We will be studying, on the one hand, more and more general ways of *generating* languages, beginning with ways similar to the ones used in the definition of L_1, and on the other hand, corresponding methods of greater and greater sophistication for *recognizing* strings in these languages. In the second approach, it will be useful to think of the algorithm for recognizing the language as being embodied in an *abstract machine*, and a precise description of the machine will effectively give us a precise way of specifying which strings are in the language. Initially these abstract machines will be fairly primitive, since it turns out that languages like L_1 can be recognized easily. A language like L_2 will require a more powerful type of abstract machine to recognize it, as well as a more general method of generating it. Before we are through, we will study machines equivalent in power to the most sophisticated computer.

EXERCISES

1.1. Describe each of these infinite sets precisely, using a formula that does not involve "...". if you wish, you can use $\mathcal{N}$, $\mathcal{R}$, $\mathcal{Z}$, and other sets discussed in the chapter.

a. $\{0, -1, 2, -3, 4, -5, \ldots\}$

b. $\{1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, 3/16, 5/16, 7/16, \ldots\}$

c. $\{10, 1100, 111000, 11110000, \ldots\}$ (a subset of $\{0, 1\}^*$)

d. $\{\{0\}, \{1\}, \{2\}, \ldots\}$

e. $\{\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 3\}, \ldots\}$

f. $\{\{0\}, \{0, 1\}, \{0, 1, 2, 3\}, \{0, 1, 2, 3, 4, 5, 6, 7\}, \{0, 1, \ldots, 15\}, \{0, 1, 2, \ldots, 31\}, \ldots\}$

1.2. Label each of the eight regions in Figure 1.2, using A, B, C, and appropriate set operations.

1.3. Use Venn diagrams to verify each of the set identities (1.1)–(1.19).

1.4. Assume that A and B are sets. In each case, find a simpler expression representing the given set. The easiest way is probably to use Venn diagrams, but also practice manipulating the formulas using the set identities (1.1)–(1.19).

a. $A - (A - B)$

b. $A - (A \cap B)$

c. $(A \cup B) - A$

d. $(A - B) \cup (B - A) \cup (A \cap B)$

e. $(A' \cap B')'$

f. $(A' \cup B')'$

g. $A \cup (B \cap (A - (B - A)))$

h. $A' \cup (B - (A \cup (B' - A)))$

1.5. Show using Venn diagrams that the symmetric difference operation $\oplus$ satisfies the associative property $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

1.6. In each case, find a simpler statement (one not involving the symmetric difference operation) equivalent to the given one. Assume in each case that

A and B are subsets of U.

a. $A \oplus B = A$

b. $A \oplus B = A - B$

c. $A \oplus B = A \cup B$

d. $A \oplus B = A \cap B$

e. $A \oplus B = A'$

1.7. In each case, find an expression for the indicated set, involving A, B, C, and the three operations $\cup$, $\cap$, and $'$.

a. $\{x | x \in A \text{ or } x \in B \text{ but not both}\}$

b. $\{x | x$ is an element of exactly one of the three sets A, B, and $C\}$

c. $\{x | x$ is an element of at most one of the three sets A, B, and $C\}$

d. $\{x | x$ is an element of exactly two of the three sets A, B, and $C\}$

e. $\{x | x$ is an element of at least one and at most two of the three sets A, B, and $C\}$

1.8. For each integer n, denote by C_n the set of all real numbers less than n, and for each positive number n let D_n be the set of all real numbers less than $1/n$. Express each of the following sets in a simpler form not involving unions or intersections. (For example, the answer to (a) is C_{10}.) Since ∞ is not a number, the expressions C_∞ and D_∞ do not make sense and should not appear in your answer.

a. $\bigcup_{n=1}^{10} C_n$

b. $\bigcup_{n=1}^{10} D_n$

c. $\bigcap_{n=1}^{10} C_n$

d. $\bigcap_{n=1}^{10} D_n$

e. $\bigcup_{n=1}^{\infty} C_n$

f. $\bigcup_{n=1}^{\infty} D_n$

g. $\bigcap_{n=1}^{\infty} C_n$

h. $\bigcap_{n=1}^{\infty} D_n$

i. $\bigcup_{n=-\infty}^{\infty} C_n$

j. $\bigcap_{n=-\infty}^{\infty} C_n$

1.9. One might think that an empty set of real numbers and an empty set of sets are two different objects. Show that according to our definitions, there is only one set containing no elements.

