# ECON20010 NOTES

ADEN CHEN

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## 1. The Envelope Theorem

**Theorem 1.1.** *Consider the constraint maximization problem*

$$v(\alpha) \coloneqq \max_x f(x;\alpha) \quad s.t. \quad g(x;\alpha) = 0.$$

*The associated Lagrangian is*

$$\mathcal{L}(x,\lambda,\alpha) = f(x;\alpha) + \lambda g(x;\alpha).$$

*We have that*

$$\frac{\partial v(\alpha)}{\partial \alpha_k} = \frac{\partial \mathcal{L}(x^*,\lambda^*,\alpha)}{\partial \alpha_k}.$$

**Proof.** Note that $v(\alpha) \equiv \mathcal{L}(x^*,\lambda^*,\alpha)$ and thus

$$\frac{\partial v}{\partial \alpha_k} = \frac{\partial \mathcal{L}}{\partial x}\frac{\partial x^*}{\partial \alpha_k} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda^*}{\partial \alpha_k} + \frac{\partial \mathcal{L}}{\partial \alpha_k},$$

where the first two terms are both 0 by the first order conditions. □

*Remark* 1.2.

- By the implicit function theorem, we need $f_{xx} \neq 0$.
- Think "all indirect effects vanish" at the optimum. That is, if we think $v(\alpha) = f(x^*;\alpha)$, then

$$\frac{\partial f}{\partial x}\frac{\partial x^*}{\partial \alpha_k} = 0.$$

*Example* 1.3. Consider the value function

$$\begin{aligned} v(p_x,p_y,m) = U(x^*,y^*) &= U(x^*,y^*) + \lambda^*[m - p_x x - p_y y] \\ &= \mathcal{L}(x^*,y^*,\lambda^*;p_x,p_y,m). \end{aligned}$$

By the envelope theorem,

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

Similarly,

$$\frac{\partial v}{\partial p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

## 2. Scarcity: the Budget Constraint

**Definition 2.1.**

- The **budget set** consists of all feasible consumption bundles.
- The **budget constraint** exactly exhausts the consumer's income.

### 2.1. Budget Set.
The relative price:

$$\frac{p_x}{p_y}$$

- Mnemonic: fractions of this form ($p_x/p_y, U_x/U_y$) is always the price of $x$ in units of $y$.

To stay on the budget constraint,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p_x}{p_y}.$$

- Think "the rate at which the market *allows* the consumer to exchange good $x$ for good $y$."
- Think the opportunity cost of good $x$.

### 2.2. Preference.
Basic axioms:

- **Completeness**. For any pair of consumption of bundles, say $c_1$ and $c_2$, either $c_1 \succeq c_2$, $c_2 \succeq c_1$, or both.
  - Requires an answer and assumes no framing effects.
- **Transitivity**. If $c_1 \succeq c_2$ and $c_2 \succeq c_3$, then $c_1 \succeq c_3$.
  - When transitivity fails, we have a "money pump."

A preference ordering is **rational** if it satisfies completeness and transitivity. They are the minimal requirement for the existence of a utility function representation. We also typically assume the following:

- **Continuity**. If $c_1 \succ c_2$ then there are neighborhoods $N_1$ and $N_2$ around $c_1$ and $c_2$ such that

  $$x \succ y, \quad \forall x \in N_1, \quad y \in N_2.$$

  This implies that if $c_1 \succ c_2$ then there exists $c_3$ such that

  $$c_1 \succ c_3 \succ c_2.$$

- **Monotonicity**.
  - **Monotone**. If $c_1 \gg c_2$[1] then $c_1 \succ c_2$.
  - **Strongly monotone**. If $c_1 \geq c_2$[2] and $c_1 \neq c_2$ then $c_1 \succ c_2$.
  - **Local non-satiation**. If for every bundle $c$ and every $\epsilon > 0$, there exists $x \in N_\epsilon(c)$ such that $x \succ c$.

