MATH20510 (S25): Analysis in Rn III (accelerated)

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1 Integration of Differential Forms

1.1 Integration on a Cell

Definition 1.1. A **k-cell** in $\mathbb{R}^k$ is a set of the form $I^k \coloneqq \{x \in \mathbb{R}^k : a_i \leq x_i \leq b_i, i = 1, \ldots, k\}$.

Definition 1.2. Let $f \in C(I^k)$ be real valued and write $f_k \coloneqq f$. Define for each $i = k, \ldots, 1$

$$f_{i-1}(x_1, \ldots, x_{k-1}) \coloneqq \int_{a_i}^{b_i} f_i(x_1, \ldots, x_i) \, \mathrm{d}x_i.$$

We define

$$\int_{I_k}^{f(x)} \mathrm{d}x \coloneqq \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} f_k(x_1, \ldots, x_k) \, \mathrm{d}x_k \ldots \, \mathrm{d}x_1 = f_0.$$

Remark 1.3.

- Since f is continuous on a compact set, it is uniformly continuous. Thus all iterated integrals are well-defined and uniformly continuous on I^i ($1 \leq i \leq k$).

- The integral over a k-cell is independent of the order of integration, by the following result:

Theorem 1.4. *If $f \in C(I^k)$, then $L(f) = L'(f)$, where $L(f)$ is the integral of f over I^k as defined above, and $L'(f)$ is the integral of f over the same domain with a different order of integration.*

Proof. If $h(x) = f_1(x_1) \ldots h_k(x_k)$, where $h_j \in C([a_j, b_j])$, then

$$L(h) = \prod_{i=1}^{k} \int_{a_i}^{b_i} h_i(x_i) \, \mathrm{d}x_i = L'(h).$$

If $\mathcal{A}$ is the set of all finite sums of such functions h, it follows that $L(g) = L'(g)$ for all $g \in \mathcal{A}$. The Stone-Weierstrass theorem shows that $\mathcal{A}$ is dense in $C(I^k)$. Put $V = \prod_{i=1}^{k}(b_i - a_i)$. If $f \in C(I^k)$ and $\epsilon > 0$, there exists $g \in \mathcal{A}$ such that $\|f - g\| < \epsilon/V$, where $\|f\|$ is defined as $\max_{x \in I^k} |f(x)|$. Then $|L(f - g)| < \epsilon$, $L'(f - g) < \epsilon$, and since

$$L(f) - L'(f) = L(f - g) + L'(g - f),$$

we conclude that $|L(f) - L'(f)| < 2\epsilon$. $\square$

Definition 1.5. The **support** of function f on $\mathbb{R}^k$ is the closure of the set of all points $x \in \mathbb{R}^k$ at which $f(x) \neq 0$. We write $f \in C_c(\mathbb{R}^k)$ if f is a continuous function with compact support, that is, if $K \coloneqq \operatorname{supp} f \subset I^k$ for some k-cell I^k. In this case we define

$$\int_{\mathbb{R}^k} f(x)\,\mathrm{d}x \coloneqq \int_{I^k} f(x)\,\mathrm{d}x.$$

Definition 1.6. Let $G : \mathbb{R}^n \supset E \to \mathbb{R}^n$, where E is open. If there is an integer m and a real function g with domain E such that for all $x \in E$ we have

$$G(x) = \sum x_i e_i + g(x) e_m,$$

then we call G **primitive**.

Remark 1.7.

- In other words, G changes only one coordinate.
- If g is differentiable at $x \in E$, then so if G. The matrix $\mathrm{D}G(x)$ has

  $$(\partial_1 g)(x), \dots, (\partial_m g)(x), \dots, (\partial_n g)(x)$$

  as its mth row. On the jth row, where $j \neq m$, we have the jth unit vector. Thus the Jacobian of G at a is

  $$J_G(a) = \det \mathrm{D}G(a) = (\partial_m g)(a)$$

  and so $G'(a)$ is invertible if and only if $(\partial_m g)(a) \neq 0$.

Definition 1.8. A linear operator B on $\mathbb{R}^n n$ that interchanges some pair of members of the standard basis and leaves the others fixed will be called a **flip**.

Theorem 1.9. *Suppose $F : \mathbb{R}^n \supset E \to \mathbb{R}^n$ is C^1, $0 \in E$, $F(0) = 0$, and $F'(0)$ is invertible. Then there is a neighborhood of 0 in $\mathbb{R}^n$ in which a representation*

$$F(x) = B_1 \dots B_{n-1} G_n \circ \cdots \circ G_1(x)$$

is valid. Each G_i is a primitive C^1 mapping in some neighborhood of 0; $G_i(0) = 0$, $G_i'(0)$ is invertible, and each B_i is either a flip or the identity.

