Day 1 Lecture 3

Backward Propagation

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Learning

Purely Supervised

Typically Backpropagation + Stochastic Gradient Descent (SGD)
Good when there are lots of labeled data

Layer-wise Unsupervised + Supervised classifier

Train each layer in sequence, using regularized auto-encoders or Restricted Boltzmann Machines (RBM)
Hold the feature extractor, on top train linear classifier on features
Good when labeled data is scarce but there are lots of unlabeled data

Layer-wise Unsupervised + Supervised Backprop

Train each layer in sequence
Backprop through the whole system
Good when learning problem is very difficult

Slide Credit: Lecun

From Lecture 3

L Hidden Layers

Hidden pre-activation ($k>0$)

$$\mathbf{a}^{(k+1)}(\mathbf{x}) = \mathbf{b}^{(k)} + \mathbf{W}^{(k)}\mathbf{h}^{(k)}(\mathbf{x})$$

$$\mathbf{h}^{(1)}(\mathbf{x}) = \mathbf{x}$$

Hidden activation ($k=1,\ldots L$)

$$\mathbf{h}^{(k)}(\mathbf{x}) = \mathbf{g}(\mathbf{a}^{(k)}(\mathbf{x}))$$

Output activation ($k=L+1$)

$$\mathbf{h}^{(L+1)}(\mathbf{x}) = \mathbf{o}(\mathbf{a}^{(L+1)}(\mathbf{x})) = \mathbf{f}(\mathbf{x})$$

$$o(\mathbf{a}) = \text{softmax}(\mathbf{a}) = \left[\frac{\exp(a_1)}{\sum_c \exp(a_c)} \cdots \frac{\exp(a_C)}{\sum_c \exp(a_c)}\right]^T$$

