ECON20010 NOTES

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1. The Envelope Theorem

Theorem 1.1. *Consider the constraint maximization problem*

$$v(\alpha) \coloneqq \max_x f(x;\alpha) \quad s.t. \quad g(x;\alpha) = 0.$$

The associated Lagrangian is

$$\mathcal{L}(x,\lambda,\alpha) = f(x;\alpha) + \lambda g(x;\alpha).$$

We have that

$$\frac{\partial v(\alpha)}{\partial \alpha_k} = \frac{\partial \mathcal{L}(x^*,\lambda^*,\alpha)}{\partial \alpha_k}.$$

Proof. Note that $v(\alpha) \equiv \mathcal{L}(x^*,\lambda^*,\alpha)$ and thus

$$\frac{\partial v}{\partial \alpha_k} = \frac{\partial \mathcal{L}}{\partial x}\frac{\partial x^*}{\partial \alpha_k} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda^*}{\partial \alpha_k} + \frac{\partial \mathcal{L}}{\partial \alpha_k},$$

where the first two terms are both 0 by the first order conditions. □

Remark 1.2.

- By the implicit function theorem, we need $f_{xx} \neq 0$.
- Think "all indirect effects vanish" at the optimum. That is, if we think $v(\alpha) = f(x^*;\alpha)$, then
$$\frac{\partial f}{\partial x}\frac{\partial x^*}{\partial \alpha_k} = 0.$$

Example 1.3. Consider the value function

$$\begin{aligned} v(p_x,p_y,m) = U(x^*,y^*) &= U(x^*,y^*) + \lambda^*[m - p_x x - p_y y] \\ &= \mathcal{L}(x^*,y^*,\lambda^*;p_x,p_y,m). \end{aligned}$$

By the envelope theorem,

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

Similarly,

$$\frac{\partial v}{\partial p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

2. Scarcity: the Budget Constraint

Definition 2.1.
- The **budget set** consists of all feasible consumption bundles.
- The **budget constraint** exactly exhausts the consumer's income.

2.1. Budget Set.
The relative price:

$$\frac{p_x}{p_y}$$

- Mnemonic: fractions of this form $(p_x/p_y, U_x/U_y)$ is always the price of x in units of y.

To stay on the budget constraint,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p_x}{p_y}.$$

- Think "the rate at which the market *allows* the consumer to exchange good x for good y."
- Think the opportunity cost of good x.

2.2. Preference.
Basic axioms:
- **Completeness**. For any pair of consumption of bundles, say c_1 and c_2, either $c_1 \succeq c_2$, $c_2 \succeq c_1$, or both.
  - Requires an answer and assumes no framing effects.
- **Transitivity**. If $c_1 \succeq c_2$ and $c_2 \succeq c_3$, then $c_1 \succeq c_3$.
  - When transitivity fails, we have a "money pump."

A preference ordering is **rational** if it satisfies completeness and transitivity. They are the minimal requirement for the existence of a utility function representation. We also typically assume the following:
- **Continuity**. If $c_1 \succ c_2$ then there are neighborhoods N_1 and N_2 around c_1 and c_2 such that

$$x \succ y, \quad \forall x \in N_1, \quad y \in N_2.$$

  This implies that if $c_1 \succ c_2$ then there exists c_3 such that

$$c_1 \succ c_3 \succ c_2.$$

- **Monotonicity**.
  - **Monotone**. If $c_1 \gg c_2$[1] then $c_1 \succ c_2$.
  - **Strongly monotone**. If $c_1 \geq c_2$[2] and $c_1 \neq c_2$ then $c_1 \succ c_2$.
  - **Local non-satiation**. If for every bundle c and every $\epsilon > 0$, there exists $x \in N_\epsilon(c)$ such that $x \succ c$.

[1] We write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i, \forall i$.

[2] We write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i, \forall i$.

