# ASSET PRICING: MATHEMATICAL FOUNDATIONS

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## Contents



## 1. Derivatives: A One-Dimensional Recap

**Definition 1.1.** Let $f : \mathbb{R} \supset \Omega \to \mathbb{R}$. The derivative of $f$ at $x \in \Omega$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

*Remark* 1.2.

- Think "first order approximation."
- Note that $f'$ is also a function, with the same domain as $f$ (when $f$ is enough regular).

## 2. Derivatives: Partial and Total

**Definition 2.1.** Let $f : \mathbb{R}^n \supset \Omega \to \mathbb{R}$. The partial derivative of $f$ at $\mathbf{x} \in \Omega$ with respect to the $i$th variable is defined as

$$\frac{\partial f}{\partial x_i} = f_{x_i} = f_i(x) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h},$$

if the limit exists.

*Remark* 2.2.

- Desmos 3d Demo.
- Think derivative with respect to the $i$th position, not to $x_i$.
- "First order approximation in the $i$th direction."

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- With enough regularity imposed on $f$, we can write

$$\mathrm{d}f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_i}.$$

"Think first order approximation."

## 3. The Lagrangian

*Example* 3.1.

$$\max_{\{x,y\}} U(x,y), \quad \text{s.t.} \begin{cases} p_x x + p_y y \leq m \\ x, y \geq 0 \end{cases}$$

- For ease of mathematics, we often consider the constraint $p_x x + p_y y = m$.
- For a more general case, see Karush–Kuhn–Tucker conditions.

**Theorem 3.2.** *More abstractly, the above problem can be thought of as*

$$\max f(x_1, x_2) \quad \textit{s.t.}\ \ g(x_1, x_2) = c$$

*for a constant $c$. With enough regularity, the optima occurs at the critical points of the Lagrangian, defined as*

$$\mathcal{L}(x_1, x_2, \lambda) \coloneqq f(x_1, x_2) + \lambda[c - g(x_1, x_2)].$$

*That is, the optima subject to given constraint satisfies*

$$\begin{aligned} [x_1]: &\qquad f_1(x_1^*, x_2^*) = \lambda g_1(x_1^*, x_2^*) \\ [x_2]: &\qquad f_2(x_1^*, x_2^*) = \lambda g_2(x_1^*, x_2^*) \\ [\lambda]: &\qquad g(x_1^*, x_2^*) = c. \end{aligned}$$

## 4. Lagrangian: An Example

### 4.1. **The Utility Maximization Problem.** The problem:

$$v(p_x, p_y, m) \coloneqq \max_{x,y} U(x,y) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

### 4.2. **Interpretation.** We want to maximize

$$\mathrm{d}U = U_x \mathrm{d}x + U_y \mathrm{d}y$$

such that

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0 \implies \mathrm{d}y = -\frac{p_x}{p_y}\mathrm{d}x.$$

This gives

$$\mathrm{d}U = \left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x.$$

We can rewrite these two expressions in the following forms:

- Set $\mathrm{d}x > 0$ if $U_x/U_y > p_x/p_y$.

$$\left[\frac{U_x}{U_y} - \frac{p_x}{p_y}\right] U_y \ \mathrm{d}x$$

"Take advantage of all trading opportunities."

- Set $\mathrm{d}x > 0$ if $U_x/p_x > U_y/p_y$. Note that $U_x/p_y$ is marginal utility of money *spent on $x$*.
$$\left[\frac{U_x}{p_x} - \frac{U_y}{p_y}\right] p_x \,\mathrm{d}x$$
"Bang for your buck."
- Set $\mathrm{d}x > 0$ if $U_x > U_y \cdot p_x/p_y$. Note that $U_x$ is the marginal benefit of buying $x$ and $U_y \cdot p_x/p_y$ is the marginal cost of buying $x$.
$$\left[U_x - U_y \cdot \frac{p_x}{p_y}\right] \mathrm{d}x$$
"Trade until marginal cost equals marginal benefit."

In the last expression, if we write
$$\lambda = \frac{U_y}{p_y},$$
(think marginal utility of income) we have that at optimum,
$$(U_x - \lambda p_x)\mathrm{d}x = 0,$$
$$\lambda = \frac{U_y}{p_y} \iff U_y - \lambda p_y = 0,$$
$$p_x x + p_y y = m.$$
These three equalities describe precisely the critical points of the following
$$\mathcal{L}(p_x, p_y, \lambda) \coloneqq U(x, y) + \lambda \left[m - p_x x - p_y y\right],$$
called the **Lagrangian**. That is, setting
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$
recovers the above three equations.

*Remark* 4.1.
- We are not maximizing the Lagrangian but utility level (subject to given constraint).
- $\lambda$ might be negative or zero. Think bliss point.

## 5. Taylor Expansion

**Definition 5.1.** The Taylor polynomial of degree $n$ of the function $f$ around point $a$ is given by
$$P(a + x) = \sum_{k=1}^{n} \frac{f^{(n)}(a)}{k!} \cdot x^k.$$
It has the same $k$ derivatives as $f$. Think "$k$th order approximation.“

*Remark* 5.2. We will often use the first or second order approximation starting from a given point $a$:
$$f(a + h) \approx f(a) + f'(a)h,$$
$$f(a + h) \approx f(a) + f'(a)h + f''(a)h^2.$$

## 6. Probability

**Definition 6.1.** A discrete random variable $X$ can be described by the (at most countable) values it can attain and the probability of attaining them.

- The expectation of $X$ is defined as
$$\mathbb{E}(X) = \sum x \cdot \mathbb{P}(X = x).$$
Think weighted average.
- The variance of $X$ is defined as
$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2].$$
- For discrete random variables $X$ and $Y$, the covariance is defined by
$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X)) \cdot (Y - \mathbb{E}(Y))].$$

**Proposition 6.2.**

- $\mathbb{E}$ *is linear. That is,* $\mathbb{E}[a + bX] = a + b\,\mathbb{E}[X]$ *for* $a, b \in \mathbb{R}$.
- $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ *and* $\mathrm{Var}(a + bX) = b^2\,\mathrm{Var}(X)$.
- $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$.