[1] We write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i, \forall i$.

[2] We write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i, \forall i$.

- **Convexity**. If $c_1 \succeq c_2$, then
$$\theta c_1 + (1-\theta)c_2 \succeq c_2, \quad \forall \theta \in (0,1).$$
If convexity is satisfied, the **upper contour set**, the "at least as good as" set, is convex.

Additional axioms place even more structures on the utility function:

- **Homotheticity**. If $c_1 \succeq c_2$, then
$$tc_1 \succeq tc_2, \quad \forall t > 0.$$
- **Quasilinearity** in good $i$. If $\mathbf{c_1} \succeq \mathbf{c_2}$, then
$$\mathbf{c_1} + t\mathbf{e}_i \succeq \mathbf{c_2} + t\mathbf{e}_i, \quad \forall t > 0.$$

2.3. **Translating preference ordering to the utility function:**

**Theorem 2.2** (Utility Representation Theorem)**.** *If a preference ordering is rational, then it admits a utility function representation. Moreover, the utility function is unique up to a monotonically increasing transformation.*

*Remark* 2.3. The additional assumption of monotonicity, though not required, allows for a very simple proof: simply send each consumption bundle to the size of the unique bundle on $t\sum \mathbf{e}$ equivalent to the given bundle.

**Proposition 2.4.** *If a preference ordering satisfies convexity, then the corresponding utility function representation will be quasi-concave. The indifference curves (level sets) will have non-increasing marginal rate of substitution (slopes).*

**Proposition 2.5.** *A preference ordering is homothetic if and only if its utility representation has MRS that is homogeneous of degree* 1.

2.4. **The Marginal Rate of Substitution.** The MRS
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y}$$
is the quantity of $y$ the consumer is willing to sacrifice in exchange for an additional unit of $x$. (Think $p_x/p_y$.) It measures an individual's **willingness to pay** for $x$ in terms of $y$.

## 3. Utility Maximization

The problem:

$$v(p_x, p_y, m) \coloneqq \max_{x,y} U(x, y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

### 3.1. **Interpretation.**

We want to maximize

$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$

such that

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$

This gives

$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x.$$

We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.

  $$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \; \mathrm{d}x$$

  "Take advantage of all trading opportunities."
- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that $U_x/p_y$ is marginal utility of money *spent on* $x$.

  $$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \; \mathrm{d}x$$

  "Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that $U_x$ is the marginal benefit of buying $x$ and $U_y \cdot p_x/p_y$ is the marginal cost of buying $x$.

  $$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$

  "Trade until marginal cost equals marginal benefit."

In the last expression, if we write

$$\lambda = \frac{U_y}{p_y},$$

(think marginal utility of income) we have that at optimum,

$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$

$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$

$$p_x x + p_y y = m.$$

These three equalities describe precisely the critical points of the following

$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right],$$

called the **Lagrangian**. That is, setting

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

recovers the above three equations.

*Remark* 3.1.

- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- $\lambda$ might be negative or zero. Think bliss point.

### 3.2. The Indirect Utility Function.

**Proposition 3.2.**

$$\frac{\partial v}{\partial m} = \lambda^*.$$

**Proof.** Noting

$$v = U(x^*, y^*) + \lambda^*[m - p_x x^* - p_y y^*] = \mathcal{L}^*,$$

we have

$$\begin{aligned}
\frac{\partial v}{\partial m} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} \\
&= U_x^* \frac{\partial x}{\partial m} + U_y^* \frac{\partial y}{\partial m} + \lambda^* \left[1 - p_x \frac{\partial x}{\partial m} - p_y \frac{\partial y}{\partial m}\right] + \frac{\partial \lambda}{\partial m}\left[m - p_x x^* - p_y y^*\right] \\
&= \left(U_x^* - \lambda^* p_x\right) \frac{\partial x}{\partial m} + \left(U_y^* - \lambda^* p_y\right) \frac{\partial y}{\partial m} + \frac{\partial \lambda^*}{\partial m}\left(m - p_x x^* - p_y y^*\right) + \lambda^* \\
&= \lambda^*.
\end{aligned}$$