- **Convexity**. If $c_1 \succeq c_2$, then
$$\theta c_1 + (1-\theta)c_2 \succeq c_2, \quad \forall \theta \in (0,1).$$
If convexity is satisfied, the **upper contour set**, the "at least as good as" set, is convex.

Additional axioms place even more structures on the utility function:

- **Homotheticity**. If $c_1 \succeq c_2$, then
$$tc_1 \succeq tc_2, \quad \forall t > 0.$$
- **Quasilinearity** in good i. If $\mathbf{c_1} \succeq \mathbf{c_2}$, then
$$\mathbf{c_1} + t\mathbf{e}_i \succeq \mathbf{c_2} + t\mathbf{e}_i, \quad \forall t > 0.$$

2.3. **Translating preference ordering to the utility function:**

Theorem 2.2 (Utility Representation Theorem)**.** *If a preference ordering is rational, then it admits a utility function representation. Moreover, the utility function is unique up to a monotonically increasing transformation.*

Remark 2.3. The additional assumption of monotonicity, though not required, allows for a very simple proof: simply send each consumption bundle to the size of the unique bundle on $t\sum \mathbf{e}$ equivalent to the given bundle.

Proposition 2.4. *If a preference ordering satisfies convexity, then the corresponding utility function representation will be quasi-concave. The indifference curves (level sets) will have non-increasing marginal rate of substitution (slopes).*

Proposition 2.5. *A preference ordering is homothetic if and only if its utility representation has MRS that is homogeneous of degree* 1.

2.4. **The Marginal Rate of Substitution.** The MRS
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y}$$
is the quantity of y the consumer is willing to sacrifice in exchange for an additional unit of x. (Think p_x/p_y.) It measures an individual's **willingness to pay** for x in terms of y.

3. Utility Maximization

The problem:

$$v(p_x, p_y, m) \coloneqq \max_{x,y} U(x, y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

3.1. Interpretation. We want to maximize

$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$

such that

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$

This gives

$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x.$$

We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.
  $$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \ \mathrm{d}x$$
  "Take advantage of all trading opportunities."
- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that U_x/p_y is marginal utility of money *spent on* x.
  $$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \ \mathrm{d}x$$
  "Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that U_x is the marginal benefit of buying x and $U_y \cdot p_x/p_y$ is the marginal cost of buying x.
  $$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$
  "Trade until marginal cost equals marginal benefit."

In the last expression, if we write

$$\lambda = \frac{U_y}{p_y},$$

(think marginal utility of income) we have that at optimum,

$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$

These three equalities describe precisely the critical points of the following

$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right],$$

called the **Lagrangian**. That is, setting

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

recovers the above three equations.

Remark 3.1.

- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- λ might be negative or zero. Think bliss point.

3.2. **The Indirect Utility Function.**

Proposition 3.2.

$$\frac{\partial v}{\partial m} = \lambda^*.$$

Proof. Noting

$$v = U(x^*, y^*) + \lambda^*[m - p_x x^* - p_y y^*] = \mathcal{L}^*,$$

we have

$$\begin{aligned}
\frac{\partial v}{\partial m} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} \\
&= U_x^* \frac{\partial x}{\partial m} + U_y^* \frac{\partial y}{\partial m} + \lambda^* \left[1 - p_x \frac{\partial x}{\partial m} - p_y \frac{\partial y}{\partial m}\right] + \frac{\partial \lambda}{\partial m} \left[m - p_x x^* - p_y y^*\right] \\
&= \left(U_x^* - \lambda^* p_x\right) \frac{\partial x}{\partial m} + \left(U_y^* - \lambda^* p_y\right) \frac{\partial y}{\partial m} + \frac{\partial \lambda^*}{\partial m} \left(m - p_x x^* - p_y y^*\right) + \lambda^* \\
&= \lambda^*.
\end{aligned}$$

The last equality follows by noting that at the optimum,

$$U_x^* - \lambda^* p_x = U_y^* - \lambda^* p_y = m - p_x x^* - p_y y^* = 0.$$

Alternatively, one may use the envelope theorem:

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

□

Note that

$$\frac{\partial v}{\partial m} = \lambda^* = \frac{U_x^*}{p_x} = \frac{U_x^*}{p_x}.$$

So when the individual is not satiated ($U_x, U_y \neq 0$), marginal utility of income is positive. When the budget constraint does not require binding, the marginal utility of income is generally nonnegative.

