Notes: MATH273 (F25) Basic Theory of Ordinary Differential Equations

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Contents

1 Motivation, Preview of Application

Example 1.1 ((Stochastic) gradient descent). We are interested in solving

$$\min_{x \in D} g(x)$$

where g represents a cost, and $D \subset \mathbb{R}^n$. The FOC is $\nabla g(x) = 0$. If g is nonlinear and $n \gg 1$, this is a very hard problem. We can however always consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\nabla g(x(t)),$$

where g is given and $x(t)$ is unknown. If $\mathbb{R}^{n \times n} \ni \nabla^2 g > 0$ (is positive definite) and $x(t_0) = x_0$, then $x(t) \to x_*$, where $x_* \coloneqq \arg\min_{x \in D} g(x)$.

2 Basic Definitions and Examples

Definition 2.1 (Differential Equation)**.** A **differential equation** is an equation that relates a function y and its derivatives. A general representation is

$$F\left[x, y, \partial_i y, \partial_i^2 y, \dots, \partial_i^{(n)} y\right] = 0.$$

Problem 2.2 (Heat Equation). Consider $u(t,x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

$$\partial_t u(t,x) = \Delta u(t,x) = \sum_{i=1}^{n} \partial_{x_i}^2 u(t,x).$$

This is a second order differential equation.

Problem 2.3.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u + \frac{\mathrm{d}}{\mathrm{d}t} u = u.$$

This is a second order ODE.

- **ODEs** contain only derivatives on one variable.
- **PDEs** can contain multiple partial derivatives.

Definition 2.4 (Order of a Differential Equation)**.** The **order** of a differential equation is the order of the highest order derivative that appears in the equation.

Definition 2.5 (Linear and Nonlinear ODEs)**.** We say an ODE is **linear** if

$$F\left[x, y(x), y'(x), \dots, y^{(n)}(x)\right]$$

depends on $y, y', \dots, y^{(n)}$ linearly. Note that we allow F to depend on x nonlinearly.
An **nonlinear** ODE is one that is not linear.

Note that we may always represent a linear ODE as

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = b(x),$$

where each $a_i(x)$ can be nonlienar in x.

In general, linear ODEs are fully solvable by hand. For nonlinear ODEs, we may only be able to solve some special cases.

Problem 2.6.

$$\frac{\mathrm{d}}{\mathrm{d}t} y = c,$$

where c is a constant. Integrating, we get $y = ct + b$, where b is an arbitrary constant.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y = 0.$$

Integrating twice, we get $y = at + b$, where a, b are arbitrary constants.

These (non unique) are called **general solutions**. We thus sometimes prescribe also an initial condition (IC) to further determine the solution. An example of an IC for the second order ODE above is

$$y(0) = y_0, \quad y'(0) = v_0,$$

In these examples, note in particular that we have uniqueness results given the ICs.

2.1 First Order Linear ODEs

We may represent a first order linear ODE as

$$a_1(x)y'(x) + a_0(x)y(x) = b(t).$$

The general method is to rewrite the ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)] = f(t)$$

and integrate both sides.

When $a_1 \neq 0$, we can rewrite the ODE as

$$y'(x) + p(x)y(x) = g(x),$$

with p, g given. The particular case $b = 0$ ($g = 0$) is considered first:

2.1.1 The case b = 0

Problem 2.7. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = a(t)y(t). \tag{1}$$

Assuming $y(t) \neq 0$, we may rewrite this as

$$a(t) = \frac{1}{y(t)}\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\log|y(t)|].$$

Integrating, we get

$$\log|y(t)| = \int a(t)\,\mathrm{d}t + C,$$

and so

$$y(t) = \pm e^C \exp\left(\int a(t)\,\mathrm{d}t\right) = C' \exp\left(\int a(t)\,\mathrm{d}t\right),$$

where C' is an arbitrary constant (the case $C' = 0$ is attained when $y = 0$).

2.1.2 The Integrating Factor

Problem 2.8. Consider

$$y'(x) + p(x)y(x) = g(x). \tag{2}$$

Observe that for each $\mu(t) \neq 0$, the ODE is equivalent to

$$\mu y' + \mu p y = \mu g.$$

Let's guess that the left hand side can be written as $\frac{\mathrm{d}}{\mathrm{d}t}[a(t)y(t)]$ for some a. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}[a(t)y(t)] = a'(t)y(t) + a(t)y'(t) = \mu y' + \mu p y \implies \begin{cases} a = \mu, \\ \mu' = \mu p. \end{cases}$$

The function μ is called the **integrating factor**. It suffices to find one μ such that $\mu' = \mu p$. A μ is given by the previous case:

$$\mu(t) = \exp\left(\int_{t_0}^{t} p(s)\,\mathrm{d}s\right).$$

We now reduced the ODE to

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu(t)y(t)] = \mu(t)g(t),$$

which can be solved by integrating and dividing by μ:

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\,\mathrm{d}s + C\right).$$

Example 2.9.

$$y' + y = e^t.$$

We seek a μ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu y] = \mu y' + \mu y = \mu e^t.$$

This gives

$$\begin{cases} \mu' = \mu, \\ \mu = e^t. \end{cases}$$

Using this choice of μ we rewrite the ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^t y] = e^t e^t = e^{2t}$$

$$e^t y = \frac{1}{2}e^{2t} + C,$$

from which $y = \frac{1}{2}e^t + ce^{-t}$. ✍

Determining C: Suppose we are given

$$y(t_0) = y_0.$$

Then

$$\mu y(t) = \int_{t_0}^{t} \mu g(s)\, \mathrm{d}s + c.$$

Taking $t = t_0$, we get

$$\mu y(t_0) = c$$

Since (given our choice of μ)

$$\mu(t_0) = 1,$$

we have $C = y(t_0) = y_0$, which we can plug back in the general solutions obtained above.

3 Separation of Variables

Recall that first order ODEs can be represented as

$$y'(t) + p(t)y(t) = g(t).$$

Using the implicit function theorem, we can in principle rewrite this as

$$y'(x) = f(x, y)$$

and then

$$M(x, y) + N(x, y)y' = 0.$$

Question: for which M, N can we solve this ODE?

Recall that last lecture we solved

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)] = g(t)$$

with y unknown.

A first special case (separation of variables) is when $M = M(x)$ and $N = N(y)$:

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Proof *(Formal Derivation)***.** If we formally treat dx and dy as differentials, we can rewrite the above as

$$N(y)\mathrm{d}y = -M(x)\mathrm{d}x.$$

In this view the variables x and y are separated. Integrating both sides, we obtain

$$\int N(y)\mathrm{d}y = -\int M(x)\mathrm{d}x + C.$$

If we can find n and m such that $n' = N$ and $m' = M$, then we have

$$n(y) = -m(x) + C,$$

from which we can solve for y. □

Proof *(Rigorous Derivation)***.** We integrate over x to get

$$\int M(x)\,\mathrm{d}x + \int N(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = 0.$$

With a change of variables we have

$$\int M(x)\,\mathrm{d}x + \int N(y)\,\mathrm{d}y = C.$$

□

3.0.1 Examples

Example 3.1.

$$x + y\frac{dy}{dx} = 0.$$

We have $M = 1$ and $N =$ y. Integrating, we get

$$\frac{x^2}{2} + C + \int y\frac{dy}{dx}\,dx = 0$$

and so

$$\frac{x^2}{2} + \frac{y^2}{2} = C.$$

With additional initial conditions we can determine $y(x)$.

Example 3.2.

$$y + e^x\frac{dy}{dx} = 0.$$

Dividing by ye^x (assuming $y \neq 0$), we get

$$\frac{1}{y}\frac{dy}{dx} = -e^{-x}.$$

and then

$$-e^{-x} + C + \log|y| = 0.$$

More generally, suppose the dependence of M and N on (x, y) can be separated in the following sense:

$$M_1(x)M_2(y) + N_1(x)N_2(y)\frac{dy}{dx} = 0.$$

Again dividing both sides, we get

$$\frac{M_1}{N_1} + \frac{N_2}{M_2}\frac{dy}{dx} = 0.$$

Example 3.3.

$$e^{x+y} + xy\frac{dy}{dx} = 0.$$

Use above.

Warning: this method does not work for the following ODE:

$$M_1(x)M_2(y) + Z_1(x)Z_2(y) + N_1(x)N_2(y)\frac{dy}{dx} = 0$$

3.1 Generalization

The method of integrating both sides cannot be pushed much further beyond the following case:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\varphi(x, y(x))\right] = g(x).$$

Integration gives

$$\varphi(x, y(x)) = \int g(x)\,\mathrm{d}x + c.$$

In principle we can solve for y by the implicit function theorem.

The ODE above can be written equivalently as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tilde{\varphi}(x, y(x))\right] \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\left[\varphi(x, y(x)) - \int g(x)\,\mathrm{d}x\right] = 0.$$

Thus we can without loss of generality suppose $g = 0$. We turn next thus to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x, y(x)) = 0.$$

For which M, N can we convert $M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ to the above form?

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x, y(x)) = \partial_1\varphi + \partial_2\varphi\frac{\mathrm{d}y}{\mathrm{d}x}.$$

This implies

$$\begin{cases} M(x, y) = \partial_1\varphi(x, y), \\ N(x, y) = \partial_2\varphi(x, y). \end{cases}$$

Definition 3.4. We say $M + Ny' = 0$ is an **exact equation** if there exists φ such that

$$M = \partial_1\varphi, \quad N = \partial_2\varphi.$$

What is the minimum requirement for M, N to be an exact equation? If φ is sufficiently smooth (C^2), then we would expect

$$\partial_2 M = \partial_2\partial_1\varphi = \partial_1\partial_2\varphi = \partial_1 N.$$

This in fact is also sufficient:

Theorem 3.5. *Suppose $M, N, \partial_2 M, \partial_1 N$ are continuous in the box $B = [a, b] \times [c, d]$ and $(x, y) \in B$. Then the equation $M(x, y) + N(x, y)y' = 0$ is exact if and only if*

$$\partial_2 M(x, y) = \partial_1 N(x, y).$$

That is, there exists φ such that

$$M = \partial_1\varphi, \quad N = \partial_2\varphi.$$

Example 3.6. The equation $M(x)\ \mathrm{d}x + N(y)\ \mathrm{d}y = 0$ is exact, with $\partial_2 M = 0 = \partial_1 N$. But observe also that

$$\partial_2[M(x) + y] = 1 = \partial_1[N(y) + x].$$

So we can solve the ODE

$$(M(x) + y) + (N(y) + x)\, y' = 0,$$

which is not separable.

Proof. That exactness implies $\partial_2 M = \partial_1 N$ is easy and shown above.

It remains to prove that $\partial_2 M = \partial_1 N$ implies exactness. To that end we construct φ as follows:
Step 1: construct φ so that $\partial_1 \varphi = M(x, y)$. We set

$$\varphi(x, y) = \int_{x_0}^{x} M(s, y)\ \mathrm{d}s + h(y),$$

where h is to be determined.
Step 2: determine h so that $\partial_2 \varphi = N(x, y)$. Note that

$$\begin{aligned} \partial_2 \varphi(x, y) &= \int_{x_0}^{x} \partial_2 M(s, y)\ \mathrm{d}s + h'(y) \\ &= \int_{x_0}^{x} \partial_1 N(s, y)\ \mathrm{d}s + h'(y) \\ &= N(x, y) - N(x_0, y) + h'(y), \end{aligned}$$

from which we can specify h by

$$h(y) = \int_{y_0}^{y} N(x_0, s)\ \mathrm{d}s + C.$$

In sum, φ is given by

$$\varphi(x, y) = \int_{x_0}^{x} M(s, y)\ \mathrm{d}s + \int_{y_0}^{y} N(x_0, s)\ \mathrm{d}s + C.$$

This completes the proof. □

Example 3.7.

$$y^2 x + (1 + x^2 y) y' = 0.$$

We have

$$\partial_2 M = 2xy = \partial_1 N,$$

and may thus set

$$\varphi(x, y) = \int y^2 x\ \mathrm{d}x = \frac{x^2 y^2}{2} + h(y).$$

$$\partial_2 \varphi = x^2 y + h'(y) = 1 + x^2 y \implies h(y) = y + C.$$

Finally, we can rewrite the original ODE as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2y^2}{2}+y\right]=0 \implies \frac{x^2y^2}{2}+y=C,$$

with

$$\varphi(x,y)=\frac{x^2y^2}{2}+y+C'.$$

Suppose we have the IC $y(0)=1$. Then

$$C=\frac{0^2 1^2}{2}+1=1.$$

3.2

We consider further ODEs of the term

$$M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}=0.$$

We can equivalently write this as

$$\mu M+\mu N\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

for some $\mu \neq 0$. This is exact when

$$\partial_2(\mu M)=\partial_1(\mu N).$$

The goal, thus, is to find μ such that the above is true, when the original ODE might not be exact. If $\mu(x,y)=\mu(x)$ or $\mu(x,y)=\mu(y)$, then we need not deal with mixed partials.

