ECON20210 (S25): The Elements of Economic Analysis III Honors

Lecturer: Kotaro Yoshida

Notes by: Aden Chen

Monday 26th May, 2025

Contents

1 Difference Equations

We start with the difference equation

$$x_{t+1} = f(x_t).$$

Let $\overline{x}$ be a steady state: $f(\overline{x}) = \overline{x}$.

1.1 Linearization

First order Taylor approximation gives

$$x_{t+1} - \overline{x} \approx f'(\overline{x})(x_t - \overline{x}).$$

Then,

$$\begin{aligned} x_t &= (f'(\overline{x}))^t (x_0 - \overline{x}) + \overline{x} \\ &= (f'(\overline{x}))^t x_0 + (1 - (f'(\overline{x}))^t)\overline{x}. \end{aligned}$$

The steady state is locally stable if $|f'(x)| < 1$.

1.2 Log Linearization

We write

$$x_t = \overline{x} \exp(u_t).$$

Note that $u_t = \log x_t - \log \overline{x}$ is the approximate percentage deviation of x_t from $\overline{x}$. Then we have

$$\overline{x} \exp(u_{t+1}) = f(\overline{x} \exp(u_t)).$$

Linearization gives

$$\overline{x}(1 + u_{t+1}) \approx \overline{x} + f'(\overline{x})\overline{x}u_t.$$

Then,

$$u_{t+1} = f'(\overline{x})u_t.$$

So if $|f'(\overline{x})| < 1$ we have local stability and

$$\frac{x_t - \overline{x}}{\overline{x}} = (f'(\overline{x}))^t \left(\frac{x_0 - \overline{x}}{\overline{x}}\right).$$

2 Measurement

2.1 GDP

- Nominal GDP values goods and services at current prices.

$$Y_t^n := \sum_i P_{i,t} Q_{i,t}.$$

- Real GDP values goods and services at constant prices.

$$Y_t^r := \sum_i P_{i,0} Q_{i,t}.$$

- The GDP deflator is the ratio of nominal to real GDP.

$$P_t := \frac{Y_t^n}{Y_t^r}.$$

  This is also the Paasche index.

GDP can be thought of as a measurement of expenditure, income, and output:

$$\begin{aligned}
Y &= \underbrace{C}_{\sim 70\%} + \underbrace{I}_{\sim 15\%} + G + \underbrace{EX - IM}_{\text{net exports}} \\
&= \underbrace{wL}_{\sim 66\%} + \underbrace{\pi + rK}_{\sim 33\%} + T \\
&= f(A, K, L, X).
\end{aligned}$$

2.2 Price Index

- The Laspeyres index uses base-period quantities:

$$P_L \coloneqq \frac{\sum_i P_{i,t} Q_{i,0}}{\sum_i P_{i,0} Q_{i,0}}.$$

- The Paasche index uses current-period quantities:

$$P_P \coloneqq \frac{\sum_i P_{i,t} Q_{i,t}}{\sum_i P_{i,0} Q_{i,t}}.$$

Remark 2.1.

- The Laspeyres index is an upper bound on the true cost of living index, while the Paasche index is a lower bound.
- In the Laspeyre index, for example, we are ignoring the possibility of substitution between goods, or new goods being introduced, or old goods being made better.
- A decomposition of the Laspeyres index is given by

$$P_L = \frac{\sum_i P_{i,t} Q_{i,0}}{\sum_i P_{i,0} Q_{i,0}} = \sum_i \frac{\sum P_{i,0} Q_{i,0}}{\sum P_{i,0} Q_{i,t}} \frac{P_{i,t}}{P_{i,0}}$$

- The “ture” inflation rate should be given by

$$X_t = \frac{C(u_0, P_t)}{C(u_0, P_0)}.$$

Proposition 2.2. *The numerator of the Laspeyres index is a Taylor approximation of $C(u_0, P_t)$. The denominator of the Paasche index is a Taylor approximation of $C(u_t, P_0)$. Since C is concave in P, the Laspeyres index overestimates the true inflation rate while the Paasche index underestimates it.*

Proof. We prove the first statement. Note that $P_t \cdot x(u_0, P_0)$ is a first order Taylor approximation to C starting at P_0:

$$\begin{aligned} C(u_0, P_t) &\approx C(u_0, P_0) + \sum (P_{i,t} - P_{i,0}) \frac{\partial C(u_0, P_{t,0})}{\partial P_i} \\ &= \sum P_{i,0} Q_{i,0} + \sum (P_{i,t} - P_{i,0}) Q_{i,0}, \end{aligned}$$

where the second line follows from Shephard's lemma. □

The Fisher ideal index is a middle ground:

Definition 2.3 (Fisher Ideal Index)**.**

$$\left(\frac{\sum_i p_{i,t} q_{i,0}}{\sum_i p_{i,0} q_{i,0}} \right)^{\frac{1}{2}} \left(\frac{\sum_i p_{i,t} q_{i,t}}{\sum_i p_{i,0} q_{i,t}} \right)^{\frac{1}{2}}.$$

Note that this geometric average is equivalent to the arithmetic average of the net inflation rates. (To see this, take log on both sides).

3 Solow Growth Model

We hope to explain the following facts:

(i) $wN/Y \approx 2/3$, $r^k K/Y \approx 1/3$.[1]

(ii) $k := K/N$; $\Delta k/k \approx 2 - 3\%$.

(iii) $y := Y/N$; $\Delta y/y \approx 2 - 3\%$.

(iv) K/Y constant, r constant.

(v) Conditional convergence.

Assuming the production function $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$, we have

$$\frac{\Delta Y_t}{Y_t} \approx \underbrace{\frac{\Delta A_t}{A_t}}_{\text{Solow Residual}} + \alpha \frac{\Delta K_t}{K_t} + (1 - \alpha) \frac{\Delta N_t}{N_t}.$$

3.1 Model Setup

- Law of motion of capital:

$$K_{t+1} = sY_t + (1 - \delta) K_t.$$

Phase diagram: investment and BEI against K or k.

[1] r^k is the price of capital.

3.2 Model Setup: Version I

- Exogenous saving rate.
- $Y_t = AN^{1-\alpha}K_t^\alpha$, so
$$\Delta K_t = sAN^{1-\alpha}K_t^\alpha - \delta K_t.$$

3.3 Steady State

Set $\Delta K_t = 0$ to get
$$K_{ss} = \left[\frac{sAN^{1-\alpha}}{\delta}\right]^{\frac{1}{1-\alpha}}.$$

Phase diagram (K_t, K_{t+1}).

