ECON20010 NOTES

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Contents

- Last updated: Wednesday 4th December, 2024.
- See here for the most recent version of this document.

Textbook recommendations from Lima (with increasingly more math):

- Perloff; Varian; Besanko and Braeutigam.
- Jehle-Reny.
- Mas-Colell, Whinston, and Green.
- Kreps.

1. The Envelope Theorem

Theorem 1.1. *Consider the constraint maximization problem*

$$v(\alpha) \coloneqq \max_x f(x;\alpha) \quad \textit{s.t.} \quad g(x;\alpha) = 0.$$

The associated Lagrangian is

$$\mathcal{L}(x,\lambda,\alpha) = f(x;\alpha) + \lambda g(x;\alpha).$$

We have that

$$\frac{\partial v(\alpha)}{\partial \alpha_k} = \frac{\partial \mathcal{L}(x^*,\lambda^*,\alpha)}{\partial \alpha_k}.$$

Proof. Note that $v(\alpha) \equiv \mathcal{L}(x^*,\lambda^*,\alpha)$ and thus

$$\frac{\partial v}{\partial \alpha_k} = \frac{\partial \mathcal{L}}{\partial x}\frac{\partial x^*}{\partial \alpha_k} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda^*}{\partial \alpha_k} + \frac{\partial \mathcal{L}}{\partial \alpha_k},$$

where the first two terms are both 0 by the first order conditions. □

Remark 1.2.

- By the implicit function theorem, we need $f_{xx} \neq 0$.
- Think "all indirect effects vanish" at the optimum. That is, if we think $v(\alpha) = f(x^*;\alpha)$, then

$$\frac{\partial f}{\partial x}\frac{\partial x^*}{\partial \alpha_k} = 0.$$

Example 1.3. Consider the value function

$$\begin{aligned} v(p_x,p_y,m) = U(x^*,y^*) &= U(x^*,y^*) + \lambda^*[m - p_x x - p_y y] \\ &= \mathcal{L}(x^*,y^*,\lambda^*;p_x,p_y,m). \end{aligned}$$

By the envelope theorem,

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

Similarly,

$$\frac{\partial v}{\partial p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

2. Scarcity: the Budget Constraint

Definition 2.1.

- The **budget set** consists of all feasible consumption bundles.
- The **budget constraint** exactly exhausts the consumer's income.

2.1. Budget Set.

The relative price:

$$\frac{p_x}{p_y}$$

- Mnemonic: fractions of this form $(p_x/p_y, U_x/U_y)$ is always the price of x in units of y.

To stay on the budget constraint,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p_x}{p_y}.$$

- Think "the rate at which the market *allows* the consumer to exchange good x for good y."
- Think the opportunity cost of good x.

2.2. Preference.

Basic axioms:

- **Completeness**. For any pair of consumption of bundles, say c_1 and c_2, either $c_1 \succeq c_2$, $c_2 \succeq c_1$, or both.
  - Requires an answer and assumes no framing effects.
- **Transitivity**. If $c_1 \succeq c_2$ and $c_2 \succeq c_3$, then $c_1 \succeq c_3$.
  - When transitivity fails, we have a "money pump."

A preference ordering is **rational** if it satisfies completeness and transitivity. They are the minimal requirement for the existence of a utility function representation. We also typically assume the following:

- **Continuity**. If $c_1 \succ c_2$ then there are neighborhoods N_1 and N_2 around c_1 and c_2 such that

  $$x \succ y, \quad \forall x \in N_1, \quad y \in N_2.$$

  This implies that if $c_1 \succ c_2$ then there exists c_3 such that

  $$c_1 \succ c_3 \succ c_2.$$

- **Monotonicity**.
  - **Monotone**. If $c_1 \gg c_2$[1] then $c_1 \succ c_2$.
  - **Strongly monotone**. If $c_1 \geq c_2$[2] and $c_1 \neq c_2$ then $c_1 \succ c_2$.
  - **Local non-satiation**. If for every bundle c and every $\epsilon > 0$, there exists $x \in N_\epsilon(c)$ such that $x \succ c$.

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[1] We write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i, \forall i$.

[2] We write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i, \forall i$.

- **Convexity**. If $c_1 \succeq c_2$, then
$$\theta c_1 + (1-\theta)c_2 \succeq c_2, \quad \forall \theta \in (0,1).$$
If convexity is satisfied, the **upper contour set**, the "at least as good as" set, is convex.

Additional axioms place even more structures on the utility function:

- **Homotheticity**. If $c_1 \succeq c_2$, then
$$tc_1 \succeq tc_2, \quad \forall t > 0.$$
- **Quasilinearity** in good i. If $\mathbf{c_1} \succeq \mathbf{c_2}$, then
$$\mathbf{c_1} + t\mathbf{e}_i \succeq \mathbf{c_2} + t\mathbf{e}_i, \quad \forall t > 0.$$

2.3. **The utility function.**

Theorem 2.2 (Utility Representation Theorem)**.** *If a preference ordering is rational, then it admits a utility function representation. Moreover, the utility function is unique up to a monotonically increasing transformation.*

Remark 2.3. The additional assumption of monotonicity, though not required, allows for a very simple proof: simply send each consumption bundle to the size of the unique bundle on $t\sum \mathbf{e}$ equivalent to the given bundle.

Proposition 2.4. *If a preference ordering satisfies convexity, then the corresponding utility function representation will be quasi-concave. The indifference curves (level sets) will have non-increasing marginal rate of substitution (slopes).*

Proposition 2.5. *A preference ordering is homothetic if and only if its utility representation has MRS that is homogeneous of degree* 1.

2.4. **Marginal Rate of Substitution.** The marginal rate of substitution (MRS)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y}$$
is the quantity of y the consumer is willing to sacrifice in exchange for an additional unit of x. It measures an individual's **willingness to pay** for x in terms of y.[3]

[3]Recall the mnemonic: U_x/U_y is the price of x in terms of y.

3. Utility Maximization

The problem:

$$v(p_x, p_y, m) \coloneqq \max_{x,y} U(x, y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

3.1. **Interpretation.**

We want to maximize

$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$

such that

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$

This gives

$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x.$$

We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.
$$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \,\mathrm{d}x$$
"Take advantage of all trading opportunities."
- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that U_x/p_x is marginal utility of money *spent on x*.
$$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \,\mathrm{d}x$$
"Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that U_x is the marginal benefit of buying x and $U_y \cdot p_x/p_y$ is the marginal cost of buying x.
$$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$
"Trade until marginal cost equals marginal benefit."

In the last expression, if we write

$$\lambda = \frac{U_y}{p_y},$$

(think marginal utility of income) we have that at optimum,

$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$

These three equalities describe precisely the critical points of the **Lagrangian**:

$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right].$$

That is, setting

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

recovers the above three equations.

Remark 3.1.

- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- λ might be negative or zero. Think bliss point.

