Trigonometry

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§1 The Trigonometric ratios

The trigonometric ratios are defined for angles. In this handout, we will limit ourselves to acute angles. There are three main trigonometric ratios and they have their corresponding reciprocal ratios. Please note: In the following example, we have labelled the sides relative to $\angle C$ If we calculate the t-ratios for $\angle B$, AC, for example, will not be the Adjacent side as it is, on the contrary Opposite to $\angle B$. This applies to all triangles. Looking at Figure 1, we can define the ratios to be

$$\sin C = \frac{BA}{BC} = \frac{OppositeSide}{Hypotenuse}$$

$$\cos C = \frac{AC}{BC} = \frac{AdjacentSide}{Hypotenuse}$$

$$\tan C = \frac{\sin C}{\cos C} = \frac{\frac{BA}{BC}}{\frac{AC}{BC}} = \frac{BA}{AC} = \frac{OppositeSide}{AdjacentSide}$$

These are the three main ratios. The other three are derived by taking the reciprocal of these three.

$$\csc C = \frac{1}{\sin C}$$

$$\sec C = \frac{1}{\cos C}$$

$$\cot C = \frac{1}{\tan C}$$

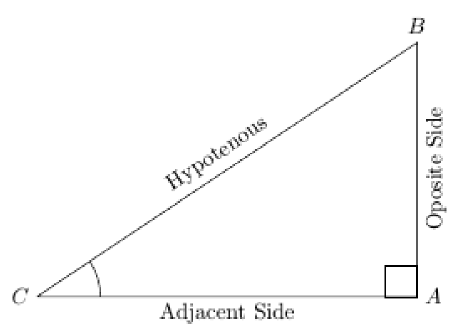

Figure 1: A triangle $\triangle ABC$ right-angled at A[1]

An important observation is:

$$\cot\theta \times \tan\theta = \sin\theta \times \csc\theta = \cos\theta \times \sec\theta = 1$$

§2 Some Trigonometric identities

There are many trigonometric identities, however, we will discuss only 3. The following identities can be easily derived using the Pythagorean theorem. We will prove only one identity, however, the other identities have very very similar proofs.

§2.1 Identity involving $\sin$ and $\cos$

Consider the triangle in Figure 2. By the Pythagorean theorem, we have

$$a^2 + b^2 = c^2$$

Now we divide the whole equation by c^2 so we will get

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

The RHS simplifies to become 1 but the LHS has some familiar fractions. Notice that $\sin\alpha = \frac{a}{c}$ so $\frac{a^2}{c^2} = \sin^2\alpha$ similarly, $\frac{b^2}{c^2} = \cos^2\alpha$. If we substitute these values back into the equation, we get

$$\sin^2\alpha + cos^2\alpha = 1$$

And we have arrived at our first identity. To prove the other identities, start out with the Pythagorean theorem and divide by a leg (other than the hypotenuse, as we have used the hypotenuse to prove the above identity) each time. We play around with the identity above to get $\cos^2\alpha = 1 - \sin^2\alpha$ and $\sin^2\alpha = 1 - \cos^2\alpha$

§2.2 Identity involving $\tan$ and $\sec$

$$1 + \tan^2\alpha = \sec^2\alpha$$

Rearranging, we get $\sec^2\alpha - \tan^2\alpha = 1$ Now using the identity $[a^2 - b^2 = (a+b)(a-b)]$, we can rewrite the trig identity as

$$(\sec\alpha + \tan\alpha)(\sec\alpha - \tan\alpha) = 1$$

dividing both sides by $(\sec\alpha + \tan\alpha)$ we get

$$\sec\alpha - \tan\alpha = \frac{1}{\sec\alpha + \tan\alpha}$$

[1]Source: https://tex.stackexchange.com/questions/294795/problem-with-right-angle-triangle/294798

Rearranging the equation above gives us

$$\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$$

These two 'sub'-formulae are useful in problems where one is asked to prove a given trigonometric expression

§2.3 Identity involving csc and cot

$$1 + \cot^2\theta = \csc^2\theta$$

We can use the same methods we applied to the second identity to get

$$\csc^2\theta - \cot^2\theta = 1$$

$$\csc\theta + \cot\theta = \frac{1}{\csc\theta - \cot\theta}$$

$$\csc\theta - \cot\theta = \frac{1}{\csc\theta + \cot\theta}$$

§3 Complementary angles

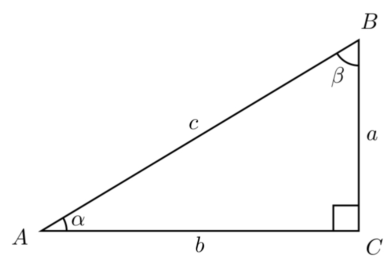

Figure 2: $\triangle ABC$ right angled at C[2]

