Notes: MATH262 (F25) Point-Set Topology

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1 The Basics

1.1 The Basis of a Topology

Definition 1.1. A collection $\mathcal{B}$ of subsets of a set X is a **basis** for a topology on X if

(i) $\cup_{B \in \mathcal{B}} B = X$,

(ii) For each $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Proposition 1.2. *Let $\mathcal{B}$ be a basis for a topology on X. Call a set $U \subset X$ open if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Then, the collection of open sets is a topology on X. Note that in the topology, open sets are arbitrary unions of sets in the basis. This topology is said to be **generated** by the basis $\mathcal{B}$.*

Proof. Let $\mathcal{T}$ be the collection of open sets defined as above. That $\varnothing, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions is immediate from the definition. For finite intersections, note that

$$\bigcup B_\alpha \cap \bigcup B_\beta = \bigcup_{\alpha,\beta} (B_\alpha \cap B_\beta),$$

where each $B_\alpha \cap B_\beta \in \mathcal{B}$ is open by the basis property. □

Proposition 1.3. *A collection of sets $\mathcal{C}$ such that $\cup_{C \in \mathcal{C}} C = X$ can be completed to a basis for a topology on X by adding all finite intersections of sets in $\mathcal{C}$.*

1.2 Product Topology

Definition 1.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis $\{U \times V | U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$.

Remark 1.5. The product topology is the natural topology that makes $f : Z \to X \times Y$ continuous if and only if both $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

Example 1.6. X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) | x \in X\} \subset X \times X$ is closed.

To see this, suppose X is Hausdorff and fix $(x, y) \in \Delta^c$ (so $x \neq y$). Pick disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$ and observe that $(x, y) \in U \times V \subset \Delta^c$. The other direction is similar.

Example 1.7. $\mathbb{R}_{\text{Zar}} \times \mathbb{R}_{\text{Zar}} \not\cong \mathbb{R}^2_{\text{Zar}}$. The latter contains curves.

1.3 Subspace Topology

Definition 1.8. Let $(X, \mathcal{T})$ be a topological space and fix subset $Y \subset X$. The **subspace topology** on Y is $\{Y \cap U | U \in \mathcal{T}\}$.

Some compatibility results:

Proposition 1.9.

(i) If $Y \subset X$ is open, then any open set $U \subset Y$ in the subspace topology is also open in X.

(ii) Let $A \subset Y \subset X$ The topology A inherits as a subspace of Y is the same as the topology A inherits as a subspace of X.

Proof. Immediate. $\square$

Proposition 1.10. *Let $A \subset X$ and $B \subset Y$. The following two topologies on $A \times B$ coincide:*

(i) Take product topology on $X \times Y$ and then subspace topology on $A \times B$.

(ii) Take subspace topologies on A and B first, then take product topology on $A \times B$.

1.4 Ordered Sets

Definition 1.11.

- A **relation** on a set X is a subset $R \subset X \times X$.
- A **partially-ordered** set is a set X with a relation $\leq$ such that for any $x, y, z \in X$:
  - (i) $x \leq x$ (reflexive),
  - (ii) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric),
  - (iii) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive).

Example 1.12. Let X be a set. The power set 2^X is partially-ordered by inclusion.

Example 1.13. The set of all topologies on a set X is partially-ordered by inclusion. $\mathcal{T}_1 \leq \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case, we say $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$ and $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$.

Definition 1.14. For a partially ordered set P,

- $a \in P$ is **maximal (minimal)** if there is no $b \in P$ such that $a < b$ ($b < a$).
- $a \in P$ is the **greatest (least)** element if for all $b \in P$, $b \leq a$ ($a \leq b$).

A partially ordered set P is **linearly ordered** if for any $a, b \in P$, either $a \leq b$ or $b \leq a$.

Definition 1.15. Let X and Y be linearly ordered sets. The linear order on $X \times Y$ defined by $(x_1, y_1) \leq (x_2, y_2)$ if

(i) $x_1 < x_2$, or

(ii) $x_1 = x_2$ and $y_1 \leq y_2$

is called the **dictionary order**.