Let's begin with $\mu=\mu(x)$: We would like to solve

$$\partial_2(\mu M)=\mu\partial_2 M=\mu' N+\mu\partial_1 N=\partial_1(\mu N),$$

or equivalently

$$\frac{\mu'}{\mu}=\frac{\partial_2 M-\partial_1 N}{N}.$$

This approach works when the right hand side is a function of x only.

A similar condition can be derived for $\mu=\mu(y)$.

4 Second Order Linear ODEs

A second order linear ODE can be written as

$$F(t, y, y', y'') = 0,$$

and by the inverse function theorem, as

$$y'' = f(t, y, y').$$

Definition 4.1. We say this ODE is **linear** if F depends on y, y', y'' linearly (note again that we do not require linearity in t). Thus a second order linear ODE can be written as

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

In the case that $P(t) \neq 0$, we can rewrite this as

$$y'' + p(t)y' + q(t)y = g(t),$$

Example 4.2. $y'' = 0$. The general solution is $y(t) = c_1 t + c_2$. We need 2 ICs to determine c_1, c_2, for examele

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = v_0 \end{cases}.$$

In general, for an n^{th} order ODE we need n ICs to determine a unique solution.

Definition 4.3. We say the ODE is **homogeneous** if $G = 0$ and **nonhomogeneous** otherwise.

Remark 4.4 (Property of homogeneous ODEs). If y solves $y'' + p(t)y' + q(t)y = 0$, then ay solves the same ODE for any $a \in \mathbb{Z}$.

We start with the homogeneous case.

4.1 Homogeneous Second Order Linear ODEs with Constant Coefficients

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}.$$

4.1.1 The Ansatz of Polynomials

We assume first that $y(t) = \sum_{j=0}^{n} a_j t_j$. Plugging this into the ODE, we get terms involving t^n which cannot be canceled.

4.1.2 Recall

If $a \equiv 0$, then this reduces to $by' + cy = 0$. This can be written as one of the following:

$$y' + \frac{c}{b}y = 0, \quad b\frac{y'}{y} + c = 0.$$

And in either case we will get $y(t) = e^{-\frac{c}{b}t} \cdot c_0$.

4.1.3 The Ansatz of Exponentials

Inspired by the first order case, we now try the ansatz $y(t) = c_0 e^{\lambda t}$. Plugging into the ODE, we get

$$c_0 \left[a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t}\right] = 0,$$

which reduces the original ODE to the following:

$$a\lambda^2 + b\lambda + c = 0.$$

4.1.4 The Operator L

Define the operator L as

$$(Ly)(t) = P(t)y'' + Q(t)y' + R(t)y.$$

Example 4.5. For any constants c_1, c_2 and functions y_1, y_2. Note that

$$\begin{aligned} L\left[c_1 y_1 + c_2 y_2\right] &= P(t)\left[c_1 y_1 + c_2 y_2\right]'' + Q(t)\left[c_1 y_1 + c_2 y_2\right]' + R(t)\left[c_1 y_1 + c_2 y_2\right] \\ &= c_1 L[y_1] + c_2 L[y_2], \end{aligned}$$

and so the operator L is linear.

A solution y to the ODE $P(t)y'' + Q(t)y' + R(t)y = 0$ then can equivalently be written as $Ly = 0$. Now note that by linearity, we have if $Ly_1 = Ly_2 = 0$ for two "different solutions" y_1 and y_2, then since

$$L[c_1 y_1 + c_2 y_2] = c_1 L y_1 + c_2 L y_2 = 0,$$

the general solution can be obtained as

$$y = c_1 y_1 + c_2 y_2.$$

This technique of obtaining the general solution is called **linear superposition**. It turns out that the correct notion of solutions being "different" is linear independence.

Example 4.6.

$$y'' - 5y' + 6y = 0.$$

We can solve $\lambda^2 - 5\lambda + 6 = 0$ to get

$$\lambda_1 = 2, \quad \lambda_2 = 3.$$

Thus the first solution is $y_1 = e^{2t}$ and the second solution is $y_2 = e^{3t}$.

4.1.5 Three Cases of Obtaining the General Solution

The solution of the characteristic polynomial can be classified into three cases:

(i) Two real roots. In this case the general solution is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

(ii) Two different complex roots that are complex conjugates $\lambda \pm \mathrm{i}\mu$ (since all coefficients are real). Recall that

$$e^z := \sum_{k\geq 0} \frac{z^k}{k!}, \quad z \in \mathbb{C}, \qquad e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}.$$

Thus noting that

$$\begin{aligned} e^{(\lambda+\mathrm{i}\mu)t} &= e^{\lambda t}\left[\cos\mu t + \mathrm{i}\sin\mu t\right] \\ e^{(\lambda-\mathrm{i}\mu)t} &= e^{\lambda t}\left[\cos\mu t - \mathrm{i}\sin\mu t\right], \end{aligned}$$

we see two ways to obtain a real solution:

- Choose $c_1 = c_2 \in \mathbb{R}$ to get a multiple of $y(t) = e^{\lambda t}\cos\mu t$.
- Choose $c_1 = -c_2 \in \mathrm{i}\mathbb{R}$ to get a multiple of $y(t) = e^{\lambda t}\sin\mu t$.

The general solution is then a linear combination of the above two:

$$y(t) = e^{\lambda t}\left[c_1\cos\mu t + c_2\sin\mu t\right], \quad c_1, c_2 \in \mathbb{R}.$$

Example 4.7.

$$y'' + y = 0.$$

Solving the characteristic polynomial gives $\lambda_1 = \mathrm{i}$ and $\lambda_2 = -\mathrm{i}$. Two real solutions are $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

(iii) One real root with a multiplicity two. For the characteristic polynomial $a\lambda^2 + b\lambda + c = 0$, we have $\lambda = \lambda_1 = \lambda_2 = -b/2a$ and $4ac = b^2$. This gives a solution $y(t) = e^{\lambda}t$. We seek another solution y_2 using the so called **reduction of order** method. We try the ansatz $y_2(t) = \mu(t)y_1(t)$.

Claim 4.8. μ solves a first order ODE.

Proof. Suppose y_1 solves $Py_1'' + Qy' + Ry = 0$. If $y_2 = \mu y_1$ satisfies the same ODE, then

$$P(\mu y_1)'' + Q(\mu y_1)' + R(\mu y_1) = 0.$$

In general after expanding the left hand side, we get $\sum_{i=0}^{2} a_i(t)\mu^{(i)}(t) = 0$. We will show $a_0 = 0$. Expanding, we get

$$P\left[\mu'' y_1 + 2\mu' y_1' + \mu y_1''\right] + Q\left[\mu' y_1 + \mu y_1'\right] + R\mu y_1 = 0.$$

Note that the μ-terms sum to $\mu\left[Py_1'' + Qy_1' + Ry_1\right] = 0$. Thus μ solves the ODE involving μ' and μ''

$$\mu'' Py_1 + \mu'\left[2Py_1' + Qy_1\right] = 0.$$

This can be solved by separation of variables:

$$\frac{\mu''}{\mu'} = -\frac{2Py_1' + Qy_1}{Py_1}.$$

□

Example 4.9. Suppose $P \equiv a$, $Q \equiv b$, and $R \equiv c$. We have $y_1 = e^{\lambda t}$ where $\lambda := -b/2a$. We have that μ defined above solved

$$\mu'' a e^{\lambda t} + \mu'\left[2a(e^{\lambda t})' + be^{\lambda t}\right] = 0.$$

The term in the bracket evaluates to 0 by $2a\lambda + b = 0$. Thus $\mu'' = 0$ and so $\mu(t) = t + C$. Thus the general solution is

$$y(t) = (c_1 t + c_2)e^{\lambda t}.$$

4.2 Series Solution to Homogeneous Second Order Linear ODEs

Consider the ODE

$$Py'' + Qy' + Ry = 0.$$

We will use the ansatz $y(x) = \sum_{n\geq 0}^{\infty} a_n (x - x_0)^n$.

Remark 4.10. Recall the following facts about power series $\sum_{n\geq 0} a_n (x - x_0)^n$:

- The root test for convergence: Let $\mu := \limsup_{n\to\infty} |a_n|^{1/n}$. If $|x - x_0| < \mu^{-1}$, then the power series converges absolutely.
- Within the radius of convergence, we can differentiate and integrate the power series term by term.

  $$y'(x) = \sum_{n\geq 1} n a_n (x - x_0)^{n-1}.$$

  Similarly we can compute the k^{th} derivative:

  $$y^{(k)}(x) = \sum_{n\geq k} n(n-1)\ldots(n-k+1) a_n (x - x_0)^{n-k}.$$

  The resulting power series have the same radius of convergence.
- We say y is analytic in $B(x_0, \mu^{-1})$.

Example 4.11 (Airy's Equation). We seek the solution to the ODE $y'' - xy = 0$ near $x_0 = 0$.

$$\begin{aligned}
y &= \sum_{n\geq 0} a_n x^n, \\
y'' &= \sum_{n\geq 2} a_n n(n-1)x^{n-2} = \sum_{n\geq 0} a_{n+2}(n+2)(n+1)x^n, \\
xy &= \sum_{n\geq 0} a_n x^{n+1} = \sum_{n\geq 1} a_{n-1}x^n.
\end{aligned}$$

We collect terms to get

$$\begin{aligned}
x^0 : &\qquad a_2 \cdot 2 \cdot 1 = 0 \\
x_n, n \geq 1 : &\qquad a_{n+2}(n+2)(n+1) - a_{n-1} = 0.
\end{aligned}$$

This gives $a_2 = 0$ and

$$a_{m+3} = \frac{a_m}{(m+3)(m+2)}, \quad m := n-1 \geq 0$$

Equivalently, for $i = 0, 1, 2$, with $m + 3 = 3k + i$ we have

$$a_{3k+i} = \frac{a_{3k+i-3}}{(3k+i)(3k+i-1)}.$$

This can be solved using iterative substitution (See remark below). With

$$b_k = a_{3k+i}, \quad c_k = \frac{1}{(3k+i)(3k+i-1)},$$

we get

$$a_{3k+i} = a_i \prod_{j=1}^{k} \frac{1}{(3j+i)(3j+i-1)}$$

with $a_2 = 0$ and a_0, a_1 free. Thus the general solution is

$$y(x) = a_0 \sum_{i\geq 0} A_k x^{3k} + a_1 \sum_{k\geq 0} B_k x^{3k+1},$$

where

$$A_k = \prod_{j=1}^{k} \frac{1}{(3j)(3j-1)}, \quad B_k = \prod_{j=1}^{k} \frac{1}{(3j+1)(3j)}.$$

Note in particular that $|A_k|, |B_k| \leq 1$ for each k, and thus $\mu := \limsup[\cdot]^{1/n} \leq 1$. Thus the solution is analytic on a neighborhood of 0 with radius of convergence at least 1.

Remark 4.12. Suppose $b_k = c_k b_{k-1}$ for $k \geq 1$ and c_k and b_0 are given. Then we have the following solution by iterative substitution:

$$\begin{aligned} b_k &= c_k \left[c_{k-1} b_{k-2}\right] = c_k c_{k-1} \left[c_{k-2} b_{k-3}\right] = \cdots \\ &= c_k c_{k-1} \ldots c_1 b_0 = b_0 \prod_{i=1}^{k} c_i. \end{aligned}$$

Example 4.13 (Airy's Equation, around $x_0 = 1$). We seek the solution to the ODE $y'' - xy = 0$ near $x_0 = 1$.

$$\begin{aligned} y &= \sum_{n\geq 0} a_n (x-1)^n, \\ y'' &= \sum_{n\geq 2} a_n n(n-1)(x-1)^{n-2} = \sum_{n\geq 0} a_{n+2}(n+2)(n+1)(x-1)^n, \\ xy &= (x-1+1)y \\ &= \sum_{n\geq 0} a_n (x-1)^{n+1} + \sum_{n\geq 0} a_n (x-1)^n = \sum_{n\geq 1} a_{n-1}(x-1)^n + \sum_{n\geq 0} a_n (x-1)^n. \end{aligned}$$

We collect terms:

$$\begin{aligned} (x-1)^0 : & \qquad a_2 \cdot 2 \cdot 1 - a_0 = 0 \\ (x-1)^n, n \geq 1 : & \qquad a_{n+2}(n+2)(n+1) - a_{n-1} - a_n = 0. \end{aligned}$$

This gives $a_2 = a_0/2$ and

$$a_3 = \frac{a_0 + a_2}{3 \cdot 2}, \quad a_4 = \frac{a_2 + a_1}{4 \cdot 3} = \frac{a_0}{2 \cdot 3 \cdot 4} + \frac{a_1}{3 \cdot 4}, \quad \ldots$$

In this case it is hard to obtain a closed form for a_n. We note that the general solution is determined by a_0, a_1.

Remark 4.14. This power series is useful for numerical approximation of the solution. We can truncate the series at some N and use the first N terms to approximate the solution.

Theorem 4.15 (5.3.1, BDM 9th edition)**.** *Consider $P(x)y'' + Q(x)y' + R(x)y = 0$, where we assume P, Q, R are analytic near x_0 with convergence radius R and write*

$$P(x) = \sum P_j (x - x_0)^j, \forall |x - x_0| < R$$

and similarly for Q and R. If $P(x_0) \neq 0$ in the ball $B(x_0, R)$, we can consider

$$y'' + \frac{Q}{P} y' + \frac{R}{P} y = 0.$$

Then there exists a power series solution y to the ODE of the form

$$y(x) = \sum_{n\geq 0} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2$$

with convergence radius at least R.