3.2. **The Indirect Utility Function.**

Proposition 3.2.

$$\frac{\partial v}{\partial m} = \lambda^*.$$

Proof. Noting

$$v = U(x^*, y^*) + \lambda^*[m - p_x x^* - p_y y^*] = \mathcal{L},$$

we have

$$\begin{aligned}
\frac{\partial v}{\partial m} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} \\
&= U_x \frac{\partial x^*}{\partial m} + U_y \frac{\partial y^*}{\partial m} + \lambda^* \left[1 - p_x \frac{\partial x}{\partial m} - p_y \frac{\partial y}{\partial m}\right] + \frac{\partial \lambda}{\partial m}\left[m - p_x x^* - p_y y^*\right] \\
&= (U_x - \lambda^* p_x) \frac{\partial x^*}{\partial m} + \left(U_y - \lambda^* p_y\right) \frac{\partial y^*}{\partial m} + \frac{\partial \lambda^*}{\partial m}\left(m - p_x x^* - p_y y^*\right) + \lambda^* \\
&= \lambda^*.
\end{aligned}$$

The last equality follows by noting that at the optimum,

$$U_x - \lambda^* p_x = U_y - \lambda^* p_y = m - p_x x^* - p_y y^* = 0.$$

Alternatively, one may use the envelope theorem:

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

□

Note that

$$\frac{\partial v}{\partial m} = \lambda^* = \frac{U_x}{p_x} = \frac{U_y}{p_y}.$$

So when the individual is not satiated ($U_x, U_y \neq 0$), marginal utility of income is positive. When the budget constraint does not require binding, the marginal utility of income is generally nonnegative.

Again using the Envelope Theorem, we have

$$\frac{\partial v}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

This value is generally nonpositive, and only zero when one does not consume good x or when the marginal utility of x is 0.

4. Expenditure Minimization

The problem:

$$e(p_x, p_y, \overline{U}) \coloneqq \max_{x,y} p_x x + p_y y \quad \text{s.t.} \quad U(x, y) = \overline{U}.$$

The Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \eta \left[\overline{U} - U(x, y)\right]$$

$$\begin{aligned} &[x] && p_x = \eta^* U_x(x^*, y^*) \\ &[y] && p_y = \eta^* U_y(x^*, y^*) \\ &[\eta] && \overline{U} = U(x^*, y^*). \end{aligned}$$

Properties 4.1. Properties of Hicksian demand functions:

(i) Homogeneous of degree 0 in prices:

$$x_j^h(\alpha \mathbf{p}, U) = x_j^h(\mathbf{p}, U).$$

Differentiating with respect to α gives

$$\sum \frac{\partial x_j^h}{\partial p_i} p_i = 0 \implies \sum \epsilon_{ji}^h = 0.$$

This property is called "adding up."

(ii) Cross-price effects on Hicksian demand are symmetric:

$$\frac{\partial x_i^h}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial x_j^h}{\partial p_i}.$$

From this we have

$$p_i x_i \frac{p_j}{x_i} \frac{\partial x_i^h}{\partial p_j} = p_j x_j \frac{p_i}{x_j} \frac{\partial x_j^h}{\partial p_i}.$$

That is,

$$s_i \epsilon_{ij}^h = s_j \epsilon_{ji}^h \implies \frac{\epsilon_{ij}^h}{\epsilon_{ji}^h} = \frac{s_j}{s_i}.$$

The more important good impacts the less important good more.

(iii) Differentiating $U(\mathbf{x}^h(\mathbf{p}, U)) = U$ with respect to p_j gives

$$\sum \frac{\partial U}{\partial x_i} \frac{\partial x_i^h}{\partial p_j} = 0 \implies \sum p_i \frac{\partial x_i^h}{\partial p_j} = 0 \implies \sum \epsilon_{ij}^h s_i = 0.$$

Remark 4.2. Symmetry and homogeneity [adding up] gives adding up [homogeneity]. In case where there are two goods only, the latter two also gives symmetry.

Proposition 4.3. *Properties of the expenditure function:*

- *Homogeneous of degree* 1 *in prices.*
- *Non-decreasing in prices.* $\partial e/\partial p_i = x_i^h \geq 0$.
- *Increasing in utility.* $\partial e/\partial U = \eta^* > 0$.
- *Concave in prices.*

$$\frac{\partial^2 e}{\partial p_i^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0,$$

where the last inequality follows from the law of demand. Alternatively, note that the price of the original bundle, which grows at most linearly, is an upper bound of the expenditure function.

Remark 4.4. Implication of adding up: for any good x at least one other good is a net substitute with x.

5. Changes in Behavior

Consider a price increase from p_x^o to p_x^f. Let o be the original consumption, f be the final consumption, and s be the optimal consumption after an income transfer such that the individual stays on the same indifference curve as before (has the same purchasing power). We may decompose $x^f - x^o$:

$$x^f - x^o = x^f - x^s + x^s - x^o.$$

- $x^f - x^o$: the Marshallian price effect (the total effect).
- $x^f - x^s$: the effect due to compensation (the income effect).
- $x^s - x^o$: the Hicksian price effect (substitution effect).

The Slutsky equation is an continuous analogue of this decomposition.

5.1. The Slutsky Equation.

Recall from duality that

$$x^h(p_x, p_y, \overline{U}) = x^m(p_x, p_y, m = e(p_x, p_y, \overline{U})).$$

As price changes, changes in $e(p_x, p_y, \overline{U})$ ensures that purchasing power does not change.

By differentiating, we get the **Slutsky equation**:

Proposition 5.1.

$$\frac{\partial x^h}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot \frac{\partial e}{\partial p_x}.$$

We may rewrite the Slutsky equation using the envelop theorem as

$$\frac{\partial x^h}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot x^m.$$

This shows that we can recover the unobservable $\partial x^h / \partial p_x$ from the observables.

We can also rewrite the Slutsky equation as

$$\frac{\partial x^m}{\partial p_x} = \frac{\partial x^h}{\partial p_x} + \left(-\frac{\partial x^m}{\partial m} \cdot x^m\right),$$

where

- $\partial x^h / \partial p_x$ is the substitution effect,
- $-\partial x^m / \partial m \cdot x^m$ is the income effect.

5.2. Compensation.

- **Slutsky transfer** keeps the original bundle affordable.

$$T_S = \Delta p_x \cdot x^o.$$

- **Hicks transfer** keeps the original utility level affordable.
  $$T_H = e(p_x^f, p_y, v^o) - m = e(p_x^f, p_y, v^o) - e(p_x^o, p_y, v^o).$$
  In the Slutsky equation, the term
  $$\partial e/\partial p_x = x^m = x^h$$
  is the continuous analogue of the Hicks transfer.
- **Frisch transfer**: the transfer that restores the purchasing power by making the price of utility λ constant.

Remark 5.2. Note that $T_S \geq T_H$ when $p_x^f > p_x^o$. When government impose taxes, revenue equals T_S, but it need only pay T_H to the citizens to make them equally well off.

5.3. **The Law of Demand.** The substitution effect is always nonpositive by the law of demand:
$$\partial x^h/\partial p_x \leq 0.$$
More generally, we have:

Proposition 5.3 (Generalized law of demand)**.**
$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) \leq 0.$$

Proof. Note that
$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) = \left(\mathbf{x}^1\mathbf{p}^1 - \mathbf{x}^0\mathbf{p}^1\right) + \left(\mathbf{x}^0\mathbf{p}^0 - \mathbf{x}^1\mathbf{p}^0\right).$$
The last two terms are both nonpositive. □

Remark 5.4.

- Note that this gives $\partial x_i^h/\partial p_i \leq 0$ (if the derivative exists).
- But think also graphs for the case of two goods.
- Note that the law of demand holds not only when indifference curves are concave. Remember the following two examples as well:
  - When indifference curves are concave, the expenditure minimizing points occur at the edges.
  - In the perfect complement case, we have that $\partial x^h/\partial p_x = 0$.