Again using the Envelope Theorem, we have

$$\frac{\partial v}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

This value is generally nonpositive, and only zero when one does not consume good x or when the marginal utility of x is 0.

4. Expenditure Minimization

The problem:

$$e(p_x, p_y, \overline{U}) \coloneqq \max_{x,y} p_x x + p_y y \quad \text{s.t.} \quad U(x,y) = \overline{U}.$$

The Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \eta\left[\overline{U} - U(x,y)\right]$$

$$\begin{aligned}
&[x] && p_x = \eta^* U_x(x^*, y^*) \\
&[y] && p_y = \eta^* U_y(x^*, y^*) \\
&[\eta] && \overline{U} = U(x^*, y^*).
\end{aligned}$$

Properties 4.1. Properties of Hicksian demand functions:

(i) Homogeneous of degree 0 in prices:

$$x_i^h(\alpha\mathbf{p}, U) = x_i^h(\mathbf{p}, U).$$

Differentiating with respect to α gives

$$\sum \frac{\partial x_i^h}{\partial p_j} p_j = 0 \implies \sum \epsilon_{ij}^h = 0.$$

(ii) Cross-price effects on Hicksian demand are symmetric:

$$\frac{\partial x_i^h}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial x_j^h}{\partial p_i}.$$

From this we have

$$p_i x_i \frac{p_j}{x_i}\frac{\partial x_i^h}{\partial p_j} = p_j x_j \frac{p_i}{x_j}\frac{\partial x_j^h}{\partial p_i}.$$

That is,

$$s_i \epsilon_{ij}^h = s_j \epsilon_{ji}^h \implies \frac{\epsilon_{ij}^h}{\epsilon_{ji}^h} = \frac{s_j}{s_i}.$$

The more important good impacts the less important good more.

(iii) Differentiating $U(\mathbf{x}^h(\mathbf{p}, U)) = U$ with respect to p_i gives

$$\sum \frac{\partial U}{\partial x_j}\frac{\partial x_j^h}{\partial p_i} = 0 \implies \sum p_j \frac{\partial x_j^h}{\partial p_i} = 0 \implies \sum s_j \epsilon_{ji}^h = 0.$$

Remark 4.2. Symmetry and homogeneity [adding up] gives adding up [homogeneity]. In case where there are two goods only, the latter two also gives symmetry.

Proposition 4.3. *Properties of the expenditure function:*

- *Homogeneous of degree* 1 *in prices.*
- *Non-decreasing in prices.* $\partial e / \partial p_i = x_i^h \geq 0$.
- *Increasing in utility.* $\partial e / \partial U = \eta^* > 0$.
- *Concave in prices.*
$$\frac{\partial^2 e}{\partial p_i^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0,$$
where the last inequality follows from the law of demand. Alternatively, note that the price of the original bundle, which grows linearly, is an upper bound of the expenditure function.

5. Changes in Behavior

Consider a price increase from p_x^o to p_x^f. Let o be the original consumption, f be the final consumption, and s be the optimal consumption after an income transfer such that the individual stays on the same indifference curve as before (has the same purchasing power). We may decompose $x^f - x^o$:

$$x^f - x^o = x^f - x^s + x^s - x^o.$$

- $x^f - x^o$: the Marshallian price effect (the total effect).
- $x^f - x^s$: the effect due to compensation (the income effect).
- $x^s - x^o$: the Hicksian price effect (substitution effect).

The Slutsky equation is an continuous analogue of this decomposition.

