ECON20110 (W25): The Elements of Economic Analysis II Honors

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1 On Mathematics

1.1 Constrained Maximization

E.g.,

$$\max_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad \text{s.t.} \quad G(\mathbf{x}, \boldsymbol{\theta}) \geq 0.$$

Solving a whole class of optimization problems parameterized by $\tilde{\theta}$ generates two functions:

- The solution function
- The Value function

Results like the envelope theorem relates these two functions.

1.2 The Kuhn-Tucker Theorem

Consider the maximization function $\max_x f(x)$. The first order condition gives $f'(x^*) = 0$. Now suppose that x_1 is such that $f'(x_1) > 0$. We may be temped to argue that x_1 is not a solution since we can increase f by increasing the value of x, but this assumes that x is in the interior of the domain. Thus the first order condition considers only interior solutions. The Kuhn-Tucker theorem addresses this issue.

Theorem 1.1 (Kuhn-Tucker)**.** *Consider the constrained maximization problem*

$$\max_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad \textit{s.t.} \quad G(\mathbf{x}, \boldsymbol{\theta}) \geq 0, \quad \mathbf{x} \geq 0.$$

The Lagrangian is

$$\mathcal{L} \coloneqq U(\mathbf{x}, \boldsymbol{\theta}) + \lambda G(\mathbf{x}, \boldsymbol{\theta}).$$

The FOCs are:

- *for each i, $\partial \mathcal{L} / \partial x_i \leq 0$ and $x_i \geq 0$, with complementary slackness; That is, at most one of the two conditions can be a strict inequality.*
- *$\partial \mathcal{L} / \partial \lambda \geq 0$ and $\lambda \geq 0$, with complementary slackness.*

Remark 1.2.

- For the direction of the inequalities on $\partial \mathcal{L} / \partial x_i \leq 0$ and $\partial \mathcal{L} / \partial \lambda \geq 0$, remember the picture. In minimization problems they are flipped.

- Often, we can rule out many of these cases. For example, when u is strictly increasing, we have that $\lambda > 0$; and $\lim_{x_1 \to 0} \partial u / \partial x_i = \infty$ gives $x_i > 0$.

- Think of λ as a penalty of not satisfying the constraint. Note that we need negative penalty for maximization problems, and positive penalty for maximization problems. Thus for the minimization problem we write

$$\mathcal{L} \coloneqq U(\mathbf{x}, \boldsymbol{\theta}) + \lambda[0 - G(\mathbf{x}, \boldsymbol{\theta})].$$

Example 1.3. Cost minimization problem:

- The problem:
$$\min_{\mathbf{x}} \boldsymbol{\omega} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y.$$

- The Lagrangian:
$$\mathcal{L} \coloneqq \boldsymbol{\omega} \cdot \mathbf{x} + \lambda[y - f(\mathbf{x})].$$

- FOCs: $\partial \mathcal{L} / \partial x_i \geq 0$, $x_i \geq 0$, CS; $\partial \mathcal{L} / \partial \lambda \leq 0$, $\lambda \geq 0$, CS. CHECK

1.3 Elasticity of Substitution

Elasticities are of the form

$$-\frac{\mathrm{d} \log y}{\mathrm{d} \log x} = -\frac{\mathrm{d}y / y}{\mathrm{d}x / x}.$$

- Elasticities gives the proportion response of x as y changes proportionately.

- Knowing the elasticities gives information on how the product xy changes as y changes. For example, if $\sigma > 1$, then xy decreases as y increases.

The elasticity of substitution captures how the (optimal) relative consumption level between two goods responds to changes of the corresponding price ratio:

$$\sigma_{ij} = -\frac{\mathrm{d} \log(x_i^* / x_j^*)}{\mathrm{d} \log(p_i / p_j)} = \frac{\mathrm{d} \log(x_j / x_i)}{\mathrm{d} \log(MU_i / MU_j)}.$$

Remark 1.4.

- We think of relative prices as exogenous. The last formula is often used as the definition because it can be computed straight from definition.
- If $\sigma_{ij} > 1$, then relative expenditure $(p_i x_i)/(p_j x_j)$ decreases as p_i/p_j increases, etc.
- Larger values of σ_{ij} means it is "easier to substitute i for j".

1.4 Sets and Mapping

Notation 1.5. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

Definition 1.6. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we write:

- $\mathbf{x} \geq \mathbf{y}$ if $\forall i : x_i \geq y_i$;
- $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.
- $\mathbf{x} \gg \mathbf{y}$ (read strongly greater than) if $\forall i : x_i > y_i$;

Definition 1.7. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **strictly increasing** if $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{x} \gg \mathbf{y}$. It is **strongly increasing** if $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{x} > \mathbf{y}$.

Example 1.8. Strongly increasing implies strictly increasing.

- Cobb-Douglas is strongly increasing when $\mathbf{x} \gg \mathbf{0}$ but is only strictly increasing when $x_i = 0$ for some i.
- The linear production function $f(\mathbf{x}) = \sum x_i$ is strongly increasing.
- The Leontief production function $f(\mathbf{x}) = \min x_i$ is strictly increasing but not strongly increasing.

Definition 1.9. The $N \times N$ matrix M is **negative semidefinite** (NSD) if

$$\forall z \in \mathbb{R}^N : z \cdot Mz \leq 0$$

and **positive semidefinite** (PSD) if

$$\forall z \in \mathbb{R}^N : z \cdot Mz \geq 0.$$

If the inequality is strict for all $z \neq 0$, then M is **negative definite** (ND) (resp., **positive definite** (PD)).

Proposition 1.10.

(i) *M is PSD (PD) $\iff$ $-M$ is NSD (ND).*

(ii) *M is ND (PD) $\iff$ M is NSD (PSD), but the converse is not true.*

(iii) *M is ND (PD) $\iff$ M' is ND (PD).*

(iv) *M is ND (PD) $\iff$ M^{-1} is ND (PD).*

Proof. The first three statements are immediate. For the last, note that

$$z'Mz = (z'Mz)' = z'M'z = z'MM^{-1}M'z = (M'z)'M^{-1}M'z.$$

□

1.5 Concave and Convex Functions

Notation 1.11. Let $\mathbf{x}^1, \mathbf{x}^2 \in X$ and $t \in [0, 1]$. We often denote $\mathbf{x}^t = t\mathbf{x}^1 + (1-t)\mathbf{x}^2$.

Definition 1.12. A function $f : X \to \mathbb{R}$ is convex (resp., strictly convex) if epi f is convex (resp., strictly convex). The function f is concave (resp., strictly concave) if $-f$ is concave (resp., strictly concave).

Proposition 1.13. *A function $f : X \to \mathbb{R}$ is convex if and only if for all $x_1, \ldots, x_k \in X$ and $\alpha_1, \ldots, \alpha_n$ such that $\sum \alpha_i = 1$, we have $f\left(\sum \alpha_i x_i\right) \leq \sum \alpha_i f(x_i)$.*

We may think of α_i as probability masses. The following result generalizes this to probability densities:

Proposition 1.14 (Jensen's Inequality)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable, and X is a random variable such that $\mathrm{E}[X]$ and $f(\mathrm{E}\,X)$ exist, then $f(\mathrm{E}\,X) \leq \mathrm{E}\,f(X)$.*

Proof. From convexity of f we know $f(x) \geq f(y) + f'(y)(x-y)$ for any x and y. Setting $y = \mathrm{E}\,X$ gives

$$f(X) \geq f(\mathrm{E}\,X) + f'(\mathrm{E}\,X)(X-\mu), \quad \forall x.$$

Taking expectation on both sides gives the desired result. □

Proposition 1.15. *The C^1 function $f : X \to \mathbb{R}$ is convex if and only if*

$$f(x+t) \geq f(x) + \nabla f(x) \cdot t$$

for all $x \in X$ and $t \in \mathbb{R}^N$ such that $x + t \in X$.

