ECON20010 PSET 1

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Contents

1. The Lagrangian

Note:

- The Lagrangian multiplier of the UMP can be non-positive. Think bliss points.

2. The Envelope Theorem

Theorem 2.1. *Let*

$$v(a) = \max_x f(x; a).$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}a} = \frac{\mathrm{d}f(x^*; a)}{\mathrm{d}a} = \left.\frac{\partial f}{\partial a}\right|_{x=x^*}.$$

Remark 2.2. Intuition:

$$\frac{\mathrm{d}f(x^*; a)}{\mathrm{d}a} = \sum \frac{\partial f}{\partial x_i^*} \cdot \frac{\partial x_i^*}{\partial a} + \frac{\partial f}{\partial a},$$

and at optimum, each $\partial f / \partial x_i^* = 0$. "All indirect effects vanish." Note that by the implicit function theorem, we need $f_{xx} \neq 0$.

Example 2.3. Consider the value function

$$\begin{aligned} v(p_x, p_y, m) &= U(x^*, y^*) = U(x^*, y^*) + \lambda^*[m - p_x x - p_y y] \\ &= \mathcal{L}(x^*, y^*, \lambda^*; p_x, p_y, m) \\ &=: \mathcal{L}^*(p_x, p_y, m). \end{aligned}$$

We then have by the envelope theorem, $\mathrm{d}\mathcal{L}/\mathrm{d}m = \partial\mathcal{L}/\partial m$ and thus

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}^*}{\partial m} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

Similarly,

$$\frac{\partial v}{\partial p_x} = \frac{\partial \mathcal{L}^*}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

3. Scarcity

Definition 3.1.

- The **budget set** consists of all feasible consumption bundles.
- The **budget constraint** exactly exhausts the consumer's income.

3.1. Budget Set.

The relative price:

$$\frac{p_x}{p_y}$$

- Mnemonic: this is always the price of x in units of y.

  To stay on the budget constraint,

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p_x}{p_y}.$$

- Think "the rate at which the market *allows* the consumer to exchange good x for good y."
- Think the opportunity cost of good x.

3.2. Preference.

Axioms:

- **Completeness**. For any pair of consumption of bundles, say c_1 and c_2, either $c_1 \succeq c_2$, $c_2 \succeq c_1$, or both.
  - Requires a answer and assumes no framing effects.
- **Transitivity**. If $c_1 \succeq c_2$ and $c_2 \succeq c_3$ then $c_1 \succeq c_3$.
  - Money pump.
- A preference ordering is **rational** if it satisfies completeness and transitivity. They are the minimal requirement for the existence of a utility function representation.

We also typically assume the following:

- **Continuity**. If $c_1 \succ c_2$ then there are neighborhoods N_1 and N_2 around c_1 and c_2 such that

  $$x \succ y, \quad \forall x \in N_1, \quad y \in N_2.$$

  This implies that if $c_1 \succ c_2$ then there exists c_3 such that

  $$c_1 \succ c_3 \succ c_2.$$

- **Monotonicity**.
  - **Monotone**. If $c_1 \gg c_2$[1] then $c_1 \succ c_2$.
  - **Strongly monotone**. If $c_1 \geq c_2$[2] and $c_1 \neq c_2$ then $c_1 \succ c_2$.
  - **Local non-satiation**. If for every bundle c and every $\epsilon > 0$, there exists $x \in N_\epsilon(c)$ such that $x \succ c$.

[1] We write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i, \forall i$.

[2] We write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i, \forall i$.

- **Convexity**. If $c_1 \succeq c_2$, then
$$\theta c_1 + (1-\theta)c_2 \succeq c_2, \quad \forall \theta \in (0,1).$$
If convexity is satisfied, the **upper contour set**, the "at least as good as" set, is convex.

Additional axioms place even more structures on the utility function:

- **Homotheticity**. If $c_1 \succeq c_2$, then
$$tc_1 \succeq tc_2, \quad \forall t > 0.$$
- **Quasilinearity** in good i. If $\mathbf{c_1} \succeq \mathbf{c_2}$, then
$$\mathbf{c_1} + t\mathbf{e}_i \succeq \mathbf{c_2} + t\mathbf{e}_i, \quad \forall t > 0.$$

3.3. **Translating preference ordering to the utility function:**

Theorem 3.2.