1.10. List the elements of $2^{2^{\{0,1\}}}$.

1.11. Denote by p, q, and r the statements $a = 1$, $b = 0$, and $c = 3$, respectively. Write each of the following statements symbolically, using p, q, r, $\wedge$, $\vee$, $\neg$, and $\rightarrow$.

a. Either $a = 1$ or $b \neq 0$.

b. $b = 0$ but neither $a = 1$ nor $c = 3$. (Note: in logic, unlike English, "but" and "and" are interchangeable.)

c. It is not the case that both $a \neq 1$ and $b = 0$.
d. If $a \neq 1$ then $c = 3$, but otherwise $c \neq 3$.
e. $b = 0$ only if either $a = 1$ or $c = 3$.
f. If it is not the case that either $a = 1$ or $b = 0$, then only if $c = 3$ is $a \neq 1$.

1.12. Which of these statements are true?
a. If $1 + 1 = 2$, then $2 + 2 = 4$.
b. $1 + 1 = 3$ only if $2 + 2 = 6$.
c. ($1 = 2$ and $1 = 3$) if and only if $1 = 3$.
d. If $1 + 1 = 3$ then $1 + 2 = 3$.
e. If $1 = 2$, then $2 = 3$ and $2 = 4$.
f. Only if $3 - 1 = 2$ is $1 - 2 = 0$.

1.13. In each case, construct a truth table for the statement and use the result to find a simpler statement that is logically equivalent.
a. $(p \to q) \land (p \to \neg q)$
b. $p \lor (p \to q)$
c. $p \land (p \to q)$
d. $(p \to q) \land (\neg p \to q)$
e. $p \leftrightarrow (p \leftrightarrow q)$
f. $q \land (p \to q)$

1.14. A principle of classical logic is *modus ponens*, which asserts that the proposition $(p \land (p \to q)) \to q$ is a tautology, or that $p \land (p \to q)$ logically implies q. Show that this result requires the truth table for the conditional statement $r \to s$ to be defined exactly as we defined it in the two cases where r is false.

1.15. Suppose m_1 and m_2 are integers representing months ($1 \leq m_i \leq 12$), and d_1 and d_2 are integers representing days (d_i is at least 1 and no larger than the number of days in month m_i). For each i, the pair (m_i, d_i) can be thought of as representing a date. We wish to write a logical proposition involving the four integers that says (m_1, d_1) comes before (m_2, d_2) in the calendar.
a. Find such a proposition that is a disjunction of two propositions.
b. Find such a proposition that is a conjunction of two propositions.

1.16. Show that the statements $p \lor q \lor r \lor s$ and $(\neg p \land \neg q \land \neg r) \to s$ are equivalent.

1.17. In each case, say whether the statement is a tautology, a contradiction, or neither.
a. $p \lor \neg(p \to p)$
b. $p \land \neg(p \to p)$
c. $p \to \neg p$
d. $(p \to \neg p) \lor (\neg p \to p)$
e. $(p \to \neg p) \land (\neg p \to p)$

1.18. Consider the statement "Everybody loves somebody sometime." In order to express this precisely, let $L(x, y, t)$ be a proposition involving the three free variables x, y, and t that expresses the fact that x loves y at time t. (Here x and y are humans, and t is a time.) Using this notation, express the original statement using quantifiers.

1.19. Let $F(x, t)$ be the proposition: You can fool person x at time t. Using this notation, write a quantified statement to formalize Abraham Lincoln's statement: "You can fool all the people some of the time, and you can fool some of the people all the time, but you can not fool all the people all of the time." Give at least two different answers (not equivalent), representing different possible interpretations of the statement.

