MATH20510 (S25): Analysis in Rn III (accelerated)

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1 Integration of Differential Forms

1.1 Integration on a Cell

Definition 1.1. A ***k*-cell** in $\mathbb{R}^k$ is a set of the form $I^k \coloneqq \{x \in \mathbb{R}^k : a_i \leq x_i \leq b_i, i = 1, \ldots, k\}$.

Definition 1.2. Let $f \in C(I^k)$ be real valued and write $f_k \coloneqq f$. Define for each $i = k, \ldots, 1$

$$f_{i-1}(x_1, \ldots, x_{k-1}) \coloneqq \int_{a_i}^{b_i} f_i(x_1, \ldots, x_i)\, \mathrm{d}x_i.$$

We define

$$\int_{I_k}^{f(x)} \mathrm{d}x \coloneqq \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} f_k(x_1, \ldots, x_k)\, \mathrm{d}x_k \ldots\, \mathrm{d}x_1 = f_0.$$

Remark 1.3.

- Since f is continuous on a compact set, it is uniformly continuous. Thus all iterated integrals are well-defined and uniformly continuous on I^i $(1 \leq i \leq k)$.

- The integral over a k-cell is independent of the order of integration, by the following result:

Theorem 1.4. *If $f \in C(I^k)$, then $L(f) = L'(f)$, where $L(f)$ is the integral of f over I^k as defined above, and $L'(f)$ is the integral of f over the same domain with a different order of integration.*

Proof. If $h(x) = f_1(x_1) \ldots h_k(x_k)$, where $h_j \in C([a_j, b_j])$, then

$$L(h) = \prod_{i=1}^{k} \int_{a_i}^{b_i} h_i(x_i)\, \mathrm{d}x_i = L'(h).$$

If $\mathcal{A}$ is the set of all finite sums of such functions h, it follows that $L(g) = L'(g)$ for all $g \in \mathcal{A}$. The Stone-Weierstrass theorem shows that $\mathcal{A}$ is dense in $C(I^k)$. Put $V = \prod_{i=1}^{k}(b_i - a_i)$. If $f \in C(I^k)$ and $\epsilon > 0$, there exists $g \in \mathcal{A}$ such that $\|f - g\| < \epsilon/V$, where $\|f\|$ is defined as $\max_{x \in I^k} |f(x)|$. Then $|L(f - g)| < \epsilon$, $L'(f - g) < \epsilon$, and since

$$L(f) - L'(f) = L(f - g) + L'(g - f),$$

we conclude that $|L(f) - L'(f)| < 2\epsilon$. □

Definition 1.5. The **support** of function f on $\mathbb{R}^k$ is the closure of the set of all points $x \in \mathbb{R}^k$ at which $f(x) \neq 0$. We write $f \in C_c(\mathbb{R}^k)$ if f is a continuous function with compact support, that is, if $K \coloneqq \operatorname{supp} f \subset I^k$ for some k-cell I^k. In this case we define

$$\int_{\mathbb{R}^k} f(x)\,\mathrm{d}x \coloneqq \int_{I^k} f(x)\,\mathrm{d}x.$$

Theorem 1.6. *Let T be a one-to-one C^1 mapping from an open set $E \in \mathbb{R}^k$ into $\mathbb{R}^k$ such that $J_T(x) \neq 0$ for all $x \in T$. If f is a continuous function on $\mathbb{R}^k$ whose support is compact and lies in $T(E)$, then*

$$\int_{\mathbb{R}^k} f(y)\,\mathrm{d}y = \int_{\mathbb{R}^k} f(T(x))|J_T(x)|\,\mathrm{d}x.$$