The last equality follows by noting that at the optimum,

$$U_x^* - \lambda^* p_x = U_y^* - \lambda^* p_y = m - p_x x^* - p_y y^* = 0.$$

Alternatively, one may use the envelope theorem:

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

□

Note that

$$\frac{\partial v}{\partial m} = \lambda^* = \frac{U_x^*}{p_x} = \frac{U_x^*}{p_x}.$$

So when the individual is not satiated ($U_x, U_y \neq 0$), marginal utility of income is positive. When the budget constraint does not require binding, the marginal utility of income is generally nonnegative.

Again using the Envelope Theorem, we have

$$\frac{\partial v}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

This value is generally nonpositive, and only zero when one does not consume good $x$ or when the marginal utility of $x$ is 0.

## 4. Expenditure Minimization

The problem:

$$e(p_x, p_y, \overline{U}) \coloneqq \max_{x,y} p_x x + p_y y \quad \text{s.t.} \quad U(x, y) = \overline{U}.$$

The Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \eta \left[\overline{U} - U(x, y)\right]$$

$$\begin{aligned} &[x] & p_x &= \eta^* U_x(x^*, y^*) \\ &[y] & p_y &= \eta^* U_y(x^*, y^*) \\ &[\eta] & \overline{U} &= U(x^*, y^*). \end{aligned}$$

*Properties* 4.1. Properties of Hicksian demand functions:

(i) Homogeneous of degree 0 in prices:

$$x_i^h(\alpha \mathbf{p}, U) = x_i^h(\mathbf{p}, U).$$

Differentiating with respect to $\alpha$ gives

$$\sum \frac{\partial x_i^h}{\partial p_j} p_j = 0 \implies \sum \epsilon_{ij}^h = 0.$$

(ii) Cross-price effects on Hicksian demand are symmetric:

$$\frac{\partial x_i^h}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial x_j^h}{\partial p_i}.$$

From this we have

$$p_i x_i \frac{p_j}{x_i} \frac{\partial x_i^h}{\partial p_j} = p_j x_j \frac{p_i}{x_j} \frac{\partial x_j^h}{\partial p_i}.$$

That is,

$$s_i \epsilon_{ij}^h = s_j \epsilon_{ji}^h \implies \frac{\epsilon_{ij}^h}{\epsilon_{ji}^h} = \frac{s_j}{s_i}.$$

The more important good impacts the less important good more.

(iii) Differentiating $U(\mathbf{x}^h(\mathbf{p}, U)) = U$ with respect to $p_i$ gives

$$\sum \frac{\partial U}{\partial x_j} \frac{\partial x_j^h}{\partial p_i} = 0 \implies \sum p_j \frac{\partial x_j^h}{\partial p_i} = 0 \implies \sum s_j \epsilon_{ji}^h = 0.$$

*Remark* 4.2. Symmetry and homogeneity [adding up] gives adding up [homogeneity]. In case where there are two goods only, the latter two also gives symmetry.

**Proposition 4.3.** *Properties of the expenditure function:*

- *Homogeneous of degree* 1 *in prices.*
- *Non-decreasing in prices.* $\partial e/\partial p_i = x_i^h \geq 0$.
- *Increasing in utility.* $\partial e/\partial U = \eta^* > 0$.
- *Concave in prices.*

$$\frac{\partial^2 e}{\partial p_i^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0,$$

*where the last inequality follows from the law of demand. Alternatively, note that the price of the original bundle, which grows linearly, is an upper bound of the expenditure function.*

## 5. Changes in Behavior

Consider a price increase from $p_x^o$ to $p_x^f$. Let $o$ be the original consumption, $f$ be the final consumption, and $s$ be the optimal consumption after an income transfer such that the individual stays on the same indifference curve as before (has the same purchasing power). We may decompose $x^f - x^o$:

$$x^f - x^o = x^f - x^s + x^s - x^o.$$

- $x^f - x^o$: the Marshallian price effect (the total effect).
- $x^f - x^s$: the effect due to compensation (the income effect).
- $x^s - x^o$: the Hicksian price effect (substitution effect).

