ECON20110 (W25): The Elements of Economic Analysis II Honors

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1 On Mathematics

1.1 Constrained Maximization

E.g.,

$$\max_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad \text{s.t.} \quad G(\mathbf{x}, \boldsymbol{\theta}) \geq 0.$$

Solving a whole class of optimization problems parameterized by $\tilde{\theta}$ generates two functions:

- The solution function
- The Value function

Results like the envelope theorem relates these two functions.

1.2 The Kuhn-Tucker Theorem

Consider the maximization function $\max_x f(x)$. The first order condition gives $f'(x^*) = 0$. Now suppose that x_1 is such that $f'(x_1) > 0$. We may be temped to argue that x_1 is not a solution since we can increase f by increasing the value of x, but this assumes that x is in the interior of the domain. Thus the first order condition considers only interior solutions. The Kuhn-Tucker theorem addresses this issue.

Theorem 1.1 (Kuhn-Tucker)**.** *The FOCs for the constrained minimization problem*

$$\max_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad s.t. \quad G(\mathbf{x}, \boldsymbol{\theta}) \geq 0.$$

are:

- *for each i, $\partial\mathcal{L}/\partial x_i \leq 0$ and $x_i \geq 0$, with complementary slackness; That is, at most one of the two conditions can be a strict inequality.*
- *$\partial\mathcal{L}/\partial\lambda \geq 0$ and $\lambda \geq 0$, with complementary slackness.*

Remark 1.2.

- For the direction of the inequalities on $\partial\mathcal{L}/\partial x_i \geq 0$ and $\partial\mathcal{L}/\partial\lambda \geq 0$, remember the picture. In minimization problems they are flipped.
- Often, we can rule out many of these cases. For example, when u is strictly increasing, we have that $\lambda > 0$; and $\lim_{x_1 \to 0} \partial u/\partial x_i = \infty$ gives $x_i > 0$.

- Negative sign in front so that we have a positive parameter.
- Think of λ as a penalty of not satisfying the constraint (note that we need negative penalty for maximization problems, and positive penalty for maximization problems).

1.3 Elasticity of Substitution

Elasticities are of the form

$$-\frac{\mathrm{d}\log y}{\mathrm{d}\log x} = -\frac{\mathrm{d}y/y}{\mathrm{d}x/x}.$$

- Elasticities gives the proportion response of x as y changes proportionately.
- Knowing the elasticities gives information on how the product xy changes as y changes. For example, if $\sigma > 1$, then xy decreases as y increases.

The elasticity of substitution captures how the (optimal) relative consumption level between two goods responds to changes of the corresponding price ratio:

$$\sigma_{ij} = -\frac{\mathrm{d}\log(x_i^*/x_j^*)}{\mathrm{d}\log(p_i/p_j)} = \frac{\mathrm{d}\log(x_j/x_i)}{\mathrm{d}\log(MU_i/MU_j)}.$$

Remark 1.3.

- We think of relative prices as exogenous. The last formula is often used as the definition because it can be computed straight from definition.
- If $\sigma_{ij} > 1$, then relative expenditure $(p_i x_i)/(p_j x_j)$ decreases as p_i/p_j increases, etc.
- Larger values of σ_{ij} means it is "easier to substitute i for j".

1.4 Sets and Mapping

Notation 1.4. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

Definition 1.5. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we write:

- $\mathbf{x} \geq \mathbf{y}$ if $\forall i : x_i \geq y_i$;

- $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

- $\mathbf{x} \gg \mathbf{y}$ (read strongly greater than) if $\forall i : x_i > y_i$;

Definition 1.6. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **strictly increasing** if $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{x} \gg \mathbf{y}$. It is **strongly increasing** if $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{x} > \mathbf{y}$.

Example 1.7. Strongly increasing implies strictly increasing.

- Cobb-Douglas is strongly increasing when $\mathbf{x} \gg \mathbf{0}$ but is only strictly increasing when $x_i = 0$ for some i.

- The linear production function $f(\mathbf{x}) = \sum x_i$ is strongly increasing.

- The Leontief production function $f(\mathbf{x}) = \min x_i$ is strictly increasing but not strongly increasing.

