Draft 2024/09/13

SPECTRAL THEORY AND THE MIN-MAX THEOREM

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Abstract. We develop the necessary tools and provide a proof of the min-max theorem, which offers a variational characterization of the discrete eigenvalues that lie below the essential spectrum of a self-adjoint operator.

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Introduction

This paper offers a brief introduction to spectral theory, covering key concepts such as the spectrum, resolvent, functional calculus, and spectral projections, and ultimately proving the min-max theorem.

Section 1 reviews the concept of Hilbert spaces and their structures. Section 2 introduces the adjoint, unitarity, and closure, and presents the multiplication operator. We then define the spectrum and resolvent set in Section 3, discussing their properties and calculating the spectra of different sets of operators. These foundations are essential for Section 4, where we prove the spectral theorem and, along the way, construct a functional calculus. This leads to the definition of spectral projectors and a discussion on the discrete and essential spectrum in Section 5. Finally, in Section 6, we prove the min-max theorem.

We assume the reader is familiar with analysis in $\mathbb{R}^n$.

1. Hilbert Spaces

Recall that a Hilbert space is an inner product space that is complete with respect to the metric induced by its inner product. We assume that the inner product is linear in the first variable and conjugate linear in the second. In what follows, we use the symbol $\mathcal{H}$ to denote a separable complex Hilbert space, that is, a complex Hilbert space that admits a countable dense subset.

One fundamental property of Hilbert spaces is that every closed subspace admits a complement. That is,

Theorem 1.1. Let M be a closed subspace of a Hilbert space $\mathcal{H}$. Then $H = M \oplus M^\perp$.

Proof. See [4, pp. 137–138, Remark 5]. □

In fact, this property characterizes Hilbert spaces: a Banach space X is isomorphic to a Hilbert space if and only if every closed subspace of X is complemented [4, p. 39].

We start with two consequences of this theorem concerning orthogonal complements and density of subspaces. They will be important when dealing with unbounded linear operators, frequently defined only on a dense linear subspace for reasons that will become clear.

Proposition 1.2. Let $M \subset \mathcal{H}$. Then $\operatorname{span} M$ is dense in $\mathcal{H}$ if and only if $M^\perp = \{0\}$.

Proof. Let $x \in M^\perp$ with $\operatorname{span} M$ dense in $\mathcal{H}$. Thus there exists a sequence (x_n) in $\operatorname{span} M$ converging to x. By continuity of the inner product, $0 = \langle x_n, x\rangle \to \langle x, x\rangle$ and we have $x = 0$. Conversely, if $M^\perp = \{0\}$, then a fortiori $(\overline{\operatorname{span} M})^\perp = \{0\}$. Since $\operatorname{span} M$ is a subspace of $\mathcal{H}$, Theorem 1.1 yields $H = \overline{\operatorname{span} M}$. □

Proposition 1.3. Let $W \subset \mathcal{H}$ be a linear subspace. Then $(W^\perp)^\perp = \overline{W}$.

Proof. By unwrapping the definition, we have $W \subset (W^\perp)^\perp$. Since $(W^\perp)^\perp$ is closed, the inclusion can be extended to $\overline{W} \subset (W^\perp)^\perp$ by continuity of the inner product. To prove the opposite inclusion, let $x \in (W^\perp)^\perp$ be arbitrary and suppose $x \notin \overline{W}$. By the Hahn-Banach theorem[1], there exists a continuous linear functional f on $\mathcal{H}$ that strictly separates x and $\overline{W}$. That is, $f(w) = 0$ for all $w \in \overline{W}$ and $f(x) \neq 0$. Let $v_f \in \mathcal{H}$ be the Riesz representation of f (see Theorem 2.1) and we have

$$\left\langle v_f, w\right\rangle = 0, \quad \forall w \in W.$$

So $v_f \in W^\perp$. But from $x \in (W^\perp)^\perp$ we know $\left\langle v_f, x\right\rangle = 0$, contradicting $f(x) \neq 0$. □

Recall, next, the direct sum of Hilbert spaces:

[1] See, e.g., [4, Chapter 1] or REU paper [9].

Definition 1.4. For the sequence of Hilbert spaces $\{\mathcal{H}_j\}_{j\in\mathbb{N}}$, we define their direct sum as
$$\bigoplus_{j=1}^{\infty} \mathcal{H}_j := \left\{(u_1, u_2, \cdots) : u_j \in \mathcal{H}_j, \sum \left\|u_j\right\|^2_{\mathcal{H}_j} < \infty\right\}.$$

The assumption on norms guarantees convergence of the inner product defined by
$$\langle (u_1, u_2, \cdots), (v_1, v_2, \cdots)\rangle := \sum_{j=1}^{\infty} \left\langle u_j, v_j\right\rangle_{\mathcal{H}_j}.$$
Note that the direct sum of Hilbert spaces is a Hilbert space.[2]

2. Operators

We recall the operator norm: $\|T\| := \sup_{u\in\mathcal{H}\setminus\{0\}} \|Tu\|/\|u\|$, which one can understand as the "maximum stretch" of an operator. From this view, it is not hard to see that operators with finite operator norm, **bounded operators**, are continuous.

In this paper, we will focus primarily on unbounded linear operators, that is, linear operators whose operator norm are *not necessarily* finite. Specification of the domain is important when dealing with unbounded operators. While it is often assumed that bounded operators are defined everywhere, unbounded operators are frequently defined only on a subset of a Hilbert space. As we will see, the adjoint does not exist for everywhere defined operators that are not bounded.

The notation $\mathcal{D}(T)$ will be used to denote the domain of the operator T. We say T is an operator **on** $\mathcal{H}$ if it is defined everywhere on $\mathcal{H}$, and an operator **in** $\mathcal{H}$ if $\mathcal{D}(T) \subset \mathcal{H}$. Extensions of operators will also be frequently considered: An operator S is an **extension** of T if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $S|_{\mathcal{D}(T)} = T$, that is, the restriction of S to the set $\mathcal{D}(T)$ coincides with T. We denote this relation as $T \subset S$.

In what follows, we present the notions of the adjoint, closure, and the graph of an operator. We introduce unitary and self-adjoint operators and the multiplication operator, and discuss their characterizations and key properties.

2.1. Adjoints.

Recall the Riesz representation theorem:

Theorem 2.1 (Riesz Representation Theorem). For each continuous linear functional f on $\mathcal{H}$, there exists a unique $v_f \in \mathcal{H}$, called the **Riesz representation** of f, such that
$$f(u) = \left\langle v_f, u\right\rangle, \quad \forall u \in \mathcal{H}.$$
Furthermore, $\|f\| = \left\|v_f\right\|$.

Proof. See [4, p. 135], or REU paper [1] for a more detailed explanation and background information. □

[2] See [5, p. 24] for a proof.

Definition 2.2. Let T be a linear operator in $\mathcal{H}$ (with dense domain). Then its **adjoint** T^* is defined as follows: The domain $\mathcal{D}(T^*)$ consists of vectors $v \in \mathcal{H}$ for which the map

$$\mathcal{D}(T) \ni u \longmapsto \langle v, Tu \rangle \in \mathbb{C}$$

is bounded. For such v, the functional extends to all of $\mathcal{H}$ by continuity, with the extension remains being bounded. Thus, by Theorem 2.1, there exists a unique vector which we denote T^*v such that

$$\langle v, Tu \rangle = \langle T^*v, u \rangle , \quad \forall u \in \mathcal{D}(T).$$

Since there are no Riesz representations for unbounded functionals, as maps of the form $u \mapsto \langle w, u \rangle$ are always bounded by Cauchy-Schwarz, the adjoint as defined above is the operator with the largest domain such that

$$\langle v, Tu \rangle = \langle T^*v, u \rangle , \quad \forall u \in \mathcal{D}(T), \quad \forall v \in \mathcal{D}(T^*).$$

For bounded operators, this domain is, by Theorem 2.1, the entirety of $\mathcal{H}$.

The assumption that the operator is densely defined is important. If $\overline{\mathcal{D}(T)} \neq \mathcal{H}$, there exists, by Proposition 1.2, a nonzero $w \in \mathcal{D}(T)^{\perp}$. That is, we have $\langle w, v \rangle = 0$ for all $v \in \mathcal{D}(T)$, and T^*u is not unique as we can add to it w.

Note also the linearity of the adjoint.

Definition 2.3. A linear operator T in $\mathcal{H}$ is

- **symmetric** (or **Hermitian**) if $\langle u, Tv \rangle = \langle Tu, v \rangle$ for all $u, v \in \mathcal{D}(T)$.
- **self-adjoint** if $T = T^*$.

It is not hard to see that an operator T being symmetric is equivalent to $T \subset T^*$. Thus, self-adjoint operators are symmetric. The converse is not, in general, true, as T^* may be a proper extension of T. However, in the case that T is defined everywhere on $\mathcal{H}$, we will see that the Hellinger-Toeplitz theorem (Corollary 2.14) gives the boundedness of T, and Theorem 2.1 then implies $\mathcal{D}(T^*) = \mathcal{H}$. Everywhere defined symmetric operators are, thus, bounded and self-adjoint.