Definition 1.16 (Order Topology)**.** Let L be a linearly ordered set. The **order topology** on L is the topology generated by the basis consisting of all open intervals $(a, b) = \{x \in L | a < x < b\}$ along with $[s, b)$ and $(b, l]$, where s and l are the smallest and largest elements of L (if they exist).

Proposition 1.17. *The order topology is Hausdroff.*

Proof. Take $x, y \in L$ with $x < y$. If $x \neq s$ and $y \neq l$, then there exists a, b such that

$$a < x < y < b.$$

If there is an z such that $x < z < y$, then take (x, z) and (z, y). Otherwise, take (a, y) and (x, b). □

1.5 Closure, Interior, and Limit Points

Definition 1.18. The **closure** of $A \subset X$ is the intersection of all closed sets containing A, denoted $\mathrm{Cl}(A)$. The **interior** of $A \subset X$ is the union of all open sets contained in A, denoted $\mathrm{Int}(A)$.

Note that $\mathrm{Cl}(A)$ is the unique closed set such that $A \subset \mathrm{Cl}(A)$ and for any closed set C with $A \subset C$, we have $\mathrm{Cl}(A) \subset C$.

Definition 1.19. We say x is a **limit point** of $A \subset X$ if for any open set U containing x, U intersects A by some point other than x itself. The set of limit points of A is denoted A'.

Proposition 1.20. $\mathrm{Cl}(A) = A \cup A'$.

Proof. Note that $x \in A \cup A'$ if and only if each open neighborhood of x intersects A. Thus,

$$A \cup A' = \bigcap \left\{U^{\mathsf{c}} : U \text{ open}, U \cap A = \varnothing\right\} = \bigcap \{S : A \subset S, S \text{ closed}\} = \mathrm{Cl}(A).$$

□

2 Continuity

Definition 2.1. A function $f : X \to Y$ between topological spaces is **continuous** if the preimage of every open set in Y is open in X.

Remark 2.2. Continuity can be checked on the basis of the topology on X and Y.

Corollary 2.3. *Continuity in the metric sense coincides with continuity in the topological sense when both X and Y are equipped with the metric topology.*

Proposition 2.4. *$f : X \to Y$ is continuous if and only if it is continuous at every point $x \in X$, i.e., for any open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subset V$.*

Proof. The forward direction is immediate from the definition. For the backward direction, fix open set $V \subset Y$. The preimage $f^{-1}(V)$ is open since for any $x \in f^{-1}(V)$, there exists an open neighborhood U of x such that $f(U) \subset V$, so $U \subset f^{-1}(V)$. □

Proposition 2.5. *If $f : X \to Y$ is continuous, then $f(x_n) \to f(x)$ whenever $x_n \to x$.*

Proof. Fix open set $V \subset Y$ such that $f(x) \in V$. Then, there exists an open neighborhood U of x such that $f(U) \subset V$. Since $x_n \to x$, there exists N such that for all $n \geq N$, $x_n \in U$. Thus, for all $n \geq N$, we have $f(x_n) \in f(U) \subset V$. □

Proposition 2.6. *Properties of continuous functions:*

(i) The identity map $\mathrm{Id} : X \to X$ *is continuous.*

(ii) If $f : X \to Y$, $g : Y \to Z$ are continuous, then so is $g \circ f$. (If particular, topological spaces and continuous maps form a category.)

(iii) Let X be a topological space, and let $A \subset X$ be equipped with the subspace topology. Then, the inclusion map $i : A \hookrightarrow X$ is continuous.

(iv) Consider $X \times Y$ with the product topology. Then, the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

(v) A map $f : Z \to X \times Y$ is continuous if and only if both $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.

Proof. For the last statement, let $\pi_X \circ f, \pi_Y \circ f$ be continuous and note that

$$f^{-1}(U \times V) = (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(V).$$

□

2.1 Sheaf Theory

Let X, Y be topological spaces and suppose $f : X \to Y$ be continuous. Suppose $A \overset{i}{\hookrightarrow} X$ is a subspace. Then the restriction $f|_A : A \to Y$ is continuous, since $f|_A = f \circ i$.