5.4. **Giffen Goods.** Marshallian demand does not always comply with the law of demand. A good whose Marshallian demand does not comply with the law of demand is called a **giffen good**. Their existence is theoretically possible, but not empirically supported.

Looking back at the Slutsky equation, we see that for a good x to be a giffen good, we need the following three conditions:

(i) the individual buys a large amount of x,
(ii) good x is inferior,

(iii) the demand for good x is elastic.

These three conditions do not occur together often: narrowly defined categories usually has 0 income elasticity, but broad categories are usually normal goods.

5.5. **Normal & Inferior Goods.**

Definition 5.5. Good x is said to be a

- **normal** good if $\eta_x > 0$,
- **inferior** good if $\eta_x < 0$,
- **necessity** if $\eta_x < 1$,
- **luxury** if $\eta_x > 1$.

Remark 5.6. We can equivalently define normality using Hicksian demands. From

$$\frac{\partial x^h}{\partial U} \cdot \frac{\partial v}{\partial m} = \frac{\partial x^m}{\partial m}$$

we know that $\partial x^h / \partial U$ and $\partial x^m / \partial m$ have the same sign. Note that $\partial v / \partial m = \lambda^* > 0$ is the utility of a dollar.

5.6. **An experiment for testing normality.** We fix the consumption of x and vary income m. For normal goods, the willingness to pay for x increases as income increase. Thus x is normal if

$$\frac{\partial (U_x / U_y)}{\partial y} > 0.$$

Think graphs.

Example 5.7. With the quasilinear utility function $U(x, y) = v(x) + y$, the good x is neither normal nor inferior. The willingness to pay

$$\frac{U_x}{U_y} = \frac{v'(x)}{1}$$

does not change as we vary y (by varying income).

5.7. **Cross Effects.**

Definition 5.8. We say y is

- a **substitute** of x if $\partial y / \partial p_x > 0$,
- a **complement** of x if $\partial y / \partial p_x < 0$,
- **unrelated** with x if $\partial y / \partial p_x = 0$.

If we use $y = y^h$ in the definition above, we say **gross** substitutes/complements; if we use y^m, we say **net** substitutes/complements.[4]

[4]Think Hicksian demand "nets out" the income effect.

Remark 5.9. The property of being substitutes is not transitive. That is, if x and y are substitutes and y and z, then x and z are not necessarily substitutes. Consider $U = \min\{x + y, y + z\}$.

Remark 5.10.

- Cross price effects for Hicksian demands are symmetric:
$$\frac{\partial x^h}{\partial p_y} = \frac{\partial^2 e}{\partial p_x \partial p_y} = \frac{\partial y^h}{\partial p_x}.$$
This does not hold in general for Marshallian demands; see the cross-price Slutsky equation.
- For any good x, at least one other good is a net substitute with x. If not, as price of x increase, consumption and thus utility level strictly decreases (note that the consumption of x strictly decreases by the law of demand). In particular, when there are only two goods, the two goods cannot be net complements.

Proposition 5.11. *We have the* ***cross-price Slutsky equation****:*
$$\frac{\partial y^h}{\partial p_x} = \frac{\partial y^m}{\partial p_x} + \frac{\partial y^m}{\partial m} \cdot \frac{\partial e}{\partial p_x}$$
where
$$\frac{\partial e}{\partial p_x} = x^m = x^h.$$

6. Elasticities and Aggregation

Proposition 6.1. *The Slutsky equation in elasticity form:*

$$\epsilon_{ij}^m = \epsilon_{ij}^h - \eta_i s_j.$$

From this we have the following:

Proposition 6.2 (Symmetry of Marshallian Demands)**.** *We have*

$$s_i \epsilon_{ij}^m = s_j \epsilon_{ji}^m + s_i s_j (\eta_j - \eta_i).$$

Symmetry holds when two goods have equal income elasticities.

Proposition 6.3.

- *Engel aggregation:*
$$\sum s_i \eta_i = 1.$$
- *Cournot aggregation:*
$$\sum_i \epsilon_{ij} s_i = -s_j.$$
- *Homogeneity:*
$$\sum_i \epsilon_{ji} = -\eta_j.$$

Proof.

- From $m = \sum p_i x_i$ we have
$$1 = \frac{\partial m}{\partial m} = \sum p_i \frac{\partial x_i}{\partial m} = \sum \eta_i s_i.$$
- Differentiating the same identity with respect to p_j gives
$$0 = \frac{\partial m}{\partial p_x} = x_j + \sum \frac{\partial x_i}{\partial p_j} p_i = x_j + \sum \epsilon_{ij} \frac{p_i x_i}{p_j}.$$
- Differentiating the identity $x^m(\mathbf{x}, m) = x^m(t\mathbf{x}, m)$ with respect to t gives
$$0 = \sum \frac{\partial x_j}{\partial p_i} p_i + \frac{\partial x_j}{\partial m} m.$$

□

Remark 6.4. Some implications:

- Engel aggregation: It will never the case that all goods are inferior, all goods are necessities, or all goods are luxuries.

- Cournot aggregation: A Giffen good has many complements:
$$\sum_{i \neq x} \epsilon_{ix} s_i = -s_x - \epsilon_{xx} s_x < 0.$$
The budget share of good j is small if it has many substitutes, and large if it has many complements, or when their respective budget shares are large. If the number of substitutes for a good decrease, the demand for that good becomes more inelastic.
- Restriction of homogeneity: A Giffen good is the substitute of many goods:
$$\sum_{i \neq x} \epsilon_{xi} + \epsilon^h_{xx} = \eta_x s_x - \eta_x > 0.$$

Proposition 6.5. *If $s_i \geq 0$ is such that $\sum s_i = 1$, and η_i and ϵ_{ij} are such that Engel and Cournot aggregations and the restrict of homogeneity hold, then there exists a utility function U, income m, and prices* $\mathbf{p}$ *that generate these parameters.*

Proof. From Afriat's theorem. □

7. Welfare

7.1. Exact Measures of Welfare Change.

The difference in utility

$$\Delta v = v^f - v^o.$$

- Gets direction right, but magnitude depends on the specific utility representation chosen — it is not invariant to the utility representation.
- Difficulty in the interpretation of units.

Definition 7.1.

- **Compensating variation**: the income transfer that induces the consumer accept the change in price voluntarily.
- **Equivalent variation**: the income transfer that induces the consumer to reject the change in price voluntarily.

7.2. The Compensating Variation.

Proposition 7.2. *There holds*

$$\begin{aligned} CV|_{p_x^o}^{p_x^f} := -T_H|_{p_x^o}^{p_x^f} &= -\left[e(\mathbf{p}^f, v^o) - m\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^f, v^f)\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^o, v^o)\right]. \end{aligned}$$

Remark 7.3. Do not think as the bounds on Hicks transfers as bounds of an integral. They indicate only the direction of a price change.

Remark 7.4.

- CV is the utility change in dollars:

  $$CV = e(\mathbf{p}^f, v^f) - e(\mathbf{p}^f, v^o).$$

  Think of $e(\mathbf{p}^f, \cdot)$ as a monotonic transformation of $U(x, y)$, an equivalent utility representation in units of dollars.[5] This is called the **money metric utility function**.
- CV is invariant.

  $$CV = e(\mathbf{p}^o, v^o) - e(\mathbf{p}^f, v^o)$$

  is the cost of two different bundles on the same indifference curve, which does not vary according to the utility representation.