Two angles are said to be complementary if their sum is 90°. We can say that α and β (looking at Figure 2) are complementary angles as $\alpha + \beta = 90°$(By Angle Sum Property of Triangles). We see that if we compute $\sin\alpha$ and $\cos\beta$ we get

$$\sin\alpha = \frac{a}{c} = \cos\beta$$

Similarly, we get

$$\tan\alpha = \frac{a}{b} = \cot\beta$$

and

$$\sec\alpha = \frac{c}{b} = \csc\beta$$

We can frame general equations after this observation. For some angle θ, the following equations hold

$$\sin(90 - \theta) = \cos\theta$$

$$\cos(90 - \theta) = \sin\theta$$

$$\sec(90-\theta) = \csc\theta$$
$$\csc(90-\theta) = \sec\theta$$

$$\tan(90-\theta) = \cot\theta$$
$$\cot(90-\theta) = \tan\theta$$

§4 Values of T-ratios of some specific angles

ratio \ angle θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
$\text{cosec}\,\theta$	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
$\cot\theta$	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Figure 3: The T-ratios for $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$[3]

§5 Basic Applications

Trigonometry can be used to solve real-life problems. For example, say, we have a straight, rigid ladder which makes an angle 35° with the ground at a point 12m away from the wall. What is the length of the ladder? Firstly, we consider the fact that the wall and the ground are perpendicular, so any line segment connecting the ground and the wall will form a right-angled triangle with the wall and ground. In this problem, the ladder acts as the line segment that forms a right-angled triangle. A diagram should help with this. Let us name the triangle. Let A be the meeting point of the wall and the ground. Obviously

[2]Source:https://tex.stackexchange.com/questions/346613/how-to-label-angles-in-a-right-triangle
[3]Source:https://www.onlinemath4all.com/trigonometric-ratios-of-special-angles.html

$\angle A = 90°$. Let the endpoints of the ladder be B (where it meets the wall) and C (where it meets the ground). It is given that $\angle C = 35°$. Now we need to find a ratio which relates BC (the length of the ladder) and AC (distance from wall). This is

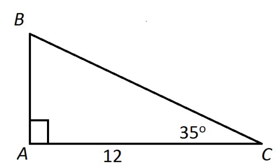

Figure 4: The triangle formed by the ladder, ground and wall.[4]

$$\cos 35° = \frac{AC}{BC} = \frac{12}{BC}$$

After rearranging, we get

$$BC = 12 \sec 35°$$

The value of $\sec 35°$ can be derived through calculations beyond the scope of this handout, however, one can easily find its value with a simple search! $\sec 35° \approx 1.74$

$$\boxed{12 \sec 35° = 20.92 = BC}$$

The same principles are used to solve other problems.

§6 Examples

Prove that

$$\sqrt{\sec^2\theta + \csc^2\theta} = \tan\theta + \cot\theta$$

Solution: We use the trig identities involving all 4 of the ratios in this problem (refer to Section 2). Namely, $\sec^2\theta = 1 + \tan^2\theta$ and $\csc^2\theta = 1 + \cot^2\theta$. We substitute this into the LHS to get

$$LHS = \sqrt{tan^2\theta + 1 + \cot^2\theta + 1}$$

$$LHS = \sqrt{tan^2\theta + cot^2\theta + 2\tan\theta\cot\theta}$$

We replaced 2 with $2\tan\theta\cot\theta$ as $\tan\theta\cot\theta = 1$ so there is no change in the value of the expression.

$$LHS = \sqrt{(\tan\theta + \cot\theta)^2}$$

$$LHS = \tan\theta + \cot\theta = RHS$$

If $\alpha + \beta = 90°$, Evaluate

$$\tan\alpha\csc\alpha\sin^2\beta\cos\beta\cos\alpha\tan\beta\sec^3\alpha$$

[4]Source:https://www.varsitytutors.com/hiset-math-help/tangent

Solution: The thing about these types of 'complex-looking' problems is that, most of the time, you will get a numerical answer independent of any trigonometric ratio. Keeping that in mind, we notice complementary angles!! We will use the identities mentioned in Section 3 to solve this problem. We will try to convert all the trigonometric ratios in terms of t-ratios of one angle. For this problem, let us convert everything into a ratio involving α. So we will get

$$\tan\alpha \csc\alpha \cos^2\alpha \sin\alpha \cos\alpha \cot\alpha \sec^3\alpha$$

