ECON20010 PSET 1

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Contents

1. The Lagrangian

Note:

- The Lagrangian multiplier of the UMP can be non-positive. Think bliss points.

2. The Envelope Theorem

Theorem 2.1. *Let*

$$v(a) = \max_x f(x; a).$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}a} = \frac{\mathrm{d}f(x^*; a)}{\mathrm{d}a} = \left.\frac{\partial f}{\partial a}\right|_{x=x^*}.$$

Remark 2.2. Intuition:

$$\frac{\mathrm{d}f(x^*; a)}{\mathrm{d}a} = \sum \frac{\partial f}{\partial x_i^*} \cdot \frac{\partial x_i^*}{\partial a} + \frac{\partial f}{\partial a},$$

and at optimum, each $\partial f/\partial x_i^* = 0$. "All indirect effects vanish." Note that by the implicit function theorem, we need $f_{xx} \neq 0$.

Example 2.3. Consider the value function

$$\begin{aligned} v(p_x, p_y, m) &= U(x^*, y^*) = U(x^*, y^*) + \lambda^*[m - p_x x - p_y y] \\ &= \mathcal{L}(x^*, y^*, \lambda^*; p_x, p_y, m) \\ &=: \mathcal{L}^*(p_x, p_y, m). \end{aligned}$$

We then have by the envelope theorem, $\mathrm{d}\mathcal{L}/\mathrm{d}m = \partial\mathcal{L}/\partial m$ and thus

$$\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}^*}{\partial m} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

Similarly,

$$\frac{\partial v}{\partial p_x} = \frac{\partial \mathcal{L}^*}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

3. Scarcity

Definition 3.1.

- The **budget set** consists of all feasible consumption bundles.
- The **budget constraint** exactly exhausts the consumer's income.

3.1. Budget Set.

The relative price:

$$\frac{p_x}{p_y}$$

- Mnemonic: this is always the price of x in units of y.

  To stay on the budget constraint,

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p_x}{p_y}.$$

- Think "the rate at which the market *allows* the consumer to exchange good x for good y."
- Think the opportunity cost of good x.

3.2. Preference.

Axioms:

- **Completeness**. For any pair of consumption of bundles, say c_1 and c_2, either $c_1 \succeq c_2$, $c_2 \succeq c_1$, or both.
  - Requires a answer and assumes no framing effects.
- **Transitivity**. If $c_1 \succeq c_2$ and $c_2 \succeq c_3$ then $c_1 \succeq c_3$.
  - Money pump.
- A preference ordering is **rational** if it satisfies completeness and transitivity. They are the minimal requirement for the existence of a utility function representation.

We also typically assume the following:

- **Continuity**. If $c_1 \succ c_2$ then there are neighborhoods N_1 and N_2 around c_1 and c_2 such that

  $$x \succ y, \quad \forall x \in N_1, \quad y \in N_2.$$

  This implies that if $c_1 \succ c_2$ then there exists c_3 such that

  $$c_1 \succ c_3 \succ c_2.$$

- **Monotonicity**.
  - **Monotone**. If $c_1 \gg c_2$[1] then $c_1 \succ c_2$.

[1] We write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i, \forall i$.

- **Strongly monotone**. If $c_1 \geq c_2$[2] and $c_1 \neq c_2$ then $c_1 \succ c_2$.
- **Local non-satiation**. If for every bundle c and every $\epsilon > 0$, there exists $x \in N_\epsilon(c)$ such that $x \succ c$.

- **Convexity**. If $c_1 \succeq c_2$, then
$$\theta c_1 + (1-\theta)c_2 \succeq c_2, \quad \forall \theta \in (0,1).$$

If convexity is satisfied, the **upper contour set**, the "at least as good as" set, is convex.

Additional axioms place even more structures on the utility function:

- **Homotheticity**. If $c_1 \succeq c_2$, then
$$tc_1 \succeq tc_2, \quad \forall t > 0.$$
- **Quasilinearity** in good i. If $\mathbf{c_1} \succeq \mathbf{c_2}$, then
$$\mathbf{c_1} + t\mathbf{e}_i \succeq \mathbf{c_2} + t\mathbf{e}_i, \quad \forall t > 0.$$

3.3. Translating preference ordering to the utility function:

Theorem 3.2.

- *If a preference ordering is rational, then it admits a utility function representation. (Representation Theorem;*[3]*) The utility function is unique up to a monotonically increasing transformation.*
- *If a preference ordering satisfies convexity, then the corresponding utility function representation will be quasi-concave. The indifference curves (level sets) will have non-increasing marginal rate of substitution (slopes).*

3.4. The Marginal Rate of Substitution.

The MRS
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y}$$
is the quantity of y the consumer is willing to sacrifice in exchange for an additional unit of x. (Think p_x/p_y.) It measures an individual's **willingness to pay** for x in terms of y.

