Notes: ECMA33220 (F25) Introduction to Advanced Macroeconomic Analysis

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1 Time Series Basics

Definition 1.1. An $\mathbf{AR}(k)$ (autoregressive) process is a time series $\{y_t\}$ satisfying

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \cdots + \varphi_k y_{t-k} + \varepsilon_t, \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

Here, y_{t-k} is called a **lag** of order k. A $\mathbf{MA}(k)$ (moving-average) process is a time series $\{y_t\}$ satisfying

$$y_t = \varepsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_k \epsilon_{t-k}, \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

1.1 Forecasting

Consider the AR(1) process $y_t = \rho y_{t-1} + \varepsilon_t$ where $\epsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$. Observe that $\mathbb{E}_t[y_{t+k}] = \rho^k y_t$. This can be generalized to the AR(n) case:

Consider the AR(n) process

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \cdots + \varphi_n y_{t-n} + \varepsilon_t, \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

We have $x_t = Bx_{t-1} + \varepsilon_t A$, where

$$x_t \coloneqq \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-n+1} \end{bmatrix}, \quad B = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_{n-1} & \varphi_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Note that B is called the **companion matrix**. As above,

$$\mathbb{E}_t[x_{t+k}] = B^k x_t \in \mathbb{R}^{n \times 1}.$$

Corollary 1.2. *For an* AR(m) *process* $\{y_t\}$*, we have* $\mathbb{E}_t[y_{t+k}] = \left(B^k x_t\right)_1$.

1.2 Impulse Response Functions

Definition 1.3. The **impulse response** $r(k)$ to a shock one standard deviation in size is the forecast for $y_k - \mathbb{E}[y_k]$, given $\varepsilon_0 = \sigma$ and $y_t = 0, t < 0$.

Remark 1.4. $\mathbb{E}[y_k]$ is the *unconditional expectation* of y_k, i.e., its *deterministic* component.

Proposition 1.5. *For an* AR(m) *process* $\{y_t\}$*, the impulse response function is given by*

$$r(k) = \sigma \left(B^k e_1\right)_1,$$

where B is the companion matrix.

Proposition 1.6. *For a* AR(1) *process* $y_t = \rho y_{t-1} + \varepsilon_t$*,*

- *The impulse response converges to zero if* $|\rho| < 1$,
- *The absolutely value of the impulse response converges to a nonzero value if* $|\rho| = 1$.
- *The absolutely value of the impulse response explodes if* $|\rho| > 1$.

If we denote $\theta_k := r(k)$, then if μ_t is the deterministic piece component of y_t, we have

$$y_t - \mu_t = \sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k}$$

is an infinite-order MA representation of the AR process. In fact, we have the

Theorem 1.7 (Wold Representation Theorem)**.** *Any covariance stationary time series can be represented as*

$$y_t = \mu_t + \sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k},$$

where the ε_t*'s are the one-step ahead linear forecast errors for the* y_t*'s, given information on lagged values of* y_t*, and where* μ_t *is a deterministic function of time.*

Remark 1.8.

- The one-step ahead linear forecast error is defined as $\varepsilon_t = y_t - \mathbb{E}_{t-1}[y_t]$.
- Loosely speaking, a process is **covariance stationary** if it fluctuates around some point, rather than diverges, once deterministic time trends μ_t are removed.

Note that $\mathbb{E}_t[\varepsilon_{t+k}] = 0$ for each $k > 0$. Thus we have the

Corollary 1.9.

$$\mathbb{E}_t[y_{t+k}] = \mu_{t+k} + \sum_{j=k}^{\infty} \theta_j \mathbb{E}_t[\varepsilon_{t+k-j}].$$

1.3 Autocovariances

Definition 1.10. Let $\{y_t\}$ be a time series.

- The kth **autocovariance** at time t is defined as the covariance between y_t and y_{t-k}.
- The kth **autocorrelation** at time t is defined as the correlation between y_t and y_{t-k}.

