ECON20010 NOTES

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1. The Envelope Theorem

Theorem 1.1. *Let*

$$v(a) = \max_x f(x; a).$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}a} = \frac{\mathrm{d}f(x^*; a)}{\mathrm{d}a} = \left.\frac{\partial f}{\partial a}\right|_{x=x^*}.$$

Remark 1.2. Intuition:

$$\frac{\mathrm{d}f(x^*; a)}{\mathrm{d}a} = \sum \frac{\partial f}{\partial x_i^*} \cdot \frac{\partial x_i^*}{\partial a} + \frac{\partial f}{\partial a},$$

and at optimum, each $\partial f / \partial x_i^* = 0$. "All indirect effects vanish." Note that by the implicit function theorem, we need $f_{xx} \neq 0$.

Example 1.3. Consider the value function

$$\begin{aligned} v(p_x, p_y, m) &= U(x^*, y^*) = U(x^*, y^*) + \lambda^*[m - p_x x - p_y y] \\ &= \mathcal{L}(x^*, y^*, \lambda^*; p_x, p_y, m) \\ &=: \mathcal{L}^*(p_x, p_y, m). \end{aligned}$$

We then have by the envelope theorem, $\mathrm{d}\mathcal{L}/\mathrm{d}m = \partial\mathcal{L}/\partial m$ and thus

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}^*}{\partial m} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

Similarly,

$$\frac{\partial v}{\partial p_x} = \frac{\partial \mathcal{L}^*}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

2. Scarcity: the Budget Constraint

Definition 2.1.

- The **budget set** consists of all feasible consumption bundles.
- The **budget constraint** exactly exhausts the consumer's income.

2.1. Budget Set.

The relative price:

$$\frac{p_x}{p_y}$$

- Mnemonic: this is always the price of x in units of y.

To stay on the budget constraint,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p_x}{p_y}.$$

- Think "the rate at which the market *allows* the consumer to exchange good x for good y."
- Think the opportunity cost of good x.

2.2. Preference.

Basic axioms:

- **Completeness**. For any pair of consumption of bundles, say c_1 and c_2, either $c_1 \succeq c_2$, $c_2 \succeq c_1$, or both.
  - Requires a answer and assumes no framing effects.
- **Transitivity**. If $c_1 \succeq c_2$ and $c_2 \succeq c_3$ then $c_1 \succeq c_3$.
  - Money pump.
- A preference ordering is **rational** if it satisfies completeness and transitivity. They are the minimal requirement for the existence of a utility function representation.

We also typically assume the following:

- **Continuity**. If $c_1 \succ c_2$ then there are neighborhoods N_1 and N_2 around c_1 and c_2 such that

  $$x \succ y, \quad \forall x \in N_1, \quad y \in N_2.$$

  This implies that if $c_1 \succ c_2$ then there exists c_3 such that

  $$c_1 \succ c_3 \succ c_2.$$

- **Monotonicity**.
  - **Monotone**. If $c_1 \gg c_2$[1] then $c_1 \succ c_2$.
  - **Strongly monotone**. If $c_1 \geq c_2$[2] and $c_1 \neq c_2$ then $c_1 \succ c_2$.
  - **Local non-satiation**. If for every bundle c and every $\epsilon > 0$, there exists $x \in N_\epsilon(c)$ such that $x \succ c$.

[1] We write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i, \forall i$.

[2] We write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i, \forall i$.

- **Convexity**. If $c_1 \succeq c_2$, then
$$\theta c_1 + (1-\theta)c_2 \succeq c_2, \quad \forall \theta \in (0,1).$$
If convexity is satisfied, the **upper contour set**, the "at least as good as" set, is convex.

Additional axioms place even more structures on the utility function:

- **Homotheticity**. If $c_1 \succeq c_2$, then
$$tc_1 \succeq tc_2, \quad \forall t > 0.$$
- **Quasilinearity** in good i. If $\mathbf{c_1} \succeq \mathbf{c_2}$, then
$$\mathbf{c_1} + t\mathbf{e}_i \succeq \mathbf{c_2} + t\mathbf{e}_i, \quad \forall t > 0.$$

2.3. Translating preference ordering to the utility function:

Theorem 2.2.