The Slutsky equation is an continuous analogue of this decomposition.

### 5.1. The Slutsky Equation.

Recall from duality that

$$x^h(p_x, p_y, \overline{U}) = x^m(p_x, p_y, m = e(p_x, p_y, \overline{U})).$$

As price changes, changes in $e(p_x, p_y, \overline{U})$ ensures that purchasing power does not change.

By differentiating, we get the **Slutsky equation**:

**Proposition 5.1.**

$$\frac{\partial x^h}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot \frac{\partial e}{\partial p_x}.$$

We may rewrite the Slutsky equation using the envelop theorem as

$$\frac{\partial x^h}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot x^m.$$

This shows that we can recover the unobservable $\partial x^h / \partial p_x$ from the observables.

We can also rewrite the Slutsky equation as

$$\frac{\partial x^m}{\partial p_x} = \frac{\partial x^h}{\partial p_x} + \left(-\frac{\partial x^m}{\partial m} \cdot x^m\right),$$

where

- $\partial x^h / \partial p_x$ is the substitution effect,
- $\partial x^m / \partial m \cdot x^m$ is the income effect.

### 5.2. Compensation.

- **Slutsky transfer** keeps the original bundle affordable.

$$T_S = \Delta p_x \cdot x^o.$$

- **Hicks transfer** keeps the original utility level affordable.
  $$T_H = e(p_x^f, p_y, v^o) - m = e(p_x^f, p_y, v^o) - e(p_x^o, p_y, v^o).$$
  In the Slutsky equation, the term
  $$\partial e/\partial p_x = x^m = x^h$$
  is the continuous analogue of the Hicks transfer.
- **Frisch transfer**: the transfer that restores the purchasing power by making the price of utility $\lambda$ constant.

*Remark* 5.2. Note that $T_S \geq T_H$.

5.3. **The Law of Demand.** The substitution effect follows the law of demand:
$$\partial x^h/\partial p_x \leq 0.$$
More generally, we have:

**Proposition 5.3** (Generalized law of demand)**.**
$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) \leq 0.$$

**Proof.** Note that
$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) = \left(\mathbf{x}^1\mathbf{p}^1 - \mathbf{x}^0\mathbf{p}^1\right) + \left(\mathbf{x}^0\mathbf{p}^0 - \mathbf{x}^1\mathbf{p}^0\right).$$
The last two terms are both non-positive. □

*Remark* 5.4.

- Note that this gives $\partial x_i^h/\partial p_i \leq 0$ (if the derivative exists).
- But think also graphs for the case of two goods.
- Note that the law of demand holds not only when indifference curves are concave. Remember the following two examples as well:
  - When indifference curves are concave, the expenditure minimizing points occur at the edges.
  - In the perfect complement case, we have that $\partial x^h/\partial p_x = 0$.

5.4. **Giffen Goods.** Marshallian demand does not always comply with the law of demand. A good whose Marshallian demand does not comply with the law of demand is called a **giffen good**. Their existence is theoretically possible, but not empirically supported.

Looking back at the Slutsky equation, we see that for a good $x$ to be a giffen good, we need the following three conditions:

(i) the individual buys a large amount of $x$,
(ii) good $x$ is inferior,
(iii) the demand for good $x$ is elastic.

These three conditions do not occur together often: narrowly defined categories usually has 0 elasticity, but broad categories are usually normal goods.

### 5.5. Normal & Inferior Goods.

**Definition 5.5.** Good $x$ is said to be **normal** if $\partial x^m / \partial m > 0$ and **inferior** if $\partial x^m / \partial m < 0$.