- *If a preference ordering is rational, then it admits a utility function representation. (Representation Theorem;*[3]*) The utility function is unique up to a monotonically increasing transformation.*
- *If a preference ordering satisfies convexity, then the corresponding utility function representation will be quasi-concave. The indifference curves (level sets) will have non-increasing marginal rate of substitution (slopes).*

3.4. **The Marginal Rate of Substitution.** The MRS
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y}$$
is the quantity of y the consumer is willing to sacrifice in exchange for an additional unit of x. (Think p_x/p_y.) It measures an individual's **willingness to pay** for x in terms of y.

4. The Utility Maximization Problem

The problem:
$$v(p_x, p_y, m) := \max_{x,y} U(x,y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

4.1. **Interpretation.** We want to maximize
$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$
such that
$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$
This gives
$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right]\mathrm{d}x.$$
We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.
$$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \ \mathrm{d}x$$
"Take advantage of all trading opportunities."

[3]For a simple version of this, think assigning the size of the unique bundle on $t \sum \mathbf{e}$ equivalent to a given consumption bundle.

- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that U_x/p_y is marginal utility of money *spent on* x.
$$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \,\mathrm{d}x$$
"Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that U_x is the marginal benefit of buying x and $U_y \cdot p_x/p_y$ is the marginal cost of buying x.
$$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$
"Trade until marginal cost equals marginal benefit."

In the last expression, if we write
$$\lambda = \frac{U_y}{p_y},$$
(think marginal utility of income) we have that at optimum,
$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$
These three equalities describe precisely the critical points of the following
$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right],$$
called the **Lagrangian**. That is, setting
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$
recovers the above three equations.

Remark 4.1.
- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- λ might be negative or zero. Think bliss point.

4.2. **The Indirect Utility Function.**

Proposition 4.2.
$$\frac{\partial v}{\partial m} = \lambda^*.$$

Proof. Noting
$$v = U(x^*, y^*) + \lambda^*[m - p_x x^* - p_y y^*] = \mathcal{L}^*,$$
we have
$$\begin{aligned}
\frac{\partial v}{\partial m} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} \\
&= U_x^* \frac{\partial x}{\partial m} + U_y^* \frac{\partial y}{\partial m} + \lambda^* \left[1 - p_x \frac{\partial x}{\partial m} - p_y \frac{\partial y}{\partial m}\right] + \frac{\partial \lambda}{\partial m}\left[m - p_x x^* - p_y y^*\right] \\
&= \left(U_x^* - \lambda^* p_x\right) \frac{\partial x}{\partial m} + \left(U_y^* - \lambda^* p_y\right) \frac{\partial y}{\partial m} + \frac{\partial \lambda^*}{\partial m}\left(m - p_x x^* - p_y y^*\right) + \lambda^* \\
&= \lambda^*.
\end{aligned}$$

The last equality follows by noting that at the optimum,

$$U_x^* - \lambda^* p_x = U_y^* - \lambda^* p_y = m - p_x x^* - p_y y^* = 0.$$

Alternatively, one may use the envelope theorem:

$$\frac{\partial v}{\partial m} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

□

Note that

$$\frac{\partial v}{\partial m} = \lambda^* = \frac{U_x^*}{p_x} = \frac{U_x^*}{p_x}.$$

So when not satiated ($U_x, U_y \neq 0$), marginal utility of income is positive. When budget constraint does not require to bind, the marginal utility of income is generally nonnegative.

Again using the Envelope Theorem, we have

$$\frac{\partial v}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

This value is generally nonpositive, and only zero when one does not consume the specific good or when the marginal utility of that good is 0.

5. Expenditure Minimization

The problem:

$$e(p_x, p_y, \overline{U}) := \max_{x,y} p_x x + p_y y \quad \text{s.t.} \quad U(x, y) = \overline{U}.$$

The Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \eta\left[\overline{U} - U(x, y)\right]$$

$$\begin{aligned}
&[x] && p_x = \eta^* U_x(x^*, y^*) \\
&[y] && p_y = \eta^* U_y(x^*, y^*) \\
&[\eta] && \overline{U} = U(x^*, y^*).
\end{aligned}$$