Proof. Suppose f is convex. For any $\alpha \neq 0$,

$$f\big(\alpha(x+t) + (1-\alpha)x\big) \leq \alpha f(x+t) + (1-\alpha) f(x),$$

giving

$$f(x+\alpha t) - f(x) \leq \alpha\big(f(x+t) - f(x)\big)$$

and then

$$f(x) + \frac{f(x+\alpha t) - f(x)}{\alpha} \leq f(x+t).$$

Taking $\alpha \to 0$ gives the desired result.

For the reverse direction, consider arbitrary $x, y \in X$ and $\lambda \in [0,1]$. Write $z = \lambda x + (1-\lambda) y$ By assumption we have

$$\begin{aligned} f(x) &\geq f(z) + \nabla f(z)(1-\lambda)(x-y), \\ f(y) &\geq f(z) + \nabla f(z)\lambda(y-x), \end{aligned}$$

together giving

$$\lambda f(x) + (1-\lambda) f(y) \geq f(z).$$

□

Proposition 1.16. *The C^2 function $f : X \to \mathbb{R}$ is convex if and only if $\mathrm{D}^2 f(x)$ is PSD for every $x \in X$.*

Proof. Suppose f is convex. Fix some $x \in X$. For any $t \neq 0$, the second-order Taylor expansion is

$$f(x+\alpha t) = f(x) + \nabla f(x) \cdot (\alpha t) + \frac{\alpha^2}{2} t \cdot \mathrm{D}^2 f(x+\beta t) t.$$

By Proposition 1.15,

$$\frac{\alpha^2}{2} t \cdot \mathrm{D}^2 f(x+\beta t) t \geq 0.$$

And conversely. □

1.6 Quasi-Concavity and Quasi-Convexity

Definition 1.17. Let $X \subset \mathbb{R}^n$ be convex. A function $f : X \to \mathbb{R}$ is **quasi-concave** if for all $\mathbf{x}^1, \mathbf{x}^2 \in X$ and $t \in [0, 1]$, we have $f(\mathbf{x}^t) \geq \min\{f(\mathbf{x}^1), f(\mathbf{x}^2)\}$. The function f is **strictly quasi-concave** if the inequality is strict for all $\mathbf{x}^1 \neq \mathbf{x}^2$.

Remark 1.18.

- Thus (strictly) concave functions are (strictly) quasi-concave. The converse is not true; consider $x \mapsto x^2$, $x > 0$.
- Quasi-concavity is a ordinal property that is preserved by monotone transformations, by the following result:

Proposition 1.19. *Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. If $f : X \to \mathbb{R}$ is quasi-concave, so is $g \circ f$. If, in addition, g is strictly increasing, then $g \circ f$ is strictly quasi-concave for all f that is strictly quasi-concave.*

Definition 1.20. For $f : X \to \mathbb{R}$ and $\mathbf{x}^0 \in X$, the **level set** relative to $f(\mathbf{x}^0)$ is the set $L(\mathbf{x}^0) \coloneqq \{\mathbf{x} \in X : f(\mathbf{x}) = f(\mathbf{x}^0)\}$; the **superior set** (or the upper contour set) is the set $S(\mathbf{x}^0) \coloneqq \{\mathbf{x} \in X : f(\mathbf{x}) \geq f(\mathbf{x}^0)\}$; the **inferior set** (or the lower contour set) is the set $I(\mathbf{x}) \coloneqq \{\mathbf{x} \in X : f(\mathbf{x}) \leq f(\mathbf{x}^0)\}$.

The following results is more or less immediate:

Proposition 1.21. *The function $f : X \to \mathbb{R}$ is quasi-concave if and only if for all $\mathbf{x}^0 \in X$, $S(\mathbf{x}^0)$ is convex.*

Proposition 1.22. *If $f : X \to \mathbb{R}$ is (strictly) quasi-concave then $-f$ is (strictly) quasi-convex.*

Just like convexity, quasi-convexity of a function can be related to its Hessian, using the following results:

Lemma 1.23. *The C^1 function $f : X \to \mathbb{R}$ is quasi-convex if and only if for each $x, y \in X$ such that $f(y) \geq f(x)$ we have*

$$\nabla f(x) \cdot (y - x) \geq 0.$$

Proof. Similar to Proposition 1.15. □

Proposition 1.24. *The C^2 function $f : X \to \mathbb{R}$ is quasiconvex if and only if for each $x \in X$, the Hessian matrix $\mathrm{D}^2 f(x)$ is PSD in the subspace $\{x \in \mathbb{R}^N : \nabla f(x) \cdot y = 0\}$.*

Proof. Similar to Proposition 1.16. □

1.7 Homogeneous Functions

Definition 1.25. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **homogeneous of degree k** if for all $\mathbf{x} \in \mathbb{R}^n$ and all $\lambda > 0$ we have

$$f(\lambda \mathbf{x}) = \lambda^k f(\mathbf{x}).$$

Proposition 1.26. *If $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k and differentiable, its first order partial derivatives are homogeneous of degree $k-1$. Thus the slops of the isoquants at $\mathbf{x}$ is always the same at $\lambda\mathbf{x}$ for any $\lambda > 0$.*

Proof. Differentiating both sides, we get

$$\frac{\partial f}{\partial x_i}(\lambda \mathbf{x}) = \lambda^{k-1} \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

$\square$

2 Production Technology

"Firm" simply refers to actors on the supply side. They transform resources (inputs) into goods and services (outputs), while constrained by the production technology. The only difference between firms and consumers is the problems they are solving: the former solves a profit maximization problem, the latter a utility maximization problem.

We identify the firm's input choices with members of $X \subset \mathbb{R}_+^m$ and output choices with members of $Y \subset \mathbb{R}_+^n$. We can describe a firm's technology by specifying its **production possibility set**, $F \subset X \times Y$, each member of which is called a production plan. In this course we assume $n = 1$. The upper contour of the production possibility set is called the production possibility frontier, which can be described by a production function. Most of the times there is no loss of generality in considering only the production function. Think when.

Definition 2.1. Let $F \subset \mathbb{R}_+^m \times \mathbb{R}_+$ be a production possibility set. The **production function** $f : \mathbb{R}_+^m \to \mathbb{R}_+$ is defined by

$$f(\mathbf{x}) \coloneqq \sup\{y \in \mathbb{R}_+ : (\mathbf{x}, y) \in F\}.$$

2.1 Placing Structure on the Production Function

Assumption 2.2. We typically assume that the production function $f : \mathbb{R}_+^m \to \mathbb{R}_+$ is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}_+^m$ and $f(\mathbf{0}) = 0$.

- Strict quasiconcavity gives strictly convex upper contours. It can be thought of as there being some complementarities in the inputs.
- These assumptions guarantees that the firm's production optimization (cost minimization) problem is well-defined and has a unique solution.

Definition 2.3. The **martial product** of input i at input vector $\mathbf{x}$ is

$$\mathrm{MP}_i(\mathbf{x}) \coloneqq \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

The **marginal rate of technical substitution (MRTS)** between inputs i and j is

$$MRTS_{ij}(\mathbf{x}) \coloneqq \frac{\mathrm{MP}_i(\mathbf{x})}{\mathrm{MP}_j(\mathbf{x})}.$$

For $y \geq 0$, the y-level **isoquant** of $f : \mathbb{R}_+^m \to \mathbb{R}_+$ is

$$Q(y) \coloneqq \{\mathbf{x} \in \mathbb{R}_+^m : f(\mathbf{x}) = y\}.$$

Remark 2.4. Recall the notions of marginal utility, marginal rate of substitution, and indifference curves in consumer theory.

Proposition 2.5. *Under Assumption 2.2 and when $n = 2$,*

(i) The slope of isoquant (at a point $\mathbf{x}$*) is given by the MRTS.*

(ii) Isoquant are always downward sloping.