*Remark* 5.6. We can equivalently define it using Hicksian demands. From

$$\frac{\partial x^h}{\partial U} \cdot \frac{\partial v}{\partial m} = \frac{\partial x^m}{\partial m}$$

we know that $\partial x^h / \partial U$ and $\partial x^m / \partial m$ have the same sign. Note that $\partial v / \partial m = \lambda^* > 0$ is the utility of a dollar.

*Remark* 5.7. We say good $i$ is

- **normal** if $\eta_i > 0$,
- **inferior** if $\eta_i < 0$,
- **necessity** if $\eta_i < 1$,
- **luxury** if $\eta_i > 1$.

### 5.6. An experiment for testing normality.

We fix the consumption of $x$ and vary income $m$. For normal goods, the willingness to pay for $x$ increases as income increase. Thus $x$ is normal if

$$\frac{\partial (U_x / U_y)}{\partial y} > 0.$$

*Example* 5.8. With the quasilinear utility function $U(x, y) = v(x) + y$, the good $x$ is neither normal nor inferior. The willingness to pay

$$\frac{U_x}{U_y} = \frac{v'(x)}{1}$$

does not change as we vary $y$ (by varying income).

### 5.7. Cross Effects.

**Definition 5.9.** We say $y$ is

- a **substitute** of $x$ if $\partial y / \partial p_x > 0$,
- a **complement** of $x$ if $\partial y / \partial p_x < 0$,
- **unrelated** with $x$ if $\partial y / \partial p_x = 0$.

If we use $y = y^h$ in the definition above, we say **gross** substitutes/complements; if we use $y^m$, we say **net** substitutes/complements.[3]

*Remark* 5.10.

[3] Think Hicksian demand "nets out" the income effect.

- Cross price effects for Hicksian demands are symmetric:
$$\frac{\partial x^h}{\partial p_y} = \frac{\partial^2 e}{\partial p_x \partial p_y} = \frac{\partial y^h}{\partial p_x}.$$
This does not hold in general for Marshallian demands; see the cross-price Slutsky equation.
- For any good $x$, at least one other good is a net substitute with $x$. If not, as price of $x$ increase, consumption and thus utility level strictly decreases (note that consumption of $x$ also decreases by the law of demand). In particular, when there are only two goods, the two goods cannot be net complements.

**Proposition 5.11.** *We have the **cross-price Slutsky equation**:*
$$\frac{\partial y^h}{\partial p_x} = \frac{\partial y^m}{\partial p_x} + \frac{\partial y^m}{\partial m} \cdot \frac{\partial e}{\partial p_x}$$
*where*
$$\frac{\partial e}{\partial p_x} = x^m = x^h.$$

## 6. Elasticities and Aggregation

**Proposition 6.1.** *The Slutsky equation in elasticity form:*

$$\epsilon_{ij}^m = \epsilon_{ij}^h - \eta_i s_j.$$

From this we have the following:

**Proposition 6.2** (Symmetry of Marshallian Demands)**.** *We have*

$$s_i \epsilon_{ij}^m = s_j \epsilon_{ji}^m + s_i s_j (\eta_j - \eta_i).$$

*Symmetry holds when two goods have equal income elasticities.*

**Proposition 6.3.**

- *Engel aggregation:*
$$\sum s_i \eta_i = 1.$$
- *Cournot aggregation:*
$$\sum_i \epsilon_{ij} s_i = -s_j.$$
- *Implication of homogeneity:*
$$\sum_i \epsilon_{ji} = -\eta_j.$$

**Proof.**

- From $m = \sum p_i x_i$ we have
$$1 = \frac{\partial m}{\partial m} = \sum p_i \frac{\partial x_i}{\partial m} = \sum \eta_i s_i.$$
- Differentiating the same identity with respect to $p_j$ gives
$$0 = \frac{\partial m}{\partial p_x} = x_j + \sum \frac{\partial x_i}{\partial p_j} p_i = x_j + \sum \epsilon_{ij} \frac{p_i x_i}{p_j}.$$
- Differentiating the identity $x^m(\mathbf{x}, m) = x^m(t\mathbf{x}, m)$ with respect to $t$ gives
$$0 = \sum \frac{\partial x_j}{\partial p_i} p_i + \frac{\partial x_j}{\partial m} m.$$

□

*Remark* 6.4.

