Introduction to Asset Pricing

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What is asset pricing?

- An **asset** is anything that promises future cash flows.
- Asset pricing investigates how much an asset is worth, accounting for time and risk.
- By pricing assets, we want to answer
  - How do asset prices change with parameters in the macroeconomy?
  - What compensates for risk from the perspective of a representative buyer?
- As in microeconomic price theory, we assume:
  - Optimizing agents: Traders are utility maximizing with rational expectations.
  - Market clearing: We try to explain asset prices *after* the market has arrived at equilibrium.

Autumn objectives

- Derive the **equity risk premium**, an expression for the difference between prices of risky and risk-free assets;
- Understand the equity premium puzzles;
- Evaluate current solutions:
    - Alternative utility curves;
    - Heterogeneous agents;
    - Behavioral economics;
- Introduce numerical solution methods for macroeconomic models.

The representative agent

- The utility function should capture **temporal discounting** and **risk aversion**.
- The **present discount factor** β converts utility in the next period into present utility.

$$\max_{c_t, c_{t+1}} U(c_t) + \beta U(c_{t+1})$$

- The concavity of the utility function captures the agent's risk aversion.

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

- The agent faces **diminishing marginal utility** in consumption.

$$U'(c) = c^{-\gamma}$$

- γ is the agent's **relative risk aversion**

$$\gamma = -\frac{U''(c)c}{U'(c)}$$

The economy

- The agent can only spend her income on consumption or assets. Whatever income remains will perish by the next period.
- That is, the only way to transfer wealth into the future is by purchasing assets.
- There are two types of assets,
  - risk-free bonds;
  - stocks with **exogenous** and **stochastic** dividends.
- In the beginning (period 0), the agent's only endowments are b_0 bonds and s_0 stocks.
- Returns from assets is the only source of income for the agent.
- The risk-free bond and consumption are **numeraires**. That is, its price equals 1.
- The dividends follow a first-order **Markov process**. That is, only dividends from the present period (and not any time before) affects dividends in the next period.

The agent's problem

- In the first period of a two-period problem (the agent dies at the end of period 1), the agent solves
$$\max_{c_0, c_1} \mathbb{E}_0[U(c_0) + \beta U(c_1) | d_0].$$
- She is subject to the constraints:
  - $c_0 + p_0 s_1 + b_1 = (p_0 + d_0)s_0 + (1 + r_0^f)b_0$ in period 0;
  - $c_1 + p_1 s_2 + b_2 = (p_1 + d_1)s_1 + (1 + r_1^f)b_1$ in period 1.
    - $\mathbb{E}_t(x|s)$ denotes the expectation for variable x given information s in period t;
    - $b_t :=$ bond holdings in period t;
    - $s_t :=$ stock holdings in period t;
    - $r_t^f :=$ risk-free interest rate of bonds;
    - $p_t :=$ price stock in period t;
    - $d_t :=$ stochastic dividends from stock in period t.

Simplified constraints

- Since stock and bond holdings in period 2 contributes nothing to the agent's utility, s_2 and b_2 are automatically set to zero. The period-1 constraint becomes

$$c_1 = (p_1 + d_1)s_1 + r_1^f b_1.$$

- We rewrite the agent's problem as

$$\max_{c_0, b_1, s_1} U(c_0) + \beta \mathbb{E}_0\{U[(p_1 + d_1)s_1 + (1 + r_1^f)b_1] | d_0\}$$

such that

$$c_0 + p_0 s_1 + b_1 = (p_0 + d_0)s_0 + (1 + r_0^f)b_0.$$

- It is now clear that the agent is optimizing over products purchased in the initial period.

The Expectation Operator

- **Conditional expectation** is computed like unconditional expectation, except with **conditional probability**:
$$\mathbb{E}(X|s) = \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i|s).$$
- $\mathbb{E}(aX + Y) = a\,\mathbb{E}(X) + \mathbb{E}(Y)$. We call this result the **linearity of expectation**.

The Lagrange Multiplier Theorem

- If x^* is an extremum of $f(x)$ over the set $\{x | g_1(x), g_2(x), \ldots, g_m(x) = 0\}$, then there are $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ such that

$$f'(x^*) + \lambda_1 g_1'(x^*) + \lambda_2 g_2'(x^*) + \cdots + \lambda_m g_m'(x^*) = 0$$

- To solve for the constrained extrema, we can instead find the the critical points of

$$L(x, \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \cdots + \lambda_m g_m(x),$$

  which satisfies the first order conditions:

$$\begin{array}{ll} [x^*] & f'(x^*) + \lambda_1 g_1'(x^*) + \lambda_2 g_2'(x^*) + \cdots + \lambda_m g_m'(x^*) = 0 \\ [\lambda_i] & g_i(x) = 0 \end{array}$$

