MINIMUM VALUE PROBLEM

Over all real numbers, find the minimum value of a positive real number, y such that

$$y = \sqrt{(x+6)^2+25} + \sqrt{(x-6)^2+121}$$

At stationary point, the rate of change in x will equal zero, and the second rate of change in x will result in a positive value.

Hence,

$$\frac{dy}{dx} = 0 \; and \; \frac{d^2y}{dx^2} \; is \; positive$$

Thus, we need to find the first derivative of y

Recall that

$$y = \sqrt{(x+6)^2+25} + \sqrt{(x-6)^2+121}$$

$\Rightarrow y = ((x+6)^2+25)^{\frac{1}{2}} + ((x-6)^2+141)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{x+6}{\sqrt{(x+6)^2+25}} + \frac{x-6}{\sqrt{(x-6)^2+121}}$$

Equating $\frac{dy}{dx}$ to zero and solving then gives

$$0 = \frac{x+6}{\sqrt{(x+6)^2+25}} + \frac{x-6}{\sqrt{(x-6)^2+121}}$$

Then

$$\frac{x+6}{\sqrt{(x+6)^2+25}} = \frac{6-x}{\sqrt{(x-6)^2+121}}$$

Squaring both sides

$$\frac{(x+6)^2}{(x+6)^2+25} = \frac{(x-6)^2}{(x-6)^2+121}$$

Dividing the numerator and denominator of the L.H.S all through with $(x+6)^2$ with and numerator and denominator of the R.H.S all through with $(x-6)^2$.

This gives:

$$\frac{1}{1+\frac{25}{(x+6)^2}}=\frac{1}{1+\frac{121}{(x-6)^2}}$$

From the equation above, the equation can only be true if and only if:

$$\frac{25}{(x+6)^2}=\frac{121}{(x-6)^2}$$

Taking the square root of both sides, we have:

$$\frac{5}{x+6}=\pm\frac{11}{x-6}$$

This implies that:

$$\frac{5}{x+6}=\frac{11}{x-6}\ or\ \frac{5}{x+6}=-\frac{11}{x-6}$$

Taking the first case:

$$\frac{5}{x+6}=\frac{11}{x-6}$$

This implies:

$5(x-6)=11(x+6)\ \Rightarrow 5x-30=11x+66 \Rightarrow 11x-5x=-66-30$

$\Rightarrow 6x=-96 \quad \Rightarrow x=-16$

Taking the second case:

$$\frac{5}{x+6}=-\frac{11}{x-6}$$

This implies:

$5(x-6)=-11(x+6) \qquad \Rightarrow 5x-30=-11x-66 \Rightarrow 11x+5x=-66+30$

$\Rightarrow 16x=-36\ \ \Rightarrow x=-\frac{9}{4}$

The maximum point gives the minimum value, hence $x=-\frac{9}{4}$ would give the maximum value y, hence;

When x = $\frac{-9}{4}$,

$$y=\sqrt{\left(\frac{-9}{4}+6\right)^2+25}+\sqrt{\left(\frac{-9}{4}-6\right)^2+121}=\sqrt{\left(\frac{15}{4}\right)^2+25}+\sqrt{\left(\frac{-33}{4}\right)^2+121}$$

$$y = \sqrt{\frac{625}{16}} + \sqrt{\frac{3025}{16}} = \frac{15}{4} + \frac{55}{4} = \frac{70}{4} = 17.5$$

So, the minimum value of y is given as:

$$y = 17.5$$