- *If a preference ordering is rational, then it admits a utility function representation. (Representation Theorem;[3]) The utility function is unique up to a monotonically increasing transformation.*
- *If a preference ordering satisfies convexity, then the corresponding utility function representation will be quasi-concave. The indifference curves (level sets) will have non-increasing marginal rate of substitution (slopes).*

2.4. The Marginal Rate of Substitution.

The MRS
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y}$$
is the quantity of y the consumer is willing to sacrifice in exchange for an additional unit of x. (Think p_x/p_y.) It measures an individual's **willingness to pay** for x in terms of y.

[3] For a simple version of this, think assigning the size of the unique bundle on $t\sum \mathbf{e}$ equivalent to a given consumption bundle.

3. Utility Maximization

The problem:

$$v(p_x, p_y, m) \coloneqq \max_{x,y} U(x, y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

3.1. **Interpretation.**

We want to maximize

$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$

such that

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$

This gives

$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x.$$

We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.
$$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \ \mathrm{d}x$$
"Take advantage of all trading opportunities."
- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that U_x/p_y is marginal utility of money *spent on* x.
$$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \ \mathrm{d}x$$
"Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that U_x is the marginal benefit of buying x and $U_y \cdot p_x/p_y$ is the marginal cost of buying x.
$$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$
"Trade until marginal cost equals marginal benefit."

In the last expression, if we write

$$\lambda = \frac{U_y}{p_y},$$

(think marginal utility of income) we have that at optimum,

$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$

These three equalities describe precisely the critical points of the following

$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right],$$

called the **Lagrangian**. That is, setting

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

recovers the above three equations.

Remark 3.1.

- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- λ might be negative or zero. Think bliss point.

3.2. **The Indirect Utility Function.**

Proposition 3.2.

$$\frac{\partial v}{\partial m} = \lambda^*.$$

Proof. Noting

$$v = U(x^*, y^*) + \lambda^*[m - p_x x^* - p_y y^*] = \mathcal{L}^*,$$

we have

$$\begin{aligned}
\frac{\partial v}{\partial m} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} \\
&= U_x^* \frac{\partial x}{\partial m} + U_y^* \frac{\partial y}{\partial m} + \lambda^* \left[1 - p_x \frac{\partial x}{\partial m} - p_y \frac{\partial y}{\partial m}\right] + \frac{\partial \lambda}{\partial m}\left[m - p_x x^* - p_y y^*\right] \\
&= \left(U_x^* - \lambda^* p_x\right) \frac{\partial x}{\partial m} + \left(U_y^* - \lambda^* p_y\right) \frac{\partial y}{\partial m} + \frac{\partial \lambda^*}{\partial m}\left(m - p_x x^* - p_y y^*\right) + \lambda^* \\
&= \lambda^*.
\end{aligned}$$

The last equality follows by noting that at the optimum,

$$U_x^* - \lambda^* p_x = U_y^* - \lambda^* p_y = m - p_x x^* - p_y y^* = 0.$$

Alternatively, one may use the envelope theorem:

$$\frac{\partial v}{\partial m} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

□

Note that

$$\frac{\partial v}{\partial m} = \lambda^* = \frac{U_x^*}{p_x} = \frac{U_x^*}{p_x}.$$

So when not satiated ($U_x, U_y \neq 0$), marginal utility of income is positive. When budget constraint does not require to bind, the marginal utility of income is generally nonnegative.

Again using the Envelope Theorem, we have

$$\frac{\partial v}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

This value is generally nonpositive, and only zero when one does not consume the specific good or when the marginal utility of that good is 0.