Proof. Omitted. □

Example 4.16. Examples of analytic functions:

- Polynomials. $R = +\infty$.
- e^x. $R = +\infty$.
- $\log(1 + x)$.
- $\sin x$, $\cos x$.

Examples of non-analytic functions:

- Any non-smooth function. E.g., $|x|^{1/2}$.

4.3 Non-Homogeneous Second Order Linear ODEs

$$Py'' + Qy' + Ry = G.$$

Observation: if φ and ψ are two solutions to the ODE above, then $\varphi - \psi$ solves the corresponding homogeneous ODE $Py'' + Qy' + Ry = 0$.

Proposition 4.17. *Suppose that y_0 solves $Py'' + Qy' + Ry = G$ and y_1, y_2 are two different solutions to $Py'' + Qy' + Ry = 0$. Then the general solution to the non-homogeneous ODE is*

$$y = y_0 + c_1 y_1 + c_2 y_2, \quad \forall c_1, c_2 \in \mathbb{R}.$$

That is, the general solution is a particular solution plus the general solution to the corresponding homogeneous ODE.

With this in mind, we see that we need only find one solution y_0 to the non-homogeneous ODE.

4.3.1 Variation of Parameters / Constants

Assume that y_1, y_2 are two different solutions to the corresponding homogeneous ODE $y''+py'+qy = 0$. Recall that the goal is a particular solution y_0 to the non-homogeneous ODE $y'' + py' + qy = g$. We try the ansatz

$$y_0(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t),$$

where functions μ_1, μ_2 are to be determined. Plugging into the $y'' + py' + qy$, we will get in general terms involving μ_i, μ_i', μ_i''. We will select μ_i in a way that the μ_i and μ_i'' terms vanish.

Note that

$$y' = \mu_1' y_1 + \mu_1 y_1' + \mu_2' y_2 + \mu_2 y_2' = \mu_1 y_1' + \mu_2 y_2',$$

where the last equality results after we *impose the restriction* $\mu_1' y_1 + \mu_2' y_2 = 0$. Now,

$$y'' = \mu_1' y_1' + \mu_2 y_2' + \mu_1 y_1'' + \mu_2 y_2''$$

and so

$$y'' + py' + qy = \mu_1' y_1' + \mu_2' y_2' + \mu_1 (y_1'' + py_1' + qy_1) + \mu_2 (y_2'' + py_2' + qy_2),$$

where y_1 and y_2 solve the homogeneous ODE, and so the last two terms vanish. If we set

$$\begin{cases} \mu_1' y_1 + \mu_2' y_2 = 0, \\ \mu_1' y_1' + \mu_2' y_2' = g, \end{cases} \iff \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

we get a 2×2 linear system for μ_1', μ_2'. This is solvable if and only the matrix if invertible:

$$\begin{pmatrix} \mu_2 \\ \mu_2 \end{pmatrix} (x) = \int_{x_0}^{x} A^{-1}(t) \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \, \mathrm{d}t.$$

5 First order ODE system

5.1 Motivation and Setup

The ODE

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = g(t)$$

can be rewritten as a first order system of ODEs. Write for each j, $x_j(t) \coloneqq y^{(j-1)}(t)$ so that we have

$$y^{(j)} = \frac{\mathrm{d}}{\mathrm{d}t} x_j.$$

Now the original ODE can be rewritten as

$$a_n x_n'(t) + a_{n-1} x_n(t) + \cdots + a_1 x_2(t) + a_0 x_1(t) = g(t).$$

This gives the system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = \frac{1}{a_n}\left[g(t) - a_{n-1}x_n - \cdots - a_1 x_2 - a_0 x_1\right] \end{cases}.$$

This can be summarized as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-1}}{a_n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{g(t)}{a_n} \end{pmatrix}.$$

Let's for now abstract away from the above construction and consider the general first order system. With $F_1, \dots, F_n$ given and $x_1(t), \dots, x_n(t)$ unknown, we consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1(t, x_1, \dots, x_n) \\ F_2(t, x_1, \dots, x_n) \\ \vdots \\ F_n(t, x_1, \dots, x_n) \end{pmatrix},$$

which can be summarized as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{F}(t, \mathbf{x}).$$

This is a linear system if we can write

$$\mathbf{F}(t, \mathbf{x}) = A(t)\mathbf{x} + \mathbf{b}(t)$$

for $A(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{b}(t) \in \mathbb{R}^{n\times 1}$.

Example 5.1. The first order ODE system that arises from the n^{th} order ODE is linear.

Definition 5.2. We say the system $\mathbf{F}(t, \mathbf{x}) = A(t)\mathbf{x} + \mathbf{b}(t)$ is **homogeneous** if $\mathbf{b} \equiv 0$ and **non-homogeneous** otherwise.

Example 5.3.

$$y'' + py' + qy = g.$$

Let $x_1 := y$ and $x_2 := y'$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Initial conditions for the first order system can be written as

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^{n\times 1}.$$

Theorem 5.4 (Existence and Uniqueness)**.** *Suppose F_j is continuous and $|\partial_{x_i} F_j| \leq M$ (Lipschitz) for $t \in (a_0, b_0) =: I_0$ and $x_i \in (a_i, b_i) =: I_i$. Then for each $t_0 \in I_0$, $x_{0,i} \in I_i$ there exists $\delta > 0$ and a unique solution $\mathbf{x}(t)$ to the ODE system on $(t_0 - \delta, t_0 + \delta)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| < \delta$.*

Note that for the special case

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A(t)\mathbf{x} + \mathbf{b}, \quad \mathbf{x}, \mathbf{b} \in \mathbb{R}^{n\times 1}, \quad A \in \mathbb{R}^{n\times n},$$

it is important to restrict A and $\mathbf{b}$ to be real, since complex numbers can roughly be identified as two real numbers, and in those cases solutions may not be unique.

Example 5.5. Consider $F = A(\mathbf{x})\mathbf{x} + \mathbf{b}$. If $|A| \leq C$, then there exists unique solution in a small neighborhood.

5.2 Solving the Homogeneous First Order Linear ODE System

Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A(t)\mathbf{x}(t)$$

Suppose $\mathbf{x}^{(1)} = \left(x_1^{(1)}, \ldots, x_n^{(1)}\right)', \ldots, \mathbf{x}^{(j)} = \left(x_1^{(2)}, \ldots, x_n^{(2)}\right)'$ are j solutions to the ODE. Then by linearity, so are $\sum_j c_j \mathbf{x}^{(j)}$.

How do we differentiate different solutions?

Definition 5.6. We say $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linear independent if the **Wronski matrix** is not singular, i.e., if the **Wronskian**

$$W(t) := \det\left(x^1, \ldots, x^n\right)$$

is not identically zero.

Note that if $W(t_0) = 0$, then $x^{(j_0)}(t_0) = \sum_{j \neq j_0} c_j x^j(t_0)$.

Theorem 5.7.

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = \operatorname{tr}(A(t))\, W(t).$$

Proof. Write

$$M(t) = \begin{pmatrix} x^1(t) & x^2(t) & \dots & x^n(t) \end{pmatrix} = \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \dots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \dots & x_2^{(n)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \dots & x_n^{(n)}(t) \end{pmatrix} =: \begin{pmatrix} \mathbf{b}_1(t) \\ \mathbf{b}_2(t) \\ \vdots \\ \mathbf{b}_n(t) \end{pmatrix}.$$

We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix} = \det \begin{pmatrix} b_1' \\ b_2 \\ \dots \\ b_n \end{pmatrix} + \det \begin{pmatrix} b_1 \\ b_2' \\ \dots \\ b_n \end{pmatrix} + \dots + \det \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n' \end{pmatrix}.$$

To see this we recall

$$\frac{\mathrm{d}}{\mathrm{d}t}(C_1 \dots C_n) = \sum C_1 \dots C_{k-1} C_k' C_{k+1} \dots C_n$$

and

$$\det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \sum a_{1i_1} a_{2i_2} \dots a_{ni_n} \operatorname{sgn}(\sigma).$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ b_k' \\ \dots \\ b_n \end{pmatrix}.$$

Recalling $\frac{\mathrm{d}}{\mathrm{d}t} x^i = A x^i$ for each i and fixing k, we have $\frac{\mathrm{d}}{\mathrm{d}t} x_k^i = [A x^i]_k = \sum_j A_{kj} x_j^i$ for each i. Thus by stacking we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b_k = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} X_k^{(1)} & \dots & X_k^{(n)} \end{pmatrix} = \sum_j A_{kj} \begin{pmatrix} X_j^{(1)} & \dots & X_j^{(n)} \end{pmatrix} = \sum_j A_{kj} b_j.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ b_k' \\ \dots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ \sum_j A_{kj} b_j \\ \dots \\ b_n \end{pmatrix}$$

$$= \sum_{k=1}^{n} \det \begin{pmatrix} b_1 \\ \dots \\ A_{kk} b_k \\ \dots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} A_{kk} \det \begin{pmatrix} b_1 \\ \dots \\ b_k \\ \dots \\ b_n \end{pmatrix} = \sum_{k} A_{kk} W(t).$$

□

Corollary 5.8.

- $W(t_0) = 0$ *for some* t *if and only if* $W(t) = 0$ *for each* $t \in I$.
- $W(t_0) \neq 0$ *for some* $t_0 \in I$ *if and only if* $W(t) \neq 0$ *for each* $t \in I$.

In particular, it suffices to check $W(t_0)$ *at any* t_0.

Proof. Note that

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\, \mathrm{d}s \right),$$

where the last term is never zero. Write $\mu := \operatorname{tr} \circ A$. We have if $\mu(t) \neq 0$ for each t, then $W(t_0) = 0$ if and only if $W(t) = 0$ for each $t \in I$.

$W(t_0) \neq 0$ if and only if $W(t) \not\equiv 0$. □

Theorem 5.9. *Suppose that* $x^1, \dots, x^n$ *are n linearly independent* real *solutions to the homogeneous ODE system* $\frac{\mathrm{d}}{\mathrm{d}t}x = A(t)x$*. Then any solution x to the ODE can be written uniquely as*

$$x(t) = \sum_{j=1}^{n} c_j x^{(j)}(t), \quad c_j \in \mathbb{R}.$$

In particular, this tells us that the space of solutions to the homogeneous ODE system (or any n^{th} order ODE) is an n-dimensional vector space.

Proof. For each c write

$$y_c(t) = x(t) - \sum_i c_i x^i(t).$$

By previous results, y_c solves the homogeneous ODE.

Now fix t_0, we seek c_i such that

$$\begin{pmatrix} x^1(t_0) & x^2(t_0) & \dots & x^n(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = x(t_0).$$

Since $x^1, \dots, x^n$ are linearly independent, the Wronski matrix

$$M(t_0) := \begin{pmatrix} x^1(t_0) & x^2(t_0) & \dots & x^n(t_0) \end{pmatrix}$$

is invertible, and we can obtain $\mathbf{c}$ as $M^{-1}(t_0)x(t_0)$.

In particular we have $y_c(t_0) = 0$ and y_c solves the homogeneous ODE. Since 0 is also a solution to the homogeneous ODE, by uniqueness we have $y_c(t) \equiv 0$, which gives $x(t) \equiv \sum c_i x^i$. $\square$

Theorem 5.10. *Suppose that $X^{(i)}$ solves $\frac{\mathrm{d}}{\mathrm{d}t}X^{(i)} = A(t)X^{(i)}$ with the IC $X^{(i)}(t_0) = \mathbf{e}_i$ for $1 \le i \le n$. Then $X^{(i)}$ are linearly independent solutions.*

Proof. The existence and uniqueness theorem guarantees that $X^{(i)}$ exist and are unique. To check that they are independent, we need to check that

$$W(t) = \det \begin{pmatrix} X^{(1)}(t) & X^{(2)}(t) & \dots & X^{(n)}(t) \end{pmatrix}$$

is not identically zero. Recall that to do this it suffices to check $W(t_0) \ne 0$:

$$W(t_0) = \det I = 1 \ne 0.$$

But note of course that we can pick any n linearly independent initial conditions in $\mathbb{R}^n$. $\square$

Remark 5.11. The preceding two theorems together implies that there are exactly n linearly independent solutions to the homogeneous ODE system.

Note that the theorem above also gives a method to find n linearly independent solutions.

Example 5.12.

$$y'' + py' + qy = 0.$$

This is a second order ODE which can be rewritten as a 2×2 first order ODE system. The results above imply that there are exactly two linearly independent solutions to the ODE. The linear independence turns out to be equivalent to $y_1 \ne cy_2$ in this case.

5.3 Finding Solutions Explicitly

Recall that for $n = 1$, we have that the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = a(t)x(t), \quad \in \mathbb{R}$$

has solution

$$x(t) = x(t_0) \exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right).$$

In the special case that $A(t)$ is diagonal with $n \ge 2$, we may easily generalize the result above: $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = A\mathbf{x}$ has solution

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i = A_{ii}X_i, \quad 1 \le i \le n.$$

We say that the solution is **decoupled**.