(iii) We have diminishing $MRTS$.

Proof. (i) Clear.

(ii) From f being strictly increasing, we know $f_i > 0$.

(iii) From f being strictly quasiconcave, isoquants bend towards the origin.

□

2.2 Return to Scale

Definition 2.6. We say the production function f exhibits (globally)

- **constant return to scale** if $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $x \in \mathbb{R}_+^n$ and all $t > 0$.
- **increasing return to scale** if $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $x \in \mathbb{R}_+^n$ and all $t > 1$.
- **decreasing return to scale** if $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $x \in \mathbb{R}_+^n$ and all $t > 1$.

Example 2.7. The Cobb-Douglas production function $f(\mathbf{x}) = A \prod x_i^{\alpha_i}$ with $\sum \alpha_i = 1$ is homogeneous of degree one and thus exhibits constant return to scale.

Example 2.8. A firm with increasing return to scale will enjoying decreasing average cost. The reverse is not always true.

Definition 2.9. For a production function $f(\mathbf{x})$, the **elasticity of Substitution between inputs i and j** (at point $\mathbf{x}$) is defined as

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) &:= -\frac{\mathrm{d}\log(x_i/x_j)}{\mathrm{d}\log(\mathrm{MP}_i/\mathrm{MP}_j)}\\ &= \frac{\mathrm{d}\log(x_j/x_i)}{\mathrm{d}\log(\mathrm{MP}_i/\mathrm{MP}_j)}.\end{aligned}$$

Remark 2.10. A larger σ_{ij} means it is easier to substitute i for j. To see this, consider the CES production function

$$f(x_1, x_2) = \left(\alpha x_1^{\frac{\sigma-1}{\sigma}} + (1-\alpha)x_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

As $\sigma \to \infty$, $f(\mathbf{x}) \to \alpha x_1 + (1-\alpha)x_2$; as $\sigma \to 0$, $f(\mathbf{x}) \to \min\{x_1, x_2\}$.

Example 2.11. Cobb-Douglas has constant and unit elasticity of substitution, this directly gives constant expenditure share. In general, if $u(\mathbf{x}) = x_1^\alpha x_2^\beta$ and $\alpha, \beta > 0$, then

$$p_1 x_1 = \frac{\alpha m}{\alpha + \beta}, \quad p_2 x_2 = \frac{\beta m}{\alpha + \beta}$$

and thus

$$x_1^* = \frac{\alpha}{\alpha + \beta}\frac{m}{p_1}, \quad x_2^* = \frac{\beta}{\alpha + \beta}\frac{m}{p_2}.$$

3 Cost Minimization

We assume both the product market and the factor (input) markets are perfectly competitive. In doing so, we are ignoring any influence of any player in the market on the prices. The firm thus solves the following profit-maximizing problem:

$$\max_{y,\mathbf{x}} py - \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}),$$

where $\boldsymbol{\omega}$ contains the prices of the inputs.

We may rewrite it as a two part problem:

$$\max_{y} \max_{\mathbf{x}} py - \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}),$$

which is equivalent to

$$\max_{y} py - \min_{\mathbf{x}} \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}).$$

Note the resemblance with the notion of backward induction. We consider first the cost minimization problem, that is:

$$\min_{\mathbf{x}} \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y.$$

The Lagrangian is

$$\mathcal{L} = \boldsymbol{\omega}\mathbf{x} + \lambda(y - f(\mathbf{x})).$$

The Kuhn-Tucker FOCs are

- for $i = 1, \ldots, m$: $\partial\mathcal{L}/\partial x_i = \omega_i - \lambda\partial f(\mathbf{x})/\partial x_i \geq 0$ and $x_i \geq 0$, with C.S.;
- $\partial\mathcal{L}/\partial\lambda = y - f(\mathbf{x}) \leq 0$ and $\lambda \geq 0$, with C.S.

Assuming interior solution and $f(\mathbf{x}) = y$, the FOCs reduce to

$$\boldsymbol{\omega} = \lambda\mathbf{MP}, \quad f(\mathbf{x}) = y$$

We have thus for all i, j that

$$\frac{\omega_i}{\omega_j} = \frac{\text{MP}_i}{\text{MP}_j},$$

that is, the MRTS between i and j equals their price ratio.

Definition 3.1. The **conditional input demand function** is

$$\mathbf{x}(\boldsymbol{\omega}, y) \equiv \underset{x \in f^{-1}(\{y\})}{\arg\min}\ \boldsymbol{\omega} \cdot \mathbf{x},$$

and the **cost function** is the minimized value function.

Remark 3.2.

- The continuity of the production function guarantees the existence of a solution to the cost-minimization problem (when $\omega \gg 0$).
- Strict quasi-concavity of the production function guarantees that the solution to the cost-minimization problem is unique and that the conditional input demand function is thus well defined.

Definition 3.3. We define the **marginal cost of production** as $\mathrm{MC}(\boldsymbol{\omega}, y) \coloneqq \partial c(\boldsymbol{\omega}, y)/\partial y$ and the **average cost** as $\mathrm{AC}(\boldsymbol{\omega}, y) \coloneqq \partial c(\boldsymbol{\omega}, y)/\partial y$.

3.1 Comparative Statics

Proposition 3.4. *The conditional input demand functions are homogeneous of degree zero in* $\boldsymbol{\omega}$*; the cost functions are homogeneous of degree one in* $\boldsymbol{\omega}$*.*

Theorem 3.5. *If* $\omega \gg 0$*, then* $c(\boldsymbol{\omega}, y)$ *is strictly increasing in* y*. Thus, assuming differentiability,* $\partial x_i(\boldsymbol{\omega}, y)/\partial y \geq 0$*; conditional input demand is never inferior,*

Proof. Assume $c(\boldsymbol{\omega}, y_1) \geq c(\boldsymbol{\omega}, y_2)$ with $y_1 < y_2$. Then we can also produce y_1 with the input that is optimal for producing y_2, which gives a contradiction. □

Proof *(Assuming differentiability)***.** By the Envelope theorem, $\mathrm{MC} = \partial c/\partial y = \partial \mathcal{L}/\partial y = \lambda$. If f is strictly increasing, then an increase in output can only be achieved with increases in inputs. With $f(\mathbf{x}) = 0$, we thus have $x_i > 0$ if $y > 0$. Thus from the FOCs, we have $\omega_i = \lambda\,\mathrm{MP}_i$, from which we get and $\mathrm{MP}_i > 0$ and $\lambda > 0$. □

Remark 3.6. Thus, when there is no free input, marginal cost of production is always positive.

Theorem 3.7 (Shephard's Lemma)**.** *If* f *is strictly quasiconcave and* $\boldsymbol{\omega} \gg 0$*, then*

$$\frac{\partial c(\boldsymbol{\omega}, y)}{\partial \omega_i} = x_i(\boldsymbol{\omega}, y).$$

Proof. Strict quasiconcavity gives continuity of c. The results follows from Envelope theorem. $\square$

Corollary 3.8. *The cost function $c(\omega, y)$ is (weakly) increasing in $\boldsymbol{\omega}$.*

Theorem 3.9. *$c(\boldsymbol{\omega}, y)$ is increasing and concave in $\boldsymbol{\omega}$.*

Proof. Note that $c(t\omega, y)$ is bounded above by the linear function $tc(\boldsymbol{\omega}, y)$, since one can always choose the original bundle. Alternatively, fix y and $t \in (0, 1)$. Let

$$\mathbf{x}^1 = \mathbf{x}(\boldsymbol{\omega}^1, y), \quad \mathbf{x}^2 = \mathbf{x}(\boldsymbol{\omega}^2, y), \quad \mathbf{x}^t = \mathbf{x}\big(t\boldsymbol{\omega}^1 + (1-t)\boldsymbol{\omega}^2, y\big).$$

