MATH20510 (S25): Analysis in Rn III (accelerated)

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1 Integration of Differential Forms

1.1 Integration on a Cell

Definition 1.1. A ***k*-cell** in $\mathbb{R}^k$ is a set of the form $I^k := \{x \in \mathbb{R}^k : a_i \leq x_i \leq b_i, i = 1, \ldots, k\}$.

Definition 1.2. Let $f \in C(I^k)$ be real valued and write $f_k := f$. Define for each $i = k, \ldots, 1$

$$f_{i-1}(x_1, \ldots, x_{k-1}) := \int_{a_i}^{b_i} f_i(x_1, \ldots, x_i) \, \mathrm{d}x_i.$$

We define

$$\int_{I_k} f(x) \, \mathrm{d}x := \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} f_k(x_1, \ldots, x_k) \, \mathrm{d}x_k \ldots \, \mathrm{d}x_1 = f_0.$$

Remark 1.3.

- Since f is continuous on a compact set, it is uniformly continuous. Thus all iterated integrals are well-defined and uniformly continuous on I^i $(1 \leq i \leq k)$.
- The integral over a k-cell is independent of the order of integration, by the following result:

Theorem 1.4. *If $f \in C(I^k)$, then $L(f) = L'(f)$, where $L(f)$ is the integral of f over I^k as defined above, and $L'(f)$ is the integral of f over the same domain with a different order of integration.*

Proof. If $h(x) = h_1(x_1) \ldots h_k(x_k)$, where $h_j \in C([a_j, b_j])$, then

$$L(h) = \prod_{i=1}^{k} \int_{a_i}^{b_i} h_i(x_i) \, \mathrm{d}x_i = L'(h).$$

If $\mathcal{A}$ is the set of all finite sums of such functions h, it follows that $L(g) = L'(g)$ for all $g \in \mathcal{A}$. The Stone-Weierstrass theorem shows that $\mathcal{A}$ is dense in $C(I^k)$. Put $V = \prod_{i=1}^{k}(b_i - a_i)$. If $f \in C(I^k)$ and $\epsilon > 0$, there exists $g \in \mathcal{A}$ such that

$\|f - g\| < \epsilon/V$, where $\|f\|$ is defined as $\max_{x \in I^k}|f(x)|$. Then $|L(f - g)| < \epsilon$, $L'(f - g) < \epsilon$, and since

$$L(f) - L'(f) = L(f - g) + L'(g - f),$$

we conclude that $|L(f) - L'(f)| < 2\epsilon$. □

Definition 1.5. The **support** of function f on $\mathbb{R}^k$ is the closure of the set of all points $x \in \mathbb{R}^k$ at which $f(x) \neq 0$. We write $f \in C_c(\mathbb{R}^k)$ if f is a continuous function with compact support, that is, if $K := \operatorname{supp} f \subset I^k$ for some k-cell I^k. In this case we define

$$\int_{\mathbb{R}^k} f(x)\, \mathrm{d}x := \int_{I^k} f(x)\, \mathrm{d}x.$$

Definition 1.6. Let $G : \mathbb{R}^n \supset E \to \mathbb{R}^n$, where E is open. If there is an integer m and a real function g with domain E such that for all $x \in E$ we have

$$G(x) = \sum x_i e_i + g(x) e_m,$$

then we call G **primitive**.

Remark 1.7.

- In other words, G changes only one coordinate.
- If g is differentiable at $x \in E$, then so if G. The matrix $\mathrm{D}G(x)$ has

  $$(\partial_1 g)(x), \dots, (\partial_m g)(x), \dots, (\partial_n g)(x)$$

  as its mth row. On the jth row, where $j \neq m$, we have the jth unit vector. Thus the Jacobian of G at a is

  $$J_G(a) = \det \mathrm{D}G(a) = (\partial_m g)(a)$$

  and so $G'(a)$ is invertible if and only if $(\partial_m g)(a) \neq 0$.

Definition 1.8. A linear operator B on $\mathbb{R}^n$ that interchanges some pair of members of the standard basis and leaves the others fixed will be called a **flip**.

Theorem 1.9. *Suppose $F : \mathbb{R}^n \supset E \to \mathbb{R}^n$ is C^1, $0 \in E$, $F(0) = 0$, and $F'(0)$ is invertible. Then there is a neighborhood of 0 in $\mathbb{R}^n$ in which a representation*

$$F(x) = B_1 \dots B_{n-1} G_n \circ \dots \circ G_1(x)$$

is valid. Each G_i is a primitive C^1 mapping in some neighborhood of 0; $G_i(0) = 0$, $G_i'(0)$ is invertible, and each B_i is either a flip or the identity.

Theorem 1.10 (Partition of Unity)**.** *Let K be a compact subset of $\mathbb{R}^n$. Let $\{V_\alpha\}$ be an open cover of K. Then there exists function $\psi_1, \dots, \psi_k \in C(\mathbb{R}^n)$ such that*

- $0 \le \psi_i \le 1$ *for* $1 \le i \le s$,
- $\operatorname{supp} \psi_i \subset V_\alpha$ *for some* α[1], *and*
- $\sum_i \psi_i = 1$ *for each* $x \in K$.

Corollary 1.11. *If $f \in C(\mathbb{R}^n)$ and the support of f lies in K, then*

$$f = \sum \psi_i f.$$

Each $\psi_i f$ has support in some V_α.

Remark 1.12. This is a representation of f using functions with "small" supports. We represent global information using local information.

Theorem 1.13 (Change of Variables)**.** *Let T be a one-to-one C^1 mapping from an open set $E \in \mathbb{R}^k$ into $\mathbb{R}^k$ such that $J_T(x) \ne 0$ for all $x \in T$. If $f \in C_c(\mathbb{R}^n)$ and* $\operatorname{supp} f \in T(E)$, *then*

$$\int_{\mathbb{R}^k} f(y)\, \mathrm{d}y = \int_{\mathbb{R}^k} f(T(x)) |J_T(x)|\, \mathrm{d}x.$$

Proof. If T is a primitive mapping, then the theorem is true by the one dimensional change of variable theorem. If T is a flip, the theorem reduces to the case in the first theorem of this section.

If the theorem is true for transformations P, Q, and if $S = P \circ Q$, then

$$\begin{aligned} \int f(z)\, \mathrm{d}z &= \int f(P(y)) |J_P(y)|\, \mathrm{d}y \\ &= \int f(P(Q(x))) |J_P(Q(x))| |J_Q(x)|\, \mathrm{d}x = \int f(S(x)) |J_S(x)|\, \mathrm{d}x, \end{aligned}$$

[1] This is sometimes expressed by saying that $\{\psi_i\}$ is subordinate to the cover $\{V_\alpha\}$.

where we used the fact that

$$\begin{aligned} J_P(Q(x)) &= \det \mathrm{D}P(Q(x)) \det \mathrm{D}Q(x) \\ &= \det \mathrm{D}P(Q(x))\mathrm{D}Q(x) = \det \mathrm{D}S(x) = J_S(x). \end{aligned}$$

This follows from the chain rule and the fact that the determinant of a product of matrices is the product of the determinants.

Now, for each $a \in E$ there exists a neighborhood $U \subset E$ of a in which

$$T(x) = T(a) + B_1 \dots B_{k-1} G_k \circ \cdots \circ G_1(x - a).$$

It follows that the theorem holds if the support of f lies in $T(U)$.

That is, each point $y \in T(E)$ lies in an open set $V_y \subset T(E)$ such that the theorem holds for all continuous functions whose support lies in V_y.

For an arbitrary function f, we need only write it as a sum of functions with compact support using the partition of unity. □

2 Differential Forms

Definition 2.1 (k-surface)**.** Suppose E in an open set in $\mathbb{R}^n$. A $\boldsymbol{k}$**-surface** in E is a C^1 mapping Φ from a compact set $D \subset \mathbb{R}^k$ into E.

Definition 2.2 (k-form)**.** Let $E \subset \mathbb{R}^n$ be open. A **differential form of order** $\boldsymbol{k \geq 1}$ in E is a function ω, symbolically represented by

$$\omega = \sum a_{i_1 \ldots i_k} \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \, \mathrm{d}x_{i_k},$$

(where the indices $i_1, \ldots, i_k$ range independently from 1 to n), which assigns to each k-surfaces Φ in E a number $\omega(\Phi) = \int_\Phi \omega$ according to the rule

$$\int_\Phi \omega = \int_D \sum a_{i_1 \ldots i_k}(\Phi(u)) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)} \, \mathrm{d}u_1 \ldots \, \mathrm{d}u_k,$$

where D is the parameter domain of Φ.

Definition 2.3.

- We write $\omega_1 = \omega_2$ if and only if $\omega_1(\Phi) = \omega_2(\Phi)$ for every k-surface Φ in E. In particular, $\omega = 0$ means that $\omega(\Phi) = 0$ for every k-surface Φ in E.
- A k-form is said to be of class C^n if the functions $a_{i_1 \ldots i_k}$ are all of class C^n.
- A 0-form in E is defined to be a continuous function in E.
- We write $\Lambda^k(D)$ for the set of all k-forms in D.

Proposition 2.4.

- $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. *In particular,* $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$.
- $\mathrm{d}x_I = -\mathrm{d}x_J$ *if J is obtained by interchanging two subscripts in I.*

Definition 2.5. If $i_1, \ldots, i_k$ be integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and if I, and if I is the k-tuple $\{i_1, \ldots, i_k\}$, then we call I and **increasing** $\boldsymbol{k}$**-index**, and we use the brief notation

$$\mathrm{d}x_I := \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \, \mathrm{d}x_{i_k}.$$

These are the **basic** $\boldsymbol{k}$**-forms** in $\mathbb{R}^n$.

Remark 2.6. Every k-form can be written as

$$\omega = \sum_I b_I(x)\, \mathrm{d}x_I,$$

where each I is increasing. We call this the **standard presentation** of ω.

Theorem 2.7. *Suppose $\omega = \sum_I b_I\, \mathrm{d}x_I$ is the standard presentation of a k-form ω in an open set $E \subset \mathbb{R}^n$. If $\omega = 0$ in E, then $b_I(x) = 0$ for ever increasing k-index I and for every $x \in E$.*

Definition 2.8 (products)**.** The product of the basic forms $\mathrm{d}x_I$ and $\mathrm{d}x_J$, where $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$, is the $(p+q)$-form

$$\mathrm{d}x_I \wedge \mathrm{d}x_J \coloneqq \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}.$$

If ω and λ are respectively p- and q-forms on an open set $E \subset \mathbb{R}^n$ with standard representations

$$\omega = \sum_I b_I(x)\mathrm{d}x_I, \quad \lambda = \sum_J c_J(x)\mathrm{d}x_J,$$

then their product is defined to be the $(p+q)$-form

$$\omega \wedge \lambda \coloneqq \sum_{I,J} b_I(x) c_J(x) \mathrm{d}x_I \wedge \mathrm{d}x_J.$$

Proposition 2.9 (Properties of the Product)**.**

- *Distributive Laws:*

$$(\omega_1 + \omega_2) \wedge \lambda = (\omega_1 \wedge \lambda) + (\omega_2 \wedge \lambda),$$
$$\omega \wedge (\lambda_1 + \lambda_2) = (\omega \wedge \lambda_1) + (\omega \wedge \lambda_2).$$

- *Associativity:*

$$(\omega \wedge \lambda) \wedge \sigma = \omega \wedge (\lambda \wedge \sigma).$$

2.1 Differentiation

Definition 2.10. The differentiation operator d associates a $(k+1)$-form $\mathrm{d}\omega$ to each k-form ω of class C^1 in an open set $E \subset \mathbb{R}^n$.

A 0-form of class C^1 is just a continuous function $f \in C^1(E)$. We define

$$\mathrm{d}f := \sum_i (\mathrm{D}_i f)(x)\mathrm{d}x_i.$$

If $\omega = \sum_I b_I(x)\mathrm{d}x_I$ is the standard presentation of a k-form ω and each $b_i \in C^1(E)$, then we define

$$\mathrm{d}\omega := \sum_I (\mathrm{d}b_I) \wedge \mathrm{d}x_I.$$

Remark 2.11.

- Since $\mathrm{D}_i(fg) = g\mathrm{D}_i f + f\mathrm{D}_i$, we have $\mathrm{d}(fg) = g\ \mathrm{d}f + f\ \mathrm{d}g$.
- $\mathrm{d}^2 x_I = 0$.
- Using the same trick as in the proof for the first part of the next theorem, we see that d is linear.

☕

Theorem 2.12.

(i) If ω and λ are k- and m-forms of class C^1 in E, then

$$\mathrm{d}(\omega \wedge \lambda) = (\mathrm{d}\omega) \wedge \lambda + (-1)^k \omega \wedge (\mathrm{d}\lambda).$$

(ii) (Poincaré lemma) If ω is of class C^2 in E, then $\mathrm{d}^2\omega = 0$.

Proof.