Definition 1.8. The $N \times N$ matrix M is **negative semidefinite** (NSD) if

$$\forall z \in \mathbb{R}^N : z \cdot Mz \leq 0$$

and **positive semidefinite** (PSD) if

$$\forall z \in \mathbb{R}^N : z \cdot Mz \geq 0.$$

If the inequality is strict for all $z \neq 0$, then M is **negative definite** (ND) (resp., **positive definite** (PD)).

Proposition 1.9.

(i) *M is PSD (PD) $\iff$ $-M$ is NSD (ND).*

(ii) *M is ND (PD) $\iff$ M is NSD (PSD), but the converse is not true.*

(iii) *M is ND (PD) $\iff$ M' is ND (PD).*

(iv) *M is ND (PD) $\iff$ M^{-1} is ND (PD).*

Proof. The first three statements are immediate. For the last, note that

$$z'Mz = (z'Mz)' = z'M'z = z'MM^{-1}M'z = (M'z)'M^{-1}M'z.$$

□

1.5 Concave and Convex Functions

Notation 1.10. Let $\mathbf{x}^1, \mathbf{x}^2 \in X$ and $t \in [0,1]$. We often denote $\mathbf{x}^t = t\mathbf{x}^1 + (1-t)\mathbf{x}^2$.

Definition 1.11. A function $f : X \to \mathbb{R}$ is convex (resp., strictly convex) if epi f is convex (resp., strictly convex). The function f is concave (resp., strictly concave) if $-f$ is concave (resp., strictly concave).

Proposition 1.12. *A function $f : X \to \mathbb{R}$ is convex if and only if for all $x_1, \ldots, x_k \in X$ and $\alpha_1, \ldots, \alpha_n$ such that $\sum \alpha_i = 1$, we have $f\left(\sum \alpha_i x_i\right) \le \sum \alpha_i f(x_i)$.*

We may think of α_i as probability masses. The following result generalizes this to probability densities:

Proposition 1.13 (Jensen's Inequality)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable, and X is a random variable such that* $\mathrm{E}[X]$ *and $f(\mathrm{E}\,X)$ exist, then $f(\mathrm{E}\,X) \le \mathrm{E}\,f(X)$.*

Proof. From convexity of f we know $f(x) \ge f(y) + f'(y)(x-y)$ for any x and y. Setting $y = \mathrm{E}\,X$ gives

$$f(X) \ge f(\mathrm{E}\,X) + f'(\mathrm{E}\,X)(X - \mu), \quad \forall x.$$

Taking expectation on both sides gives the desired result. $\square$

Proposition 1.14. *The C^1 function $f : X \to \mathbb{R}$ is convex if and only if*

$$f(x+t) \ge f(x) + \nabla f(x) \cdot t$$

for all $x \in X$ and $t \in \mathbb{R}^N$ such that $x + t \in X$.

Proof. Suppose f is convex. For any $\alpha \ne 0$,

$$f\big(\alpha(x+t) + (1-\alpha)x\big) \le \alpha f(x+t) + (1-\alpha) f(x),$$

giving

$$f(x + \alpha t) - f(x) \le \alpha\big(f(x+t) - f(x)\big)$$

and then

$$f(x) + \frac{f(x+\alpha t) - f(x)}{\alpha} \le f(x+t).$$

Taking $\alpha \to 0$ gives the desired result.

For the reverse direction, consider arbitrary $x, y \in X$ and $\lambda \in [0,1]$. Write $z = \lambda x + (1-\lambda)y$ By assumption we have

$$f(x) \geq f(z) + \nabla f(z)(1-\lambda)(x-y),$$
$$f(y) \geq f(z) + \nabla f(z)\lambda(y-x),$$

together giving

$$\lambda f(x) + (1-\lambda) f(y) \geq f(z).$$

□

Proposition 1.15. *The C^2 function $f : X \to \mathbb{R}$ is convex if and only if $\mathrm{D}^2 f(x)$ is PSD for every $x \in X$.*

Proof. Suppose f is convex. Fix some $x \in X$. For any $t \neq 0$, the second-order Taylor expansion is

$$f(x + \alpha t) = f(x) + \nabla f(x) \cdot (\alpha t) + \frac{\alpha^2}{2} t \cdot \mathrm{D}^2 f(x + \beta t) t.$$

By Proposition 1.14,

$$\frac{\alpha^2}{2} t \cdot \mathrm{D}^2 f(x + \beta t) t \geq 0.$$

And conversely. □

1.6 Quasi-Concavity and Quasi-Convexity

Definition 1.16. Let $X \subset \mathbb{R}^n$ be convex. A function $f : X \to \mathbb{R}$ is **quasi-concave** if for all $\mathbf{x}^1, \mathbf{x}^2 \in X$ and $t \in [0,1]$, we have $f(\mathbf{x}^t) \geq \min\{f(\mathbf{x}^1), f(\mathbf{x}^2)\}$. The function f is **strictly quasi-concave** if the inequality is strict for all $\mathbf{x}^1 \neq \mathbf{x}^2$.

