

# MATH20410 (W25): Analysis in $\mathbb{R}^n$ II (accelerated)

Lecturer: Joe Jackson

Notes by: Aden Chen

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# 1 Single-Variable Differential Calculus

In this chapter, we consider mainly functions of the form  $f : I \rightarrow \mathbb{R}$ , where  $I$  is an interval, e.g.,  $(a, b)$ ,  $[a, b]$ ,  $(a, \infty)$ ,  $\mathbb{R}$ . This is the function we have in mind unless otherwise stated.

**Definition 1.1** (Differentiability). We say  $f$  is **differentiable** at  $x \in I$  if the limit

$$f'(x) := \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

exists. In this case, we call  $f'(x)$  the derivative of  $f$  at  $x$ . Moreover:

- We say that  $f$  is **differentiable** if  $f'(x)$  exists for each  $x \in I$ .
- We say  $f$  is **continuously differentiable** ( $f \in C^1$ ) if  $f' : I \rightarrow \mathbb{R}$  is continuous.

*Example 1.2.*

- $f(x) = |x|$ . Differentiable on  $\mathbb{R} \setminus \{0\}$ .
- $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ . Continuous but not differentiable at 0.
- $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ . Differentiable everywhere (in particular at 0), but  $f \notin C^1$ .

**Proposition 1.3** (Rules for computing derivatives).

- (i) *Linearity.*  $(af + bg)' = af' + bg'$  (if  $f'$  and  $g'$  exist, such requirements are hereafter omitted).
- (ii) *Product rule.*  $(fg)' = f'g + fg'$ .
- (iii) *Quotient rule.*  $(f/g)' = (f'g - fg')/g^2$ .<sup>1</sup>
- (iv) *Chain rule.*  $(f \circ g)' = (f' \circ g) \cdot g'$ .

<sup>1</sup>Low dhigh minus high dlow. Not Haidilao...

**Proof.** We prove the quotient rule; the remaining are left as exercises. Starting from the definition

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{t \rightarrow x} \frac{\frac{f}{g}(t) - \frac{f}{g}(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{\frac{f(t)}{g(t)} + \frac{f(x)}{g(t)} - \frac{f(x)}{g(t)} + \frac{f(x)}{g(x)}}{t - x}.\end{aligned}$$

Note that

$$\frac{\frac{f(x)}{g(t)} + \frac{f(x)}{g(x)}}{t - x} = \frac{f(x)}{g(x)g(t)} \frac{g(x) - g(t)}{t - x}$$

and we have

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)}$$

□

**Theorem 1.4.** *If  $f$  is differentiable at  $x$  then  $f$  is continuous at  $x$ .*

**Proof.** Note that

$$\lim_{t \rightarrow x} f(t) - f(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} (t - x) = f'(x) \cdot 0 = 0.$$

□

## 1.1 The Mean Value Theorem

**Lemma 1.5.** *Suppose  $f : [a, b] \rightarrow \mathbb{R}$  has a local maximum or minimum at  $x \in (a, b)$ . If  $f'(x)$  exists, then  $f'(x) = 0$ .*

**Proof.** From the definition of the derivative, consider the limits from the left and right; one is non-positive and the other is non-negative. □

**Theorem 1.6** (Rolle's Theorem). *Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and such that  $f(a) = f(b)$ . Then there exists  $x \in (a, b)$  such that  $f'(x) = 0$ .*

**Proof.** Consider the global maximum or minimum (exist since  $f$  is a continuous function defined on a compact set) and apply the previous lemma. (If both the maximum and minimum is at  $a$  or  $b$ ,  $f$  is constant.) □

**Theorem 1.7** (Mean Value Theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $x \in (a, b)$  such that  $f(b) - f(a) = f'(x)(b - a)$ .*

**Proof.** Apply Rolle's to  $\tilde{f} = f - [f(b) - f(a)] \cdot \frac{x-a}{b-a}$ . □

**Theorem 1.8.** *Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable.*

(a) *if  $f' = 0$ , then  $f$  is constant.*

(b) *if  $f' \geq 0$ , then  $f$  is increasing.*

(c) *if  $f' \leq 0$ , then  $f$  is decreasing.*

**Proof.** Apply the mean value theorem. □

**Theorem 1.9** (The Intermediate Value Property of Derivatives). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable<sup>2</sup> and suppose  $f'(a) < \lambda < f'(b)$ . Then there exists  $x \in (a, b)$  such that  $f'(x) = \lambda$ .* <sup>2</sup> $f$  need not be  $C^1$ !