4. Expenditure Minimization

The problem:

$$e(p_x, p_y, \overline{U}) \coloneqq \max_{x,y} p_x x + p_y y \quad \text{s.t.} \quad U(x, y) = \overline{U}.$$

The Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \eta \left[\overline{U} - U(x, y)\right]$$

$$\begin{aligned} &[x] && p_x = \eta^* U_x(x^*, y^*) \\ &[y] && p_y = \eta^* U_y(x^*, y^*) \\ &[\eta] && \overline{U} = U(x^*, y^*). \end{aligned}$$

Properties 4.1. Properties of Hicksian demand functions:

(i) Homogeneous of degree 0 in prices:

$$x_i^H(\alpha\mathbf{p}, U) = x_i^H(\mathbf{p}, U).$$

Differentiating with respect to α gives

$$\sum \frac{\partial x_i^H}{\partial p_j} p_j = 0 \implies \sum \epsilon_{ij}^H = 0.$$

(ii) Cross-price effects on Hicksian demand are symmetric:

$$\frac{\partial x_i^H}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial x_j^H}{\partial p_i}.$$

From this we have

$$p_i x_i \frac{p_j}{x_i} \frac{\partial x_i^H}{\partial p_j} = p_j x_j \frac{p_i}{x_j} \frac{\partial x_j^H}{\partial p_i}.$$

That is,

$$s_i \epsilon_{ij}^H = s_j \epsilon_{ji}^H \implies \frac{\epsilon_{ij}^H}{\epsilon_{ji}^H} = \frac{s_j}{s_i}.$$

The more important good impacts the less important good more.

(iii) Differentiating $U(\mathbf{x}^H(\mathbf{p}, U)) = U$ with respect to p_i gives

$$\sum \frac{\partial U}{\partial x_j} \frac{\partial x_j^H}{\partial p_i} = 0 \implies \sum p_j \frac{\partial x_j^H}{\partial p_i} = 0 \implies \sum s_j \epsilon_{ji}^H = 0.$$

Symmetry and homogeneity [adding up] gives adding up [homogeneity]. In case where there are two goods only, the latter two also gives symmetry.

5. Changes in Behavior

Consider a price increase from p_x^o to p_x^f. Let o be the original consumption, f be the final consumption, and s be the optimal consumption after an income transfer such that the individual stays on the same indifference curve as before (has the same purchasing power). We may decompose $x^f - x^o$:

$$x^f - x^o = x^f - x^s + x^s - x^o.$$

- $x^f - x^o$: the Marshallian price effect (the total effect).
- $x^f - x^s$: effect due to compensation (the income effect).
- $x^s - x^o$: the Hicksian price effect (substitution effect).

5.1. **The Slutsky Equation.**

Recall from duality that

$$x^h(p_x, p_y, \overline{U}) = x^m(p_x, p_y, m = e(p_x, p_y, \overline{U})).$$

As price changes, changes in $e(p_x, p_y, \overline{U})$ ensures that purchasing power does not change.

By differentiating, we get the **Slutsky equation**:

$$\frac{\partial x_h}{\partial p_x} = \frac{\partial x_m}{\partial p_x} + \frac{\partial x_m}{\partial m} \cdot \frac{\partial e}{\partial p_x},$$

which we can rewrite using the envelop theorem as

$$\frac{\partial x_h}{\partial p_x} = \frac{\partial x_m}{\partial p_x} + \frac{\partial x_m}{\partial m} \cdot x_m.$$

This shows that we can recover the unobservable $\partial x_h / \partial p_x$ from the observables.

We can also rewrite the Slutsky equation as

$$\frac{\partial x_m}{\partial p_x} = \underbrace{\frac{\partial x_h}{\partial p_x}}_{\text{Substitution Effect}} + \underbrace{\left(-\frac{\partial x_m}{\partial m} \cdot x_m\right)}_{\text{Income Effect}}.$$

5.2. **Two Ways to think about compensation.**

(The setting is a increase in the price of x.)