(i) By how the product and the derivative is defined, we need only prove the statement for the special case

$$\omega = f\ \mathrm{d}x_I, \quad \lambda = g\ \mathrm{d}x_J,$$

where $f, g \in C^1(E)$, $\mathrm{d}x_I$ is a basic k-form, and $\mathrm{d}x_J$ is a basic m-form. (If k or m is 0, omit $\mathrm{d}x_I$ or $\mathrm{d}x_J$.) Then we have

$$\omega \wedge \lambda = fg\ \mathrm{d}x_I \wedge \mathrm{d}x_J.$$

If I and J have no common indices, then the desired statement is proved, with each of the three terms being 0. Let's thus suppose otherwise. We may write

$$\begin{aligned}
\mathrm{d}(\omega \wedge \lambda) &= \mathrm{d}(fg \ \mathrm{d}x_I \wedge \mathrm{d}x_J) \\
&= (-1)^{\alpha} \ \mathrm{d}(fg \ \mathrm{d}x_{[I,J]}) \\
&= (-1)^{\alpha}(f \ \mathrm{d}g + g \ \mathrm{d}f) \wedge \mathrm{d}x_{[I,J]} \\
&= (f \ \mathrm{d}g + g \ \mathrm{d}f) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J.
\end{aligned}$$

Here, α is the number of interchanges of indices needed to make (I, J) increasing, and $[I, J]$ denotes the increasing $(k + m)$-tuple obtained by combining the indices in I and J. Now note that

$$\mathrm{d}g \wedge \mathrm{d}x_I = (-1)^k \mathrm{d}x_I \wedge \mathrm{d}g,$$

and so

$$\begin{aligned}
\mathrm{d}(\omega \wedge \lambda) &= (\mathrm{d}f \wedge \mathrm{d}x_I) \wedge (g \ \mathrm{d}x_J) + (-1)^k (f \ \mathrm{d}x_I) \wedge (\mathrm{d}g \wedge \mathrm{d}x_J) \\
&= (\mathrm{d}\omega) \wedge \lambda + (-1)^k \omega \wedge \mathrm{d}\lambda.
\end{aligned}$$

(ii) We first prove the statement for a 0-form $f \in C^2$. We have

$$\begin{aligned}
\mathrm{d}^2 f &= \mathrm{d}\left(\sum_j (\mathrm{D}_j f)(x)\mathrm{d}x_j\right) = \sum_j \mathrm{d}(\mathrm{D}_j f) \wedge \mathrm{d}x_j \\
&= \sum_j \sum_i (\mathrm{D}_{ij} f)(x)\mathrm{d}x_i \wedge \ \mathrm{d}x_j.
\end{aligned}$$

Since $\mathrm{D}_{ij} f = \mathrm{D}_{ji} f$ and $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$, we have $\mathrm{d}^2 f = 0$. If $\omega = f \ \mathrm{d}x_I$ then $\mathrm{d}\omega = (\mathrm{d}f) \wedge \mathrm{d}x_I$. Then, the first part of this theorem shows that

$$\mathrm{d}^2\omega = (\mathrm{d}^2 f) \wedge \mathrm{d}x_I + (-1)^{|I|}(\mathrm{d}f) \wedge \mathrm{d}^2 x_I.$$

It remains to recall that $\mathrm{d}^2 x_I = \mathrm{d}^2 f = 0$.

□

Example 2.13. Let $f \in \Lambda^0(\mathbb{R}^3)$. Let

$$\mathrm{d}f = \frac{\partial f}{\partial x} \ \mathrm{d}x + \frac{\partial f}{\partial y} \ \mathrm{d}y + \frac{\partial f}{\partial z} \ \mathrm{d}z,$$

and write

$$F := \nabla f = \operatorname{grad} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =: \langle A, B, C \rangle .$$

We have

$$\mathrm{d}(\mathrm{d}f) = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial y} \right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y,$$

Note that the three functions are the three coordinates of $\nabla \times F = \operatorname{curl} F$. That is, $\mathrm{d}(\mathrm{d}f)$ has the three coordinates of

$$\operatorname{curl}(\operatorname{grad} f) = \nabla \times (\nabla f).$$

Consider now the 2-form in $\mathbb{R}^3$

$$\mathrm{d}^2 f = P \,\mathrm{d}y \wedge \mathrm{d}z + Q \,\mathrm{d}z \wedge \mathrm{d}x + R \,\mathrm{d}x \wedge \mathrm{d}y =: \omega$$

and

$$G = \langle P, Q, R \rangle .$$

Then we have

$$\mathrm{d}\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

which, note, contains the three coordinates of $\operatorname{div} G := \nabla \cdot G$. We have then that $\mathrm{d}(\mathrm{d}^2 f)$ has the same three coordinates as

$$\operatorname{div}(\operatorname{curl} F) = \nabla \cdot (\nabla \times F).$$

2.2 Change of Variables

Let $T : \mathbb{R}^n \supset E \to V \in \mathbb{R}^m$ be C^1. Let ω be a k-form in V with standard presentation

$$\omega = \sum_I b_I(y) \,\mathrm{d}y_I.$$

Write $y = (y_1, \ldots, y_m) = T(x) = (t_i(x), \ldots, t_m(x))$. Since

$$\mathrm{d}t_i = \sum_j (\mathrm{D}_j t_i)(x) \mathrm{d}x_j,$$

each $\mathrm{d}t_i$ is a 1-form in E. We may define the **pullback**

$$\omega_T := \sum_I b_I(T(x))\,\mathrm{d}t_{i_1}(x) \wedge \cdots \wedge \mathrm{d}t_{i_k}(x).$$

Note that ω_T is a k-form in E.

Theorem 2.14. *With E and T as above, let ω and λ be k- and m-forms in V. Then,*

(i) $(\omega + \lambda)_T = \omega_T + \lambda_T$ *if $k = m$;*

(ii) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$*;*

(iii) $\mathrm{d}(\omega_T) = (\mathrm{d}\omega)_T$ *if ω is of class C^1 and T is of class C^2.*

Proof. (i) follows from the definition. To prove (ii), note that

$$(\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_r})_T = \mathrm{d}t_{i_1}(x) \wedge \cdots \wedge \mathrm{d}t_{i_r}(x)$$

regardless of whether $\{i_1, \ldots, i_r\}$ is increasing, since the same number of minus signs are needed on each side of the equation to produce increasing rearrangements. We turn to (iii).

If f is a 0-form of class C^1 in V, then

$$f_T(x) = f(T(x)), \quad \mathrm{d}f = \sum_i (\mathrm{D}_i f)(y)\,\mathrm{d}y_i.$$

Using the chain rule, we have

$$\begin{aligned}
\mathrm{d}(f_T) &= \sum_j (\mathrm{D}_j f_T)(x)\,\mathrm{d}x_j \\
&= \sum_j \sum_i \mathrm{D}_i f(T(x))(\mathrm{D}_j t_i) T(x)\,\mathrm{d}x_j \\
&= \sum_i \mathrm{D}_i f(T(x))\mathrm{d}t_i = (\mathrm{d}f)_T.
\end{aligned}$$

If $\mathrm{d}y_I = \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{i_k}$, then $(\mathrm{d}y_I)_T = \mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_k}$ is a basic k-form. Thus $\mathrm{d}((\mathrm{d}y_I)_T) = 0$. Suppose now that $\omega = f\,\mathrm{d}y_I$. Then $\omega_T = f_T(x)(\mathrm{d}y_I)_T$ and, by the discussion above, we have

$$\begin{aligned}
\mathrm{d}(\omega_T) &= \mathrm{d}(f_T) \wedge (\mathrm{d}y_I)_T = (\mathrm{d}f)_T \wedge (\mathrm{d}y_I)_T \\
&= ((\mathrm{d}f) \wedge \mathrm{d}y_I)_T = (\mathrm{d}\omega)_T.
\end{aligned}$$

By applying Part (i), we can prove the general case. □

2.3 A Geometric Perspective

2.3.1 Tangent Space

Definition 2.15 (Tangent Space)**.** Given a curve C and a point p on the curve, the **tangent space** to C at p is defined as

$$T_pC \coloneqq \operatorname{span}\{\text{vectors tangent to } C \text{ at } p\}.$$

Example 2.16. For a 2 surface $r(u,v) = (x(u,v), y(u,v), z(u,v))$, where $\mathbb{R}^2 \supset D \to \mathbb{R}^3$, we define

$$T_pS = \operatorname{span}\{r_u(p), r_v(p)\},$$

where

$$r_u \coloneqq \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle.$$

Definition 2.17. A **1-form** in $\mathbb{R}^n$ is a linear function $\omega : T_p\mathbb{R}^n \to \mathbb{R}$. That is, $\Lambda^1(\mathbb{R}^n) = (T_p\mathbb{R}^n)^*$.

Proposition 2.18. *Any* 1*-form* ω *in* $\mathbb{R}^n$ *is a linear combination of* $\langle \mathrm{d}x_1, \dots, \mathrm{d}x_n \rangle$, *where* $\mathrm{d}x_i : x \mapsto x_i$.

Example 2.19. If $\omega = a\,\mathrm{d}x + b\,\mathrm{d}y$, then $\omega(\langle 1,2\rangle) = a \cdot 1 + b \cdot 2$.

Definition 2.20. An ***k*-form** on $\mathbb{R}^n$ is a function $\omega : (T_p\mathbb{R}^n)^k \to \mathbb{R}$ that is multilinear and alternating.

Proposition 2.21. $\Lambda^k(\mathbb{R}^n)$ *has basis*

$$\left\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : 1 \le i_1 < \cdots < i_k \le n\right\}.$$

Definition 2.22. For $v^{(1)}, \dots, v^{(k)} \in T_p\mathbb{R}^n$ and $\omega = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ we define

$$\omega(v^{(1)}, \dots, v^{(k)}) \coloneqq \det\left(v^{(j)}_{x_{i_m}}\right)_{1 \le j,m \le k} = \det\begin{pmatrix} v^{(1)}_{x_{i_1}} & \dots & v^{(1)}_{x_{i_k}} \\ \vdots & \ddots & \vdots \\ v^{(k)}_{x_{i_1}} & \dots & v^{(k)}_{x_{i_k}} \end{pmatrix}$$

Example 2.23. Let $v^{(1)} = \langle 1, -1, 3, 5\rangle, v^{(2)} = \langle 0, 1, -1, 4\rangle \in T_p\mathbb{R}^4$. We have

$$\mathrm{d}x \wedge \mathrm{d}y(v^{(1)}, v^{(2)}) = \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1,$$
$$\mathrm{d}z \wedge \mathrm{d}w(v^{(1)}, v^{(2)}) = \det\begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix} = 17.$$

Let $\omega = 3\ \mathrm{d}x \wedge \mathrm{d}y + 5\ \mathrm{d}z \wedge \mathrm{d}w \in \Lambda^2(\mathbb{R}^2)$. We have

$$\omega(v^{(1)}, v^{(2)}) = 3 \cdot 1 + 5 \cdot 17 = 88.$$

Example 2.24. Let $\omega = \sum_I a_I\ \mathrm{d}x_I$. We have

$$\omega(v^{(1)}, \dots, v^{(k)}) = \sum_I a_I \det\left(v_{i_m}^{(j)}\right)_{1\le j,m\le k}.$$

Definition 2.25. A **differential k-form** on $\mathbb{R}^n$ is denoted

$$\omega = \sum_I f_I(x)\ \mathrm{d}x_I,$$

where $f_I : \mathbb{R}^n \to \mathbb{R}$ are infinitely differentiable. Thus for each $p \in \mathbb{R}^n$, ω_p is a k-form in $T_p\mathbb{R}^n$.

Example 2.26. Let $\omega = x^2\ \mathrm{d}x \wedge \mathrm{d}y - x^3 z\ \mathrm{d}y \wedge \mathrm{d}z \in \Lambda^2(\mathbb{R}^3)$. To evaluate ω, we need a base point p, and k vectors based at p, say $v^{(1)}, \dots, v^{(k)} \in T_p\mathbb{R}^n$. Suppose $p = (2, 1, -1)$, $v^{(1)} = \langle 1, -2, 3\rangle$, and $v^{(2)} = \langle 2, 0, 1\rangle$. We have

$$\omega_p = 4\ \mathrm{d}x \wedge \mathrm{d}y + 8\ \mathrm{d}y \wedge \mathrm{d}z = 4\det\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} + 8\det\begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix}.$$

2.3.2 Integration of Differential Forms

Definition 2.27 (Integration of differential 2-forms)**.** Let $\phi : \mathbb{R}^2 \supset D \to S \subset \mathbb{R}^n$ be a 2-surface and let $\omega = \sum_I f_I(x)\ \mathrm{d}x_I$ be a differential 2-form in $\mathbb{R}^n$. We define

$$\int_S \omega \coloneqq \iint_D \omega_{\phi(u,v)}\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right)\ \mathrm{d}u\ \mathrm{d}v.$$

Now recall that

$$\mathrm{d}x_i \wedge \mathrm{d}x_j(v^{(1)}, v^{(2)}) = \det\begin{pmatrix} v_i^{(1)} & v_j^{(1)} \\ v_i^{(2)} & v_j^{(2)} \end{pmatrix}.$$

Example 2.28. Let

$$\begin{aligned} \omega &\coloneqq xy\, \mathrm{d}x \wedge \mathrm{d}y + x^2\, \mathrm{d}x \wedge \mathrm{d}z \in \Lambda^2(\mathbb{R}^3) \\ S : \phi(u, v) &= \left\langle u, v, u^2 + v^2 \right\rangle \\ D &= \left\{(u, v) : u^2 + v^2 \le 1\right\}. \end{aligned}$$

Note that

$$\begin{aligned} \omega_{\phi(u,v)}\left(\phi_u, \phi_v\right) &= \omega_{\left\langle u,v,u^2+v^2\right\rangle}\left(\langle 1, 0, 2u\rangle, \langle 0, 1, 2v\rangle\right) \\ &= uv\, \mathrm{d}x \wedge \mathrm{d}y\left(\phi_u, \phi_v\right) + u^2\, \mathrm{d}x \wedge \mathrm{d}z\left(\phi_u, \phi_v\right) \\ &= uv \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u^2 \det\begin{pmatrix} 1 & 2u \\ 0 & 2v \end{pmatrix} \\ &= uv + 2u^2 v. \end{aligned}$$

Then,

$$\begin{aligned} \int_S \omega &= \iint_D (uv + 2u^2 v)\, \mathrm{d}u\, \mathrm{d}v \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 \cos\theta \sin\theta + 2r^3 \cos^2\theta \sin\theta\right) r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^1 r^3 \cos\theta \sin\theta + 2r^4 \cos^2\theta \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{1}{4} \sin\theta \cos\theta + \frac{2}{5} \cos^2\theta \sin\theta\, \mathrm{d}\theta = 0, \end{aligned}$$

since the integrand is 2π-periodic. Note that we switched to polar coordinates in the second equality:

$$u \coloneqq r\cos\theta, \quad v \coloneqq r\sin\theta, \quad \mathrm{d}u\, \mathrm{d}v = r\, \mathrm{d}r\, \mathrm{d}\theta.$$