- From Engel aggregation: some goods must be normal. It will never the case that all goods are inferior, all goods are necessities, or all goods are luxuries.
- The budget share of good $j$ is small if it has many substitutes, and large if it has many complements, or when their respective budget shares are large.

- From implication of homogeneity: Normal goods have many substitutes; inferior goods have many complements; strongly inferior goods have strong complements. “No good can be a Giffen good ($\epsilon_{ii} > 0$) unless it is strong complements with other goods.”

## 7. Welfare

**7.1. Exact Measures of Welfare Change.** The difference in utility

$$\Delta v = v^f - v^o.$$

- Gets direction right, but magnitude depends on the specific utility representation chosen — it is not invariant to the utility representation.
- Difficulty in the interpretation of units.

**Definition 7.1.**

- **Compensating variation**: the income transfer that induces the consumer accept the change in price voluntarily.
- **Equivalent variation**: the income transfer that induces the consumer to reject the change in price voluntarily.

**7.2. The Compensating Variation.**

**Proposition 7.2.** *There holds*

$$\begin{aligned} CV := -T_H &= -\left[e(\mathbf{p}^f, v^o) - m\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^f, v^f)\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^o, v^o)\right]. \end{aligned}$$

*Remark* 7.3. Do not think as the bounds on Hicks transfers as bounds of an integral. They indicate only the direction of price change.

*Remark* 7.4.

- $CV$ is the utility change in dollars:
$$CV = e(\mathbf{p}^f, v^f) - e(\mathbf{p}^f, v^o).$$
Think of $e(\mathbf{p}^f, \cdot)$ as a monotonic transformation of $U(x, y)$, an equivalent utility representation in units of dollars.[4] This is called the **money metric utility function**.
- $CV$ is invariant.
$$CV = e(\mathbf{p}^o, v^o) - e(\mathbf{p}^f, v^o)$$
is the cost of two different bundles on the same indifference curve, which does not vary according to the utility representation.

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[4] This is a valid transformation since
$$\frac{\partial e}{\partial U} = \eta^* = \frac{p_x}{U_x(x, y)} > 0.$$

## 7.3. **The Equivalent Variation.**

**Proposition 7.5.** *There holds*

$$\begin{aligned} EV &= e(\mathbf{p}^o, v^f) - m \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^f, v^f) \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^o, v^o) \\ &= T_H|_{p_x^f}^{p_x^o} \end{aligned}$$

*Moreover, just as the CV, the EV is an invariant measure of utility change in dollars.*

## 7.4. **The Surplus form.**

**Proposition 7.6.** *For a price change of $p_x^o$ to $p_x^f$, we may write the CV as*

$$CV|_{p_x^o}^{p_x^f} := -T_H|_{p_x^o}^{p_x^f} = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\, \mathrm{d}p_x.$$

*Similarly, we may write the EV as*

$$EV = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^f)\, \mathrm{d}p_x.$$

*These are the **surplus forms**.*

**Proof.** Note that

$$CV = -\int_{p_x^o}^{p_x^f} \frac{\partial e(p_x, p_y, v^o)}{\partial p_x}\, \mathrm{d}p_x = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\, \mathrm{d}p_x,$$

where the last equality follows from Shepherd's lemma. □

*Remark* 7.7. We can use the surplus forms to analyzes the relative magnitudes of the $CV$ and $EV$. Recall that $\partial x^h/\partial U$ and $\partial x^m/\partial m$ have the same sign. This leads to the following:

**Proposition 7.8.** *If good x is normal, then the Hicksian demand $x^h$ shifts in the same direction as utility level.*

*Example* 7.9. If $v^f < v^o$ and $x$ is a normal good, then $x^h(\mathbf{p}, v^f) < x^h(\mathbf{p}, v^o)$ and

$$|CV| > |CS| > |EV|,$$

where the comparison of the magnitude of $CS$ comes from noting the points where $x^h$ intersect with $x^m$.