1.20. In each case below, say whether the given statement is true for the universe $(0, 1) = \{x \in \mathcal{R} \mid 0 < x < 1\}$, and say whether it is true for the universe $[0, 1] = \{x \in \mathcal{R} \mid 0 \le x \le 1\}$.

a. $\forall x(\exists y(x > y))$
b. $\forall x(\exists y(x \ge y))$
c. $\exists y(\forall x(x > y))$
d. $\exists y(\forall x(x \ge y))$

1.21. Suppose A and B are finite sets, A has n elements, and $f : A \to B$.

a. If f is one-to-one, what can you say about the number of elements of B?
b. If f is onto, what can you say about the number of elements of B?

1.22. In this problem, as usual, $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^+$ the set of nonnegative real numbers, $\mathcal{N}$ the set of natural numbers (nonnegative integers), and $2^{\mathcal{R}}$ the set of subsets of $\mathcal{R}$. $[0, 1]$ denotes the set $\{x \in \mathcal{R} \mid 0 \le x \le 1\}$. In each case, say whether the indicated function is one-to-one, and say what its range is.

a. $f_a : \mathcal{R}^+ \to \mathcal{R}^+$ defined by $f_a(x) = x + a$ (where a is some fixed element of $\mathcal{R}^+$)
b. $d : \mathcal{R}^+ \to \mathcal{R}^+$ defined by $d(x) = 2x$
c. $t : \mathcal{N} \to \mathcal{N}$ defined by $t(x) = 2x$
d. $g : \mathcal{R}^+ \to \mathcal{N}$ defined by $g(x) = \lfloor x \rfloor$ (the largest integer $\le x$)
e. $p : \mathcal{R}^+ \to \mathcal{R}^+$ defined by $p(x) = x + \lfloor x \rfloor$
f. $i : 2^{\mathcal{R}} \to 2^{\mathcal{R}}$ defined by $i(A) = A \cap [0, 1]$
g. $u : 2^{\mathcal{R}} \to 2^{\mathcal{R}}$ defined by $u(A) = A \cup [0, 1]$
h. $m : \mathcal{R}^+ \to \mathcal{R}^+$ defined by $m(x) = \min(x, 2)$
i. $M : \mathcal{R}^+ \to \mathcal{R}^+$ defined by $M(x) = \max(x, 2)$
j. $s : \mathcal{R}^+ \to \mathcal{R}^+$ defined by $s(x) = \min(x, 2) + \max(x, 2)$

1.23. Suppose A and B are sets, $f : A \to B$, and $g : B \to A$. If $f(g(y)) = y$ for every $y \in B$, then f is a _____ function and g is a _____ function. Give reasons for your answers.

1.24. Let $A = \{2, 3, 4, 6, 7, 12, 18\}$ and $B = \{7, 8, 9, 10\}$.

a. Define $f : A \to B$ as follows: $f(2) = 7$; $f(3) = 9$; $f(4) = 8$; $f(6) = f(7) = 10$; $f(12) = 9$; $f(18) = 7$. Find a function $g : B \to A$ so that for every $y \in B$, $f(g(y)) = y$. Is there more than one such g?

b. Define $g : B \to A$ as follows: $g(7) = 6$; $g(8) = 7$; $g(9) = 2$; $g(10) = 18$. Find a function $f : A \to B$ so that for every $y \in B$, $f(g(y)) = y$. Is there more than one such f?

1.25. Let f_a, d, t, g, i, and u be the functions defined in Exercise 1.22. In each case, find a formula for the indicated function, and simplify it as much as possible.

a. $g \circ d$
b. $t \circ g$
c. $t \circ t$
d. $d \circ f_a$
e. $f_a \circ d$
f. $g \circ f_a$
g. $u \circ i$
h. $i \circ u$

1.26. In each case, show that f is a bijection and find a formula for f^{-1}.

a. $f : \mathcal{R} \to \mathcal{R}$ defined by $f(x) = x$
b. $f : \mathcal{R}^+ \to \{x \in \mathcal{R} \mid 0 < x \leq 1\}$ defined by $f(x) = 1/(1 + x)$
c. $f : \mathcal{R} \times \mathcal{R} \to \mathcal{R} \times \mathcal{R}$ defined by $f(x, y) = (x + y, x - y)$