- **Slutsky transfer** keeps the original bundle affordable.

$$T_S = \Delta p_x \cdot x^o.$$

- **Hicks transfer** keeps the original utility level affordable.

$$T_H = e(p_x^f, p_y, v^o) - m = e(p_x^f, p_y, v^o) - e(p_x^o, p_y, v^o).$$

In the Slutsky equation,

$$\frac{\partial e}{\partial p_x} = x_m = x_h$$

is the continuous analogue of the Hicks transfer.

Remark 5.1. Note that $T_S \geq T_H$.

5.3. **Law of Demand.** The substitution effect follows the law of demand:

$$\partial x^h / \partial p_x \leq 0.$$

Proposition 5.2 (Generalized law of demand)**.**

$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) \leq 0.$$

Proof. Note that

$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) = \left(\mathbf{x}^1\mathbf{p}^1 - \mathbf{x}^0\mathbf{p}^1\right) + \left(\mathbf{x}^0\mathbf{p}^0 - \mathbf{x}^1\mathbf{p}^0\right).$$

The last two terms are both non-positive. □

Remark 5.3.

- Note that this gives $\partial x_i^h / \partial p_i \leq 0$ (if the derivative exists).
- But think also graphs for the case of two goods.
- Note that the law of demand holds not only when indifference curves are concave. In the concave case, the expenditure minimizing points occur at the edges. In the perfect complement case, we have that $\partial x^h / \partial p_x = 0$.

5.4. **Giffen Goods.** Marshallian demand does not always comply with the law of demand. A good whose Marshallian demand does not comply with the law of demand is called a **giffen good**. Their existence is theoretically possible, but not empirically supported.

Looking back at the Slutsky equation, we see that for a good x to be a giffen good, we need the following three conditions

(i) the individual buys a large amount of x,
(ii) good x is inferior,
(iii) the demand for good x is elastic.

These three conditions do not occur together often: narrowly defined categories usually has 0 elasticity, but broad categories are usually normal goods.

5.5. **Normal & Inferior Goods.** The experiment for testing normality: We fix the consumption of x and vary income m. For normal goods, the willingness to pay for x increases as income increase. Thus x is normal if

$$\frac{\partial U_x / U_y}{\partial y} > 0.$$

Example 5.4. With the quasilinear utility function $U(x, y) = v(x) + y$, the good x is neither normal nor inferior. The willingness to pay

$$\frac{U_x}{U_y} = \frac{v'(x)}{1}$$

does not change as we vary y (by varying income).

5.6. **Cross Effects.**

Definition 5.5. We say y is

- a **substitute** of x if
$$\partial y / \partial p_x > 0$$
- a **complement** of x if
$$\partial y / \partial p_x < 0$$
- **unrelated** with x if
$$\partial y / \partial p_x = 0.$$

If we use $y = y_h$ in the definition above, we say **gross** substitutes/complements; if we use y_m, we say **net** substitutes/complements.[4]

Remark 5.6.

- Cross price effects for Hicksian demands are symmetric:
$$\frac{\partial x_h}{\partial p_y} = \frac{\partial^2 e}{\partial p_x \partial p_y} = \frac{\partial y_h}{\partial p_x}.$$
This does not hold in general for Marshallian demands; see the cross-price Slutsky equation.
- For any good x, at least one other good is a net substitute with x. If not, as price of x increase, consumption and thus utility level strictly decreases (note that consumption of x also decreases by the law of demand). In particular, when there are only two goods, the two goods cannot be net complements.

Proposition 5.7. *The **cross-price Slutsky equation**:*
$$\frac{\partial y_h}{\partial p_x} = \frac{\partial y_m}{\partial p_x} + \frac{\partial y_m}{\partial m} \cdot \frac{\partial e}{\partial p_x}$$
where
$$\frac{\partial e}{\partial p_x} = x_m = x_h.$$

[4]Think Hicksian demand “nets out” the income effect.

6. Elasticities and Aggregation

Proposition 6.1.