- No growth in k or y.
- Assuming competitive market, factor price equals marginal product, and
  - $wN = (1-\alpha)Y$ and $r^k K = \alpha Y$.
  - Y/K constant, r constant (since K constant at SS).
- No conditional convergence.

3.4 Model Setup: Version II

- Exogenous growth in A and N:
$$\frac{A_{t+1}}{A_t} = 1 + g, \quad \frac{N_{t+1}}{N_t} = 1 + n.$$
- Technology augmented labor production: $Y_t = (A_t N_t)^{1-\alpha} K_t^\alpha$.
- Define $k_t := K_t/(A_t N_t)$ as the effective-labor-normalized capital. Define y_t similarly.

3.5 Steady State

From the law of motion we have
$$\frac{\Delta K_t}{K_t} = sk_t^{\alpha-1} - \delta.$$

Then use
$$\frac{\Delta K_t}{K_t} = \frac{\Delta A_t N_t k_t}{A_t N_t k_t} \approx \frac{\Delta A_t}{A_t} + \frac{\Delta N_t}{N_t} + \frac{\Delta k_t}{k_t}$$
to deduce $\Delta k_t / k_t = sk_t^{\alpha-1} - \delta - g - n$ and
$$\Delta k_t = sk_t^\alpha - (\delta + g + n)k_t.$$

Thus

$$k_{ss} = \left[\frac{s}{\delta + n + g}\right]^{\frac{1}{1-\alpha}}.$$

Phase diagram: BEI and investment as a function of k_t.

- $y_t = k_t^{\alpha}$, so at steady state, y and k do not grow.
- Interest rate is

$$1 + r = \frac{\partial Y_t}{\partial K_t} = \alpha \left(\frac{A_t N_t}{K_t}\right)^{1-\alpha} - \delta = \alpha \cdot \frac{n + g + \delta}{s} - \delta.$$

- From approximation above, K and Y (and thus I and C) grow at the rate of $n + g$.
- Can show that $\tilde{y} := Y_t/N_t = A_t y_t$ (and similarly k_t) grows at the rate of g using the approximation

$$\frac{\Delta \tilde{k}_t}{\tilde{k}_t} \approx \frac{\Delta A_t}{A_t} + \frac{\Delta k_t}{k_t}.$$

- Condition convergence can be explained by the fact that conditional on the same parameters, countries converge over time. (Be uneasy of such explanations.)

3.6 Golden Rule

The golden rule saving rate can be found by solving $s_{gr} = \arg\max_s y_{ss} - (\delta + n + g)k_{ss}$, or equivalently by maximizing over k_{ss}:

$$k_{gr} := \arg\max_k k^{\alpha} - (\delta + n + g)k = \left[\frac{\alpha}{\delta + n + g}\right]^{\frac{1}{1-\alpha}}.$$

Compare with the expression for k_{ss} to see that

$$s_{gr} = \alpha.$$

- Economies that over save is dynamically inefficient since it can increase consumption at each time period by reducing saving rates to that dictated by the golden rule.

4 Neoclassical Growth Model

4.1 Setup

$$\max_{\{C_t, K_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(C_t) \quad \text{s.t.} \quad f(K_t) = C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{I_t}.$$

Lagrangian:

$$\mathcal{L} := \sum_{t=0}^{T} \beta^t u(C_t) + \sum_{t=0}^{T} \lambda_t \left[f(K_t) - C_t - K_{t+1} + (1-\delta)K_t \right].$$

FOCs:

$$\begin{aligned} \{C_t\}_{t=0}^{T} : \qquad & \beta^t u'(C_t) = \lambda_t \\ \{K_{t+1}\}_{t=0}^{T} : \qquad & \lambda_{t+1} \left[f'(K_{t+1}) + 1 - \delta \right] = \lambda_t . \end{aligned}$$

4.2 Euler Equation and the Steady State

Proposition 4.1 (Euler Equation)**.**

$$u'(C_t) = \beta u'(C_{t+1})[f'(K_{t+1}) + 1 - \delta].$$

Remark 4.2. The marginal utility of consuming an addition unit today equals the time discounted marginal utility of saving to the next period. Thus $1 + r = f'(K_{t+1}) + 1 - \delta$ is the exchange rate. ☕

At steady state, K and thus Y and C is constant. By the Euler equation,

$$1 = \beta[f'(K_{ss}) + 1 - \delta].$$

4.3 Transversality

In finite horizon we impose $K_{T+1} \geq 0$ and have the additional term $\mu_T K_{T+1}$ in the Lagrangian, with the following additional FOCs

$$\begin{aligned} \{K_{T+1}\} : \qquad & \lambda_T = \mu_T \\ \{\mu_T\} : \qquad & \mu_T K_{T+1} = 0. \end{aligned}$$

In the infinite horizon problem, we impose the

Proposition 4.3 (Transversality Condition)**.**

$$\lim_{T\to\infty} \underbrace{\beta^T u'(C_T)}_{\lambda_T} K_{T+1} = 0.$$

Remark 4.4. The present value of the capital stock should converge to zero along the optimal path. The term $\beta^t u'(C_t) K_{t+1}$ is the marginal utility of converting one unit of capital in time $t + 1$ into consumption in time t. ☕

The model can be summarized using the following four pieces of information:

(i) Euler equation:

$$u'(C_t) = \beta u'(C_{t+1})[1 + f'(K_{t+1}) - \delta].$$

(ii) Law of motion:

$$K_{t+1} - K_t = \underbrace{f(K_t) - C_t}_{I_t} - \delta K_t.$$

(iii) K_0 (given).

(iv) TVC.

5 Decentralized Neoclassical Model

- Identical households endowed with capital and bonds in period 0, and labor in each period.
- Households owns capital.