[5] This is a valid transformation since

$$\frac{\partial e}{\partial U} = \eta^* = \frac{p_x}{U_x(x, y)} > 0.$$

7.3. **The Equivalent Variation.**

Proposition 7.5. *There holds*

$$\begin{aligned} EV &= e(\mathbf{p}^o, v^f) - m \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^f, v^f) \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^o, v^o) \\ &= T_H|_{p_x^f}^{p_x^o} \end{aligned}$$

Moreover, just as the CV, the EV is an invariant measure of utility change in dollars.

7.4. **The Surplus form.**

Proposition 7.6. *For a price change of p_x^o to p_x^f, we may write the CV as*

$$CV = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\, \mathrm{d}p_x.$$

Similarly, we may write the EV as

$$EV = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^f)\, \mathrm{d}p_x.$$

*These are the **surplus forms**.*

Proof. Note that

$$CV = -\int_{p_x^o}^{p_x^f} \frac{\partial e(p_x, p_y, v^o)}{\partial p_x}\, \mathrm{d}p_x = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\, \mathrm{d}p_x,$$

where the last equality follows from Shepherd's lemma. □

Remark 7.7. We can use the surplus forms to analyzes the relative magnitudes of the CV and EV. Recall that $\partial x^h/\partial U$ and $\partial x^m/\partial m$ have the same sign. This leads to the following:

Proposition 7.8. *If good x is normal, then the Hicksian demand x^h shifts in the same direction as utility level.*

Example 7.9. If $v^f < v^o$ and x is a normal good, then $x^h(\mathbf{p}, v^f) < x^h(\mathbf{p}, v^o)$ and

$$|CV| > |\Delta CS| > |EV|,$$

where the comparisons with the magnitude of ΔCS comes from noting the points at which the Hicksian demand functions intersect x^m.

Remark 7.10. We can use the Slutsky transfer to approximate CV and EV:

- $CV \approx -\Delta p_x \cdot x^o$.

- $EV \approx -\Delta p_x \cdot x^f$.

How well the approximations are depends on how willing the individual is to substitute. Think perfect complements and perfect substitutes.

Definition 7.11. The **change in consumer's surplus** is given by

$$\Delta CS = -\int_{p_x^o}^{p_x^f} x^m(p_x, p_y, m)\,\mathrm{d}p_x.$$

Remark 7.12.

- The change in consumer surplus picks up not only the change in welfare (utility) but also change in purchasing power. It is not an exact welfare measure.
- We can decompose changes in ΔCS into two effects: change in consumption (in unit of dollars) and change in expenditure on goods being purchased.

7.5. **Efficiency.**

Definition 7.13. A change is called a **Pareto improvement** if it leaves everyone in a society better-off (or at least as well-off as they were before). A situation is called **Pareto efficient** or **Pareto optimal** if all possible Pareto improvements have already been made.

Definition 7.14. A re-allocation is a **Kaldor–Hicks improvement** if those that are made better off could hypothetically compensate those that are made worse off and lead to a Pareto-improving outcome. A situation is said to be **Kaldor–Hicks efficient**, or equivalently is said to satisfy the **Kaldor–Hicks criterion**, if no potential Kaldor–Hicks improvement from that situation exists.

8. An Endowment Economy

We generalize the parameter m, which reflects command over resources. We assume that an individual owns the following endowments (in units of x and y) that may be converted into income:

$$E = \{\omega_x, \omega_y\}.$$

The constraint becomes

$$E \coloneqq p_x x + p_y y \leq m \coloneqq p_x \omega_x + p_y \omega_y.$$

Since we assume monotonic preference, we may set $E = m$.

The UMP becomes:

$$v^* = \max_{x,y} U(x, y) \quad \text{s.t.} \quad p_x x + p_y y = p_x \omega_x + p_y \omega_y.$$

Note that the budget line always passes through the initial bundle (ω_x, ω_y).

Definition 8.1. We call the consumer a

- **net seller** of x if $x < \omega_x$,
- **net buyer** of x if $x > \omega_x$.

In an endowment economy, when prices change, the effective income also change. The budget line pivots on the initial bundle. We call the difference in income

$$w \coloneqq m^f - m^o = \Delta p_x \cdot \omega_x.$$

the **windfall** gain / loss [6] and the resulting change in welfare the **endowment effect**. Because of the endowment effect, the consumer might gain utility even when price increases.

8.1. Price Effect.

To decompose the price effect, we consider a tax that perfectly offsets the windfall. This reduces the analysis to the same case as when there is no endowment. We have

$$x^F - x^O = x^S - x^O + x^I - x^S + x^F - x^I,$$

where

- $x^S - x^O$ is the substitution effect,
- $x^I - x^S$ is the income effect,
- $x^F - x^I$ is the endowment effect.

We have from this decomposition that:

Proposition 8.2. *The welfare effect in CV equals the CV with income fixed at the initial level plus windfall, and the welfare effect in EV equals the EV with income fixed at the initial level plus windfall.*

[6] Note the resemblance with the Slutsky compensation.

Proof. For CV, let $v^I := v(\mathbf{p}^f, m^o)$.

$$\begin{aligned} CV &= e(\mathbf{p}^f, v^f) - e(\mathbf{p}^f, v^I) + e(\mathbf{p}^f, v^I) - e(\mathbf{p}^f, v^o) \\ &= m^f - m^o + CV|_{m^o}. \end{aligned}$$

For EV, let $v^I := v(\mathbf{p}^o, m^o)$.

$$\begin{aligned} EV &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^o, v^I) + e(\mathbf{p}^o, v^I) - e(\mathbf{p}^o, v^f) \\ &= m^f - m^o + EV|_{m^o}. \end{aligned}$$

□

8.2. **Slutsky Equation.**

Definition 8.3. The function

$$x^* = x^m(p_x, p_y, p_x\omega_x, p_y\omega_y)$$

is called the **Walrasian** demand function.

Remark 8.4. The Walrasian demand function is not compensated.

We have that

$$\frac{\mathrm{d}x^*}{\mathrm{d}p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot \frac{\partial m}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot \omega_x.$$

Note that the term $\partial x^m/\partial p_x$ is taken with $m = m^o$. Recall that

$$\frac{\partial x^m}{\partial p_x} = \frac{\partial x^h}{\partial p_x} - \frac{\partial x^m}{\partial m} \cdot x^*.$$

Combining the above gives

Proposition 8.5.

$$\frac{\mathrm{d}x^*}{\mathrm{d}p_x} = \frac{\partial x^h}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot (\omega_x - x^*)\,.$$

Remark 8.6. Thus, the combined income and endowment effects depends on whether the individual is a net seller or a net buyer. Endowment and income effects move opposite each other.

Proposition 8.7. *There is always a set of prices* $\mathbf{p}$ *that will induce the consumer to keep her endowed bundle. With this set of prices, the utility level of the individual is the lowest. This is the "worst bundle."*

Remark 8.8. Intuition: think picking p_x such that the budge line is tangent to the indifference curve. With higher p_x, the individual can get to a higher indifference curve by selling x; with lower p_x, the individual sells x. Think the $v - p_x$ curve!

Remark 8.9. Implications: If the individual starts off as a seller and the price increases, the individual gains utility; if the price decreases such that the individual remains a seller, she is worse off; if the price decreases to an extent such that the individual changes to a buyer, the welfare implication is not clear.