We define also the following partial ordering on the set of self-adjoint operators:

Definition 2.4. For a self-adjoint operator T in $\mathcal{H}$ and a constant $c \in \mathbb{R}$, we write $T \geq c$ if

$$\langle u, Tu \rangle \geq c \langle u, u \rangle , \quad \forall u \in \mathcal{D}(T).$$

For self-adjoint operators S and T, we write $S \leq T$ if $T - S \geq 0$.

2.2. **Unitary Operators.**

Definition 2.5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A linear operator $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is said to be an **isometry** if it preserves norm, and **unitary** if it is a surjective isometry.

Operators $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ are **unitarily equivalent** if there exists a unitary map $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that $T_2 = UT_1U^{-1}$.

Note that by the polarization identity, isometries also preserve the inner product. That is, for an isometry $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, there holds

$$\langle u, v \rangle = \langle Tu, Tv \rangle , \quad \forall u, v \in \mathcal{H}_1.$$

We state also another characterization of unitary operators:

Proposition 2.6. A linear operator U on a Hilbert space $\mathcal{H}$ is unitary if and only if $U^*U = UU^* = I$.

Proof. Since U preserves norm, we know $\|U\| = 1$ and U is bounded. Theorem 2.1 then gives $\mathcal{D}(U^*) = \mathcal{D}(U) = \mathcal{H}$ and thus

$$\langle u, v\rangle = \langle Uu, Uv\rangle = \langle u, U^*Uv\rangle, \quad \forall u, v \in \mathcal{H}.$$

This leads to $U^*U = I$. To prove $UU^* = I$, note that Theorem 2.1 also gives $\|U^*v\| = \|u \mapsto \langle v, Uu\rangle\|$. The surjectivity of U then gives

$$\|U^*v\| = \sup_{\|u\|=\|Uu\|=1} \langle v, Uu\rangle = \sup_{\|u\|=1} \langle v, u\rangle = \|v\|, \quad \forall v \in \mathcal{H}.$$

Thus U^* is also unitary. Similar as above, we have

$$\langle u, v\rangle = \langle U^*u, U^*v\rangle = \langle u, UU^*v\rangle, \quad \forall u, v \in \mathcal{H},$$

giving $UU^* = I$.

Conversely, if $U^*U = UU^* = I$, a similar argument proves U is an isometry. To see that U is surjective, note that for any $u \in \mathcal{H}$, we have $U(U^*u) = u$. $\square$

2.3. **The Multiplication Operator.**

Definition 2.7. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For a measurable function $f : X \to \mathbb{C}$, the **multiplication operator** on $L^2(X, \mu)$ is defined as

$$M_f : v \longmapsto fv, \quad \mathcal{D}(M_f) \coloneqq \{v \in L^2(X, \mu) : fv \in L^2(X, \mu)\}.$$

The multiplication operator will serve as the continuous analogue of diagonal matrices in linear algebra. We prove a few basic properties of it:

Proposition 2.8. Let M_f, X, and μ be as above. Then

(a) M_f is bounded if and only if $f \in L^\infty(X, \mu)$, with $\|M_f\| \le \|f\|_\infty$,
(b) $M_f^* = M_{\overline{f}}$. Thus M_f is self-adjoint if and only if f is real-valued.

Proof.

(a) Let $f \in L^\infty(X, \mu)$. For any $u \in \mathcal{D}(M_f)$, we have

$$\|fu\| = \int |f|^2|u|^2 \, d\mu \le \int \|f\|_\infty |u|^2 \, d\mu = \|f\|_\infty \|u\|.$$

Therefore $\|M_f\| \le \|f\|_\infty$. On the other hand, for any $a < \|f\|_\infty$, let $A \coloneqq \{x : |f(x)| \ge a\}$. We then have

$$\|f\mathbb{1}_A\| \ge \|a\mathbb{1}_A\| = a\|\mathbb{1}_A\|.$$

Since $\|\mathbb{1}_A\| = \mu(A) > 0$, we may conclude that $\|M_f\| \ge a$. It then follows that $\|M_f\| = \|f\|_\infty$. Conversely, if $f \notin L^\infty(X, \mu)$, then the same argument shows that M_f is not bounded.

(b) We first check that M_f is densely defined (so that its adjoint exists): Note that

$$|f| \le \frac{1}{2}(|f|^2 + 1) \implies \frac{|f|}{|f|^2 + 1} \le \frac{1}{2}.$$

Thus for any $u \in \mathcal{D}(M_f)^\perp$, we have $\frac{fu}{|f|^2+1} \in L^2(X,\mu)$. This gives $\frac{u}{|f|^2+1} \in \mathcal{D}(M_f)$, which implies

$$0 = \left\langle \frac{u}{|f|^2+1}, u \right\rangle = \frac{1}{|f|^2+1} \int |u|^2 \, \mathrm{d}\mu.$$

Therefore, $u = 0$ a.e. and M_f is densely defined by Proposition 1.2.

Next, notice that for any $u \in \mathcal{D}(M_f)$ and any $v \in \mathcal{D}(M_{\overline{f}})$, we have

$$\langle fu, v \rangle = \int fu\overline{v} \, \mathrm{d}\mu = \int u\overline{\overline{f}v} \, \mathrm{d}\mu = \left\langle u, \overline{f}v \right\rangle.$$

It follows that $M_{\overline{f}} \subset M_f^*$.

To complete the proof, we need only show $\mathcal{D}(M_f^*) \subset \mathcal{D}(M_{\overline{f}})$. Let $v \in \mathcal{D}(M_f^*)$. For any $u \in \mathcal{D}(M_f)$, we know from above that $\frac{fu}{|f|^2+1} \in L^2(X,\mu)$. Thus, by definition of the adjoint,

$$\left\langle \frac{fu}{|f|^2+1}, v \right\rangle = \left\langle \frac{u}{|f|^2+1}, M_f^* v \right\rangle,$$

which gives

$$\left\langle u, \overline{f}v \right\rangle = \left\langle u, M_f^* v \right\rangle, \quad \forall u \in \mathcal{D}(M_f).$$

It then follows that $\overline{f}v = M_{f^*}v \in L^2(X,\mu)$, and thus $v \in \mathcal{D}(M_{\overline{f}})$.

□

2.4. **Closed Operators.**

Definition 2.9. We say an operator T is **closed** if its graph

$$\mathrm{G}(T) \coloneqq \{(u, Tu) : u \in \mathcal{D}(T)\}$$

is closed as a subspace of $\mathcal{H} \times \mathcal{H}$. We say T is **closable** if $\overline{\mathrm{G}(T)}$ is the graph of an operator, and define its closure $\overline{T}$ by $\mathrm{G}(\overline{T}) = \overline{\mathrm{G}(T)}$.

Thus an operator is closable if and only if it admits a closed extension, in which case the closure is its closed extension with the smallest domain. The property of being closed can be thought of as a weak form of continuity. Not all closed operators are bounded, but all bounded operators are closed.

Recall, however, that in the case where an operator is defined everywhere, being closed is equivalent to being bounded by the following:

Theorem 2.10 (Closed Graph Theorem). Let T be an operator with domain $\mathcal{H}$. Then T is bounded if and only if it is closed.

Proof. See [4, p. 37]. □

Using the graph of a linear operator, we can reformulate the definition of the adjoint and easily see that it is always closed:

Proposition 2.11. Let T be a linear operator in $\mathcal{H}$. Then the adjoint of T is closed and

$$\mathrm{G}(T^*) = J(\mathrm{G}(T)^\perp),$$

where J is defined as $J(x, y) \mapsto (y, -x)$.

Proof. Note that $(u, w) \in \mathrm{G}(T^*)$ if and only if $\langle v, Tu\rangle = \langle w, v\rangle$ for all $v \in \mathcal{D}(T)$. This happens precisely when

$$(u, w) \perp (Tv, -v) = J(v, Tv), \quad \forall v \in \mathcal{D}(T).$$

So $(u, w) \in \mathrm{G}(T^*)$ if and only if $(u, w) \in J(\mathrm{G}(T))^\perp$. Thus

$$\mathrm{G}(T^*) = J(\mathrm{G}(T))^\perp = J(\mathrm{G}(T)^\perp).$$

Since orthogonal complements are always closed and J preserves closed sets, the adjoint T^* is closed. □

Immediately, we have the following:

Corollary 2.12. Self-adjoint operators are closed.

Theorem 2.13. An operator T is closable if and only if $\mathcal{D}(T^*)$ is dense, in which case

$$\overline{T} = T^{**}.$$

Proof. Suppose $\mathcal{D}(T^*)$ is dense. Then T^{**} exists. Applying Proposition 2.11 twice and using Proposition 1.3 gives

$$G(T^{**}) = J(G(T^*)^\perp) = \overline{G(T)}.$$

Thus T is closable with $\overline{T} = T^{**}$.