1.27. Show that if $f : A \to B$ is a bijection, then f^{-1} is also a bijection, and $(f^{-1})^{-1} = f$.

1.28. In each case, a relation on the set $\{1, 2, 3\}$ is given. Of the three properties, reflexivity, symmetry, and transitivity, determine which ones the relation has. Give reasons.

a. $R = \{(1, 3), (3, 1), (2, 2)\}$
b. $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$
c. $R = \emptyset$

1.29. For each of the eight lines of the table below, construct a relation on $\{1, 2, 3\}$ that fits the description.

reflexive	symmetric	transitive
true	true	true
true	true	false
true	false	true
true	false	false
false	true	true
false	true	false
false	false	true
false	false	false

1.30. Three relations are given on the set of all nonempty subsets of $\mathcal{N}$. In each case, say whether the relation is reflexive, whether it is symmetric, and whether it is transitive.

a. R is defined by: ARB if and only if $A \subseteq B$.

b. R is defined by: ARB if and only if $A \cap B \neq \emptyset$.

c. R is defined by: ARB if and only if $1 \in A \cap B$.

1.31. How would your answer to Exercise 1.30 change if in each case R were the indicated relation on the set of *all* subsets of $\mathcal{N}$?

1.32. Let R be a relation on a set S. Write three quantified statements (the domain being S in each case), which say, respectively, that R is not reflexive, R is not symmetric, and R is not transitive.

1.33. In each case, a set A is specified, and a relation R is defined on it. Show that R is an equivalence relation.

a. $A = 2^S$, for some set S. An element X of A is related via R to an element Y if there is a bijection from X to Y.

b. A is an arbitrary set, and it is assumed that for some other set B, $f : A \to B$ is a function. For $x, y \in A$, xRy if $f(x) = f(y)$.

c. Suppose U is the set $\{1, 2, \ldots, 10\}$. A is the set of all statements over the universe U—that is, statements involving at most one free variable, which can have as its value an element of U. (Included in A are the statements *false* and *true*.) For two elements r and s of A, rRs if $r \Leftrightarrow s$.

d. A is the set $\mathcal{R}$, and for $x, y \in A$, xRy if $x - y$ is an integer.

e. A is the set of all infinite sequences $x = x_0x_1x_2\cdots$ of 0's and 1's. For two such sequences x and y, xRy if there exists an integer k so that $x_i = y_i$ for every $i \geq k$.

1.34. In Exercise 1.33*a*, if S has exactly 10 elements, how many equivalence classes are there for the relation R? Describe them. What are the elements of the equivalence class containing $\{a, b\}$ (assuming a and b are two elements of S)?

1.35. In Exercise 1.33*b*, if A and B are both the set of real numbers, and f is the function defined by $f(x) = x^2$, describe the equivalence classes.

1.36. In Exercise 1.33*b*, suppose A has n elements and B has m elements.

a. If f is one-to-one (and not necessarily onto), how many equivalence classes are there?

b. If f is onto (and not necessarily one-to-one), how many equivalence classes are there?

1.37. In Exercise 1.33*c*, how many equivalence classes are there? List some elements in the equivalence class containing the statement $(x = 3) \vee (x = 7)$. List some elements in the equivalence class containing the statement *true*, and some in the equivalence class containing *false*.

1.38. Let L be a language. It is clear from the definitions that $L^+ \subseteq L^*$. Under what circumstances are they equal?

1.39. a. Find a language L over $\{a, b\}$ that is neither $\{\Lambda\}$ nor $\{a, b\}^*$ and satisfies $L = L^*$.

b. Find an infinite language L over $\{a, b\}$ for which $L \neq L^*$.

1.40. In each case, give an example of languages L_1 and L_2 satisfying $L_1L_2 = L_2L_1$ as well as the additional conditions indicated.

a. Neither language is a subset of the other, and neither language is $\{\Lambda\}$.

b. L_1 is a proper nonempty subset of L_2 (*proper* means $L_1 \neq L_2$), and $L_1 \neq \{\Lambda\}$.