9. Equilibrium in an Endowment Economy

9.1. **A Pure Exchange Economy.** In this section, we examine a pure exchange economy[7], which can be graphically represented using an **Edgeworth box**. Unlike a competitive market, which we examine next, the "prices" in a barter economy need not stay fixed as quantity vary. In a sense, the theory developed in this section is more general than that of competitive equilibrium. Resulting differences in implications should be noted.

It is reasonable to expect the final allocation to satisfy the following:

- **Feasibility**: $\sum \mathbf{x}^i \leq \sum \boldsymbol{\omega}^i$.
- **Participation constraints**: $U^i(\mathbf{x}) \geq U^i(\boldsymbol{\omega})$.
- Utility maximizing: the indifference curves of any pair of individuals are tangent. If the indifference curves of any two individuals cross, a Pareto dominating allocation can be reached through further exchange.

We may visualize the final allocation in a two good, two individual economy using an **Edgeworth box**, in which we have the following:

- Each feasible and market clearing allocation can be represented by one point in Edgeworth box.
- The **core** of the economy is the set of allocations that satisfy the participation constraints. In a Edgeworth box, it is the area between the indifference curves passing through the initial endowment $\boldsymbol{\omega}$.
- The indifference curves of the two individuals passing through $\boldsymbol{\omega}$ are tangent. The **contract curve** is the set of all such allocations.

The above requirements on the final allocations of a pure exchange economy can thus be rephrased as follows:

Proposition 9.1. *In an Edgeworth box, the final allocation is in the intersection of the core of the economy and the contract curve.*

Note that there might be multiple points in the intersection; the equilibrium is not unique.

---

[7]The poem *Jack Sprat* describes a perfect example of trade:

Jack Sprat could eat no fat,
His wife could eat no lean.
And so between them both, you see,
They licked the platter clean.

10. Pareto Optima and the Benevolent Social Planner

The benevolent social planner picks the bundle that maximizes the utility level of a person i given set utility levels of other persons:

$$\max U^i(\mathbf{x}^i) \quad \text{s.t.} \quad \begin{cases} \sum_k \mathbf{x}^k = \sum_k \omega^k \\ U^j(\mathbf{x}^j) = \overline{U}^j, \qquad \forall j \neq i \end{cases}$$

This problem is called the **Pareto Problem**, and the solutions are Pareto optimal in the following sense:

Definition 10.1. A feasible allocation $\{\mathbf{x}^1, \dots, \mathbf{x}^N\}$ is **Pareto optimal** is there is no other allocation $\{\mathbf{y}^1, \dots, \mathbf{y}^N\}$ such that $U^i(\mathbf{y}^i) \geq U^i(\mathbf{x}^i)$ for each individual i and $U^j(\mathbf{y}^j) > U^j(\mathbf{x}^j)$ for at least one j.

If an allocation $\{\mathbf{y}^i\}$ exists, we say it **Pareto dominates** the allocation $\{\mathbf{x}^i\}$. Two distinct Pareto optimal allocations are **Pareto non-comparable**.

Example 10.2. In a one good economy, any split that exhausts the total amount of goods is Pareto optimal, assuming monotonicity.

In a two good two individual economy, the Lagrangian for individual i's Pareto problem is

$$\begin{aligned} \mathcal{L} \coloneqq U^i(x^i, y^i) &+ \gamma_x \left[\omega_x^i + \omega_x^j - x^i - x^j\right] + \gamma_y \left[\omega_y^i + \omega_y^j - y^i - y^j\right] \\ &+ \mu_j \left[U^j(x^j, y^j) - \overline{U}^j\right] \end{aligned}$$

We have the following first order conditions:

$$\begin{aligned} [x^i]: &\quad U_x^i(x^i, y^i) = \gamma_x & [y^i]: &\quad U_y^i(x^i, y^i) = \gamma_y \\ [x^j]: &\quad U_x^j(x^j, y^j) = \gamma_x & [y^j]: &\quad U_y^j(x^j, y^j) = \gamma_y \\ [\gamma_x]: &\quad \omega_x^i + \omega_x^j = x^i + x^j & [\gamma_y]: &\quad \omega_y^i + \omega_y^j = y^i + y^j \\ [\mu_j]: &\quad U^j(x^j, y^j) = \overline{U}^j, \end{aligned}$$

which implies the following:

- At optimum, the MRS for each individual is equal, and equals the relative price:
$$\frac{U_x^i}{U_y^i} = \frac{\gamma_x}{\gamma_y} = \frac{U_x^j}{U_y^j}$$
- The optimal allocation satisfies the feasibility constraints $[\gamma_x]$ and $[\gamma_y]$.

Thus we have that the solutions of the Pareto problem is the contract curve.

The social planner's problem can be equivalently characterized as:

$$\max \sum_i \theta_i U^i(\mathbf{x}^i) \quad \text{s.t.} \quad \sum \mathbf{x}^i = \sum \boldsymbol{\omega}^i,$$

where $\boldsymbol{\theta}$ are the weights placed on each individual's utility function.

Remark 10.3. When all θ_i are equal, it is not necessarily true that all U^i are the same, or that all $\mathbf{x}^i$ are the same.

Proposition 10.4. *The followings are the same sets:*

- *The set of solutions of the social planner's problem.*
- *The contract curve.*
- *The set of Pareto optimal allocations.*

11. Equilibrium in an Competitive Endowment Economy

We now analyze a competitive economy, in which exchange are mediated by pries that is assumed to be unaffected by quantities bought or sold. This may be thought of as a particular case of a pure exchange economy.

Note an important mental shift: We think of individuals as both buyers and sellers who deposit all endowments in the market, collect income, and purchase goods with their income. Prices $\mathbf{p}$ *adjust* to make the behaviors on the supply and demand sides consistent — to clear the market and create an equilibrium that may be characterized as follows:[8]

Definition 11.1. A **competitive equilibrium** (CE) in a k good, n person economy is *relative* prices $\mathbf{p}$ and an allocation $\{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$ such that given the relative prices, the allocation satisfies the following two conditions:

(1) utility maximizing, that is, it solves
$$[C]: \quad \max_{\mathbf{x}^i} U^i(\mathbf{x}^i) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \boldsymbol{\omega}^i$$
for each individual i.

(2) market clearing, that is, it satisfies
$$[E]: \quad \sum \mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \boldsymbol{\omega}^i) = \sum \boldsymbol{\omega}^i.$$

Remark 11.2.

- The foundation of economics is scarcity, at both an individual level and a market level. Purposive behavior is often not required (think law of demand, for example).
- Graphically, at equilibrium, the budget constraint for each individual passes through their endowment and is tangent to their indifference curves.

Remark 11.3. The consumer need only know the relative prices to make a decision: Take $k = 2$ as an example. We may characterize $[C]$ as satisfying
$$p \coloneqq \frac{p_x}{p_y} = \frac{U_x(\mathbf{x})}{U_y(\mathbf{x})},$$
where p is the relative price of x with y as the numeraire, and
$$\mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \boldsymbol{\omega} \iff px + y = p\omega_x + \omega_y.$$

Remark 11.4. Note that we have $k - 1$ relative prices and, by the following result, also $k - 1$ equations from $[E]$.

Theorem 11.5 (Walras' Law)**.** *If the consumer spends his/her entire income, and if all but one market clear, then so must the last.*

[8] The fictional **Walrasian auctioneer** is the price adjuster. He is the invisible hand.