**Proof** (*à la Pugh*). Slide a small secant of length so small that the slope around  $a$  and  $b$  is separated also by  $\lambda$ . By continuity of the slope, there exists a secant between  $a$  and  $b$  with slope  $\lambda$ . Apply the mean value theorem to this slope. □

**Proof** (*à la Joe/Rudin*). We start with  $\lambda = 0$ . Then  $f'(a), f'(b) \neq 0$  and the global min/max of  $f$  cannot be at the endpoints. At the global extrema we have the desired result. When  $\lambda \neq 0$ , consider  $\tilde{f} := f - \lambda x$ . □

*Example 1.10.* Consider

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

We have

$$f(x) = \begin{cases} 2x \sin(1/x) = \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

which has the intermediate value property.

**Theorem 1.11** (Generalized Mean Value Theorem). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $x \in (a, b)$  such that*

$$(f(a) - f(b))g'(x) = (g(a) - g(b))f'(x).$$

*Remark 1.12.* When the above is not zero,

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(x)}{g'(x)}.$$

**Proof.** Define

$$h(t) := (f(b) - f(a))g(t) - (g(b) - g(a))f(t).$$

Note that

$$h(a) = f(b)g(a) - f(a)g(b) = h(b)$$

and apply Rolle's. □

## 1.2 L'Hôpital's Rule

**Theorem 1.13** (L'Hôpital's Rule, a particular case). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $g(x) \neq 0$  in a neighborhood of  $a$  and  $f(x) = g(x) = 0$ , then*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

*if the last limit exists.*

**Proof.** Consider some small  $\delta > 0$ . The generalized MVT gives some  $x \in (a, a+\delta)$  such that

$$\frac{f(a+\delta)}{g(a+\delta)} = \frac{f'(x)}{g'(x)} \approx \lim_{t \rightarrow a} \frac{f'(t)}{g'(t)},$$

where the last approximation follows from the existence of the limit. Note that as  $\delta \rightarrow 0$ ,  $x \rightarrow a$ , and the approximation error shrinks to 0. □

Refer to Rudin or something for the general case.

## 1.3 Higher Derivatives

If  $f : I \rightarrow \mathbb{R}$  is differentiable, then we can define the second derivative  $f'' := (f')'$  if  $f'$  is differentiable. Higher derivatives can be defined similarly. We usually write  $f^{(n)}$  for the  $n$ -th derivative of  $f$ .

*Example 1.14.*  $L(x) = f(x_0) + f'(x_0)(x - x_0)$  is a (first order) linear approximation of  $f$  at  $x_0$ . How good is this approximation? A first answer is

$$f(x) = L(x) + o(|x - x_0|),$$

since we have as  $x \rightarrow x_0$  that

$$\frac{f(x) - L(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \rightarrow 0.$$

But can we say even more about the quality of the approximation? – Yes, if  $f$  is twice differentiable.

**Proposition 1.15** (First-order Taylor's Theorem). *Suppose  $f'$  exists and is continuous on  $[a, b]$  and  $f''$  exists on  $(a, b)$ . Let  $x_0, x \in [a, b]$  with  $x_0 \neq x$ . Then*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(y)(x - x_0)^2,$$

where  $y$  is between  $x_0$  and  $x$ . In particular, we have

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \frac{1}{2} \sup_{y \in (a, b)} |f''(y)| \cdot |x - x_0|^2.$$

**Proof.** Find  $M$  such that we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{M}{2}(x - x_0)^2.$$

We need only find  $y$  such that  $M = f''(y)$ . Define

$$g(t) := f(t) - f(x_0) - f'(x_0)(t - x_0) - \frac{M}{2}(t - x_0)^2.$$

Note that  $g''(t) = f''(t) - M$ , so we need only find a point at which  $g''$  vanishes. Since  $g(x_0) = g(x) = 0$ , by the MVT there exists  $y'$  between  $x_0$  and  $x$  such that  $g'(y') = 0$ . Observe that  $g'(x_0) = 0$ , and so by the MVT again, there exists  $y$  between  $x_0$  and  $y'$  (and by extension between  $x_0$  and  $x$ ) such that  $g''(y) = 0$ .  $\square$