For more general $A(t)$, we can only restrict to the special case $A(t) \equiv A$ is constant. To see why further generalization is hard, consider the case $n = 2$ and

$$y'' + py' + qy = 0.$$

Recall that we can only solve this explicitly if p, q are constant.

5.3.1 The case A is constant

Recall that for $n = 1$, we know that the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = ax(t), \quad a \in \mathbb{R}$$

has solution $x(t) = ce^{t}$.

We seek to generalized this to the case

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{x}(t) = A\mathbf{x}(t), \quad \mathbf{a} \in \mathbb{R}^{n\times n}.$$

The examples above for $n = 1$ motivates the ansatz

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^{n\times 1}, \lambda \in \mathbb{R}.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\mathbf{v}\right) = \lambda e^{\lambda t}\mathbf{v}$$
$$A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v}.$$

Thus $\mathbf{x}(t)$ solves the ODE system if and only if $\lambda\mathbf{v} = A\mathbf{v}$, or if and only if $(\lambda, \mathbf{v})$ is an eigenvalue-eigenvector pair of A.

Remark 5.13.

- Special linear combinations of $x_i(t)$ are solutions to a corresponding 1×1 ODE.
- After projecting in the direction of the eigenvectors, the ODE system decouples.

We have the following cases:

(i) A has n distinct real eigenvalues.

(ii) A has complex eigenvectors, but all eigenvalues are distinct.

(iii) Repeated eigenvalues.

(iv) n linearly independent real eigenvectors.

The first three cases are disjoint, while the last case can overlap with the first three.

5.3.2 Case (i): A has n distinct real eigenvalues

Recall that when eigenvalues are real, so are the eigenvectors. Then we have the following n solutions:

$$\mathbf{x}^{(i)} = e^{\lambda_i t}\mathbf{v}_i, \quad 1 \leq i \leq n.$$

It can be shown that if $\lambda_1, \ldots, \lambda_n$ are distinct, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. This gives $W(0) = \det\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right) \neq 0$ and so the solutions above are linearly independent.

5.3.3 Case (iv): A has n linearly independent real eigenvectors

Note that case (iv) includes case (i). This is a strict subset:

Example 5.14.

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}.$$

Since $\mathbf{v}_i$ are real, so are λ_i. We have

$$A\begin{pmatrix}\mathbf{v}_1 & \ldots & \mathbf{v}_n\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1 & \ldots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} \iff AP = P\Lambda.$$

We have $\mathbf{v}_i$ are linearly independent if and only if P is invertible. In such case we have $A = P\Lambda P^{-1}$, or A is diagonalizable.

Recall the following:

Theorem 5.15 (Symmetric Matrix)**.** *If $A = A^\top \in \mathbb{R}^{n\times n}$, then $A = Q\Lambda Q^\top$, where $Q \in \mathbb{R}^{n\times n}$, $QQ^\top = Q^\top Q = I_n$, $Q^{-1} = Q^\top$, and Λ is diagonal with real eigenvalues.*

Thus if A is symmetric, we can find n linearly independent solutions to the ODE system.

5.3.4 Case (ii): A has complex eigenvalues, but all eigenvalues are distinct

Note that if $(\lambda, \mathbf{v})$ is an eigenvalue-eigenvector pair, then so is $(\overline{\lambda}, \overline{\mathbf{v}})$, since $A\mathbf{v} = \lambda\mathbf{v} \iff A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$ (Note that this relies on A being real). We seek two real solutions from $e^{\lambda t}v$ and $e^{\overline{\lambda}t}\overline{v}$. Suppose then that $\lambda = \alpha + \mathrm{i}\beta$ and $v = a + \mathrm{i}b$ for $\alpha, \beta \in \mathbb{R}$, $a, b \in \mathbb{R}^{n\times 1}$. Then

$$\begin{aligned} e^{\lambda t}v &= e^{(\alpha+\mathrm{i}\beta)t}(a + \mathrm{i}b) \\ &= e^{\alpha t}(\cos(\beta t) + \mathrm{i}\sin(\beta t))(a + \mathrm{i}b) \\ &= e^{\alpha t}\left[(\cos(\beta t)a - \sin(\beta t)b) + \mathrm{i}(\sin(\beta t)a + \cos(\beta t)b)\right] \end{aligned}$$

and similarly,

$$e^{\overline{\lambda}t}\overline{v} = e^{\alpha t}\left[(\cos(\beta t)a - \sin(\beta t)b) - \mathrm{i}(\sin(\beta t)a + \cos(\beta t)b)\right].$$

From this we see that

$$\begin{aligned}
\operatorname{Re} e^{\lambda t} v &= \frac{e^{\lambda t} v + e^{\overline{\lambda} t}\overline{v}}{2} = e^{\alpha t}(\cos(\beta t) a - \sin(\beta t) b)\\
\operatorname{Im} e^{\lambda t} v &= \frac{e^{\lambda t} v - e^{\overline{\lambda} t}\overline{v}}{2\mathrm{i}} = e^{\alpha t}(\sin(\beta t) a + \cos(\beta t) b)
\end{aligned}$$

are two real solutions to the ODE system.

Thus each complex eigenvalue-eigenvector pair gives two real solutions to the ODE system.

5.3.5 Case (iii): A has Repeated eigenvalues

Example 5.16.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}\mathbf{x}.$$

The eigenvalue is λ with multiplicity 3. This matrix is not diagonalizable, since

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff v_2 = v_3 = 0,$$

and so each eigenvector is of the form $\mathbf{v} = \begin{pmatrix} v_1 & 0 & 0 \end{pmatrix}^\top$.

The methods discussed above thus does not work. Note, however, that x_3 can be solved easily: $x_3(t) = c_3 e^{\lambda t}$. Then the restriction on x_2 becomes $x_2' = \lambda x_2 + c_3 e^{\lambda t}$. Using the integrating factor $e^{-\lambda t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda t} x_2\right) = c_3,$$

which gives $x_2(t) = e^{\lambda t}(c_2 + c_3 t)$. Finally, the restriction on x_1 becomes $x_1' = \lambda x_1 + e^{\lambda t}(c_2 + c_3 t)$. Using the integrating factor $e^{-\lambda t}$ again, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda t} x_1\right) = c_2 + c_3 t,$$

which gives $x_1(t) = e^{\lambda t}\left(c_1 + c_2 t + \frac{c_3 t^2}{2}\right)$. Thus the general solution to the ODE system is

$$\mathbf{x}(t) = e^{\lambda t}\begin{pmatrix} c_1 + c_2 t + \frac{c_3 t^2}{2} \\ c_2 + c_3 t \\ c_3 \end{pmatrix} = c_1 e^{\lambda t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t}\begin{pmatrix} \frac{t^2}{2} \\ t \\ 1 \end{pmatrix}.$$

5.4 Matrix Exponential

Consider the $n = 1$ case $x' = ax$, $a \in \mathbb{R}$. We have solution $x(t) = e^{at}x_0$. We seek to generalize this to the case $n \geq 2$ where

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

But of course we need first to make sense of e^{At} for $A \in \mathbb{R}^{n\times n}$. The hope is that a definition will be consistent with $(e^{At})' = Ae^{At}$.

Definition 5.17.

$$\exp[A] := \sum_{k=0}^{\infty} \frac{A^k}{k!} \in \mathbb{R}^{n\times n},$$

if the series converges.

Example 5.18.

- If $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, we have

$$\exp[A] = \operatorname{diag}\left(e^{\lambda_1}, \dots, e^{\lambda_n}\right).$$

- $A = \begin{pmatrix} & \beta \\ -\beta & \end{pmatrix} = \beta J$, where $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. We have $A^k = \beta^k J^k$. Note that $J^2 = -I$, $J^3 = -J$, $J^4 = I$. Thus we have

$$A^{4k+i} = \beta^{4k+i}J^i, \quad i = 0, 1, 2, 3,$$

  and then

$$\exp[A] = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

Remark 5.19. In general $AB \neq BA$. If however $A = P^a$ and $B = P^b$, then $AB = BA$.

For $\exp[A]$ to be well-defined, we require $\sum_{k\geq 0}(A^k)ij/k!$ to converge for each $1 \leq i, j \leq n$. Let's suppose $\max_{i,j}|A_{ij}| \leq a$.

Proposition 5.20.

$$\left|(A^k)_{ij}\right| \leq (na)^{k-1}a.$$

Proof. We use induction. The case $k = 1$ is clear. Now suppose the condition holds for $k \geq 1$. From $(A^{k+1})_{ij} = (A^kA)_{ij} = \sum_{l=1}^n (A^k)_{il}A_{lj}$ we have

$$|(A^{k+1})_{ij}| \leq \sum_{l=1}^{n}|(A^k)_{il}||A_{lj}| \leq \sum_{l=1}^{n}(na)^{k-1}a^2 = (na)^k a.$$

□

In light of the proposition above, we have for each N.

$$\sum_{k=0}^{N} \frac{|(A^k)_{ij}|}{k!} \leq \sum_{k=0}^{\infty} \frac{(na)^{k-1}a}{k!} = \sum_{k=0}^{\infty} \frac{1}{n}\frac{(na)^k}{k!} = \frac{e^{na}}{n} < \infty$$

Thus $\exp[A]$ is well-defined for all $A \in \mathbb{R}^{n\times n}$.

Lemma 5.21.

(i) *If* $B = T^{-1}AT$, *then* $\exp[B] = T^{-1}\exp[A]T$.

(ii) *If* $AB = BA$, *then* $\exp[A+B] = \exp[A]\cdot\exp[B]$.

(iii) $(\exp[A])^{-1} = \exp[-A]$.

Proof.

(i) Note only that $B^k = T^{-1}A^kT$.

(ii) Note that

$$\begin{aligned}\exp[A+B] &= \sum_{k=0}^{\infty}\frac{(A+B)^k}{k!} = \sum_{k\geq 0}\frac{1}{k!}\left(\sum_j \binom{k}{j}A^jB^{k-j}\right)\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{A^jB^{k-j}}{j!(k-j)!} = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{A^j}{j!}\frac{B^{k-j}}{(k-j)!} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{A^j}{j!}\frac{B^k}{k!}\\ &= \exp[A]\exp[B].\end{aligned}$$

(iii) Applying property (ii) with $B = -A$ gives $\exp[A]\exp[-A] = \exp[0] = I$. Similarly, $\exp[-A]\exp[A] = I$.

□

Proposition 5.22. *Suppose* $Av = \lambda v$. *Then* $\exp[A]v = e^{\lambda}v$.

Proof. Use the fact that $A^kv = \lambda^kv$. □

Proposition 5.23.

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp[tA] = A\exp[tA] = \exp[tA]A.$$

Proof. A previous calculation shows that if $\max|(A)_{ij}| \leq a$, then $\sum(\dots) \leq e^{nat}/n < \infty$. So $t \mapsto \exp[tA]$ behaves like a power series with radius of convergence ∞.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty}\frac{(tA)^k}{k!}\right) &= \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{(tA)^k}{k!} = \sum_{k=1}^{\infty}\frac{A^k}{k!}kt^{k-1} = \sum_{k\geq 1}\frac{A^k}{(k-1)!}t^{k-1}\\ &= A\sum_{k=1}^{\infty}\frac{(tA)^{k-1}}{(k-1)!} = A\exp[tA].\end{aligned}$$

But in the last line we may as well place A at the end to obtain $\frac{\mathrm{d}}{\mathrm{d}t} = \exp[tA]A$. □

Theorem 5.24. *Suppose that $A \in \mathbb{R}^{n\times n}$. Then the solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax, \quad x(t_0) = x_0 \in \mathbb{R}^{n\times 1}$$

is given by

$$x(t) = e^{A(t-t_0)}x_0.$$

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A(t-t_0)}x_0\right) = Ae^{A(t-t_0)}x_0 = Ax(t).$$

At $t = t_0$ we have

$$x(t_0) = e^{A\cdot 0}x_0 = Ix_0 = x_0.$$

By the existence and uniqueness theorem, this is the unique solution. □

Remark 5.25.

(i) e^{At} is called the **Fundamental matrix** of the ODE system.