Conversely, suppose $\mathcal{D}(T^*)$ is not dense. From Proposition 1.2 we know there exists a nonzero vector $v \in \mathcal{D}(T^*)^\perp$. Thus $(v, 0) \perp \mathrm{G}(T^*)$ and we have, by Proposition 2.11 and Proposition 1.3,

$$(0, -v) \in J(\mathrm{G}(T^*)^\perp) = \overline{G(T)}.$$

As $v \neq 0$, $\overline{G(T)}$ cannot be a graph of an operator. Thus T is not closable. □

Combined with the Closed graph theorem (Theorem 2.10) , Theorem 2.13 directly leads to the Hellinger-Toeplitz Theorem:

Corollary 2.14 (Hellinger-Toeplitz Theorem). A self-adjoint operator with domain $\mathcal{H}$ is bounded.

3. Spectrum and Resolvent

Definition 3.1. Let T be a linear operator in $\mathcal{H}$. A **regular value** of T is a complex number λ such that the operator

$$T - \lambda : \mathcal{D}(T) \longrightarrow \mathcal{H}, \quad u \longmapsto Tu - \lambda u$$

is surjective and has a bounded inverse.[3] The **resolvent set** of T, $\operatorname{res} T$, is the set of all regular values of T.

Thus a regular value is a complex number λ such that the linear problem

$$(T - \lambda)x = v$$

is always, uniquely, and continuously solvable.

[3] Some, e.g. [7], require only that the inverse is densely defined and bounded.

Definition 3.2. Let T be a densely defined linear operator in $\mathcal{H}$.

- The **spectrum** of T, $\operatorname{spec} T$, is the complement of the resolvent set in $\mathbb{C}$. That is,
$$\operatorname{spec} T \coloneqq \mathbb{C} \setminus \operatorname{res} T.$$
- A complex number λ is an **eigenvalue** of T if there exists a corresponding **eigenvector** $\phi \in \mathcal{H} \setminus \{0\}$ such that
$$(T - \lambda)\phi = 0.$$
The null space $\mathcal{N}(T - \lambda)$ is the corresponding **eigenspace**.
- The set of all eigenvalues is called the **point spectrum**, denoted $\operatorname{spec}_p T$.

Note that $\operatorname{spec}_p T \subset \operatorname{spec} T$.

3.1. **Properties of the Resolvent.**

Theorem 3.3. Let T be a densely defined operator in $\mathcal{H}$. The resolvent set $\operatorname{res} T$ is open (thus the spectrum is closed), and the **resolvent** of T
$$R(\cdot; T) : \operatorname{res} T \longrightarrow \mathcal{L}(\mathcal{H}), \quad \lambda \longmapsto (T - \lambda)^{-1}$$
is holomorphic.

Proof. If $\operatorname{res} T = \varnothing$, it is closed, so we assume otherwise and pick arbitrary point $z_0 \in \operatorname{res} T$. Note that on $\mathcal{D}(T)$ we have
$$T - z = \Big(I - (z - z_0)R(z_0; T)\Big)(T - z_0).$$
When z is such that $|z - z_0| < 1/\|R(z_0; T)\|$, there holds
$$\big(I - (z - z_0)R(z_0; T)\big)^{-1} = \sum_{j=0}^{\infty} (z - z_0)^j R(z_0; T)^j,$$
with the right side converging absolutely in $\mathcal{L}(\mathcal{H})$. Thus we have
$$R(z; T) = R(z_0; T)\big(I - (z - z_0)R(z_0; T)\big)^{-1} = \sum_{j=0}^{\infty} (z - z_0)^j R(z_0; T)^{j+1}.$$
Hence, $R(\cdot; T)$ is holomorphic. □

Theorem 3.4. Let T be an linear operator in $\mathcal{H}$ and $z, w \in \operatorname{res} T$. Then $R(z; T)$ commutes with $R(w; T)$ and
$$R(z; T) - R(w; T) = (z - w)R(z; T)R(w; T).$$

Proof. Note that on $\mathcal{D}(T)$ we have $(T - w) - (T - z) = z - w$. Evaluating at $R(z; T)v$, where $v \in \mathcal{D}(T)$ is arbitrary, gives
$$(T - w)R(z; T)v - v = (z - w)R(z; T)v.$$
Apply $R(w; T)$ on both sides and we have, using the linearity of $R(w; T)$, that
$$R(z; T)v - R(w; T)v = (z - w)R(w; T)R(z; T)v.$$
This proves the desired identity. Noting that the same argument proves
$$R(w; T) - R(z; T) = (w - z)R(w; T)R(z; T),$$

we obtain commutativity. □

We discuss, next, the spectra of bounded, unitary, self-adjoint, and multiplication operators.

3.2. **Spectrum of a Bounded Operator.**

Theorem 3.5. For a bounded operator T, there holds $\operatorname{spec} T \subset \left\{z \in \mathbb{C} : |z| \leq \|T\|\right\}$.

Proof. For $|z| \geq \|T\|$, the series

$$S \coloneqq \sum_{j=0}^{\infty} z^{-j} T^j$$

converges absolutely in $\mathcal{L}(\mathcal{H})$. Note that

$$TS = \sum_{j=0}^{\infty} z^{-j} T^{j+1} = \sum_{j=1}^{\infty} z^{-j+1} T^j = z(S - I),$$

which gives $(T - z)\left(-z^{-1} S\right) = I$. Thus $z \in \operatorname{res} T$. □

Informally, when $|z|$ is larger than $\|T\|$, the operator T becomes "small enough" relative to the "nice" operator $-zI$. As a result, the addition of T to $-zI$ will not undermine the injectivity of the latter.

3.3. **Spectrum of a Unitary Operator.** If U is unitary, then $U^{-1} = U^*$ exists, so $0 \in \operatorname{res} U$. The same argument in the proof of Theorem 3.3 shows that all z with

$$|z| < \frac{1}{\|R(0; U)\|} = \frac{1}{\|U^*\|} = 1$$

is contained in the resolvent. Combining this with Theorem 3.5, we have:

Corollary 3.6. The spectrum of a unitary operator on $\mathcal{H}$ is a closed subset of the unit circle $\mathbb{S} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$.

Finally, we note that for unitarily equivalent operators T and $S = UTU^*$, where U is unitary, we have

$$S - \lambda = UTU^* - \lambda = U(T - \lambda)U^*$$

for arbitrary $\lambda \in \mathbb{C}$. From here we see that $S - \lambda$ has bounded inverse if and only if $T - \lambda$ has bounded inverse, and similarly, $S - \lambda$ is injective if and only if $T - \lambda$ is injective. Thus we have:

Proposition 3.7. Unitarily equivalent operators share the same spectrum and eigenvalues.

3.4. **Spectrum of a Self-adjoint Operator.** We use $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively to denote the null space and the range of the operator T.

Proposition 3.8. Let T be a densely defined linear operator and $z \in \mathbb{C}$. Then

$$\mathcal{N}(T^* - \overline{z}) = \mathcal{R}(T - z)^{\perp}, \quad \mathcal{N}(T^* - \overline{z})^{\perp} = \overline{\mathcal{R}(T - z)}.$$

Proof. Note that the second statement can be obtained from the first by taking orthogonal complements on both sides and using Proposition 1.3. We prove the first equality. Since $\mathcal{D}(T)$ is dense, the condition $f \in \mathcal{N}(T^* - \overline{z})$ is equivalent to

$$\langle (T^* - \overline{z}) f, g \rangle = 0, \quad \forall g \in \mathcal{D}(T).$$

That is,

$$\langle f, (T - z) g \rangle = 0, \quad \forall g \in \mathcal{D}(T).$$

This means precisely that $f \perp \mathcal{R}(T - z)$. ◻

Theorem 3.9. Let T be a self-adjoint operator in $\mathcal{H}$. Then $\operatorname{spec} T \subset \mathbb{R}$ and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$\langle u, (T - z)u \rangle = \langle u, Tu \rangle - \operatorname{Re} \overline{z} \langle u, u \rangle - \mathrm{i} \operatorname{Im} \overline{z} \langle u, u \rangle .$$

Noting that $\langle u, Tu \rangle = \overline{\langle Tu, u \rangle}$ is real, we have

$$\|u\| \|(T - z)u\| \geq |\operatorname{Im} \langle u, (T - z)u \rangle| = |\operatorname{Im} \overline{z} \langle u, u \rangle| = |\operatorname{Im} z| \|u\|^2.$$

Hence,

$$\|(T - z)u\| \geq |\operatorname{Im} z| \|u\|.$$

This gives the injectivity of $T - z$, and thus $\mathcal{N}(T - z) = \{0\}$ for any $z \in \mathbb{C} \setminus \mathbb{R}$. By Proposition 3.8, we know $\mathcal{R}(T - z)$ is dense in $\mathcal{H}$.