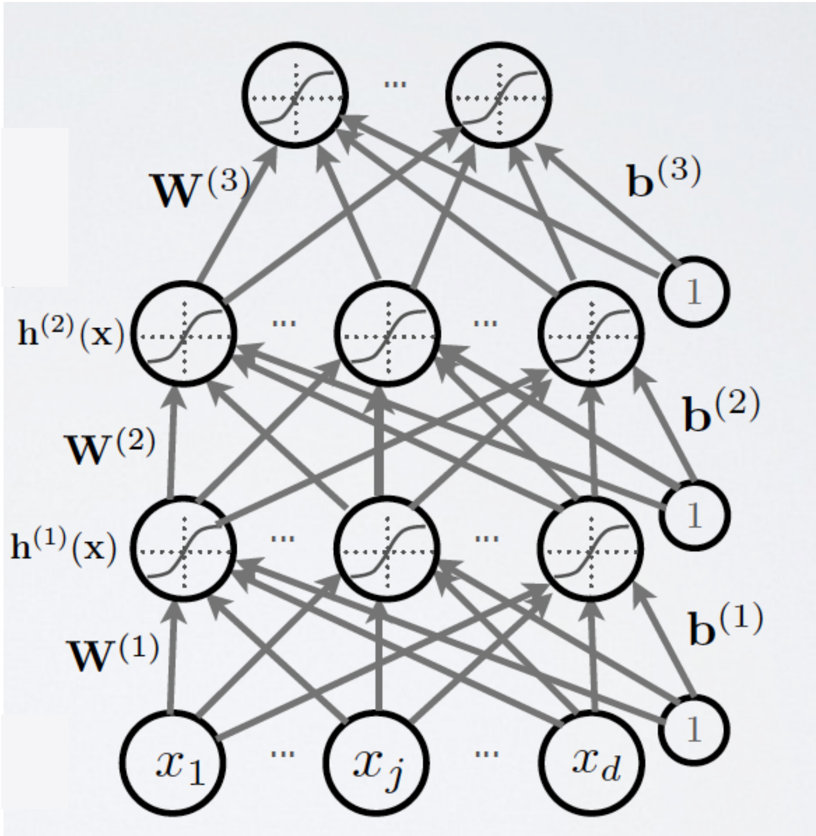


Figure Credit: Hugo Laroche NN course

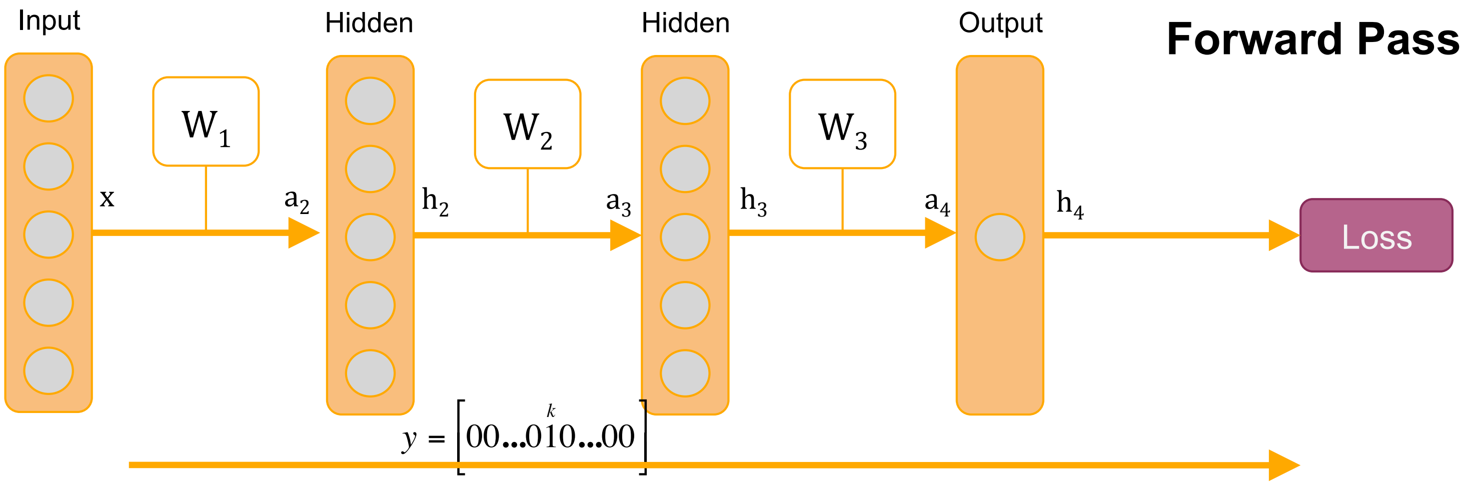
Backpropagation algorithm

The output of the Network gives class **scores** that depens on the input and the parameters

$$f(\mathbf{x}) = \mathbf{h}^{(L+1)}(\mathbf{x}) = \mathbf{o}(\mathbf{a}^{(L+1)}(\mathbf{x})) = \mathbf{o}(\mathbf{b}^{(L)} + \mathbf{W}^{(L)}\mathbf{h}^{(L)}(\mathbf{x}))$$

- Define a **loss function** that quantifies our unhappiness with the scores across the training data.
- Come up with a way of efficiently finding the parameters that minimize the loss function (**optimization**)

Forward Pass



Probability Class given an input (softmax)

$$p(c_k = 1|\mathbf{x}) = \frac{\exp(a_k)}{\sum_c \exp(a_c)}$$

Figure Credit: Kevin McGuiness

Loss function; e.g., negative log-likelihood (good for classification)

$$L(\mathbf{x}, y; \mathbf{W}) = -\sum_j y_j \log(p(c_j|\mathbf{x}))$$

Regularization term (L2 Norm) aka as weight decay

$$L(\mathbf{x}, y; \mathbf{W}) = -\sum_j y_j \log(p(c_j|\mathbf{x})) + \frac{\lambda}{2}\|\mathbf{W}\|_2^2$$