Definition 2.29. Now let ω be a differential k-form on $\mathbb{R}^k$, say

$$\omega = \sum_I f_I(x)\, \mathrm{d}x_I,$$

and a k-surface $\phi : \mathbb{R}^k \supset D \to S \subset \mathbb{R}^n$, we define

$$\int_S \omega := \int \cdots \int_D \omega_{\phi(u_1,\ldots,u_k)} \left(\frac{\partial \phi}{\partial u_1}, \ldots, \frac{\partial \phi}{\partial u_k} \right) \mathrm{d}u_1 \ldots \mathrm{d}u_k.$$

Recall that

$$\mathrm{d}x_I(v^{(1)}, \ldots, v^{(k)}) = \det \begin{pmatrix} v_{i_1}^{(1)} & \cdots & v_{i_k}^{(1)} \\ \vdots & \ddots & \vdots \\ v_{i_1}^{(k)} & \cdots & v_{i_k}^{(k)} \end{pmatrix}.$$

Example 2.30. Let

$$\omega = x_1 \,\mathrm{d}x_1 + (x_1^2 + x_2)\mathrm{d}x_2 + x_3 x_4 \,\mathrm{d}x_4 \in \Lambda^1(\mathbb{R}^4)$$

and consider the curve $C : \phi : [0, 3\pi] \to \mathbb{R}^4$ defined by

$$\phi(t) := \langle \cos t, \sin t, t, -t \rangle .$$

Note that

$$\phi'(t) = \langle -\sin t, \cos t, 1, -1 \rangle .$$

We then have

$$\begin{aligned} \int_C \omega &= \int_0^{3\pi} \cos t(-\sin t) + (\cos^2 t + \sin t) \cos t + t^2 \,\mathrm{d}t \\ &= \int_0^{3\pi} \cos^3 t + t^2 \,\mathrm{d}t = \int_0^{3\pi} \cos t(1 - \sin^2 t) + t^2 \,\mathrm{d}t \\ &= \sin t + \frac{1}{3} \sin^3 t + \frac{t^3}{3} \bigg|_0^{3\pi} = 9\pi^2. \end{aligned}$$

Integration is independent of the parameterization we choose. As an example, consider a 2-form

$$\omega = f(x, t) \,\mathrm{d}x \wedge \mathrm{d}y$$

over $D \subset \mathbb{R}^2$. We have

$$\begin{aligned} \int_D \omega &= \iint_D \omega_{\mathrm{Id}} \left(\frac{\partial \,\mathrm{Id}}{\partial x}, \frac{\partial \,\mathrm{Id}}{\partial y} \right) \mathrm{d}A \\ &= \iint_d f(x, y) \,\mathrm{d}x \wedge \mathrm{d}y \begin{pmatrix} \langle 1, 0 \rangle \\ \langle 0, 1 \rangle \end{pmatrix} \mathrm{d}A = \iint_D f(x, y) \mathrm{d}A. \end{aligned}$$

Now, with another parameterization of D:

$$\phi : D' \ni (u, v) \longmapsto (x, y) \in D.$$

We have

$$\begin{aligned}
\int_D \omega &= \iint_{D'} \omega_{\phi(u,v)} \left(\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v} \right) \, \mathrm{d}A' \\
&= \iint_{D'} f(\phi(u, v)) \, \mathrm{d}x \wedge \mathrm{d}y \left(\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v} \right) \, \mathrm{d}A' \\
&= \iint_{D'} f(\phi(u, v)) \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \, \mathrm{d}A' \\
&= \iint_D f(x, y) \, \mathrm{d}A.
\end{aligned}$$

More generally, for a k-surface $\phi : (u_1, \ldots, u_k) \mapsto \mathbb{R}^m$, we have

$$\int_S \omega = \int \cdots \int_D \omega(\phi(u_1, \ldots, u_k)) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)} \, \mathrm{d}u_1 \ldots \mathrm{d}u_k.$$

2.3.3 Pushforward, Pullback

Definition 2.31. Let T be a C'-mapping from E to V:

$$\mathbb{R}^n \supset E \xrightarrow{T} V \subset \mathbb{R}^m.$$

The mapping T induces **pushforward** tangent spaces:

$$T_p\mathbb{R}^n \xrightarrow{T_*} T_{T(p)}\mathbb{R}^m, \quad p \in E.$$

Let ω be a k-form on V. The mapping T induces the **pullback**

$$T^*\omega(v_1, \ldots, v_k) = \omega(T_*v_1, \ldots, T_*v_k).$$

Note that $T^*\omega$ is also sometimes written as ω_T (e.g., in Rudin).

Proposition 2.32. *Let ω be a k-form and λ be an l-form.*

(i) $T^*(a\omega + b\lambda) = aT^*\omega + bT^*\lambda.$

(ii) $T^*(\omega \wedge \lambda) = T^*\omega \wedge T^*\lambda.$

(iii) $\mathrm{d}(T^*\omega) = T^*(\mathrm{d}\omega)$.

(iv) *Let* $E \xrightarrow{E} V \xrightarrow{S} W$. *Then* $T^*(S^*\omega) = (ST)^*\omega$.

Proposition 2.33. *Let* Φ *be a* k*-surface in* E.

$$\mathbb{R}^k \supset D \xrightarrow{\Phi} \Phi(D) = S \subset E, \quad \mathbb{R}^n \supset E \xrightarrow{T} V \subset \mathbb{R}^m.$$

Then,

$$\int_{T\Phi} \omega = \int_{\Phi} T^*\omega.$$

3 Simplexes and Chains

Definition 3.1. Consider the mapping $f : X \to Y$, where X and Y are vector spaces. We say f is **affine** if $f - f(0)$ is linear. That is, if $f(x) = f(0) + Ax$ for some linear transformation $A \in L(X, Y)$.

Definition 3.2. The **standard simplex** is defined as

$$Q^k := \left\{ u \in \mathbb{R}^k : u = \sum_{i=1}^{k} \alpha_i e_i, \alpha_i \geq 0, \sum \alpha_i \leq 1 \right\}.$$

Definition 3.3. If $p_0, \dots, p_k$ are points of $\mathbb{R}^n$, the **oriented affine k-simplex** $\sigma = [p_0, \dots, p_k] : Q^k \to \mathbb{R}^n$ is defined by the affine mapping

$$\sigma(\alpha_1 e_1 + \cdots + \alpha_k e_k) := p_0 + \sum \alpha_i (p_i - p_0).$$

Note that σ is characterized by $\sigma(0) = p_0$, $\sigma(e_i) = p_i$, and

$$\sigma(u) = p_0 + Au, \quad Ae_i = p_i - p_0, \quad \forall u \in Q^k, 1 \leq i \leq k.$$

Definition 3.4. An **affine k-chain** Γ in an open set $E \subset \mathbb{R}^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \dots, \sigma_r$ in $\mathbb{E}$. These need not be distinct. If ω is a k-form in E, we define

$$\int_\Gamma \omega := \sum_{i=1}^{r} \int_{\sigma_i} \omega.$$

Definition 3.5. For $k \geq 1$, the **boundary** of the oriented affine k-simplex $\sigma = [p_0, \dots, p_k]$ is defined to be the affine $(k-1)$-chain

$$\partial\sigma := \sum_{j=0}^{k} (-1)^j [p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_k].$$

Example 3.6. The boundary of $\sigma = [p_0, p_1, p_2]$ is

$$\partial\sigma := [p_1, p_2] - [p_0, p_2] + [p_0, p_1] = [p_0, p_1] + [p_1, p_2] + [p_2, p_0].$$

Definition 3.7. A **differentiable simplex** Let $T : \mathbb{R}^n \supset E \to V \subset \mathbb{R}^m$ be a C^2 mapping. If σ is an oriented affine k-simplex in E, then the composite mapping $\Phi := T \circ \sigma$ is a k-surface in V with parameter domain Q^k. We call Φ an **oriented differentiable k-simplex** in V.

A finite collection Ψ of oriented k-simplexes $\Phi_1, \ldots, \Phi_r$ of class C^2 in V is called a **k-chain of class C^2** in V. If ω is a k-form in V, we define

$$\int_\Psi \omega := \sum_{i=1}^r \int_{\Phi_i} \omega.$$

The **boundary** of the oriented k-simplex $\Phi = T \circ \sigma$ is the $(k-1)$ chain

$$\partial\Phi = T(\partial\sigma).$$

Similarly, we define

$$\partial\Psi = \sum_i \partial\Phi_i.$$

Let Q^n be the standard simplex in $\mathbb{R}^n$, let σ_0 be the identity mapping with domain Q^n. That is,

$$\sigma_0 = [0, e_1, \ldots, e_n].$$

Then, its boundary

$$\partial\sigma_0 = [e_1, e_2, \ldots, e_n] - [0, e_2, \ldots, e_n] + \cdots + (-1)^n[0, e_1, \ldots, e_{n-1}]$$

is called the **positively oriented boundary** of the set Q^n.

Now let T be a one-to-one mapping of Q^n into $\mathbb{R}^n$ of class C^2 whose Jacobian is positive. The inverse function theorem implies $E = T(Q^n)$ is the closure of an open subset of $\mathbb{R}^n$. We define the **positively oriented boundary** of E to be the $(n-1)$-chain

$$\partial E := \partial T = T(\partial\sigma_0).$$

Theorem 3.8 (Generalized Stokes' Theorem)**.** *If Ψ is a k-chain of class C^2 in an open set $V \in \mathbb{R}^m$ and if ω is a $(k-1)$-form of class C^1, then*

$$\int_\Psi \mathrm{d}\omega = \int_{\partial\Psi} \omega.$$

Example 3.9. If $k = 1, m = 1$ we have the fundamental theorem of calculus:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

If $k = 2, m = 2$ we have Green's theorem: If

$$\omega = A \, \mathrm{d}x + B \, \mathrm{d}y, \quad \mathrm{d}\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, \mathrm{d}x \wedge \mathrm{d}y,$$

then

$$\int_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, \mathrm{d}x \wedge \mathrm{d}y = \oint_{\partial D} A \, \mathrm{d}x + B \, \mathrm{d}y.$$

If $k = 3, m = 3$, we have the so-called "divergence theorem": If

$$\omega = A \, \mathrm{d}y \wedge \mathrm{d}z + B \, \mathrm{d}z \wedge \mathrm{d}x + C \, \mathrm{d}x \wedge \mathrm{d}y, \quad \mathrm{d}\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

then

$$\int_D \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \int_{\partial D} A \, \mathrm{d}y \wedge \mathrm{d}z + B \, \mathrm{d}z \wedge \mathrm{d}x + C \, \mathrm{d}x \wedge \mathrm{d}y,$$

We may rewrite the previous equation as

$$\int_D \operatorname{div} F \, \mathrm{d}V = \int_S F \cdot n \, \mathrm{d}S,$$

where $F = \langle A, B, C \rangle$ and n is the unit normal vector to the surface $\partial D = S$. Note that $n \, \mathrm{d}S = \langle \mathrm{d}x\mathrm{d}z, \mathrm{d}z\mathrm{d}y, \mathrm{d}x\mathrm{d}y \rangle$.

If $k = 2, m = 3$, we have the original theorem discovered by Stokes: If

$$\omega = A \, \mathrm{d}x + B \, \mathrm{d}y + C \, \mathrm{d}z,$$
$$\mathrm{d}\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) \, \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \right) \, \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, \mathrm{d}x \wedge \mathrm{d}y,$$

then

$$\int_{\partial S} \omega = \int_S \mathrm{d}\omega.$$

That is,

$$\int Fn \, \mathrm{d}s = \int (\operatorname{curl} F) \cdot n \, \mathrm{d}S.$$

Sometimes $n \, \mathrm{d}s$ is written as $\mathrm{d}v$.

Proof *(Sketch)*. We want to prove $\int_\Psi \mathrm{d}\omega = \int_{\partial\Psi} \omega$. Since $\Psi = \sum \Phi_i = \sum T \circ \sigma$ (by the pullback commuting with exterior differentiation $\mathrm{d}\omega_T = (\mathrm{d}\omega)_T$ and the change of variables formula $\int_{T\sigma} \mathrm{d}\omega = \int_\sigma (\mathrm{d}\omega)_T = \int_\sigma \mathrm{d}(\omega_T)$), we need only show

$$\int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \omega.$$

It suffices to consider some special case, and then some Ouyang magique gives the desired result, I guess. □

4 Closed Forms and Exact Forms

Theorem 4.1. *Let $D \subset \mathbb{R}^3$ and let $F = \langle A(z,y,z), B(z,y,z), C(z,y,z)\rangle$ be continuous in D. Then the following are equivalent:*

*(1) F is a **conservative/gradient**. That is, there exists a differentiable function $u(x,y,z)$ such that $\mathrm{d}u = A\ \mathrm{d}x + B\ \mathrm{d}y + C\ \mathrm{d}z$ (that is, u is the anti-derivative of $\omega = A\ \mathrm{d}x + B\ \mathrm{d}y + C\ \mathrm{d}z$).*

(2) For any closed curve $\Lambda \subset D$ we have

$$\oint_\Lambda A\ \mathrm{d}x + B\ \mathrm{d}y + C\ \mathrm{d}z = 0.$$

(3) The line integral $\int_C A\ \mathrm{d}x + B\ \mathrm{d}y + C\ \mathrm{d}z$ is independent of path.

