aligned} x(t) &= e^{\lambda t}\Big(c_1 v_1 + c_2 t v_2\Big) \\ &= e^{\lambda t} t\left(\frac{1}{t} c_1 v_1 + c_2 v_2\right). \end{aligned}$$

For large t, the trajectory is approximately along the line spanned by v_2.

(i) If $\lambda > 0$, $|x| \to \infty$ exponentially fast.

(ii) If $\lambda = 0$, $|x| \sim t$.

(iii) $\lambda < 0$, $|x| \to 0$ exponentially fast.

Q: How smooth is x_2 as a function of x_1? Obviously Case 3 is C^∞.
Nonlinear phase plane/portrait is much harder:

Example 8.5.

$$\begin{cases} x_1' = f_1(x_1, x_2) \\ x_2' = f_2(x_1, x_2) \end{cases}.$$

Suppose for example that f_i are polynomials. We may have a few critical points, say $(0,0)$ and x_c. A hard problem: find a C^∞ trajectory starting from 0 and passing x_c.

Note that by existence and uniqueness, trajectories cannot cross (except at critical points).

8.2 Stability

We say the critical point x_* is

- **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - x_*| < \delta$, $|x(t) - x_*| < \varepsilon$ for each $t \geq t_0$.
- **asymptotically stable** if there exists $\delta > 0$ such that if $|x_0 - x_*| < \delta$, then $\lim_{t\to\infty} x(t) = x_*$.
- **unstable** if it is not stable.

We summarize the stability of the critical point $(0,0)$ for the 2×2 linear systems:

(i) If $\operatorname{Re}\lambda_i < 0$ for $i = 1, 2$, then both stable and asymptotically stable (sink).

(ii) If $\operatorname{Re}\lambda_i > 0$ for some i, then unstable.

(iii) If $\operatorname{Re}\lambda_i \leq 0$ for all i and $\operatorname{Re}\lambda_i = 0$ for some i, then not asmptotically stable. Moreover,

    (i) if $\lambda_1 \neq \lambda_2$, then stable ($|e^{\lambda_1} t| = |e^{\lambda_2 t}| \leq 1$).

(ii) if $0 = \lambda_1 = \lambda_2$, 2 eigenvectors, stable.

(iii) $0 = \lambda_1 = \lambda_2$, where λ_2 is the generalized eigen-vector, then unstable (since $|x(t)| \sim t$).

Example 8.6. Consider the pendulum equation

$$\theta'' + \gamma\theta' + w^2 \sin\theta = 0.$$

It has two critical points: $(0, 0)$ and $(\pi, 0)$. The latter corresponds to the inverted pendulum and is unstable.

Lemma 8.7. *For each $\varepsilon > 0$ and integer $n \geq 0$, there exists a constant $C > 0$ such that*

$$t^n \leq Ce^{\varepsilon t}, \quad t \geq 0.$$

Proof. Note that

$$t^n = e^{\frac{n}{\varepsilon}\cdot\varepsilon \log t}, \quad Ce^{\varepsilon t} = e^{\varepsilon C + \varepsilon t}.$$

The first exponent is concave. □

Theorem 8.8 (Linear Stability)**.** *Consider the ODE system $x' = Ax$, $A \in \mathbb{R}^{n\times n}$, with $x_* = 0$ as a critical point. Let λ_i be the eigen-values of A.*

(i) If $\operatorname{Re}\lambda_i < 0$ for each i, then x_* is stable and asymptotically stable.

(ii) If $\operatorname{Re}(\lambda_i) > 0$ for some i, then x_* is unstable.

(iii) If $\operatorname{Re}(\lambda_i) \geq 0$ for some i, then x_* is not asymptotically stable.

(iv) If $\operatorname{Re}(\lambda_i) \leq 0$ for each i and if the Jordan block J_i associated with λ_i, $\operatorname{Re}(\lambda_i) = 0$, is 1×1, then x_* is stable. If the block is larger than 1×1, then x_* is unstable.