Proof. Assume first that there is only one individual. We have

$$\sum p_j x_j = \sum p_j \omega_j$$

Thus if $x_j = \omega_j$ for all $j \neq j_0$, we know that $x_{j_0} = \omega_{j_0}$. When there are multiple individuals, simply sum the demand of each individual up and repeat the above analysis. □

Example 11.6. In a 2 goods, 1 person, Cobb-Douglas economy, we seek p such that

$$[C]: \quad \max_{\mathbf{x}} U_i(\mathbf{x}) \quad \text{s.t.} \quad px + y = p\omega_x + \omega_y$$

$$[E]: \quad \sum \mathbf{x}(p, 1, p\omega_x^i + \omega_y^i) = \sum \boldsymbol{\omega}.$$

That is, we seek p such that

$$x^* = \frac{\alpha(p\omega_x + \omega_y)}{p} = \omega_x, \quad y^* = (1-\alpha)(p\omega_x + \omega_y) = \omega_y.$$

The first condition gives

$$p = \frac{\alpha}{1-\alpha}\frac{\omega_y}{\omega_x}.$$

By Walrus' law, the second condition is also satisfied. When the individual values x (relative to y), or when x is scarce (relative to y), then p is high.

Remark 11.7. For any given individual, we may think of going from an autarky to free trade as a price change (from the price that sustains the consumption of the individual's endowment to the equilibrium price), and the welfare change can be measured accordingly.

Noting that the final budget line (the budget line for each individual coincide in an Edgeworth box) must pass through the initial endowment $\boldsymbol{\omega}$, we have:

Proposition 11.8. *In an Edgeworth box, the final allocation of a two good, two person competitive endowment economy satisfies the following:*

- *Feasibility.*
- *Participation constraint: it is in the core.*
- *Utility maximizing: it in on the contract curve and tangent to indifference curves of both individuals.*
- *Mediated through price: it is on the budget line (which is tangent to both indifference curves).*

Remark 11.9. Remember also the following Edgeworth boxes that do not represent final allocations:

- Two points on a budget line will cause excess demand of one good and excess supply of the other good.

- A common point with indifference curves not tangent to the budget line is not optimizing.

11.1. **The Welfare Theorems.**

Theorem 11.10 (First Welfare Theorem)**.** *The competitive equilibrium is Pareto optimal.*

Proof. At a competitive equilibrium, the indifference curves are also tangent, all being tangent to the budget line. It thus lie on the contract curve. ◻

Remark 11.11. The market solution is called "decentralized," whereas the solution by the social planner is called "centralized." The first welfare theorem tells us that decentralized information is sufficient to achieve efficiency (in the Pareto sense)!

Theorem 11.12 (Second Welfare Theorem)**.** *If the indifference curves are convex, then every Pareto optimum can be supported as a competitive equilibrium with transfers (i.e., redistribution of income, e.g., through lump sum taxes).*

Remark 11.13. Markets are designed for efficient, not equality (since initial distributions might be unequal). For that we need redistribution.

Remark 11.14. The difference between communism and a market economy is, then, the cost of information.

11.2. **Decentralization of the Pareto Optimal Allocation.** Here we are "implementing" the second welfare theorem.

Definition 11.15. Getting a centralized Pareto optimal solution through market means is called **decentralizing** the market.

To decentralize to obtain any Pareto optimal allocation PO, we want

$$\hat{p} = \frac{U_x^i}{U_y^i}, \quad m_i^{PO} = \hat{p}x_i^{PO} + y_i^{PO}.$$

Note that at the income m_i^{PO}, the competitive equilibrium results in $p = \hat{p}$, since at that price by construction we have feasibility and optimizing. Thus we need only redistribute income to mach m_i^{PO}. To implement, we set

$$T_i := m_i^{PO} - m_i^{CE}.$$

Remark 11.16. Think of relative prices as amounts of goods, and transfers as redistributing the initial endowment. Note that all prices and income are in units of a certain good (the numeraire).

12. Labor Economics

In the following sections, we will go from descriptive models of several problems to the canonical models developed in the preceding sections. We begin with the descriptive model of the labor supply.

The individual solves

$$\max_{C,R,L} U(C,R) \quad \text{s.t.} \quad \begin{cases} L + R = T \\ pC = wL \end{cases},$$

where

- C is consumption in unit;
- R is leisure (rest), L is labor, and T is the time endowment;
- p is price of 1 unit of consumption and w is wage.

The solution is functions C^*, R^*, and L^*, all with parameters (p, w, T). The above is called the **standard form** of the individual's UMP.

12.1. An Alternative Formulation.

We may also write the UMP as

$$v^* = \max_L U\left(\frac{w}{p}L, T - L\right),$$

which clearly demonstrates the nature of the trade-off. The first order condition gives

$$[C]: \quad U_C \cdot \frac{w}{p} + U_R \cdot (-1) = 0 \iff \frac{U_R}{U_C} = \frac{w}{p}.$$

12.2. The Canonical Problem.

We want, however, to rephrase the problem and solutions in terms of familiar language, using the tools we have developed. To do this we note that the constraints of standard form can be equivalently written as

$$pC + wR = wT.$$

This constraint is called the canonical constraint. We having the following:

- Wage w is the opportunity cost of leisure.
- $pC + wR$ is the expenditure of the bundle and wT is the value of the endowment, called the individual's **full income** (economists generally refer to this quantity when they use the term "income").
- The constraint in the standard form $pC = wL$ gives the rate of exchange of C and L.

We have then the **canonical problem**:

$$\max_{C,R} U(C,R) \quad \text{s.t.} \quad pC + wR = wT,$$

from which we get the Walrasian demand functions C^*, R^*, and L^*, all with parameters (p, w, wT).

Note that everything that we derived before continues to hold! We have, for example, the Slutsky equation

$$\frac{\partial R}{\partial w} = \frac{\partial R^h}{\partial w} + (T - R)\frac{\partial R}{\partial m}.$$

Note that $T - R \geq 0$, so the individual is always a seller of their time and can thus never be worse off in face of a wage increase.

Example 12.1. Suppose

$$U = C^{\alpha} R^{1-\alpha}.$$

We have then

$$C = \frac{\alpha w T}{p}, \quad R = \frac{(1-\alpha)wT}{w} = (1-\alpha)T.$$

Note that in the descriptive problem, from $\partial R/\partial w = 0$ we get only the full effect, whereas in the canonical problem, we may use $\partial R/\partial w$ to decompose the change into substitution, income, and endowment effects.

12.3. **The Theory of Household Production (Becker, 1965).** Individuals likely care not just about consumption and leisure, but also about household produced goods, **commodities**, that require consumption and leisure as inputs. We define commodity as

$$Z \coloneqq \max\left\{\frac{C}{a}, \frac{R}{b}\right\},$$

where a and b are **input coefficients**. Note that this gives

$$C = aZ, \quad R = bZ.$$

Example 12.2. Take watching a two-hour movie as a commodity We may define $Z \coloneqq \min\{c, R/2\}$, where c is the number of movie tickets.