Now, note that for any converging sequence $(T - z)u_n$, the estimate above ensures the convergence of u_n. Since $T - \lambda$ is closed by Corollary 2.12, this implies

$$\lim \left(u_n, (T - z)u_n \right) \in \mathrm{G}(T - z).$$

In particular, $\lim (T - z)u_n \in \mathcal{R}(T - z)$ for any converging sequence $(T - z)u_n$. That is, $\mathcal{R}(T - z)$ is closed and

$$\mathcal{R}(T - z) = \overline{\mathcal{R}(T - z)} = \mathcal{H}.$$

So $T - z$ is surjective, with inverse bounded by the same estimate. It follows that $\operatorname{spec} T \subset \mathbb{R}$.

Finally, let ϕ_1 and ϕ_2 be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2. We have

$$\lambda_1 \langle \phi_1, \phi_2 \rangle = \langle T\phi_1, \phi_2 \rangle = \langle \phi_1, T\phi_2 \rangle = \lambda_2 \langle \phi_1, \phi_2 \rangle .$$

This gives $\langle \phi_1, \phi_2 \rangle = 0$. ◻

3.5. **Spectrum of a Multiplication Operator.** The following will be useful tool for computing the spectrum:

Proposition 3.10. Let T be a self-adjoint operator and $z \in \mathbb{C}$. If there exists a sequence $u_n \subset \mathcal{D}(T)$ with $\|u_n\| = 1$ and

$$\lim_{n \to \infty} \|(T - z)u_n\| = 0,$$

then $z \in \operatorname{spec} T$.

Such a sequence (u_n) is called a **Weyl sequence** for z.

Proof. If $T - z$ is not injective, then $z \in \operatorname{spec} T$. We assume otherwise and set $x_n \coloneqq (T - z)u_n \neq 0$. Then

$$\|(T - z)u_n\| = \frac{\|(T - z)u_n\|}{\|u_n\|} = \frac{\|x_n\|}{\left\|(T - z)^{-1} x_n\right\|} \to 0.$$

Hence, $(T - z)^{-1}$ is not bounded. □

We are now equipped to calculate the spectrum of the multiplication operator:

Proposition 3.11. For the multiplication operator M_f on $L^2(X, \mu)$ (see Definition 2.7),
(a) $\operatorname{spec}_{\mathrm{p}} M_f = \left\{\lambda \in \mathbb{C} : \mu\{x : f(x) = \lambda\} > 0\right\}$,
(b) $\operatorname{spec} M_f = \operatorname{ess\,ran} f := \left\{\lambda \in \mathbb{C} : \mu\left\{x : |f(x) - \lambda| < \epsilon\right\} > 0 \text{ for all } \epsilon > 0\right\}$.

One may think of ess ran f, the **essential range** of the function f, as the areas on which the range of f is "concentrated."

Proof.
(a) If λ is such that $\mu\{x : f(x) = \lambda\} = 0$, then the equation
$$(M_f - \lambda)\phi = 0, \quad \phi \in L^2(X, \mu)$$
implies $\phi = 0$ a.e. Thus, $\mathcal{N}(M_f - \lambda) = \{0\}$ and $\lambda \notin \operatorname{spec}_{\mathrm{p}} M_f$.

If $\mu\{x : f(x) = \lambda\} > 0$, then there exists a subset $A \subset \{x : f(x) = \lambda\}$ with $0 < \mu(A) < \infty$. From $(M_f - \lambda)\mathbb{1}_A = 0$, we know $\lambda \in \operatorname{spec}_{\mathrm{p}} M_f$.

(b) Let $\lambda \notin \operatorname{ess\,ran} f$. Then for some $\epsilon > 0$, there holds $|f - \lambda| > \epsilon$ a.e., so $1/(f - \lambda)$ is defined a.e. and $\|1/(f - \lambda)\|_\infty < \infty$. Since $\left\|M_{1/(f-\lambda)}\right\|$ is bounded by Proposition 2.8, we have $\lambda \in \operatorname{res} M_f$.

It remains to show that $\operatorname{ess\,ran} f \subset \operatorname{spec} M_f$. For a $\lambda \in \operatorname{ess\,ran} f$, if $\mu\{x : f(x) = \lambda\} > 0$, then we have shown that $\lambda \in \operatorname{spec}_{\mathrm{p}} M_f \subset \operatorname{spec} M_f$. Thus assume otherwise. Then, as above, $1/(f - \lambda)$ is defined a.e. and $M_{1/(f-\lambda)}$ is the inverse of $(M_f - \lambda)$. From $\lambda \in \operatorname{ess\,ran} f$, we have $|f - \lambda| < \epsilon$ on a set of positive measure for all $\epsilon > 0$. This gives $\|1/(f - \lambda)\|_\infty = \infty$, so $M_{1/(f-\lambda)}$ is not bounded and $\lambda \in \operatorname{spec} M_f$.

□

4. The Spectral Theorem

In this section, we prove the spectral theorem, which states that every self-adjoint operator is unitarily equivalent to a multiplication operator in some measure space. We will first develop the spectral theorem for unitary operators and then pass it to self-adjoint operators using the Cayley transform.

Along the way, we also introduce a functional calculus using the Fourier transform, which allows us to apply functions defined on the spectrum of an operator to the operator itself. This will set the stage for the spectral resolution for self-adjoint operators in the next section.

4.1. **Spectral Theory for Unitary Operators.** We start with unitary operators, whose spectra, we recall by Corollary 3.6, lie in the unit circle. We use the Fourier transform to construct a functional calculus for applying complex-valued functions acting on the unit circle to unitary operators:

For a smooth function $f : \mathbb{S} \to \mathbb{C}$, we have
$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n, \quad z \in \mathbb{S},$$
with the right side converging uniformly and where
$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta} \, \mathrm{d}\theta, \quad n \in \mathbb{Z}$$

are the discrete Fourier coefficients. We can thus define $f(U)$ using the convergent series

$$f(U) \coloneqq \sum_{n\in\mathbb{Z}} \hat{f}(n)U^n.$$

As $C^\infty(\mathbb{S})$ is dense in $C(\mathbb{S})$ by the Weierstrass approximation theorem, we can extend the map $f \mapsto f(U)$ constructed above to $C(\mathbb{S})$ by means of density. We have the following:

Proposition 4.1. Let U be a unitary operator. The map $C^\infty(\mathbb{S}) \ni f \mapsto f(U) \in \mathcal{L}(\mathcal{H})$ defined above extends uniquely to a linear map $C(\mathbb{S}) \to \mathcal{L}(\mathcal{H})$ such that for any $f, g \in C(\mathbb{S})$ one has the following:

(a) $f(U)^* = \overline{f}(U)$.
(b) $f(U)g(U) = (fg)(U)$.
(c) If $f \geq 0$, then $f(U) \geq 0$.
(d) $\|f(U)\| \leq \|f\|_\infty$.

Proof. We first prove the properties for $f, g \in C^\infty(\mathbb{S})$.

(a) From definition, we have $\overline{\hat{f}(n)} = \hat{\overline{f}}(-n)$ for any $n \in \mathbb{Z}$. Thus

$$f(U)^* = \sum_{n\in\mathbb{Z}} \overline{\hat{f}(n)}U^{-n} = \sum_{n\in\mathbb{Z}} \hat{\overline{f}}(-n)U^{-n} = \overline{f}(U).$$

(b) Note that

$$(fg)(e^{\mathrm{i}\theta}) = \left(\sum_{j\in\mathbb{Z}} \hat{f}(j)e^{\mathrm{i}j\theta}\right)\left(\sum_{k\in\mathbb{Z}} \hat{g}(k)e^{\mathrm{i}k\theta}\right) = \sum_{n\in\mathbb{Z}} c_n e^{\mathrm{i}n\theta},$$

where $c_n \coloneqq \sum_{j\in\mathbb{Z}} \hat{f}(j)\hat{g}(n-j)$. Using the orthogonality of $e^{\mathrm{i}n\theta}$, we have $\widehat{fg}(n) = c_n$. Thus

$$f(U)g(U) = \sum_{n\in\mathbb{Z}} c_n = (fg)(U).$$

(c) Let $f \in C^\infty(\mathbb{S})$ with $f \geq 0$. Consider the function $h_\epsilon(z) \coloneqq \sqrt{f(z)+\epsilon}$ where $\epsilon > 0$. As h_ϵ is smooth, $h_\epsilon(U)$ is defined by the above, and since h_ϵ is real-valued, $h_\epsilon(U)$ is self-adjoint by (a). Property (b) gives $f(U) = h_\epsilon(U)^2 - \epsilon$, and thus we have

$$\langle u, f(U)u\rangle = \|h_\epsilon(U)u\|^2 - \epsilon\|u\|^2, \quad \forall u \in \mathcal{H}.$$

Since $\epsilon > 0$ is arbitrary, we obtain $f(U) \geq 0$.