Remark 1.17.

- Thus (strictly) concave functions are (strictly) quasi-concave. The converse is not true; consider $x \mapsto x^2$, $x > 0$.
- Quasi-concavity is a ordinal property that is preserved by monotone transformations, by the following result:

Proposition 1.18. *Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. If $f : X \to \mathbb{R}$ is quasi-concave, so is $g \circ f$. If, in addition, g is strictly increasing, then $g \circ f$ is strictly quasi-concave for all f that is strictly quasi-concave.*

Definition 1.19. For $f : X \to \mathbb{R}$ and $\mathbf{x}^0 \in X$, the **level set** relative to $f(\mathbf{x}^0)$ is the set $L(\mathbf{x}^0) := \{\mathbf{x} \in X : f(\mathbf{x}) = f(\mathbf{x}^0)\}$; the **superior set** (or the upper contour set) is the set $S(\mathbf{x}^0) := \{\mathbf{x} \in X : f(\mathbf{x}) \geq f(\mathbf{x}^0)\}$; the **inferior set** (or the lower contour set) is the set $I(\mathbf{x}) := \{\mathbf{x} \in X : f(\mathbf{x}) \leq f(\mathbf{x}^0)\}$.

The following results is more or less immediate:

Proposition 1.20. *The function $f : X \to \mathbb{R}$ is quasi-concave if and only if for all $\mathbf{x}^0 \in X$, $S(\mathbf{x}^0)$ is convex.*

Proposition 1.21. *If $f : X \to \mathbb{R}$ is (strictly) quasi-concave then $-f$ is (strictly) quasi-convex.*

Just like convexity, quasi-convexity of a function can be related to its Hessian, using the following results:

Lemma 1.22. *The C^1 function $f : X \to \mathbb{R}$ is quasi-convex if and only if for each $x, y \in X$ such that $f(y) \geq f(x)$ we have*

$$\nabla f(x) \cdot (y - x) \geq 0.$$

Proof. Similar to Proposition 1.14. □

Proposition 1.23. *The C^2 function $f : X \to \mathbb{R}$ is quasiconvex if and only if for each $x \in X$, the Hessian matrix $\mathrm{D}^2 f(x)$ is PSD in the subspace $\{x \in \mathbb{R}^N : \nabla f(x) \cdot y = 0\}$.*

Proof. Similar to Proposition 1.15. □

1.7 Homogeneous Functions

Definition 1.24. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **homogeneous of degree k** if for all $\mathbf{x} \in \mathbb{R}^n$ and all $\lambda > 0$ we have

$$f(\lambda \mathbf{x}) = \lambda^k f(\mathbf{x}).$$

Proposition 1.25. *If $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k and differentiable, its first order partial derivatives are homogeneous of degree $k - 1$. Thus the slops of the isoquants at $\mathbf{x}$ is always the same at $\lambda\mathbf{x}$ for any $\lambda > 0$.*

Proof. Differentiating both sides, we get

$$\frac{\partial f}{\partial x_i}(\lambda \mathbf{x}) = \lambda^{k-1} \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

□

2 Production Technology

"Firm" simply refers to actors on the supply side. They transform resources (inputs) into goods and services (outputs), while constrained by the production technology. The only difference between firms and consumers is the problems they are solving: the former solves a profit maximization problem, the latter a utility maximization problem.

We identify the firm's input choices with members of $X \subset \mathbb{R}_+^m$ and output choices with members of $Y \subset \mathbb{R}_+^n$. We can describe a firm's technology by specifying its **production possibility set**, $F \subset X \times Y$, each member of which is called a production plan. In this course we assume $n = 1$. The upper contour of the production possibility set is called the production possibility frontier, which can be described by a production function. Most of the times there is no loss of generality in considering only the production function. Think when.