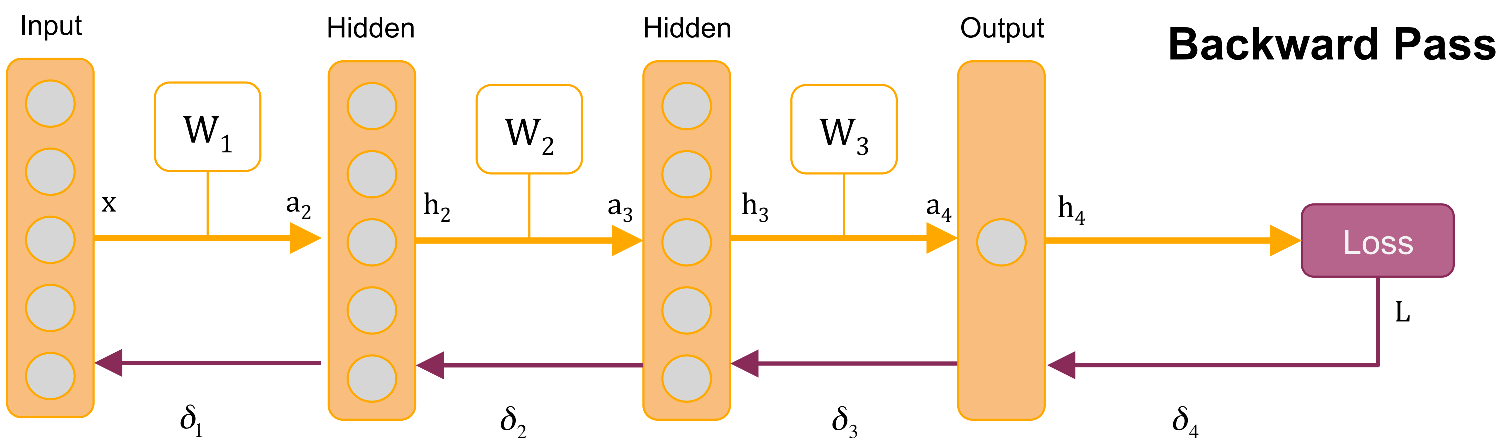
Minimize the loss (plus some regularization term) w.r.t. Parameters over the whole training set.

$$\mathbf{W}^* = argmin_\theta \sum_j L(\mathbf{x}^n, y^n; \mathbf{W})$$

Backpropagation algorithm

- We need a way to fit the model to data: find parameters (**W**$^{(k)}$, **b**$^{(k)}$) of the network that (locally) minimize the loss function.
- We can use **stochastic gradient descent**. Or better yet, mini-batch stochastic gradient descent.
- To do this, we need to find the gradient of the loss function with respect to all the parameters of the model (**W**$^{(k)}$, **b**$^{(k)}$)
- These can be found using the **chain rule** of differentiation.
- The calculations reveal that the gradient wrt. the parameters in layer k only depends on the error from the above layer and the output from the layer below.
- This means that the gradients for each layer can be computed iteratively, starting at the last layer and propagating the error back through the network. This is known as the **backpropagation** algorithm.

Slide Credit: Kevin McGuiness



1. Find the error in the top layer:

$$\delta_K = \frac{\partial L}{\partial a_K}$$

$$\delta_K = \frac{\partial L}{\partial h_K}\frac{\partial h_K}{\partial a_K}$$

$$\delta_K = \frac{\partial L}{\partial h_K} \bullet g'(a_K)$$

Figure Credit: Kevin McGuiness

2. Compute weight updates

$$\frac{\partial L}{\partial W_k} = \frac{\partial L}{\partial a_{k+1}}\frac{\partial a_{k+1}}{\partial W_k}$$

$$\frac{\partial L}{\partial W_k} = \frac{\partial L}{\partial a_{k+1}} \bullet h_k$$

$$\frac{\partial L}{\partial W_k} = \delta_{k+1} \bullet h_k$$

3. Backpropagate error to layer below

$$\delta_k = \frac{\partial L}{\partial a_k}$$

$$\delta_k = \frac{\partial L}{\partial ha_{k+1}}\frac{\partial a_{k+1}}{\partial h_k}\frac{\partial h_k}{\partial a_k}$$

$$\delta_k = W_k^T \frac{\partial L}{\partial ha_{k+1}} \bullet g'(a_k)$$

$$\delta_k = W_k^T \delta_{k+1} \bullet g'(a_k)$$

Optimization

Stochastic Gradient Descent

$$\mathbf{W} \longleftarrow \mathbf{W} - \eta \frac{\partial L}{\partial W}$$

Stochastic Gradient Descent with momentum

$$\mathbf{W} \longleftarrow \mathbf{W} - \eta \Delta$$

$$\Delta \longleftarrow 0.9\Delta + \frac{\partial L}{\partial \mathbf{W}}$$

Stochastic Gradient Descent with L2 regularization

$$\mathbf{W} \longleftarrow \mathbf{W} - \eta \frac{\partial L}{\partial W} - \lambda |\mathbf{W}|$$