The more general story: given  $f : [a, b] \rightarrow \mathbb{R}$  and  $x_0 \in [a, b]$ , we may define

$$P_0(x) := f(x_0),$$

$$P_1(x) := f(x_0) + f'(x_0)(x - x_0),$$

$$P_2(x) := f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2,$$

$\vdots$

$$P_n(x) := \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k,$$

when the corresponding derivatives exist. Note that  $P_n(x)$  is the unique degree  $n$  polynomial such that  $P_n^{(k)}(x_0) = f^{(k)}(x_0)$  for  $k = 1, \dots, n$ .

**Theorem 1.16** (Taylor's Theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that*

- $f^{(k)}$  exists on  $[a, b]$  for  $k = 1, \dots, n$ ; and
- $f^{(n+1)}$  exists on  $(a, b)$ .

*Then, for any  $x_0, x \in [a, b]$  with  $x_0 \neq x$ , there exists  $y$  between  $x_0$  and  $x$  such that*

$$f(x) = P_n(x) + \frac{f^{(n+1)}(y)}{(n+1)!} (x - x_0)^{n+1}.$$

*for some  $y$  between  $x_0$  and  $x$ .*

We proof the case  $n = 2$ , the same idea can be used to prove the general case.

**Proof.** Define

$$g(t) = f(t) - P_2(t) - \frac{M}{6} (t - x_0)^3.$$

Since  $g''' = f''' - M$ , we need only find  $y$  such that  $g'''(y) = 0$ . Note that  $g(x_0) = g(x) = 0$ , and so by the MVT there exists  $y'$  between  $x_0$  and  $x$  such that  $g'(y') = 0$ . Next, note that  $g'(x_0) = 0$ , and so by the MVT there exists  $y''$  between  $x_0$  and  $y'$  such that  $g''(y'') = 0$ . Finally, note that  $g''(x_0) = 0$ , and so by the MVT there exists  $y$  between  $x_0$  and  $y''$  such that  $g'''(y) = 0$ .  $\square$



## 2 Multivariable Differential Calculus

Some remainders about  $\mathbb{R}^n$ :

- $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}\}$ .
- $\mathbb{R}^n$  is a vector space, with canonical basis  $\{e_1, \dots, e_n\}$ .
- $\mathbb{R}^n$  comes with an inner product  $\langle x, y \rangle = x \cdot y = \sum x_i y_i$ , a norm  $|x| = \sqrt{x \cdot x} = (\sum x_i y_i)^{1/2}$ , and a metric  $d(x, y) = |x - y|$ .

### 2.1 Higher Dimensional Codomains

Consider a function  $f : \mathbb{R} \supset I \rightarrow \mathbb{R}^n$ .

**Definition 2.1.**  $f$  is differentiable at  $x$  if the limit

$$f'(x) := \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$$

exists.

*Remark 2.2.* We may write  $f(t) = (f_1(t), \dots, f_n(t))$ , and  $f'(x) = (f'_1(x), \dots, f'_n(x))$ , since a sequence  $x \in \mathbb{R}^n$  converges if and only if each of its components converges.

**Theorem 2.3.** *We have the following analog of the MVT:*

$$|f(b) - f(a)| \leq |f'(t)| \cdot |b - a|.$$

*for some  $t$  between  $a$  and  $b$ .*

**Proof.** Assume  $a < b$ . Define

$$h(t) := \langle f(b) - f(a), f(t) \rangle.$$

The MVT gives

$$\begin{aligned} h(b) - h(a) &= h'(t)(b - a) = \langle f(b) - f(a), f'(t) \rangle (b - a) \\ &\leq (b - a) |f(b) - f(a)| |f'(t)|, \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality. Noting that

$$h(b) - h(a) = |f(b) - f(a)|^2,$$

we have the desired result. □

## 2.2 Higher Dimensional Domain

We next consider functions  $f : U \rightarrow \mathbb{R}$ , where  $U \subset \mathbb{R}^n$  is open.