If in addition F is C^1 and D is simply-connected, then the above three statements are equivalent to the following one as well:

(4) $\operatorname{curl} F = 0$*, that is*

$$\begin{cases} \partial C/\partial y = \partial B/\partial z \\ \partial A/\partial z = \partial C/\partial x \\ \partial B/\partial x = \partial A/\partial y \end{cases}$$

Proof. (2) $\implies$ (3): Write $C_1 + C_2^- = \Lambda$. We have $\int_{C_1} + \int_{C_2^-} = \oint_\Lambda = 0$ and so

$$\int_{C_1} = -\int_{C_2^-} = \int_{C_2}.$$

(3) $\implies$ (2) is essentially the same as (2) $\implies$ (3). (3) $\implies$ (1): Define

$$u(x,y,z) = \int_C A\ \mathrm{d}x + B\ \mathrm{d}y + C\ \mathrm{d}z,$$

where C is a path from a fixed point P_0 to (x,y,z). Then, it is easy to check that

$$\begin{cases} A = \partial u/\partial x \\ B = \partial u/\partial y \\ C = \partial u/\partial z \end{cases}.$$

(1) $\implies$ (3): We have by the fundamental theorem of calculus

$$I = \int_C \omega = u(Q) - u(P).$$

(1) $\implies$ (4): Suppose ω has anti-derivative u. Then

$$A = \frac{\partial u}{\partial x}, \quad B = \frac{\partial u}{\partial y}, \quad C = \frac{\partial u}{\partial z}$$

and then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

(4) $\implies$ (1): For any closed curve $\Lambda \subset D$ there exists (by D being simply-connected) a 2-surface S such that $\partial S = \Lambda$. By Stokes' theorem, we have

$$\oint_\Lambda F \, \mathrm{d}r = \int_S (\operatorname{curl} F) \cdot n \, \mathrm{d}S = 0.$$

□

Remark 4.2. (4) $\implies$ (1)–(3) requires D to be simply-connected. (1)–(3) $\implies$ (4) does not. ☕

Definition 4.3. Let ω be a k-form in D in an open set $E \subset \mathbb{R}^m$.

(i) We say ω is **exact in E** if there is a $(k-1)$-form λ in E such that $\omega = \mathrm{d}\lambda$ (that is, λ is an anti-derivative / primitive of ω).

(ii) We say ω is **closed** if $\omega \in C^1$ and $\mathrm{d}\omega = 0$.

Remark 4.4. Let $\omega = \sum_i f_i(x) \, \mathrm{d}x_i$ be a 1-form. Then ω is closed if and only if $\partial_j f_i(x) = \partial_i f_j(x)$ for any $1 \leq i, j \leq n$ and any $x \in E$. ☕

Since $\mathrm{d}^2\omega = 0$ always, we have the following:

Proposition 4.5. *Let ω be a k-form of class C^1. If ω is exact then it is closed.*

In general closed does not imply exact. The following result gives a sufficient condition for closed forms to be exact:

Theorem 4.6 (Poincaré lemma)**.** *If $E \subset \mathbb{R}^n$ is simply-connected, then we have ω is closed implies ω is exact.*

Proposition 4.7.

(i) If ω is exact, and Ψ is a k-chain in E such that $\partial\Psi = 0$ (Ψ is "closed"), then

$$\int_{\Psi} \omega = 0.$$

(ii) If ω is closed, and $\Phi = \partial\Psi$ (Φ is "exact"), then

$$\int_{\Phi} \omega = 0.$$

Example 4.8. Let $E = \mathbb{R}^2 \setminus \{0\}$ and set

$$\omega := \frac{x \, \mathrm{d}y - y \, \mathrm{d}x}{x^2 + y^2} \in \Lambda^1(E).$$

Write $\omega = A \, \mathrm{d}x + B \, \mathrm{d}y$. Then,

(i) ω is closed. To check this, we need only verify that $\partial B/\partial x = \partial A/\partial y$:

$$\frac{\partial A}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial B}{\partial x}.$$

(ii) ω is not exact. By the remark above, we need only show that $\int_{\Lambda} \omega \neq 0$ for some closed curve Λ: Let Λ be the unit circle and D be the unit disk. We have by Stokes' theorem

$$\oint_{\Lambda} \omega = \oint_{\Lambda} \frac{x \, \mathrm{d}y - y \, \mathrm{d}x}{x^2 + y^2} = \oint_{\Lambda} x \, \mathrm{d}y - y \, \mathrm{d}x = \iint_{D} 2 \, \mathrm{d}x \wedge \mathrm{d}y = 2\pi \neq 0.$$

(iii) ω, however, is exact on $\{(x, y) : y > 0\}$. Note that we can verify $\int_{\partial\Psi} \omega = 0$ from Stokes' theorem and (i). This does not work for the example in (ii) since Λ is not the boundary of a 2-chain in E.

Now consider $D = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], y = 0\}$ and set $E = \mathbb{R}^2 \setminus D$. Let Γ be a closed curve around D. Then

$$\int_{\Gamma} \frac{x \, \mathrm{d}y - y \, \mathrm{d}x}{x^2 + y^2} - \int_{\Gamma} \frac{(x-1)\mathrm{d}y - y \, \mathrm{d}x}{(x-1)^2 + y^2} = 2\pi - 2\pi = 0.$$

Thus ω is exact in E.

Now consider $D = \{(0,0),(1,0)\}$ and set $E = \mathbb{R}^2 \setminus D$. Let Γ be a closed curve that goes between $(0,0)$ and $(1,0)$. Then

$$\int_\Gamma \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} - \int_\Gamma \frac{(x-1)\mathrm{d}y - y\,\mathrm{d}x}{(x-1)^2+y^2} = 2\pi - 0 = 2\pi.$$

Example 4.9. Let $E = \mathbb{R}^3$ and set

$$\begin{aligned}\omega &= (yze^{xyz} + 3x^2)\mathrm{d}x + (xze^{xyz} + \sin y)\mathrm{d}y + (xye^{xyz} + 2z)\mathrm{d}z \\ &:= A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z.\end{aligned}$$

Then $\operatorname{curl}(\langle A, B, C\rangle) = 0$ and so ω is closed and exact (since the domain is simply connected).

But what is the anti-derivative of ω, $u(x,y,z)$? We want

$$\frac{\partial u}{\partial x} = A = yze^{xyz} + 3x^2,$$

which gives

$$u = e^{xyz} + 3x^3 + \phi(y,z)$$

for some function ϕ. Differentiating with respect to y gives

$$\frac{\partial u}{\partial y} = xze^{xyz} + \frac{\partial \phi}{\partial y} = B = xze^{xyz} + \sin y,$$

and so

$$\phi(y,z) = -\cos(y) + \psi(z), \quad u = e^{xyz} + x^3 - \cos y + \psi(z).$$

Differentiating with respect to z gives

$$\frac{\partial u}{\partial z} = xye^{xyz} + \frac{\partial \psi}{\partial z} = C = xye^{xyz} + 2z,$$

and so $\psi(z) = z^2 + c$. Thus,

$$u(x,y,z) = e^{xyz} + 3x^3 - \cos y + z^2 + c.$$

Another way to find u is to integrate: Consider $P = (0,0,0)$ and $Q = (\xi, \eta, \zeta)$. Consider the path

$$(0,0,0) \to (\xi,0,0) \to (\xi,\eta,0) \to (\xi,\eta,\zeta).$$

Since ω is exact, line integrals are independent of paths.

$$\begin{aligned}
u(\xi,\eta,\zeta)-u(0,0,0) &= \int_{(0,0,0)}^{(\xi,\eta,\zeta)} \omega \\
&= \int_0^{\xi} A(x,0,0)\mathrm{d}x + \int_0^{\eta} B(\xi,y,0)\mathrm{d}y + \int_0^{\zeta} B(\xi,\eta,z)\mathrm{d}z \\
&= \int_0^{\xi} 3x^2\mathrm{d}x + \int_0^{\eta} \sin y\mathrm{d}y + \int_0^{\zeta} (\xi\eta e^{\xi\eta z}2z)\mathrm{d}z \\
&= \xi^3 - \cos\eta + 1 + e^{\xi\eta\zeta} - 1 + \zeta^2.
\end{aligned}$$

Thus

$$u(x,y,z) = x^3 + e^{xyz} - \cos y + z^2 + c.$$

Example 4.10. Let $f(r) \in C(0,+\infty)$ where $r = \sqrt{x^2+y^2+z^2}$. Let $D \subset \mathbb{R}^3 \setminus \{0\}$. Show that $\omega = f(r)(x\ \mathrm{d}x + y\ \mathrm{d}y + z\ \mathrm{d}z)$ is exact in D.

- If D is convex (contractible), then by Poincaré lemma, ω is exact if we can show it is closed (i.e., if if we can show curl $f(r)\,\langle x,y,z\rangle = 0$).

Denote $A = f(r)x$, $B = f(r)y$, and $C = f(r)z$. Then,

$$\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = f'(r)\frac{x}{r}y - f'(r)\frac{y}{r}x = 0,$$

and analogously.

- If D is not convex, we want to find an antiderivative u such that $\mathrm{d}u = \omega$:

$$\int rf(r)\ \mathrm{d}r = u(r), \quad u'(r) = rf(r).$$

$$\begin{aligned}
\mathrm{d}u &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}\mathrm{d}z \\
&= rf(r)\frac{x}{r}\ \mathrm{d}x + \cdots + \dots \\
&= f(r)x\ \mathrm{d}x + \cdots + \cdots = \omega.
\end{aligned}$$

4.1 Vector Analysis

Consider vector field $F = \langle A, B, C \rangle$. We can identify it with the 1-form in $\mathbb{R}^3$

$$\lambda_F = A \, \mathrm{d}x + B \, \mathrm{d}y + C \, \mathrm{d}z$$

and the 2-form in $\mathbb{R}^3$

$$\omega_F = A \, \mathrm{d}y \wedge \mathrm{d}z + B \, \mathrm{d}z \wedge \mathrm{d}x + C \, \mathrm{d}x \wedge \mathrm{d}y.$$

From $\mathrm{d}^2\omega = 0$, we know

Theorem 4.11.

- *If $F = \nabla u$, then $\nabla \times F = 0$.*
- *If $F = \nabla \times G$, then $\nabla \cdot F = 0$.*

Proof. Note that

$$\begin{gathered} F \doteq \lambda_F \doteq \omega_F \\ \nabla F \doteq \mathrm{d}F \\ \nabla \times F \doteq \mathrm{d}\lambda_F \\ \nabla \cdot F \doteq \mathrm{d}\omega_F \end{gathered}$$

□

Theorem 4.12 (Poincaré lemma)**.** *Assume E is contractible (or, in particular, convex)*

- *If $\nabla \times F = 0$, then $F = \nabla u$ for some $u \in C^1$.*
- *If $\nabla \cdot F = 0$, then $F = \nabla \times G$ for some vector field G in E.*

Line integrals are just integrals of 1-forms. Let

$$\gamma(t) = \langle \gamma_1(t), \gamma_2(t), \gamma_3(t) \rangle .$$

Then γ' is the tangent vector. We have

$$\int_\gamma \lambda_F = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \mathrm{d}t.$$

We may write

$$\gamma'(t)\mathrm{d}t = T(t)|\gamma'(t)|\mathrm{d}t = T(t)\mathrm{d}s,$$

where $\mathrm{d}s$ is the arc length element.

Let Φ be a 2-surface in $\mathbb{R}^3$ with

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \\ z &= z(u, v). \end{aligned}$$

Then

$$\int_\Phi \omega_F = \int_D F(\Phi(u, v)) \cdot N(u, v)\, \mathrm{d}u\mathrm{d}v.$$

We may write

$$\begin{aligned} N(u, v) = \Phi_u \times \Phi_v &= \left\langle \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right\rangle \\ &= n(u, v)|N(u, v)|\, \mathrm{d}u\mathrm{d}v \\ &= n(u, v)\, \mathrm{d}A, \end{aligned}$$

where $\mathrm{d}A$ is the surface area element.

Proposition 4.13. *Let γ be a C^1-curve in an open set $E \subset \mathbb{R}^3$ with parameter interval $[0, 1]$. We may write*

$$\int_\gamma \lambda_F = \int_0^1 F \cdot \gamma'\, \mathrm{d}u = \int_0^1 F \cdot t\, |\gamma'|\, \mathrm{d}u = \int_\gamma F \cdot t\, \mathrm{d}s.$$

Proposition 4.14. *Let Φ be a 2-surface in an open set $E \subset \mathbb{R}^3$ of class C^2 with parameter domain $D \subset \mathbb{R}^2$. We may write*

$$\begin{aligned} \int_\Phi \omega_F &= \int_D F(\Phi(u, v)) \cdot N(u, v)\, \mathrm{d}u\mathrm{d}v \\ &= \int_D F(\Phi(u, v)) \cdot n(u, v)\, |N(u, v)|\, \mathrm{d}u\mathrm{d}v \\ &= \int_\Phi (F \cdot n)\, \mathrm{d}A, \end{aligned}$$

where

$$N(u, v) \coloneqq \left\langle \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right\rangle.$$

Stoke's original theorem can now be written as

Proposition 4.15. *If F is a vector field of class C^2 in an ope set $E \subset \mathbb{R}^3$ and Φ is a 2-surface of class C^2 in E, then*

$$\int_{\Phi} (\nabla \times F) \cdot n \,\mathrm{d}A = \int_{\partial\Phi} F \cdot t \,\mathrm{d}s.$$

The divergence theorem can be written as

Proposition 4.16. *If F is a vector field of class C^1 in an open set $E \subset \mathbb{R}^3$ and Ω is a closed subset of E with positively oriented boundary $\partial\Omega$, then*

$$\int_{\Omega} \nabla \cdot F \,\mathrm{d}V = \int_{\partial\Omega} F \cdot n \,\mathrm{d}S.$$

5 Differential Forms Practice Problem

Problem 5.1. Let $\omega = x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$ in $\mathbb{R}^3$. Consider the curve $\Gamma = \{x^2+y^2+z^2 = 1\} \cap \{x+y+z=0\}$. Choose the orientation to be counter-clockwise when viewed from the positive x-axis. Compute $\int_\Gamma \omega$.

Solution 1: We have for $t \in [0, 2\pi]$ that

$$\begin{cases} x = \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{6}}\sin t \\ y = -\frac{1}{\sqrt{2}}cost + \frac{1}{\sqrt{6}}\sin t \\ z = -\frac{2}{\sqrt{6}}\sin t \end{cases}$$

Then,

$$\int_\Gamma \omega = \int_0^{2\pi} x(t)x'(t) + y(t)y'(t) + z(t)z'(t)\ \mathrm{d}t.$$

Solution 2: Observe that $F = \langle x, y, z\rangle = n$, the unit normal vector to S^2. We may thus use

$$\int_\Gamma \omega = \int_\Gamma F \cdot t\ \mathrm{d}s = 0,$$

where the last equality follows by observing that t is always tangent to n.

