ASSET PRICING: MATHEMATICAL FOUNDATIONS

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Contents

1. Derivatives: A One-Dimensional Recap

Definition 1.1. Let $f : \mathbb{R} \supset \Omega \to \mathbb{R}$. The derivative of f at $x \in \Omega$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

Remark 1.2.

- Think "first order approximation."
- Note that f' is also a function, with the same domain as f (when f is enough regular).

2. Derivatives: Partial and Total

Definition 2.1. Let $f : \mathbb{R}^n \supset \Omega \to \mathbb{R}$. The partial derivative of f at $\mathbf{x} \in \Omega$ with respect to the ith variable is defined as

$$\frac{\partial f}{\partial x_i} = f_{x_i} = f_i(x) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h},$$

if the limit exists.

Remark 2.2.

- Desmos 3d Demo.
- Think derivative with respect to the ith position, not to x_i.
- "First order approximation in the ith direction."

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- With enough regularity imposed on f, we can write

$$\mathrm{d}f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_i}.$$

"Think first order approximation."

3. The Lagrangian

Example 3.1.

$$\max_{\{x,y\}} U(x,y), \quad \text{s.t.} \begin{cases} p_x x + p_y y \leq m \\ x, y \geq 0 \end{cases}$$

- For ease of mathematics, we often consider the constraint $p_x x + p_y y = m$.
- For a more general case, see Karush–Kuhn–Tucker conditions.

Theorem 3.2. *More abstractly, the above problem can be thought of as*

$$\max f(x_1, x_2) \quad \textit{s.t.}\ g(x_1, x_2) = c$$

for a constant c. With enough regularity, the optima occurs at the critical points of the Lagrangian, defined as

$$\mathcal{L}(x_1, x_2, \lambda) \coloneqq f(x_1, x_2) + \lambda[c - g(x_1, x_2)].$$

That is, the optima subject to given constraint satisfies

$$\begin{aligned} [x_1]: \qquad & f_1(x_1^*, x_2^*) = \lambda g_1(x_1^*, x_2^*) \\ [x_2]: \qquad & f_2(x_1^*, x_2^*) = \lambda g_2(x_1^*, x_2^*) \\ [\lambda]: \qquad & g(x_1^*, x_2^*) = c. \end{aligned}$$

4. Lagrangian: An Example

4.1. The Utility Maximization Problem.

The problem:

$$v(p_x, p_y, m) \coloneqq \max_{x,y} U(x,y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

4.2. Interpretation.

We want to maximize

$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$

such that

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$

This gives

$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x.$$

We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.

$$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y\ \mathrm{d}x$$

"Take advantage of all trading opportunities."

- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that U_x/p_y is marginal utility of money *spent on* x.

$$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \,\mathrm{d}x$$

"Bang for your buck."

- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that U_x is the marginal benefit of buying x and $U_y \cdot p_x/p_y$ is the marginal cost of buying x.

$$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$

"Trade until marginal cost equals marginal benefit."

In the last expression, if we write

$$\lambda = \frac{U_y}{p_y},$$

(think marginal utility of income) we have that at optimum,

$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$

These three equalities describe precisely the critical points of the following

$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda\left[m - p_x x - p_y y\right],$$

called the **Lagrangian**. That is, setting

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

recovers the above three equations.

Remark 4.1.

- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- λ might be negative or zero. Think bliss point.

5. Taylor Expansion

Definition 5.1. The Taylor polynomial of degree n of the function f around point a is given by

$$P(a + x) = \sum_{k=1}^{n} \frac{f^{(n)}(a)}{k!} \cdot x^k.$$

It has the same k derivatives as f. Think "kth order approximation."

Remark 5.2. We will often use the first or second order approximation starting from a given point a:

$$f(a + h) \approx f(a) + f'(a)h,$$
$$f(a + h) \approx f(a) + f'(a)h + f''(a)\frac{h^2}{2}.$$

6. Probability

Definition 6.1. A discrete random variable X can be described by the (at most countable) values it can attain and the probability of attaining them.

- The expectation of X is defined as
$$\mathbb{E}(X) = \sum x \cdot \mathbb{P}(X = x).$$
Think weighted average.
- The variance of X is defined as
$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2].$$
- For discrete random variables X and Y, the covariance is defined by
$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X)) \cdot (Y - \mathbb{E}(Y))].$$

Proposition 6.2.

- $\mathbb{E}$ *is linear. That is,* $\mathbb{E}[a + bX] = a + b\,\mathbb{E}[X]$ *for* $a, b \in \mathbb{R}$.
- $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ *and* $\mathrm{Var}(a + bX) = b^2\,\mathrm{Var}(X)$.
- $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$.