4. The Utility Maximization Problem

The problem:
$$v(p_x, p_y, m) := \max_{x,y} U(x,y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

4.1. Interpretation.

We want to maximize
$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$
such that
$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$
This gives
$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right]\mathrm{d}x.$$
We can rewrite these two expressions in the following forms:

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[2] We write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i, \forall i$.

[3] For a simple version of this, think assigning the size of the unique bundle on $t\sum \mathbf{e}$ equivalent to a given consumption bundle.

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.
$$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \, \mathrm{d}x$$
"Take advantage of all trading opportunities."
- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that U_x/p_y is marginal utility of money *spent on* x.
$$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \, \mathrm{d}x$$
"Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that U_x is the marginal benefit of buying x and $U_y \cdot p_x/p_y$ is the marginal cost of buying x.
$$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$
"Trade until marginal cost equals marginal benefit."

In the last expression, if we write
$$\lambda = \frac{U_y}{p_y},$$
(think marginal utility of income) we have that at optimum,
$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$

These three equalities describe precisely the critical points of the following
$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right],$$
called the **Lagrangian**. That is, setting
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$
recovers the above three equations.

Remark 4.1.

- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- λ might be negative or zero. Think bliss point.

4.2. **The Indirect Utility Function.**

Proposition 4.2.
$$\frac{\partial v}{\partial m} = \lambda^*.$$

Proof. Noting
$$v = U(x^*, y^*) + \lambda^*[m - p_x x^* - p_y y^*] = \mathcal{L}^*,$$

we have

$$\begin{aligned}\frac{\partial v}{\partial m} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m}\\ &= U_x^* \frac{\partial x}{\partial m} + U_y^* \frac{\partial y}{\partial m} + \lambda^* \left[1 - p_x \frac{\partial x}{\partial m} - p_y \frac{\partial y}{\partial m}\right] + \frac{\partial \lambda}{\partial m}\left[m - p_x x^* - p_y y^*\right]\\ &= \left(U_x^* - \lambda^* p_x\right) \frac{\partial x}{\partial m} + \left(U_y^* - \lambda^* p_y\right) \frac{\partial y}{\partial m} + \frac{\partial \lambda^*}{\partial m}\left(m - p_x x^* - p_y y^*\right) + \lambda^*\\ &= \lambda^*.\end{aligned}$$

The last equality follows by noting that at the optimum,

$$U_x^* - \lambda^* p_x = U_y^* - \lambda^* p_y = m - p_x x^* - p_y y^* = 0.$$

Alternatively, one may use the envelope theorem:

$$\frac{\partial v}{\partial m} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^*.$$

□

Note that

$$\frac{\partial v}{\partial m} = \lambda^* = \frac{U_x^*}{p_x} = \frac{U_x^*}{p_x}.$$

So when not satiated ($U_x, U_y \neq 0$), marginal utility of income is positive. When budget constraint does not require to bind, the marginal utility of income is generally nonnegative.

Again using the Envelope Theorem, we have

$$\frac{\partial v}{\partial p_x} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_x} = \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda^* x^*.$$

This value is generally nonpositive, and only zero when one does not consume the specific good or when the marginal utility of that good is 0.

5. Expenditure Minimization

The problem:

$$e(p_x, p_y, \overline{U}) \coloneqq \max_{x,y} p_x x + p_y y \quad \text{s.t.} \quad U(x, y) = \overline{U}.$$

The Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \eta\left[\overline{U} - U(x, y)\right]$$

$$\begin{aligned} &[x] && p_x = \eta^* U_x(x^*, y^*)\\ &[y] && p_y = \eta^* U_y(x^*, y^*)\\ &[\eta] && \overline{U} = U(x^*, y^*).\end{aligned}$$

6. Duality

7. Changes in Behavior

Consider a price increase from p_x^o to p_x^f. Let o be the original consumption, f be the final consumption, and s be the optimal consumption after an income transfer such that the individual stays on the same indifference curve as before (has the same purchasing power). We may decompose $x^f - x^o$:

$$x^f - x^o = x^f - x^s + x^s - x^o.$$

- $x^f - x^o$: the Marshallian price effect (the total effect).

- $x^f - x^s$: effect due to compensation (the income effect).
- $x^s - x^o$: the Hicksian price effect (substitution effect).