1.41. Let L_1 and L_2 be subsets of $\{0, 1\}^*$, and consider the two languages $L_1^* \cup L_2^*$ and $(L_1 \cup L_2)^*$.

a. Which of the two is always a subset of the other? Why? Give an example (i.e., say what L_1 and L_2 are) so that the opposite inclusion does not hold.

b. If $L_1^* \subseteq L_2^*$, then $(L_1 \cup L_2)^* = L_2^* = L_1^* \cup L_2^*$. Similarly if $L_2^* \subseteq L_1^*$. Give an example of languages L_1 and L_2 for which $L_1^* \not\subseteq L_2^*$, $L_2^* \not\subseteq L_1^*$, and $L_1^* \cup L_2^* = (L_1 \cup L_2)^*$.

1.42. Show that if A and B are languages over Σ and $A \subseteq B$, then $A^* \subseteq B^*$.

1.43. Show that for any language L, $L^* = (L^*)^* = (L^+)^* = (L^*)^+$.

1.44. For a finite language L, denote by $|L|$ the number of elements of L. (For example, $|\{\Lambda, a, ababb\}| = 3$.) Is it always true that for finite languages A and B, $|AB| = |A||B|$? Either prove the equality or find a counterexample.

1.45. List some elements of $\{a, ab\}^*$. Can you describe a simple way to recognize elements of this language? In other words, try to find a proposition $p(x)$ so that

a. $\{a, ab\}^*$ is precisely the set of strings x satisfying $p(x)$; and

b. for any x, there is a simple procedure to test whether x satisfies $p(x)$.

1.46. a. Consider the language L of all strings of a's and b's that do not end with b and do not contain the substring bb. Find a finite language S so that $L = S^*$.

b. Show that there is no language S so that the language of all strings of a's and b's that do not contain the substring bb is equal to S^*.

1.47. Let L_1, L_2, and L_3 be languages over some alphabet Σ. In each part below, two languages are given. Say what the relationship is between them. (Are they always equal? If not, is one always a subset of the other?) Give reasons for your answers, including counterexamples if appropriate.

a. $L_1(L_2 \cap L_3)$, $L_1L_2 \cap L_1L_3$

b. $L_1^* \cap L_2^*$, $(L_1 \cap L_2)^*$

c. $L_1^*L_2^*$, $(L_1L_2)^*$

d. $L_1^*(L_2L_1^*)^*$, $(L_1^*L_2)^*L_1^*$

MORE CHALLENGING PROBLEMS

1.48. Show the associative property of symmetric difference (see Exercise 1.5) without using Venn diagrams.

1.49. Suppose that for a finite set S, $|S|$ denotes the number of elements of S, and let A, B, C, and D be finite sets.

a. Show that $|A \cup B| = |A| + |B| - |A \cap B|$.

b. Show that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. (An element x can be in none, one, two, or three of the sets A, B, and C. For each case, consider the contribution of x to each of the terms on the right side of the formula.)

c. Find a formula for $|A \cup B \cup C \cup D|$.

1.50. a. How many elements are there in the following set?

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}\}\}$$

b. Describe precisely the algorithm you used to answer part (a).

1.51. Simplify the given set as much as possible in each case. Assume that all the numbers involved are real numbers.

a.

$$\bigcap_{r>0} \{x \mid |x - a| < r\}$$

b.

$$\bigcup_{r<1} \{x \mid |x - a| \leq r\}$$

1.52. Is it possible for two distinct, nonempty sets A and B to satisfy $A \times B \subseteq B \times A$? Give either an example of sets A and B for which this is true or a general reason why it is impossible.

1.53. Suppose that A and B are subsets of a universal set U.

a. What is the relationship between $2^{A \cup B}$ and $2^A \cup 2^B$? (Under what circumstances are they equal? If they are not equal, is one necessarily a subset of the other, and if so, which one?) Give reasons for your answers.

b. What is the relationship between $2^{A \cap B}$ and $2^A \cap 2^B$? Give reasons.

c. What is the relationship between $2^{A \oplus B}$ and $2^A \oplus 2^B$? (The operator $\oplus$ is symmetric difference, as in Exercise 1.5.) Give reasons.

d. What is the relationship between $2^{(A')}$ and $(2^A)'$? (Both are subsets of 2^U.) Give reasons.