Moreover, a time series $\{y_t\}$ is called **covariance stationary** if its mean of y_t as well as all autocovariances are finite and do not depend on t.

Example 1.11. Consider the mean zero AR(1) process $y_t = \rho y_{t-1} + \varepsilon_t$. The variance is given by

$$\mathbb{E}[y_t y_t] = \rho^2 \mathbb{E}[y_{t-1} y_{t-1}] + \sigma^2.$$

- If $|\rho| \geq 1$, then $\mathbb{V}(y_t) = \infty$.
- If $|\rho| < 1$, we have
$$\mathbb{V}(y_t) = \frac{\sigma^2}{1-\rho^2}.$$
In this case the autocovariance is given by
$$\begin{aligned}\mathbb{E}[y_t y_{t-k}] &= \mathbb{E}[\varepsilon_t y_{t-k}] + \rho \mathbb{E}[y_{t-1} y_{t-k}] + \cdot + \rho^k \mathbb{E}[y_{t-k} y_{t-k}] \\ &= \frac{\sigma^2 \rho^k}{1-\rho^2},\end{aligned}$$
since $y_t = \rho^k y_{t-k} + \rho^{k-1} \varepsilon_{t-k-1} + \cdots + \rho \varepsilon_{t-1} + \varepsilon_t$. Thus the kth autocorrelation is given by ρ^k.

From this discussion we have the

Proposition 1.12. *An* AR(1) *process with constant and finite mean and iid shocks is covariance stationary if and only if* $|\rho| < 1$.

To derive the autocovariances of a general AR(n) process, we first recall the following:

Let A, B be two matrices. The **Kronecker product** $A \otimes B$ is defined as the block matrix

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix},$$

where each element of A is multiplied with the entire matrix B. Let vec($\cdot$) denote the column wise vectorization operator. We have the following:

Proposition 1.13. *Let A, B, C be three matrices such that the matrix product ABC is well-defined. Then,*

$$\operatorname{vec}(ABC) = (C' \otimes A) \operatorname{vec}(B).$$

Now consider the mean zero covariance stationary AR(m) process $\{y_t\}$ and write $x_t = Bx_{t-1} + \varepsilon_t A$, $\mathbb{E}[\varepsilon_t^2] = \sigma^2$ as before. Define the autocovariances $\Gamma_k = \mathbb{E}[x_t x'_{t-k}]$. We have

$$\begin{aligned}\Gamma_0 := \mathbb{E}[x_t x'_t] &= B\mathbb{E}[x_{t-1} x'_{t-1}]B' + A\sigma^2 A' \\ &= B\Gamma_0 B' + A\sigma^2 A'.\end{aligned}$$

Applying the $\mathrm{vec}(\cdot)$ operator and using the Kronecker product property, we have

$$\mathrm{vec}(\Gamma_0) = (B \otimes B)\,\mathrm{vec}(\Gamma_0) + (A \otimes A)\sigma^2.$$

It turns out that if all eigenvalues of B lie strictly within the unit circle, then $(I - B \otimes B)$ is invertible and we have

Proposition 1.14. *For a covariance stationary* $\mathrm{AR}(m)$ *process* $\{y_t\}$*, the variance-covariance matrix of* x_t *is given by*

$$\mathrm{vec}(\Gamma_0) = (I_{m^2} - B \otimes B)^{-1}\big((A \otimes A)\sigma^2\big).$$

Recall here that $(\Gamma_0)_{1,1}$ is the variance of y_t, and $(\Gamma_0)_{1,j}$ is the $(j-1)$th autocovariance of y_t. Now observe the

Proposition 1.15 (Yule-Walker Equation)**.**

$$\Gamma_k := \mathbb{E}[x_t x'_{t-k}] = B\mathbb{E}[x_{t-1}x'_{t-k}] = B\Gamma_{k-1}.$$

With this, we can iteratively compute the kth autocovariance of the $\mathrm{AR}(m)$ process by $(B^k\Gamma_0)_{1,1}$.