First-order conditions

The agent solves

$$\max_{c_0, b_1, s_1} U(c_0) + \beta\, \mathbb{E}_0\{U[(p_1 + d_1)s_1 + r_1^f b_1]|d_0\}$$

$$\text{s.t.} \quad (p_0 + d_0)s_0 + (1 + r_0^f)b_0 - (c_0 + p_0 s_1 + b_1) = 0,$$

or equivalently,

$$\max_{c_0, b_1, r_1, \lambda} U(c_0) + \beta\, \mathbb{E}_0\{U[(p_1 + d_1)s_1 + (1 + r_1^f)b_1]|d_0\} + \lambda[(p_0 + d_0)s_0 + (1 + r_0^f)b_0 - (c_0 + p_0 s_1 + b_1)]$$

$$\begin{aligned}
&[c_0] \quad U'(c_0) - \lambda = 0 \\
&[b_1] \quad \beta(1 + r_1^f)\, \mathbb{E}\{U'[(p_1 + d_1)s_1 + (1 + r_1^f)b_1]|d_0\} - \lambda = 0 \\
&[s_1] \quad \beta\, \mathbb{E}\{(p_1 + d_1)U'[(p_1 + d_1)s_1 + (1 + r_1^f)b_1]\} - p_0\lambda = 0 \\
&[\lambda] \quad (p_0 + d_0)s_0 + (1 + r_0^f)b_0 - (c_0 + p_0 s_1 + b_1) = 0
\end{aligned}$$

No arbitrage

- The present-discount expected marginal increase to the agent's future utility brought by bonds is equal to the marginal decrease in present utility brought by purchasing bonds (not consuming instead):

$$\beta(1 + r_1^f)\,\mathbb{E}(U'(c_1)|d_0) = U'(c_0).$$

- The same for stock:

$$\beta\,\mathbb{E}[(p_1 + d_1)U'(c_1)] = p_0 U'(c_0).$$

- The price of equity equals expected marginal contribution to future consumption, discounted by the impatience factor and the relative marginal utility of present consumption.

$$p_0 = \beta\,\mathbb{E}\left[(p_1 + d_1)\frac{U'(c_1)}{U'(c_0)}\bigg|\, d_0\right] = \beta\,\mathbb{E}\left[(p_1 + d_1)\left(\frac{c_1}{c_0}\right)^{-\gamma}\Bigg|\, d_0\right].$$

Infinite time horizon

- The agent, now immortal because she is representing her country, solves the problem

$$\max_{\{c_t\}_{t=0}^{\infty},\{s_t\}_{t=1}^{\infty},\{b_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_0) \quad \text{s.t.} \quad \forall t, \quad c_t + p_t s_{t+1} + b_{t+1} = p_t s_t + (1 + r_t^f) b_t$$

- Note that the agent faces the same problem in each period because the number of future periods remain the same; the only change is the income she is given.
- We introduce the **indirect utility fucntion** of period T, which equals U evaluated at its optimizing arguments and defines utility in terms of income and prices instead of goods:

$$V(b_T, s_T) = \max_{\{c_t\}_{t=T}^{\infty},\{s_t\}_{t=T}^{\infty},\{b_t\}_{t=T}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t.} \quad \forall t, \quad c_t + p_t s_{t+1} + b_{t+1} = p_t s_t + (1 + r_t^f) b_t,$$

Only the arguments in the indirect utility function, not its shape, change over periods.

The Bellman Equation

- The **Bellman Equation** of the problem us

$$
\begin{aligned}
V(b_T, s_T) &= \max_{c_T, b_{T+1}, s_{T+1}} [U(c_T) + \beta V(b_T, s_T)] \quad \text{s.t.} c_T + b_{T+1} + p_T s_{T+1} = (1 + r_T^f) b_T + p_T s_T \\
&= \max_{c_T, b_{T+1}, s_{T+1}} \{U[(1 + r_T^f) b_T + p_T s_T - b_{T+1} - p_T s_{T+1}] + \beta V(b_T, s_T)\}
\end{aligned}
$$

- After a lot of algebra (to be posted upon request), we generalize the equity premium derived in the last slide:

$$
\begin{aligned}
p_T &= \beta\,\mathbb{E}_T \left[(p_{T+1} + d_{T+1}) \frac{U'(c_{T+1})}{U'(c_T)} \right] = \beta\,\mathbb{E}_T \left[(p_{T+1} + d_{T+1}) \left(\frac{c_{T+1}}{c_T} \right)^{-\gamma} \right] \\
&= \mathbb{E}_T(m_{T+1} X_{T+1}), \text{ where } m_T = \beta \frac{U'(c_{T+1})}{U'(c_T)} \text{ and } X_T = p_{T+1} + d_{T+1}.
\end{aligned}
$$

- The **stocahstic discount factor** m_t represents how much the agent values a marginal increase in future income relative to present consumption in period t.