The individual solves the UMP

$$v \coloneqq \max U(Z) \quad \text{s.t.} \quad \begin{cases} L + R = T \\ pC = wL \end{cases}$$

We can rewrite the constraints as $pC + wR = wT$ and then

$$p(aZ) + w(bZ) = wT.$$

Thus $\pi_Z \coloneqq ap + bw$ is the shadow price of Z and we have the canonical model:

$$\max_Z U(Z) \quad \text{s.t.} \quad \pi_Z \cdot Z = wT.$$

When there are multiple commodities

$$Z_i \coloneqq \min\left\{\frac{C_i}{a_i}, \frac{R_i}{b_i}\right\},$$

we have the descriptive problem

$$\max U(\mathbf{Z}) \quad \text{s.t.} \quad \begin{cases} L + R = T \\ pC = wL \end{cases}$$

and, by the same method as before, the canonical problem

$$\max_{\mathbf{Z}} U(\mathbf{Z}) \quad \text{s.t.} \quad \boldsymbol{\pi} \cdot \mathbf{Z} = S.$$

Remark 12.3. Note that prices are now much more general. The price for watching movies, for example, include not just the price of the tickets, but also the opportunity costs of not working.

13. Intertemporal Choice

A consumption bundle is a sequence of consumption levels $\{c_t\}_{t=0}^T$. We assume that the utility function can be written as

$$U\left(\{c_t\}_{t=0}^T\right) = \sum_{t=0}^T \beta^t u(c_t).$$

- Think of the c_t's as the same object but different goods.
- The constant β is the subjective **discount factor**. We typically assume $\beta < 1$. This captures impatience and the fact the survival into the next period is not guaranteed.[9] It is analogous to the interest rate in the financial market.

Assumptions:

- We assume **time-separability**, or **"independent wants"**:
$$\frac{\partial^2 U}{\partial c_t \partial c_{t+k}} = 0, \quad \forall k \neq 0.$$
We assume that past consumption do not affect marginal utility in the future, that there is no possibility of addiction.
- We assume
$$\frac{\partial^2 U}{\partial c_t^2} = u''(c_t) < 0,$$
that is, diminishing marginal utility. This gives convex indifference curves. To see this, differentiate the MRS.

Note that we have the MRS:

$$\mathrm{MRS}_{c_t, c_{t+1}} = \frac{u'(c_t)}{\beta u'(c_{t+1})}$$

When β is large, we care more about the future.

Proposition 13.1. *With the above assumptions, each c_t is normal.*

13.1. **Endowment Economy.** We assume an endowment economy with exogenous endowments $E = \{\omega_t\}_{t=0}^T$. We assume access to a perfect credit market or a storage technology.

13.2. **A Storage Market.** The descriptive two-period problem:

$$v = \max_{c_t, c_{t+1}} U(c_t, c_{t+1}) \quad \text{s.t.} \quad \begin{cases} c_t + s_{t+1} = \omega_t, \\ c_{t+1} = \omega_{t+1} + (1-\delta) s_{t+1}, \\ s_{t+1} \geq 0, \end{cases}$$

where δ is the depreciation rate.

[9] The perpetual youth model uses precisely this to model survival.

The canonical two-period problem: We may combine the constraints to get

$$\begin{cases}(1-\delta)c_t + c_{t+1} = (1-\delta)\omega_t + \omega_{t+1}\\ c_t \le \omega_t.\end{cases}$$

- Think of the first constraint as a budget constraint where prices are denoted in units of the second good ($1-\delta = p_t/p_{t+1}$).
- These two constraints together is a truncated budget line.

13.3. **A Perfect Credit Market.** The descriptive two-period problem:

$$v = \max_{c_t, c_{t+1}} U(c_t, c_{t+1}) \quad \text{s.t.} \quad \begin{cases} c_t + b_{t+1} = \omega_t,\\ c_{t+1} = \omega_{t+1} + (1+r)b_{t+1},\end{cases}$$

where r is the interest rate.

The canonical two-period problem: we may combine the constraints to get

$$(1+r)c_t + c_{t+1} = (1+r)\omega_t + \omega_{t+1}.$$

Note that from the first order conditions, we have an Euler equation:

Proposition 13.2.

$$u'(c_t) = (1+r)\beta u'(u_{t+1}).$$

Remark 13.3. The constant $(1+r)\beta$ is a "effective discount factor".

Example 13.4. If $(1+r)\beta = 1$ we have

$$c^* = \frac{1+r}{2+r}\omega_t + \frac{1}{2+r}\omega_{t+1}.$$

Note that consumption depends also on future consumption — Friedman's permanent income hypothesis.

14. Choice Under Uncertainty

Throughout this section, we use $A = \{a_1, \dots, a_n\}$ to denote a finite set of **outcomes**. Our objects of interest in this section are gambles or lotteries over these outcomes.

Definition 14.1. A **simple gamble** assigns a probability, p_i, to each of the outcomes a_i in A. We denote a simple gamble as $(p_1 \circ a_1, \dots, p_n \circ a_n)$ and use $\mathcal{G}_S$ to denote the set of simple gambles on A. A **compound gamble**, by contrast, has other gambles as prizes. We denote the set of all gambles, simple and compound, using $\mathcal{G}$.

Remark 14.2. We may also denote a gamble as

$$L = \left[I_i, I_j; \pi_i, \pi_j\right] = \left(\pi_i \circ I_i, \pi_j \circ I_j\right).$$

We call the stage at which one makes the decision ***ex ante*** and the stage after the lottery is realized and the uncertainty resolved ***ex post***.

We have the following **axioms of choice under uncertainty** for an individual's preference relation over the set of all gambles $\mathcal{G}$:

- G1. Completeness.
- G2. Transitivity. We can thus order the finite set of outcomes A as follows:
$$a_1 \succeq a_2 \succeq \cdots \succeq a_n.$$
- G3. Continuity. For any gamble $g \in \mathcal{G}$, there is some probability $\alpha \in [0, 1]$ such that
$$g \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n).$$
- G4. Monotonicity. For all probabilities $\alpha, \beta \in [0, 1]$,
$$(\alpha \circ a_1, (1 - \alpha) \circ a_n) \succeq (\beta \circ a_1, (1 - \beta) \circ a_n)$$
if and only if $\alpha \geq \beta$.
- G5. Substitution. If $g = (p_1 \circ g^1, \dots, p_k \circ g^k)$ and $h = (p_1 \circ h^1, \dots, p_k \circ h^k)$ are in $\mathcal{G}$, and if $g^i \sim h^i$ for every i, then $h \sim g$.
- G6. Reduction to Simple Gambles. For any gamble $g \in \mathcal{G}$, if $(p_1 \circ a_1, \dots, p_n \circ a_n)$ is the simple gamble **induced** by g, then $(p_1 \circ a_1, \dots, p_n \circ a_n) \sim g$ We say $g \in \mathcal{G}$ induces a simple bundle g' if they assign the same effective probabilities to each outcome in A.

Definition 14.3. The utility function $U : \mathcal{G} \to \mathbb{R}$ has the **expected utility property** if, for every $g \in \mathcal{G}$,

$$U(g) = \sum_{i=1}^{n} p_i u(a_i),$$

where $(p_1 \circ a_1, \dots, p_n \circ a_n)$ is the simple gamble induced by g. When the above is true, $U : \mathcal{G} \to \mathbb{R}$ is called a **von Neumann-Morgenstern (VNM)** utility functions and $u : A \to \mathbb{R}$ a **state sub-utility function**.

Remark 14.4. One problem of using expected income is that it is not always defined. Remember the St. Petersburg paradox, responding to which Bernoulli claims that it is more reasonable to have $u'(I) = \mathrm{d}I/I$, which gives $u(I) = \log I$. Economists, however, ignored this — until VNM.