Definition 2.1. Let $F \subset \mathbb{R}_+^m \times \mathbb{R}_+$ be a production possibility set. The **production function** $f : \mathbb{R}_+^m \to \mathbb{R}_+$ is defined by

$$f(\mathbf{x}) \coloneqq \sup\{y \in \mathbb{R}_+ : (\mathbf{x}, y) \in F\}.$$

2.1 Placing Structure on the Production Function

Assumption 2.2. We typically assume that the production function $f : \mathbb{R}_+^m \to \mathbb{R}_+$ is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}_+^m$ and $f(\mathbf{0}) = 0$.

- Strict quasiconcavity gives strictly convex upper contours. It can be though of as there being some complementarities in the inputs.
- These assumptions guarantees that the firm's production optimization (cost minimization) problem is well-defined and has a unique solution.

Definition 2.3. The **martial product** of input i at input vector $\mathbf{x}$ is

$$\mathrm{MP}_i(\mathbf{x}) \coloneqq \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

The **marginal rate of technical substitution (MRTS)** between inputs i and j is

$$MRTS_{ij}(\mathbf{x}) \coloneqq \frac{\mathrm{MP}_i(\mathbf{x})}{\mathrm{MP}_j(\mathbf{x})}.$$

For $y \geq 0$, the y-level **isoquant** of $f : \mathbb{R}_+^m \to \mathbb{R}_+$ is

$$Q(y) \coloneqq \{\mathbf{x} \in \mathbb{R}_+^m : f(\mathbf{x}) = y\}.$$

Remark 2.4. Recall the notions of marginal utility, marginal rate of substitution, and indifference curves in consumer theory.

Proposition 2.5. *Under Assumption 2.2 and when $n = 2$,*

(i) The slope of isoquant (at a point $\mathbf{x}$) is given by the MRTS.

(ii) Isoquant are always downward sloping.

(iii) We have diminishing $MRTS$.

Proof. (i) Clear.

(ii) From f being strictly increasing, we know $f_i > 0$.

(iii) From f being strictly quasiconcave, isoquants bend towards the origin.

□

2.2 Return to Scale

Definition 2.6. We say the production function f exhibits (globally)

- **constant return to scale** if $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $x \in \mathbb{R}^n_+$ and all $t > 0$.
- **increasing return to scale** if $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $x \in \mathbb{R}^n_+$ and all $t > 1$.
- **decreasing return to scale** if $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $x \in \mathbb{R}^n_+$ and all $t > 1$.

Example 2.7. The Cobb-Douglas production function $f(\mathbf{x}) = A \prod x_i^{\alpha_i}$ with $\sum \alpha_i = 1$ is homogeneous of degree one and thus exhibits constant return to scale.

Example 2.8. A firm with increasing return to scale will enjoying decreasing average cost. The reverse is not always true.

Definition 2.9. For a production function $f(\mathbf{x})$, the **elasticity of Substitution between inputs i and j** (at point $\mathbf{x}$) is defined as

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) &\coloneqq -\frac{\mathrm{d}\log(x_i/x_j)}{\mathrm{d}\log(\mathrm{MP}_i/\mathrm{MP}_j)} \\ &= \frac{\mathrm{d}\log(x_j/x_i)}{\mathrm{d}\log(\mathrm{MP}_i/\mathrm{MP}_j)}.\end{aligned}$$

Remark 2.10. A larger σ_{ij} means it is easier to substitute i for j. To see this, consider the CES production function

$$f(x_1, x_2) = \left(\alpha x_1^{\frac{\sigma-1}{\sigma}} + (1-\alpha)x_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

As $\sigma \to \infty$, $f(\mathbf{x}) \to \alpha x_1 + (1-\alpha)x_2$; as $\sigma \to 0$, $f(\mathbf{x}) \to \min\{x_1, x_2\}$.

Example 2.11. Cobb-Douglas has constant and unit elasticity of substitution, this directly gives constant expenditure share. In general, if $u(\mathbf{x}) = x_1^\alpha x_2^\beta$ and $\alpha, \beta > 0$, then

$$p_1 x_1 = \frac{\alpha m}{\alpha+\beta}, \quad p_2 x_2 = \frac{\beta m}{\alpha+\beta}$$

and thus

$$x_1^* = \frac{\alpha}{\alpha+\beta}\frac{m}{p_1}, \quad x_2^* = \frac{\beta}{\alpha+\beta}\frac{m}{p_2}.$$

3 Cost Minimization

We assume both the product market and the factor (input) markets are perfectly competitive. In doing so, we are ignoring any influence of any player in the market on the prices. The firm thus solves the following profit-maximizing problem:

$$\max_{y,\mathbf{x}} py - \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}),$$

where $\boldsymbol{\omega}$ contains the prices of the inputs.