If we rearrange a bit, we get

$$\tan\alpha \cot\alpha \csc\alpha \sin\alpha \cos^3\alpha \sec^3\alpha$$

Notice how we can simplify it to 1 as reciprocal ratios are being multiplied here. So the answer is **1**

Prove that

$$\cot^2\theta\left[\frac{\sec\theta-1}{\sin\theta+1}\right]+\tan^2\theta\csc^2\theta\left[\frac{\sin\theta-1}{\sec\theta+1}\right]=0$$

Solution: We split this expression into two halves, solve them separately, then combine them. For the first half, we see $\cot^2\theta$ and $\sec\theta-1$, so we try to generate $\tan^2\theta$ to cancel $\cot^2\theta$ as $\cot^2\theta\times\tan^2\theta=1$. And we can! Since $\sec^2\theta-1=\tan^2\theta$ (check Section 2) we can write $\sec\theta-1=\frac{\tan^2\theta}{\sec\theta+1}$. Now, if we substitute this value into the expression, we get

$$\cot^2\theta\left[\frac{\frac{\tan^2\theta}{\sec\theta+1}}{\sin\theta+1}\right]=\cot^2\theta\left[\frac{\tan^2\theta}{(1+\sec\theta)(\sin\theta+1)}\right]$$

Since $\cot^2\theta\times\tan^2\theta=1$, we can write it as

$$\frac{1}{(1+\sec\theta)(\sin\theta+1)}$$

Any further simplification would be a waste of time. So we move on to the second half

$$\tan^2\theta\csc^2\theta\left[\frac{\sin\theta-1}{\sec\theta+1}\right]$$

Since $\frac{1}{\sec\theta+1}$ is common in both halves, we will leave it undisturbed (as we can factor it out later on). So we focus on the other terms. We represent them in terms of sin and cos to help us simplify.

$$\frac{\sin^2\theta}{\cos^2\theta}\times\frac{1}{\sin^2\theta}\left[\frac{\sin\theta-1}{\sec\theta+1}\right]=\frac{1}{\cos^2\theta}\left[\frac{\sin\theta-1}{\sec\theta+1}\right]$$

Now we add both the halves to get

$$\frac{1}{(1+\sec\theta)(\sin\theta+1)}+\frac{1}{\cos^2\theta}\left[\frac{\sin\theta-1}{\sec\theta+1}\right]$$

Factoring the common term, we get

$$\frac{1}{\sec\theta+1}\left[\frac{1}{\sin\theta+1}+\frac{\sin\theta-1}{\cos^2\theta}\right]$$

Adding the fraction inside the square brackets, it becomes,

$$\frac{1}{\sec\theta+1}\left[\frac{\cos^2\theta+\sin^2\theta-1}{(\sin\theta+1)\cos^2\theta}\right]$$

Since $\sin^2\theta+\cos^2\theta=1$, we get

$$LHS=\frac{1}{\sec\theta+1}\left[\frac{1-1}{(\sin\theta+1)\cos^2\theta}\right]=0=RHS$$

§7 Problems

§7.1 Prove the following

§7.1.1

$$1+\tan^2\theta=\sec^2$$

§7.1.2

$$\csc^2-\cot^2=1$$

§7.2 Find the values asked

§7.2.1

$$\cos\theta\sin(90-\theta)+\sin\theta\cos(90-\theta)$$

§7.2.2 If $x=a\sin\theta$ and $y=b\cos\theta$, find

$$b^2x^2+a^2y^2+1$$

§7.2.3 If $\tan\theta=\sqrt{3}$, find the value of the value of

$$\frac{(\sin\theta+\cos\theta)^2-2\sin\theta\cos\theta}{\sec^2\theta-1}$$

§7.2.4 If $5x=\sec\theta$ and $\frac{5}{x}=\tan\theta$, then find the value of

$$5\left(x^2-\frac{1}{x^2}\right)$$

§7.2.5 Find the value of

$$\frac{\cos^2\theta}{\sin^2\theta}-\tan^2\theta\csc^4\theta\cos^2\theta$$

§7.2.6 If $\tan\theta+\cot\theta=2$ Find

$$\tan^{n+m}\theta+\cot^{p+l}$$

where $(n+m)$ and $(p+l)$ are real numbers

§7.2.7 If $3\cos\theta=2\sin\theta$ Find

$$\tan^2\theta+cot^2\theta+\sec^2\theta-1$$