**Definition 2.4** (Partial Derivatives).

$$\frac{\partial f}{\partial x_i}(x) = D_i f(x) := \lim_{h \rightarrow 0} \frac{f(x + h e_i) - f(x)}{h}.$$

**Definition 2.5** (Directional Derivatives). Fix  $u \in \mathbb{R}^n$ .

$$= D_u f(x) := \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h}.$$

### 2.2.1 The Derivative

Intuition: A function is differentiable if a first-order Taylor expansion holds. That is, if  $f$  is “well-approximated” by a linear function.

**Definition 2.6.** We denote the set of all linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}$  as  $L(\mathbb{R}^n, \mathbb{R})$ .

**Definition 2.7** (The Derivative). A function  $f$  is differentiable at  $x$  if there exists a linear map  $T \in L(\mathbb{R}^n, \mathbb{R})$  such that

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x) - T(h)}{|h|} = 0.$$

In this case we write  $Df(x) = T$ . In other words,  $f(x + h) = f(x) + Df(x)(h) + o(|h|)$ .

*Remark 2.8.*

- If  $f$  is differentiable, then

$$Df : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}).$$

- It is easy to check that  $Df$  is well defined, that is, there is at most one  $T$  such that the limit holds.

We may think of the linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$T(u) = \langle u, v \rangle, \tag{1}$$

where  $v := (Te_1, \dots, Te_n)$ .

**Definition 2.9** (The Gradient). If  $f$  is differentiable at  $x$ , we define  $\nabla f(x) = v$ , where  $v$  satisfies (1). In other words,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - \langle \nabla f(x), h \rangle}{|h|} = 0.$$

**Theorem 2.10.** If  $f$  is differentiable at  $x$ , then  $D_u f(x)$  exists for all  $u \in \mathbb{R}^n$  and  $D_u f(x) = Df(x)u = \langle \nabla f(x), u \rangle$ .

**Proof.** Note that as  $t \rightarrow 0$ , we have

$$\begin{aligned} \left| \frac{f(x+tu) - f(x)}{t} - Df(x)u \right| &= \left| \frac{f(x+tu) - f(x) - Df(x)(tu)}{t} \right| \\ &= \left| \frac{f(x+tu) - f(x) - Df(x)(tu)}{|tu|} \right| \cdot |u| \rightarrow 0. \end{aligned}$$

□

*Remark 2.11.* In particular we have  $D_i f(x) = D_{e_i} f(x) = Df(x)e_i = \langle \nabla f(x), e_i \rangle$ . In other words, if  $f$  is differentiable, then  $\nabla f(x) = (D_1 f, \dots, D_n f)$ .

*Remark 2.12.*

- Differentiability holds if and only if the gradient exists.
- Differentiability implies the existence of directional derivatives, which then implies the existence of partial derivatives. The converse implications are not true.

*Example 2.13.* Consider

$$f(x_1, x_2) := \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

It is easy to see that  $D_1 f(0) = D_2 f(0) = 0$  but  $D_{(1,1)} f(0)$  does not exist. Indeed,  $f$  is not even continuous on the line  $t(1, 1)$ .

*Example 2.14.* Consider

$$f(x_1, x_2) := \begin{cases} \frac{x_1^3}{x_1^2 + x_2^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Note that

$$D_u f(0) = \lim_{t \rightarrow 0} \frac{t^3 u_1^3}{t^2(u_1^2 + u_2^2)} \cdot \frac{1}{t} = \frac{u_1^3}{u_1^2 + u_2^2}.$$

However,  $Df(0)$  cannot exist, since the above mapping is not linear.

**Theorem 2.15.** *If the partial derivatives  $D_1 f, \dots, D_n f$  exist and are continuous (in a neighborhood of  $x$ ), then  $f$  is differentiable at  $x$ .*

**Proof.** Fix arbitrary  $x \in E$  and define  $Ah = \sum D_i f(x) h_i$ . We write  $\omega_k := \sum_{i=1}^k h_i e_i$  for  $k = 1, \dots, n$  and  $\omega_0 := x$ . Note that  $\omega_n = h$ . By the MVT we can find  $\delta_k$  between 0 and  $h_k$  such that

$$\begin{aligned} f(x+h) - f(x) - Ah &= \sum_{k=1}^n f(x+\omega_k) - f(x+\omega_{k-1}) - D_k f(x) h_k \\ &= \sum_{k=1}^n h_k [D_k(x+\omega_k + \delta_k e_i) - D_k f(x)], \end{aligned}$$

which by continuity of  $D_i$  is sublinear. □

### 2.3 Extension to Functions with Higher Dimensional Codomains

Immediate.