We may rewrite it as a two part problem:

$$\max_{y} \max_{\mathbf{x}} py - \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}),$$

which is equivalent to

$$\max_{y} py - \min_{\mathbf{x}} \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}).$$

Note the resemblance with the notion of backward induction. We consider first the cost minimization problem, that is:

$$\min_{\mathbf{x}} \boldsymbol{\omega}\mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y.$$

The Lagrangian is

$$\mathcal{L} = \boldsymbol{\omega}\mathbf{x} + \lambda(y - f(\mathbf{x})).$$

The Kuhn-Tucker FOCs are

- for $i = 1, \ldots, m$: $\partial\mathcal{L}/\partial x_i = \omega_i - \lambda\partial f(\mathbf{x})/\partial x_i \geq 0$ and $x_i \geq 0$, with C.S.;
- $\partial\mathcal{L}/\partial\lambda = y - f(\mathbf{x}) \leq 0$ and $\lambda \geq 0$, with C.S.

Assuming interior solution and $f(\mathbf{x}) = y$, the FOCs reduce to

$$\boldsymbol{\omega} = \lambda\mathbf{MP}, \quad f(\mathbf{x}) = y$$

We have thus for all i, j that

$$\frac{\omega_i}{\omega_j} = \frac{\mathrm{MP}_i}{\mathrm{MP}_j},$$

that is, the MRTS between i and j equals their price ratio.

Definition 3.1. The **conditional input demand function** is

$$\mathbf{x}(\boldsymbol{\omega}, y) \equiv \operatorname*{arg\,min}_{x \in f^{-1}(\{y\})} \boldsymbol{\omega} \cdot \mathbf{x},$$

and the **cost function** is the minimized value function.

Remark 3.2.

- The continuity of the production function guarantees the existence of a solution to the cost-minimization problem (when $\omega \gg 0$).

- Strict quasi-concavity of the production function guarantees that the solution to the cost-minimization problem is unique. The conditional input demand function is thus well defined.

Definition 3.3. We define the **marginal cost of production** as $\mathrm{MC}(\boldsymbol{\omega}, y) := \partial c(\boldsymbol{\omega}, y)/\partial y$ and the **average cost** as $\mathrm{AC}(\boldsymbol{\omega}, y) := \partial c(\boldsymbol{\omega}, y)/\partial y$.

3.1 Comparative Statics

Proposition 3.4. *The conditional input demand functions and thus the cost functions are homogeneous of degree zero in $\boldsymbol{\omega}$.*

Theorem 3.5. *If f is continuous and strictly increasing and $\boldsymbol{\omega} \gg 0$, then $c(\boldsymbol{\omega}, y)$ is strictly increasing in y.*

Proof. Assume $c(\boldsymbol{\omega}, y_1) \geq c(\boldsymbol{\omega}, y_2)$ with $y_1 < y_2$. Then we can also produce y_1 with the input that is optimal for producing y_2, which gives a contradiction. □

Proof *(Assuming differentiability)***.** By the Envelope theorem, $\mathrm{MC} = \partial c/\partial y = \partial \mathcal{L}/\partial y = \lambda$. If f is strictly increasing, then an increase in output can only be achieved with increases in inputs. With $f(\mathbf{x}) = 0$, we thus have $x_i > 0$ if $y > 0$. Thus from the FOCs, we have $\omega_i = \lambda\,\mathrm{MP}_i$, from which we get and $\mathrm{MP}_i > 0$ and $\lambda > 0$. □

Remark 3.6. Thus, when there is no free input, marginal cost of production is always positive.