By definition we have

$$\boldsymbol{\omega}^t \mathbf{x}^1 \leq \boldsymbol{\omega}^t \mathbf{x}^t, \quad \boldsymbol{\omega}^t \mathbf{x}^t \leq \boldsymbol{\omega}^t \mathbf{x}^t,$$

which gives

$$t\boldsymbol{\omega}^t \mathbf{x}^1 + (1-t)\boldsymbol{\omega}\mathbf{x}^2 \leq \boldsymbol{\omega}^t \mathbf{x}^t.$$

Therefore we have

$$c(\boldsymbol{\omega}, y) = \boldsymbol{\omega}^t \mathbf{x}^t \geq tc(\boldsymbol{\omega}^1, y) + (1-t)c(\boldsymbol{\omega}^2, y).$$

Since $\mathbf{x}^1$, $\mathbf{x}^2$, and $t \in (0, 1)$ are arbitrary, we have shown that c is convex. $\square$

Definition 3.10. We define the substitution matrix

$$\boldsymbol{\sigma}^*(\boldsymbol{\omega}, y) \coloneqq \left[\frac{\partial x_i(\omega, y)}{\partial \omega_j}\right].$$

Theorem 3.11. *$\boldsymbol{\sigma}^*$ is symmetric and negative semidefinite. In particular, we have $\partial x_i(\omega, y)/\partial\omega_i \leq 0$ for all i, the law of demand.*

Proof. By Shephard's Lemma, $\boldsymbol{\sigma}^*$ is the Hessian of $c(\omega, y)$, which is concave in $\boldsymbol{\omega}$. $\square$

Proposition 3.12. *If the production function is CRS [IRS, DRS], then its average cost function is constant [decreasing, increasing].*

Proof. Let f be a CRS production function. Fix $y > 0$ and $t > 1$. Denote as $\mathcal{P}(y)$ the set of input plans viable for producing y units of output. Note that we have

$$\begin{aligned}\mathcal{P}(ty) = \left\{\mathbf{x} \in \mathbb{R}^n_+ : f(\mathbf{x}) \geq ty\right\} &= \left\{t\mathbf{x} \in \mathbb{R}^n_+ : f(t\mathbf{x}) \geq ty\right\} \\ &= \left\{t\mathbf{x} \in \mathbb{R}^n_+ : f(\mathbf{x}) \geq y\right\} = t\mathcal{P}(y)\end{aligned}$$

where the second to last equality follows from f being CRS. Thus,

$$\mathbf{x}(ty) = \arg\min_{\mathbf{x}\in\mathcal{P}(ty)} \boldsymbol{\omega}\mathbf{x} = t \arg\min_{\mathbf{x}\in\mathcal{P}(y)} \boldsymbol{\omega}\mathbf{x} = t\mathbf{x}(y),$$

from which it is immediate that $c(ty) = tc(y)$. Then,

$$\frac{c(ty)}{ty} = \frac{c(y)}{y}.$$

Since y and $t > 1$ are arbitrary, we know that the average cost is constant. The case of IRS and DRS is completely similar. □

We summarize the properties below:

Proposition 3.13 (Properties of the Cost Function)**.** *If f is continuous and strictly increasing, then $c(\boldsymbol{\omega}, y)$ is*

- *Zero when $y = 0$,*
- *Continuous on its domain,*
- *For all $\omega \gg 0$, strictly increasing and unbounded above in y; thus $\partial x_i(\boldsymbol{\omega}, y)/\partial y \geq 0$, and conditional input demand is never inferior,*
- *Increasing in $\boldsymbol{\omega}$,*
- *Homogeneous of degree one in $\boldsymbol{\omega}$,*
- *Concave in $\boldsymbol{\omega}$.*

Moreover, if f is strictly quasiconcave we have Shephard's lemma: $c(\boldsymbol{\omega}, y)$ is differentiable in $\boldsymbol{\omega}$ at $(\boldsymbol{\omega}_0, y_0)$ whenever $\boldsymbol{\omega}_0 \gg 0$ and

$$\frac{\partial c(\boldsymbol{\omega}_0, y_0)}{\partial \omega_i} = x_i(\boldsymbol{\omega}_0, y_0), \quad i = 1, \ldots, n.$$

Proposition 3.14 (Properties of the Conditional Input Demand Function)**.** *Suppose the production function satisfies Assumption 2.2 and that the associated cost function is twice continuously differentiable. Then*

(i) $\mathbf{x}(\boldsymbol{\omega}, y)$ *is homogeneous of degree zero in $\boldsymbol{\omega}$,*

(ii) The substitution matrix, defined and denoted

$$\boldsymbol{\sigma}^*(\omega, y) \coloneqq \begin{bmatrix} \frac{\partial x_1(\omega,y)}{\partial \omega_1} & \cdots & \frac{\partial x_1(\omega,y)}{\partial \omega_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(\omega,y)}{\partial \omega_1} & \cdots & \frac{\partial x_n(\omega,y)}{\partial \omega_n} \end{bmatrix},$$

is symmetric and negative semidefinite. In particular, the negative semidefiniteness property implies that $\partial x_i / \partial \omega_i \leq 0$ for each i. Recall law of demand in consumer theory.

3.2 Short-run Vs. Long-run Cost Minimization

In the short-run, some inputs can be fixed. We can model it thus as

$$\min_{\tilde{x}} \tilde{\boldsymbol{\omega}}\tilde{\mathbf{x}} + \overline{\boldsymbol{\omega}\mathbf{x}} \quad \text{s.t.} \quad f(\tilde{\mathbf{x}}, \overline{\mathbf{x}}) \geq y,$$

where $\overline{\mathbf{x}}$ is a vector of fixed inputs and $\overline{\boldsymbol{\omega}}$ and $\overline{\tilde{\omega}}$ the corresponding price vectors.

In the short-run, there is more constraint, thus:

Proposition 3.15. *For every $(\boldsymbol{\omega}, y)$, we have*

$$sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}) \geq c(\boldsymbol{\omega}, y),$$

where $\boldsymbol{\omega} = (\tilde{\omega}, \overline{\boldsymbol{\omega}})$ is the vector of all input prices. Moreover, assuming differentiability, we have $sc(\boldsymbol{\omega}, y, \overline{\mathbf{x}}(\boldsymbol{\omega}, y))$ and $c(\boldsymbol{\omega}, y)$ are tangent to teach other at y, where $\overline{\mathbf{x}}(\boldsymbol{\omega}, y)$ is the long-run conditional input demand function.

Proof. Consider the identity

$$sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y)) = c(\boldsymbol{\omega}, y).$$

Differentiating the identity by y, we get

$$\begin{aligned} \frac{\mathrm{d}c(\boldsymbol{\omega}, y)}{\mathrm{d}y} &= \frac{\mathrm{d}sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\mathrm{d}y} \\ &= \frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial y} + \sum_j \frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial \overline{x}_j} \frac{\partial \overline{x}_j}{\partial y} \\ &= \frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial y}, \end{aligned}$$

where the last equality follows from noting that

$$\frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial \overline{x}_j} = 0,$$

also from differentiating the identity. $\square$

Remark 3.16. The long-run cost curve is the lower envelope of the entire family of short-run total cost curves!

4 Profit Maximization

4.1 Direct profit maximizing

We solve

$$\max_{\mathbf{x}} pf(\mathbf{x}) - \boldsymbol{\omega} \cdot \mathbf{x}.$$

The FOC gives $p\partial f(\mathbf{x})/\partial x_i = \omega_i$ for each i, where $p\partial f(\mathbf{x})/\partial x_i$ is called the **marginal revenue product** of input i, or MRP_i.

