MATH20410 (W25): ANALYSIS IN RN II (ACCELERATED)

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CONTENTS

1. Single-Variable Differential Calculus

In this chapter, we consider mainly functions of the form $f : I \to \mathbb{R}$, where I is an interval, e.g., (a, b), $[a, b]$, $(a, b]$, (a, ∞), $\mathbb{R}$. This is the function we have in mind unless otherwise stated.

Definition 1.1 (Differentiability)**.** We say f is **differentiable at** $x \in I$ if the limit

$$f'(x) \coloneqq \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

exists. In this case, we call $f'(x)$ the derivative of f at x. Moreover:

- We say that f is **differentiable** if $f'(x)$ exists for each $x \in I$.
- We say f is **continuously differentiable** ($f \in C^1$) if $f' : I \to \mathbb{R}$ is continuous.

Example 1.2.

- $f(x) = |x|$. Differentiable on $\mathbb{R} \setminus \{0\}$.
- $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Continuous but not differentiable at 0.
- $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Differentiable everywhere (in particular at 0), but $f \notin C^1$.

Proposition 1.3 (Rules for computing derivatives)**.**

(i) Linearity. $(af + bg)' = af' + bg'$ *(if f' and g' exist, such requirements are hereafter omitted).*
(ii) Product rule. $(fg)' = f'g + fg'$.
(iii) Quotient rule. $(f/g)' = (f'g - fg')/g^2$.[1]
(iv) Chain rule. $(f \circ g)' = (f' \circ g) \cdot g'$.

[1] Low dhigh minus high dlow. Not Haidilao...

Proof. We prove the quotient rule; the remaining are left as exercises. Starting from the definition

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{t\to x} \frac{\frac{f}{g}(t) - \frac{f}{g}(x)}{t - x} \\ &= \lim_{t\to x} \frac{\frac{f(t)}{f(t)} + \frac{f(x)}{g(t)} - \frac{f(x)}{g(t)} + \frac{f(x)}{g(x)}}{t - x}.\end{aligned}$$

Note that

$$\frac{\frac{f(x)}{g(t)} + \frac{f(x)}{g(x)}}{t - x} = \frac{f(x)}{g(x)g(t)} \frac{g(x) - g(t)}{t - x}$$

and we have

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)}$$

□

Theorem 1.4. *If f is differentiable at x then f is continuous at x.*

Proof. Note that

$$\lim_{t\to x} f(t) - f(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}(t-x) = f'(x)\cdot 0 = 0.$$

□

1.1. **The Mean Value Theorem.**

Lemma 1.5. *Suppose $f : [a,b] \to \mathbb{R}$ has a local maximum or minimum at $x \in (a,b)$. If $f'(x)$ exists, then $f'(x) = 0$.*

Proof. From the definition of the derivative, consider the limits from the left and right; one is non-positive and the other is non-negative. □

Theorem 1.6 (Rolle's Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), and such that $f(a) = f(b)$. Then there exists $x \in (a,b)$ such that $f'(x) = 0$.*

Proof. Consider the global maximum or minimum (exist since f is continuous defined on a compact set) and apply the previous lemma. (If both the maximum and minimum is at a or b, f is constant.) □

Theorem 1.7 (Mean Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be such that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x \in (a,b)$ such that $f(b) - f(a) = f'(x)(b-a)$.*

Proof. Apply Rolle's to $\tilde{f} = f - [f(b) - f(a)] \cdot \frac{x-a}{b-a}$. □

1.2. **Applications of the MVT.**

Theorem 1.8. *Let $f : (a,b) \to \mathbb{R}$ be differentiable.*

(a)) if $f' = 0$, then f is constant.
(b)) if $f' \geq 0$, then f is increasing.
(c)) if $f' \leq 0$, then f is decreasing.

Proof. Apply the mean value theorem. □

Theorem 1.9 (The Intermediate Value Property of Derivatives)**.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable*[2] *and suppose $f'(a) < \lambda < f'(b)$ Then there exists $x \in (a,b)$ such that $f'(x) = \lambda$.*

[2] f need not be C^1!

Proof. Slide a small secant of length so small that the slope around a and b is separated also by λ. By continuity of the slope, there exists a secant between a and b with slope λ. Apply the mean value theorem to this slope. □