Theorem 14.5. *If preference $\succeq$ over gambles in $\mathcal{G}$ satisfy axioms* G1 *to* G6*, then there exists a utility function $U : \mathcal{G} \to \mathbb{R}$ representing $\succeq$ on $\mathcal{G}$ such that U has the expected utility property. Moreover, U is unique up to positive affine transformation.*

14.1. **Risk Aversion.** We consider, now, gambles over wealth ($A = \mathbb{R}_+$) and assume $u' > 0$.

Definition 14.6. For the simple gamble g with expected income $\mathbb{E}\, I$, an individual is said to be

- **risk averse** at g is $u(\mathbb{E}\, I) > U(g)$,
- **risk neutral** at g is $u(\mathbb{E}\, I) = U(g)$,
- **risk loving** at g is $u(\mathbb{E}\, I) < U(g)$.

If an individual is risk averse every non-degenerate simple gamble g, the individual is said simply to be risk averse.

Proposition 14.7. *Let u be an individual's VNM utility function. The individual is risk averse if and only if u is strictly concave, is risk neutral if and only if u is linear, and is risk loving if and only if u is strictly convex.*

Definition 14.8. The **certainty equivalent (CE)** of any simple gamble g over wealth levels is an amount of wealth, CE, offered with certainty, such that $u(g) = u(CE)$. The **risk premium** is an amount of wealth, P, such that $u(g) = u(\mathbb{E}\, I - P)$.

Remark 14.9. In the I-u space, the point corresponding to the expected utility $(CE, u(CE))$ has the same u value as the point

$$\pi_i\big(I_i, u(I_i)\big) + \pi_j\big(I_j, u(I_j)\big).$$

Remember the picture!

Proposition 14.10. *If the individual is risk averse, then $CE < u(\mathbb{E}\, I)$.*

14.2. **Application in Criminal Behavior.** We may model the decision to commit a crime as a gamble. If not caught, the individual retains income I, but if caught, the individual faces a fine F and has income $I - F$. The expected income of the gamble is then

$$\mathbb{E}\, I = \pi_{NC} I + \pi_C (I - F) = I - \pi_C F.$$

- Policy A: raise π_C by p%.
- Policy B: raise fine F by p%.

Both policies raises the expected fine $\pi_C F$ by the same percentage, but see visually that $u(CE_B) < u(CE_A)$. That is, the expected utility of policy under policy B is less than under policy A, and the deterrence effect of policy B is stronger!

Appendix A: Utility Functions

14.3. Perfect Complements.

$$U(x, y) = \min\left\{\frac{x}{a}, \frac{y}{b}\right\}.$$

$$\frac{x^m}{a} = \frac{y^m}{b} = v = \frac{m}{ap_x + bp_y}.$$

$$x^h = aU, \quad y^h = bU, \quad e = (ap_x + bp_y)U.$$

- This is also known as a **Leontief utility function**.
- Think a units of x "pairs" with b units of y.
- All income effect; no substitution effect.

14.4. Perfect Substitutes.

$$U(x, y) = ax + by.$$

$$e = \min\left\{\frac{p_x}{a}, \frac{p_y}{b}\right\} U.$$

- Think of p_x/a as the price of obtaining one utils by buying x.
- All substitution effects; no income effect.

14.5. Cobb-Douglas for Two Goods.

$$U(x, y) = x^\alpha y^{1-\alpha}.$$

$$x^m = \frac{\alpha m}{p_x}, \quad y^m = \frac{(1-\alpha)m}{p_y}, \quad \lambda^* = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha},$$

$$v = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \cdot m.$$

$$e = U \cdot \left(\frac{p_x}{\alpha}\right)^\alpha \left(\frac{p_y}{1-\alpha}\right)^{1-\alpha},$$

$$x^h = U \cdot \left(\frac{\alpha}{p_x} \cdot \frac{p_y}{1-\alpha}\right)^{1-\alpha}, \quad y^h = U \cdot \left(\frac{\alpha}{p_x} \cdot \frac{p_y}{1-\alpha}\right)^{-\alpha},$$

- Proportion of income spent on each good is constant.
  - Income elasticity of demand is 1 for each good.
  - Substitution effects are offsetted exactly by the income effected.
  - All goods are unrelated (neither substitutes nor complements).
- Homothetic.
- For a two person Cobb-Douglas competitive economy, the equilibrium price is 1 if

$$\alpha = \gamma = \frac{\sum_i \omega_x^i}{\sum_i \omega_x^i + \sum_i \omega_y^i}.$$

14.6. **Cobb-Douglas.**

$$U(x) = \prod x_i^{\alpha_i}, \quad \text{where} \quad \sum \alpha_i = 1.$$

$$x_j^m = \frac{\alpha_j m}{p_j}, \quad v = m \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}.$$

$$x_j^h = u \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i} \cdot \frac{\alpha_i}{p_i}, \quad e = u \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i}.$$

- We have $s_i = \alpha_i$ for each i.

14.7. **Constant Elasticity of Substitution Utility Function.**

$$U(x, y) = \left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{-1/\rho}.$$

$$x_1^m = \frac{m}{p_1 + \kappa p_2}, \quad x_2^m = \frac{\kappa m}{p_1 + \kappa p_2}, \quad \kappa = \left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}.$$

- Shares spent on each good is constant $x_2^m = \kappa x_1^m$.
- Income elasticities of demand is 1.
- Indirect utility function is proportional to income. $v = \lambda \cdot m$.
- Constant elasticity of substitution:

$$\sigma = \frac{\mathrm{d}\log\left(\frac{x_1}{x_2}\right)}{\mathrm{d}\log\left(\frac{U_2}{U_1}\right)} = \frac{1}{\rho + 1}.$$

$$x_1^h = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot U$$

$$x_2^h = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot \kappa U$$

$$e = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot U \cdot [p_1 + \kappa p_2].$$

- Hicksian demands are proportional to utility level.

14.8. **Quasilinear Utility Functions.**

$$U(x, y) = V(x) + y.$$

- Good x is neither normal nor inferior: The willingness to pay for x

$$\frac{U_x}{U_y} = V'(x)$$

  does not change as we vary consumption of y (by varying income). The consumption of x does not vary as income vary.

- The optimality condition is
$$\frac{U_x}{U_y} = V'(x) = \frac{p_x}{p_y}.$$
- Consider edge cases.

14.9. **Quadratic Utility Function.**
$$U(x, y) = -\frac{1}{2}(x - b_x)^2 - \frac{1}{2}(y - b_y)^2.$$
$$x^m = b_x + \frac{p_x}{p_x^2 + p_y^2}\left(m - p_x b_x - p_y b_y\right),$$
$$y^m = b_y + \frac{p_y}{p_x^2 + p_y^2}\left(m - p_x b_x - p_y b_y\right),$$
$$v = -\frac{1}{2} \cdot \frac{(m - p_x b_x - p_y b_y)^2}{p_x^2 + p_y^2}.$$

- Think bliss point.

Appendix B: Models

14.10. **The Baumol–Tobin model.**

- Exhausts monthly income Y.
- Interest rate i.
- Goes to ATM N times a month, each time withdrawing W with a direct cost of F incurred.
- Assume constant rate of spending, and money demand is average holding of money $M = W/2$.

$$\begin{aligned} e(F, Y, i) &= \min_{W,N} NF + \frac{Wi}{2} \quad \text{s.t.} \quad NW = Y \\ &= \min_{N} NF + \frac{Yi}{2N}. \end{aligned}$$

Solving it gives

$$N^* = \sqrt{\frac{Yi}{2F}}.$$