Proposition 2.7. *Suppose $U_\alpha \subset X$ are open sets covering X. Then a function $f : X \to Y$ is continuous if and only if each $f|_{U_\alpha}$ is continuous.*

Proof. The forward direction is clear. For the backward direction, fix open set $V \subset Y$. Then,

$$f^{-1}(V) = f^{-1}(V) \cap \bigcup U_\alpha = \bigcup \left[f^{-1}(V) \cap U_\alpha \right] = \bigcup_\alpha f|_{U_\alpha}^{-1}(V),$$

which is open since each $f|_{U_\alpha}$ is continuous. □

Proposition 2.8 (Gluing)**.** *Let $X = U_1 \cup U_2$ and Y be topological spaces. Let $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ be continuous functions that agree on $U_1 \cap U_2$. Then, there exists a unique continuous function $f : X \to Y$ such that $f|_{U_1} = f_1$ and $f|_{U_2} = f_2$.*

Proof. Define

$$f(x) = \begin{cases} f_1(x), & x \in U_1, \\ f_2(x), & x \in U_2. \end{cases}$$

This is well-defined since f_1 and f_2 agree on $U_1 \cap U_2$. To see that f is continuous, fix open set $V \subset Y$. Then,

$$f^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V),$$

which is open since both f_1 and f_2 are continuous. □

2.2 Homeomorphisms

Definition 2.9. We say X and Y are **homeomorphic** if there exists a bijective continuous map $f : X \to Y$ such that f^{-1} is also continuous. Such a map f is called a **homeomorphism**.

Example 2.10. It is not sufficient to impose bijectivity and continuity. Consider $f : [0, 1) \to S^1 \subset \mathbb{R}^2$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$.

However, bijective continuous maps are homeomorphisms in some cases, e.g., when the domain is compact and the codomain is Hausdorff. We postpone the discussion of this result.

2.3 Quotients

Definition 2.11. A map $f : X \to Y$ is a **quotient map** if $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X.

Note that all quotient maps are continuous. Further, if $f : X \to Y$ is surjective, then there exists a unique topology on Y such that f is a quotient map. (Uniqueness is immediate from the definition; for existence note that the subsets of Y whose preimage is open forms a topology.)

This motivates the following definition:

Definition 2.12. If X is a topological space and $\sim$ is an equivalence relation on X, then the **quotient topology** on $X/\sim$ is the unique topology such that the natural projection map $\pi : X \to X/\sim$ is a quotient map.

Example 2.13. Let $X = [0, 1]$ and let $\sim$ be the equivalence relation such that the classes are $\{0, 1\}$ and $\{x\}$ for all $x \in (0, 1)$. Then, $X/\sim \cong S^1$.

Definition 2.14. Define the n-sphere as

$$S^n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}.$$

Define the n-disk as

$$D^n = \{x \in \mathbb{R}^n | |x| \leq 1\}.$$

Proposition 2.15.

- $D^2 \cong [0, 1]^2 \cong \mathbb{R}^2$.
- *On D^2, take the minimal equivalence relation such that $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. Then, $D^2/\sim \cong S^1 \times S^1$, a torus.*
- *D^2 with the minimal equivalence relation such that $(0, y) \sim (0, 1 - y)$ is homeomorphic to the Möbius strip.*
- *D^2 with the minimal equivalence relation such that $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$ is homeomorphic to the Klein bottle.*

2.4 Topological Groups

Definition 2.16. A topological group is a group G equipped with a topology such that the multiplication map $m : G \times G \to G$ defined by $m(g, h) = gh$ and the inversion map $i : G \to G$ defined by $i(g) = g^{-1}$ are continuous.

3 Connectedness

Definition 3.1. A **separation** of a topological space X is a pair of disjoint nonempty open sets $U, V \subset X$ such that $X = U \cup V$. The space X is **connected** if there does not exist a separation of X. For a subset $A \subset X$, we say A is connected if it is connected in the subspace topology.

Remark 3.2. X is connected if and only if its only clopen subsets are $\varnothing$ and X. Note also that connectedness is intrinsic topological property; it does not depend on the ambient space.