Solution 3: Note that $\nabla \times F = 0$, and so $\mathrm{d}\omega = 0$. By Poincaré lemma, ω is exact. Thus

$$\oint_\Gamma \omega = 0.$$

Problem 5.2. Let S be the unit sphere in $\mathbb{R}^3$. Let the orientation be outward. Compute

(i) $I_1 = \iint_S x\ \mathrm{d}y\mathrm{d}z + y\ \mathrm{d}z\mathrm{d}x + z\ \mathrm{d}x\mathrm{d}y$.

(ii) $I_2 = \iint_S x^2\ \mathrm{d}y\mathrm{d}z + y^2\ \mathrm{d}z\mathrm{d}x + z^2\ \mathrm{d}x\mathrm{d}y$.

Solution 1: Using spherical coordinates and symmetry, we can write

$$I_1 = 3\iint_S z\ \mathrm{d}x\mathrm{d}y, \quad I_2 = 3\iint_S z^2\ \mathrm{d}x\mathrm{d}y.$$

Solution 2:

$$I_1 = \iint_S \omega_F = \iint_S \langle x, y, z\rangle \cdot n\ \mathrm{d}S = \iint_S 1\ \mathrm{d}S = 4\pi$$

and, similarly,

$$I_2 = \iint_S x^3 + y^3 + z^3 \, \mathrm{d}S = 0,$$

where the last equality follows from symmetry.

Solution 3: Let D be the unit ball. By Gauss / Divergence Theorem, we have

$$I_1 = \iint_S \omega_F = \iiint_D \nabla \cdot F \, \mathrm{d}V = \iiint_D 3 \, \mathrm{d}V = 3 \cdot \frac{4}{3}\pi = 4\pi.$$

Similarly,

$$I_2 = \iiint_D 2x + 2y + 2z \, \mathrm{d}V = 0.$$

The last equality once again follows from symmetry.

Problem 5.3. Let

$$I = \oint_{\Gamma_h} (y^2 - z^2) \, \mathrm{d}x + (z^2 - x^2) \, \mathrm{d}y + (x^2 - y^2) \, \mathrm{d}z,$$

where

$$\Gamma_h = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \cap \{x + y + z = h\}, \quad h \in (-1, 1)$$

with counter-clockwise orientation when viewed from the positive z-axis.

Solution 1: Let S_h be the disk in the plane $\{x + y + z = h\}$ enclosed by Γ_h. By Stokes' theorem,

$$\oint_{\Gamma_h} \omega = \oint_{\Gamma_h} F \cdot t \, \mathrm{d}s = \iint_{S_h} (\nabla \times F) \cdot n \, \mathrm{d}S.$$

We have

$$\nabla \times F = \langle -2y - 2x, -2z - 2x, -2x - 2y \rangle$$

and

$$n = \langle 1, 1, 1 \rangle / \sqrt{3}.$$

Thus

$$(\nabla \times F) \cdot n = -\frac{4}{\sqrt{3}}(x + y + z) = -\frac{4h}{\sqrt{3}}.$$

We finally have

$$I = \iint_{S_h} -\frac{4h}{\sqrt{3}} \, \mathrm{d}S = -\frac{4h}{\sqrt{3}} \iint_{S_h} \mathrm{d}S.$$

It remains to find the area of S_h: we have $r_h = \sqrt{1-d^2} = \sqrt{1-h^2/3}$, where we get d by $d = |\langle 0,0,h\rangle \cdot n| = |h|/\sqrt{3}$. Thus $\iint_{S_h} \mathrm{d}S = \pi(1-h^2/3)^2$ and so

$$I = -\frac{4h}{\sqrt{3}}\pi\left(1-\frac{h^2}{3}\right)^2.$$

6 Lebesgue Theory

6.1 Set Functions

Definition 6.1. A family $\mathscr{R}$ of sets is called a **ring** if $A, B \in \mathscr{R}$ implies

$$A \cup B \in \mathscr{R}, \quad A \setminus B \in \mathscr{R}.$$

If, in addition to $\mathcal{R}$ being a ring, we also have $A_n \in \mathcal{R}$ for $n \in \mathbb{N}$ implies

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{R},$$

then $\mathcal{R}$ is called a **σ-ring**.

Remark 6.2.

- Rings are closed under finite intersections:

$$A \cap B = A \setminus (A \setminus B) \in \mathscr{R}.$$

- σ-rings are closed under countable intersections:

$$\bigcap_{n=1}^{\infty} A_n = A_1 \setminus \left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n) \right).$$

Definition 6.3. We say ϕ is a **set function** on $\mathscr{R}$ if ϕ assigns to every $A \in \mathscr{R}$ a number $\phi(A)$ of the extended real number system.

- We say ϕ is **additive** if $A \cap B = \varnothing$ implies

$$\phi(A \cup B) = \phi(A) + \phi(B).$$

- We say ϕ is **countably additive** if $A_i \cap A_j = \varnothing$ for $i \neq j$ implies

$$\phi\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \phi(A_n).$$

Remark 6.4. Note that since $\phi(\cup_{n=1}^{\infty} A_n)$ is independent of the order in which the A_n's are arranged, $\sum_{n=1}^{\infty} \phi(A_n)$ converges absolutely if it converges at all.

Proposition 6.5. *If ϕ is additive, then*

(i) If we assume that the range of ϕ is not only $+\infty$ or $-\infty$, then $\phi(\varnothing) = 0$.

(ii) ϕ is finitely additive.

(iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.

(iv) If $\phi(A) \geq 0$ for all A and $A_1 \subset A_2$, then $\phi(A_1) \leq \phi(A_2)$.

(v) If $B \subset A$ and $|\phi(B)| < +\infty$, then $\phi(A \setminus B) = \phi(A) - \phi(B)$.

Proof.

(i) Note that
$$\phi(A \sqcup \varnothing) = \phi(A) + \phi(\varnothing).$$

(ii) Induction.

(iii) Write
$$\begin{aligned} A \sqcup B &= (A \setminus B) \sqcup (B \setminus A) \sqcup (A \cap B), \\ A &= (A \setminus B) \sqcup (A \cap B) \\ B &= (B \setminus A) \sqcup (A \cap B). \end{aligned}$$

(iv) Note that
$$A_2 = A_1 \sqcup (A_2 \setminus A_1)$$

(v) Note that
$$A = B \sqcup (A \setminus B).$$

□

Definition 6.6. Let ϕ be a set function on a ring $\mathscr{R}$. Suppose $A_1 \subset A_2 \subset \ldots$ and $A := \cup A_n \in \mathscr{R}$. We say ϕ is **lower continuous** if $\lim_{n\to\infty} \phi(A_n) = \phi(A)$.

Proposition 6.7. *If ϕ is countably additive, then it is lower continuous.*

Proof. Set $B_1 := A_1$ and $B_n := A_n \setminus A_{n-1}$. Note that B_n is pairwise disjoint and $A = \cup B_n$. Thus,
$$\phi(A) = \sum_{i=1}^{\infty} \phi(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \phi(B_i).$$

□

6.2 Construction of the Lebesgue Measure

Definition 6.8. Let $\mathbb{R}^p$ denote the p-dimensional Euclidean space.

- By an **interval** in $\mathbb{R}^p$ we mean the set of points x such that

$$a_i \le x_i \le b_i, \quad 1 \le i \le p,$$

  or the set of points characterized by the display above with some of the inequalities being strict.

- If I is an interval, we define

$$m(I) := \prod (b_i - a_i).$$

  If $A \in \mathscr{E}$, and $A = \sqcup_{i=1}^{n} A_i$ where each A_i is an interval, then

$$m(A) := \sum_{i=1}^{n} m(A_i).$$

- An **elementary set** A is the union of a finite number of intervals. The family of all elementary subsets of $\mathbb{R}^p$ is denoted $\mathscr{E}$.

Proposition 6.9.

- *$\mathscr{E}$ is a ring but not a σ-ring (consider, e.g., $\mathbb{N}^p$).*
- *Any $A \in \mathscr{E}$ can be written as the union of finitely many disjoint intervals.*
- *m is well defined on $\mathscr{E}$.*
- *m is additive on $\mathscr{E}$.*

Our goal now is to extend m on $\mathscr{E}$ to a σ-ring.

Definition 6.10 (Regular)**.** A nonnegative additive set function ϕ defined on $\mathscr{E}$ is said to be **regular** if the following is true: To every $A \in \mathscr{E}$ and to every $\epsilon > 0$ there exist sets $F \in \mathscr{E}$ and $G \in \mathscr{E}$ such that F is closed, G is open, $F \subset A \subset G$, and

$$\phi(G) - \epsilon \le \phi(A) \le \phi(F) + \epsilon.$$

Example 6.11.

- m is regular on $\mathscr{E}$.
- Let α be a monotonically increasing function on $\mathbb{R}$. Put

$$\begin{aligned}
\mu([a,b)) &= \alpha(b-) - \alpha(a-) \\
\mu([a,b]) &= \alpha(b+) - \alpha(a-) \\
\mu((a,b]) &= \alpha(b+) - \alpha(a+) \\
\mu((a,b)) &= \alpha(b-) - \alpha(a+).
\end{aligned}$$

  Note that m is the special case of setting $\alpha(x) = x$. We have μ is regular on $\mathscr{E}$ for the same reason.

We next show that every regular set function on $\mathscr{E}$ can be extended to a countably additive set function on a σ-ring containing $\mathscr{E}$.

Definition 6.12. Let μ m be additive, regular, nonnegative, and finite on $\mathscr{E}$. Consider countable covering s of any set $E \subset \mathbb{R}^p$ by open elementary sets A_n:

$$E \subset \bigcup_{n=1}^{\infty} A_n.$$

Define for any $E \subset \mathbb{R}^p$,

$$\mu^*(E) := \inf \sum_{n=1}^{\infty} \mu(A_n),$$

where the inf is taken over all countable coverings of E by open elementary sets. μ^* is called the **outer measure** of E, corresponding to μ.

Proposition 6.13 (Properties of the Outer Measure)**.**

(i) $\mu^* \geq 0$ *and* $\mu^*(E_1) \leq \mu^*(E_2)$ *if* $E_1 \subset E_2$.

(ii) μ^* *is an extension of* μ *from* $\mathscr{E}$ *to* $\mathcal{P}(\mathbb{R}^n)$. *That is, for any* $A \in \mathscr{E}$, $\mu^*(A) = \mu(A)$.

(iii) μ^* *is* ***countably subadditive****: If* $E = \cup_{n=1}^{\infty} E_n$, *then*

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

Proof. (i) Immediate.

(ii) Fix $A \in \mathscr{E}$ and $\epsilon > 0$. Find $A \subset G$ open such that

$$\mu(G) \leq \mu(A) + \epsilon.$$

Since $\mu^*(A) \leq \mu(G)$ and since ϵ was arbitrary, we have $\mu^*(A) \leq \mu(A)$.

We next prove the reverse inequality. By the definition of μ^*, there exists a sequence $\{A_n\}$ of open elementary sets whose union contains A such that $\sum \mu(A_n) \leq \mu^*(A) + \epsilon$. The regularity of μ shows that A contains a closed elementary set F such that $\mu(F) \geq \mu(A) - \epsilon$. Since F is compact, we have $F \subset A_1 \cup \cdots \cup A_N$ for some N. Then,

$$\begin{aligned}\mu(A) &\leq \mu(F) + \epsilon \leq \mu(A_1 \cup \cdots \cup A_N) + \epsilon \\ &\leq \sum_{n=1}^{N} \mu(A_n) + \epsilon \leq \mu^*(A) + 2\epsilon.\end{aligned}$$

By sending ϵ to 0 we have $\mu(A) \leq \mu^*(A)$.

(iii) Suppose $E = \cup E_n$ with $\mu^*(E) < +\infty$ for each n. Given $\epsilon > 0$, there are coverings $\{A_{nk}\}$, $k = 1, 2, \ldots$ of E_n by open elementary sets such that

$$\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(E_n) + 2^{-n}\epsilon.$$

Then,

$$\mu^*(A) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \epsilon.$$

□

Our goal now is to find some family of sets $\mathfrak{M}$ such that $\mu^*|_{\mathfrak{M}}$ is countably additive.

Definition 6.14. For any $A, B \subset \mathbb{R}^p$, we define

$$\begin{aligned}S(A, B) &:= (A \setminus B) \cup (B \setminus A) \\ d(A, B) &:= \mu^*(S(A, B)).\end{aligned}$$

- We write $A_n \to A$ if $d(A_n, A) \to 0$, and in such case, we say A is **finitely μ-measurable** and write $A \in \mathfrak{M}_F(\mu)$.

- If A is the union of a countable collection of finitely μ-measurable sets, we say A is **μ-measurable** and write $A \in \mathfrak{M}(\mu)$.

Remark 6.15. Note that d is nonnegative, symmetric, and satisfies the triangle inequality. However, it is not positive definite and as such does not define a metric.

Example 6.16. Let $A = \varnothing$ and B be countable. We have $d(A, B) = 0$.

We may, however, quotient $\mathbb{R}^p$ by the equivalence relation $A \sim B$ if $d(A, B) = 0$. In this quotient space, d define a metric.

Remark 6.17.

- $\mathfrak{M}_F(\mu)$ is obtained as the closure of $\mathscr{E}$, $\mathfrak{M}_F(\mu) := \overline{\mathscr{E}}$ is a ring.
- $\mathfrak{M} := \sigma(\mathfrak{M}_F) = \sigma(\overline{\mathscr{E}})$ is a σ-ring.

Theorem 6.18. *$\mathfrak{M}(\mu)$ is a σ-ring, and μ^* is countably additive on $\mathfrak{M}(\mu)$.*

Example 6.19. When $\mu = m$, we have μ is the Lebesgue measure.

Remark 6.20.

(i) If A is open, then $A \in \mathfrak{M}(\mu)$ for any μ, since every open sets in $\mathbb{R}^p$ is the union of a countable collection of open intervals. By taking complements, we see that $\mathfrak{M}(\mu)$ contains also all closed sets.

(ii) We say E is a **Borel** set if E can be obtained by a countable number of operations starting from open sets, each operation being either a union, intersection, or complement. The Borel σ-ring is denoted $\mathscr{B}$. Note that $\mathscr{B}$ is the smallest σ-ring containing all open sets and so by (i), $\mathscr{B} \subset \mathfrak{M}$.

