MATH20410 (W25): Analysis in Rn II (accelerated)

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Contents

1 Single-Variable Differential Calculus

In this chapter, we consider mainly functions of the form $f : I \to \mathbb{R}$, where I is an interval, e.g., (a, b), $[a, b]$, $(a, b]$, (a, ∞), $\mathbb{R}$. This is the function we have in mind unless otherwise stated.

Definition 1.1 (Differentiability)**.** We say f is **differentiable at** $x \in I$ if the limit

$$f'(x) \coloneqq \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists. In this case, we call $f'(x)$ the derivative of f at x. Moreover:

- We say that f is **differentiable** if $f'(x)$ exists for each $x \in I$.
- We say f is **continuously differentiable** ($f \in C^1$) if $f' : I \to \mathbb{R}$ is continuous.

Example 1.2.

- $f(x) = |x|$. Differentiable on $\mathbb{R} \setminus \{0\}$.
- $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Continuous but not differentiable at 0.
- $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Differentiable everywhere (in particular at 0), but $f \notin C^1$.

Proposition 1.3 (Rules for computing derivatives)**.**

(i) Linearity. $(af + bg)' = af' + bg'$ *(if f' and g' exist, such requirements are hereafter omitted).*

(ii) Product rule. $(fg)' = f'g + fg'$.

(iii) Quotient rule. $(f/g)' = (f'g - fg')/g^2$.[1]

(iv) Chain rule. $(f \circ g)' = (f' \circ g) \cdot g'$.

[1] Low dhigh minus high dlow. Not Haidilao...

Proof. We prove the quotient rule; the remaining are left as exercises. Starting from the definition

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{t\to x}\frac{\frac{f}{g}(t)-\frac{f}{g}(x)}{t-x}\\ &= \lim_{t\to x}\frac{\frac{f(t)}{f(t)}+\frac{f(x)}{g(t)}-\frac{f(x)}{g(t)}+\frac{f(x)}{g(x)}}{t-x}.\end{aligned}$$

Note that

$$\frac{\frac{f(x)}{g(t)}+\frac{f(x)}{g(x)}}{t-x}=\frac{f(x)}{g(x)g(t)}\frac{g(x)-g(t)}{t-x}$$

and we have

$$\left(\frac{f}{g}\right)'(x)=\frac{f'(x)}{g(x)}-\frac{f(x)g'(x)}{g^2(x)}$$

□

Theorem 1.4. *If f is differentiable at x then f is continuous at x.*

Proof. Note that

$$\lim_{t\to x} f(t)-f(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}(t-x)=f'(x)\cdot 0=0.$$

□

1.1 The Mean Value Theorem

Lemma 1.5. *Suppose $f : [a,b] \to \mathbb{R}$ has a local maximum or minimum at $x \in (a,b)$. If $f'(x)$ exists, then $f'(x)=0$.*

Proof. From the definition of the derivative, consider the limits from the left and right; one is non-positive and the other is non-negative. □

Theorem 1.6 (Rolle's Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), and such that $f(a)=f(b)$. Then there exists $x \in (a,b)$ such that $f'(x)=0$.*

Proof. Consider the global maximum or minimum (exist since f is continuous defined on a compact set) and apply the previous lemma. (If both the maximum and minimum is at a or b, f is constant.) □

Theorem 1.7 (Mean Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be such that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x \in (a,b)$ such that $f(b) - f(a) = f'(x)(b-a)$.*

Proof. Apply Rolle's to $\tilde{f} = f - [f(b) - f(a)] \cdot \frac{x-a}{b-a}$. □

Theorem 1.8. *Let $f : (a,b) \to \mathbb{R}$ be differentiable.*

(a) if $f' = 0$, then f is constant.

(b) if $f' \geq 0$, then f is increasing.

(c) if $f' \leq 0$, then f is decreasing.

Proof. Apply the mean value theorem. □

Theorem 1.9 (The Intermediate Value Property of Derivatives)**.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable[2] and suppose $f'(a) < \lambda < f'(b)$ Then there exists $x \in (a,b)$ such that $f'(x) = \lambda$.*

[2] f need not be C^1!

Proof *(à la Pugh)***.** Slide a small secant of length so small that the slope around a and b is separated also by λ. By continuity of the slope, there exists a secant between a and b with slope λ. Apply the mean value theorem to this slope. □

Proof *(à la Joe/Rudin)***.** We start with $\lambda = 0$. Then $f'(a), f'(b) \neq 0$ and the global min/max of f cannot be at the endpoints. At the global extrema we have the desired result. When $\lambda \neq 0$, consider $\tilde{f} \coloneqq f - \lambda x$. □

Example 1.10. Consider

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

We have

$$f(x) = \begin{cases} 2x \sin(1/x) = \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

which has the intermediate value property.

Theorem 1.11 (Generalized Mean Value Theorem)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x \in (a,b)$ such that*

$$\big(f(a) - f(b)\big)g'(x) = \big(g(a) - g(b)\big)f'(x).$$

Remark 1.12. When the above is not zero,

$$\frac{f(a)-f(b)}{g(a)-g(b)}=\frac{f'(x)}{g'(x)}.$$

Proof. Define

$$h(t) \coloneqq \big(f(b)-f(a)\big)g(t)-\big(g(b)-g(a)\big)f(t).$$

Note that

$$h(a)=f(b)g(a)-f(a)g(b)=h(b)$$

and apply Rolle's. $\square$

1.2 L'Hôpital's Rule

Theorem 1.13 (L'Hôpital's Rule, a particular case)**.** *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $g(x)\neq 0$ in a neighborhood of a and $f(x)=g(x)=0$, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

if the last limit exists.

Proof. Consider some small $\delta>0$. The generalized MVT gives some $x\in(a,a+\delta)$ such that

$$\frac{f(a+\delta)}{g(a+\delta)}=\frac{f'(x)}{g'(x)}\approx\lim_{t\to a}\frac{f'(t)}{g'(t)},$$

where the last approximation follows from the existence of the limit. Note that as $\delta\to 0$, $x\to a$, and the approximation error shrinks to 0. $\square$

Refer to Rudin or something for the general case.

1.3 Higher Derivatives

If $f:I\to\mathbb{R}$ is differentiable, then we can define the second derivative $f''\coloneqq(f')'$ if f' is differentiable. Higher derivatives can be defined similarly. We usually write $f^{(n)}$ for the n-th derivative of f.

Example 1.14. $L(x) = f(x_0) + f'(x_0)(x - x_0)$ is a (first order) linear approximation of f at x_0. How good is this approximation? A first answer is

$$f(x) = L(x) + o(|x - x_0|),$$

since as $x \to x_0$, we have

$$\frac{f(x) - L(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \to 0.$$