ASSET PRICING LECTURE 01: MATHEMATICAL FOUNDATIONS

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Contents

1. Derivatives: A One-Dimensional Recap

Definition 1.1. Let $f : \mathbb{R} \supset \Omega \to \mathbb{R}$. The derivative of f at $x \in \Omega$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

Remark 1.2.

- Think "first order approximation."
- Note that f' is also a function, with the same domain as f (when f is enough regular).

2. Derivatives: Partial and Total

Definition 2.1. Let $f : \mathbb{R}^n \supset \Omega \to \mathbb{R}$. The partial derivative of f at $\mathbf{x} \in \Omega$ with respect to the ith variable is defined as

$$\frac{\partial f}{\partial x_i} = f_{x_i} = f_i(x) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h},$$

if the limit exists.

Remark 2.2.

- Think derivative with respect to the ith position, not to x_i.
- "First order approximation in the ith direction."
- With enough regularity imposed on f, we can write

$$\mathrm{d}f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_i}.$$

"Think first order approximation."

3. The Lagrangian

Example 3.1.

$$\max_{\{x,y\}} U(x,y), \quad \text{s.t.} \begin{cases} p_x x + p_y y \le m \\ x, y \ge 0 \end{cases}$$

- For ease of mathematics, we often consider the constraint $p_x x + p_y y = m$.
- For a more general case, see Karush–Kuhn–Tucker conditions.

Theorem 3.2. *More abstractly, the above problem can be thought of as*

$$\max f(x_1, x_2) \quad \textit{s.t. } g(x_1, x_2) = c$$

for a constant c. With enough regularity, the optima occurs at the critical points of the Lagrangian, defined as

$$\mathcal{L}(x_1, x_2, \lambda) \coloneqq f(x_1, x_2) + \lambda[c - g(x_1, x_2)].$$

That is, the optima subject to given constraint satisfies

$$\begin{aligned} [x_1]: &\quad f_1(x_1^*, x_2^*) = \lambda g_1(x_1^*, x_2^*) \\ [x_2]: &\quad f_2(x_1^*, x_2^*) = \lambda g_2(x_1^*, x_2^*) \\ [\lambda]: &\quad g(x_1^*, x_2^*) = c. \end{aligned}$$

4. Taylor Expansion

Definition 4.1. The Taylor polynomial of degree n of the function f around point a is given by

$$f(a+x) = \sum_{k=1}^{n} \frac{f^{(n)}(a)}{k!} \cdot x^k.$$

It has the same k derivatives as f. Think "kth order approximation.“

5. Probability

Definition 5.1. A discrete random variable X can be described by the (at most countable) values it can attain and the probability of attaining them.

- The expectation of X is defined as
$$\mathbb{E}(X) = \sum x \cdot \mathbb{P}(X = x).$$
Think weighted average.
- The variance of X is defined as
$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2].$$
- For discrete random variables X and Y, the covariance is defined by
$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X)) \cdot (Y - \mathbb{E}(Y))].$$

Proposition 5.2.

- $\mathbb{E}$ *is linear.*
- $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ *and* $\text{Var}(a + bX) = b^2 \text{Var}(X)$.
- $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.