(iii) μ is regular on $\mathfrak{M}$. For any $A \in \mathfrak{M}(\mu)$ and $\epsilon > 0$, there exists F closed and G open such that

$$F \subset A \subset G, \quad \mu(G \setminus A) < \epsilon, \quad \mu(A \setminus F) < \epsilon.$$

The first inequality holds since μ^* was defined by coverings of open sets. The second inequality holds by taking complements.

(iv) If $A \in \mathfrak{M}(\mu)$, there exists $F, G \in \mathscr{B}$, $F \subset A \subset B$, such that

$$\mu(G \setminus A) = \mu(A \setminus F) = 0.$$

This follows from (iii) by sending $\epsilon := 1/n \to 0$. Thus any $A \in \mathfrak{M}(\mu)$ is the union of a Borel set and a set of measure zero.

6.3 Measure Space

Definition 6.21. Suppose X is a set. We say X is a **measure space** if there exists a σ-ring $\mathfrak{M}$ of subsets of X (called measurable sets) and a nonnegative countably additive set function μ (called a measure) defined on $\mathfrak{M}$. If, in addition, $X \in \mathfrak{M}$, then X is said to be a **measurable space** and $\mathfrak{M}$ is called a $\boldsymbol{\sigma}$**-field**.

Example 6.22.

(i) $X = \mathbb{R}^p$, $\mathfrak{M} = \mathfrak{M}(m)$, $\mu = m$ is called the Lebesgue measure.

(ii) $X \simeq \mathbb{N}$, $\mathfrak{M} = \mathcal{P}(\mathbb{N})$, $\mu(A) = |A|$ is called the counting measure.

(iii) Probability theory: $\mathfrak{M}$ is the set of all events, $\mu = \mathbb{P}$ is the probability of an event, $\mathbb{P}(E) = 1$, and $0 \leq \mathbb{P}(A) \leq 1$ for each $A \in \mathfrak{M}$.

6.4 Measurable Function

Let X denote a measurable space throughout this chapter.

Definition 6.23. A function $f : X \to \overline{\mathbb{R}}$ is called **measurable** if

$$f^{-1}((a, +\infty]) := \{x | f(x) > a\}$$

is measurable for each $a \in \mathbb{R}$.

Example 6.24. $X = \mathbb{R}^p$ and $\mathfrak{M} = \mathfrak{M}(m)$. If f is continuous, then f is measurable.

Proposition 6.25. *We can replace "$f^{-1}(a, +\infty]$" with any of the following:*

(i) $f^{-1}[a, +\infty]$,

(ii) $f^{-1}[-\infty, a)$,

(iii) $f^{-1}[-\infty, a]$.

Proof. (i) We have if each $f^{-1}(a, +\infty]$ is measurable that

$$f^{-1}[a, +\infty] = \bigcup_{n=1}^{\infty} f^{-1}\left(a - \frac{1}{n}, +\infty\right] \in \mathfrak{M}.$$

Similarly, assuming each $f^{-1}[a, +\infty]$ is measurable,

$$f^{-1}(a, +\infty] = \bigcup_{n=1}^{\infty} f^{-1}\left[a + \frac{1}{n}, +\infty\right] \in \mathfrak{M}.$$

(ii), (iii)

$$f^{-1}[-\infty, a) = X \setminus f^{-1}[a, +\infty]$$

and so on.

□

Theorem 6.26. *If f is measurable, then $|f|$ is measurable.*

Proof. We have

$$|f|^{-1}[-\infty, a) = f^{-1}[-\infty, a) \bigcap f^{-1}(-a, +\infty] \in \mathfrak{M}.$$

□

Theorem 6.27. *Let $\{f_n\}$ be a sequence of measurable functions. Then* $\sup f_n$*,* $\inf f_n$*,* $\limsup f_n$*, and* $\liminf f_n$ *are measurable.*

Proof. For any a,

$$\{x : \sup f_n(x) > a\} = \bigcup \{x : f_n(x) > a\}.$$

The same proof works for inf. For lim sup, note that

$$\limsup f_n = \inf_m (\sup_{n \geq m} f_n).$$

Alternatively, note that

$$\{\limsup f_n > a\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{f_n > a\}.$$

The lim inf case is of course similar. □

Corollary 6.28.

(i) *If f and g are measurable, then* $\min(f, g)$ *and* $\max(f, g)$ *are measurable. In particular,*

$$f^+ := \max(f, 0), \quad f^- := -\min(f, 0) = \max(-f, 0)$$

are measurable.

(ii) *If a sequence of measurable functions f_n converges pointwise to f, then f is measurable.*

Theorem 6.29. *Let $f, g : X \to \mathbb{R}$ be measurable and $F : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be continuous. Then,*

$$h(x) := F(f(x), g(x))$$

is measurable. In particular, $f + g$, $f - g$, fg, and f/g are measurable.

Proof. Let

$$G_a := F^{-1}(a, +\infty).$$

Then G_a is open and we can write

$$G_a = \bigcup I_n,$$

where I_n is a sequence of open intervals in $\mathbb{R}^2$. For each $I_n = (a_n, b_n) \times (c_n, d_n)$, we have

$$\{x : (f(x), g(x)) \in I_n\} = \{x : a_n < f(x) < b_n\} \cap \{x : c_n < f(x) < d_n\}$$

is measurable. Hence the same is true of $\{x : F(x) > a\} = \{x : (f(x), g(x)) \in G_a\}$. □

Remark 6.30. Measurability of a function does not depend on the measure, but the σ-ring.

Definition 6.31. If $f^{-1}(a, +\infty)$ is always a Borel set, then f is said to be **Borel-measurable**.

6.5 Simple Functions

Definition 6.32. Let s be a real-valued function defined on X. If the range of s is finite, we say s is a **simple function**.

Definition 6.33. Let $E \subset X$ and put

$$K_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

Then, K_E is called the **characteristic function** of E.

Remark 6.34. Every simple function can be written as a finite linear combination of characteristic functions. ☕

Every function can be approximated by simple functions:

Theorem 6.35. *Let $f : X \to \mathbb{R}$. Then there exists a sequence $\{s_n\}$ of simple functions such that $s_n \to f$ pointwise.*

(i) If f is measurable, then $\{s_n\}$ can be chosen to be measurable.

(ii) If f is nonnegative, then $\{s_n\}$ can be chosen to be monotonically increasing.

(iii) If f is bounded, the convergence is uniform.

Proof. For $f \geq 0$, define

$$E_{ni} := \left\{ x : \frac{i-1}{2^n} \leq f(x) \leq \frac{i}{2^n} \right\}, \quad F_n = \{x : f(x) \geq n\}$$

and put

$$s_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mathbb{1}_{E_{ni}}(x) + n\mathbb{1}_{F_n}(x).$$

For each x, if $f(x) = \infty$, then $s_n(x) = n$ for each n. If $f(x) < \infty$, then there exists N such that $|f(x)| \leq N$. Then, for $n \geq N$,

$$|s_n(x) - f(x)| \leq \frac{1}{2^n} \longrightarrow 0.$$

For the general case, write $f = f^+ - f^-$. □

6.6 Integration

Definition 6.36. Let $(X, \mathfrak{M}, \mu)$ be a measurable space. Let $E \in \mathfrak{M}$.

(i) For a nonnegative measurable simple function $s = \sum c_i \mathbb{1}_{E_i}$ with each $c_i \geq 0$, we define
$$I_E(s) := \sum c_i \mu(E \cap E_i).$$

(ii) If f is measurable and nonnegative, we define
$$\int_E f \, \mathrm{d}\mu := \sup I_E(s),$$
where the sup is taken over all measurable simple functions f such that $0 \leq s \leq f$.

(iii) For a general measurable function f, we define
$$\int_E f \, \mathrm{d}\mu := \int_E f^+ \, \mathrm{d}\mu - \int_E f^- \, \mathrm{d}\mu,$$
provided at least of the two terms is finite. If both terms are finite, we say f is **integrable** and write $f \in \mathscr{L}(\mu)$ on E. In this case, we define
$$\int_E |f| \, \mathrm{d}\mu := \int_E f^+ \, \mathrm{d}\mu + \int_E f^- \, \mathrm{d}\mu < \infty.$$
Note that $\int f \, \mathrm{d}\mu \leq \int |f| \, \mathrm{d}\mu$.

Proposition 6.37.

(1) It is easy to verify that
$$\int_E s \, \mathrm{d}\mu = I_E(s).$$

(2) If f is bounded on E, $\mu(\mu) < \infty$, then $f \in \mathscr{L}(E)$.

(3) If $a \leq f(x) \leq b$, $\mu(E) < \infty$, then
$$a\mu(E) \leq \int_E f \, \mathrm{d}\mu \leq b\mu(E).$$

(4) *If* $f, g \in \mathscr{L}(\mu)$, $f \leq g$ *on* E, *then*

$$\int_E f \,\mathrm{d}\mu \leq \int_E g \,\mathrm{d}\mu.$$

(5) *If* $\mu(E) = 0$, *then* $\int_E f \,\mathrm{d}\mu = 0$.

(6) *If* $f \in \mathscr{L}(\mu)$ *on* E, *then* $f \in \mathscr{L}(\mu)$ *on* $A \subset E$ *for any* $A \subset E$.

(7) *If* $E = \cup_{n=1}^{\infty} E_n$ *with* E_n *pairwise disjoint, then if either (1)* f *is nonnegative and measurable, or (2)* $f \in \mathscr{L}(\mu)$, *then*

$$\int_E f \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{E_n} f \,\mathrm{d}\mu.$$

Thus the function $\phi(E) := \int_E f \,\mathrm{d}\mu$ *is countably additive on* $\mathfrak{M}$.

(8) *If* $B \subset A$ *and* $\mu(B \setminus A) = 0$, *then*

$$\int_A f \,\mathrm{d}\mu = \int_B f \,\mathrm{d}\mu.$$

(9) *Thus, if* $\mu(S(A, B)) = 0$, *then* $\int_A f \,\mathrm{d}\mu = \int_B f \,\mathrm{d}\mu$.

(10) *If* $|f| \leq g$ *ae on* E *and* $g \in \mathscr{L}(\mu)$ *on* E, *then* $f \in \mathscr{L}(\mu)$ *on* E.

Definition 6.38. We write $f \sim g$ if the set $\{x : f(x) \neq g(x)\} \cap E$ has measure zero and say $f = g$ **almost everywhere** (ae) on E.

Remark 6.39.

- $\sim$ as defined above is an equivalence relation.
- If $f \sim g$ on E we have
$$\int_A f \,\mathrm{d}\mu = \int_A g \,\mathrm{d}\mu$$
provided the integrals exist, for every measurable subset A of E.

6.7 Convergence Theorems

Theorem 6.40 (Monotone Convergence Theorem)**.** *Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that*

$$0 \le f_1(x) \le f_2(x) \le \dots$$

and let f be the pointwise limit of f_n. Then,

$$\int_E f_n \, \mathrm{d}\mu \longrightarrow \int_E f \, \mathrm{d}\mu.$$

Proof. Note first that $\int_E f_n \, \mathrm{d}\mu$ is increasing and so converges to some limit $\alpha \in [0, +\infty]$. Since $\int f_n \le \int f$ for each n, we have $\alpha \le \int f$. It remains to establish the reverse inequality. To that end choose $c \in (0, 1)$ and let s be a simple function such that $0 \le s \le f$. Put $E_n := \{x | f_n(x) \ge cs(x)\}$ for each n. Note that we have $E_1 \subset E_2 \subset \dots$ and $E = \cup E_n$. For each n, then,

$$\int_E f_n \, \mathrm{d}\mu \ge \int_{E_n} f_n \, \mathrm{d}\mu \ge c \int_{E_n} s \, \mathrm{d}\mu$$

By countable additivity and lower continuity (which is a consequence of countable additivity) of $A \mapsto \int_A g$, we have

$$\int_{E_n} s \, \mathrm{d}\mu \longrightarrow \int_E s \, \mathrm{d}\mu.$$

We thus have $\alpha \ge c \int_E s \, \mathrm{d}\mu$. Sending $c \to 1$, this gives $\alpha \ge \int_E s \, \mathrm{d}\mu$. Taking sup over all simple functions s such that $0 \le s \le f$, we have

$$\alpha \ge \int_E f \, \mathrm{d}\mu.$$

□

Corollary 6.41. *Let $\{f_n\} \ge 0$. We have*

$$\int \sum_{n=1}^{\infty} f_n \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_E f_n \, \mathrm{d}\mu.$$

Theorem 6.42 (Linearity)**.**

$$\int_E \sum_{i=1}^{n} c_i f_i = \sum_{i=1}^{n} c_i \int_E f_i.$$

Proof *(Idea)***.** For any f, there exists simple functions $\{\phi_n\}$ such that $0 \le \phi_n \uparrow f$. Use linearity for simple functions and the decomposition $f = f^+ - f^-$. $\square$

Theorem 6.43 (Fatou's Theorem)**.** *Let* $\{f_n\} \ge 0$. *Then*

$$\int \liminf f_n \le \liminf \int f_n.$$

Proof. Define $g_n \coloneqq \liminf_{1 \le i \le n} f_n$ and observe that $g_n \uparrow \liminf f_n =: g$. From each $g_n \le f_n$ we have $\int g_n \le \int f_n$. Taking $\liminf$ on both sides we have (using the monotone convergence theorem)

$$\underbrace{\lim \int g_n = \int \lim g_n = \int \liminf f_n}_{\text{MCT}} \le \liminf \int f_n.$$

$\square$

Theorem 6.44 (Dominated Convergence Theorem)**.** *Let* $\{f_n\}$ *be such that*

- $|f_n| \le g$ *for some* $g \in \mathscr{L}(\mu)$ *on* E*, and*
- $f_n \to f$ *pointwise ae on* E*.*

Then, $f \in \mathscr{L}(\mu)$ *and*

$$\lim_{n \to \infty} \int_E f_n \, \mathrm{d}\mu = \int_E \lim_{n \to \infty} f_n \, \mathrm{d}\mu.$$

Proof. Note first that $|f| \le g$. Also, since $g \in \mathscr{L}(\mu)$, we know

$$f_n \in \mathscr{L}(\mu), \quad f \in \mathscr{L}(\mu).$$

Note in particular that we have $g + f_n \ge 0$. Fatou's Lemma gives

$$\int g + f = \int \liminf(g + f_n) \le \liminf \int (g + f_n) = \int g + \lim f_n.$$

Since $g \in \mathscr{L}$, we have $|\int g| < \infty$ and thus

$$\int f \le \liminf \int f_n.$$

We may similarly apply Fatou's lemma to $g - f_n$ to get

$$\int g - f = \int \liminf(g - f_n) \leq \int g + \liminf - \int f_n \leq \int g - \limsup \int f_n$$

and thus

$$-\int f \leq -\limsup \int f_n.$$

$\square$

Corollary 6.45. *Suppose $\mu(E) < \infty$, $\{f_n\}$ is uniformly bounded on E, and $f_n \to f$ ae on E. Then, $f \in \mathscr{L}(\mu)$ on E and* $\lim_{n\to\infty} \int f_n = \int \lim f_n = \int f$.