(ii) If $x_0 = v$, $Av = \lambda v$, then $x(t) = e^{At}v = e^{\lambda t}v$. In this connection we see that the eigenvalue-eigenvector method is a special case of the matrix exponential method. More generally, see next point:

(iii) Suppose now A has n linearly independent eigenvectors $v_1, \dots, v_n \in \mathbb{C}^n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ (where note that we allow $\mathbb{C}$). Then by writing $P = (v_1, \dots, v_n)$, we have $AP = P\Lambda$, and so A is diagonalizable with $A = P\Lambda P^{-1}$. In particular,

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\mathrm{diag}\left(e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right)P^{-1}.$$

Thus

$$e^{At}Pe_i = e^{\lambda_i t}v_i.$$

☕

Theorem 5.26 (Jordan Normal Form)**.** *Suppose that $A \in \mathbb{C}^{n\times n}$. Then there exists $U, J \in \mathbb{C}^{n\times n}$ such that*

$$A = UJU^{-1},$$

where J is given by

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}, \quad J_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} \in \mathbb{C}^{m_j\times m_j}, \quad \sum m_j = n.$$

The columns of U are called generalized eigenvectors *and satisfy*

$$(A - \lambda I)^k u_i = 0, \quad k \le n.$$

Given a Jordan normal form decomposition $A = UJU^{-1}$, we have

$$e^{At} = U \exp[Jt] U^{-1},$$

Here,

$$J^k = \begin{pmatrix} J_1^k & & \\ & \ddots & \\ & & J_m^k \end{pmatrix}, \quad J_j^k = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}^k .$$

If $J_i = \alpha_i I + N$, where $N \in \mathbb{R}^{l \times l}$ is the nilpotent matrix with 1 on the superdiagonal and 0 elsewhere, then

$$e^{J_i t} = e^{\alpha_i t} e^{Nt} = \exp[\alpha_i t I] \exp(Nt),$$

since $IN = NI$. We have $\exp(\alpha_i t I) = e^{\alpha_i t} I$ and N^k is the matrix with 1 on the k^{th} superdiagonal and 0 elsewhere (Ex.). Thus

$$\exp[Nt] = I + Nt + \frac{(Nt)^2}{2!} + \cdots + \frac{(Nt)^{l-1}}{(l-1)!} \qquad = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{l-1}}{(l-1)!} \\ 0 & 1 & t & \dots & \frac{t^{l-2}}{(l-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Example 5.27.

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} x$$

has solution

$$e^{At} = e^{\lambda t} e^{Nt} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

5.5 Nonhomogeneous ODE Systems

Consider the nonhomogeneous ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t} x = A(t)x + G(t), \quad x(t_0) = x_0.$$

Recall that in the $n = 1$ case ($y'' + py' + qy = g(t)$), we first find solutions y_1 and y_2 to the corresponding homogeneous ODE, and then use the ansatz $y = \mu_1 y_1 + \mu_2 y_2$ with some clever restrictions on μ_1, μ_2 (variation of parameters/constants).

To generalize to nonhomogeneous ODE systems, we first recall the case $n = 1$

$$\frac{\mathrm{d}}{\mathrm{d}t} x = ax + g(t)$$

with a constant. Using the integrating factor e^{-at}, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-at}x\right) = e^{-at}g(t)$$

and so

$$e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^{t} e^{-as}g(s)\,\mathrm{d}s,$$

giving

$$x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^{t} e^{a(t-s)}g(s)\,\mathrm{d}s$$

or equivalently

$$x(t) = e^{a(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-a(s-t_0)}g(s)\,\mathrm{d}s\right].$$

Note that the first term $e^{a(t-t_0)}x_0$ solves the homogeneous ODE with the given IC, and for fixed s, the term $e^{a(t-s)}g(s)$ solves the homogeneous ODE with IC $g(s)$ at $t = s$. We may think of them as solutions to the following two ODEs:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_1 = ax_1, & x_1(t_0) = x_0 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 = g(t), & x_2(t_0) = 0 \end{cases}$$

Theorem 5.28. *Fur $n \geq 2$, the solution to $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + G(t)$, $x(t_0) = x_0$ is given by*

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}G(s)\,\mathrm{d}s \\ &= e^{A(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-A(s-t_0)}G(s)\,\mathrm{d}s\right]. \end{aligned}$$

Proof. Define

$$c(t) \coloneqq x_0 + \int_{t_0}^{t} e^{-A(s-t_0)}G(s)\,\mathrm{d}s.$$

Then the proposed solution is $x(t) = e^{A(t-t_0)}c(t)$. In like of existence and uniqueness, we just need to verify that this solves the ODE with the IC. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A(t-t_0)}c(t)\right] = Ae^{A(t-t_0)}c(t) + e^{A(t-t_0)}\dot{c}(t).$$

Now, using the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t_0}^{t} b(s)\,\mathrm{d}s\right) = b(t)$, we have

$$\dot{c}(t) = e^{-A(s-t_0)}G(t)\Big|_{s=t} = e^{-A(t-t_0)}G(t),$$

Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A(t-t_0)}c(t)\right] &= Ae^{A(t-t_0)}c(t) + e^{A(t-t_0)}\dot{c}(t) \\ &= Ax + G(t). \end{aligned}$$

□

Remark 5.29. From this result we have **Duhamel's formula**: The differential equation

$$\partial_t f = Lf + g(t, x), \qquad f|_{t=0} = f_0$$

where L is a t-independent linear operator (e.g., ∂_x), has solution

$$f = e^{Lt} f_0 + \int_0^t e^{L(t-s)} g(s, x) \, \mathrm{d}s.$$

☕

6 The Theory of Existence and Uniqueness

We focus on the case $n = 1$, but the proof generalizes to $n \geq 2$ easily.

Theorem 6.1 (Existence and Uniqueness)**.** *Consider the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} y = f(t, y(t)), \qquad y(t_0) = y_0$$

in the region $R : |t - t_0| \leq a$, $|y - y_0| \leq b$.

We assume that f is continuous in R and Lipschitz in y with Lipschitz constant L. Then, there exists $h > 0$ such that the ODE admits a unique C^1 solution for $|t - t_0| \leq h$.

Proof. The idea is to construct a sequence $\varphi_0, \varphi_1, \ldots$ so that $\varphi_n \to \varphi$. One idea is to define $\varphi_0 = y_0$, $\varphi_1(t) = y_0 + f(t_0, y_0)(t - t_0)$, and so on. But this requires f to be differentiable.

Alternatively, we may integrate:

$$\varphi(t) \coloneqq t_0 + \int_{t_0}^{t} f(s, \varphi(s)) \, \mathrm{d}s.$$

Note that differencing both sides gives $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = f(t, \varphi(t))$. Again we set $\varphi_0 = y_0$. For $n \geq 0$, we define

$$\varphi_{n+1}(t) \coloneqq y_0 + \int_{t_0}^{t} f(s, \varphi_n(s)) \, \mathrm{d}s$$

and show that φ_n converges. This method is called **Picaro iteration**.[1]

We will show first that $|\varphi_n(t) - y_0| \leq b$ in a small neighborhood of t_0. From f being continuous, we know that $|f| \leq M$ in R. Thus

$$|\varphi_{n+1}(t) - y_0| \leq \int_{t_0}^{t} |f(s, \varphi_n(s))| \, \mathrm{d}s \leq M|t - t_0|.$$

And so by setting $h \leq b/M$, we have $|\varphi_n(t) - y_0| \leq b$ for all n and $|t - t_0| \leq h$.

Next, we show that $|\varphi_{n+1} - \varphi_n| \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} |\varphi_{n+1}(t) - \varphi_n(t)| &= \left| \int_{t_0}^{t} f(s, \varphi_n(s)) - f(s, \varphi_{n-1}(s)) \, \mathrm{d}s \right| \\ &\leq L \int_{t_0}^{t} |\varphi_n(s) - \varphi_{n-1}(s)| \, \mathrm{d}s. \end{aligned}$$

Note recall that we have the uniform in time bound

$$|\varphi_1 - \varphi_0| = |\varphi_1 - y_0| \leq Mh.$$

Iterating the inequality above gives

$$|\varphi_{n+1}(t) - \varphi_n(t)| \leq (Lh)^n Mh$$

[1]Another way is to use a fixed point theorem on a suitable function space.

which converges if we choose h such that $Lh < 1$. We define thus

$$\varphi := \lim_{n\to\infty} ved\varphi_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} (\varphi_{i+1} - \varphi_i) + \varphi_0.$$

Note that the series on the right converges uniformly in light of the bound above. In particular,

$$|\varphi - \varphi_n| \le \sum_{i\ge n} |\varphi_{i+1} - \varphi_i| \le Mh\frac{(Lh)^n}{1 - Lh}.$$

To see φ solves the ODE, note that from (exercise)

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = f(t, \varphi(t)) \implies \varphi \in C^1.$$

It remains to show uniqueness. Suppose that ψ is another solution to the ODE with the same IC. Note that

$$\varphi(t) = t_0 + \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s, \quad \psi(t) = t_0 + \int_{t_0}^{t} f(s, \psi(s))\, \mathrm{d}s.$$

Thus

$$d(t) := \varphi(t) - \psi(t) = \int_{t_0}^{t} \left[f(s, \varphi(s)) - f(s, \psi(s))\right]\, \mathrm{d}s.$$

The Lipschitz condition gives

$$|d(t)| \le L\int_{t_0}^{t} |d(s)|\, \mathrm{d}s \implies |d|'(t) \le LD'(t), \quad \text{where } D(t) := \int_{t_0}^{t} |d(s)|\, \mathrm{d}s.$$

Using the integrating factor e^{-Lt} we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-Lt}D(t)\right) \le 0, \quad D(t) \ge 0$$

which gives

$$e^{-Lt}D(t) \le e^{-Lt_0}D(t_0) = 0$$

and in turn

$$D(t) = 0, d = 0, \varphi = \psi.$$

□

Remark 6.2. The assumption $f \in C^0$ already gives existence (Lipschitz is not required). The Lipschitz assumption gives uniqueness (no continuity required). ☕

7 Quantitative Estimates

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(t,x), \quad x(t_0) = x_0.$$

Recall that

- $f \in C^0 \implies$ Existence
- f is Lipschitz in $x \implies$ Uniqueness

Recall that with

$$|\varphi(t) - \psi(t)| \leq L \int_{t_0}^{t} |\varphi(s) - \psi(s)| \, \mathrm{d}s,$$

we have

$$d(t) := |\varphi(t) - \psi(t)| \leq L \int_{t_0}^{t} d(s) \, \mathrm{d}s.$$

We generalize this result:

Lemma 7.1 (Gronwall's Inequality)**.** *For given α, β, assume $\beta(t) \geq 0$, $\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)\,\mathrm{d}s$ and define*

$$B(s) := \exp\left(\int_0^s \beta(t)\,\mathrm{d}t\right).$$

Then,

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\frac{B(t)}{B(s)}\,\mathrm{d}s.$$

If $\alpha \in C^1$, the right hand side may be written by integration by parts as

$$\alpha(0)B(t) + \int_0^t \alpha'(s)\frac{B(t)}{B(s)}\,\mathrm{d}s.$$

Proof. We write

$$A(t) := \int_0^t \beta(s)\psi(s)\,\mathrm{d}s$$

and seek an ODE inequality for A. The idea is to solve an ODE inequality for A of the form $\frac{\mathrm{d}}{\mathrm{d}t}[\text{unknown}] \leq \text{known}$ (note that this works only if the RHS is known). Note that we have $A'(t) = \beta(t)\psi(t)$. Using the assumptions that $\psi(t) \leq \alpha(t) + A(t)$ and $\beta \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \psi\beta \leq \alpha\beta + A\beta.$$

This is a first order linear "ODE inequality" in A. Using the integrating factor $1/B(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A(t)}{B(t)}\right) \leq \alpha(t)\beta(t)\frac{1}{B(t)},$$

where we note that

$$B^{-1} = \exp\left(-\int_0^t \beta \, \mathrm{d}t\right), \quad \dot{B}^{-1} = -\beta B^{-1}.$$

Integrating both sides from 0 to t gives

$$A(t)B^{-1}(t) \leq \int_0^t \alpha(s)\beta(s)B^{-1}(s) \, \mathrm{d}s + A(0)B^{-1}(0),$$

where the last term is zero since $A(0) = 0$ by definition. Now we have

$$\psi(t) \leq \alpha(t) + A(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\frac{B(t)}{B(s)} \, \mathrm{d}s.$$

□

Remark 7.2. Applications:

- $\alpha = 0$, $\beta = L$, $\psi = d$ and we recover the motivating example.
- $\alpha = 0$ implies $\psi \leq 0$.
- $\alpha(t) = \alpha_0 + \alpha_1 t$, $\beta = L$. Then

  $$B(s) = \exp\left(\int_0^s L \, \mathrm{d}s\right) = e^{Ls}$$

  and so

  $$\psi(t) \leq \alpha_0 e^{Lt} + \alpha_1 \int_0^t e^{L(t-s)} \, \mathrm{d}s.$$

  In some sense this is similar to the Duhamel's formula for the ODE system

  $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x = Lx + \alpha_1 \\ x(0) = \alpha_0 \end{cases} \implies x(t) = e^{Lt}\alpha_0 + \alpha_1 \int_0^t e^{L(t-s)} \, \mathrm{d}s.$$

☕

Consider now a small perturbation in x_0 or f in the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(t, x), \quad x(t_0) = x_0$$

so that it becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}y = g(t, y), \quad y(t_0) = y_0.$$

How do these errors affect x? The hope is that x does not change too much. Otherwise the model is not robust.

Theorem 7.3 (Continuous Dependence in IC, parameters, etc.)**.** *Suppose f is Lipschitz in the domain D of interest with Lipschitz constant L and denote*

$$M := \max_{(t,x)\in D} |f(t,x) - g(t,x)|.$$

Then,

$$|x(t) - y(t)| \leq e^{L|t-t_0|}|x_0 - y_0| + \frac{M}{L}\left(e^{L|t-t_0|} - 1\right),$$

where the first part is due to the IC error and the second part is due to the ODE error.

