ECON20110 (W25): The Elements of Economic Analysis II Honors

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1 Review

1.1 Constrained Maximization

E.g.,

$$\max_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad \text{s.t.} \quad G(\mathbf{x}, \boldsymbol{\theta}) \geq 0.$$

Solving a whole class of optimization problems parameterized by $\tilde{\theta}$ generates two functions:

- The solution function
- The Value function

Results like the envelope theorem relates these two functions.

1.2 The Kuhn-Tucker Theorem

Consider the maximization function $\max_x f(x)$. The first order condition gives $f'(x^*) = 0$. Now suppose that x_1 is such that $f'(x_1) > 0$. We may be temped to argue that x_1 is not a solution since we can increase f by increasing the value of x, but this assumes that x is in the interior of the domain. Thus the first order condition considers only interior solutions. The Kuhn-Tucker theorem addresses this issue.

Theorem 1.1 (Kuhn-Tucker)**.** *The FOCs for the constrained optimization problem*

$$\max_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad \textit{s.t.} \quad G(\mathbf{x}, \boldsymbol{\theta}) \geq 0.$$

are:

- *for each i, $\partial\mathcal{L}/\partial x_i \leq 0$ and $x_i \geq 0$, with complementary slackness; That is, at most one of the two conditions can be a strict inequality.*
- *$\partial\mathcal{L}/\partial\lambda \geq 0$ and $\lambda \geq 0$, with complementary slackness.*

Remark 1.2.

- For the direction of the inequalities on $\partial\mathcal{L}/\partial x_i \geq 0$ and $\partial\mathcal{L}/\partial\lambda \geq 0$, remember the picture. In minimization problems they are flipped.
- Often, we can rule out many of these cases. For example, when u is strictly increasing, we have that $\lambda > 0$; and $\lim_{x_1 \to 0} \partial u/\partial x_i = \infty$ gives $x_i > 0$.
- Negative sign in front so that we have a positive parameter.

1.3 Elasticity of Substitution

Remark 1.3.

$$-\frac{\mathrm{d}\log y}{\mathrm{d}\log x} = -\frac{\mathrm{d}y/y}{\mathrm{d}x/x}.$$

- Elasticities gives the proportion response of x as y changes proportionately.
- Knowing the elasticities gives information on how the product xy changes as y changes. For example, if $\sigma > 1$, then xy decreases as y increases.

The elasticity of substitution captures how the (optimal) relative consumption level between two goods responds to changes of the corresponding price ratio:

$$\sigma_{ij} = -\frac{\mathrm{d}\log(x_i^*/x_j^*)}{\mathrm{d}\log(p_i/p_j)} = \frac{\mathrm{d}\log(x_j/x_i)}{\mathrm{d}\log(MU_i/MU_j)}.$$

Remark 1.4.

- We think of relative prices as exogenous. The last formula is often used as the definition because it can be computed straight from definition.
- If $\sigma_{ij} > 1$, then relative expenditure $(p_i x_i)/(p_j x_j)$ decreases as p_i/p_j increases, etc.
- Larger values of σ_{ij} means it is "easier to substitute i for j".

2 Production Technology

"Firm" simply refers to actors on the supply side. They transform resources (inputs) into goods and services (outputs), while constrained by the production technology. The only difference between firms and consumers is the problems they are solving: the former solves a profit maximization problem, the latter a utility maximization problem.

Notation 2.1. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

We identify the firm's input choices with members of $X \subset \mathbb{R}^m_+$ and output choices with members of $Y \subset \mathbb{R}^n_+$. We can describe a firm's technology by specifying its **production possibility set**, $F \subset X \times Y$, each member of which is called a production plan. In this course we assume $n = 1$. The upper contour of the production possibility set is called the production possibility frontier, which can be described by a production function. Most of the times there is no loss of generality in considering only the production function. Think when.

Definition 2.2. Let $F \subset \mathbb{R}^m_+ \times \mathbb{R}_+$ be a production possibility set. The **production function** $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ is defined by

$$f(\mathbf{x}) \coloneqq \sup\{y \in \mathbb{R}_+ : (\mathbf{x}, y) \in F\}.$$

2.1 Placing Structure on the Production Function

Definition 2.3. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we write:

- $\mathbf{x} \geq \mathbf{y}$ if $\forall i : x_i \geq y_i$;
- $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.
- $\mathbf{x} \gg \mathbf{y}$ (read strongly greater than) if $\forall i : x_i > y_i$;

Definition 2.4. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **strictly increasing** if $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{x} \gg \mathbf{y}$. It is **strongly increasing** if $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{x} > \mathbf{y}$.

Assumption 2.5. We typically assume that the production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^m_+$ and $f(\mathbf{0}) = 0$.

This assumes that the firm's production optimization (cost minimization) problem is well-defined and has a unique solution.

Definition 2.6. The **martial product** of input i at input vector $\mathbf{x}$ is

$$MP_i(\mathbf{x}) \coloneqq \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

The **marginal rate of technical substitution (MRTS)** between inputs i and j is

$$MRTS_{ij}(\mathbf{x}) \coloneqq \frac{MP_i(\mathbf{x})}{MP_j(\mathbf{x})}.$$

For $y \geq 0$, the y-level **isoquant** of $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ is

$$Q(y) \coloneqq \{\mathbf{x} \in \mathbb{R}^m_+ : f(\mathbf{x}) = y\}.$$

Remark 2.7. Recall the notions of marginal utility, marginal rate of substitution, and indifference curves in consumer theory.

Proposition 2.8. *Under Assumption 2.5 and when $n = 2$,*

(i) The slope of isoquant (at a point $\mathbf{x}$) is given by the MRTS.

(ii) Isoquant are always downward sloping.

(iii) We have diminishing $MRTS$.

Proof. (i) Clear.

(ii) From f being strictly increasing, we know $f_i > 0$.

(iii) From f being strictly quasiconcave, isoquants bend towards the origin. more detail?

□

2.2 Return to Scale

Definition 2.9. We say the production function f exhibits (globally)

- **constant return to scale** if $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $x \in \mathbb{R}^n_+$ and all $t > 0$.
- **increasing return to scale** if $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $x \in \mathbb{R}^n_+$ and all $t > 1$.
- **decreasing return to scale** if $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $x \in \mathbb{R}^n_+$ and all $t > 1$.

Example 2.10. The Cobb-Douglas production function $f(\mathbf{x}) = A \prod x_i^{\alpha_i}$ with $\sum \alpha_i = 1$ is homogeneous of degree one and thus exhibits constant return to scale.

Definition 2.11. For a production function $f(\mathbf{x})$, the **elasticity of Substitution between inputs i and j** (at point $\mathbf{x}$) is defined as

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) &\coloneqq -\frac{\mathrm{d}\log(x_i/x_j)}{\mathrm{d}\log(MP_i/MP_j)} \\ &= \frac{\mathrm{d}\log(x_j/x_i)}{\mathrm{d}\log(MP_i/MP_j)}.\end{aligned}$$

Remark 2.12. A larger σ_{ij} means it is easier to substitute i for j. To see this, consider the CES production function

$$f(x_1, x_2) = \left(\alpha x_1^{\frac{\sigma-1}{\sigma}} + (1-\alpha) x_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

As $\sigma \to \infty$, $f(\mathbf{x}) \to \alpha x_1 + (1-\alpha)x_2$; as $\sigma \to 0$, $f(\mathbf{x}) \to \min\{x_1, x_2\}$.