1.54. Find a statement logically equivalent to $p \leftrightarrow q$ that is in the form of a disjunction, and simplify it as much as possible. (One approach is to use the last paragraph of Section 1.2.2, from which it follows that $p \leftrightarrow q$ is equivalent to $(\neg p \vee q) \wedge (p \vee \neg q)$, and then to use distributive laws.)

1.55. In each case, write a quantified statement, using the formal notation discussed in the chapter, that expresses the given statement. In both cases the set A is assumed to be a subset of the domain, not necessarily the entire domain.

a. There is exactly one element x in the set A satisfying the condition P—that is, for which the proposition $P(x)$ holds.

b. There are at least two distinct elements in the set A satisfying the condition P.

1.56. Below are four pairs of statements. In all cases, the universe for the quantified statements is assumed to be the set $\mathcal{N}$. We say one statement logically implies the other if, for any choice of statements $p(x)$ and $q(x)$ for which the first is true, the second is also true. In each case, say whether the first statement logically implies the second, and whether the second logically implies the first. In each case where the answer is no, give an example of statements $p(x)$ and $q(x)$ to illustrate.

a.

$$\forall x(p(x) \vee q(x))$$
$$\forall x(p(x)) \vee \forall x(q(x))$$

b.

$$\forall x(p(x) \wedge q(x))$$
$$\forall x(p(x)) \wedge \forall x(q(x))$$

c.

$$\exists x(p(x) \vee q(x))$$
$$\exists x(p(x)) \vee \exists x(q(x))$$

d.

$$\exists x(p(x) \wedge q(x))$$
$$\exists x(p(x)) \wedge \exists x(q(x))$$

1.57. Suppose A and B are sets and $f : A \to B$. Let S and T be subsets of A.

a. Is the set $f(S \cup T)$ a subset of $f(S) \cup f(T)$? If so, give a proof; if not, give a counterexample (i.e., specify sets A, B, S, and T and a function f).

b. Is the set $f(S) \cup f(T)$ a subset of $f(S \cup T)$? Give either a proof or a counterexample.

c. Repeat part (a) with intersection instead of union.

d. Repeat part (b) with intersection instead of union.

e. In each of the first four parts where your answer is no, what extra assumption on the function f would make the answer yes? Give reasons for your answer.

1.58. Suppose n is a positive integer and $X = \{1, 2, \ldots, n\}$. Let $A = 2^X$; let B be the set of all functions from X to $\{0, 1\}$, and let $C = \{0, 1\}^n = \{0, 1\} \times \{0, 1\} \times \cdots \times \{0, 1\}$ (where there are n factors).

a. Describe an explicit bijection f from A to B. (In other words, define a function $f : A \to B$, by saying for each subset S of X what function from X to $\{0, 1\}$ $f(S)$ is, and show that f is a bijection.

b. Describe an explicit bijection g from B to C. (In this case, you have to say, for each function $t : X \to \{0, 1\}$, what n-tuple $g(t)$ is, and then show g is a bijection.)

c. Describe an explicit bijection h from C to A. (Here you have to start with an n-tuple $N = (i_1, i_2, \ldots, i_n)$ and say what set $h(N)$ is, and then show h is a bijection.)

1.59. Let E be the set $\{1, 2, 3, \ldots\}$, S the set of nonempty subsets of E, T the set of nonempty proper subsets of $\mathcal{N}$, and $\mathcal{P}$ the set of partitions of $\mathcal{N}$ into two nonempty subsets.

a. Suppose $f : T \to \mathcal{P}$ is defined by the formula $f(A) = \{A, \mathcal{N} - A\}$ (in other words, for a nonempty subset A of $\mathcal{N}$, $f(A)$ is the partition of $\mathcal{N}$ consisting of the two subsets A and $\mathcal{N} - A$). Is f a bijection from T to $\mathcal{P}$? Why or why not?

b. Suppose $g : S \to \mathcal{P}$ be defined by $g(A) = \{A, \mathcal{N} - A\}$. Is g a bijection from S to $\mathcal{P}$? Why or why not?