1.4 Lag Operator Calculus

Definition 1.16. The **lag operator** L shifts a time series back by one period: $(Ly)_t = y_{t-1}$.

We may use the lag operator to rewrite an $\mathrm{AR}(1)$ process $y_t = \rho y_{t-1} + \varepsilon_t$ as $(1 - \rho L)y_t = \varepsilon_t$. More generally, an $\mathrm{AR}(m)$ process can be written as

$$\left(1 - \varphi_1 L - \varphi_2 L^2 - \cdots - \varphi_m L^m\right) y_t = \varepsilon_t.$$

Definition 1.17. For an $\mathrm{AR}(m)$ process $(1 - \rho(L))y_t = \varepsilon_t$, the **characteristic polynomial** is defined as

$$\begin{aligned} p(\lambda) &:= \lambda^m(1 - \varphi(\lambda^{-1})) \\ &= \lambda^m - \varphi_1\lambda^{m-1} - \varphi_2\lambda^{m-2} - \cdots - \varphi_m. \end{aligned}$$

The complex-valued solutions $\lambda_1, \ldots, \lambda_m$ of $p(\lambda) = 0$ are called the **roots** of the characteristic polynomial.

Proposition 1.18. *An* $\mathrm{AR}(m)$ *process is covariance stationary if and only if all roots of its characteristic polynomial lie strictly within the unit circle.*

Remark 1.19. Real-valued roots induce impulse response dynamics as seen in the $\mathrm{AR}(1)$ case.

- > 1: explosive.
- $= 1$: unit root; shock never dies out.
- < 1: shock eventually dies out.

Complex valued roots always come in conjugate pairs $a \pm b\mathrm{i}$ and induce (damped or explosive) oscillatory impulse response dynamics with frequency $\arctan(b/a)$. ☕

2 Vector Autoregressions

Definition 2.1. A **vector autoregression** of order k for an m-dimensional vector y_t is a vector time series $\{y_t\}$ satisfying

$$y_t = B_1 y_{t-1} + \cdots + B_k y_{t-k} + u_t, \quad u_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma),$$

where $B_j \in \mathbb{R}^{m \times m}$ and where u_t is called the **one-step ahead linear forecast error**.

As before, we will ignore deterministic components such as time trends.

Definition 2.2. Let a be some m-dimensional vector. The **impulse response** $r_a(k)$ to the vector a is the forecast for y_k given $u_0 = a$ and $y_t = 0, t < 0$.

Example 2.3. For a VAR(1) process, $r_a(k) = B^k a$.

Again, we can write a VAR(k) process as a VAR(1) process in companion form, by defining

$$\mathbf{x}_t := \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-k+1} \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} B_1 & B_2 & \cdots & B_{k-1} & B_k \\ I_m & 0 & \cdots & 0 & 0 \\ 0 & I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} I_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We thus will next only consider VAR(1) processes.

2.1 Diagonalization and Decoupling

Fix a covariance stationary VAR(1) process $y_t = By_{t-1} + u_t$, $u_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma)$. If $B = VDV^{-1}$ is diagonalizable, then $B^k = VD^kV^{-1}$ and

$$r_a(k) = B^k a = VD^kV^{-1}a.$$

Remark 2.4. Not all matrices are diagonalizable, but all matrices admit a Jordan normal form, which can be used similarly.

Write

$$V = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m \end{bmatrix}.$$

Note that the impulse response to $a = v_j$ is given by

$$r_{v_j}(k) = \lambda_j^k v_j.$$

Thus, the dynamics of a VAR(1) with a diagonalizable B can be described by the dynamics along each eigenvector v_j. The dynamics has been **decoupled**.

Proposition 2.5. *The roots $\lambda_1, \dots, \lambda_m$ can be calculated as the roots of the* ***characteristic polynomial***

$$p(\lambda) := \det(\lambda I_m - B).$$

Remark 2.6. One can show that this is equivalent to the definition of characteristic polynomial we gave before.