Remark 8.9. To gain some intuition of why only the real parts matter, note that

$$|e^{(\alpha+\mathrm{i}\beta)t}| = |e^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t)| = e^{\alpha t}.$$

If $\alpha > 0$, $|x(t)| \to \infty$ and x_* is unstable. If $\alpha = 0$, $|x(t)| \equiv |x_0|$ and x_* is stable. If $\alpha < 0$, $|x(t)| \to 0$ and x_* is asymptotically stable.

Proof. Recall that the solution is $x(t) = e^{At}x_0$. Write $A = UJU^{-1}$ and note that

$$e^{At} = Ue^{Jt}U^{-1}, \quad e^{Jt} = \begin{pmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_k t} \end{pmatrix},$$

where J_j are the Jordan blocks of A.

We start from case (ii). Note that $x(t) = e^{\lambda_i} v_i$ solves the ODE (if λ_i is complex, take the real or complex parts of x). We have

$$|x(t)| = |cv_i|e^{\operatorname{Re}(\lambda_i)t} \longrightarrow \infty.$$

Thus x_* is unstable.

Next, consider case (iii). If $|\lambda_i| \geq 0$, we have with a small perturbation of size c that

$$|x(t)| = |cv_i| e^{\mathrm{Re}(\lambda_i)t} \geq |cv_i|,$$

which does not converge to 0.

Now consider (i). Recall that

$$e^{J_j t} = \begin{pmatrix} e^{\lambda_j t} & te^{\lambda_j t} & \frac{t^2}{2} e^{\lambda_j t} & \cdots & \frac{t^{m_j-1}}{(m_j-1)!} e^{\lambda_j t} \\ 0 & e^{\lambda_j t} & te^{\lambda_j t} & \cdots & \frac{t^{m_j-2}}{(m_j-2)!} e^{\lambda_j t} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_j t} \end{pmatrix}.$$

We claim (Ex.) that $|t^m| \leq e^{\varepsilon t} C_{m,\varepsilon}$ for each $t \geq 0$. (E.g., $t^{1000} \leq e^{0.1t} C$ for each $t \geq 0$.) We have the estimate

$$|e^{\lambda_i t} t^m| \leq |e^{\mathrm{Re}\,\lambda_i t} t^m| \leq C_A e^{-(2/3)\lambda t},$$

where we pick $\varepsilon = \lambda/3$. This gives

$$|(e^{J_j t})_{kl}| \leq C_A e^{-(2/3)\lambda t} \implies |(e^{Jt})_{ij}| \leq C_A e^{-(2/3)\lambda t}.$$

Now note that

$$|(e^{At})_{ij}| \leq \sum_{k,l} |U_{ik}| |(e^{Jt})_{kl}| |(U^{-1})_{lj}|.$$

We thus have

$$|(e^{At})_{ij}| \leq C_a n^2 C_a e^{-(2/3)\lambda t} \leq C_A e^{-(2/3)\lambda t}.$$

Then,

$$|e^{At} x_0| \leq C_A e^{-(2/3)\lambda t} |x_0| \to 0.$$

Finally, consider (iv). Recall that

$$e^{J_i t} = \begin{pmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} e^{\lambda_i t} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_i t} \end{pmatrix}.$$

If $n_i \geq 2$ then x_* is unstable. Ex: find x_0 cleverly such that $e^{At} x_0 \to \infty$. If $n_i = 1$ for each n_i, then $|e^{J_i t}| \approx 1$. □

Example 8.10. $x' = \begin{pmatrix} -2 & \\ & -1 \end{pmatrix} x$ is asymptotically stable and stable. $x' = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} x$ has eigen-values ±i and is stable but not asymptotically stable.

8.3 Nonlinear Stability

Consider

$$x' = F(x), \quad F(x_*) = 0, \quad F \in C^2.$$

We linearize around x_*:

$$x' = F(x) = F(x_*) + \nabla F(x_*)(x - x_*) + G(x - x_*).$$

If $|x - x_*| < 1$, we have $G(x - x_*) \le M|x - x_*|^2$ for some $M > 0$. Write $z(t) = x(t) - x_*$ and note that

$$z'(t) = x'(t) = Az + G(z), \quad A := \nabla F(x_*),$$

where

$$|G(z)| \le M|z|^2, \quad \forall |z| < 1.$$

This system has the equilibrium $z_* = 0$. The hope is that for small z, $|G(z)| \le M|z|^2 \ll |z|$, and the solution of the nonlinear system behaves like that of the linear system $z' = Az$.