5.1. The Slutsky Equation.

Recall from duality that

$$x^h(p_x, p_y, \overline{U}) = x^m(p_x, p_y, m = e(p_x, p_y, \overline{U})).$$

As price changes, changes in $e(p_x, p_y, \overline{U})$ ensures that purchasing power does not change.

By differentiating, we get the **Slutsky equation**:

Proposition 5.1.

$$\frac{\partial x^h}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot \frac{\partial e}{\partial p_x}.$$

We may rewrite the Slutsky equation using the envelop theorem as

$$\frac{\partial x^h}{\partial p_x} = \frac{\partial x^m}{\partial p_x} + \frac{\partial x^m}{\partial m} \cdot x^m.$$

This shows that we can recover the unobservable $\partial x^h / \partial p_x$ from the observables.

We can also rewrite the Slutsky equation as

$$\frac{\partial x^m}{\partial p_x} = \frac{\partial x^h}{\partial p_x} + \left(-\frac{\partial x^m}{\partial m} \cdot x^m\right),$$

where

- $\partial x^h / \partial p_x$ is the substitution effect,
- $\partial x^m / \partial m \cdot x^m$ is the income effect.

5.2. Compensation.

- **Slutsky transfer** keeps the original bundle affordable.

$$T_S = \Delta p_x \cdot x^o.$$

- **Hicks transfer** keeps the original utility level affordable.
  $$T_H = e(p_x^f, p_y, v^o) - m = e(p_x^f, p_y, v^o) - e(p_x^o, p_y, v^o).$$
  In the Slutsky equation, the term
  $$\partial e/\partial p_x = x^m = x^h$$
  is the continuous analogue of the Hicks transfer.
- **Frisch transfer**: the transfer that restores the purchasing power by making the price of utility λ constant.

Remark 5.2. Note that $T_S \geq T_H$.

5.3. **The Law of Demand.** The substitution effect follows the law of demand:
$$\partial x^h/\partial p_x \leq 0.$$
More generally, we have:

Proposition 5.3 (Generalized law of demand)**.**
$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) \leq 0.$$

Proof. Note that
$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) = \left(\mathbf{x}^1\mathbf{p}^1 - \mathbf{x}^0\mathbf{p}^1\right) + \left(\mathbf{x}^0\mathbf{p}^0 - \mathbf{x}^1\mathbf{p}^0\right).$$
The last two terms are both non-positive. □

Remark 5.4.
- Note that this gives $\partial x_i^h/\partial p_i \leq 0$ (if the derivative exists).
- But think also graphs for the case of two goods.
- Note that the law of demand holds not only when indifference curves are concave. Remember the following two examples as well:
  - When indifference curves are concave, the expenditure minimizing points occur at the edges.
  - In the perfect complement case, we have that $\partial x^h/\partial p_x = 0$.

5.4. **Giffen Goods.** Marshallian demand does not always comply with the law of demand. A good whose Marshallian demand does not comply with the law of demand is called a **giffen good**. Their existence is theoretically possible, but not empirically supported.

Looking back at the Slutsky equation, we see that for a good x to be a giffen good, we need the following three conditions:

(i) the individual buys a large amount of x,
(ii) good x is inferior,
(iii) the demand for good x is elastic.

These three conditions do not occur together often: narrowly defined categories usually has 0 elasticity, but broad categories are usually normal goods.

5.5. **Normal & Inferior Goods.**

Definition 5.5. Good x is said to be **normal** if $\partial x^m/\partial m > 0$ and **inferior** if $\partial x^m/\partial m < 0$.

Remark 5.6. We can equivalently define it using Hicksian demands. From

$$\frac{\partial x^h}{\partial U} \cdot \frac{\partial v}{\partial m} = \frac{\partial x^m}{\partial m}$$

we know that $\partial x^h/\partial U$ and $\partial x^m/\partial m$ have the same sign. Note that $\partial v/\partial m = \lambda^* > 0$ is the utility of a dollar.

Remark 5.7. We say good i is

- **normal** if $\eta_i > 0$,
- **inferior** if $\eta_i < 0$,
- **necessity** if $\eta_i < 1$,
- **luxury** if $\eta_i > 1$.