Proof. Write $B = VDV^{-1}$ and observe that

$$\begin{aligned} p(\lambda) &= \det(\lambda I - B) = \det(V)\det(\lambda I - D)\det(V^{-1}) \\ &= \det(\lambda I - D) = \prod_{j=1}^{m}(\lambda - \lambda_j). \end{aligned}$$

□

Proposition 2.7. *; A* VAR(1) *process is covariance stationary if and only if all roots of its characteristic polynomial lie strictly within the unit circle.*

2.2 Cointegration

We next analyzes the case where some roots, say $\lambda_1, \dots, \lambda_r$ are stable, while all others are exactly equal to 1: $\lambda_{r+1} = \cdots = \lambda_m = 1$. Furthermore, we assume that these **unit roots** affect every entry of y_t. That is, that the last $m - r$ columns of V have at least one nonzero entry in each row.

Definition 2.8. Define the **difference operator Δ** as $\Delta y_t := y_t - y_{t-1} = (1 - L)y_t$. We say a univariate time series y_t is **integrated** if y_t is not stationary but the first difference Δy_t is stationary.

Proposition 2.9. *A univariate* AR(k) *process* $\{y_t\}$ *is integrated if and only if exactly one of the roots of its characteristic polynomial is equal to* 1 *and all others are stable.*

Definition 2.10. The vector time series y_t is said to be **cointegrated of rank *r*** if each of the series taken individually is integrated, and r linear combination of the series, $\beta' y_t$, is stationary for some $\beta \in \mathbb{R}^{m\times r}$ of rank $r \geq 1$.

Now, split the matrices V, D, and V^{-1} (where $B = VDV^{-1}$) in the following way:

$$V = \begin{bmatrix} v_r & v_\perp \end{bmatrix}, \quad D = \begin{bmatrix} D_r & 0 \\ 0 & I_{m-r} \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} \beta' \\ \beta'_\perp \end{bmatrix},$$

where v_r, D_r, and β correspond to the stable eigenvalues, while $v_\perp$, and $\beta_\perp$ correspond to the unit roots. (Note that $\perp$ here only differentiates the two parts; it does not imply orthogonality.)

We have the

Proposition 2.11. *Let $|\lambda_1|, \dots, |\lambda_r| < 1$ be the stable roots and let $\lambda_{r+1} = \cdots = \lambda_m = 1$ be the unit roots. The matrix β as defined above is a cointegrating matrix of rank r for the* VAR(1) *process $y_t = By_{t-1} + u_t$.*

Note also that using the notation above, we have

$$\begin{aligned} B^k a &= \begin{bmatrix} \nu_r & \nu_\perp \end{bmatrix} \begin{bmatrix} D_r^k & 0 \\ 0 & I_{m-r} \end{bmatrix} \begin{bmatrix} \beta' \\ \beta'_\perp \end{bmatrix} a \\ &= \underbrace{\nu_r D_r^k \beta' a}_{\text{transitory part}} + \underbrace{\nu_\perp \beta'_\perp a}_{\text{permanent part}} . \end{aligned}$$

In particular,

Proposition 2.12. *The long-run impulse response is given by* $\lim_{k\to\infty} B^k a = \nu_\perp \beta'_\perp a$.

2.3 Error Correction

Define $\alpha := \nu(I_r - D_r)$ and note that

$$\Delta y_t - u_t = (B - I)y_{t-1} = V(D - I)V^{-1} y_{t-1} = -\alpha\beta' y_{t-1},$$

where the last equality follows from the fact that $D - I$ is a diagonal matrix with the first r diagonal entries equal to $\lambda_j - 1$ and the last $m - r$ diagonal entries equal to 0. This is called the **error correction representation** of the $\mathrm{VAR}(1)$ process.

Remark 2.13.