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - y(t)] = f(t, x(t)) - g(t, y(t)).$$

Integrating,

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_{t_0}^{t} |f(s, x(s)) - g(s, y(s))| \,\mathrm{d}s.$$

The integrand may be bounded as

$$\begin{aligned} |f(s,x(s)) - g(s,y(s))| &\leq |f(s,x(s)) - f(s,y(s))| + |f(s,y(s)) - g(s,y(s))| \\ &\leq L|x(s) - y(s)| + M. \end{aligned}$$

Observe that

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_{t_0}^{t} \left[L|x(s) - y(s)| + M\right] \,\mathrm{d}s.$$

Setting $\alpha := |x_0 - y_0| + M(t - t_0)$ and $\psi(t) := |x(t) - y(t)|$, we may apply Gronwall's inequality with $\beta = L$ to get

$$\begin{aligned} |x(t) - y(t)| &\leq e^{L(t-t_0)}|x_0 - y_0| + \int_{t_0}^{t} MLe^{L(t-s)} \,\mathrm{d}s \\ &\leq e^{L|t-t_0|}|x_0 - y_0| + \frac{M}{L}\left(e^{L|t-t_0|} - 1\right). \end{aligned}$$

□

Corollary 7.4.

- *If $f = g$, $x_0 = y_0$, then $x = y$.*
- *If $f = g$ then*

  $$|x(t) - y(t)| \leq e^{L|t-t_0|}|x_0 - y_0|.$$

  That is, we have continuous (and in fact Lipschitz) dependence on the IC. In other words, if we define $\Phi(t, x_0)$ to be the solution of

  $$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = f(t, \Phi), \quad \Phi(t_0, x_0) = x_0,$$

  then Φ is Lipschitz in x_0.

- *If* $x_0 = y_0$ *and say* $|f - g| \le \varepsilon$, *then*
$$|x(t) - y(t)| \le \frac{\varepsilon}{L}\left(e^{L|t-t_0|} - 1\right).$$
In particular, if $f_\alpha(t, x) \coloneqq \alpha f_0 + (1-\alpha) f_1$, *then*
$$|f_\alpha - f_\beta| \le |\alpha - \beta| \left(|f_1| + |f_0|\right).$$

Example 7.5. We perturb the ODE
$$\frac{\mathrm{d}}{\mathrm{d}t} x = \frac{x^2}{1 + t^2 + x^2}, \quad x(0) = 0$$
to get
$$\frac{\mathrm{d}}{\mathrm{d}t} y = \frac{y^2 + \varepsilon}{1 + t^2 + y^2} + \varepsilon, \quad y(0) = \varepsilon.$$
We have $|f - g| \le \varepsilon$ and $|x_0 - y_0| \le \varepsilon$, and we may check that $|\partial_x f(t, x)| \le L$ for any t, x. Thus
$$|x(t) - y(t)| \le e^{Lt}\varepsilon + \frac{\varepsilon}{L}\left(e^{Lt} - 1\right) = \varepsilon\left(e^{Lt} + \frac{e^{Lt} - 1}{L}\right).$$

Now consider
$$y' \le F(t, y(t)), \quad t \in [a, b].$$

Theorem 7.6 (Comparison)**.** *Suppose F is Lipschitz. Let* $f, g \in C^1$ *are such that*
$$f' \le F(t, f(t)), \quad g' = F(t, g(t)).$$
If $f(a) \le g(a)$, *then*
$$f(t) \le g(t), \quad t \in [a, b].$$

Proof. Suppose for contradiction that $f(t_1) > g(t_1)$. Let $\Omega \coloneqq \{t : f(t) \le f(t)\}$. We have $a \in \Omega$. Let $t_0 \coloneqq \max\{t \in [a, t_0] \cap \Omega\}$. We know $f(t_0) = g(t_0)$ and $f(t) \le g(t)$ for each $t \in \Omega$. Moreover, $f(t) > g(t)$ for each $t \in (t_0, t_1]$. But this and continuity implies that $f(t_0) \ge g(t_0)$. Thus $f(t_0) = g(t_0)$.

Now, from assumption we have
$$f'(t) - g'(t) \le F(t, f(t)) - F(t, g(t))$$
By integrating both sides,
$$\begin{aligned} f(t) - g(t) &\le f(t_0) - g(t_0) + \int_{t_0}^{t} F(s, f(s)) - F(s, g(s))\, \mathrm{d}s \\ &\le \int_{t_0}^{t} L(f(s) - g(s))\, \mathrm{d}s, \end{aligned}$$
where the second inequality is justified from $f(t) \ge g(t)$ on $(t_0, t_1]$. By Gronwall's inequality, we have
$$f(t) - g(t) \le 0, \quad t \in [t_0, t_1]$$
a contradiction. □

Now, if F is not Lipschitz, this no longer holds true:

Example 7.7.

$$\frac{\mathrm{d}}{\mathrm{d}t} y = y^{\frac{1}{3}}, \quad y(0) = 0$$

has solution $g(t) \equiv 0$. Now try the ansatz $f(t) = ct^{\alpha}$. We have

$$f'(t) = c\alpha t^{\alpha-1}, \quad f^{\frac{1}{3}}(t) = c^{\frac{1}{3}} t^{\frac{\alpha}{3}},$$

giving

$$\alpha = \frac{3}{2}, \quad c = \pm \left(\frac{3}{2}\right)^{\frac{3}{2}}.$$

In particular, a solution is

$$f(t) = \begin{cases} ct^{\frac{3}{2}}, & t \geq 0 \\ -c(-t)^{\frac{3}{2}}, & t < 0 \end{cases},$$

which crosses g at $t = 0$ and is C^1 (since in particular both parts of f is C^1 at 0).

Remark 7.8. Another version of the comparison theorem is as follows: $f' < F(t, f)$ with strict inequality and $f(a) < g(a)$, f is continuous (Lipschitz not required??).

Theorem 7.9.

$$\frac{\mathrm{d}}{\mathrm{d}t} x = f(t, x), \quad \frac{\mathrm{d}}{\mathrm{d}t} y = g(t, y).$$

If $f(t, z) \leq g(t, z)$ for any t, z in the domain of interest D, and if f or g is Lipschitz in D, and that $x(a) \leq y(a)$, then $x(t) \leq y(t)$ for each $t \in [a, b]$.

Proof. Suppose g is Lipschitz (the other case is similar). We have

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(t, x) \leq g(t, x(t)), \quad \frac{\mathrm{d}}{\mathrm{d}t} y(t) = g(t, y).$$

By the comparison theorem, we have $x(t) \leq y(t)$ for each $t \in [a, b]$. □

Example 7.10. Consider

$$\frac{\mathrm{d}}{\mathrm{d}y} x_{\alpha}(t) = f(t, x_{\alpha}(t)) + \alpha, \quad x_{\alpha}(t_0) = x_0.$$

Then the solution satisfies

$$x_{\alpha}(t) \leq x_{\beta}(t), \quad t \in [a, b], \quad \text{if } \alpha \leq \beta.$$

Example 7.11.

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = P_n(x, t), \quad t \in [0, 1], \quad x(0) = x_0,$$

where P_n is a polynomial such that $P_n(x, t) \leq Ce^x$ for some constant $C > 0$. We know x is bounded above by the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} y = Ce^y, \quad y(0) = x_0.$$

Of course, this works for any upper bounding function.

7.1 Extension

In this section we will always assume that $f \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz in the second argument in any bounded region.

Example 7.12 (of Blowup).

$$\frac{\mathrm{d}}{\mathrm{d}t} y = y^2, \quad y(0) = 1.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} y^{-1} = -1 \implies y^{-1}(t) = 1 - t \implies y = \frac{1}{1-t}.$$

But observe that the solution exists up to 1^-: $\lim_{t \to 1^-} y(t) = +\infty$.

How can we detect/characterize blowup?

Lemma 7.13 (Gluing)**.** *Suppose $f \in C(\mathbb{R}, \mathbb{R})$ and $f(t,x)$ is Lipschitz in any bounded region:*

$$|f(t,x_1) - f(t,x_2)| \le L_D |x_1 - x_2|, \quad (t,x_1),(t,x_2) \in D \subset \mathbb{R}^2.$$

Suppose $\varphi_i(t)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t} x = f(t,x), \quad x(t_0) = x_0$$

on $t \in I_i := (a_i, b_i)$. If $t_0 \in I_1 \cap I_2 = (a,b)$ and $\varphi_1(t_0) = \varphi_2(t_0)$. Then $\varphi_1(t) = \varphi_2(t), t \in I_1 \cap I_2$ and

$$\varphi(t) := \begin{cases} \varphi_1(t), & t \in I_1 \\ \varphi_2(t), & t \in I_2 \end{cases}$$

is a C^1 solution to the ODE on $I_1 \cup I_2$.

Proof. The hard part is to show $\varphi_1 = \varphi_2$ on $I_1 \cap I_2$.

Apply existence and uniqueness on to the ODE with IC at t_0 to get $\varphi_1 = \varphi_2$ on $t_0 \pm h$. Write

$$J_- := \{p : \varphi_1(t) = \varphi_2(t), t \in [p, t_0 + h] \cap (a,b)\}.$$

Then we can prove that $a = \inf J_- =: A$. Suppose not, then $A > a$. There exists a sequence $P_n \to A$, each $P_n \in J$, so that $\varphi_1 = \varphi_2$ on each $[P_n, t_0 + h]$. By continuity, $\varphi_1(P_n) = \varphi_2(P_n)$. Again by continuity, $\varphi_1(A) = \varphi_2(A)$. But by existence and uniqueness, $\varphi_1 = \varphi_2$ on $[A - h', t_0 + h]$ for some $h' > 0$, contradicting the definition of A.

Alternatively:

Set $J := \{t \in I_1 \cap I_2 : \varphi_1(t) = \varphi_2(t)\}$. The goal is to show $J = I_1 \cap I_2, J \neq \varnothing$. By continuity, J is relatively closed. But since it is also relatively open (by local existence and uniqueness), we have $J = I_1 \cap I_2$.

Yet another alternative: Using Gronwall's inequality, from

$$|\varphi_1(t) - \varphi_2(t)| \le \int_{t_0}^{t} L |\varphi_1(s) - \varphi_2(s)| \, \mathrm{d}s,$$

we have $\varphi_1 = \varphi_2$ on $I_1 \cap I_2$.

We now know that φ is well-defined. It is C^1 since each φ_i is C^1 on I_i, and it solves the ODE since each φ_i does so on I_i. $\square$

What happens if f is not Lipschitz? Well then φ_1 may not be identically equal to φ_2 on $I_1 \cap I_2$. Then there are multiple ways to choose φ, and the second statement is ambiguous.

Theorem 7.14 (Blowup Criterion)**.** *Suppose $f \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz in x in any bounded region. Suppose that φ is a C^1 solution to $\frac{\mathrm{d}}{\mathrm{d}t}x = f(t,x)$ on (t_-, t_+). The solution can be extended to $(t_-, t_+ + \varepsilon)$ for some $\varepsilon > 0$ if and only if one of the following equialent conditions hold:*

(i) There exists a sequence $\{t_n\}$ such that $t_n \longrightarrow (t_+)^-$ and $\varphi(t_n) \longrightarrow \varphi_0 \neq \pm\infty$. Note that this is true for oscillating behavior but not monotonic blowup.

(ii) $\liminf_{t \to (t_+)^-} |\varphi(t)| < +\infty$.

In particular, the solution cannot be extended beyond t_+ if $\lim_{t \to (t_+)^-} |\varphi(t)| = +\infty$.

Proof. Note that the forward direction is implied by the gluing lemma. For the converse, suppose $|\varphi(t_n) - \varphi_0| \le 1$ for each $n \ge N$. The idea is to apply existence and uniqueness at $(t_n, \varphi(t_n))$ and hope that we get a solution on $(t_n - h, t_n + h)$ with $t_n + h > t_+$.

Write

$$S := [t_+ - 1, t_+ + 1] \times [\varphi_0 - 1, \varphi_0 + 1]$$
$$D := [t_+ - 2, t_+ + 2] \times [\varphi_0 - 2, \varphi_0 + 2]$$

We will control $(t_n, \varphi(t_n))$ to be in S, and φ to be in D. By f being continuous and Lipschitz in x in D, we have

$$\begin{cases} |f| \le M \\ |f(t,x) - f(t,y)| \le L|x - y| \end{cases}$$

in D for some $L, M > 0$. In particular, $|\varphi'| \le M$ in D. Applying existence and uniqueness at $(t_n, \varphi(t_n))$ gives a unique solution ψ such that $\psi(t_n) = \varphi(t_n)$ and with domain $t_n - h, t_n + h$, where h can be chosen as

$$0 < h < \min\left\{1, \frac{1}{M}, \frac{1}{L}\right\}.$$

Since h is independent of n, we may choose n large enough so that $t_n + h > t_+$. $\square$

Q: which ODEs admit global solutions?