- *Engel aggregation:*
$$\sum s_i \eta_i = 1.$$
- *Cournot aggregation:*
$$\sum \sigma_i \epsilon_{ij} = -s_j.$$
- *Implication of homogeneity:*
$$\eta_1 = -\sum \epsilon_{1,i}.$$

Proof.

- From $m = \sum p_i x_i$ we have
$$1 = \frac{\partial m}{\partial m} = \sum p_i \frac{\partial x_i}{\partial m} = \sum \eta_i s_i.$$
- Differentiating the same identity with respect to p_1 gives
$$0 = \frac{\partial m}{\partial p_x} = x_1 + \sum p_i \frac{\partial x_i}{\partial p_1}. = x_1 + \sum \epsilon_{i,1} \cdot \frac{p_i x_i}{p_1},$$
which simplifies to the desired result.
- Differentiating the identity $x^m(\mathbf{x}, m) = x^m(t\mathbf{x}, m)$ with respect to t gives
$$\sum \frac{\partial x_1}{\partial p_i} \cdot p_i + \frac{\partial x_1}{\partial m} \cdot m = 0.$$

□

Remark 6.2.

- From Engel aggregation: some goods must be normal. Moreover, it will never the case that all goods are inferior, all goods are necessities, or all goods are luxuries.
- From implication of homogeneity: "no good can be a Giffen good $\epsilon_{x_i,p_i} > 0$ unless it is strong complements with other goods."

7. Welfare

7.1. Exact Measures of Welfare Change.

The difference in utility
$$\Delta v = v^f - v^o.$$

- Gets direction right, but magnitude depends on the specific utility representation chosen — it is not invariant to the utility representation.
- Difficulty in the interpretation of units.

Definition 7.1.

- **Compensating variation**: the income transfer that induces the consumer accept the change in price voluntarily.
- **Equivalent variation**: the income transfer that induces the consumer to reject the change in price voluntarily.

7.2. The Compensating Variation.

Proposition 7.2. *There holds*
$$\begin{aligned} CV := -T_H &= -\left[e(\mathbf{p}^f, v^o) - m\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^f, v^f)\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^o, v^o)\right]. \end{aligned}$$

Remark 7.3.

- CV is the utility change in dollars:
$$CV = e(\mathbf{p}^f, v^f) - e(\mathbf{p}^f, v^o).$$
Think of $e(\mathbf{p}^f, \cdot)$ as a monotonic transformation of $U(x, y)$, an equivalent utility representation in units of dollars.[5] This is called the **money metric utility function**.
- CV is invariant.
$$CV = e(\mathbf{p}^o, v^o) - e(\mathbf{p}^f, v^o)$$
is the cost of two different bundles on the same indifference curve, which does not vary according to the utility representation.
- Using the Slutsky transfer as an approximation, we have
$$CV = -T_H \approx -\Delta p_x \cdot x^o.$$

[5] This is a valid transformation since
$$\frac{\partial e}{\partial U} = \eta^* = \frac{p_x}{U_x(x, y)} > 0.$$

7.3. **The Equivalent Variation.**

Proposition 7.4. *There holds*

$$\begin{aligned} EV &= e(\mathbf{p}^o, v^f) - m \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^f, v^f) \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^o, v^o) \end{aligned}$$

Moreover, just as the CV, the EV is an invariant measure of utility change in dollars.

7.3.1. *Equivalent Variation.* The maximum amount of income the consumer is willing to sacrifice to avoid the increase in price is called the **equivalent variation**.

$$T_H = e(p_x^o, p_y, v^f) - m$$

7.4. **The Surplus form.**

Proposition 7.5. *For a price change of p_x^o to p_x^f, we may write the CV as*

$$CV|_{p_x^o}^{p_x^f} := -T_H|_{p_x^o}^{p_x^f} = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\, \mathrm{d}p_x.$$

Similarly, we may write the EV as

$$EV = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^f)\, \mathrm{d}p_x.$$