*Remark* 7.10. We can use the Slutsky transfer to approximate $CV$ and $EV$:

- $CV \approx -\Delta p_x \cdot x^o$.

- $EV \approx -\Delta p_x \cdot x^f$.

How well the approximation is depends on how willing the individual is willing to substitute. Think perfect complements and perfect substitutes.

**Definition 7.11.** The **change in consumer's surplus** is given by

$$\Delta CS = -\int_{p_x^o}^{p_x^f} x^m(p_x, p_y, m)\, \mathrm{d}p_x.$$

*Remark* 7.12.

- The change in consumer surplus picks up not only the change in welfare (utility) but also change in purchasing power. It is not an exact welfare measure.
- We can decompose changes in $CS$ into two effects: change in consumption (in unit of dollars) and change in expenditure on goods being purchased.

**Definition 7.13.** The **inverse demand function** measures the consumer's willingness to pay for the marginal unit of good $x$. It is given by

$$p_x = p_x(\overline{x}, p_y, \overline{U}),$$

where $\overline{x}$ is the marginal unit of good $x$.

Total surplus:

$$TS = \int_{p_x}^{\infty} x^h(\rho, p_y, \overline{U})\, \mathrm{d}\rho.$$

7.5. **Price Indices.**

**Definition 7.14.**

- The **ideal index**:
$$I = \frac{e(\mathbf{p}^f, \overline{U})}{e(\mathbf{p}^o, \overline{U})}.$$
- The **Laspeyres price index**:
$$P_L = \frac{\mathbf{p}^f\mathbf{x}^o}{\mathbf{p}^o\mathbf{x}^o}.$$
- The **Paasche price index**:
$$P_P = \frac{\mathbf{p}^f\mathbf{x}^f}{\mathbf{p}^o\mathbf{x}^f}.$$

## Appendix A: Utility Functions

### 7.6. **Perfect Complements.**

$$U(x, y) = \min\left\{\frac{x}{a}, \frac{y}{b}\right\}.$$

*Remark* 7.15. This is also known as a **Leontief utility function**.

7.6.1. *Solution of EMP.*

$$x^h = aU, \quad y^h = bU, \quad e = (p_x a + p_y b)U.$$

*Remark* 7.16.

- Think $a$ units of $x$ "pairs" with $b$ units of $y$.
- All income effect; no substitution effect.

### 7.7. **Perfect Substitutes.**

$$U(x, y) = ax + by.$$

7.7.1. *Solution of EMP.*

$$e = \min\left\{\frac{p_x}{a}, \frac{p_y}{b}\right\} U.$$

*Remark* 7.17.

- Think of $p_x/a$ as the price of obtaining one utils by buying $x$.
- All substitution effects; no income effect.

### 7.8. **Cobb-Douglas for Two Goods.**

$$U(x, y) = x^\alpha y^{1-\alpha}.$$

7.8.1. *Solution of UMP.*

$$x^m = \frac{\alpha m}{p_x}, \quad y^m = \frac{(1-\alpha)m}{p_y}, \quad \lambda^* = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha},$$

$$v = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \cdot m.$$

*Remark* 7.18.

- Proportion of income spent on each good is constant.
  - Income elasticities of demand is 1.
  - Substitution effects are offsetted exactly by the income effected.
  - All goods are unrelated (neither substitutes nor complements).
- Homothetic.