Lemma 3.3. *A separation of $Y \subset X$ exists if and only if there exists a pair of nonempty disjoint sets A, B such that $Y = A \cup B$ and $A' \cap B = A \cap B' = \varnothing$, where the limits are taken in X.*

Proof. For the forward direction, let $Y = A \cup B$ and note that $\mathrm{Cl}_Y A = \mathrm{Cl}_X \cap Y$. Thus, $\mathrm{Cl}_X A \cap B = \mathrm{Cl}_Y A \cap B = \varnothing$. Conversely, if the condition holds, then $(A \cup A') \cap B = \varnothing$ implies $(A \cup A') \cap Y = A$. $\square$

Example 3.4. $\mathbb{Q} \subset \mathbb{R}$ is **totally disconnected**. That is, each connected nonempty $A \subset \mathbb{Q}$ is a singleton.

Proof. Suppose A has at least two points $p < q$. Take an irrational number $\alpha \in (p, q)$ and observe

$$A = (A \cap (-\infty, \alpha)) \cup (A \cap (\alpha, \infty))$$

has a separation, where both sets are open in $\mathbb{Q}$ since $(-\infty, \alpha)$ and (α, ∞) are open in $\mathbb{R}$. $\square$

Proposition 3.5. *Suppose X is connected and $X = U \cup V$ is a separation. If $A \subset X$ is connected, then $A \subset U$ or $A \subset V$.*

Proof. Note that $(A \cap U) \cup (A \cap V)$ is a separation of A unless one of the intersections is empty. $\square$

Proposition 3.6. *Let $A \subset X$ be connected. If $A \subset B \subset \mathrm{Cl}\, A$, then B is connected.*

Proof. Suppose not. Then, there exists a separation $B = (E \cap B) \cup (F \cap B)$, where E, F are disjoint nonempty closed sets in X. Since $A \subset B$, we have either $A \subset E$ or $A \subset F$ by the previous lemma. Without loss of generality, suppose $A \subset E$. Then, $B \subset \mathrm{Cl}\, A \subset E$, contradicting $F \cap B$ being nonempty. $\square$

Proposition 3.7. *Let $A_\alpha \subset X$ be connected for all α in some index set J. If $a \in \cap A_\alpha$ for some $a \in X$, then $\cup_\alpha A_\alpha$ is connected.*

Proof. If not, then there exists a separation $\cup A_\alpha = [\cup A_\alpha \cap U] \cup [\cup A_\alpha \cap V]$. Suppose without loss that $a \in U$. Then, for each α, we have $A_\alpha \subset U$ by the previous lemma, and V is empty. $\square$

Proposition 3.8. *If X, Y are connected, then so is $X \times Y$ with the product topology.*

Proof. Fix $x \in X$, $y \in Y$ and note that $\{x\} \times Y \cong Y$ is connected. Thus, each cross $(\{x\} \times Y) \cup (X \times \{y\})$ is connected. Apply the previous proposition, noting that $X \times Y$ is the union of all such crosses passing through (x, y). $\square$

Proposition 3.9. *If X is connected and $f : X \to Y$ is continuous and surjective, then $f(X) \subset Y$ is connected.*

Proof. If not, say if $f(X) = U \cup V$ is a separation, then $X = f^{-1}(U) \cup f^{-1}(V)$ is a separation of X, contradicting connectedness. $\square$

Proposition 3.10. *Any convex subset $C \subset \mathbb{R}$ is connected.*

Proof. We deal with the case where $C = [a, b]$ is closed and bounded. Other cases are similar.

Suppose not. Then, there exists a separation $[a, b] = U \cup V$. Without loss of generality, suppose $a \in U$. Define

$$s = \sup\{x \in [a, b] | [a, x] \subset U\}.$$

Note that $s \in [a, b]$. If $s \in U$, then there exists $\epsilon > 0$ such that $(s, s + \epsilon) \subset U$, contradicting the definition of s. If $s \in V$, then there exists $\epsilon > 0$ such that $(s - \epsilon, s) \subset V$, again contradicting the definition of s. $\square$

Alternatively, one can use the intermediate value theorem, noting that X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.