5.6. **An experiment for testing normality.** We fix the consumption of x and vary income m. For normal goods, the willingness to pay for x increases as income increase. Thus x is normal if

$$\frac{\partial (U_x/U_y)}{\partial y} > 0.$$

Example 5.8. With the quasilinear utility function $U(x, y) = v(x) + y$, the good x is neither normal nor inferior. The willingness to pay

$$\frac{U_x}{U_y} = \frac{v'(x)}{1}$$

does not change as we vary y (by varying income).

5.7. **Cross Effects.**

Definition 5.9. We say y is

- a **substitute** of x if $\partial y/\partial p_x > 0$,
- a **complement** of x if $\partial y/\partial p_x < 0$,
- **unrelated** with x if $\partial y/\partial p_x = 0$.

If we use $y = y^h$ in the definition above, we say **gross** substitutes/complements; if we use y^m, we say **net** substitutes/complements.[3]

[3] Think Hicksian demand "nets out" the income effect.

Remark 5.10. The property of being substitutes is not transitive. That is, if x and y are substitutes and y and z, then x and z are not necessarily substitutes. Consider $U = \min\{x + y, y + z\}$.

Remark 5.11.

- Cross price effects for Hicksian demands are symmetric:
$$\frac{\partial x^h}{\partial p_y} = \frac{\partial^2 e}{\partial p_x \partial p_y} = \frac{\partial y^h}{\partial p_x}.$$
This does not hold in general for Marshallian demands; see the cross-price Slutsky equation.
- For any good x, at least one other good is a net substitute with x. If not, as price of x increase, consumption and thus utility level strictly decreases (note that consumption of x also decreases by the law of demand). In particular, when there are only two goods, the two goods cannot be net complements.

Proposition 5.12. *We have the* ***cross-price Slutsky equation****:*
$$\frac{\partial y^h}{\partial p_x} = \frac{\partial y^m}{\partial p_x} + \frac{\partial y^m}{\partial m} \cdot \frac{\partial e}{\partial p_x}$$
where
$$\frac{\partial e}{\partial p_x} = x^m = x^h.$$

6. Elasticities and Aggregation

Proposition 6.1. *The Slutsky equation in elasticity form:*

$$\epsilon_{ij}^m = \epsilon_{ij}^h - \eta_i s_j.$$

From this we have the following:

Proposition 6.2 (Symmetry of Marshallian Demands)**.** *We have*

$$s_i \epsilon_{ij}^m = s_j \epsilon_{ji}^m + s_i s_j (\eta_j - \eta_i).$$

Symmetry holds when two goods have equal income elasticities.

Proposition 6.3.

- *Engel aggregation:*

$$\sum s_i \eta_i = 1.$$

- *Cournot aggregation:*

$$\sum_i \epsilon_{ij} s_i = -s_j.$$

- *Implication of homogeneity:*

$$\sum_i \epsilon_{ji} = -\eta_j.$$

Proof.

- From $m = \sum p_i x_i$ we have

$$1 = \frac{\partial m}{\partial m} = \sum p_i \frac{\partial x_i}{\partial m} = \sum \eta_i s_i.$$

- Differentiating the same identity with respect to p_j gives

$$0 = \frac{\partial m}{\partial p_x} = x_j + \sum \frac{\partial x_i}{\partial p_j} p_i = x_j + \sum \epsilon_{ij} \frac{p_i x_i}{p_j}.$$

- Differentiating the identity $x^m(\mathbf{x}, m) = x^m(t\mathbf{x}, m)$ with respect to t gives

$$0 = \sum \frac{\partial x_j}{\partial p_i} p_i + \frac{\partial x_j}{\partial m} m.$$

□

Remark 6.4.

- From Engel aggregation: some goods must be normal. It will never the case that all goods are inferior, all goods are necessities, or all goods are luxuries.
- The budget share of good j is small if it has many substitutes, and large if it has many complements, or when their respective budget shares are large.