5.1 Model Setup: Firms

$$\max_{\{K_t, N_t\}} F(K_t, N_t, A_t) - w_t N_t - R_t K_t$$

FOC:

$$\begin{aligned} \{K_t\}: &\qquad F_K(K_t, N_t, A_t) = R_t \\ \{N_t\}: &\qquad F_N(K_t, N_t, A_t) = w_t \end{aligned}$$

5.2 Model Setup: Households

$$\begin{aligned} &\max_{\{C_t, K_{t+1}, B_{t+1}, N_t\}} \sum_{t=0}^{T} \beta^t u(C_t) \\ \text{s.t.} \quad & C_t + \underbrace{K_{t+1} - (1-\delta)K_t + B_{t+1}}_{I_t} = w_t N_t + R_t K_t + (1+r_t)B_t + \pi_t, \\ & B_0, K_0 \text{ given.} \end{aligned}$$

Assuming perfectly inelastic labor supply $N_t = 1$, we can rewrite the budget constraint as

$$C_t + K_{t+1} + B_{t+1} = w_t + (1 + R_t - \delta)K_t + (1+r_t)B_t + \pi_t.$$

Lagrangian:

$$\mathcal{L} := \sum_t \beta^t u(C_t) + \sum_t \lambda_t \left[w_t + (1 + R_t - \delta)K_t + (1+r_t)B_t + \pi_t - C_t - K_{t+1} - B_{t+1} \right].$$

FOC:

$$\begin{aligned}
\{C_t\}: && \beta^t u'(C_t) - \lambda_t = 0 \\
\{K_{t+1}\}: && -\lambda_t + \lambda_{t+1}(1 + R_{t+1} - \delta) = 0 \\
\{B_{t+1}\}: && -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0 \\
\{\lambda_t\}: && w_t + (1 + R_t - \delta)K_t + (1 + r_t)B_t + \pi_t - C_t - K_{t+1} - B_{t+1} = 0
\end{aligned}$$

5.3 The Market Clearing Conditions

$$\begin{aligned}
\text{Goods}: \quad & C_t + K_{t+1} - (1-\delta)K_t = F(K_t, N_t, A_t) \\
\text{Bonds}: \quad & B_{t+1} = 0 \\
\text{Capital}: \quad & K^s_{t+1} = K^d_{t+1} \qquad \text{or} \quad f_K(A_{t+1}, K_{t+1}, N_{t+1}) = R_{t+1} \\
\text{Labor}: \quad & N^s_t = N^d_t \qquad \text{or} \quad f_K(A_t, K_t, N_t) = w_t.
\end{aligned}$$

Remark 5.1. The constraints can be equivalently expressed as supply equals demand or prices being equal.

Definition 5.2. The **competitive equilibrium** of this economy consists of a sequence of allocations $\{C_t, K_{t+1}, B_{t+1}, N_t\}$ and prices $\{R_t, w_t, r_t\}$ such that

(i) The household maximizes its utility by choosing $\{C_t, K_{t+1}, B_{t+1}, N_t\}$ given $\{R_t, w_t, r_t\}$ and K_0, B_0.

(ii) The firm maximizes profits in every period t at $\{K_t, N_t\}$ given $\{w_t, R_t\}$.

(iii) All markets clear.

5.4 Characterization

From $\{K_{t+1}\}$ and $\{B_{t+1}\}$ we get the "no arbitrage" condition:

$$r_{t+1} = R_{t+1} - \delta.$$

From $\{C_t\}$, $\{K_{t+1}\}$, and $\{B_{t+1}\}$ we get the Euler equations:

$$\begin{aligned}
u'(C_t) &= \beta u'(C_{t+1})(1 + R_{t+1} - \delta) \\
u'(C_t) &= \beta u'(C_{t+1})(1 + r_{t+1})
\end{aligned}$$

We have the TVCs:

$$\begin{aligned}
\lim_{t\to\infty} \beta^t u'(C_t)K_{t+1} &= 0 \\
\lim_{t\to\infty} \beta^t u'(C_t)B_{t+1} &= 0
\end{aligned}$$

The condition $\lim_{t\to\infty} \beta^t u'(C_t)B_{t+1} \geq 0$ is known as the "no ponzi" condition while the opposite inequality is the usual optimizing restriction.

Proposition 5.3. *A complete characterization of the CE is thus given by*

$$\begin{aligned} u'(C_t) &= \beta u'(C_{t+1})(1 + R_{t+1} - \delta) \\ R_{t+1} - \delta &= r_{t+1} \\ \lim_{t\to 0} \beta^t u'(C_t) K_{t+1} &= 0 \\ R_t &= f_K(A_t, K_t, N_t) \\ w_t &= f_N(A_t, K_t, N_t) \\ N_t &= 1 \\ B_{t+1} &= 0 \\ C_t + K_{t+1} - (1-\delta)K_t &= f(A_t, K_t, N_t) \end{aligned}$$

Remark 5.4. The first, third, and last conditions characterizes the decentralized NCG.

6 Dynamic Programming

Example 6.1 (NCG). Recall that $f(K_t) = K_{t+1} - (1-\delta)K_t + C_t$.

$$\begin{aligned} V(K_0) &:= \max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\underbrace{f(K_t) - K_{t+1} + (1-\delta)K_t}_{C_t}) \\ &= \max_{K_1} \{u(f(K_0) - K_1 + (1-\delta)K_0) + \beta V(K_1)\}. \end{aligned}$$

We may derive the policy function $K_t \mapsto K_{t+1}$ and then the function $K_t \mapsto C_t$. This is the saddle path (since gives the level of consumption to which the economy will jump in face of a new shock).

Example 6.2 (Brock and Mirman (1972)).

$$\max \sum \beta^t \log c_t \quad \text{s.t.} \quad k_{t+1} = k_t^\alpha - c_t$$

with k_0 given. We have the Bellman equation

$$V(k) = \max_{k'} \log(k^\alpha - k') + \beta V(k').$$

Algorithm 6.3 (Numerical Value Function Iteration)**.**

(i) Discretize the state space.

(ii) For each possible x, guess $V_0(x)$ (e.g.,$V_0(x) = 0$).

(iii) For each x, maximize the rand hand size of the Bellman equation.

Repeat until $V_i = V_{i+1}$.

7 Real Business Cycle

We are interested in fluctuations in output, consumption, and investment.

Definition 7.1 (Hodrick-Prescott Filter)**.** The trend $\{\tau_t\}$ is found by solving

$$\max_{\{\tau_t\}} \sum_t (x_t - \tau_t)^2 + \lambda \sum_t [(\tau_{t+1} - \tau_t) - (\tau_t - t_{t-1})]^2,$$

where λ is the smoothing parameter.