Alternatively, we can use a two-step approach where we first find the cost function c and maximize

$$\max_{y} py - c(\boldsymbol{\omega}, y).$$

- This is sometimes called the optimal scale problem.
- We have the FOC $p = \partial c(\boldsymbol{\omega}, y)/\partial y$. That is, marginal revenue is equal to marginal cost.
- The SOC requires $\partial^2 c(\boldsymbol{\omega}, y)/\partial y^2 \geq 0$. That is, marginal cost must be increasing at the optimal y.

Proposition 4.1. *Strict concavity of production functions guarantees that the solution to the profit-maximization problem, if it exists, is unique.*

Definition 4.2. Whenever the profit-maximization solution exists, we can define the **profit function** as

$$\pi(p, \boldsymbol{\omega}) := \max_{y,\mathbf{x}} py - \boldsymbol{\omega} \cdot \mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}).$$

If the profit-maximization solutions are unique, we call the firm's optimal output and input choices its **output supply function** $y^*(y, \boldsymbol{\omega})$ and **input demand function** $\mathbf{x}^*(p, \boldsymbol{\omega})$.

Theorem 4.3. *If f is continuous, strictly increasing, and strictly quasi-concave, then, for all $p \geq 0$ and $\boldsymbol{\omega} \geq \mathbf{0}$, the profit function $\pi(p, \boldsymbol{\omega})$, if well-defined, is:*

- *continuous;*
- *increasing in p;*
- *decreasing in $\boldsymbol{\omega}$;*

- *homogeneous of degree one in* $(p, \boldsymbol{\omega})$;
- *convex in* $(p, \boldsymbol{\omega})$;
- *differentiable in* $(p, \boldsymbol{\omega}) \gg 0$.

Theorem 4.4 (Hotelling's Lemma)**.** *We have* $\partial\pi(p, \boldsymbol{\omega})/\partial p = y(p, \boldsymbol{\omega})$ *and* $\partial\pi(p, \boldsymbol{\omega})/\partial\omega_i = -x_i^*(p, \boldsymbol{\omega})$.

Combined with convexity, we have:

Proposition 4.5. *The substitution matrix*

$$\begin{bmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial \omega_1} & \cdots & \frac{\partial y}{\partial \omega_n} \\ -\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial \omega_1} & \cdots & -\frac{\partial x_1}{\partial \omega_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial x_n}{\partial p} & -\frac{\partial x_n}{\partial \omega_1} & \cdots & -\frac{\partial x_n}{\partial \omega_n} \end{bmatrix}$$

is symmetric and positive semidefinite. In particular, $\partial y/\partial p \geq 0$ *and* $\partial x_i/\partial\omega_i \leq 0$. *The input demand cannot be inferior.*

Proposition 4.6. $y^*(p, \boldsymbol{\omega})$ *and* $x_i^*(p, \boldsymbol{\omega})$ *are homogeneous of degree zero.*

From the identity $\mathbf{x}(p, \boldsymbol{\omega}) = \mathbf{x}(\boldsymbol{\omega}, y(p, \boldsymbol{\omega}))$, we obtain a Slutsky equation for the input demand function:

$$\begin{aligned} \frac{\partial x_i(p, \boldsymbol{\omega})}{\partial \omega_j} &= \frac{\partial x_i(\boldsymbol{\omega}, y)}{\partial \omega_j} + \frac{\partial x_i(\boldsymbol{\omega}, y)}{\partial y} \cdot \frac{\partial y(p, \boldsymbol{\omega})}{\partial \omega_j} \\ &= \frac{\partial x_i(\boldsymbol{\omega}, y)}{\partial \omega_j} - \frac{\partial x_i(\boldsymbol{\omega}, y)}{\partial y} \cdot x_j(p, \boldsymbol{\omega}), \end{aligned}$$

where the second term is called the **scale effect**.

- Recall that we have always that $\partial x_i(\boldsymbol{\omega}, y)/\partial y > 0$; thus the scale effect is always negative.
- For $i = j$, recall that $\partial x_i(\boldsymbol{\omega}, y)/\partial\omega_i \leq 0$. Thus input demand always satisfies the law of demand and is never Giffen.

4.2 Marginal and Average Cost

We can define the marginal cost $\mathrm{MC}(y) \coloneqq \partial c(\omega, y)/\partial y$ and the average cost $\mathrm{AC}(y) \coloneqq c(\omega, y)/y$. Note that we have

$$\frac{\mathrm{d}\,\mathrm{AC}(y)}{\mathrm{d}y} = \frac{y \cdot \mathrm{MC}(y) - c}{y^2} = \frac{1}{y}(\mathrm{MC}(y) - \mathrm{AC}(y)).$$

Thus the average cost is increasing [decreasing] if and only if it is strictly lower [higher] than the marginal cost. Moreover, L'Hopital shows that $\lim_{h \to 0^+} \mathrm{MC}(h) = \mathrm{AC}(h)$.

4.3 Firm Behavior

If a firm is allowed to shut-down at no cost, that is, if $f(\mathbf{0}) = 0$, then its profit is bounded below by zero, and the firm will only supply a positive quantity if total revenue exceeds total cost, that is, if $p \geq \mathrm{AC}$. We know thus that the (positive part of the) supply curve is exactly portion of the firm's marginal cost curve that (1) is increasing, and (2) lies above the firm's average cost curve.

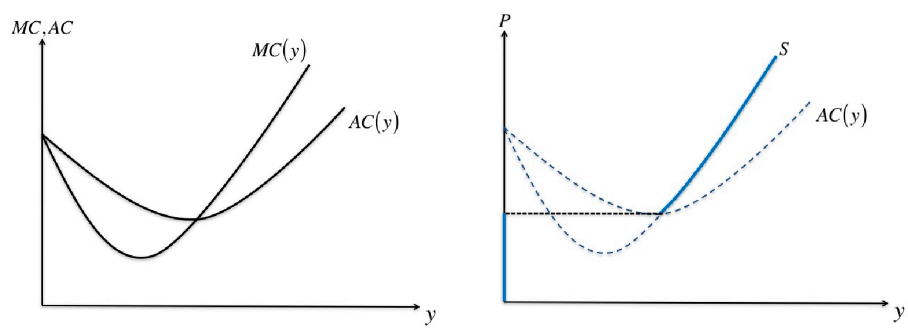

Figure 1: Average Cost, Marginal Cost, Supply

Note also that the supply curve includes part of the p axis when $\mathrm{MC} < \mathrm{AC}$, which corresponds to the occasions when the firm shuts down.