Theorem 3.7 (Shephard's Lemma)**.** *If f is strictly quasiconcave, then:*

$$\frac{\partial c(\boldsymbol{\omega}, y)}{\partial \omega_i} = x_i(\boldsymbol{\omega}, y).$$

Proof. Strict quasiconcavity gives continuity of c. The results follows from Envelope theorem. $\square$

Corollary 3.8. *The cost function $c(\omega, y)$ is (weakly) increasing in $\boldsymbol{\omega}$.*

Theorem 3.9. *$c(\boldsymbol{\omega}, y)$ is increasing and concave in $\boldsymbol{\omega}$.*

Proof. Note that $c(t\omega, y)$ is bounded above by the linear function $tc(\boldsymbol{\omega}, y)$, since one can always choose the original bundle. Alternatively, fix y and $t \in (0, 1)$. Let

$$\mathbf{x}^1 = \mathbf{x}(\boldsymbol{\omega}^1, y), \quad \mathbf{x}^2 = \mathbf{x}(\boldsymbol{\omega}^2, y), \quad \mathbf{x}^t = \mathbf{x}\big(t\boldsymbol{\omega}^1 + (1-t)\boldsymbol{\omega}^2, y\big).$$

By definition we have

$$\boldsymbol{\omega}^t\mathbf{x}^1 \le \boldsymbol{\omega}^t\mathbf{x}^t, \quad \boldsymbol{\omega}^t\mathbf{x}^t \le \boldsymbol{\omega}^t\mathbf{x}^t,$$

which gives

$$t\boldsymbol{\omega}^t\mathbf{x}^1 + (1-t)\boldsymbol{\omega}\mathbf{x}^2 \le \boldsymbol{\omega}^t\mathbf{x}^t.$$

Therefore we have

$$c(\boldsymbol{\omega}, y) = \boldsymbol{\omega}^t\mathbf{x}^t \ge tc(\boldsymbol{\omega}^1, y) + (1-t)c(\boldsymbol{\omega}^2, y).$$

Since $\mathbf{x}^1$, $\mathbf{x}^2$, and $t \in (0, 1)$ are arbitrary, we have shown that c is convex. $\square$

Definition 3.10. We define the substitution matrix

$$\boldsymbol{\sigma}^*(\boldsymbol{\omega}, y) := \left[\frac{\partial x_i(\omega, y)}{\partial \omega_j}\right].$$

Theorem 3.11. *$\boldsymbol{\sigma}^*$ is symmetric and negative semidefinite. In particular, we have $\partial x_i(\omega, y)/\partial\omega_i \le 0$ for all i, the law of demand.*

Proof. By Shephard's Lemma, $\boldsymbol{\sigma}^*$ is the Hessian of $c(\omega, y)$, which is concave in $\boldsymbol{\omega}$. $\square$

Proposition 3.12. *If the production function is CRS [IRS, DRS], then its average cost function is constant [decreasing, increasing].*

Proof. Let f be a CRS production function. Fix $y > 0$ and $t > 1$. Denote as $\mathcal{P}(y)$ the set of input plans viable for producing y units of output. Note that we have

$$\begin{aligned}\mathcal{P}(ty) = \big\{\mathbf{x} \in \mathbb{R}^n_+ : f(\mathbf{x}) \ge ty\big\} &= \big\{t\mathbf{x} \in \mathbb{R}^n_+ : f(t\mathbf{x}) \ge ty\big\} \\ &= \big\{t\mathbf{x} \in \mathbb{R}^n_+ : f(\mathbf{x}) \ge y\big\} = t\mathcal{P}(y)\end{aligned}$$

where the second to last equality follows from f being CRS. Thus,

$$\mathbf{x}(ty) = \arg\min_{\mathbf{x}\in\mathcal{P}(ty)} \boldsymbol{\omega}\mathbf{x} = t \arg\min_{\mathbf{x}\in\mathcal{P}(y)} \boldsymbol{\omega}\mathbf{x} = t\mathbf{x}(y),$$

from which it is immediate that $c(ty) = tc(y)$. Then,

$$\frac{c(ty)}{ty} = \frac{c(y)}{y}.$$

Since y and $t > 1$ are arbitrary, we know that the average cost is constant. The case of IRS and DRS is completely similar. □

We summarize the properties below:

Proposition 3.13 (Properties of the Cost Function)**.** *If f is continuous and strictly increasing, then $c(\omega, y)$ is*

- *Zero when $y = 0$,*
- *Continuous on its domain,*
- *For all $\omega \gg 0$, strictly increasing and unbounded above in y,*
- *Increasing in $\boldsymbol{\omega}$,*
- *Homogeneous of degree one in $\boldsymbol{\omega}$,*
- *Concave in $\boldsymbol{\omega}$.*