Remark 7.2.

- As $\lambda \to 0$ we have $\tau_t = y_t$. As $\lambda \to \infty$, we have regression on a linear time trend.
- Common proactive is to use $\lambda = 100$ for annual data and $\lambda = 1600$ for quarterly data.
- Alternatives include: Band Path and the Kalman filter.

7.1 Modern business cycle facts

(i) Business cycles do not exhibit periodic regularity

(ii) Most components of GDP are pro-cyclical.

(iii) Real wage is mildly pro-cyclical.

(iv) Consumption is less volatile relation to output, while investment and durable goods consumption is highly volatile relative to output.

(v) The effects of temporary shocks on output is highly persistent (lasting up to 3 quarters).

(vi) The response of output to a shock is hump-shaped. The effect of a shock on output peaks in the fourth or fifth quarter.

7.2 Features of RBC Theory

- Founded on microeconomics optimization.
- No frictions.
- No need for government intervention to smooth business cycle fluctuations.
- Can generate business cycles that reasonable mimic data.
- Propagating mechanism:

- Investment changes transmit current chock to the next period.
- Labor supply changes amplify the investment changes.
- Shocks are persistent.

- Nominal shocks have no role (money is neutral).

Difficulties:

- Difficult to interpret negative technology shock.
- Does technology fluctuate so frequently?
- Exogenous shocks themselves are not explained by the model.
- Money is empirically known to affect output.
- No involuntary unemployment.

7.3 Model 1: Brock-Mirman with IID Shock

A representative household solves

$$\max_{\{c_t,k_{t+1}\}} \mathbb{E}_0 \sum \beta^t \log c_t \quad \text{s.t.} \quad k_{t+1} = A_t k_t^\alpha - c_t,$$

where k_0 is given and $A_t = e^{\epsilon_t}$ is exogenous, with ϵ_t being iid with mean zero and variance σ^2.

Bellman equation:

$$V(A_t, k_t) = \max_{k_{t+1}} \left\{ u(A_t k_t^\alpha - k^{t+1}) + \beta \mathbb{E}_t V(A, k_{t+1}) \right\}.$$

FOC:

$$\frac{1}{c_t} = \beta \mathbb{E}_t V_k(A, k_{t+1}).$$

Envelope condition:

$$V_k(A_t, k_t) = \frac{\alpha A_t k_t^{\alpha-1}}{c_t}$$

Euler equation:

$$\frac{1}{c_t} = \alpha\beta \mathbb{E}_t \left[\frac{\alpha A_{t+1} k_{t+1}^{\alpha-1}}{c_{t+1}} \right].$$

We may guess $k_{t+1} = (1-\mu) A_t k_t^\alpha$ with μ unknown to find

$$k_{t+1} = \alpha\beta A_t k_t^\alpha,$$

which gives

$$\log k_{t+1} = \log\alpha + \log\beta + \epsilon_t + \alpha \log k_t.$$

Thus with a shock $\epsilon_t = 0$ and $\epsilon_{t+i} = 0$ for $i \geq 1$ we have

$$k_{t+1} \text{ increases by } 1$$
$$k_{t+2} \text{ increases by } \alpha$$
$$k_{t+3} \text{ increases by } \alpha^2,$$

and so on. Recall that in the phase diagram for the neoclassical growth model, when $\Delta k = 0$ shifts up for 1 period, the economy responds by increasing $c \ldots$

The effect of technology shocks dies down quite quickly and cannot replicate the persistence observed in the data. Moreover, it cannot capture the hump-shaped response of output to a shock.

7.4 Model 2: Elastic Labor Supply

$$\max \mathbb{E}_0 \left[\sum \beta^t (\log c_t + \log(1 - n_t)) \right] \quad \text{s.t.} \quad c_t + k_{t+1} = A_t k_t^\alpha n_t^{1-\alpha},$$

where A_t follows the same iid process as before. By guessing the policy function as $k_{t+1} = \mu A_t k_t^\alpha n_t^{1-\alpha}$, we get

$$k_{t+1} = \alpha\beta A_t k_t^\alpha n_t^{1-\alpha},$$
$$n_t = \frac{1-\alpha}{2 - \alpha\beta - \alpha}.$$

In particular, n_t is fixed rather than being pro-cyclical. Note that in face of a positive shock in A_t, the following happens:

- Y_t increases, which cases N_t to decrease (income effect),
- marginal product of labor increases, which causes N_t to increase (substitution effect).

With log utility, the substitution and income effects cancel out exactly. To get pro-cyclical labor supply, we need to modify preference so that the substitution effect dominates the income effect:

7.5 Model 3

$$\max_{\{c_t, h_t, i_t, k_{t+1}\}} \mathbb{E}_0 \sum \beta^t \left[\log c_t - \theta \frac{h_t^{1+\psi}}{1+\psi} \right]$$
$$\text{s.t.} \quad c_t + i_t = \exp(a_t) k_t^\alpha h_t^{1-\alpha}$$
$$k_{t+1} = i_t + (1-\delta) k_t$$
$$k_0 \text{ given.}$$

Assume first $a_{t+1} = \rho a_t + \epsilon_t$ where $0 < \rho < 0$ and ϵ_t's are iid, mean zero, and has variance σ^2. It turns out that under a shock to a_t, we have

- $Y_t \uparrow \implies h_t \downarrow$ (IE),
- MPL $\uparrow \implies h_t \uparrow$ (SE).

This time, however, the substitution effect dominates the income effect, causing $h_t \uparrow$. We then have MPK $\uparrow \implies k \uparrow$.

Without any persistence in ϵ, we could reproduce persistence in k. However, the hump-shaped response of output to a shock is not captured. We thus add a second shock to investment:

7.6 Model 4

$$
\begin{aligned}
\max_{\{c_t,h_t,i_t,k_{t+1}\}} \quad & \mathbb{E}_0 \sum \beta^t \left[\log c_t - \theta \frac{h_t^{1+\psi}}{1+\psi}\right] \\
\text{s.t.} \quad & c_t + i_t = \exp(a_t) k_t^{\alpha} h_t^{1-\alpha} \\
& k_{t+1} = \exp(b_t) i_t + (1-\delta) k_t \\
& k_0 \text{ given.}
\end{aligned}
$$

This time, however, we assume

$$
\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} \rho & \tau \\ \tau & \rho \end{pmatrix} \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ \nu_t \end{pmatrix}.
$$

We may think of a_t and b_t as production and investment technology.