4.3.1 Graphs

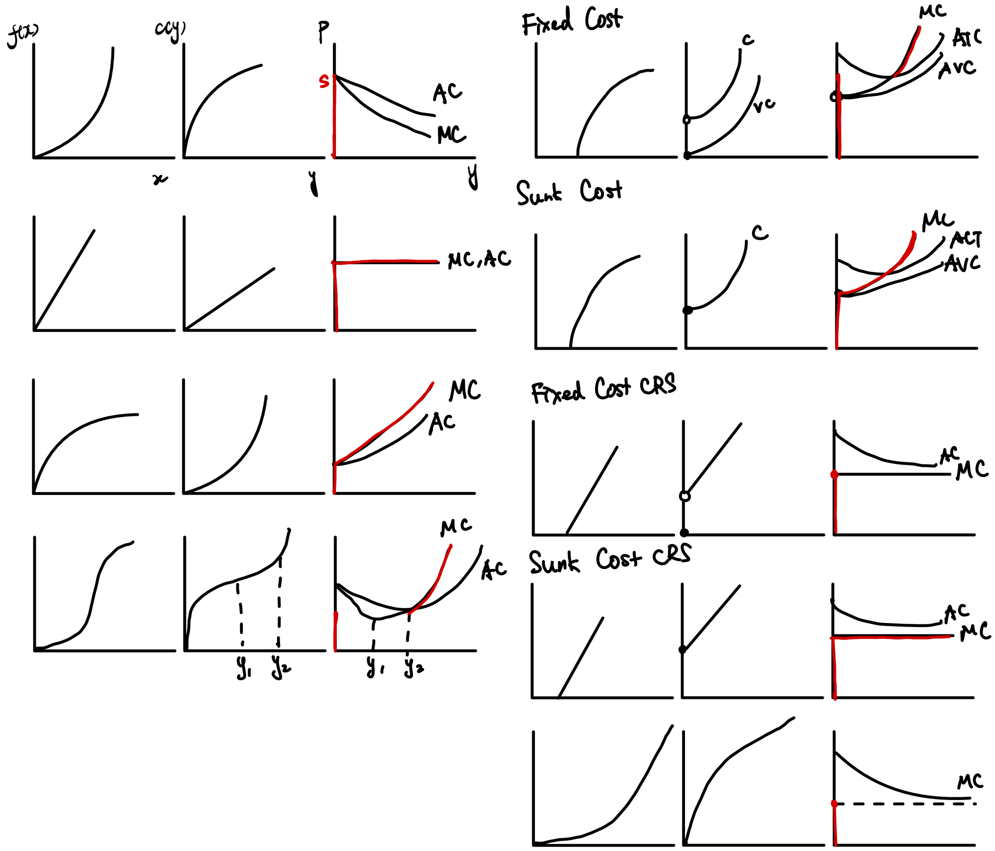

Figure 2: Curves

- AC corresponds to the slope of the secant starting from the origin and MC corresponds to the derivative.
- For the CRS case, note that the supply curve is not defined when $p > \text{MC}$, since the producer can always gain more profit by increasing supply.
- For the fixed cost plots, note that in the cost function is always a minimized value, and thus there is a discontinuity at 0.
- Note that we shutdown if and only if $p \cdot 0 - c(0) \geq py^* - c(y^*)$.
  - Recoverable fixed cost: $c(0) = 0$. Thus shutdown if $py^* - [\kappa + vc(y^*)] \leq 0$, or equivalently, if $p \leq \text{AC}(y^*)$.

– Sunk fixed cost: $c(0) = \kappa$. Thus shutdown if $py^* - [\kappa + vc(y^*)] \leq -\kappa$, or equivalently, if $p \leq \text{AVC}(y^*)$.

5 Partial Equilibrium

We assume that all individuals are price-takers and ignore the interactions across markets and the sources of consumer income, taking the prices of other goods and the incomes as exogenously determined.

The market demand can be derived by adding up the demand of each individual:

$$q^d(p) \coloneqq \sum_{i \in \mathcal{I}} x_i(p, \mathbf{p}, m^i).$$

Similarly, the market supply is the sum of the supply of each firm:

$$q^s(p) \coloneqq \sum_{j \in \mathcal{J}} y_j(p, \omega).$$

Think of these equations as summing up individual demand/supply horizontally, making the total demand and supply curves "flatter."

Definition 5.1. Fixing prices in other markets $(\mathbf{p}, \omega)$, and consumers' income levels m^i, $i = 1, \ldots, I$, the **competitive partial equilibrium** is the price p^* such that

- each consumer i maximizes her utility,
- each producer j maximizes her profit,
- (market clearing) the market clears $q^d(p^*) = q^s(p^*)$.

5.1 Short-run and Long-run Equilibrium

In the short-run, the number of firms operating in the market is fixed. In the long-run, however, free entry and exit means firms have to make zero profit, otherwise new firms would enter/exit the market. We thus impose a **zero-profit condition**.

When there are multiple types of firms (technologies), each type of firm that remains operating in the long run will make zero profit, and remaining types should make negative profit.

5.2 Welfare

The **compensating variation** is the amount of income that satisfies

$$v^i(\mathbf{p}^1, m^i + CV) = v^i(\mathbf{p}^0, m^i).$$

We may also write

$$CV = e(\mathbf{p}^1, u^0) - e(\mathbf{p}^0, u^0) = \int_{p_j^o}^{p_j^1} \frac{\partial}{\partial p_j} e(\mathbf{p}, u^0)\, \mathrm{d}p_j, = \int_{p_j^o}^{p_j^1} x_j^h(\mathbf{p}, u^0)\, \mathrm{d}p_j.$$

We similarly define the **consumer surplus** as

$$CV = \int_{p_j^o}^{p_j^1} x_j^m(\mathbf{p}, u^0)\, \mathrm{d}p_j.$$

6 General Equilibrium

6.1 The Exchange Economy

Suppose there are I consumers $\mathcal{I} = \{1, \ldots, I\}$, the ith of whom is endowed with $\mathbf{e}^i = (e^i_1, \ldots, e^i_n)$. We use superscripts to differentiate individuals and subscripts to differentiate goods. The total endowment of the economy is $\sum \mathbf{e}^i$. An **exchange economy** is defined by a vector $\mathcal{E} := (u^i, \mathbf{e}^i)_i$. We call a consumption profile $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$ an **allocation**. We denote the set of all feasible allocation in this economy by $F(\mathbf{e}) = \{\mathbf{x} : \sum \mathbf{x}^i = \sum \mathbf{e}^i\}$.

It is reasonable to expect the final allocation to satisfy the following:

- **Feasibility**: $\sum \mathbf{x}^i \leq \sum \omega^i$.
- **Participation constraints**: $U^i(\mathbf{x}) \geq U^i(\omega)$.
- Utility maximizing: the indifference curves of any pair of individuals are tangent. If the indifference curves of any two individuals cross, a Pareto dominating allocation can be reached through further exchange.

Definition 6.1. In the Edgeworth box, the **contract curve** is the set of allocations where consumers' indifference curves are tangent to each other.

We have that the contract curve is a subset of Pareto efficient allocations. For a Pareto efficient allocation, the lens shaped area does not intersect the Edgeworth box. Remember the edge cases (e.g. the lens shaped area is outside of the Edgeworth box)!

6.2 Pareto, Blocking, Core

Definition 6.2. An allocation $\mathbf{x}' \neq \mathbf{x}$ is a **Pareto improvement** to $\mathbf{x}$ if

$$u_i(\mathbf{x}'^i) \geq u_i(\mathbf{x}^i)$$

for all i and the inequality is strict for some i.

Definition 6.3. A feasible allocation $\mathbf{x} \in F(\mathbf{x})$ is **Pareto efficient** if there is no feasible Pareto improvement to it.

Definition 6.4. Let any nonempty subset $S \subset \mathcal{I}$ of consumers be a **coalition**. We say that S **blocks** $\mathbf{x} \in F(\mathbf{e})$ if there is an allocation $\mathbf{x}'$ such that:

- $\sum_{i\in S}(\mathbf{x}')^i = \sum_{i\in S}\mathbf{e}^i$; and
- $\mathbf{x}'$ is a Pareto improvement to $\mathbf{x}$ for S: $u^i(\mathbf{x}'^i) \geq u^i(\mathbf{x}^i)$ for all consumers $i \in S$ with at least one such inequality held strict.

Definition 6.5. The **core** of the exchange economy $\mathcal{E} \coloneqq (u^i, \mathbf{e}^i)_i$, denoted $C(\mathcal{E})$, is the set of feasible allocations that are not blocked by any coalition.

Note that participation constraints and Pareto efficiency is satisfied by the core.

7 General Equilibrium of Exchange Economy

We assume a competitive market.

Assumption 7.1. For each consumer $i \in \mathcal{I}$, $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous, strictly quasiconcave, and strongly increasing.

Theorem 7.2. *Under Assumption 7.1, for each* $\mathbf{p} \gg 0$*, the consumer's utility maximization problem has a unique solution. Moreover,* $\mathbf{x}$ *is continuous in* $\mathbf{p}$ *at all* $\mathbf{p} \gg 0$.