Theorem 8.11 (Nonlinear Stability). *Let λ_i be the eigenvalues of $A := \nabla F(x_*)$, where $F \in C^2$.*

(i) If $\mathrm{Re}(\lambda_i) < 0$ for each i, then x_* is asymptotic stable and stable.

(ii) If for some i we have $\mathrm{Re}\,\lambda_i > 0$, then x_* is unstable.

In other cases (say $\mathrm{Re}\,\lambda_i = 0$) it is unclear (we need some information from G.)

Example 8.12. Consider $x' = 0 + x^2$. The IC $\varepsilon > 0$ results in finite time blowup, while IC $\varepsilon < 0$ results in global solution converging to 0.

Proof. Case (ii) is left as an exercise. Consider (i).

Idea: $z' = Az + o(1)Z$. Suppose $\delta = |z_0| \ll 1$ and note that

$$|z(t)| \approx |e^{At} z_0| \le C_A e^{-(2/3)\lambda t} |z_0|$$

and

$$|G(z(t))| \le M|z(t)|^2 \le M C_A^2 \delta^2 e^{-(4/3)\lambda t}.$$

Goal: If $|z_0| = \delta \ll 1$, then $d(t) := e^{\frac{3}{2}\lambda t} |z(t)| \le \tilde{C}_A \delta$.

Step 1: the solution exists for a short time. Step 2: By Duhamel formula,

$$z(t) = \overbrace{e^{At}}^{I} z_0 + \overbrace{\int_0^t e^{A(t-s)} \underbrace{G(z(s))}_{F(s)} \, \mathrm{d}s}^{II}.$$

We hope to derive an ODE inequality for $d(t)$. Denote as I and II the two terms on the RHS. We have

$$|I| \le C_A e^{-\frac{2}{3}\lambda t} |z_0|$$

and
$$|II| \leq C_A \int e^{-\frac{2}{3}\lambda(t-s)}|G(z(s))| \leq \tilde{C}_A \int e^{-\frac{2}{3}\lambda(t-s)}|z(s)|^2.$$

Since $|z(s)| = d(s) \cdot e^{-\frac{2}{3}\lambda t}$, we have
$$|II| \leq \tilde{C}_A e^{-\frac{2}{3}\lambda t} \int_0^t \underbrace{e^{\frac{2}{3}\lambda s} e^{-\frac{4}{3}\lambda s} d(s)^2}_{|z(s)|^2} \, \mathrm{d}s.$$

This gives
$$d(t) = e^{\frac{2}{3}\lambda t}|z(t)| \leq e^{\frac{2}{3}\lambda t} \left[|I| + |II|\right] \leq C_A\delta + \tilde{C}_A \int_0^t e^{-\frac{2}{3}\lambda s} d(s)^2 \, \mathrm{d}s.$$

We now claim that $d(t) < 2C_A\delta$ for each $t \geq 0$ if $\delta \ll 1$. We induct on time (this is called a bootstrap argument). Base case $t = 0$: $d(0) = |z_0| = \delta$. Assume that the claim is true on $[0, t)$; from this we will prove that it remains true at time t:
$$\begin{aligned} d(t) &\leq C_A\delta + \tilde{C}_A \int_0^t e^{-\frac{2}{3}\lambda s} d^2(s) \, \mathrm{d}s \\ &\leq C_A\delta + \tilde{C}_A (2C_A)^2 \int_0^t e^{-\frac{2}{3}\lambda s} \delta^2 \, \mathrm{d}s \\ &\leq C_A\delta + C_{A,2}\delta^2. \end{aligned}$$

Take $\delta \leq C_A/2/C_{A,2}$ and we have
$$d(t) \leq C_A\delta + \frac{1}{2}C_A\delta < 2C_A\delta.$$

Equivalently,
$$|z(t)| < 2C_A\delta e^{-\frac{2}{3}\lambda t} \longrightarrow 0.$$

□