Remark 3.11. Let L denote a linearly ordered set with the order topology.

For convex sets to be connected, it is not sufficient for L to be linearly ordered ($\mathbb{N}$). Nor it is sufficient for L to be densely ordered ($\mathbb{Q}$). However, if L is both densely ordered and has the least upper bound property ($\mathbb{R}$), then every convex subset of L is connected.

Corollary 3.12. *$[0, 1]^n$ and $\mathbb{R}^n$ are connected.*

3.1 Path-Connectedness

Definition 3.13. A **path** in a topological space X is a continuous map $\gamma : [0, 1] \to X$. A topological space X is **path-connected** if for any $x, y \in X$, there exists a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 3.14. There are no degenerate paths in $\mathbb{Q}$.

Proposition 3.15. *Any path-connected space is connected.*

Proof. If $X = U \cup V$ is a separation, then for a path γ, $[0, 1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a separation of $[0, 1]$, contradicting connectedness of $[0, 1]$. $\square$

Example 3.16.

- $\mathbb{R} \setminus \{0\}$ is not connected.
- $\mathbb{R}^n \setminus \{0\}$ is connected for $n \geq 2$ (it is path-connected).

- S^n is connected for $n \geq 1$. There exists a continuous map $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by
$$(x_1, \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{|x|}, \ldots, \frac{x_{n+1}}{|x|} \right).$$

3.2 Connected Components and Path Components

Definition 3.17. Let X be a topological space. We say $x \sim y$ if there exists a connected subset $A \subset X$ such that $x, y \in A$. The equivalence classes of this relation are called the **connected components** of X.

The reflectivity and symmetry of $\sim$ are immediate. To check transitivity, suppose $x, y \in A$ and $y, z \in B$ for connected sets $A, B \subset X$. Then, $A \cup B$ is connected and contains x and z.

4 Separation Axioms

Definition 4.1. Fix topological space X.

(i) X is $\boldsymbol{T_1}$ if every singleton $\{x\} \subset X$ is closed. Equivalently, for any $x, y \in X$ with $x \neq y$, there exists an open set U such that $x \in U$ and $y \notin U$.

(ii) X is $\boldsymbol{T_2}$ **(Hausdorff)** if for any $x, y \in X$ with $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

(iii) X is $\boldsymbol{T_3}$ **(regular)** if it is Hausdorff and for any closed set $C \subset X$ and any point $x \in X \setminus C$, there exist open sets U, V such that $x \in U$, $C \subset V$, and $U \cap V = \emptyset$.

(iv) X is $\boldsymbol{T_4}$ **(normal)** if it is Hausdorff and for any two disjoint closed sets $C_1, C_2 \subset X$, there exist open sets U, V such that $C_1 \subset U$, $C_2 \subset V$, and $U \cap V = \emptyset$.

Note that these axioms are increasingly stronger:

$$T_4 \implies T_3 \implies T_2 \implies T_1.$$

Theorem 4.2. *Any metrizable space is T_4.*

Proof. Fix disjoint closed sets $C_1, C_2 \subset X$. For any $x \in C_1$, there exists $r_x > 0$ such that $B_{r_x}(x) \cap C_2 = \emptyset$. Similarly, for any $y \in C_2$, there exists $r_y > 0$ such that $B_{r_y}(y) \cap C_1 = \emptyset$. Define

$$U = \bigcup_{x \in C_1} B_{r_x/2}(x), \quad V = \bigcup_{y \in C_2} B_{r_y/2}(y).$$

They must be disjoint. If not, suppose $z \in B_{r_x/2}(x) \cap B_{r_y/2}$ with $r_x \leq r_y$. Then, $d(x, y) \leq d(x, z) + d(z, y) < r_x/2 + r_y/2 \leq r_y$, contradicting the choice of r_y. □

Theorem 4.3. *If X is compact and Hausdorff, then X is normal.*

Proof. Fix disjoint closed sets $C_1, C_2 \subset X$. For any $x \in C_1$, for each $y \in C_2$, there exist open sets U_y, V_y such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. The collection $\{V_y | y \in C_2\}$ is an open cover of C_2. By compactness, there exists a finite subcover $V_{y_1}, \ldots, V_{y_n}$. Define

$$U_x = \bigcap_{i=1}^{n} U_{y_i}, \quad V_x = \bigcup_{i=1}^{n} V_{y_i}.$$

Then, $x \in U_x$, $C_2 \subset V_x$, and $U_x \cap V_x = \emptyset$.