These are the ***surplus forms****.*

Proof. Note that

$$CV = -\int_{p_x^o}^{p_x^f} \frac{\partial e(p_x, p_y, v^o)}{\partial p_x}\, \mathrm{d}p_x = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\, \mathrm{d}p_x,$$

where the last equality follows from Shepherd's lemma. □

Remark 7.6. We can use the surplus forms to analyzes the relative magnitudes of the CV and EV. To see this note that from

$$\frac{\partial x^h}{\partial U} \cdot \frac{\partial v}{\partial m} = \frac{\partial x^m}{\partial m}$$

we know that $\partial x^h/\partial U$ and $\partial x^m/\partial m$ have the same sign. Thus, if $v^f < v^o$ and x is, for example, a normal good, then

$$x^h(\mathbf{p}, v^f) < x^h(\mathbf{p}, v^o)$$

and

$$|CV| > |EV|.$$

7.5. **The Surplus Approach to Welfare Measurement.**

Definition 7.7. The **inverse demand function** measures the consumer's willingness to pay for the marginal unit of good x. It is given by

$$p_x = p_x(\overline{x}, p_y, \overline{U}),$$

where $\overline{x}$ is the marginal unit of good x.

Total surplus:

$$TS = \int_{p_x}^{\infty} x_h(\rho, p_y, \overline{U})\, \mathrm{d}\rho.$$

7.6. **Consumer's Surplus.**

Definition 7.8. The **change in consumer's surplus** is given by

$$\Delta CS = -\int_{p_x^o}^{p x^f} x_m(\rho, p_y, m)\, \mathrm{d}\rho.$$

7.7. **An Alternative Definition of Normal and Inferior Goods.** From $x_h(p_x, p_y, \overline{U}) = x_m(p_x, p_y, e(p_x, (y, \overline{U})))$, we have

$$\frac{\partial x_h(p_x, p_y, \overline{U})}{\partial \overline{U}} = \frac{\partial x_m(p_x, p_y, m)}{\partial m} \eta^*(p_x, p_y, m).$$

Thus $\frac{\partial x_h}{\partial \overline{U}}$ and $\frac{\partial x_m}{\partial m}$ have the same sign and we can equivalently define x as a normal (inferior) good if x_h moves in the same (different) direction as a change in target utility.

7.8. **A ranking of CV, EV, CS.** For a normal good x and a price increase in p_x, since $x_h(\epsilon, p_y, v^o) > x_h(\rho, p_y, v^f)$, we have

$$|CV| > |EV|.$$

Noting the intersections of x_h and x_m, we have

$$|CV| > |CS| > |EV|.$$

7.9. **Price Indices.** The **ideal index**:

$$I = \frac{e(p_x^f, p_y^f, \overline{U})}{e(p_x^o, p_y^o, \overline{U}}.$$

The **Laspeyres price index**:

$$L = \frac{p_x^f x^o + p_y^f y^o}{p_x^o x^o + p_y^o y^o}.$$

The **Paasche price index**:

$$L = \frac{p_x^f x^f + p_y^f y^f}{p_x^o x^f + p_y^o y^f}.$$

Appendix A: Utility Functions

7.10. **Perfect Complements.**

$$U(x, y) = \min\left\{\frac{x}{a}, \frac{y}{b}\right\}.$$

7.10.1. *Solution of EMP.*

$$x^h = aU, \quad y^h = bU, \quad e = (p_x a + p_y b)U.$$

Remark 7.9.

- Think a units of x "pairs" with b units of y.
- All income effect; no substitution effect.

7.11. **Perfect Substitutes.**

$$U(x, y) = ax + by.$$

7.11.1. *Solution of EMP.*

$$e = \min\left\{\frac{p_x}{a}, \frac{p_y}{b}\right\} U.$$

Remark 7.10.

- Think of p_x/a as the price of obtaining one utils by buying x.
- All substitution effects; no income effect.