Moreover, if f is strictly quasiconcave we have Shephard's lemma: $c(\boldsymbol{\omega}, y)$ is differentiable in $\boldsymbol{\omega}$ at (ω_0, y_0) whenever $\boldsymbol{\omega}_0 \gg 0$ and

$$\frac{\partial c(\boldsymbol{\omega}_0, y_0)}{\partial \omega_i} = x_i(\boldsymbol{\omega}_0, y_0), \quad i = 1, \ldots, n.$$

Proposition 3.14 (Properties of the Cost Function)**.** *Suppose the production function satisfies Assumption 2.2 and that the associated cost function is twice continuously differentiable. Then*

(i) $\mathbf{x}(\boldsymbol{\omega}, y)$ *is homogeneous of degree zero in* $\boldsymbol{\omega}$*,*

(ii) The substitution matrix, defined and denoted

$$\boldsymbol{\sigma}^*(\omega, y) \coloneqq \begin{bmatrix} \frac{\partial x_1(\omega,y)}{\partial \omega_1} & \cdots & \frac{\partial x_1(\omega,y)}{\partial \omega_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(\omega,y)}{\partial \omega_1} & \cdots & \frac{\partial x_n(\omega,y)}{\partial \omega_n} \end{bmatrix},$$

is symmetric and negative semidefinite. In particular, then negative semidefiniteness property implies that $\partial x_i / \partial \omega_i \leq 0$ for each i. Recall law of demand in consumer theory.

3.2 Short-run Vs. Long-run Cost Minimization

In the short-run, some inputs can be fixed. We can model it thus as

$$\min_{\tilde{x}} \tilde{\boldsymbol{\omega}}\tilde{\mathbf{x}} + \overline{\boldsymbol{\omega}\mathbf{x}} \quad \text{s.t.} \quad f(\tilde{\mathbf{x}}, \overline{\mathbf{x}}) \geq y,$$

where $\overline{\mathbf{x}}$ is a vector of fixed inputs and $\overline{\boldsymbol{\omega}}$ and $\overline{\tilde{\omega}}$ the corresponding price vectors.

In the short-run, there is more constraint, thus:

Proposition 3.15. *For every $(\boldsymbol{\omega}, y)$, we have*

$$sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}) \geq c(\boldsymbol{\omega}, y),$$

where $\boldsymbol{\omega} = (\tilde{\omega}, \overline{\boldsymbol{\omega}})$ is the vector of all input prices. Moreover, assuming differentiability, we have $sc(\boldsymbol{\omega}, y, \overline{x}(\boldsymbol{\omega}, y))$ and $c(\boldsymbol{\omega}, y)$ are tangent to teach other at y, where $\overline{\mathbf{x}}(\boldsymbol{\omega}, y)$ is the long-run conditional input demand function.

Proof. Consider the identity

$$sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y)) = c(\boldsymbol{\omega}, y).$$

Differentiating the identity by y, we get

$$\begin{aligned}
\frac{\mathrm{d}c(\boldsymbol{\omega}, y)}{\mathrm{d}y} &= \frac{\mathrm{d}sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\mathrm{d}y} \\
&= \frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial y} + \sum_j \frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial \overline{x}_j} \frac{\partial \overline{x}_j}{\partial y} \\
&= \frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial y},
\end{aligned}$$

where the last equality follows from noting that

$$\frac{\partial sc(\boldsymbol{\omega}, y; \overline{\mathbf{x}}(\boldsymbol{\omega}, y))}{\partial \overline{x}_j} = 0,$$

also from differentiating the identity. $\square$

Remark 3.16. The long-run cost curve is the lower envelope of the entire family of short-run total cost curves!

4 Profit Maximization

4.1 Direct profit maximizing

We solve

$$\max_{\mathbf{x}} pf(\mathbf{x}) - \boldsymbol{\omega} \cdot \mathbf{x}.$$

The FOC gives $p\partial f(\mathbf{x})/\partial x_i = \omega_i$ for each i, where $p\partial f(\mathbf{x})/\partial x_i$ is called the **marginal revenue product** of input i, or MRP_i.

Alternatively, we can use a two-step approach where we first find the cost function c and maximize

$$\max_{y} py - c(\boldsymbol{\omega}, y).$$

- This is sometimes called the optimal scale problem.
- We have the FOC $p = \partial c(\boldsymbol{\omega}, y)/\partial y$. That is, marginal revenue is equal to marginal cost.
- The SOC requires $\partial^2 c(\boldsymbol{\omega}, y)/\partial y^2 \geq 0$. That is, marginal cost must be increasing at the optimal y.