6.8 Comparison of Riemann and Lebesgue Integrals

Proposition 6.46.

(i) If $f \in \mathscr{R}[a, b]$, then $f \in \mathscr{L}[a, b]$ and

$$\int_a^b f(x)\,\mathrm{d}x = \underbrace{\mathscr{R}\int_a^b}_{\textit{Riemann Integral}} f(x)\,\mathrm{d}x.$$

However, $\mathscr{R}[a, b] \subsetneq \mathscr{L}[a, b]$.

(ii) A function f is Riemann integrable if and only if it is continuous ae on the relevant interval. The converse is however not true. For example, consider the Dirichlet function $D := \mathbb{1}(\mathbb{Q})|_{[0,1]}$.

Remark 6.47.

- (i) holds for bounded intervals. For an unbounded integral $[a, +\infty)$, $f \in \mathscr{R}[a, +\infty]$ does not imply $f \in \mathscr{L}[a, +\infty]$, unless $|f|$ is also Reimann integrable. For an example, consider $f = \sin(x)/x$. Then

  $$\mathscr{R}\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

  However,

  $$\int_0^{+\infty} \left|\frac{\sin x}{x}\right| \mathrm{d}x = +\infty.$$

  This is the reason why some call the Lebesgue integral the absolute convergent integral.

Proof.

(i) Let f be Riemann integrable. There exists a sequence of partitions P_k such that

$$\lim_{k\to\infty} L(P_k, f) = \mathcal{R}\underline{\int_a^b} f \,\mathrm{d}x, \quad \lim_{k\to\infty} U(P_k, f) = \mathcal{R}\overline{\int_a^b} f \,\mathrm{d}x.$$

Define $L_k(x) \coloneqq m_i^k$ and $U_k(x) \coloneqq M_i^k$. Then

$$L(P_k, f) = \int L_k \,\mathrm{d}x, \quad U(P_k, f) = \int U_k \,\mathrm{d}x,$$
$$L_1 \le L_2 \le \cdots \le f \le \cdots \le U_2 \le U_1.$$

Thus there exists $L \coloneqq \lim L_k$ and $U \coloneqq \lim U_k$. Observe that L and U are bounded measurable functions on $[a, b]$ and

$$L \le f \le U$$
$$\int L \,\mathrm{d}x = \mathscr{R}\underline{\int} f, \quad \int U \,\mathrm{d}x = \mathscr{R}\overline{\int} f$$

by the monotone convergence theorem. To complete, note that $f \in \mathscr{R}$ if and only if

$$\int L = \int U,$$

which happens if and only if $L = U$ ae, which which case $L = f = U$ ae.

(ii) If x belongs to no P_k, it is easy to see that $U(x) = L(x)$ if and only if f is continuous at x. Since the union of the sets P_k is countable, its measure is 0, and we conclude that f is continuous ae on $[a, b]$ if and only if $L = U$ ae, and by the precious result, if and only if $f \in \mathscr{R}$.

□

Proposition 6.48. *Let $F(x) \coloneqq \int_a^x f \,\mathrm{d}t$, where $f \in \mathscr{L}$.*

(i) If $f \in \mathscr{R}[a, b]$, then $F \in C[a, b]$.

(ii) If $f \in C[a, b]$, then $F \in C'[a, b]$ and $F' = f$.

Proposition 6.49. *Let $F(x) := \int_a^x f \,\mathrm{d}t$, where $f \in \mathscr{L}$.*

(i) $F'(x) = f(x)$ ae on $[a, b]$.

(ii) If F is differentiable at every point of $[a, b]$ and if $F' \in \mathscr{L}$, then

$$F(b) - F(a) = \int_a^b F'(t).$$

In this case, F is absolutely continuous on $[a, b]$. This is stronger even than uniform continuity of f.

6.9 Integration of Complex Functions

Let $f : X \to \mathbb{C}$ and write $f = u + \mathrm{i}v$, where u, v are real functions.

Definition 6.50. We say f is **measurable** if both u and v are measurable. Equivalently, f is measurable if and only if $f^{-1}(V)$ is measurable for open subset $V \subset \mathbb{C}$.

Definition 6.51. We say f is **integrable** if $|f| \in \mathscr{L}(\mu)$, that is, if $\int |f| \,\mathrm{d}\mu < \infty$. In this case we define

$$\int_E f \,\mathrm{d}\mu := \int_E u \,\mathrm{d}\mu + \mathrm{i} \int_E v \,\mathrm{d}\mu.$$

6.10 Functions of Class $\mathscr{L}^2$

Definition 6.52.

$$\mathscr{L}^2 := \left\{ f : \int |f|^2 \,\mathrm{d}\mu < \infty \right\}.$$

We define on $\mathscr{L}^2$ the norm

$$\|f\|_2 := \left(\int |f|^2 \,\mathrm{d}\mu \right)^{1/2} \equiv \sqrt{\langle f, f \rangle}$$

and the inner product

$$\langle f, g \rangle := \int f\overline{g} \,\mathrm{d}\mu.$$

Remark 6.53. As an inner product, $\langle \cdot, \cdot \rangle$ satisfies

- $\langle f, f \rangle \geq 0$ with equality if and only if $f = 0$.

- $\langle f, g \rangle = \overline{\langle g, f \rangle}$.
- Conjugate bilinearity:

$$\langle af_1 + bf_2, g \rangle = a \langle f_1, g \rangle + b \langle f_2, g \rangle, \quad \langle f, cg_1 + dg_2 \rangle = \overline{c} \langle f, g_1 \rangle + \overline{d} \langle f, g_2 \rangle .$$

☕

Remark 6.54. $\mathscr{L}^p$ is complete. In particular, $\mathscr{L}^2$ is a Hilbert space and $\mathscr{L}^p$ is a Banach space. ☕

Proposition 6.55 (Cauchy Schwarz)**.** *Let $f, g \in \mathscr{L}^2$. We have*

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2.$$

Proof. Note that by the positive definiteness of the inner product,

$$\langle f + \lambda g, f + \lambda g \rangle \geq 0.$$

Let $\lambda = -\langle f, g \rangle / \|g\|$ (this the length of the projection of f on g) and we get the desired result by expanding the inequality above. □

This can be generalized to the Holder inequality:

Proposition 6.56 (Holder Inequality)**.** *If $1/p + 1/q = 1$, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proposition 6.57. $\|f\|_2 := \sqrt{\langle f, f \rangle}$ *is a* ***norm****. That is,*

- $\|f\| \geq 0$ *and* $\|f\| = 0$ *if and only if* $f = 0$ *ae.*
- $\|cf\| = |c| \|f\|$.
- $\|f + g\| \leq \|f\| + \|g\|$.

Proof. The first two statements are obvious. We prove the third: Note that

$$\begin{aligned}
\|f + g\|^2 = \langle f + g, f + g \rangle &= \|f\|^2 + \|g\|^2 + 2 \langle f, g \rangle \\
&\leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| \\
&= (\|f\| + \|g\|)^2.
\end{aligned}$$

□

Definition 6.58. $\|f\|_\infty := \sup_x |f(x)|$.

Proposition 6.59 (Minkowski Inequality)**.** *For $1 \le p \le \infty$ we have*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Theorem 6.60. *$C[a,b]$ is dense in $\mathscr{L}^2[a,b]$. That is, for any $f \in \mathscr{L}^2[a,b]$ and $\epsilon > 0$, there exists a continuous function g such that $\|f - g\|_2 < \epsilon$.*

Proof *(Idea)***.** Let $A \subset [a,b]$ be closed. For χ_A we can define

$$d(x;A) := \inf_{y \in A} |x - y|.$$

The function

$$g_n(x) := \frac{1}{1 + nd(x;A)}$$

is continuous, identically 1 on A and converges to 0 on $[a,b] \setminus A$. By the dominated convergence theorem, we have

$$\|g_n - \chi_A\| \longrightarrow 0.$$

By regularity we can extend the conclusion to χ_E for arbitrary measurable sets, simple functions, and then arbitrary $\mathscr{L}^2$ functions by approximation. □

Definition 6.61. Let H be an inner product space.

- We say f and g are **orthogonal** ($f \perp g$) if $\langle f, g \rangle = 0$.
- We say $S = \{\phi_n\} \subset H$ is a **orthogonal set** if $\phi_n \perp \phi_m$ for any $n \neq m$. If, in addition, $\langle \phi_n, \phi_n \rangle = 1$, then we say S is an **orthonormal set**.
- An orthonormal set S is said to be **complete** if $S^\perp = \{0\}$, that is, if $f \perp \phi_n$ for each n implies $f = 0$.
- An orthonormal set S is said to be **closed** (and form an **orthonormal basis** of H) if for each $f \in H$ we have

  $$f = \sum c_n \phi_n.$$

  Note that in this case, $\langle f, \phi_m \rangle = c_m$. We may thus write

  $$f = \sum \langle f, \phi_n \rangle \, \phi_n.$$

  The coefficients c_n are called **Fourier coefficients** and the summation is called a **Fourier series** (or Fourier expansion).

Definition 6.62. A **Hilbert space** is a complete inner product space. That is, an inner product space H is said to be complete if each Cauchy sequence (with respect to the inner product induced metric) is convergent with a limit in H.

Proposition 6.63. *All $\mathscr{L}^p$ spaces are complete normed spaces (**Banach spaces**).*

Theorem 6.64. *Let H be a Hilbert space and let $S \subset H$ be an orthonormal set. The following are equivalent:*

(i) S is closed (so S is an orthonormal basis).

(ii) S is complete ($S^\perp = 0$).

*(iii) **Parseval identity**:*

$$\|f\|_2^2 = \sum \langle f, \phi_n \rangle^2 = \sum c_i^2 =: \|(c_n)\|_{\ell^2}^2.$$

Thus, Fourier transformations ($f \mapsto \langle f, \phi_n \rangle$) are isometries between $\mathscr{L}^2$ and ℓ^2.

7 Power Series

Definition 7.1. A **power series** is a function of the form $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ defined on the real line, and where $a_n \in \mathbb{R}$. These are called **analytic functions**.

Example 7.2.

(i) $\sum_{n=0}^{\infty} n^n x^n$ converges if and only if $x = 0$.

(ii) $\sum x^n$ converges on $(-1, 1)$.

(iii) $\sum x^n/n$ converges on $[-1, 1)$ and diverges at $x = 1$.

(iv) $\sum x^n/n^2$ converges on $[-1, 1]$.

(v) $\sum x^n/n!$ converges everywhere. To see this, write $c_n := x^n/n!$ and note that

$$\frac{|c_{n+1}|}{|c_n|} = \frac{|x|}{n+1} \longrightarrow 0$$

Theorem 7.3. *If the power series $\sum a_n x^n$ converges at $x_0 \in \mathbb{R}$ and $x_0 \neq 0$, then this series converges on $(-|x_0|, |x_0|)$. Moreover, it converges uniformly on any closed interval $[a, b] \subset (-|x_0|, |x_0|)$.*

Proof. Take $\delta \in (0, 1)$ such that $[a, b] \subset (-\delta|x_0|, \delta|x_0|)$. Since $\sum a_n x_0^n$ converges, we have $|a_n x_0^n|$ converges to 0. There thus exists M such that $|a_n x_0^n| \leq M$ for each n. Then,

$$|a_n x^n| = \left| a_n x_0^n \cdot \frac{x_n}{x_0^n} \right| \leq M\delta^n \longrightarrow 0.$$

By the Weierstrass M-test, we have the series converges uniformly on $[a, b]$. □

Corollary 7.4. *The series $\sum a_n x^n$ converges at $x_0 \neq 0$ and diverges at $x_1 \neq 0$, then there exists R, $|x_0| \leq R \leq |x_1|$, such that the series converges on $(-R, R)$ and diverges if $|x| > R$. (The behavior cannot be determined at the endpoints.)*

Definition 7.5. The **radius of convergence** of a power series $\sum a_n x^n$ is defined as

$$R := \sup \left\{ |x| : \sum a_n x^n \text{ converges} \right\}.$$

R is well-defined by the theorem above.

Theorem 7.6. *For the power series $\sum a_n x^n$. Let $\rho := \limsup \sqrt[n]{|a_n|}$. Then we have $R = 1/\rho$ (if $\rho = 0$, then $R = \infty$; if $\rho = \infty$, then $R = 0$).*

The same conclusion holds if we define $\rho := \lim |a_{n+1}|/|a_n|$.

Proof. For the first definition of ρ, use the root test. For the second, use the ratio test. □

Remark 7.7. We can analogously define the radius of convergence for the series $\sum a_n (x - x_0)^n$. The results above remain valid.

Example 7.8. Find the radius and interval of convergence for $\sum n! x^{n^n}$. Recalling that $\lim x^{1/x} = \lim \exp(\log x / x) = 1$ (the last equality follows from L'Hopital's rule), we have

$$1 \leq (n!)^{\frac{1}{n^n}} \leq (n^n)^{\frac{1}{n^n}} \longrightarrow 1.$$

Thus

$$\limsup \sqrt[n]{|a_n|} = \limsup (n!)^{\frac{1}{n^n}} = 1.$$

We thus have $R = 1$. When $x = \pm 1$, it is obvious that the series diverges. Thus the interval of convergence is $(-1, 1)$.