(d) Set $M \coloneqq \sup|f|$. We have $M^2 - |f|^2 \geq 0$. Properties (a) and (b) yield

$$\left\langle u, |f|^2(U)u\right\rangle = \left\langle u, (\overline{f}f)(U)u\right\rangle = \langle u, f(U)^*f(U)u\rangle = \|f(U)u\|$$

for all $u \in \mathcal{H}$. Thus, since $\left\langle u, (M^2 - |f|^2(U))u\right\rangle \geq 0$ by (c), we have $\|Mu\|^2 \geq \|f(U)u\|^2$.

Now, note that for any uniformly converging sequence $f_n \to f$ where $\{f_n\} \subset C^\infty(\mathbb{S})$, $f_n(U)$ is Cauchy by (d). Thus we can define $f(U) \coloneqq \lim f_n(U)$. We can then check that $f(U)$ is independent of the choices of f_n and verify the properties by passing to the limit. □

Recall the Riesz-Markov-Kakutani representation theorem:

Theorem 4.2. Let X be a compact metric space. Given a positive linear functional $\beta : C(X) \to \mathbb{C}$, there exists a unique Borel measure μ on X such that

$$\beta(f) = \int_X f \, \mathrm{d}\mu$$

for $f \in C(X)$.

Proof. See [3, p. 310] or REU papers [6] and [10]. ◻

Let U be a unitary operator and $f \in C(\mathbb{S})$. Consider functionals of the form $f \mapsto \langle u, f(U)u \rangle$, where $u \in \mathcal{H}$. They are linear in f and nonnegative by Proposition 4.1 (c). Thus, by Theorem 4.2, there exists a unique Borel measure on $\mathbb{S}$, the **spectral measure** associated with u, such that

$$\langle u, f(U)u \rangle = \int_{\mathbb{S}} f \, \mathrm{d}\mu_u.$$

Lemma 4.3. For U and u as above, the map $C(\mathbb{S}) \ni f \longmapsto f(U)u \in \mathcal{H}$ has a unique continuous extension to an isometry $W_u : L^2(\mathbb{S}, \mu_u) \to \mathcal{H}$ such that

$$UW_u[f(z)] = W_u[zf(z)].$$

Proof. Let $f, g \in C(\mathbb{S})$. By Proposition 4.1 we have

$$\langle W_u f, W_u g \rangle = \Big\langle u, (\overline{f}g)(U)u \Big\rangle = \int_{\mathbb{S}} \overline{f}g \, \mathrm{d}\mu_u = \langle f, g \rangle_{L^2(\mathbb{S},\mu_u)}.$$

Thus W_u preserves the inner product for functions in $C(\mathbb{S})$. Since $C(\mathbb{S})$ is dense in $L^2(\mathbb{S}, \mu_u)$, the operator above extends uniquely by density to an isometry. Next, note that from $[zf(z)](U) = Uf(U)$, we have $UW_u[f(z)] = W_u[zf(z)]$ for $f \in C(\mathbb{S})$, which extends also to $L^2(\mathbb{S}, \mu_u)$. ◻

If W_u happens to be surjective, then W_u is unitary, and U is unitarily equivalent to the multiplication operator $M_z : f(z) \mapsto zf(z)$. If, however, this is not the case, then we can simply iterate the construction on the orthogonal complement of $\mathcal{R}(W_u)$ and repeat the process until we obtain a surjective map:

Theorem 4.4 (Spectral Theorem for Unitary Operators). Let U be a unitary operator in $\mathcal{H}$. Then there exists a subset $N \subset \mathbb{N}$, finite Borel measures ν_n on $\mathbb{S}$ where $n \in N$, and a unitary map

$$W : L^2(Y, \nu) \to \mathcal{H}, \quad Y = \mathbb{S} \times N, \quad \nu(A \times \{n\}) = \nu_n(A) \text{ for } A \subset \mathbb{S},$$

such that

$$W^{-1}UW = M_\rho,$$

where $\rho : Y \ni (y, n) \longmapsto y \in \mathbb{S} \subset \mathbb{C}$ is the identity function on each copy of $\mathbb{S}$.

Proof. We follow [3].

Let $\{w_j\}_{j \in \mathbb{N}}$ be a dense subset of $\mathcal{H}$. Consider the vector w_1. Lemma 4.3 gives a measure ν_1 and an isometry

$$W_1 : L^2(\mathbb{S}, \nu_1) \to H_1.$$

If $H_1 = \mathcal{H}$, the proof is complete. Otherwise, note that $U(H_1) = H_1$ and, by unitarity, $U(H_1^\perp) = H_1^\perp$. Thus U can be viewed as a unitary operator on $H_1^\perp$.

Next, pick the first j such that $w_j \notin H_1$. Let v_2 be the orthogonal projection of w_j onto $H_1^\perp$. Applying Lemma 4.3 on v_2 gives a measure ν_2 and an isometry W_2 with range H_2.

If $\mathcal{H} = H_1 \oplus H_2$, the proof is again complete. Otherwise, we repeat the procedure to find $w_j \notin (H_1 \oplus H_2)$ and apply Lemma 4.3 to the orthogonal projection of w_1 on $(H_1 + H_2)^\perp$.

Repeating this process gives a (possibly finite) sequence H_k such that $\mathcal{H} = \oplus_k H_k$ and a corresponding decomposition $L^2(X, \nu) = \oplus_k(\mathbb{S}, \nu_k)$. Since each W_k is unitary, we have

$$W \coloneqq \oplus_k W_k : L^2(X, \nu) \longrightarrow \mathcal{H}$$

is unitary, and $W^{-1}UW = M_\rho$. □

4.2. **Spectral Theory for Self-Adjoint Operators.** To pass the spectral theorem to self-adjoint operators, we use the **Cayley transformation**:

$$c : \mathbb{R} \ni x \longmapsto \frac{x - \mathrm{i}}{x + \mathrm{i}} \in \mathbb{S}.$$

The function c maps the real line (which contains the spectrum of self-adjoint operators) to the unit circle (which contains the spectrum of unitary operators).

The operator valued Cayley transform maps self-adjoint operators to unitary operators:

Lemma 4.5. If T is self-adjoint, then $U \coloneqq (T - \mathrm{i})(T + \mathrm{i})^{-1} \equiv I - 2\mathrm{i}(T + \mathrm{i})^{-1}$ is unitary.

Proof. By Theorem 3.9, $\operatorname{spec} T \subset \mathbb{R}$. So the inverses $(T \pm \mathrm{i})^{-1}$ exist and are bounded. Note that $(T + \mathrm{i})^* = T - \mathrm{i}$ and thus on $\mathcal{D}(T)$ we have

$$I = \left[(T + \mathrm{i})(T + \mathrm{i})^{-1}\right]^* = \left[(T + \mathrm{i})^{-1}\right]^* (T - \mathrm{i}).$$

Therefore $\left[(T + \mathrm{i})^{-1}\right]^* = (T - \mathrm{i})^{-1}$ and $U^* = I + 2\mathrm{i}(T - \mathrm{i})^{-1}$. By Theorem 3.4, $(T - \mathrm{i})^{-1}$ and $(T + \mathrm{i})^{-1}$ commute, so $U^*U = UU^*$. To complete the proof, note that again by Theorem 3.4,

$$U^*U = I + 2\mathrm{i}\Big(R(\mathrm{i}; T) - R(-\mathrm{i}; T) - 2\mathrm{i}R(\mathrm{i}; T)R(-\mathrm{i}; T)\Big) = I.$$

□

We now have the tools to prove the spectral theorem for self-adjoint operators:

Theorem 4.6 (Spectral Theorem—Multiplication Operators Form). Let T be a self-adjoint operator in $\mathcal{H}$. Then there exists a subset $N \subset \mathbb{N}$, finite Borel measures μ_n on $\mathbb{R}$ where $n \in \mathbb{N}$, and a unitary map

$$\Theta : L^2(X, \mu) \to \mathcal{H}, \quad X = \mathbb{R} \times N, \quad \mu(A \times \{n\}) = \mu_n(A) \text{ for } A \subset \mathbb{R},$$

such that

$$\Theta^{-1} T \Theta = M_h$$

where $h : X \ni (x, n) \longmapsto x \in \mathbb{R}$ is the identity function on each copy of $\mathbb{R}$.

Proof. We follow [8].

Let U be the unitary operator associated to T given by the Cayley transform. Theorem 4.4 gives $N \subset \mathbb{N}$, finite Borel measures ν_n on $\mathbb{S}$, and a unitary map

$$W : L^2(Y, \nu) \to \mathcal{H}, \quad Y = \mathbb{S} \times N, \quad \nu(A \times \{n\}) = \nu_n(A) \text{ for } A \subset \mathbb{S},$$

such that $W^{-1}UW = M_\rho$ with $\rho : (y, n) \mapsto y$. Since $I - U = 2\mathrm{i}(T + \mathrm{i})^{-1}$ is injective, the operator $I - M_\rho$ is injective, which gives $1 \notin \operatorname{spec}_{\mathrm{p}} M_\rho$. By Proposition 3.11, $\nu(\rho^{-1}(\{1\})) = 0$, and thus $\nu_n(\{1\}) = 0$ for each n. That is, the measures ν_k have no point mass at $y = 1$. We can thus use the inverse of the Cayley transformation

$$\eta \coloneqq c^{-1} : \mathbb{S} \ni y \longmapsto \mathrm{i}\frac{1 + y}{1 - y} \in \mathbb{R}$$

to define the pushforward measure

$$\mu_n \coloneqq \eta_* \nu_n \equiv \nu_n(\eta^{-1}(A)), \quad A \subset \mathbb{R}.$$

Next, note that the pullback operator

$$\Phi : L^2(X, \mu) \to L^2(Y, \nu), \quad (\Phi f)(x, n) = f(\eta(x), n), \quad \forall (x, n) \in X$$

is unitary by construction. It follows that $\Theta \coloneqq W\Phi : L^2(X, \mu) \to \mathcal{H}$ is unitary.