Equity risk premium

- We perform the same operation on risk-free bonds to obtain

$$1 = \beta(1 + r^f_{T+1})\,\mathbb{E}\left[\frac{U'(c_{T+1})}{U'(c_T)}\right]. \tag{1}$$

- We rewrite the first-order condition for stocks as

$$1 = \beta\,\mathbb{E}_T\left[(1 + \tilde{R}_{T+1})\frac{U'(c_{T+1})}{U'(c_T)}\right], \tag{2}$$

where $\tilde{R}_{T+1}$ is the rate of return for risky assets in period $T+1$: $1 + \tilde{R}_T = X_{T+1}/p_T$.

Market equilibrium

- The asset market is at equilibrium when
  - Every agent solves his or her optimization problem;
  - The market clears, or there is no excess demand or supply of bonds or stocks.
- Since all agents are the same as the representative agent, no one sells any bond or stock.
- The budget constraint becomes

$$\sum_{t=0}^{\infty} c_t = (p_0 + d_0)s_0 + (1 + r_0^f)b_0$$

Relation to the macroeconomy

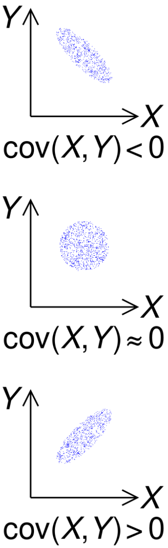
- Measured across time, the **covariance** of two variables represents the tendency at which they move in the same direction with respect to their mean:

$$\text{Cov}(X, Y) = \mathbb{E}\{[(X - \mathbb{E}(X)][(Y - \mathbb{E}(Y)]\}.$$

- Applying above to (1) and (2), we have

$$1 = \frac{\mathbb{E}_T(1 + \tilde{R}_{T+1})}{1 + r^f_{T+1}} + \text{Cov}(m_{T+1}, \tilde{R}_{T+1}) \qquad (3)$$

- Because $U'(c)$ is decreasing with consumption, the stochastic discount factor m_{T+1} is inversely correlated to economic performance.



Insurance

- Intuitively, we have (3) because the agent values cashflow more in a poorer state.
  - $\text{Cov}(m_{T+1}, 1+\tilde{R}_{T+1}) < 0$ means that the stock performs better as the macroeconomy grows, so the agent seeks higher returns since the marginal utility of cashflow is lower.
  - $\text{Cov}(m_{T+1}, 1+\tilde{R}_{T+1}) > 0$ means that the stock performs better in crises, serving as insurance, so the agent is content with lower returns.
- Recall **Taylor's Theorem**: $f(x) = \displaystyle\sum_{t=0}^{\infty} f^{(n)}(x_0)(x-x_0)^n$.
- The first-order approximation of $\dfrac{x}{1+b}$ around $(1+b)$ states that $\dfrac{1+a}{1+b} \approx 1+a-b$.
- We rewrite (3) as

$$\mathbb{E}_T(\tilde{R}_{T+1}) - r^f_{T+1} \approx -\text{Cov}(m_{T+1}, 1+\tilde{R}_{T+1}). \tag{4}$$

- The agent buys insurance despite low expected returns because it pays off in bad states.

Relation to risk aversion

- When $c_{T+1} > c_T$, m_{T+1} decreases when the relative risk aversion γ of the agent is high:

$$m_{T+1} = \beta \frac{U'(c_{T+1})}{U'(c_T)} = \beta \left(\frac{c_{T+1}}{c_T} \right)^{-\gamma}$$

- When the economy grows, the more risk-averse agents values future cashflow less.
- By first-order Taylor approximation,

$$m_{T+1} \approx \beta \left[\frac{U'(c_T)}{U'(c_T)} + \frac{U''(c_T)}{U'(c_T)} (c_T - c_{T+1}) \right] = \beta(1 - \gamma g_{T+1}),$$

  where $g_{T+1} = \dfrac{c_{T+1} - c_T}{c_T}$ is the growth in consumption.
- $$\mathbb{E}_T(\tilde{R}_{T+1}) - r^f_{T+1} \approx \beta\gamma \, \mathrm{Cov}(g_{T+1}, \tilde{R}_{T+1})$$
- The more risk-averse agents are more likely to purchase insurance.

Summary

- $$p_T = \beta\, \mathbb{E}_T \left[(p_{T+1} + d_{T+1}) \frac{U'(c_{T+1})}{U'(c_T)} \right]$$

  The price of an asset equals the present-discounted expected marginal utility of cashflow to the agent in the next period.

- $$\mathbb{E}_T(\tilde{R}_{T+1}) - r^f_{T+1} \approx a\beta\gamma\, \mathrm{Cov}(g_{T+1}, \tilde{R}_{T+1})$$

  Risk-averse agents are willing to buy assets that pays off in bad states, even if their expected returns are not as high as risk-free bonds.