The collection $\{U_x | x \in C_1\}$ is an open cover of C_1. By compactness, there exists a finite subcover $U_{x_1}, \ldots, U_{x_m}$. Define

$$U = \bigcup_{i=1}^{m} U_{x_i}, \quad V = \bigcap_{i=1}^{m} V_{x_i}.$$

Then, $C_1 \subset U$, $C_2 \subset V$, and $U \cap V = \emptyset$. □

Proposition 4.4. *Suppose X is T_1.*

(i) X is regular if and only if for any point $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $x \in V \subset \overline{V} \subset U$.

(ii) X is normal if and only if for any closed set $C \subset X$ and any neighborhood U of C, there exists a neighborhood V of C such that $C \subset V \subset \overline{V} \subset U$.

Proof. (i) Suppose X is regular. Fix x and U. Then, $X \setminus U$ is closed. By regularity, there exist open sets V, W such that $x \in V$, $X \setminus U \subset W$, and $V \cap W = \varnothing$. Thus, $x \in V \subset X \setminus W \subset U$. Note that $\overline{V} \subset X \setminus W \subset U$.

Conversely, suppose the given condition holds. Fix closed set C and point $x \notin C$. Then, $X \setminus C$ is a neighborhood of x. By assumption, there exists a neighborhood V of x such that $x \in V \subset \overline{V} \subset X \setminus C$. Let $W = X \setminus \overline{V}$. Then, $C \subset W$ and $V \cap W = \varnothing$.

(ii) Similar to (i).

□

Recall that subspace/product of Hausdorff is Hausdorff. The same is true for regular, but not for normal.

Proof *(Subspace of Regular is Regular)***.** Suppose X is regular and fix subset $Y \subset X$. Note that Y inherits T_1 from X. Take closed set $C \subset Y$ and point $y \in Y \setminus C$. Then, there exists closed set $D \subset X$ such that $C = Y \cap D$. Since $y \notin D$, by regularity of X, there exist open sets $U, V \subset X$ such that $y \in U$, $D \subset V$, and $U \cap V = \varnothing$. Then, $y \in U \cap Y$, $C \subset V \cap Y$, and $(U \cap Y) \cap (V \cap Y) = \varnothing$. □

Note that the same proof does not work for normality, since we cannot guarantee that the corresponding closed sets in the original space are disjoint.

Example 4.5. $\mathbb{R}_\ell$ is normal.

To see this, fix disjoint closed sets $C_1, C_2 \subset \mathbb{R}_\ell$. For any $x \in C_1$, since C_2 is closed, there exists $\epsilon_x > 0$ such that $[x, x + \epsilon_x) \cap C_2 = \varnothing$. Define

$$U = \bigcup_{x \in C_1} [x, x + \epsilon_x/2), \quad V = \bigcup_{y \in C_2} [y, y + \epsilon_y/2).$$

Then, $C_1 \subset U$, $C_2 \subset V$, and $U \cap V = \varnothing$.

Theorem 4.6. *If X is regular and second countable, then X is normal.*

Proof. Fix disjoint closed sets $A, C \subset X$. For each $a \in A$, choose open $U_a \subset X$ disjoint from C such that $a \in U_a$. By regularity there exists $V_a \subset \overline{V_a} \subset U_a$. But this implies the existence of $B_a \in \mathcal{B}$ (where $\mathcal{B}$ is a countable basis for X) such that $a \in B_a \subset V_a \subset U_a$. Note that $\overline{B_a} \cap C = \varnothing$.