7.8.2. *Solution of the EMP.*

$$e = U \cdot \left(\frac{p_x}{\alpha}\right)^{\alpha} \left(\frac{p_y}{1-\alpha}\right)^{1-\alpha},$$
$$x^h = U \cdot \left(\frac{\alpha}{p_x} \cdot \frac{p_y}{1-\alpha}\right)^{1-\alpha}, \quad y^h = U \cdot \left(\frac{\alpha}{p_x} \cdot \frac{p_y}{1-\alpha}\right)^{-\alpha},$$

*Remark* 7.19. Easily obtained using $e = v^{-1}$ and Shepherd's lemma.

## 7.9. **Cobb-Douglas.**

$$U(x) = \prod x_i^{\alpha_i}, \quad \text{where} \quad \sum \alpha_i = 1.$$

7.9.1. *Solutions.*

$$x_j^m = \frac{\alpha_j m}{p_j}, \quad v = m \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}.$$
$$x_j^h = u \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i} \cdot \frac{\alpha_i}{p_i}, \quad e = u \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i}.$$

*Remark* 7.20. $s_i = \alpha_i$.

## 7.10. **Constant Elasticity of Substitution Utility Function.**

$$U(x, y) = \left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{-1/\rho}.$$

7.10.1. *Solution of the UMP.*

$$x_1^m = \frac{m}{p_1 + \kappa p_2}, \quad x_2^m = \frac{\kappa m}{p_1 + \kappa p_2}, \quad \kappa = \left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}.$$

*Remark* 7.21.

- Shares spent on each good is constant $x_2^m = \kappa x_1^m$.
- Income elasticities of demand is 1.
- Indirect utility function is proportional to income. $v = \lambda \cdot m$.
- Constant elasticity of substitution:

$$\sigma = \frac{\mathrm{d}\log\left(\frac{x_1}{x_2}\right)}{\mathrm{d}\log\left(\frac{U_2}{U_1}\right)} = \frac{1}{\rho+1}.$$

7.10.2. *Solution of the EMP.*

$$x_1^h = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot U$$
$$x_2^h = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot \kappa U$$
$$e = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot U \cdot [p_1 + \kappa p_2].$$

*Remark* 7.22. Hicksian demands are proportional to utility level.

7.11. **Quasilinear Utility Functions.**

$$U(x, y) = V(x) + y.$$

*Remark* 7.23.

- Good $x$ is neither normal nor inferior: The willingness to pay for $x$
$$\frac{U_x}{U_y} = V'(x)$$
does not change as we vary consumption of $y$ (by varying income). The consumption of $x$ does not vary as income vary.
- The optimality condition is
$$\frac{U_x}{U_y} = V'(x) = \frac{p_x}{p_y}.$$
- Consider edge cases.

7.12. **Quadratic Utility Function.**

$$U(x, y) = -\frac{1}{2}(x - b_x)^2 - \frac{1}{2}(y - b_y)^2.$$

*Remark* 7.24. Think bliss point.

7.12.1. *Solution of the UMP.*

$$x^m = b_x + \frac{p_x}{p_x^2 + p_y^2}\left(m - p_x b_x - p_y b_y\right),$$
$$y^m = b_y + \frac{p_y}{p_x^2 + p_y^2}\left(m - p_x b_x - p_y b_y\right),$$
$$v = -\frac{1}{2} \cdot \frac{(m - p_x b_x - p_y b_y)^2}{p_x^2 + p_y^2}.$$

## Appendix B: Models

### 7.13. **The Baumol–Tobin model.**

- Exhausts monthly income $Y$.
- Interest rate $i$.
- Goes to ATM $N$ times a month, each time withdrawing $W$ with a direct cost of $F$ incurred.
- Assume constant rate of spending, and money demand is average holding of money $M = W/2$.

$$
\begin{aligned}
e(F, Y, i) &= \min_{W,N} NF + \frac{Wi}{2} \quad \text{s.t.} \quad NW = Y \\
&= \min_{N} NF + \frac{Yi}{2N}.
\end{aligned}
$$

Solving it gives

$$N^* = \sqrt{\frac{Yi}{2F}}.$$