This formulation makes the decay in a in face of a shock to a slower, and generates a hump-shaped response in b. Similarly, in face of a shock to ν_n, a_t will have a hump-shaped response, which then generates a hump-shaped response in y.

In particular, in face of a $b_t \uparrow$ shock, we have (by the utility function we chose) $h_t \uparrow$. Since a_t and k_t remain the same on impact, we have $w =$ MPL $\downarrow$. Afterwards, however, $a_t \uparrow \implies$ MPL $\uparrow$.

Motivation: Explanation for unemployment driven by reservation wage and for churning in the labor market.

- Most employment spells are of short duration, for about two months or less.
- Most people who are unemployed on a given date are experiencing unemployment spells with long duration.

The Beveridge curve plots unemployment rates against job vacancy rates. During recession, unemployment rises and job openings fall.

8 McCall Search Model

8.1 Model Setup

Infinitely lived risk-neutral agent starts off unemployed, receives benefit b in every period unemployed a job offer with wage drawn uniformly iid from $[0, \overline{w}]$. If accepts, she collects w in each period forever after. If rejects, she receives b in the next period and a new job offer. Thus, the agent solves

$$\max \mathbb{E}_0[U(\{c_t\}_{t=0}^{\infty})] = \max \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0[c_t]$$

Bellman equation:

$$V_U(w) = \max \left\{ \underbrace{\frac{w}{1-\beta}}_{\text{Accept}}, \quad \underbrace{b + \beta \mathbb{E}[V_U(w')]}_{\text{Reject}} \right\}.$$

The first term is linear and the second is constant. The two curves intersect at the reservation wage w_R.

8.2 Determining Reservation Wage

We have

$$\begin{aligned}
\frac{w_R}{1-\beta} &= b + \beta \int_0^{\overline{w}} V_U(w') f(w') \, \mathrm{d}w' \\
&= b + \beta \left[\int_0^{w_R} \frac{w_R}{1-\beta} f(w') \, \mathrm{d}w' + \int_{w_R}^{\overline{w}} V_U(w') d(w') \, \mathrm{d}w' \right],
\end{aligned}$$

where the last equality follows from noting that $V_U \equiv w_R/(1-\beta)$ when $w < w_R$. Subtracting from both sides $\beta \int_0^{\overline{w}} \frac{w_R}{1-\beta} f(w') \, \mathrm{d}w' = \frac{\beta w_R}{1-\beta}$ gives

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} (w' - w_R) f(w') \mathrm{d}w',$$

where the second term is "option value," the expected value of time-discounted wage premium $w' - W_R$.

8.3 Comparative Statics

Recall that the Leibniz rule gives

$$\frac{\mathrm{d} \int_a^b f(x;a) \, \mathrm{d}x}{\mathrm{d}a} = \int_a^b \frac{\partial f(x;a)}{\partial a} \, \mathrm{d}x - f(x;a).$$

We have

$$\begin{aligned}
\mathrm{d}w_R &= \mathrm{d}b - \frac{\beta}{1-\beta} \left[\int_{w_R}^{\overline{w}} f(w') \, \mathrm{d}w' \right] \mathrm{d}w_R \\
&= \mathrm{d}b - \frac{\beta}{1-\beta} (1 - F(w_R)) \mathrm{d}w_R
\end{aligned}$$

and so

$$\frac{\mathrm{d}w_R}{\mathrm{d}b} = \frac{1}{1 + \frac{\beta}{1-\beta}(1 - F(w_R))} > 0.$$

A similar derivation gives $\mathrm{d}w_R/\mathrm{d}\beta > 0$.

8.4 Extension

We may add to the model a "separation rate" p: employed individuals lose their jobs with probability p and begin searching for a new job. We have that unemployment rate follows

$$U_{t+1} = (1 - U_t)p + F(w_R)U_t$$

and so

$$U_{ss} = \frac{p}{1 + p - F(w_R)}.$$

9 Asset Pricing: The Lucas Tree Model

Identical agents endowed each with one tree. Individuals trade trees and risk free bonds. Households thus solve

$$\max \sum \beta^t u(c_t) \quad \text{s.t.} \quad c_t + p_t s_{t+1} + b_{t+1} = (d_t + p_s)s_t + (1 + r_t)b_t,$$

where s represents trees, and b bonds, and d the stochastic dividends form each tree at a given time (in each time period, each tree gives the same amount of dividends). The Bellman equation assuming iid dividends is

$$\begin{aligned} V(s_t, b_t) = \max \Big\{ & u(c_t) + \beta \mathbb{E} \left[V(s_{t+1}, b_{t+1}) \right] \\ & + \lambda_t \left[p_t s_t + d_t s_t + (1 - r_t)b_t - c_t - p_t s_{t+1} - b_{t+1} \right] \Big\}. \end{aligned}$$

We have the following FOC:

$$\begin{aligned} \{c_t\}: & \qquad \lambda_t = u'(c_t) \\ \{b_{t+1}\}: & \qquad \beta \mathbb{E}[V_b(s_{t+1}, b_{t+1})] = \lambda_t \\ \{s_{t+1}\}: & \qquad \beta \mathbb{E}[V_s(s_{t+1}, b_{t+1})] = p_t \lambda_t. \end{aligned}$$

Envelope condition:

$$\begin{aligned} \{b_t\}: & \qquad V_b(s_t, b_t) = \lambda_t(1 + r_t) \\ \{s_t\}: & \qquad V_s(s_t, b_t) = \lambda_t(p_t + d_t). \end{aligned}$$

Euler Equation:

(i) $u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1})(1 + r_{t+1}) \right]$,

(ii) $p_t u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1})(p_{t+1} + d_{t+1}) \right]$,

or

(i)

$$1 = \mathbb{E} \left[\frac{\beta u'(c_{t+1})}{u(c_t)} \right] (1 + r_{t+1}).$$

(ii)

$$1 = \mathbb{E}\left[\frac{\beta u'(c_{t+1})}{u(c_t)}\frac{p_{t+1} + d_{t+1}}{p_t}\right].$$

Remark 9.1.