1.60. Suppose U is a set, $\circ$ is a binary operation on U, and I_0 is a subset of U.

a. Let $\mathcal{S}$ be the set of subsets of U that contain I_0 as a subset and are closed under the operation $\circ$; let $T = \bigcap_{S \in \mathcal{S}} S$. Show that $I_0 \subseteq T$ and that T is closed under $\circ$.

b. Show that the set T defined in part (a) is the *smallest* subset of U that contains I_0 and is closed under $\circ$, in the sense that for any other such set T_1, $T \subseteq T_1$.

1.61. Consider the following "proof" that any relation R on a set A which is both symmetric and transitive must also be reflexive:

> Let a be any element of A. Let b be any element of A for which aRb. Then since R is symmetric, bRa. Now since R is transitive, and since aRb and bRa, it follows that aRa. Therefore, R is reflexive.

Your answer to Exercise 1.29 shows that this proof cannot be correct. What is the first incorrect statement of the proof, and why is it incorrect?

1.62. Suppose A is a set having n elements.

a. How many relations are there on A?

b. How many reflexive relations are there on A?

c. How many symmetric relations are there on A?

d. How many relations are there on A that are both reflexive and symmetric?

1.63. Suppose R is a relation on a nonempty set A.

a. Define $R^s = R \cup \{(x, y) \mid yRx\}$. Show that R^s is symmetric and is the smallest symmetric relation on A containing R (i.e., for any symmetric relation R_1 with $R \subseteq R_1$, $R^s \subseteq R_1$).

b. Define R^t to be the intersection of all transitive relations on A containing R. Show that R^t is transitive and is the smallest transitive relation on A containing R.

c. Let $R^u = R \cup \{(x, y) \mid \exists z(xRz \text{ and } zRy)\}$. Is R^u equal to the set R^t in part (b)? Either prove that it is, or give an example in which it is not.

The relations R^s and R^t are called the symmetric closure and transitive closure of R, respectively.

1.64. Let R be the equivalence relation in Exercise 1.33a. Assuming that S is finite, find a function $f : 2^S \to \mathcal{N}$ so that for any $x, y \in 2^S$, xRy if and only if $f(x) = f(y)$.

1.65. Let n be a positive integer. Find a function $f : \mathcal{N} \to \mathcal{N}$ so that for any $x, y \in \mathcal{N}$, $x \equiv_n y$ if and only if $f(x) = f(y)$.

1.66. Let A be any set, and let R be any equivalence relation on A. Find a set B and a function $f : A \to B$ so that for any $x, y \in A$, xRy if and only if $f(x) = f(y)$.

1.67. Suppose R is an equivalence relation on a set A. A subset $S \subseteq A$ is *pairwise inequivalent* if no two distinct elements of S are equivalent. S is a *maximal* pairwise inequivalent set if S is pairwise inequivalent and every element of A is equivalent to some element of S. Show that a set S is a maximal pairwise inequivalent set if and only if it contains exactly one element of each equivalence class.

1.68. Suppose R_1 and R_2 are equivalence relations on a set A. As discussed in Section 1.4, the equivalence classes of R_1 and R_2 form partitions P_1 and P_2, respectively, of A. Show that $R_1 \subseteq R_2$ if and only if the partition P_1 is *finer* than P_2 (i.e., every subset in the partition P_2 is the union of one or more subsets in the partition P_1).

1.69. Suppose Σ is an alphabet. It is obviously possible for two distinct strings x and y over Σ to satisfy the condition $xy = yx$, since this condition is always satisfied if $y = \Lambda$. Is it possible under the additional restriction that x and y are both nonnull? Either prove that this cannot happen, or describe precisely the circumstances under which it can.

1.70. Show that there is no language L so that $\{aa, bb\}^*\{ab, ba\}^* = L^*$.

1.71. Consider the language $L = \{x \in \{0, 1\}^* \mid x = yy \text{ for some string } y\}$. We know that $L = L\{\Lambda\} = \{\Lambda\}L$ (because any language L has this property). Is there any other way to express L as the concatenation of two languages? Prove your answer.