7.12. **Cobb-Douglas for Two Goods.**

$$U(x, y) = x^\alpha y^{1-\alpha}.$$

7.12.1. *Solution of UMP.*

$$x^m = \frac{\alpha m}{p_x}, \quad y^m = \frac{(1-\alpha)m}{p_y}, \quad \lambda^* = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha},$$
$$v = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \cdot m.$$

Remark 7.11.

- Proportion of income spent on each good is constant.
- Income elasticities of demand is 1.
- Substitution effects are offset ted exactly by the income effected.
- All goods are unrelated (neither substitutes nor complements).

7.12.2. *Solution of EMP.*

$$e = \left(\frac{p_x}{\alpha}\right)^{\alpha}\left(\frac{p_y}{1-\alpha}\right)^{1-\alpha},$$

$$x^h = \left(\frac{\alpha}{p_x}\cdot\frac{p_y}{1-\alpha}\right)^{1-\alpha}, \quad y^h = \left(\frac{\alpha}{p_x}\cdot\frac{p_y}{1-\alpha}\right)^{-\alpha},$$

Remark 7.12. Easily obtained using $e = v^{-1}$ and Shepherd's lemma.

7.13. **Cobb-Douglas.**

$$U(x) = \prod x_i^{\alpha_i}, \quad \text{where} \quad \sum \alpha_i = 1.$$

7.13.1. *Solutions.*

$$x_j^m = \frac{\alpha_j m}{p_j}, \quad v = m\cdot\prod\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}.$$

$$x_j^h = u\cdot\prod\left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i}\cdot\frac{\alpha_i}{p_i}, \quad e = u\cdot\prod\left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i}.$$

Remark 7.13. $s_i = \alpha_i$.

7.14. **Constant Elasticity of Substitution Utility Function.**

$$U(x,y) = \left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{-1/\rho}.$$

7.14.1. *Solution of the UMP.*

$$x_1^m = \frac{m}{p_1+\kappa p_2}, \quad x_2^m = \frac{\kappa m}{p_1+\kappa p_2}, \quad \kappa = \left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}.$$

Remark 7.14.

- Shares spent on each good is constant $x_2^m = \kappa x_1^m$.
- Income elasticities of demand is 1.
- Indirect utility function is proportional to income. $v = \lambda\cdot m$.
- Constant elasticity of substitution:

$$\sigma = \frac{\mathrm{d}\log\left(\frac{x_1}{x_2}\right)}{\mathrm{d}\log\left(\frac{U_2}{U_1}\right)} = \frac{1}{\rho+1}.$$

7.14.2. *Solution of the EMP.*

$$x_1^h = (1-\omega\kappa^{-\rho})^{\frac{1}{\rho}}\cdot U$$
$$x_2^h = (1-\omega\kappa^{-\rho})^{\frac{1}{\rho}}\cdot\kappa U$$
$$e = (1-\omega\kappa^{-\rho})^{\frac{1}{\rho}}\cdot U\cdot[p_1+\kappa p_2].$$

Remark 7.15. Hicksian demands are proportional to utility level.

7.15. **Quasilinear Utility Functions.**

$$U(x, y) = V(x) + y.$$

Remark 7.16. Good x is neither normal nor inferior: The willingness to pay for x

$$\frac{U_x}{U_y} = V'(x)$$

does not change as we vary consumption of y (by varying income). The consumption of x does not vary as income vary.

Appendix B: Models

7.16. **The Baumol–Tobin model.**

- Exhausts monthly income Y.
- Interest rate i.
- Goes to ATM N times a month, each time withdrawing W with a direct cost of F incurred.
- Assume constant rate of spending, and money demand is average holding of money $M = W/2$.

$$\begin{aligned} e(F, Y, i) &= \min_{W,N} NF + \frac{Wi}{2} \quad \text{s.t.} \quad NW = Y \\ &= \min_{N} NF + \frac{Yi}{2N}. \end{aligned}$$

Solving it gives

$$N^* = \sqrt{\frac{Yi}{2F}}.$$