- $\beta u'(c_{t+1})/u'(c_t)$ is called the **stochastic discount factor**. It is the price of consumption in $t+1$ in units of goods at time t, and as such converts units of consumption in $t+1$ into units of consumption at time t.
- The **stochastic return** of the stock/tree is defined by

$$1 + \tilde{r}_{t+1} := \frac{p_{t+1} + d_{t+1}}{p_t}.$$

Transversality condition:

$$\lim_{k\to\infty} \mathbb{E}_t[\beta^k u'(c_{t+k})b_{t+k+1}] = 0$$
$$\lim_{k\to\infty} \mathbb{E}_t[\beta^k u'(c_{t+k})p_{t+k}s_{t+k+1}] = 0.$$

Note that $s \equiv 1$ when market clears.

9.1 Pricing Assets

We have from the Euler equations that

$$\begin{aligned} p_t &= \mathbb{E}_t\left[\frac{\beta u'(c_{t+1})}{u(c_t)}\left(d_{t+1} + \underbrace{\mathbb{E}_{t+1}\left[\frac{\beta u'(c_{t+2})}{c_{t+1}}(p_{t+2} + d_{t+2})\right]}_{p_{t+1}}\right)\right] \\ &= \mathbb{E}_t\left[\frac{\beta u'(c_{t+1})}{u(c_t)}d_{t+1}\right] + \mathbb{E}_t\left[\frac{\beta u'(c_{t+2})}{u(c_t)}d_{t+2}\right] + \mathbb{E}_t\left[\frac{\beta u'(c_{t+2})}{u(c_t)}p_{t+2}\right], \end{aligned}$$

where the second line is justified by the law of iterated expectations. Repeating this procedure gives

$$p_t = \mathbb{E}_t \sum \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} + \mathbb{E}_t\left[\lim_{j\to\infty}\frac{\beta^j u'(c_{t+j})}{u'(c_t)} p_{t+j+1}\right],$$

where the last term tends to 0 by the TVC.

Now, write $m_{t+j} := \beta^j u'(c_{t+j})/u'(c_t)$ to be the stochastic discount factor in time $t+j$ to time t and $x_{t+1} := d_{t+1} + p_{t+1}$ to be the cash flow in time $t+1$. We have:

Proposition 9.2.

$$p_t = \mathbb{E}_t[m_{t+1}x_{t+1}] = \mathbb{E}_t \sum_{j=0}^{\infty} m_{t+j}d_{t+j}.$$

*The price of the risking asset is the expected present value of all future dividends. The first equality is sometimes called the **fundamental asserting pricing equation**.*

Example 9.3. Consider the case of linear utility with known dividends $\delta_{t+1} = (1+g)d_t$. Then,

$$p_t = \mathbb{E}_t \sum \beta^j d_{t+j} = \sum \beta^j (1+g)^j d_t = \sum \left(\frac{1+g}{1+\rho}\right)^j d_t = \frac{1+g}{g-\rho} d_t.$$

This is the **Gordan formula**.

9.2 Equity Premium

We are interested in $\mathbb{E}[\tilde{r}_{t+1}] - r_t$. Note that form the Euler equations we have

$$\begin{aligned} 1 &= \mathbb{E}_t \left[m_{t+1} \frac{p_{t+1} + d_{t+1}}{p_t} \right] = \mathbb{E}_t \left[m_{t+1} \right] \mathbb{E}_t \left[1 + \tilde{r}_t \right] + \mathbb{C}(m_{t+1}, 1 + \tilde{r}_t) \\ &= \frac{\mathbb{E}_t [1 + \tilde{r}_{t+1}]}{1 + r_{t+1}} + \mathbb{C}(m_{t+1}, 1 + \tilde{r}_{t+1}) \\ &\approx 1 + \mathbb{E}_t [\tilde{r}_{t+1}] - r_{t+1} + \mathbb{C}(m_{t+1}, 1 + \tilde{r}_{t+1}), \end{aligned}$$

We thus have

Proposition 9.4.

$$\begin{aligned} \mathbb{E}[\tilde{r}_{t+1}] - r_{t+1} &= -\mathbb{C}(m_{t+1}, 1 + \tilde{r}_{t+1}) \\ &= -\operatorname{Corr}(m_{t+1}, \tilde{r}_{t+1}) \sigma_{m_{t+1}} \sigma_{\tilde{r}_{t+1}}. \end{aligned}$$

Remark 9.5. Equity premium can be interpreted as a compensation for risk. Stocks that are more correlated with the state of the economy amplified the fluctuations in consumption and thus increase risk. They thus have a higher equity premium. Goods with negative correlation with the state of the economy (say gold) offers insurance against risk.

Example 9.6. Suppose there are two states of the world, good and bad. Since $c^g_{t+1} > c^b_{t+1}$ we would expect

$$m^b_{t+1} > m^g_{t+1}.$$

If the financial product is correlated with the state of the economy, then also

$$x^g > x^b.$$

From this we see that $\tilde{r}$ and m are typically negatively correlated.

Alternatively, note that

$$\begin{aligned} m_{t+1} &= \frac{\beta u'(c_{t+1})}{u'(c_t)} \\ &\approx \frac{\beta u'(c_t)}{u'(c_t)} + \frac{\beta u''(c_t)}{u'(c_t)} (c_{t+1} - c_t) \\ &= \beta \left[1 + \frac{u''(c_t) c_t}{u'(c_t)} \cdot \frac{c_{t+1} - c_t}{c_t} \right] = \beta (1 - \sigma \gamma_c), \end{aligned}$$

where σ is the coefficient of relative risk aversion and γ_c is the growth rate of consumption. We thus have

Proposition 9.7.

$$\mathbb{C}(m_{t+1}, \tilde{r}_{t+1}) = \beta\sigma\mathbb{C}(\gamma_c, \tilde{r}_{t+1})$$

Remark 9.8.