Definition 7.3. The **aggregate excess demand** function for good k is defined as

$$z_k(\mathbf{p}) \coloneqq \sum x^i_k - \sum e^i_k.$$

Theorem 7.4. *Under Assumption 7.1, for each* $\mathbf{p} \gg 0$*, we have*

- *Continuity:* $\mathbf{z}$ *is continuous at* $\mathbf{p}$*;*
- *Homogeneity:* $\mathbf{z}(\lambda \mathbf{p}) = \mathbf{z}(\mathbf{p})$ *for all* $\lambda > 0$.
- *Walras' Law:* $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.

Proof *(Walras' Law)***.** From each consumer exhausting their income we have

$$\mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i,$$

from which we have

$$0 = \sum_i \mathbf{p} \cdot (\mathbf{x}^i - \mathbf{e}^i) = \sum_i \sum_k p_k (x^i_k - e^i_k) = \sum_k p_k z_k = \mathbf{p} \cdot \mathbf{z}.$$

□

Remark 7.5.

- If all but one market clears, so does the last.

We say that the markets are in **general equilibrium** if there exists a price vector $\mathbf{p}^*$ that clear all markets simultaneously.

Definition 7.6. $\mathbf{p}^* \in \mathbb{R}^n_+$ is a **Walrasian equilibrium** if $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Theorem 7.7. *If Assumption 7.1 holds and* $\sum \mathbf{e}^i \gg \mathbf{0}$*, then there exists a* $\mathbf{p}^* \gg \mathbf{0}$ *such that* $\mathbf{z}(\mathbf{p}^*) = 0$.

7.1 Equilibrium Allocations

Definition 7.8. We denote as $X^W(\mathbf{p}^*(\mathcal{E}), \mathcal{E})$ the set of equilibrium allocations at price $\mathbf{p}^*$. We write $W(\mathcal{E}) \coloneqq \cup_{\mathbf{p}^*}(\mathbf{p}^*(\mathcal{E}), \mathcal{E})$ for the set of all equilibrium allocations.

Theorem 7.9 (First Welfare Theorem)**.** *If in an exchange economy $\mathcal{E}$, each consumer's utility function is strictly increasing, then $W(\mathcal{E}) \subset C(\mathcal{E})$. The Walrasian equilibrium allocations are in the core, and in particular, are Pareto efficient.*

Theorem 7.10 (Second Welfare Theorem)**.** *Assume for an exchange economy $\mathcal{E} = (u^i, \mathbf{e}^i)_{i\in\mathcal{I}}$ that Assumption 7.1 holds and $\sum \mathbf{e}^i \gg \mathbf{0}$. If $\overline{\mathbf{x}}$ is a Pareto efficient allocation of $\mathcal{E}$, then it is a Walrasian equilibrium allocation with some equilibrium price $\overline{\mathbf{p}}$ of an economy $(u^i, \overline{\mathbf{e}}^i)_{i\in\mathcal{I}}$ such that $\overline{\mathbf{e}}$ is a feasible allocation in $\mathcal{E}$ and $\overline{\mathbf{p}} \cdot \overline{\mathbf{e}}^i = \overline{\mathbf{p}} \cdot \overline{\mathbf{x}}^i$ for each $i \in \mathcal{I}$.*

Remark 7.11. The last equation tells us how to reallocate. Any reallocation that satisfies this equation will lead to the desired Pareto allocation, since the Walrasian equilibrium is completely determined by income.

8 General Equilibrium of Production Economy

Consider an economy where each firm organizes it production according to the production possibility set $Y^j \subset \mathbb{R}^n$. We adopt the convention that inputs in a production plan correspond to negative quantities while outputs correspond to positive quantities. The profit of firm j is then $\mathbf{p} \cdot \mathbf{y}^j$ and its profit maximization problem is

$$\max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j.$$

Assumption 8.1. The production possibility set of firm j, $Y^j \subset \mathbb{R}^n$ satisfies the following conditions:

- $\mathbf{0} \in Y^j \subset \mathbb{R}^n$,
- Y^j is closed and bounded (and thus compact),
- Y^j is **strongly convex**, i.e., for all $\mathbf{y}, \mathbf{y}' \in Y^j$ such that $\mathbf{y} \neq \mathbf{y}'$ and all $\lambda \in (0, 1)$, there exists $\overline{\mathbf{y}} \in Y^j$ such that

  $$\overline{\mathbf{y}} \geq \lambda \mathbf{y} + (1 - \lambda)\mathbf{y}'$$

  and

  $$\overline{\mathbf{y}} \neq \lambda \mathbf{y} + (1 - \lambda)\mathbf{y}'.$$

Definition 8.2. The **aggregate production possibilities** is defined as

$$Y = \left\{ \mathbf{y} | \mathbf{y} = \sum \mathbf{y}^j, \mathbf{y}^j \in Y^j \right\} = \sum Y^j.$$

Proposition 8.3. *If Y^j satisfies Assumption 8.1, then Y also satisfies this assumption.*

Theorem 8.4. *Under Assumption 8.1, then for each price vector $\mathbf{p} \gg 0$, the solution to the firm's profit maximization problem exists and in unique. Moreover, its supply function $\mathbf{y}^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_{++}$ and its profit function $\pi^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_{+}$.*

Theorem 8.5. *For any price $\mathbf{p} \geq 0$, we have*

$$\overline{y} \in \arg\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}$$

if and only if there exists $(\overline{\mathbf{y}}^j)$ such that $\overline{\mathbf{y}} \in Y^j$ for all $j \in \mathcal{J}$, $\sum \overline{\mathbf{y}}^j = \overline{y}$ and

$$\overline{\mathbf{y}}^j \in \arg\max_{\overline{\mathbf{y}}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j.$$

We will continue to assume that there is a set of consumers, $\mathcal{I}$, the members of which maximize their own utilities by buying and selling goods on the markets taking market prices as given. In addition to a bundle of endowment goods, we assume that each consumer i is also endowed with shares of the J firm. To that end, let θ^{ij} denote the share of firm j owned by consumer i. Then, consumer i receives a proportion equal to θ^{ij} of firm j's profit as part of her disposable income. Note that we have $\sum \theta^{ij} = 1$ for each j.

Consumer i's budget constraint becomes

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_j \theta^{ij} \pi^j(\mathbf{p}) = m^i(\mathbf{p}).$$

Note that m^i is continuous in $\mathbf{p}$ if π^j is continuous in $\mathbf{p}$ for all j. We thus have

Theorem 8.6. *If Y^j satisfies Assumption 8.1 for all j, then $m^i(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$. In addition, if u^i satisfies Assumption 7.1, then for all $\mathbf{p} \gg 0$, a solution to consumer i's utility maximization problem exists and is unique. Moreover, the resulting Marshallian demand function $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$ is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$.*

8.1 Walrasian Equilibrium

We can define a production economy by

$$\left(u^i, \mathbf{e}^i, \theta^{ij}, Y^j\right)_{i \in \mathcal{I}, j \in \mathcal{J}}.$$

We define the **aggregate excess demand function** for good k as

$$z_k(\mathbf{p}) = \sum_i x^i_k(\mathbf{p}, m^i(\mathbf{p})) - \sum_j y^j_k(\mathbf{p}) - \sum \mathbf{e}^i_k.$$

Theorem 8.7. *Fix a production economy. If u^i satisfies Assumption 7.1 for all i, Y^j satisfies Assumption 8.1, and there exists a $\mathbf{y} \in Y$ such that $\sum \mathbf{e}^i + \mathbf{y} \gg 0$, then there exists at least one $\mathbf{p}^* \gg 0$, such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.*

Example 8.8 (Robinson Crusoe). Robinson is endowed with a total amount of time T. He is the only producer and consumer in the economy.

As the firm, Robinson can produce coconuts with the following technology:

$$Y = \{(-h, y) | 0 \leq h \leq b, 0 \leq y \leq h^\alpha\},$$

where $\alpha \in (0, 1)$ and $b > T$, an upper bound for h, ensures that Y satisfy Assumption 8.1. We can simply describe the firm's technology by the production function $f(h) = h^\alpha$.