- From implication of homogeneity: Normal goods have many substitutes; inferior goods have many complements; strongly inferior goods have strong complements. "No good can be a Giffen good ($\epsilon_{ii} > 0$) unless it is strong complements with other goods."

7. Welfare

7.1. Exact Measures of Welfare Change. The difference in utility

$$\Delta v = v^f - v^o.$$

- Gets direction right, but magnitude depends on the specific utility representation chosen — it is not invariant to the utility representation.
- Difficulty in the interpretation of units.

Definition 7.1.

- **Compensating variation**: the income transfer that induces the consumer accept the change in price voluntarily.
- **Equivalent variation**: the income transfer that induces the consumer to reject the change in price voluntarily.

7.2. The Compensating Variation.

Proposition 7.2. *There holds*

$$\begin{aligned} CV := -T_H &= -\left[e(\mathbf{p}^f, v^o) - m\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^f, v^f)\right] \\ &= -\left[e(\mathbf{p}^f, v^o) - e(\mathbf{p}^o, v^o)\right]. \end{aligned}$$

Remark 7.3. Do not think as the bounds on Hicks transfers as bounds of an integral. They indicate only the direction of price change.

Remark 7.4.

- CV is the utility change in dollars:
$$CV = e(\mathbf{p}^f, v^f) - e(\mathbf{p}^f, v^o).$$
Think of $e(\mathbf{p}^f, \cdot)$ as a monotonic transformation of $U(x, y)$, an equivalent utility representation in units of dollars.[4] This is called the **money metric utility function**.
- CV is invariant.
$$CV = e(\mathbf{p}^o, v^o) - e(\mathbf{p}^f, v^o)$$
is the cost of two different bundles on the same indifference curve, which does not vary according to the utility representation.

[4] This is a valid transformation since
$$\frac{\partial e}{\partial U} = \eta^* = \frac{p_x}{U_x(x, y)} > 0.$$

7.3. **The Equivalent Variation.**

Proposition 7.5. *There holds*

$$\begin{aligned} EV &= e(\mathbf{p}^o, v^f) - m \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^f, v^f) \\ &= e(\mathbf{p}^o, v^f) - e(\mathbf{p}^o, v^o) \\ &= T_H\Big|_{p_x^f}^{p_x^o} \end{aligned}$$

Moreover, just as the CV, the EV is an invariant measure of utility change in dollars.

7.4. **The Surplus form.**

Proposition 7.6. *For a price change of p_x^o to p_x^f, we may write the CV as*

$$CV\Big|_{p_x^o}^{p_x^f} := -T_H\Big|_{p_x^o}^{p_x^f} = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\,\mathrm{d}p_x.$$

Similarly, we may write the EV as

$$EV = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^f)\,\mathrm{d}p_x.$$

These are the ***surplus forms****.*

Proof. Note that

$$CV = -\int_{p_x^o}^{p_x^f} \frac{\partial e(p_x, p_y, v^o)}{\partial p_x}\,\mathrm{d}p_x = -\int_{p_x^o}^{p_x^f} x^h(p_x, p_y, v^o)\,\mathrm{d}p_x,$$

where the last equality follows from Shepherd's lemma. □

Remark 7.7. We can use the surplus forms to analyzes the relative magnitudes of the CV and EV. Recall that $\partial x^h/\partial U$ and $\partial x^m/\partial m$ have the same sign. This leads to the following:

Proposition 7.8. *If good x is normal, then the Hicksian demand x^h shifts in the same direction as utility level.*

Example 7.9. If $v^f < v^o$ and x is a normal good, then $x^h(\mathbf{p}, v^f) < x^h(\mathbf{p}, v^o)$ and

$$|CV| > |\Delta CS| > |EV|,$$

where the comparisons with the magnitude of ΔCS comes from noting the points where x^h intersect with x^m.