By second countability, the collection $\{B_a | a \in A\}$ has a countable subcover $\{B_{a_i} | i \in \mathbb{N}\}$ of A, each with $\overline{B_{a_i}} \cap C = \varnothing$. Similarly choose a countable collection $\{B_{c_j} | j \in \mathbb{N}\}$ covering C with $\overline{B_{c_j}} \cap A = \varnothing$.

Now, consider sets $B'_i := B_i \setminus \cup_{j=1}^k \overline{B_j}$ and $B'_j := B_j \setminus \cup_{i=1}^k \overline{B_i}$. Note that these are all open sets, and we have

$$A \subset \bigcup_i B'_i, \quad C \subset \bigcup_j B'_j, \quad \text{and} \quad \bigcup_i B'_i \cap \bigcup_j B'_j = \varnothing.$$

Indeed, if $x \in B'_i \cap B'_j$ with say $i \leq j$, we have $x \in B'_j$, which gives $x \notin \overline{B_i}$, and then $x \notin B'_i$, a contradiction. □

Example 4.7 (Sorgenfrey Plane). Recall that $\mathbb{R}_\ell$ is the real line with the lower limit topology generated by basis of half-open intervals $[a, b)$. Recall that $\mathbb{R}_\ell$ is normal and regular. Consider the product space $\mathbb{R}_\ell \times \mathbb{R}_\ell$, called the **Sorgenfrey plane**. This is still regular as the product of regular spaces, but it is not normal:

Proposition 4.8. *The Sorgenfrey plane $\mathbb{R}_\ell^2$ is not normal.*

Proof. Let $L := \{(x, -x) | x \in \mathbb{R}\} \subset \mathbb{R}^2$ be the diagonal. Note that L is closed in $\mathbb{R}_\ell^2$, since $\mathbb{R}_\ell^2$ is finer than the standard topology. For each $(x, -x) \in L$, note that

$$L \cap \Big[[x, x+1) \times [-x, -x+1) \Big] = \{x\}.$$

Thus each singleton in L is open in the subspace topology. Thus for $A \subset L$, A is closed $\in \mathbb{R}_\ell^2$.

Now assume the Sorgenfrey plane is normal. Then for any nonempty $A \subset L$, we know that both A and $L \setminus A$ are closed. Thus there exist open nonempty sets $U_A, V_A \subset \mathbb{R}_\ell^2$ such that $A \subset U_A$, $L \setminus A \subset V_A$, and $U_A \cap V_A = \varnothing$.

Note that $\mathbb{Q}^2$ is countable and dense in $\mathbb{R}_\ell^2$. Consider function $F : 2^L \to 2^{\mathbb{Q}^2}$ defined by

$$F(\varnothing) = \varnothing, \quad F(L) = \mathbb{Q}^2, \quad F(A) = U_A \cap \mathbb{Q}^2, \quad \text{for } A \neq \varnothing, L.$$

We will show that F is injective. Since $2^{\mathbb{Q}^2}$ is in bijection with $\mathbb{R}$, and L has cardinality $\mathbb{R}$, we have a map from a set with cardinality $2^{\mathbb{R}}$ to a set with cardinality $\mathbb{R}$, which is impossible.

To see that F is injective, fix $A, B \subset L$ with $A \neq B$ and assume that $F(A) = F(B)$. If either set is empty or equal to L, then $F(A) \neq F(B)$ by definition. If not, we have

$$U_A \cap \mathbb{Q}^2 = U_B \cap \mathbb{Q}^2.$$

Without loss of generality, assume there exists $a \in A \setminus B$. We have $a \in L \setminus B$ and so $a \in V_B$. From $a \in A \subset U_A$ we conclude that $V_B \cap U_A \neq \varnothing$. Now, the density of $\mathbb{Q}^2$ gives a $q \in V_B \cap U_A \cap \mathbb{Q}^2$. In particular, $q \in U_A \cap \mathbb{Q}^2 = U_B \cap \mathbb{Q}^2$, contradicting $q \in V_B \cap \mathbb{Q}^2$. □

Lemma 4.9 (Urysohn's Lemma)**.** *X is normal if and only if for any two disjoint closed sets $A, B \subset X$, there exists a continuous function $f : X \to [0, 1]$ such that $f_A = 0$ and $f_B \equiv 1$.*

Proof. The backward direction is immediate.