- Note in particular that $\alpha\beta' = I - B$.
- With the error correction representation, we see that if $\beta' y_{t-1}$ is "large" (i.e., far from its equilibrium), then Δy_t will be "large" in the opposite direction to correct for this deviation.
- The change of y_t is driven by:
  - new shocks u_t,
  - the convergence of the stationary component $\beta' y_{t-1}$ back to zero, and
  - comovements induced by the cointegrating relationships.

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Proposition 2.14 (Engle and Granger, 1987)**.** *A* $\mathrm{VAR}(k)$ *which is cointegrated of rank r can be given an error correction representation of the form*

$$A^*(L)\Delta y_t = -\alpha\beta' y_{t-1} + u_t,$$

Remark 2.15. With cointegration, there are linear combinations $\beta' y_{t-1}$ among the variables that tend to move together and "correct the error" of moving too far from zero. ☕

3 Identification in VARs

Which vectors to choose for impulse responses? A *necessary* requirement is orthogonality. We seek to decompose u_t into mutually orthogonal components $u_t = A\varepsilon_t$, where ε_t are orthogonal shocks such that $\mathbb{E}[\varepsilon_t \varepsilon_t'] = I_m$. This is equivalent to finding A such that $\Sigma = AA'$.

Definition 3.1. An **impulse vector** a is a column of a matrix $A \in \mathbb{R}^{m\times m}$ such that $\Sigma = AA'$.

Proposition 3.2. *Let a be an impulse vector and let $\tilde{A}\tilde{A}' = \Sigma$ be some other decomposition of Σ. If* $\det \Sigma \neq 0$*, then*

$$a = \tilde{A}q,$$

for some vector q of unit length.

Proof. Let $AA' = \Sigma$ be some decomposition of Σ such that a is the jth column of A. Let $Q := \tilde{A}^{-1}A$ and note that

$$QQ' = \tilde{A}^{-1}A(A')(\tilde{A}^{-1})' = I_m.$$

Thus Q is an **orthogonal matrix** (a matrix whose columns are of unit length and mutually orthogonal; Q is orthogonal if and only if Q' is orthogonal). We have $a = \tilde{A}q$, where q is the jth column of Q. □

Corollary 3.3. *The set of decompositions of Σ is given by $\left\{\tilde{A}Q : QQ' = I_m\right\}$.*

3.1 Cholesky Decomposition

The **Cholesky decomposition** of a positive definite matrix $\Sigma = AA'$ is a decomposition where A is lower triangular with real and positive diagonal entries. In practice, we order "slow-moving" variables first and "fast-moving" variables last, so that the resulting shock from the Cholesky decomposition is consistent with the causal ordering of the variables.

3.2 Blanchard-Quah Decomposition

The **Blanchard-Quah decomposition** decomposes shocks into *permanent* and *transitory* components. To obtain such a decomposition, we seek a vector a which has no permanent effect (using say the characterization of long-run effects from before):

$$0 = \lim_{k\to\infty} r_a(k) = \lim_{k\to\infty} B^k a.$$

Typically, this gives a up to a multiplicative constant. One can then find the full decomposition $\Sigma = AA'$ in the following way:

- Find any decomposition $\Sigma = \tilde{A}\tilde{A}'$ (say the Cholesky decomposition).

- We know $a = ca_0$ is a column of A, where a_0 is some particular solution to the permanent effect restriction and c is some constant. Thus, $ca_0 = \tilde{A}q$ for some unit length vector q which satisfies

$$\frac{q}{c} = \tilde{A}^{-1}a_0.$$

  Using the unit length restriction on q, we can solve for c, q, and then a.

- The unit vector q can be completed to an orthogonal matrix Q. We then have $A = \tilde{A}Q$.

3.3 Sign Restrictions

Sample impulse responses which satisfy sign restrictions uniformly (Uhlig, 2005).

4 Business Cycles

4.1 The HP Filter

The **Hodrick-Prescott (HP) filter** decomposes a time series y_t into a trend component τ_t and a cyclical component c_t by solving

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \left((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right)^2 ,$$

Remark 4.1. For quarterly data, $\lambda = 1600$ is typically used. Otherwise, for a change in observation frequency by a factor of k, λ should be changed by a factor of k^4 (Ravn and Uhlig, 2002).