- γ_c can be thought as a measurement of the macroeconomic environment.
- σ is the inverse of the elasticity of Intertemporal substitution. A higher σ means that the agent would not change the saving behavior as much. To see this, note that using the CRRA utility function

  $$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

  we have

  $$1 = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \right] (1 + r_{t+1}).$$

  Ignoring the expectation operator and taking logs we have

  $$\log(c_{t+1}/c_t) = \frac{1}{\sigma} \log \beta + \frac{1}{\sigma} \log(1 + r_{t+1})$$

  and then

  $$\frac{\mathrm{d} \log(c_{t+1}/c_t)}{\mathrm{d}(1 + r_{t+1})} = \frac{1}{\sigma}.$$

☕

9.3 The Equity Premium Puzzle

The equity premium has been around 6 to 8 percent for the past 100 years. But this implies that σ is around 26, since $\mathbb{C}(\gamma_c, \tilde{r}_{t+1})$ is low (this is mainly because γ_c is small). This is the **equity premium puzzle**, and habit formation models try to address this (see, e.g., Epstein–Zin preferences).

Given this value of σ and the observed growth rate of c, $\gamma_c \approx 0.02$, we will have a very low elasticity of intertemporal substitution and a ridiculously high r that is inconsistent with historical data. This is the **risk-free rate puzzle**, and incomplete market models try to address this.

10 State Continent Claims

Definition 10.1. A **state contingent claim** is an asset that delivers a certain payoff in a certain state of the world. An economy is said to be **complete** if there exists a market for every state contingent claim.

Remark 10.2. State contingent claims are called primitive assets since they can be used to construct any other asset. ☕

We use x^t_{ij} to denote the number state contingent claims that pays 1 unit of good i in state i at time t and 0 otherwise. To further simplify, consider a two period model with one type of goods and a finite number of states, the jth of which occurs with probability π_j.

Individuals solve

$$\max u(c_0) + \beta \mathbb{E}[u(c_1)] = \max u(c_0) + \beta \sum \pi_j u(c_j)$$

subject to

$$y_0 = c_0 + \sum q_j x_j + b$$

and

$$c_j = y_j + x_j + (1+r)b, \quad j = 1, \dots, n.$$

We have the Lagrangian

$$\begin{aligned} \mathcal{L} &:= u(c_0) + \beta \left[\sum \pi_j u(c_j) \right] + \lambda \left[y_0 - c_0 - \sum q_j x_j - b \right] \\ &\quad + \sum \mu_j \left[y_j + x_j + (1+r)b - c_j \right]. \end{aligned}$$

and the following FOCs:

$$\begin{aligned} \{c_0\}: & \qquad u'(c_0) = \lambda \\ \{c_j\}: & \qquad \beta \pi_j u'(c_j) = \mu_j \\ \{x_j\}: & \qquad \mu_j = q_j \\ \{b\}: & \qquad \lambda = (1+r) \sum \mu_j. \end{aligned}$$

From the FOC we have the asset prices are given by

$$q_j = \pi_j \frac{\beta u'(c_j)}{u'(c_0)}.$$

Remark 10.3. We can interpret this as the probability of state j happening π_j times the stochastic discount factor in state j ($\beta u'(c_j)/u'(c_0)$) times the amount of goods delivered in that state (1). ☕

Summing across all state contingent claims gives

$$\sum q_j = \sum \pi_j \frac{\beta u'(c_j)}{u'(c_0)} = \mathbb{E}[m] = \frac{1}{1+r},$$

where the last equality follows from $\{b\}$ and is consistent with the Lucas tree model. We may interpret the left hand side as the price of a portfolio of state contingent claims that is equivalent to $1/(1+r)$ units of risk-free bond (that delivers one unit of good in the next time period no matter what state happens). John Cochrane calls it the price of a happy meal.

Combining the budget constraints give

$$c_0 + \sum q_j (c_j - y_j - b(1+r)) + b = y_0,$$

which can be simplified using the result above to yield $c_0 + \sum q_j c_j = y_0 + \sum q_j y_j$, or assuming $q_0 = 1$,

$$\sum_{j=0}^{n} q_j c_j = \sum_{j=0}^{n} q_j y_j.$$

We may interpret y_j's as endowments or goods and interpret this as the usual endowment market budget constraint.

11 Heterogeneous Agents

11.1 Complete Market

We now consider the same market as above with heterogeneous endowments (but homogeneous preference). The same derivation as above yields

$$\sum_{j=0}^{n} q_j c_j^i = \sum_{j=0}^{n} q_j y_j^i$$

$$q_j = \pi_j \frac{\beta u'(c_j^i)}{u'(c_0^i)}$$

$$\sum q_j = \frac{1}{1+r},$$

where j indexes states and i indexes individuals. Thus the stochastic discount factor in state j, q_j/π_j, is constant across individuals.

Now, if we assume CRRA utility function $u(c) = c^{1-\sigma}/(1-\sigma)$, the stochastic discount factor for state j can be expressed as

$$\beta \left(\frac{c_j^i}{c_0^i} \right)^{\sigma}.$$

Then, in particular, the consumption ratios between period/sates is the same across individuals, with levels determined by $\sum q_j y_j^i$. The rich is rich in every state. There is no mobility.

Example 11.1. If $c_0^1 > c_0^2$, then $c_j^1 > c_j^2$ for all states j.

11.2 Incomplete Market, Hugget (1993)

A motivation for introducing incomplete market is to resolve the asset premium puzzle. The intuition is that agents will want to insure themselves against bad states, but in an incomplete market, their only option is "precautionary saving."

Definition 11.2 (Precautionary Saving)**.** Consider two worlds. In world "uncertain," the individual could land with c_g or c_b with kown probabilities. In world "certain," the individual has $\mathbb{E}[c]$ for certain. Recall the Euler equations

$$u'(c_{\text{certain}}) = \beta(1+r)u'(\mathbb{E}[c])$$
$$u'(c_{\text{uncertain}}) = \beta(1+r)\mathbb{E}(u'(c)).$$

We say there is **precautionary saving** if $c_{\text{certain}} > c_{\text{uncertain}}$. By Jensen's inequality, this occurs if u' is convex or if $u''' > 0$. Intuitively, this is the case when $u'(c_{\text{very bad}})$ is "very large."

11.2.1 Model Setup

Individual i solves

$$\mathbb{E}_0\left[\sum \beta^t u(c_t)\right] \quad \text{s.t.} \quad c_t + q_t a_{t+1} = a_t + y_t, \quad a_{t+1} \geq \underline{a},$$

where q_t is the price of a unit of risk free bond that pays one unit of consumption next period and $\underline{a}$ is a borrowing constraint. We set $\underline{a} < 0$ to have the same magnitude as a year of salary.