Proposition 7.15. *Consider $x' = f(t,x)$ with $|f| \le M$ for each (t,x), $f \in C(\mathbb{R}, \mathbb{R})$ and Lipschitz in x in any bounded region. Then the solution exists globally and is unique.*

Proof. Assume the solution exists on (t_-, t_+). Note that

$$|x(t) - x(t_0)| = \left| \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s \right| \le M|t - t_0| < \infty.$$

By the blowup criterion, we know that x can be extended beyond t_+ and t_-. □

Remark 7.16. The idea of existence-uniqueness / blowup criterion applies also to some time evolution PDEs e.g., $\partial_t u = N(u, \partial u, \ldots)$. We similarly have local existence-uniqueness, and we cannot extend beyond t_+ if and only if $\lim_{t \to (t_+)^-} \|u(t, x)\|_Y = \infty$.

7.2 Finite Time Blowup

Consider the Riccatti ODE $y' = y^p$, $y(0) = 1$.

- $p = 1$. The unique solution is $y(t) = e^t$.
- $p > 1$. By the comparison principle we have $y \ge e^t$ since $y(0) \ge 1$. By separation of variables, we have

  $$\frac{y'}{y^p} = 1 \implies y^{1-p}(t) = 1 + (1-p)t.$$

  Note that

  $$y^{p-1}(t) = \frac{1}{1 - (p-1)t} \to +\infty \text{ as } t \to \left(\frac{1}{p-1}\right)^- .$$

  So we have blow up at $t_+ = 1/(p-1) < \infty$.

Example 7.17. Consider $y' = e^y \ge y^2$. By comparison with $y' = y^2$, we have finite time blowup. Consider $y' = y^n + \cdots + 1 + e^t$, $y(0) = 1$. By comparison with $y' = y^2$, we have finite time blowup.

Example 7.18. $y' = y^2$, $y(0) = y_0$.

- If $y_0 > 0$, we have finite time blowup.
- If $y_0 = 0$, we have $y \equiv 0$.
- If $y_0 < 0$, we have $y = y_0/(1 - ty_0)$. Note that $1 - ty_0 > 0$ for $t > 0$. Thus $y_0 \sim -1/t$ as $t \to +\infty$.

From this we see nonlinear stabilization: negative IC gives global solution decaying to 0, while positive IC gives finite time blowup.

Example 7.19. Consider $y' = y^2 + \varepsilon$ for $\varepsilon > 0$ and $y(0) = y_0$. Note that $y(t) \ge y_0 + \varepsilon t$ and so $y(t_0) > 0$ for t_0 large. Then, we have $\dot{y} > y^2$ and thus blowup.

Example 7.20. Consider $y' = y^2 - \varepsilon^2$. We have $y_1 \equiv \varepsilon$ and $y_2 \equiv -\varepsilon$. For $y_0 > \varepsilon$ we again have $y' \ge y^2$ and so blowup.

8 Autonomous ODE, Stability, and Phase Portrait/Plane

Recall that a general first order ODE is $x' = F(t, x)$, where $x, F \in \mathbb{R}^{n\times 1}$. An **autonomous** ODE is one where F does not depend on t:

$$x' = F(x).$$

A special solution is the constant solution$x(t) \equiv x_*$. We say x_* is in **equilibrium** or a **critical point** if $F(x_*) = 0$.

Example 8.1. $x' = x^2 - x + t$ has no constant solution.

We call $(t, x(t))$ the **trajectory** of the solution $x(t)$. For $n = 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix},$$

we call the x_1-x_2 plane the **phase plane**. We may think of x_2 as a function of x_1, $x_2 = x_2(x_1)$, to remove the dependency on t: By the chain rule,

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{\mathrm{d}x_2/\mathrm{d}t}{\mathrm{d}x_1/\mathrm{d}t} = \frac{f_1}{f_2}(x_1, x_2).$$

The representative set of trajectories is called the **phase portrait**.

Example 8.2. The ODE $x' = x^2 - x =: f$ has critical points at $x_* = 0, 1$.

Example 8.3. The system

$$\begin{cases} x' = x^2 - 2y \\ y' = x + y \end{cases}.$$

Note that $x^2 = 2y = -2x$ gives $x = 0$ or $x = -2$. This gives the critical points $(0, 0)$ and $(-2, 2)$.

8.1 Phase Portrait of 2 x 2 Systems

Consider the linear 2×2 system:

$$x' = Ax, \quad A \in \mathbb{R}^{2\times 2}.$$

The equilibrium is at $x_* = 0$ and the real eigen-vectors associated with the eigenvalue 0. But what happens as $t \to \infty$ for a general solution x?

Remark 8.4. Recall first that for

$$z(t) := e^{\alpha t}(\cos \beta t + \mathrm{i} \sin \beta t),$$

we have

(i) If $\alpha > 0$, then $|z(t)| \to \infty$ as $t \to \infty$.

(ii) If $\alpha = 0$, then $z(t)$ is bounded, periodic, and in particular does not converge.

(iii) If $\alpha < 0$, then $z(t) \to 0$ exponentially fast as $t \to \infty$.

We consider the following three cases:

Case 1:

$\lambda_1 < \lambda_2$ real. The general solution is

$$x(t) = \underbrace{e^{\lambda_t}}_{\text{amplitude}} \underbrace{\left(e^{(\lambda_1-\lambda_2)t}c_1v_1 + c_2v_2\right)}_{\text{direction}}.$$

(i) If $\lambda_1 < 0$, then any solution of the form

$$x(t) = c_1 e^{\lambda_1 t} v_1$$

will converge to 0 as $t \to \infty$, remaining on the line spanned by v_1 on the way.

(ii) (**Sink**) $\lambda_1 < \lambda_2 < 0$, then $x(t) \approx e^{\lambda_2 t} c_2 v_2$ for large t. Thus $x(t) \to 0$ as $t \to \infty$ (converging in the v_1 direction faster).

(iii) (**Saddle**) $\lambda_1 < 0 < \lambda_2$. We have $x \to 0$ only if $x \equiv 0$.

(iv) (**Source**) $0 < \lambda_1 < \lambda_2$, then $x(t) \to \infty$ (diverging in the v_2 direction faster) as $t \to \infty$ unless $x \equiv 0$.

These can be visualized in a phase portrait.

Case 2:

$\lambda = \alpha \pm \mathrm{i}\beta$, $v = a \pm \mathrm{i}b$, $b \neq 0$. The general solution is

$$\begin{aligned} x(t) &= c_1 \operatorname{Re}(e^{\lambda t} v) + c_2 \operatorname{Im}(e^{\lambda t} v) \\ &= e^{\alpha t}\left[c_1 (\cos\beta t a - \sin\beta t b) + c_2 (\sin\beta t a + \cos\beta t b)\right] \\ &= e^{\alpha t} V(t), \end{aligned}$$

where

$$V(t) \coloneqq c_1 (\cos\beta t a - \sin\beta t b) + c_2 (\sin\beta t a + \cos\beta t b)$$

is periodic in t.

(i) If $\alpha < 0$, $|x| \to 0$.

(ii) If $\alpha = 0$, then $x(t) = V(t)$ is bounded and periodic (but does not converge).

(iii) If $\alpha > 0$, then $|x| \to \infty$.

Case 3:

One real eigen-value with two linearly independent eigen-vectors. General solution

$$x(t) = e^{\lambda t} \underbrace{(c_1 v_1 + c_2(t v_1 + v_2))}_{\text{constant}}.$$

The trajectory is along a straight line.

Case 4:

One eigenvector v_1 and one generalized eigen-vector v_2 such that $(A - \lambda)^2 v_2 = 0$. General solution:

$$\begin{aligned} x(t) &= e^{\lambda t}\Big(c_1 v_1 + c_2 t v_2\Big) \\ &= e^{\lambda t} t \left(\frac{1}{t} c_1 v_1 + c_2 v_2\right). \end{aligned}$$

For large t, the trajectory is approximately along the line spanned by v_2.

(i) If $\lambda > 0$, $|x| \to \infty$ exponentially fast.

(ii) If $\lambda = 0$, $|x| \sim t$.

(iii) $\lambda < 0$, $|x| \to 0$ exponentially fast.

Q: How smooth is x_2 as a function of x_1? Obviously Case 3 is C^∞.
Nonlinear phase plane/portrait is much harder:

Example 8.5.

$$\begin{cases} x_1' = f_1(x_1, x_2) \\ x_2' = f_2(x_1, x_2) \end{cases}.$$

Suppose for example that f_i are polynomials. We may have a few critical points, say $(0, 0)$ and x_c. A hard problem: find a C^∞ trajectory starting from 0 and passing x_c.

Note that by existence and uniqueness, trajectories cannot cross (except at critical points).

8.2 Stability

We say the critical point x_* is

- **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - x_*| < \delta$, $|x(t) - x_*| < \varepsilon$ for each $t \geq t_0$.
- **asymptotically stable** if there exists $\delta > 0$ such that if $|x_0 - x_*| < \delta$, then $\lim_{t\to\infty} x(t) = x_*$.
- **unstable** if it is not stable.

We summarize the stability of the critical point $(0, 0)$ for the 2×2 linear systems:

(i) If $\operatorname{Re}\lambda_i < 0$ for $i = 1, 2$, then both stable and asymptotically stable (sink).

(ii) If $\operatorname{Re}\lambda_i > 0$ for some i, then unstable.

(iii) If $\operatorname{Re}\lambda_i \leq 0$ for all i and $\operatorname{Re}\lambda_i = 0$ for some i, then not asmptotically stable. Moreover,

    (i) if $\lambda_1 \neq \lambda_2$, then stable ($|e^{\lambda_1} t| = |e^{\lambda_2 t}| \leq 1$).

(ii) if $0 = \lambda_1 = \lambda_2$, 2 eigenvectors, stable.

(iii) $0 = \lambda_1 = \lambda_2$, where λ_2 is the generalized eigen-vector, then unstable (since $|x(t)| \sim t$).

Example 8.6. Consider the pendulum equation

$$\theta'' + \gamma\theta' + w^2 \sin\theta = 0.$$

It has two critical points: $(0, 0)$ and $(\pi, 0)$. The latter corresponds to the inverted pendulum and is unstable.

Lemma 8.7. *For each $\varepsilon > 0$ and integer $n \geq 0$, there exists a constant $C > 0$ such that*

$$t^n \leq Ce^{\varepsilon t}, \quad t \geq 0.$$

Proof. Note that

$$t^n = e^{\frac{n}{\varepsilon}\cdot\varepsilon \log t}, \quad Ce^{\varepsilon t} = e^{\varepsilon C + \varepsilon t}.$$

The first exponent is concave. □

Theorem 8.8 (Linear Stability)**.** *Consider the ODE system $x' = Ax$, $A \in \mathbb{R}^{n\times n}$, with $x_* = 0$ as a critical point. Let λ_i be the eigen-values of A.*

(i) If $\operatorname{Re}\lambda_i < 0$ for each i, then x_* is stable and asymptotically stable.

(ii) If $\operatorname{Re}(\lambda_i) > 0$ for some i, then x_* is unstable.

(iii) If $\operatorname{Re}(\lambda_i) \geq 0$ for some i, then x_* is not asymptotically stable.

(iv) If $\operatorname{Re}(\lambda_i) \leq 0$ for each i and if the Jordan block J_i associated with λ_i, $\operatorname{Re}(\lambda_i) = 0$, is 1×1, then x_* is stable. If the block is larger than 1×1, then x_* is unstable.

Remark 8.9. To gain some intuition of why only the real parts matter, note that

$$|e^{(\alpha + \mathrm{i}\beta)t}| = |e^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t)| = e^{\alpha t}.$$

If $\alpha > 0$, $|x(t)| \to \infty$ and x_* is unstable. If $\alpha = 0$, $|x(t)| \equiv |x_0|$ and x_* is stable. If $\alpha < 0$, $|x(t)| \to 0$ and x_* is asymptotically stable.

Proof. Recall that the solution is $x(t) = e^{At}x_0$. Write $A = UJU^{-1}$ and note that

$$e^{At} = Ue^{Jt}U^{-1}, \quad e^{Jt} = \begin{pmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_k t} \end{pmatrix},$$

where J_j are the Jordan blocks of A.

We start from case (ii). Note that $x(t) = e^{\lambda_i}v_i$ solves the ODE (if λ_i is complex, take the real or complex parts of x). We have

$$|x(t)| = |cv_i|e^{\operatorname{Re}(\lambda_i)t} \longrightarrow \infty.$$

Thus x_* is unstable.

Next, consider case (iii). If $|\lambda_i| \geq 0$, we have with a small perturbation of size c that

$$|x(t)| = |cv_i| e^{\mathrm{Re}(\lambda_i)t} \geq |cv_i|,$$

which does not converge to 0.