Proposition 4.1. *Strict concavity of production functions guarantee that the solution to the profit-maximization problem, if it exists, is unique.*

Definition 4.2. Whenever the profit-maximization solution exists, we can define the **profit function** as

$$\pi(p, \boldsymbol{\omega}) \coloneqq \max_{y,\mathbf{x}} py - \boldsymbol{\omega} \cdot \mathbf{x} \quad \text{s.t.} \quad y = f(\mathbf{x}).$$

If the profit-maximization solutions are unique, we call the firm's optimal output and input choices its **output supply function** $y^*(y, \boldsymbol{\omega})$ and **input demand function** $\mathbf{x}^*(p, \omega)$.

Theorem 4.3. *If f is continuous, strictly increasing, and strictly quasi-concave, then, for all $p \geq 0$ and $\boldsymbol{\omega} \geq \mathbf{0}$, the profit function $\pi(p, \boldsymbol{\omega})$, if well-defined, is:*

- *continuous;*
- *increasing in p;*
- *decreasing in $\boldsymbol{\omega}$;*

- *homogeneous of degree one in* $(p, \boldsymbol{\omega})$*;*
- *convex in* $(p, \boldsymbol{\omega})$*;*
- *differentiable in* $(p, \boldsymbol{\omega}) \gg 0$.

Theorem 4.4 (Hotelling's Lemma)**.** *We have* $\partial \pi(p, \boldsymbol{\omega})/\partial p = y(p, \boldsymbol{\omega})$ *and* $\partial \pi(p, \boldsymbol{\omega})/\partial \omega_i = -x_i^*(p, \boldsymbol{\omega})$.

Combined with convexity, we have:

Proposition 4.5. *The substitution matrix*

$$\begin{bmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial \omega_1} & \cdots & \frac{\partial y}{\partial \omega_n} \\ -\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial \omega_1} & \cdots & -\frac{\partial x_1}{\partial \omega_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial x_n}{\partial p} & -\frac{\partial x_n}{\partial \omega_1} & \cdots & -\frac{\partial x_n}{\partial \omega_n} \end{bmatrix}$$

is symmetric and positive semidefinite. In particular, $\partial y/\partial p \geq 0$ *and* $\partial x_i/\partial \omega_i \leq 0$. *The input demand cannot be Giffen.*

Proposition 4.6. $y^*(p, \boldsymbol{\omega})$ *and* $x_i^*(p, \boldsymbol{\omega})$ *are homogeneous of degree zero.*

4.2 Marginal and Average Cost

We can define the marginal cost $\mathrm{MC}(y) \coloneqq \partial c(\boldsymbol{\omega}, y)/\partial y$ and the average cost $\mathrm{AC}(y) \coloneqq c(\boldsymbol{\omega}, y)/y$. Note that we have

$$\frac{\mathrm{d}\,\mathrm{AC}(y)}{\mathrm{d}y} = \frac{y \cdot \mathrm{MC}(y) - c}{y} = \frac{1}{y}(\mathrm{MC}(y) - \mathrm{AC}(y)).$$

Thus the average cost is increasing [decreasing] if and only if it is strictly lower [higher] than the marginal cost. Moreover, L'Hopital shows that $\mathrm{MC}(0) = \mathrm{AC}(0)$.

4.3 Firm Behavior

If a firm is allowed to shut-down at no cost, that is, if $f(\mathbf{0}) = 0$, then its profit is bounded below by zero, and the firm will only supply a positive quantity if total revenue exceeds total cost, that is, if $p \geq \mathrm{AC}$. We know thus that the (positive part of the) supply curve is exactly portion of the firm's marginal cost curve that (1) is increasing, and (2) lies above the firm's average cost curve.

Appendix A

The Cobb-Douglas production function

$$f(\mathbf{x}) = Ax_1^{\alpha}x_2^{1-\alpha}$$

has conditional input demand functions

$$x_1 = \frac{y}{A}\left(\frac{\omega_2}{\omega_1}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}, \quad x_2 = \frac{y}{A}\left(\frac{\omega_1}{\omega_2}\frac{1-\alpha}{\alpha}\right)^{\alpha}$$