Individuals face idiosyncratic and exogenous labor income shock following a first order Markov process given by the transition matrix π:

$$\pi(y', y) := \mathbb{P}(y_{t+1} = y' | y_t = y).$$

11.2.2 Solution Concept

Bellman equation:

$$V(a_t, y_t) = \max_{c_t, a_{t+1}} \left\{u(c_t) + \beta\mathbb{E}[V(a_{t+1}, y_{t+1})|y] + \mu(a_{t+1} - \underline{a})\right\},$$

where

$$c_t = a_t + y_t - q_t a_{t+1}.$$

In practice, when solving the solution numerically, we may omit the last term in the Bellman equation and impose the constraint on a_{t+1} instead by setting the lower bound of the grid to $\underline{a}$.

Let $\mu(a, y)$ be the fraction of population with state (a, y) (or the unconditional probability of that state happening) and denote the policy function as g. The **aggregate asset demand** is given by

$$\iint g(a, y)\mu(a, y)\, \mathrm{d}a\, \mathrm{d}y,$$

where the motion of μ is determined by π and $g(a, y)$. Note that both functions are time-invariant.

Definition 11.3. A **stationary equilibrium** is a situation in which μ does not change over time.

Definition 11.4. A **recursive competitive equilibrium** for this economy is

- a value function V,
- an individual decision (policy) rule g,
- a stationary probability distribution μ,
- and a bond price q

such that

- Given the bond price q, the value function V and the policy function g solve the household's dynamic programming problem.
- The stationary distribution μ is induced by the exogenous Markov chain for (y, y') and the policy function g, and
- The bond market clears:

$$\sum_a \sum_y g(a, y, q)\mu(a, y, q) = 0.$$

Remark 11.5. The goods market clears in this economy when

$$\sum \sum c(a, y, q)\mu(a, y, q) = \sum \sum y(a, y, q)\mu(a, y, q).$$

The solution algorithm is as follows:

Algorithm 11.6.

(i) *Guess bond price q and solve the Bellman equation to get the policy function $g(a, y, q)$.*

(ii) *Solve the stationary distribution μ. To do this we stack μ into a* column *vector and describe the evolution of μ by a matrix Π, the ij element of which is the probability of moving from state i to state j in one period:*

$$\mu_{t+1} = \mu_t \Pi.$$

We can solve for μ_t by analytically finding the eigenvector or numerically by iterating on the equation $\mu_{t+1} = \mu_t \Pi$ until convergence.

(iii) *Calculate the aggregate asset demand $\sum \sum g(a, y, q)\mu(a, y, q)$.*

(iv) *Depending on the sign of the aggregate asset demand, update the bond price q and repeat until the bond market clears.*

The tighter the borrow constraint (the lower the absolute value of the borrowing constraint), the higher the demand for risk free bonds (and the higher q is), since agents would want to accumulate more assets to buffer consumption against bad state.

12 Continuous Time

We write
$$\dot{X}(t) \coloneqq \frac{\mathrm{d}X(t)}{\mathrm{d}t}.$$
Consider the Solow growth model. From
$$K_{t+1} - K_t = I_t - \delta K_t$$
we have the analogue
$$K_{t+\Delta t} - K_t = \underbrace{I_t}_{\text{flow per unit time}} \Delta t - \delta K_t \Delta t.$$
Sending $t \to 0$, we get
$$\dot{K}(t) = I(t) - \delta K(t).$$

Consider the second version of Solow growth model where $K = kAN$, $Y = [AN]^{1-\alpha} K^{\alpha}$, and $I = Y - C = sY$. We have at steady state
$$kAN = s\,[AN]^{1-\alpha}\,K^{\alpha} - \delta kAN.$$
Dividing taking log and dividing by K,
$$\frac{\dot{k}AN + k\dot{A}N + kA\dot{N}}{kAN} = \frac{\dot{k}}{k} + \frac{\dot{A}}{A} + \frac{\dot{N}}{N} = s\left(\frac{AN}{k}\right)^{1-\alpha} - \delta$$
and so
$$\dot{k} = sk^{\alpha} - (n + g + \delta)k.$$

12.1 Hamiltonian

It turns out that the continuous time analogue of
$$\sum \beta^t u(c_t) \quad \text{s.t.} \quad \Delta k_{t+1} = f(k_t) - c_t - \delta k_t$$
is
$$\int_0^\infty e^{\rho t} u(c(t))\,\mathrm{d}t \quad \text{s.t.} \quad \dot{k}(t) = f(k(t)) - c(t) - \delta k(t),$$
where ρ is defined by $\beta = 1/(1+\rho)$. Some intuition for why we use ρ:
$$e^r = \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n$$
and compounding interest.

Then,
$$\begin{aligned}
\mathcal{L} &= \int_0^\infty e^{\pi r} u(c(t))\,\mathrm{d}t + \int_0^\infty \lambda(t)\left[f(k(t)) - c(t) - \delta k(t) - \dot{k}(t)\right]\,\mathrm{d}t \\
&= \int_0^\infty \Big[\underbrace{e^{-\rho t} u(c(t)) - \lambda(t)\big[f(k(t)) - c(t) - \delta k(t)\big]}_{H(t,c(t),\lambda(t),k(t)),\ \text{the Hamiltonian}}\Big]\,\mathrm{d}t - \int_0^\infty \lambda(t)\dot{k}(t)\,\mathrm{d}t.
\end{aligned}$$

The cookbook recipe is

$$\begin{aligned}
\frac{\partial H}{\partial c(t)} &= 0 \\
\frac{\partial H}{\partial k} &= -\dot{k} \\
\frac{\partial H}{\partial \lambda} &= \dot{k} = f(k(t)) - c(t) - \delta k(t).
\end{aligned}$$

In our example,

$$\begin{aligned}
e^{-\rho t} u(c(t)) &= \lambda(t) \\
\lambda(t)\left[f'(k(t)) - \delta\right] &= -\dot{\lambda}(t).
\end{aligned}$$

Taking log we have

$$-\rho t = \log u'(c(t)) = \log \lambda(t)$$

Differentiate with respect to time,

$$-\rho \underbrace{\frac{u''(c(t))}{u'(c(t))c(t)}}_{\sigma} \cdot \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)}.$$

With CRRA utility, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[f'(k(t)) - \delta - \rho\right].$$

This is the continuous analogue of the Euler equation.