As the consumer, Robinson has the following utility function

$$u(y, l) = y^\beta l^{1-\beta},$$

where $\beta \in (0, 1)$ and $0 \leq l = T - h$ is his leisure time. While the Cobb-Douglas utility function is neither strongly increasing nor strictly quasiconcave at the boundary, it can be showed that all our theorems still hold with Cobb-Douglas utility functions.

An equilibrium of this economy will be given by a price $\mathbf{p}^*$ such that the firm solves

$$\max_h p^* h^\alpha - wh$$

and Robinson the consumer solves

$$\max_{y,l} y^\beta l^{1-\beta} \quad \text{s.t.} \quad p^* y + w^* l \leq w^* T + \pi(p^*, w^*),$$

and both the coconut and the labor markets clear, i.e.,

$$y^i(\mathbf{p}, w^* T + \pi(\mathbf{p})) = y^j(\mathbf{p})$$

and

$$l(\mathbf{p}) = T - h^*(\mathbf{p}).$$

Note that we have

$$z_y = y^i - y^j - e^y, \quad z_l = l - (-h^*) - T.$$

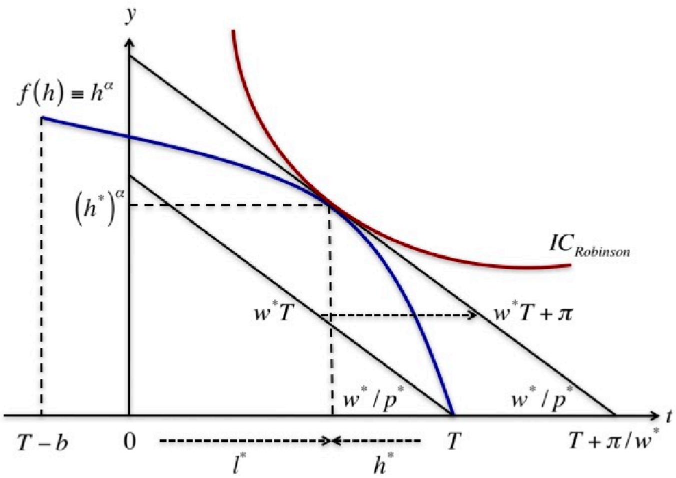

Figure 3: Robinson Crusoe

- The blue line represents the set of possible consumption of the consumer, given access to the firm's technology. It is the production possibility frontier.
- The black line tangent to the blue line is the consumer's new budget curve; it is parallel to the original budge curve. To see this, note first that the firm's revenue in units of time is py/w, where $y = h^{\alpha}$. This is the length of the lower edge of the right triangle in the figure. Thus the firm's profit is the length between T and $T + \pi/w$ in the figure. Since the firm maximizes its profit, we have that the new budget line is tangent to the line of possible consumption (the blue line).
- Prices change until equilibrium is reached, and the red indifference curve and the blue curve is tangent to the budget line at the same point.

Definition 8.9. Fix a production economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i \in \mathcal{I}, j \in \mathcal{J}}$. An allocation

$$(\mathbf{x}, \mathbf{y}) \coloneqq \Big((\mathbf{x}^1, \ldots, \mathbf{x}^I), (\mathbf{y}^1, \ldots, \mathbf{y}^J)\Big)$$

is **feasible** if for all i and j we have

$$\mathbf{x}^i \in \mathbb{R}^n_+, \quad \mathbf{y}^j \in Y^j, \quad \sum \mathbf{x}^i = \sum \mathbf{e}^i + \sum \mathbf{y}^j.$$

Definition 8.10. A feasible allocation $(\mathbf{x}, \mathbf{y})$ is **Pareto efficient** if there is no feasible Pareto improvement to it. That is, there is no feasible allocation $(\mathbf{x}', \mathbf{y}')$ such that

$$u^i(\mathbf{x}'^i) \geq u^i(\mathbf{x}^i)$$

for all $i \in \mathcal{I}$ and the inequality is strict for some i.

Theorem 8.11 (First Welfare Theorem in a Production Economy)**.** *If in a production economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$, each consumer's utility function is strictly increasing and $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ is a Walrasian equilibrium allocation for some Walrasian equilibrium $\mathbf{p}^*$, then it is a Pareto efficient allocation.*

Theorem 8.12 (Second Welfare Theorem in a Production Economy)**.** *Suppose in a production economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$, each consumer's utility function satisfied Assumption 7.1, Y_j satisfies Assumption 8.1, and $\mathbf{y} \sum \mathbf{e}^i \gg 0$ for some aggregate production vector $\mathbf{y} \in Y$. Then for each Pareto efficient allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, there exists income transfer $T_1, \ldots, T_I \in \mathbb{R}$ satisfying $\sum T_i = 0$ and a price vector $\overline{\mathbf{p}}$ such that*

- *$\hat{\mathbf{x}}_i$ maximizes $u^i(\mathbf{x}^i)$ subject to $\overline{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\overline{\mathbf{p}}) + T_i$ for all i, and*
- *$\overline{\mathbf{y}}^j$ maximizes $\overline{\mathbf{p}} \cdot \mathbf{y}^j$ such that $\mathbf{y}^j \in Y^j$ for all j.*

9 Strategic Games: Fundamentals

We are interested in models that describe strategic interactions, where

- we have multiple decision makers, and
- the outcome of each decision maker's choice depends on the choices of others.

A strategic game models a strategic situation where players make their choices simultaneously and independently. Three defining elements:

- a set of players;
- for each player, a set of actions;
- for each play, their preference / payoff function, defined over profiles of actions.

Notation:

- We typically use $I = \{1, \ldots, n\}$ to denote the set of players, where a generic member of I is denoted by i.
- For player $i \in I$, the set of actions is denoted as A_il a generic action of i is denoted a_i.
- An action profile is a vector $a = (a_1, \ldots, a_n) \in \times_i A_i$ where $a_i \in A_i$ for all i.
- We use $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ to denote an action profile of i's opponents. We also write $a = (a_i, a_{-i})$.
- We denote Player i's payoff function as $u_i : \times_{j \in I} A_j \to \mathbb{R}$.

P1 \ P2	Quiet	Fink
Quiet	2, 2	0, 3
Fink	3, 0	1, 1

Example 9.1. Player 1 chooses the row, and Player 2 chooses the column.

Example 9.2. Strictly competitive games (i.e., zero-sum games): $u_1 = -u_2$.

9.1 Solution Concepts

We assume rationality.

Definition 9.3. An action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a **Nash equilibrium** if there is no **profitable unilateral deviation** for any player i.

- A unilateral deviation is where a player changes her own action while her opponents' actions remain the same.
- A profitable unilateral deviation is a deviation that strictly incenses the player's payoff.
- Thus, a Nash equilibrium is where, fixing her opponents' actions, no player can strictly increase her payoff by only changing her own action.

Definition 9.4. An action profile $a^* \in A$ is a **Nash equilibrium** if for each $i \in I$, $u_i(a_i^*, a_{-1}^*) \geq u_i(a_i, a_{-i}^*)$ for all $a_i \in A_i$.

Example 9.5. (Fink, Fink) is a Nash equilibrium.

Appendix A

The Cobb-Douglas production function

$$f(\mathbf{x}) = Ax_1^{\alpha} x_2^{1-\alpha}$$

has conditional input demand functions

$$x_1 = \frac{y}{A} \left(\frac{\omega_2}{\omega_1} \frac{\alpha}{1-\alpha} \right)^{1-\alpha}, \quad x_2 = \frac{y}{A} \left(\frac{\omega_1}{\omega_2} \frac{1-\alpha}{\alpha} \right)^{\alpha}$$