Remark 7.10. We can use the Slutsky transfer to approximate CV and EV:

- $CV \approx -\Delta p_x \cdot x^o$.

- $EV \approx -\Delta p_x \cdot x^f$.

How well the approximation is depends on how willing the individual is to substitute. Think perfect complements and perfect substitutes.

Definition 7.11. The **change in consumer's surplus** is given by

$$\Delta CS = -\int_{p_x^o}^{p_x^f} x^m(p_x, p_y, m)\,\mathrm{d}p_x.$$

Remark 7.12.

- The change in consumer surplus picks up not only the change in welfare (utility) but also change in purchasing power. It is not an exact welfare measure.
- We can decompose changes in ΔCS into two effects: change in consumption (in unit of dollars) and change in expenditure on goods being purchased.

Definition 7.13. The **inverse demand function** measures the consumer's willingness to pay for the marginal unit of good x. It is given by

$$p_x = p_x(\bar{x}, p_y, \overline{U}),$$

where $\bar{x}$ is the marginal unit of good x.

Total surplus:

$$TS = \int_{p_x}^{\infty} x^h(\rho, p_y, \overline{U})\,\mathrm{d}\rho.$$

7.5. **Price Indices.**

Definition 7.14.

- The **ideal index**:
$$I = \frac{e(\mathbf{p}^f, \overline{U})}{e(\mathbf{p}^o, \overline{U})}.$$
- The **Laspeyres price index**:
$$P_L = \frac{\mathbf{p}^f \mathbf{x}^o}{\mathbf{p}^o \mathbf{x}^o}.$$
- The **Paasche price index**:
$$P_P = \frac{\mathbf{p}^f \mathbf{x}^f}{\mathbf{p}^o \mathbf{x}^f}.$$

7.6. **Efficiency.**

Definition 7.15. A change is called a **Pareto improvement** if it leaves everyone in a society better-off (or at least as well-off as they were before). A situation is called **Pareto efficient** or **Pareto optimal** if all possible Pareto improvements have already been made.

Definition 7.16. A re-allocation is a **Kaldor–Hicks improvement** if those that are made better off could hypothetically compensate those that are made worse off and lead to a Pareto-improving outcome. A situation is said to be **Kaldor–Hicks efficient**, or equivalently is said to satisfy the **Kaldor–Hicks criterion**, if no potential Kaldor–Hicks improvement from that situation exists.

8. An Endowment Economy

Appendix A: Utility Functions

8.1. **Perfect Complements.**

$$U(x, y) = \min\left\{\frac{x}{a}, \frac{y}{b}\right\}.$$

Remark 8.1. This is also known as a **Leontief utility function**.

$$\frac{x^m}{a} = \frac{y^m}{b} = v = \frac{m}{ap_x + bp_y}.$$

$$x^h = aU, \quad y^h = bU, \quad e = (ap_x + bp_y)U.$$

Remark 8.2.

- Think a units of x "pairs" with b units of y.
- All income effect; no substitution effect.

8.2. **Perfect Substitutes.**

$$U(x, y) = ax + by.$$

$$e = \min\left\{\frac{p_x}{a}, \frac{p_y}{b}\right\} U.$$

Remark 8.3.

- Think of p_x/a as the price of obtaining one utils by buying x.
- All substitution effects; no income effect.

8.3. **Cobb-Douglas for Two Goods.**

$$U(x, y) = x^\alpha y^{1-\alpha}.$$

$$x^m = \frac{\alpha m}{p_x}, \quad y^m = \frac{(1-\alpha)m}{p_y}, \quad \lambda^* = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha},$$

$$v = \left(\frac{\alpha}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \cdot m.$$

Remark 8.4.