We have

$$Df(x) \in L(\mathbb{R}^n, \mathbb{R}^m), \quad \mathbb{R}^n \ni h \mapsto Df(x)h \in L(\mathbb{R}^n, \mathbb{R}^m),$$

and

$$Df : \mathcal{U} \mapsto L(\mathbb{R}^n, \mathbb{R}^m).$$

Note that we may identify  $T \in L(\mathbb{R}^n, \mathbb{R}^m)$  with a unique matrix  $A = [Te_1, \dots, Te_n]$  such that we have  $Th = Ah$  for each  $h$ .

**Definition 2.16.** If  $f$  is differentiable at  $x$ , we can define  $[Df(x)] \in \mathbb{R}^{n \times m}$  to be the unique matrix such that

$$Df(x)h = [Df(x)]h.$$

More generally, for  $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ , we use  $[T]$  to denote the corresponding matrix.

**Theorem 2.17.** *If  $Df(x)$  exists, so do  $D_i f_j$ , and we have*

$$[Df(x)] = [D_i f_j] = [\nabla f_1(x) \dots \nabla f_m(x)]^\top.$$

It suffices to prove the following stronger proposition:

**Proposition 2.18.** *The function  $f$  is differentiable at  $x$  if and only if each  $f_i$  is differentiable at  $x$ . In this case,*

$$Df(x)h = (Df_1 h, \dots, Df_m(x)h) = (\langle \nabla f_1(x), h \rangle, \langle \nabla f_m(x), h \rangle) = [Df(x)]h.$$

**Proof.** Suppose  $f_i$  is differentiable. Define  $T \in L(\mathbb{R}^n, \mathbb{R}^m)$  by the formula

$$Th = (Df_1 h, \dots, Df_m(x)h).$$

Note that

$$\frac{|f(x+h) - f(x) - Th|}{|h|} = \left( \sum \frac{|f_i(x+h) - f_i(x) - Df_i(x)h|^2}{|h|} \right)^{1/2} \rightarrow 0.$$

The other direction is left as an exercise.  $\square$

**Corollary 2.19.** *If  $D_j f_i$  all exist and are continuous in a neighborhood of  $x$ , then  $f$  is differentiable at  $x$ .*

## 2.4 The Chain Rule

Consider

$$\mathbb{R}^n \supset \mathcal{U} \xrightarrow{g} \mathbb{R}^m \xrightarrow{f} \mathbb{R}^k.$$

**Theorem 2.20 (Chain Rule).** *If  $g$  is differentiable at  $x$  and  $f$  is differentiable at  $g(x)$ , then  $f \circ g$  is differentiable at  $x$  and*

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x).$$

A formal calculation:<sup>3</sup> We have

$$\begin{aligned} f \circ g(x+h) &= f \circ g(x) + Df(g(x))(g(x+h) - g(x)) + o(g(x+h) - g(x)) \\ &= f \circ g(x) + Df(g(x))(Dg(x)h + o(|h|)) + o(|h|) \\ &= f \circ g(x) + Df(g(x))(Dg(x)h) + o(|h|). \end{aligned}$$

<sup>3</sup>In math, “formal calculation” means a “systematic but without a rigorous justification.”

**Proof.** For small  $h \in \mathbb{R}^p$ , we write

$$g(x+h) = g(x) + Bh + R_g,$$

where  $B = Dg(x)$  and  $\lim_{h \rightarrow 0} R_g/h = 0$ . Similarly, we write

$$f \circ g(x+h) = f(g(x) + Bh + R_g) = f \circ g(x) + ABh + AR_g + R_f,$$

where  $A = Df(g(x))$  and  $\lim_{h \rightarrow 0} R_f/(Bh + R_g) \rightarrow 0$ . It remains to note that the last two terms are sublinear. We know that the second to last term is sublinear by recalling  $|AR_g| \leq \|A\||R_g|$ .  $\square$