For the forward direction, fix disjoint closed sets $A, B \subset X$. The idea is to construct an ascending chain of open sets separating A and B, then define f based on which open sets a point belongs to.

Let U_0 be open such that $A \subset U_0 \subset \overline{U}_0 \subset X \setminus B$, and let $U_1 := X \setminus B$. We construct sets U_p for each $p \in [0,1] \cap \mathbb{Q}$ such that for each $p < q$ in the following way: First, choose a bijection $n : \mathbb{N} \to [0,1] \cap \mathbb{Q}$ such that $n_1 = 0$ and $n_2 = 1$. Next, construct U_k inductively. Suppose $U_{n_1}, \ldots, U_{n_{k-1}}$ has been constructed such that $i < j < k$ implies $\overline{U}_{n_i} \leq U_{n_j}$. The numbers $n_1, \ldots, n_{k-1}$ is a finite subset of $\mathbb{Q}$. Let $p, q \in \{n_1, \ldots, n_{k-1}\}$ such that p, q are immediate neighbors with $p < n_k < q$. Note that $\overline{U}_{n_p} \subset U_{f(q)}$. By normality there exists some open U_{n_k} such that $\overline{U}_{n_p} \subset U_{n_k} \subset \overline{U}_{n_k} \subset U_{f(q)}$. Observe that for each $i < j \leq k$, we have $\overline{U}_{n_i} \subset U_{n_j}$.

Now, define $U_p := \varnothing$ for each rational $p < 0$ and $U_p := X$ for each rational $p > 1$. We have a collection of open sets U_p such that for arbitrary rationals $p < q$ we have $\overline{U}_p \subset U_q$.

Finally, define $f : X \to [0,1]$ by $f(x) := \inf \left\{ p \in \mathbb{Q} : x \in U_p \right\}$. First observe that $\{p \in \mathbb{Q} : x \in U_p\}$ contains only rationals in $[0,1]$. Thus the infimum is well-defined and lies in $[0,1]$. It is easy also to see that $f_A = 0$ and $f_B \equiv 1$.

To see that f is continuous, observe that if $x \in \overline{U}_s$, then

$$(s, +\infty) \subset \left\{ p \in \mathbb{Q} : x \in U_p \right\},$$

so $f(x) \leq s$. On the other hand, if $x \notin U_s$, then $f(s) \geq s$.

Now, fix some $x_0 \in X$ and $\varepsilon > 0$. Find $p, q \in \mathbb{Q}$ such that $f(x) - \varepsilon < p < f(x_0) < q < f(x) + \varepsilon$. Consider the open set $U_q \setminus \overline{U}_p$. This set contains x_0: from $f(x_0) \leq q$ we have $x_0 \in U_q$; from $f(x_0) > p$ we have $x_0 \notin \overline{U}_p$. We show that for each $y \in U_q \setminus \overline{U}_p \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, we have $f(y) \in [p, q]$. If not, say $f(y) < p$, then $y \in U_p$, contradicting $y \notin \overline{U}_p$. Similarly, if $f(y) > q$, then $y \notin U_q$, a contradiction. □

Theorem 4.10 (Urysohn Metrization Theorem)**.** *A topological space X is metrizable if and only if it is regular and second countable.*

Proof. We will embed X into **Hilbert's cube** $[0,1]^{\mathbb{N}}$ with the product topology. Recall that this space is metrizable with metric $d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|/n$. But it has another metric.

Consider $\mathcal{H}$, the Hilbert space of sequences x_n such that $\sum x_n^2 < \infty$, with metric $d(x, y) = \sqrt{\sum (x_n - y_n)^2}$. The space $\mathcal{H}$ has a subset

$$H := [0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right] \times \cdots,$$

which is homeomorphic to Hilbert's cube via the map $(x_1, x_2, \ldots) \mapsto (x_1, 2x_2, 3x_3, \ldots)$. □