Now consider (i). Recall that

$$e^{J_j t} = \begin{pmatrix} e^{\lambda_j t} & te^{\lambda_j t} & \frac{t^2}{2}e^{\lambda_j t} & \cdots & \frac{t^{m_j-1}}{(m_j-1)!}e^{\lambda_j t} \\ 0 & e^{\lambda_j t} & te^{\lambda_j t} & \cdots & \frac{t^{m_j-2}}{(m_j-2)!}e^{\lambda_j t} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_j t} \end{pmatrix}.$$

We claim (Ex.) that $|t^m| \leq e^{\varepsilon t} C_{m,\varepsilon}$ for each $t \geq 0$. (E.g., $t^{1000} \leq e^{0.1t} C$ for each $t \geq 0$.) We have the estimate

$$|e^{\lambda_i t} t^m| \leq |e^{\mathrm{Re}\,\lambda_i t} t^m| \leq C_A e^{-(2/3)\lambda t},$$

where we pick $\varepsilon = \lambda/3$. This gives

$$|(e^{J_j t})_{kl}| \leq C_A e^{-(2/3)\lambda t} \implies |(e^{Jt})_{ij}| \leq C_A e^{-(2/3)\lambda t}.$$

Now note that

$$|(e^{At})_{ij}| \leq \sum_{k,l} |U_{ik}||(e^{Jt})_{kl}||(U^{-1})_{lj}|.$$

We thus have

$$|(e^{At})_{ij}| \leq C_a n^2 C_a e^{-(2/3)\lambda t} \leq C_A e^{-(2/3)\lambda t}.$$

Then,

$$|e^{At} x_0| \leq C_A e^{-(2/3)\lambda t} |x_0| \to 0.$$

Finally, consider (iv). Recall that

$$e^{J_i t} = \begin{pmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!}e^{\lambda_i t} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_i t} \end{pmatrix}.$$

If $n_i \geq 2$ then x_* is unstable. Ex: find x_0 cleverly such that $e^{At}x_0 \to \infty$. If $n_i = 1$ for each n_i, then $|e^{J_i t}| \approx 1$. ◻

Example 8.10. $x' = \begin{pmatrix} -2 & \\ & -1 \end{pmatrix} x$ is asymptotically stable and stable. $x' = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} x$ has eigen-values ±i and is stable but not asymptotically stable.

8.3 Nonlinear Stability

Consider
$$x' = F(x), \quad F(x_*) = 0, \quad F \in C^2.$$
We linearize around x_*:
$$x' = F(x) = F(x_*) + \nabla F(x_*)(x - x_*) + G(x - x_*).$$
If $|x - x_*| < 1$, we have $G(x - x_*) \le M|x - x_*|^2$ for some $M > 0$. Write $z(t) = x(t) - x_*$ and note that
$$z'(t) = x'(t) = Az + G(z), \quad A := \nabla F(x_*),$$
where
$$|G(z)| \le M|z|^2, \quad \forall |z| < 1.$$
This system has the equilibrium $z_* = 0$. The hope is that for small z, $|G(z)| \le M|z|^2 \ll |z|$, and the solution of the nonlinear system behaves like that of the linear system $z' = Az$.

Theorem 8.11 (Nonlinear Stability)**.** *Let λ_i be the eigenvalues of $A := \nabla F(x_*)$, where $F \in C^2$.*

(i) If $\mathrm{Re}(\lambda_i) < 0$ for each i, then x_* is asymptotic stable and stable.

(ii) If for some i we have $\mathrm{Re}\,\lambda_i > 0$, then x_* is unstable.

In other cases (say $\mathrm{Re}\,\lambda_i = 0$) it is unclear (we need some information from G.)

Example 8.12. Consider $x' = 0 + x^2$. The IC $\varepsilon > 0$ results in finite time blowup, while IC $\varepsilon < 0$ results in global solution converging to 0.

Proof. Case (ii) is left as an exercise. Consider (i).

Idea: $z' = Az + o(1)Z$. Suppose $\delta = |z_0| \ll 1$ and note that
$$|z(t)| \approx |e^{At} z_0| \le C_A e^{-(2/3)\lambda t} |z_0|$$
and
$$|G(z(t))| \le M|z(t)|^2 \le M C_A^2 \delta^2 e^{-(4/3)\lambda t}.$$

Goal: If $|z_0| = \delta \ll 1$, then $d(t) := e^{\frac{3}{2}\lambda t} |z(t)| \le \tilde{C}_A \delta$.

Step 1: the solution exists for a short time. Step 2: By Duhamel formula,
$$z(t) = \overbrace{e^{At}}^{I} z_0 + \overbrace{\int_0^t e^{A(t-s)} \underbrace{G(z(s))}_{F(s)} \,\mathrm{d}s}^{II}.$$
We hope to derive an ODE inequality for $d(t)$. Denote as I and II the two terms on the RHS. We have
$$|I| \le C_A e^{-\frac{2}{3}\lambda t} |z_0|$$

and
$$|II| \le C_A \int e^{-\frac{2}{3}\lambda(t-s)} |G(z(s))| \le \tilde{C}_A \int e^{-\frac{2}{3}\lambda(t-s)} |z(s)|^2.$$
Since $|z(s)| = d(s) \cdot e^{-\frac{2}{3}\lambda t}$, we have
$$|II| \le \tilde{C}_A e^{-\frac{2}{3}\lambda t} \int_0^t \underbrace{e^{\frac{2}{3}\lambda s} e^{-\frac{4}{3}\lambda s} d(s)^2}_{|z(s)|^2} \, \mathrm{d}s.$$
This gives
$$d(t) = e^{\frac{2}{3}\lambda t} |z(t)| \le e^{\frac{2}{3}\lambda t} \left[|I| + |II| \right] \le C_A \delta + \tilde{C}_A \int_0^t e^{-\frac{2}{3}\lambda s} d(s)^2 \, \mathrm{d}s.$$

We now claim that $d(t) < 2C_A\delta$ for each $t \ge 0$ if $\delta \ll 1$. We induct on time (this is called a bootstrap argument). Base case $t = 0$: $d(0) = |z_0| = \delta$. Assume that the claim is true on $[0, t)$; from this we will prove that it remains true at time t:
$$\begin{aligned} d(t) &\le C_A \delta + \tilde{C}_A \int_0^t e^{-\frac{2}{3}\lambda s} d^2(s) \, \mathrm{d}s \\ &\le C_A \delta + \tilde{C}_A (2C_A)^2 \int_0^t e^{-\frac{2}{3}\lambda s} \delta^2 \, \mathrm{d}s \\ &\le C_A \delta + C_{A,2} \delta^2. \end{aligned}$$
Take $\delta \le C_A/2/C_{A,2}$ and we have
$$d(t) \le C_A \delta + \frac{1}{2} C_A \delta < 2C_A \delta.$$
Equivalently,
$$|z(t)| < 2C_A \delta e^{-\frac{2}{3}\lambda t} \longrightarrow 0.$$

□

Example 8.13. Consider the pendulum system
$$m \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mg}{L} \sin\theta = 0 \sin\theta = 0, \quad x = \theta, \quad y = \theta'.$$
This can be summarized as
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -\frac{g}{L} \sin x \end{pmatrix}.$$
The critical points are $(n\pi, 0)$, $n \in \mathbb{Z}$.

Note that
$$\nabla F(x_*) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} \cos(n\pi) & 0 \end{pmatrix}$$
If n is odd, we have
$$\nabla F(x_*) = \begin{pmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{g/L}$. This corresponds to the inverted pendulum and is unstable.

If n is even, we have

$$\nabla F(x_*) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix},$$

which has eigenvalues $\pm\mathrm{i}\sqrt{g/L}$. This corresponds to the hanging pendulum. We cannot conclude stability from linearization.

8.4 Liapunov

Note that the linearized method works only for $|x_0 - x_*| \ll 1$, since we are using Taylor expansion.

The Liapunov method gives a way to prove stability for larger perturbations. Accordingly, it is at times called the global method. However, while linearization applies for almost any autonomous system, the Liapunov method requires very special functions.

Definition 8.14. Consider a generalization of distance, a C^1 function $L : D \subset \mathbb{R}^n \to \mathbb{R}$. Here, D is open.

Example 8.15. $L(X) = \|x - x_*\|^2 = \sum(x_i - x_{*,i})^2$.

We now consider the evolution of $L(x(t))$:

$$\frac{\mathrm{d}}{\mathrm{d}t}L(x(t)) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) \cdot \nabla L(x(t)) = F(x(t)) \cdot \nabla L(x(t)).$$

Write $\dot{L}(x) = F \cdot \nabla L|_x$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}L(x(t)) = \dot{L}(x(t)).$$

Theorem 8.16 (Liapunov Stability)**.** *Let x_* be an equilibrium of the autonomous system $x' = F(x)$, $F \in C^1$. Assume that there exists L such that $L(x_*) = 0$ and $L(x) > 0$ for each $x \neq x_*$.*

Now, if in addition, $\dot{L}(x) \leq 0$ for each $x \in D \setminus \{x_*\}$, then x_* is stable. If $\dot{L}(x) < 0$ for each $x \neq x_*$, then x_* is asymptotically stable.

The idea is that L is like a generalized distance from x_* and $\dot{L} \leq 0$ means the distance is non-increasing along trajectories.

The potential energy of the pendulum is $U(x, y) = mgh = mgL(1 - \cos x)$, where L is the length of the pendulum. (Recall that $x = \theta$ and $y = \theta'$.)

The kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}mL^2y^2$. (Note that $v = L\dot{\theta} = Ly$.)

Define the **energy** E of the pendulum as

$$L(x, y) = \frac{1}{2}mL^2y^2 + mgL(1 - \cos x).$$

We claim that L is a Liapunov function for the hanging pendulum at $(0, 0)$:

- $L(0,0) = 0$ and $L(x,y) > 0$ for each $(x,y) \neq (0,0)$ (when we restrict the domain to $D := (-\pi, \pi) \times \mathbb{R}$).
- $\dot{L} = \nabla L \cdot F$.
$$\nabla L = \begin{pmatrix} mgL \sin x \\ mL^2 y \end{pmatrix}, \quad F = \begin{pmatrix} y \\ -\frac{g}{L} \sin x \end{pmatrix}.$$
So,
$$\dot{L} = \begin{pmatrix} mgL \sin x \\ mL^2 y \end{pmatrix} \cdot \begin{pmatrix} y \\ -\frac{g}{L} \sin x \end{pmatrix} = mgLy \sin x - mgLy \sin x = 0.$$
Thus $L(x(t)) \equiv L(x_0)$ does not change along trajectories. Thus $(0,0)$ is not asymptotically stable.

Proof *(of Liapunov Stability Theorem)***.** Recall first that $\frac{\mathrm{d}}{\mathrm{d}t} L(x(t)) = \dot{L}(t)$.

Suppose $\dot{L}(x) \leq 0$ for each $x \neq x_*$. We hope to show that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - x_*| < \delta$, then $|x(t) - x_*| < \varepsilon$ for each $t \geq t_0$.

Without loss, suppose $B(x_*, \delta) \subset D$. Let $\alpha := \min\{L(y) : |y - x_*| = \delta\} > 0$. By continuity of L we pick $\varepsilon < \delta$ small so for each $|x_0 - x_*| < \varepsilon$ we have
$$0 \leq L(x_0) = L(x_0) - L(x_*) < \alpha.$$
Note that we have $\frac{\mathrm{d}}{\mathrm{d}t} L(x(t)) \leq 0$, so $L(x(t)) \leq L(x_0) < \alpha$. We claim that $x(t) \in B(x_*, \delta)$, since otherwise we have $L(x(t)) \geq \alpha$, which contradicts $L(x(t)) < \alpha$. check notes

Now, suppose $\dot{L}(x) < 0$ for each $x \neq x_*$. Since this is a stronger condition, the above proof still works and shows that $x(t) \in B(x_*, \delta)$ for the same choice of ε. Suppose $\lim_{t \to \infty} x(t)$ does not converge to x_*. Then, $x(t_n) \to z_* \neq x_*$ for a sequence $t_n \to \infty$. But each time we visit z_*, the value $L(x(t))$ drops; after sufficiently many iterations, $L(x)$ must converge:

Note that for each $t \geq 0$, there exists large m such that
$$L(x(t)) \geq L(x(t_m)) \geq L(z_*) > 0,$$
where the second inequality comes from continuity. Consider the ODEs $z' = F(z)$, $z_0 = z_*$ and $y' = f(y)$, $y_0 = x(t_n)$. A solution to the second ODE is $y(0) = x(t_n)$, $y(1) = x(t_n + 1)$. We have $L(z_*) - L(z(1)) = \Delta > 0$. Note that
$$\begin{aligned} L(x(t_n + 1)) - L(z_*) &= L(x(t_n + 1)) - L(z_1) + L(z_1) - L(z_*) \\ &= L(y(1)) - L(z(1)) - \Delta \\ &\leq C|y(1) - z(1)| - \Delta, \end{aligned}$$
by f being Lipschitz. By continuous dependence the first term can be made less than $\Delta/2$ for large n, and we get a contradiction. $\square$

Example 8.17. The ODE system
$$\begin{aligned} x' &= -x^3 \\ y' &= -y(x^2 + z^2 + 1) \\ z' &= -\sin z \end{aligned}$$

has equilibrium $x = y = z = 0$. Define $L(v) = x^2 + y^2 + z^2$.

$$\begin{aligned} \nabla L \cdot F &= 2x(-x^3) + 2y(-y(x^2 + z^2 + 1)) + 2z(-\sin z) \\ &= -2x^4 - 2y^2(x^2 + z^2 + 1) - 2z \sin z \leq 0. \end{aligned}$$

Take $D \coloneqq \mathbb{R} \times \mathbb{R} \times (-\pi, \pi)$ and we have $\nabla L \cdot F < 0$ for each $(x, y, z) \neq (0, 0, 0)$.