- Proportion of income spent on each good is constant.
  - Income elasticities of demand is 1.
  - Substitution effects are offsetted exactly by the income effected.
  - All goods are unrelated (neither substitutes nor complements).
- Homothetic.

$$e = U \cdot \left(\frac{p_x}{\alpha}\right)^{\alpha} \left(\frac{p_y}{1-\alpha}\right)^{1-\alpha},$$

$$x^h = U \cdot \left(\frac{\alpha}{p_x} \cdot \frac{p_y}{1-\alpha}\right)^{1-\alpha}, \quad y^h = U \cdot \left(\frac{\alpha}{p_x} \cdot \frac{p_y}{1-\alpha}\right)^{-\alpha},$$

Remark 8.5. Easily obtained using $e = v^{-1}$ and Shepherd's lemma.

8.4. **Cobb-Douglas.**

$$U(x) = \prod x_i^{\alpha_i}, \quad \text{where} \quad \sum \alpha_i = 1.$$

$$x_j^m = \frac{\alpha_j m}{p_j}, \quad v = m \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}.$$

$$x_j^h = u \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i} \cdot \frac{\alpha_i}{p_i}, \quad e = u \cdot \prod \left(\frac{\alpha_i}{p_i}\right)^{-\alpha_i}.$$

Remark 8.6. $s_i = \alpha_i$.

8.5. **Constant Elasticity of Substitution Utility Function.**

$$U(x, y) = \left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{-1/\rho}.$$

$$x_1^m = \frac{m}{p_1 + \kappa p_2}, \quad x_2^m = \frac{\kappa m}{p_1 + \kappa p_2}, \quad \kappa = \left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}.$$

Remark 8.7.

- Shares spent on each good is constant $x_2^m = \kappa x_1^m$.
- Income elasticities of demand is 1.
- Indirect utility function is proportional to income. $v = \lambda \cdot m$.
- Constant elasticity of substitution:

$$\sigma = \frac{\mathrm{d}\log\left(\frac{x_1}{x_2}\right)}{\mathrm{d}\log\left(\frac{U_2}{U_1}\right)} = \frac{1}{\rho+1}.$$

$$x_1^h = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot U$$

$$x_2^h = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot \kappa U$$

$$e = (1 - \omega\kappa^{-\rho})^{\frac{1}{\rho}} \cdot U \cdot [p_1 + \kappa p_2].$$

Remark 8.8. Hicksian demands are proportional to utility level.

8.6. **Quasilinear Utility Functions.**

$$U(x, y) = V(x) + y.$$

Remark 8.9.

- Good x is neither normal nor inferior: The willingness to pay for x
$$\frac{U_x}{U_y} = V'(x)$$
does not change as we vary consumption of y (by varying income). The consumption of x does not vary as income vary.
- The optimality condition is
$$\frac{U_x}{U_y} = V'(x) = \frac{p_x}{p_y}.$$
- Consider edge cases.

8.7. **Quadratic Utility Function.**

$$U(x, y) = -\frac{1}{2}(x - b_x)^2 - \frac{1}{2}(y - b_y)^2.$$

Remark 8.10. Think bliss point.

$$\boxed{\begin{aligned} x^m &= b_x + \frac{p_x}{p_x^2 + p_y^2}\left(m - p_x b_x - p_y b_y\right), \\ y^m &= b_y + \frac{p_y}{p_x^2 + p_y^2}\left(m - p_x b_x - p_y b_y\right), \\ v &= -\frac{1}{2} \cdot \frac{(m - p_x b_x - p_y b_y)^2}{p_x^2 + p_y^2}. \end{aligned}}$$

Appendix B: Models

8.8. The Baumol–Tobin model.

- Exhausts monthly income Y.
- Interest rate i.
- Goes to ATM N times a month, each time withdrawing W with a direct cost of F incurred.
- Assume constant rate of spending, and money demand is average holding of money $M = W/2$.

$$\begin{aligned} e(F, Y, i) &= \min_{W,N} NF + \frac{Wi}{2} \quad \text{s.t.} \quad NW = Y \\ &= \min_{N} NF + \frac{Yi}{2N}. \end{aligned}$$

Solving it gives

$$N^* = \sqrt{\frac{Yi}{2F}}.$$