We verify next that $\Theta^{-1} T \Theta = M_h$. To keep track of each operator, we plot their domains and codomains below:

$$\begin{tikzcd}
L^2(X,\mu) \arrow[r, "\Phi"] \arrow[rr, bend left, "\Theta"] \arrow[loop below, "M_h"] & L^2(Y,\nu) \arrow[l, bend left, "\eta \text{ and } \eta_*"] \arrow[r, "W"] \arrow[loop below, "M_\rho"] & \mathcal{H} \arrow[loop below, "T"]
\end{tikzcd}$$

First we check the domains, proving $\Theta \mathcal{D}(M_h) = \mathcal{D}(T)$:

Let $f \in \mathcal{D}(M_h) \subset L^2(X, \mu)$. Then $g \coloneqq (M_h + \mathrm{i}) f \in L^2(X, \mu)$. Note, for any $(y, n) \in Y$, that

$$\begin{aligned}
(\Phi g)(y, n) &= \big(\Phi(h + \mathrm{i})\big)(y, n)(\Phi f)(y, n) \\
&= \Big(h\big(\eta(y), n\big) + \mathrm{i}\Big)(\Phi f)(y, n) = (\eta(y) + \mathrm{i})(\Phi f)(y, n).
\end{aligned}$$

Since $\eta(y) + \mathrm{i} = \mathrm{i}\frac{1+y}{1-y} + \mathrm{i} = \frac{2\mathrm{i}}{1-y}$, we have $(I - M_\rho)\Phi g = 2\mathrm{i}\Phi f$. Applying W on both sides and using $WM_\rho = UW$ gives $(I - U)\Theta g = 2\mathrm{i}\Theta f$. From definition, $I - U = 2\mathrm{i}(T + \mathrm{i})^{-1}$, so

$$\Theta f = (T + \mathrm{i})^{-1} \Theta g \in \mathcal{D}(T). \tag{$*$}$$

It follows that $\Theta \mathcal{D}(M_h) = \mathcal{D}(T)$.

Next, let $v \in \mathcal{D}(T)$ and $w \coloneqq (T + \mathrm{i})v$. Then

$$v = (T + \mathrm{i})^{-1} w = \frac{1}{2\mathrm{i}}(I - U)w.$$

From $W^{-1}UW = M_\rho$, we have $I - U = WM_{1-\rho}W^{-1}$. Thus $W^{-1}v = \frac{1}{2\mathrm{i}} M_{1-\rho} W^{-1} w$, and we have

$$\Theta^{-1} v = \Phi^{-1} W^{-1} v = \frac{1}{2\mathrm{i}} \Phi^{-1} M_{1-\rho} W^{-1} w. \tag{$\dagger$}$$

Now note that for $g \in L^2(Y, \nu)$ and any $(x, n) \in X$ we have

$$\begin{aligned}
(\Phi^{-1} M_{1-\rho} g)(x, n) &= (M_{1-\rho} g)\big(c(x), n\big) \\
&= \big(1 - c(x)\big) g\big(c(x), n\big) = \left(1 - \frac{x - \mathrm{i}}{x + \mathrm{i}}\right) g\big(c(x), n\big) \\
&= \frac{2\mathrm{i}}{x + \mathrm{i}} (\Phi^{-1} g)(x, n).
\end{aligned}$$

That is, $\Phi^{-1} M_{1-\rho} = 2\mathrm{i}(M_h + \mathrm{i})^{-1} \Phi^{-1}$. Combined with ($\dagger$) we get

$$\Theta^{-1} v = (M_h + \mathrm{i})^{-1} \Theta^{-1} w \in \mathcal{R}\big((M_h + \mathrm{i})^{-1}\big) = \mathcal{D}(M_h).$$

It follows that $\mathcal{D}(T) \subset \Theta \mathcal{D}(M_h)$, and thus $\mathcal{D}(T) = \Theta \mathcal{D}(M_h)$.

To finish the proof, note that for any $f \in \mathcal{D}(M_h)$, by ($*$), we have $\Theta(M_h + \mathrm{i}) f = (T + \mathrm{i})\Theta f$. This gives $\Theta M_h = T\Theta$. □

With the spectral theorem, we can now construct a functional calculus for self-adjoint operators. For a bounded Borel function $f \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}) : \mathbb{R} \to \mathbb{C}$, we define

$$f(T) \coloneqq \Theta M_{f \circ h} \Theta^{-1}$$

where Θ and h are given in Theorem 4.6. Note that since $f \circ h$ is bounded, the operator $f(T)$ is bounded by Proposition 2.8.

The map defined above has the following properties:

Proposition 4.7. Let T be a self-adjoint operator.

(a) For any $f, g \in \mathcal{B}_{\mathrm{b}}(\mathbb{R})$, we have $(fg)(T) = f(T)g(T)$ and $f(T)^* = \overline{f}(T)$.
(b) Let $f_n \to f$ pointwise and $\|f_n\|_\infty < M$ for each n. Then $f_n(T) \to f(T)$ in the strong operator topology, that is, $f_n(T) \to f(T)v$ for all $v \in \mathcal{H}$.

Proof.

(a) Let $f, g \in \mathcal{B}_{\mathrm{b}}(\mathbb{R})$. We have
$$f(T)g(T) = \Theta M_{f \circ h} M_{g \circ h} \Theta^{-1} = \Theta M_{(fg) \circ h} \Theta^{-1} = (fg)(T)$$
and
$$f(T)^* = \Theta M_{\overline{f \circ h}} \Theta^{-1} = \overline{f}(T).$$
(b) The dominated convergence theorem[4] gives
$$\lim_{n \to \infty} \|(f_n \circ h - f \circ h)v\| = \lim_{n \to \infty} \int_X |(f_n - f) \circ h|^2 |v|^2 \mathrm{d}\mu = 0$$
for all $v \in L^2(X, \mu)$. Thus we have $f_n(T)v \to f(T)v$.

□

We remark that the Spectral Theorem can be used to prove the following result showing the utility of obtaining the spectrum:

Theorem 4.8. For a self-adjoint operator T and $c \in \mathbb{R}$, we have

(a) $T \geq c$ if and only if $\operatorname{spec} T \subset [c, \infty)$.
(b) T is bounded with $\|T\| \leq c$ if and only if $\operatorname{spec} T \subset [-c, c]$.

Proof. See [8, p. 68]. □

5. Spectral Projectors

5.1. **Orthogonal Projectors.** We first present a quick review of projection operators:

Definition 5.1. A linear operator P on $\mathcal{H}$ is called a projection operator if P is idempotent. That is, $P^2 = P$. A self-adjoint projection operator is called a **orthogonal projector**.

Let P be an orthogonal projector. We have, for any $v, w \in \mathcal{H}$, that

$$\langle v - Pv, Pw \rangle = \langle v, Pw \rangle - \langle v, P^* P w \rangle = 0.$$

That is, for any $v \in \mathcal{H}$, we have $(I - P)v \perp \mathcal{R}(P)$. A similar argument shows that the converse is also true. That is, we have that a projection operator P is an orthogonal projector if and only if $\mathcal{R}(I - P) \perp \mathcal{R}(P)$.

Note also that projection operators do not increase distance; $\|P\| \leq 1$.

[4]See [2, p. 55].

Lemma 5.2. Let P_1 and P_2 be orthogonal projectors on $\mathcal{H}$ with $\mathcal{R}(P_1) \perp \mathcal{R}(P_2)$. Then $P_1P_2 = P_2P_1 = 0$ and $\mathcal{R}(P_1 + P_2) = \mathcal{R}(P_1) + \mathcal{R}(P_2)$.

Proof. It is clear that $\mathcal{R}(P_1 + P_2) \subset \mathcal{R}(P_1) + \mathcal{R}(P_2)$. We prove the opposite inclusion. For any $v \in \mathcal{H}$, since $P_2 v \perp \mathcal{R}(P_1)$, we have $P_1P_2v = 0$. Similarly, $P_2P_1 \equiv 0$. Now, pick any $v \in \mathcal{R}(P_1) + \mathcal{R}(P_2)$. Then $v = P_1w_1 + P_2w_2$ for some w_1 and w_2 in $\mathcal{H}$. From above,

$$(P_1 + P_2)(P_1w_1 + P_2w_2) = P_1^2 w_1 + P_2^2 w_2 = P_1w_1 + P_2w_2 = v.$$

Thus $v \in \mathcal{R}(P_1 + P_2)$. □

5.2. **Spectral Resolution.** Applying indicator functions on Borel sets to self-adjoint operators gives a family of orthogonal projectors:

Definition. Let $E \subset \mathbb{R}$ be a Borel subset and T a self-adjoint operator. The **spectral projector** of T on E is the orthogonal operator $P(E) \coloneqq \mathbb{1}_E(T)$.

