# MATH20410 (W25): ANALYSIS IN RN II (ACCELERATED)

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## Contents



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## 1. Single-Variable Differential Calculus

In this chapter, we consider mainly functions of the form $f : I \to \mathbb{R}$, where $I$ is an interval, e.g., $(a,b)$, $[a,b]$, $(a,b]$, $(a,\infty)$, $\mathbb{R}$. This is the function we have in mind unless otherwise stated.

**Definition 1.1** (Differentiability)**.** We say $f$ is **differentiable at** $x \in I$ if the limit

$$f'(x) := \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists. In this case, we call $f'(x)$ the derivative of $f$ at $x$. Moreover:

- We say that $f$ is **differentiable** if $f'(x)$ exists for each $x \in I$.
- We say $f$ is **continuously differentiable** ($f \in C^1$) if $f' : I \to \mathbb{R}$ is continuous.

*Example* 1.2.

- $f(x) = |x|$. Differentiable on $\mathbb{R} \setminus \{0\}$.
- $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Continuous but not differentiable at 0.
- $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Differentiable everywhere (in particular at 0), but $f \notin C^1$.

**Proposition 1.3** (Rules for computing derivatives)**.**

*(i) Linearity.* $(af+bg)' = af' + bg'$ *(if $f'$ and $g'$ exist, such requirements are hereafter omitted).*
*(ii) Product rule.* $(fg)' = f'g + fg'$.
*(iii) Quotient rule.* $(f/g)' = (f'g - fg')/g^2$.[1]
*(iv) Chain rule.* $(f \circ g)' = (f' \circ g) \cdot g'$.

[1] Low dhigh minus high dlow. Not "Hai di lao"...

**Proof.** We prove the quotient rule; the remaining are left as exercises. Starting from the definition

$$\left(\frac{f}{g}\right)'(x) = \lim_{t\to x} \frac{\frac{f}{g}(t) - \frac{f}{g}(x)}{t-x}$$
$$= \lim_{t\to x} \frac{\frac{f(t)}{f(t)} + \frac{f(x)}{g(t)} - \frac{f(x)}{g(t)} + \frac{f(x)}{g(x)}}{t-x}.$$

Note that

$$\frac{\frac{f(x)}{g(t)} + \frac{f(x)}{g(x)}}{t-x} = \frac{f(x)}{g(x)g(t)} \frac{g(x)-g(t)}{t-x}$$

and we have

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)}$$

$\square$