Remark 4.2. Two ways to get pro-cyclical productivity are TFP shocks and fixed costs.

Remark 4.3. Empirical facts about business cycles:

- Most macroeconomic time series are **procyclical**, i.e., their business cycle component is positively correlated with output.
- Strongly procyclical: consumption, investment, hours worked.
- Mildly procyclical: housing starts, exports, labor productivity, real wages, inflation, short-term interest rates.
- Acyclical: government spending, oil prices, long-term interest rates, M1.

4.2 A Benchmark Real Business Cycle Model

The social planner solves

$$\max_{c_t, n_t, k_t, y_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \quad \text{s.t.} \quad c_t + k_t = y_t + (1-\delta) k_{t-1}, \quad y_t = f\left(A_t \frac{n_t}{k_{t-1}} \right) k_{t-1}$$

and $A_t > 0$ is some exogenous productivity process (E.g., $\log A_t = \rho \log A_{t-1} + \varepsilon_t$). Note that the production function exhibits constant returns to scale.

Remark 4.4. In the decentralized economy, the representative firm is given w_t, r_t and solves

$$\max_{k_{t-1}, n_t} f\left(A_t \frac{n_t}{k_{t-1}} \right) k_{t-1} - w_t n_t - r_t k_{t-1}.$$

The FOC dictates that

$$\begin{aligned} w_t &= A_t f'\left(A_t \frac{n_t}{k_{t-1}} \right), \\ r_t &= f\left(A_t \frac{n_t}{k_{t-1}} \right) - f'\left(A_t \frac{n_t}{k_{t-1}} \right) A_t \frac{n_t}{k_{t-1}}. \end{aligned}$$

In a competitive equilibrium, profits are zero by Euler's theorem ($F(K,N) = F_K(K,N)K + F_N(K,N)N$). Thus we obtain

$$w_t n_t = f'\left(A_t \frac{n_t}{k_{t-1}}\right) A_t n_t.$$

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Remark 4.5. Some specifications of utility functions:

- Log:
$$u(c_t, n_t) = \log c_t - \chi n_t^{1+\frac{1}{\varphi}}.$$
Here, φ is the Frisch elasticity of labor supply.
- Cobb-Douglas:
$$u(c_t, n_t) = c_t^{\alpha}(1-n_t)^{1-\alpha}.$$
- Constant relative risk aversion (CRRA):
$$u(c_t, n_t) = \frac{c_t^{1-\eta} - 1}{1-\eta} - \chi n_t^{1+\frac{1}{\nu}}.$$
As $\eta \to 1$, this converges to the log utility specification.
- Constant Frisch Elasticity (CFE):
$$u(c_t, n_t) = \frac{c_t^{1-\eta}\left(1 - (1-\eta)\chi n_t^{1+\frac{1}{\varphi}}\right)^{\eta} - 1}{1-\eta}.$$
Here, φ is the Frisch elasticity of labor supply.
- Greenwood-Hercowitz-Huffman (GHH):
$$u(c_t, n_t) = \frac{\left(c_t - \chi n_t^{1+\frac{1}{\nu}}\right)^{1-\eta} - 1}{1-\eta}.$$

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Definition 4.6. The **Frisch elasticity of labor supply** is defined as the elasticity of labor supply with respect to the wage, holding the marginal utility λ of wealth constant.

Remark 4.7. From $n = \mathrm{const} \cdot w^{\varphi}$ we can easily see that the Frisch elasticity is given by φ:

$$\frac{\partial n}{\partial w} = \varphi \cdot \mathrm{const} \cdot w^{\varphi - 1} \implies \frac{\partial n}{\partial w}\frac{w}{n} = \varphi.$$

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Proposition 4.8. *Define* $w := f'(An/k)A$.

$$-u_n(c,n) = \lambda w, \quad -\frac{u_n(c,n)}{u_c(c,n)} = w.$$