The mapping $E \mapsto P(E)$ is called the **spectral resolution** of T and can be understood as a **projection-valued measure** with support $\operatorname{spec} T$. Additivity holds since disjoint Borel sets are mapped to mutually orthogonal projection operators, that is, projection operators with mutually orthogonal images (recall Lemma 5.2).

One may think of the range of $P(E)$, where E is a Borel subset of $\mathbb{R}$, as the part of the Hilbert space governed only by E, independent of the other parts of the spectrum. As we will see, the range of the projector $P\{\lambda\}$ is always the nullspace of $T - \lambda$. Thus, when λ is an eigenvalue, we have $Tu = \lambda u$ for any u in the eigenspace $\mathcal{R}(P\{\lambda\}) = \mathcal{N}(T - \lambda)$. One may thus express a self-adjoint operator T on an infinite dimensional Hilbert space as

$$Tu = \int_{\operatorname{spec} T} \lambda \, \mathrm{d}P(\mathrm{d}\lambda)u.$$

This is analogous to the finite dimensional case, where we have

$$Tu = \sum_{n=1}^{N} \lambda_n P_n,$$

with operators P_n, $1 \le n \le N$ acting as projectors onto the eigenspaces associated with λ_n.

This can be made rigorous. For our purpose, though, it suffices to view $P(\cdot)$ as orthogonal projectors and establish its finite additivity.

Proposition 5.3. Let T be a self-adjoint operator in $\mathcal{H}$, P the spectral resolution of T, and $E, F \subset \mathbb{R}$ disjoint Borel sets. Then

(a) $P(E)$ is an orthogonal projector,
(b) $P(\varnothing) = 0$ and $P(\mathbb{R}) = I$,
(c) $P(E \cup F) = P(E) + P(F)$ has range $\mathcal{R}\big(P(E \cup F)\big) = \mathcal{R}\big(P(E)\big) + \mathcal{R}\big(P(F)\big)$,
(d) $P(a, b) = 0$[5] if and only if $\operatorname{spec} T \cap (a, b) = \varnothing$,
(e) $\mathcal{R}(P\{\lambda\}) = \mathcal{N}\big(T - \lambda\big)$ for any $\lambda \in \mathbb{R}$.

Proof. We assume $\mathcal{H} = L^2(X, \mu)$ and $T = M_h$ with X, μ, h as in Theorem 4.6. Thus we have $P(E) = M_{\mathbb{1}_E \circ h}$. It is easy to check that the same statements hold for any operator T unitarily equivalent to M_h.

(a) Since $\mathbb{1}_E$ is idempotent, we have

$$P(E)^2 = \big(M_{\mathbb{1}_E \circ h}\big)^2 = M_{\mathbb{1}_E^2 \circ h} = M_{\mathbb{1}_E \circ h} = P(E).$$

Similarly, since $\mathbb{1}_E$ is real-valued, by Proposition 2.8 we have $P(E)^* = P(E)$.

(b) Note that $\mathbb{1}_\varnothing \circ h \equiv 0$ and $\mathbb{1}_\mathbb{R} \circ h \equiv 1$.

(c) The first statement follows from $\mathbb{1}_{E\cup F} = \mathbb{1}_E + \mathbb{1}_F$. To show the second statement, by Lemma 5.2, we need only show $\mathcal{R}\big(P(E)\big) \perp \mathcal{R}\big(P(F)\big)$:

Let $u \in \mathcal{R}(P(E))$ and $v \in \mathcal{R}(P(F))$. By (a), we have $u = (\mathbb{1}_E \circ h)u$. Thus, $u(x) = 0$ for all a.e. $x \notin E$. Similarly, $v(x) = 0$ for all a.e. $x \notin F$. Since $E \cap F = 0$, we have $\langle u, v\rangle = 0$.

(d) The condition $P\big((a,b)\big) = 0$, that is, $\mathbb{1}_{(a,b)} \circ h = 0$ a.e., is equivalent to $(a,b) \cap \operatorname{ess\,ran} h = \varnothing$, and $\operatorname{ess\,ran} h = \operatorname{spec} M_h$ by Proposition 3.11.

(e) Let $u \in L^2(X,\mu)$. The condition $(h-\lambda)u = 0$ is equivalent to $u = 0$ a.e. when $h - \lambda \neq 0$. This happens if and only if $u = (\mathbb{1}_{\{\lambda\}} \circ h)u = P\{\lambda\}u$, which means precisely $u \in \mathcal{R}(P\{\lambda\})$ since $P\{\lambda\}$ is an orthogonal projector.

□

By analyzing spectral projectors associated with different Borel sets, we can isolate the effect of different parts of the spectrum. Bounds of these Borel sets enable us to create useful estimates:

Proposition 5.4. Let T and P be as above, and $\lambda \in \mathbb{R}$.

(a) Let $P_\epsilon \coloneqq P(\lambda-\epsilon, \lambda+\epsilon)$. Then for all $v \in \mathcal{R}(P_\epsilon) \subset \mathcal{D}(T)$ we have

$$\|(T-\lambda)v\| \leq \epsilon.$$

(b) Let $v \in P[\lambda,\infty)\mathcal{D}(T)$. Then

$$\langle v, Tv\rangle \geq \lambda\|v\|^2.$$

Proof. Again, we assume $\mathcal{H} = L^2(X,\mu)$ and $T = M_h$, with X, μ, and h as in Theorem 4.6.

(a) Let $u \in L^2(X,\mu)$ and $v \coloneqq P_\epsilon u = (\mathbb{1}_{(\lambda-\epsilon,\lambda+\epsilon)} \circ h)u$. We check first that $v \in \mathcal{D}(T)$:

When $x \notin (\lambda-\epsilon,\lambda+\epsilon)$ we have $v = \mathbb{1}_{(\lambda-\epsilon,\lambda+\epsilon)} \circ h = 0$. Thus, for all $v \neq 0$, there holds $x \in (\lambda-\epsilon,\lambda+\epsilon)$ and then $|h| \leq |\lambda| + \epsilon$. We then have

$$\int_X |hv|^2 \,\mathrm{d}\mu \leq \int_{\{x\in X: v(x)\neq 0\}} \big(|\lambda|+\epsilon\big)^2 |v|^2 \,\mathrm{d}\mu = \big(|\lambda|+\epsilon\big)^2\|v\|^2 < \infty.$$

This gives $v \in \mathcal{D}(M_h)$.

Similarly, for all $v \neq 0$, from $x \in (\lambda-\epsilon,\lambda+\epsilon)$ we know $|h-\lambda| \leq \epsilon$. The estimate then follows:

$$\|(M_h-\lambda)v\|^2 \leq \int_{\{x\in X: v(x)\neq 0\}} \epsilon^2|v|^2 \,\mathrm{d}\mu = \epsilon^2\|v\|^2 \leq \epsilon^2\|u\|^2.$$

(b) Let $u \in \mathcal{D}(M_h)$ and $v \coloneqq P[\lambda,\infty)u = (\mathbb{1}_{[\lambda,\infty)} \circ h)u$. As above, we have $v = \mathbb{1}_{[\lambda,\infty)} \circ h = 0$ for all $x < \lambda$, and thus $h \geq \lambda$ when $v \neq 0$. Therefore,

$$\langle v, M_h v\rangle = \int_X h|v|^2 \,\mathrm{d}\mu \geq \int_{\{x\in X: v(x)\neq 0\}} \lambda|v|^2 \,\mathrm{d}\mu = \lambda\|v\|^2.$$

□

5.3. **Discrete and Essential Spectrum.**

[5]We write $P(a,b) = P((a,b))$ for clarity.

Definition 5.5. Let T be a self-adjoint operator in $\mathcal{H}$. The **discrete spectrum** $\operatorname{spec}_{\mathrm{disc}} T$ is the set of $\lambda \in \operatorname{spec} T$ such that $P(\lambda - \epsilon, \lambda + \epsilon)$ has infinite rank for all $\epsilon > 0$. The **essential spectrum** of T is defined as

$$\operatorname{spec}_{\mathrm{ess}} T \coloneqq \operatorname{spec} T \setminus \operatorname{spec}_{\mathrm{disc}} T.$$

Proposition 5.6. Let T be a self-adjoint operator in $\mathcal{H}$. A complex number λ is in the discrete spectrum of T if and only if λ is an eigenvalue of T of finite multiplicity and an isolated point of $\operatorname{spec} T$ (such eigenvalues are called **discrete eigenvalues**).