Theorem 1.10 (Partition of Unity)**.** *Let K be a compact subset of $\mathbb{R}^n$. Let $\{V_\alpha\}$ be an open cover of K. Then there exists function $\psi_1, \dots, \psi_k \in C(\mathbb{R}^n)$ such that*

- $0 \le \psi_i \le 1$ *for* $1 \le i \le s$,
- $\operatorname{supp}\psi_i \subset V_\alpha$ *for some* α[1], *and*
- $\sum_i \psi_i = 1$ *for each* $x \in K$.

Corollary 1.11. *If* $f \in C(\mathbb{R}^n)$ *and the support of* f *lies in* K*, then*

$$f = \sum \psi_i f.$$

Each $\psi_i f$ *has support in some* V_α.

Remark 1.12. This is a representation of f using functions with "small" supports. We represent global information using local information.

Theorem 1.13 (Change of Variables)**.** *Let* T *be a one-to-one* C^1 *mapping from an open set* $E \in \mathbb{R}^k$ *into* $\mathbb{R}^k$ *such that* $J_T(x) \neq 0$ *for all* $x \in T$*. If* $f \in C_c(\mathbb{R}^n)$ *and* $\operatorname{supp} f \in T(E)$*, then*

$$\int_{\mathbb{R}^k} f(y)\,\mathrm{d}y = \int_{\mathbb{R}^k} f(T(x))|J_T(x)|\,\mathrm{d}x.$$

Proof. If T is a primitive mapping, then the theorem is true by the one dimensional change of variable theorem. If T is a flip, the theorem reduces to the case in the first theorem of this section.

If the theorem is true for transformations P, Q, and if $S = P \circ Q$, then

$$\begin{aligned}\int f(z)\,\mathrm{d}z &= \int f(P(y))|J_P(y)|\,\mathrm{d}y \\ &= \int f(P(Q(x)))|J_P(Q(x))|\big|J_Q(x)\big|\,\mathrm{d}x = \int f(S(x))|J_S(x)|\,\mathrm{d}x,\end{aligned}$$

where we used the fact that

$$\begin{aligned}J_P(Q(x)) &= \det \mathrm{D}P(Q(x)) \det \mathrm{D}Q(x) \\ &= \det \mathrm{D}P(Q(x))\mathrm{D}Q(x) = \det \mathrm{D}S(x) = J_S(x).\end{aligned}$$

This follows from the chain rule and the fact that the determinant of a product of matrices is the product of the determinants.

[1] This is sometimes expressed by saying that $\{\psi_i\}$ is subordinate to the cover $\{V_\alpha\}$.

Now, for each $a \in E$ there exists a neighborhood $U \subset E$ of a in which

$$T(x) = T(a) + B_1 \dots B_{k-1} G_k \circ \cdots \circ G_1(x - a).$$

It follows that the theorem holds if the support of f lies in $T(U)$.

That is, each point $y \in T(E)$ lies in an open set $V_y \subset T(E)$ such that the theorem holds for all continuous functions whose support lies in V_y.

For an arbitrary function f, we need only write it as a sum of functions with compact support using the partition of unity. $\square$

2 Differential Forms

Definition 2.1 (k-surface)**.** Suppose E in an open set in $\mathbb{R}^n$. A ***k*-surface** in E in a C^1 mapping Φ from a compact set $D \subset \mathbb{R}^k$ into E.

Definition 2.2 (k-form)**.** Let $E \subset \mathbb{R}^n$ be open. A **differential form of order *k* ≥ 1** in E is a function ω, symbolically represented by

$$\omega = \sum a_{i_1 \ldots i_k} \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \, \mathrm{d}x_{i_k},$$

(where the indices $i_1, \ldots, i_k$ range independently from 1 to n), which assigns to each k-surfaces Φ in E a number $\omega(\Phi) = \int_\Phi \omega$ according to the rule

$$\int_\Phi \omega = \int_D \sum a_{i_1 \ldots i_k}(\Phi(u)) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)} \, \mathrm{d}u_1 \ldots \, \mathrm{d}u_k,$$

where D is the parameter domain of Φ.

Definition 2.3.

- We write $\omega_1 = \omega_2$ if and only if $\omega_1(\Phi) = \omega_2(\Phi)$ for every k-surface Φ in E. In particular, $\omega = 0$ means that $\omega(\Phi) = 0$ for every k-surface Φ in E.
- A k-form is said to be of class C^n if the functions $a_{i_1 \ldots i_k}$ are all of class C^n.
- A 0-form in E is defined to be a continuous function in E.
- We write $\Lambda^k(D)$ for the set of all k-forms in D.

Proposition 2.4.

- $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$.

Definition 2.5. If $i_1, \ldots, i_k$ be integers such that $1 \le i_1 < i_2 < \cdots < i_k \le n$ and if I, and if I is the k-tuple $\{i_1, \ldots, i_k\}$, then we call I and **increasing *k*-index**, and we use the brief notation

$$\mathrm{d}x_I \coloneqq \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \, \mathrm{d}x_{i_k}.$$

These are the **basic *k*-forms** in $\mathbb{R}^n$.