Example 7.9. Find the radius and interval of convergence for

$$\sum \frac{\log(n+1)}{n^2} (x-3)^n.$$

We have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\log(n+2)}{\log(n+1)} \cdot \frac{(n+1)^2}{n^2} \longrightarrow 1.$$

Thus $R = 1$. When $x = 3 \pm 1$, we have

$$\left| \frac{\log(n+1)}{n^2} \right| \leq \frac{\log(n+1)}{n^2} \leq \frac{1}{n^{2-\delta}}.$$

And so the series converges by the comparison test at both endpoints. The interval of convergence is $[2, 4]$.

Theorem 7.10 (Abel)**.** *Let $\sum a_n x^n$ be a power series with radius of convergence R. Then,*

(i) $\sum a_n x^n$ converges uniformly on any $[a, b] \subset (-R, R)$.

(ii) If $\sum a_n x^n$ converges at $x = R$, then it converges uniformly on any $(-R, R]$.

(iii) If $\sum a_n x^n$ converges at $x = -R$, then it converges uniformly on any $[-R, R)$.

Proof. Follows from a theorem stated above. □

Theorem 7.11 (Continuity)**.** *Let $\sum a_n x^n$ be a power series with radius of convergence R.*

(i) If the series converges at $x = x_0+R$, then $\lim_{x\to x+R^-} \sum a_n(x-x_0)^n = \sum a_n R^n$.

(ii) If the series converges at $x = x_0 - R$, then $\lim_{x\to x-R^+} \sum a_n(x - x_0)^n = \sum a_n(-R)^n$.

Consequently, the power series $\sum a_n(x - x_0)^n$ is continuous on its interval of convergence.

Theorem 7.12 (Integrability)**.** *Let $\sum a_n x^n$ be a power series with radius of convergence R. For any x_1, x_2 in the interval of convergence, we have*

$$\int_a^b \sum_{n=0}^{+\infty} a_n(x - x_0)^n \,\mathrm{d}x = \sum \int_a^b a_n(x - x_0)^n \,\mathrm{d}x.$$

Moreover,

$$\int \sum a_n(x - x_0)^n \,\mathrm{d}x = \sum_{n=0}^{+\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

The new series has the same radius of convergence as the original series:

$$\limsup \sqrt[n]{\frac{|a_{n+1}|}{n+1}} = \limsup \sqrt[n]{|a_n|}.$$

Example 7.13.

- $\sum x^n$ converges on $(-1, 1)$.
- $\sum x^{n+1}/(n + 1)$ converges on $[-1, 1)$.
- $\sum x^{n+2}/(n + 1)/(n + 2)$ converges on $[-1, 1]$.

Theorem 7.14 (Differentiability)**.** *Let $S(x) \coloneqq \sum a_n(x - x_0))^n$ be a power series with radius of convergence R. Then for $x \in (x_0 - R, x_0 + R)$, $S(x)$ is uniformly differentiable at x. Hence, any real analytic function is infinitely differentiable.*

For this series we can formally differentiate term by term:

$$S^{(k)}(x) = \sum n(n - 1)\dots(n - k + 1)a_n(x - x_0)^{n-k}.$$

Example 7.15. Find the sum of the series $\sum(n(2n+1))^{-1}$. Define

$$S(x) = \sum \frac{x^{2n+1}}{n(2n+1)}.$$

Note that S has radius of convergence $R = 1$ and converges at both endpoints. We differentiate S to get

$$S'(x) = \sum \frac{x^{2n}}{n} = -\log(1-x^2), \quad \forall x \in (-1,1).$$

Thus,

$$\begin{aligned}
S(x) - S(0) &= \int_0^x S'(t)\,\mathrm{d}t \\
&= \int_0^x -\log(1-t^2)\,\mathrm{d}t \\
&= -t\log(1-t^2)\Big|_0^x - 2\int_0^x \frac{t^2}{1-t^2}\,\mathrm{d}t \\
&= -x\log(1-x^2) + 2x + \log\frac{1-x}{1+x} \\
&= (1-x)\log(1-x) + 2x - (1+x)\log(1+x), \quad \forall x \in (-1,1).
\end{aligned}$$

Since $S(0) = 0$, we have

$$\sum \frac{1}{n(2n+1)} = \lim_{x\to 1} S(1) = 2 - 2\log 2.$$

7.1 Taylor Series

Suppose $f(x) = \sum a_n x^n$ on $(-R,R)$. By differentiating both sides n times and letting $x = 0$, we get

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Thus the power series expansion is unique.

Remark 7.16. C^∞ functions are not necessarily analytic. For example, consider

$$f(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x^2} & x > 0 \end{cases}.$$

Each derivative of f is 0 at $x = 0$, but f is not identically 0.

7.2 Complex Series

Definition 7.17. Let $S(x) = \sum a_n(z - z_0)^n$, where z_0 and a_n are complex, and $z \in \mathbb{C}$. We say S is **complex analytic** if it can be expanded in a power series in $D(z_0, \delta) \subset \mathbb{C}$.

Theorem 7.18 (M-test)**.** *If there is a sequence M_n such that $|a_n(z - z_0)^n| \leq M_n$ for each n and $z \in \Omega \subset \mathbb{C}$ and $\sum M_n < \infty$, then $\sum a_n(z - z_0)^n$ converges uniformly on* Ω*.*

Example 7.19. Consider $e^z = \sum z^n/n!$ on $D(0, R)$. We have

$$\frac{|z|^n}{n!} < \frac{R^n}{n!}$$

for each R. Thus by the M-test, the series converges uniformly on $D(0, R)$ for each R. We thus have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

Theorem 7.20 (Abel)**.** *Suppose $\sum a_n(z - z_0)^n$ converges at $z^* \pm z_0$. Then for any $0 < r < |z^* - z_0|$, the series converges uniformly on $\overline{D(z_0, r)} := \{z : |z - z_0| \leq r\}$.*

Definition 7.21. The **radius of convergence** of a complex power series $\sum a_n(z - z_0)^n$ is defined as

$$R := \sup_R \left\{ |z - z_0| : \sum a_n(z - z_0)^n \text{converges} \right\}.$$

In light of Abel's' theorem, we have again that

- For any $r \in (0, R)$, the series converges uniformly on $\overline{D(z_0, r)}$.
- For $z \notin \overline{D(0, r)}$, the series diverges.
- For $z \in \partial D(0, R)$, the behavior is not determined.

Theorem 7.22. *Let $\rho := \limsup \sqrt[n]{|a_n|}$ (or $\rho := \lim |a_{n+1}|/|a_n|$). Then, $R = 1/\rho$ (if $\rho = \infty$, then $R = 0$ and vice versa).*

Theorem 7.23. *In $D(0, R)$, $S(z) = \sum a_n(z - z_0)^n$ can be differentiated term by term:*

$$S'(z) = \sum n a_n (z - z_0)^{n-1}.$$

The new series has the same radius of convergence as the original series. Repeated application of this theorem shows that complex analytic functions are infinitely differentiable.

Theorem 7.24 (Taylor)**.** *If $S(z) = \sum a_n(z - z_0)^n$ is infinitely differentiable at on $D(0, R)$ and $a_n = f^{(n)}(z_0)/n!$, that is,*

$$S(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

Definition 7.25. We say $f(z)$ is **analytic** at $z_0 \in \mathbb{C}$ if f is complex differentiable in some neighborhood $D(z_0, g) \subset \mathbb{C}$. Equivalently, $f(z)$ is analytic in $\Omega \subset \mathbb{C}$ if it is analytic for any $z_0 \in \Omega$. If this is the case, $f(z) = \sum a_n(z - z_0)^n$ in Ω.

Example 7.26.

$$\begin{aligned} e^x &= \sum \frac{z^n}{n!}, && z \in \mathbb{C} \\ \sin z &= \sum (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}, && z \in \mathbb{C} \\ \cos z &= \sum (-1)^n \frac{z^{2n}}{(2n)!}, && z \in \mathbb{C}. \end{aligned}$$

We may read off the following:

$$\begin{aligned} \sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \\ e^{z_1 + z_2} &= e^{z_1} e^{z_2}, \\ (\sin z)' &= \cos z, \\ (\cos z)' &= -\sin z, \\ e^{iz} &= \cos z + \mathrm{i} \sin z \end{aligned}$$

Theorem 7.27. *Suppose $f(z)$ is analytic in $\Omega \subset \mathbb{C}$. If there exists $z_0 \in \Omega$ such that $f(z_0) = f'(z_0) = \cdots = 0$, then $f(z) \equiv 0$.*

Proof. Let $S \coloneqq \{z \in \Omega : f(z) = f'(z) = \cdots = 0\}$. Since $f^{(n)}$ is continuous, S is closed in Ω. Since $f(z)$ is analytic, there exists $D(z_0, \delta) \in \mathbb{C}$ such that $f(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = 0$. Thus S is open in Ω and we may conclude $S = \Omega$. □

Corollary 7.28. *If $f(z)$ is analytic and non-constant in Ω, then there exists $m \in \mathbb{N}_+$ such that $f^{(m)}(z_0) \neq 0$, $f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$. There then exists $D(z_0, \delta) \subset \mathbb{C}$ such that $f(z) = f(z_0) + (z - z_0)^m g(z)$ where $g(z_0) \neq 0$. If $f(z_0) = 0$, then z_0 is a zero of f of* ***order*** *m. Then z_0 is an* ***isolated zero****.*

Theorem 7.29 (Uniqueness of Analytic Functions)**.** *Suppose $f(z)$ and $g(z)$ are analytic in Ω. If there exists $\{z_n\} \subset \Omega$ such that $f(z_n) = g(z_n)$ and $z_n \longrightarrow z^* \in \Omega$, then f and g are identical on Ω.*

Since $f(z) = f(z_0) + (z - z_0)^m g(z)$ behaves locally like $z \mapsto z^m$, we have the following:

Theorem 7.30. *If $f'(z)$ is analytic in Ω, then f is an open mapping, that is, f maps any open set to an open set.*

7.3 The Algebraic Completeness of the Complex Field

Theorem 7.31 (Fundamental Theorem of Algebra)**.** *Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree n. Suppose $n \geq 1$ and $a_n \neq 0$ so P is non-constant. Then $P(z)$ has at least one root in $\mathbb{C}$.*

Corollary 7.32. *$P(z)$ has exactly n complex roots (counted with multiplicity).*

Recall the following: If $f(z) = \sum a_n (z - z_0)^n$ is complex analytic and non-constant, then f is an open mapping.

Proof. Since $P(z)$ is continuous, $|P(z)|$ is also continuous and thus attains minimum and maximum on $\overline{D(0, R)}$. Let's assume it attains min at $z_0 \in \overline{D_0, R}$. Since $|P(z)| \longrightarrow \infty$ as $|z| \longrightarrow \infty$, we can choose R large enough such that $z_0 \in D(0, R)$ and $\min\{|P(z)| : |z| = R\} > P(0)$.

By the open mapping theorem, $P(D(0, R))$ is open, so $P(z_0)$ is an interior point. If $|P(z_0)| \neq 0$, then $|P(z_0)|$ cannot be the min (since some point in the open set $P(D(0, R))$ will have smaller length) and we have a contradiction. □

7.4 Fourier Series

On $\mathscr{L}^2[-\pi,\pi]$ we have the orthonormal basis

$$\left\{\phi_n := \frac{1}{\sqrt{2\pi}}e^{\mathrm{i}nx} : n \in \mathbb{Z}\right\}.$$

That is,

$$\langle \phi_n, \phi_m\rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}.$$

Recall that

$$e^{\mathrm{i}nx} = \cos(nx) + \mathrm{i}\sin(nx).$$

Definition 7.33. The **Fourier series** of $f \in \mathscr{L}^2[-\pi,\pi]$ is defined as

$$\sum_{n\in\mathbb{Z}} c_n\phi_n(x),$$

where the coefficients c_n are given by

$$c_n := \langle f, \phi_n\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}nx}\,\mathrm{d}x.$$

It turns out that $\{\phi_n\}$ forms a complete orthonormal basis of $\mathscr{L}^2[-\pi,\pi]$. Thus, for each $f \in \mathscr{L}[-\pi,\pi]$, $f(x) = \sum c_n\phi_n(x)$ if and only if the Fourier series converges almost everywhere.

For $f \in \mathscr{L}[-\pi,\pi]$ we can define its Fourier series as

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{\mathrm{i}nx}.$$

Theorem 7.34 (Pointwise Convergence)**.** *Let f be a function of period 2π. If*

(i) $f(x)$ is piecewise differentiable, or

(ii) $f(x)$ is piecewise monotone,

then for any $x \in [-\pi,\pi]$, we have

$$S_n f(x) \longrightarrow \frac{1}{2}\left[f(x^-) + f(x^+)\right],$$

where

$$S_n f(x) := \sum_{k=-n}^{n} \langle f, \phi_i\rangle\,\phi_i.$$

(iii) If f is Lipschitz or α-Holder continuous at $x \in [-\pi, \pi]$, that is,

$$|f(x+y) - f(x)| \leq C|y|^{\alpha}.$$

In particular this implies f is continuous at x.

Then $S_n f(x) \to f(x)$.

Theorem 7.35 ($\mathscr{L}^2$-Convergence)**.** *If $f \in \mathscr{L}^2[-\pi, \pi]$, then $S_n f \longrightarrow f$ in the $\mathscr{L}^2$ norm.*

Theorem 7.36. *If $S_n f \to f$ in the $\mathscr{L}^p$ norm, then there exists a subsequence $S_{n_k} f \to f$ almost everywhere.*

Theorem 7.37. *For $f \in \mathscr{L}^p[-\pi, \pi]$, $S_n f \to f$ a.e., which implies the completeness of $\{\phi_n\}$.*

Theorem 7.38 (Parseval's Identity)**.**

$$\|f\|_{\mathscr{L}^2} = \|\hat{f}(x)\|_{\ell^2}.$$

7.5 Some Common Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad |x| < 1.$$

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1.$$