Proof. Let $\lambda \in \operatorname{spec}_{\mathrm{disc}} T$. By Proposition 5.3 and definition of the discrete spectrum, there exists some $\epsilon_0 > 0$ such that $N \coloneqq \dim \mathcal{R}\big(P(\lambda - \epsilon_0, \lambda + \epsilon_0)\big)$ is finite and nonzero. Now, note that the function

$$(0, \epsilon_0) \longrightarrow \{1, \cdots, N\}, \quad \epsilon \longmapsto P(\lambda - \epsilon, \lambda + \epsilon).$$

is nondecreasing (recall the partial ordering of self-adjoint operators; see Definition 2.4). There, thus, exists a $\epsilon_1 \in (0, \epsilon_0)$ such that $P_1 \coloneqq P(\lambda - \epsilon, \lambda + \epsilon)$ is constant for all ϵ in the interval $(0, \epsilon_1) \subset (0, \epsilon_0)$. We then have by Proposition 4.7 (b) that

$$P(\lambda - \epsilon, \lambda + \epsilon) \equiv P_1 \longrightarrow P\{\lambda\}$$

in the strong operator topology as $\epsilon \to 0^+$. This gives $P_1 = P\{\lambda\}$. Thus $\mathcal{R}(P\{\lambda\}) = \mathcal{R}(P_1) > 0$ and $\lambda \in \operatorname{spec}_{\mathrm{p}} T$. Next, choose a fixed $\epsilon_2 \in (0, \epsilon_1)$ and note that

$$P\{\lambda\} = P_1 = P(\lambda - \epsilon_2, \lambda + \epsilon_2).$$

Thus the spectrum is empty in $(\lambda - \epsilon_2, \lambda) \cup (\lambda, \lambda + \epsilon_2)$. That is, λ is an isolated point of the spectrum.

Conversely, let λ be an eigenvalue of finite multiplicity and an isolated point of the spectrum. Then there exists $\epsilon > 0$ such that the spectrum is empty in $(\lambda - \epsilon, \lambda) \cup (\lambda, \lambda + \epsilon)$. Thus

$$\dim \mathcal{R}\big(P(\lambda - \epsilon, \lambda + \epsilon)\big) = \dim \mathcal{R}\big(P\{\lambda\}\big) < \infty.$$

□

6. The Min-Max Theorem

The min-max theorem gives a variational characterization to the discrete eigenvalues below the essential spectrum, relating them to the **Rayleigh quotient** $\frac{\langle u, Tu\rangle}{\langle u, u\rangle}$, which one may think of as a measure of how much T stretches u in the direction of u itself. Given the constraint $u \in W$, where W is a one dimensional subspace, we can then expect the Rayleigh quotient to be maximized when u is in the eigenspace contained in W that is associated with the largest eigenvalue, and taking the minimum over all possible constraints W gives the lowest eigenvalue. Larger eigenvalues can then be obtained by increasing the dimension of W to filter out smaller ones. This provides the intuition for the following:

Theorem 6.1 (Min-Max Theorem). Let T be a self-adjoint operator whose spectrum is bounded below. Let Λ_k denote the set of subspaces of $\mathcal{D}(T)$ of dimension k and define

$$\alpha_k := \inf_{W\in\Lambda_k} \sup_{u\in W\setminus\{0\}} \frac{\langle u, Tu\rangle}{\langle u,u\rangle}$$

for $k \in \mathbb{N}$.

Then the sequence (α_k) is non-decreasing, and for each k, one and only one of the following holds:

(a) α_k is the kth eigenvalue (arranged in increasing order and counted with multiplicity) and there are at least k eigenvalues below the essential spectrum.
(b) $\alpha_k = \inf \operatorname{spec}_{\text{ess}} T$ and there are at most $k-1$ eigenvalues below the essential spectrum.

Proof. For any n, we have (think adding and relaxing constraints)

$$\inf_{W\in\Lambda_{n+1}} \sup_{u\in W\setminus\{0\}} \frac{\langle u, Tu\rangle}{\langle u,u\rangle} \geq \inf_{W\in\Lambda_{n+1}} \inf_{\substack{V\in\Lambda_n\\ V\subset W}} \sup_{u\in V\setminus\{0\}} \frac{\langle u, Tu\rangle}{\langle u,u\rangle} \geq \inf_{V\in\Lambda_n} \sup_{u\in V\setminus\{0\}} \frac{\langle u, Tu\rangle}{\langle u,u\rangle}.$$

So the sequence (α_n) is non-decreasing.

Next, let $\Sigma := \inf \operatorname{spec}_{\text{ess}} T$ (set $\Sigma := \infty$ if $\operatorname{spec}_{\text{ess}} T$ is empty). Denote the kth eigenvalue (counting multiplicities) in $(-\infty, \Sigma)$ by E_k and the associated eigenvector ϕ_k, where $k \in \{1, \cdots, N\}$ with $N \in \mathbb{N} \cup \{\infty\}$. By Theorem 3.9, we may assume that the set of ϕ_k form an orthonormal family. Let $V_n := \operatorname{span}\{\phi_1, \cdots, \phi_n\} \in \Lambda_n$. For any $u \in V_n$, there holds

$$\langle u, Tu\rangle = \left\langle \sum_{j=1}^n \langle \phi_j, u\rangle \phi_j, \sum_{j=1}^n \langle \phi_j, u\rangle E_j \phi_j \right\rangle = \sum_{j=1}^n E_j \left|\langle \phi_j, u\rangle\right|^2 \leq E_n \sum_{j=1}^n \left|\langle \phi_j, u\rangle\right|^2 = E_n \|u\|^2.$$

Thus, $\alpha_n \leq E_n$.

To prove the opposite inequality, let $W \in \Lambda_n$ and P the orthogonal projector on the subspace $\operatorname{span}\{E_1, \cdots, E_{n-1}\}$. From Proposition 5.3, we know that

$$\mathcal{R}\big(P(-\infty, E_n)\big) \subset \mathcal{R}(P) \leq n-1 < \dim W.$$

Thus we can always find a $v \in W \setminus \{0\}$ such that $Pv = 0$. Since P is an orthogonal projector, we have $v \in \mathcal{R}\big(P(-\infty, E_n)\big)$ and thus $v \in \mathcal{R}\big(P[E_n, \infty)\big)$. Proposition 5.4 then gives

$$\sup_{u\in W\setminus\{0\}} \frac{\langle u, Tu\rangle}{\langle u,u\rangle} \geq \frac{\langle v, Tv\rangle}{\langle v,v\rangle} \geq E_n.$$

Thus we have $\alpha_n = E_n$.

Now, if $N = \infty$, the proof is complete. Otherwise, it remains to show $\alpha_n = \Sigma$ for all $n \geq N+1$:

From Proposition 5.3, we have $P(-\infty, \Sigma) = \sum P\{E'_j\}$, where E_j are distinct eigenvalues in $(-\infty, \Sigma)$. Since $\mathcal{R}\big(P\{E'_j\}\big)$ are distinct, mutually orthogonal eigenspaces, we have

$$\dim \mathcal{R}\big(P(-\infty, \Sigma)\big) = \sum \dim \mathcal{R}\big(P\{E'_j\}\big) = N.$$

Let $n \geq N+1$ and $W \in \Lambda_n$. Then there exists a $v \in W \setminus \{0\}$ such that $v \perp \mathcal{R}\big(P(-\infty, \Sigma)\big)$ and a similar argument as above shows that $\alpha_n \geq \Sigma$.

To prove the opposite inequality, consider $W := \mathcal{R}\big(P(\Sigma-\epsilon, \Sigma+\epsilon)\big)$ with $\epsilon > 0$. Since $\Sigma = \inf \operatorname{spec}_{\text{ess}} T$, this is an infinite dimensional subspace. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of W and set $W_n := \operatorname{span}\{e_1, \cdots, e_n\}$. We then have, by Proposition 5.4 (a),

$$\langle u, Tu\rangle - \Sigma\|u\|^2 \leq |\langle u, (T-\Sigma)u\rangle| \leq \|u\|\|(T-\Sigma)u\| \leq \epsilon\|u\|^2$$

for any $u \in W_n$. This gives $\langle u, Tu\rangle \leq (\Sigma+\epsilon)\|u\|^2$ and thus

$$\alpha_n \leq \sup_{u\in W_n, u\neq 0} \frac{\langle u, Tu\rangle}{\langle u, u\rangle} \leq \Sigma + \epsilon.$$

We complete the proof by sending ϵ to 0. ◻

Acknowledgments

I am grateful to Andreas Stavrou for reigniting my passion for mathematics in MATH160s, and to Professor Peter May for allowing me to change my decision about writing a paper. I am deeply indebted to my mentor, DeVon Ingram, whose patience and guidance helped me realize that I am sometimes able to grasp mathematical concepts that intimidated me. Lastly, my sincere thanks to Alexandra Zhou, Hanlei Wen, Ryan Dai, Will Comess, and Joshua Johnson for unwavering support and shared laughter.

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