7.1. **The Slutsky Equation.** Recall from duality that

$$x^h(p_x, p_y, \overline{U}) = x^m(p_x, p_y, m = e(p_x, p_y, \overline{U})).$$

As price changes, changes in $e(p_x, p_y, \overline{U})$ ensures that purchasing power does not change.

By differentiating, we get the **Slutsky equation**:

$$\frac{\partial x_h}{\partial p_x} = \frac{\partial x_m}{\partial p_x} + \frac{\partial x_m}{\partial m} \cdot \frac{\partial e}{\partial p_x},$$

which we can rewrite using the envelop theorem as

$$\frac{\partial x_h}{\partial p_x} = \frac{\partial x_m}{\partial p_x} + \frac{\partial x_m}{\partial m} \cdot x_m.$$

This shows that we can recover the unobservable $\partial x_h / \partial p_x$ from the observables.

We can also rewrite the Slutsky equation as

$$\frac{\partial x_m}{\partial p_x} = \underbrace{\frac{\partial x_h}{\partial p_x}}_{\text{Substitution Effect}} + \underbrace{\left(-\frac{\partial x_m}{\partial m} \cdot x_m\right)}_{\text{Income Effect}}.$$

7.2. **Two Ways to think about compensation.** (The setting is a increase in the price of x.)

- **Slutsky transfer** keeps the original bundle affordable.

$$T_S = \Delta p_x \cdot x^o.$$

- **Hicks transfer** keeps the original utility level affordable.

$$T_H = e(p_x^f, p_y, v^o) - m = e(p_x^f, p_y, v^o) - e(p_x^o, p_y, v^o).$$

  In the Slutsky equation,

$$\frac{\partial e}{\partial p_x} = x_m = x_h$$

  is the continuous analogue of the Hicks transfer.

Remark 7.1. Note that $T_S \geq T_H$.

7.3. **Law of Demand.** The substitution effect follows the law of demand:

$$\partial x_h / \partial p_x \leq 0.$$

(Think graphs.)

Note that this holds not only when indifference curves are concave. In the concave case, the expenditure minimizing points occur at the edges. In the perfect complement case, we have that $\partial x_h / \partial p_x = 0$.

Marshallian demand, however, does not always comply with the law of demand. A good whose Marshallian demand does not comply with the law of demand is called a **giffen good**. Their existence is theoretically possible, but not empirically supported.

Looking back at the Slutsky equation, we see that for a good x to be a giffen good, we need

- the individual buys a large amount of x,
- good x is inferior,
- the demand for good x is elastic.

These three conditions do not occur together often: narrowly defined categories usually has 0 elasticity, but broad categories are usually normal goods.

7.4. **Normal & Inferior Goods.** The experiment for testing normality: We fix the consumption of x and vary income m. For normal goods, the willingness to pay for x increases as income increase. Thus x is normal if

$$\frac{\partial U_x/U_y}{\partial y} > 0.$$

Example 7.2. With the quasilinear utility function $U(x, y) = v(x) + y$, the goods x and y are unrelated. The willingness to pay

$$\frac{U_x}{U_y} = \frac{v'(x)}{1}$$

does not change as we vary y (by varying income).

7.5. **Cross Effects.**

Definition 7.3. We call y

- a substitute of x if
$$\partial y/\partial p_x > 0$$
- a complement of x if
$$\partial y/\partial p_x < 0$$
- unrelated with x if
$$\partial y/\partial p_x = 0.$$

If we use y_h above, we say **gross** substitutes/complements; if we use y_m, we say **net** substitutes/complements. (Think Hicksian demand "nets out" the income effect.)

Remark 7.4.

- If we have only two goods, it is impossible to have complements. (Think graphs.)
- If we have more than 2 goods, at least 1 is a net substitute. If not, we strictly decrease our consumption and cannot achieve the same level of utility.
- We have
$$\frac{\partial y_h}{\partial p_x} = \frac{\partial x_h}{\partial p_y}$$
but only for Hicksian demands, not Marshallian demands. See below.

The cross-price Slutsky equation:

$$\frac{\partial y_h}{\partial p_x} = \frac{\partial y_m}{\partial p_x} + \frac{\partial y_m}{\partial m} \cdot \frac{\partial e}{\partial p_x}$$

where we have

$$\frac{\partial e}{\partial p_x} = x_m = x_h.$$

Equivalently,

$$\frac{\partial y_m}{\partial p_x} = \frac{\partial y_h}{\partial p_x} - \frac{\partial y_m}{\partial m} \cdot x_m.$$

8. Utility Functions

In Cobb-Douglous, substitution effect is offset exactly by the income effect. All goods are unrelated.