# Review of Basic Properties of Vectors

This document summarizes those basic properties of **vectors** which used in the course.

A vector quantity is a physical quantity that has both size (**magnitude**) and **direction**. For example, when you move from one location to another, your movement is a vector, since you moved a certain distance (the size or magnitude of your movement) in a particular direction. If either the distance you moved or the direction in which you moved had been different, your overall movement would have been a different one. Other examples of vector quantities are forces (they have a certain size and are applied in specific directions), velocities (objects move with a particular speed in a specific direction), and so on.

To picture relationships and properties of vectors, they are often represented schematically as arrows, with the length of the arrow representing the magnitude of the vector and the direction of the arrow representing the direction of the vector. Although notation is not completely standard, vectors are usually denoted with bold-face characters, sometimes with small arrows drawn above the symbol to emphasize the "vectorness" of the quantity (as done in Figure 1 to the right). In this document, we will use the arrowhead notation in figures, but just the corresponding bold-face characters in the text. In Figure 1, there are three vectors, **a**, **b**, and **a + b**.

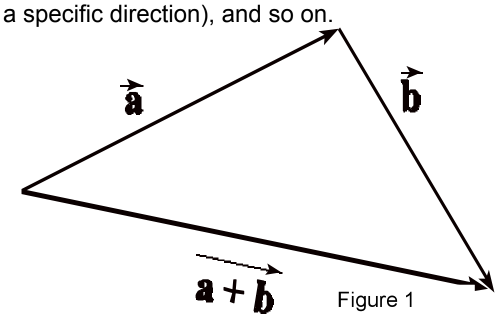

Figure 1

The **sum** or **resultant** of two vectors is a the vector you get by combining the two sequentially. In Figure 1, **a + b** is the sum of the two vectors **a** and **b**. If **a** and **b** represented two consecutive movements of the indicated distances in the indicated directions, then **a + b** is just the net effect (distance and direction) of carrying out the two original movements in sequence. Notice that the vector **a + b** is equivalent to the sequential traversal of the two individual vectors **a** and **b**.

Figure 2 to the right shows a vector **a** drawn on a coordinate plane. We would say that **a** is *"the vector <u>from</u> point $P_1$ at location ($x_1$ , $y_1$ ) <u>to</u> point $P_2$ at location ($x_2$ , $y_2$ )."* The end of the vector with the arrowhead is called its **head**, and the vector is said to go "**to**" that point. The other end of the vector is called its **tail** and the vector is said to be "**from**" that point. The vector from $P_2$ to $P_1$ in this figure is **<u>not</u>** the same as vector **a**, because it has a different direction. (You'll see soon that it is quite natural to denote the vector from $P_2$ to $P_1$ by **-a**, the negative of the vector **a**).

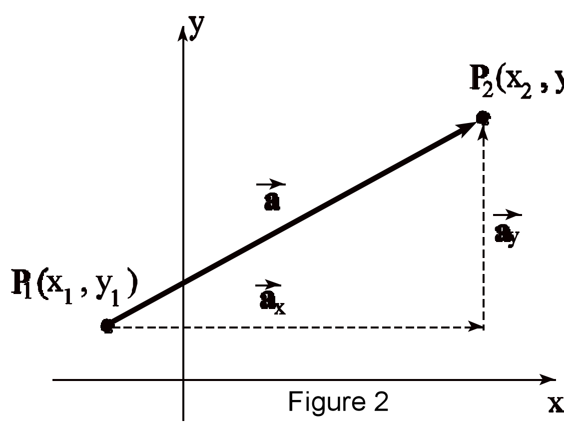

Figure 2

You can see from Figure 2 that the vector **a** can be viewed as the resultant of the two vectors $\mathbf{a}_x$ (parallel to the x-axis) and $\mathbf{a}_y$, parallel to the y-axis. It is clearly possible to **resolve** any vector in a plane into a sum of two such vectors parallel respectively to the coordinate axes. We refer to the magnitudes of these two vector, denoted here as $a_x$ and $a_y$, as the **components** of **a** with respect to x and y, or more simply as the (x, y) components of **a**. Thus, we can specify a vector by giving its magnitude and direction or by giving its components with respect to a set of coordinate axes as an ordered set of numbers, as in **a** = ($a_x$ ,$a_y$ ). Most computations involving vectors become simple arithmetic (or algebra) when you work with components rather than complicated trigonometrical computations if based on magnitudes and directions.

For the vector **a** in Figure 2, the components are easily computed:

$$a_x = x_2 - x_1 \qquad \text{(VEC-1)}$$
$$a_y = y_2 - y_1$$

If we are working in three dimensions, there will be a third coordinate axis, say z; points will have three coordinates each, say ($x_1$ , $y_1$ , $z_1$ ) and ($x_2$ , $y_2$ , $z_2$ ); and vectors will have a component along the x-, y-, and z-directions, **a** = ($a_x$ , $a_y$ , $a_z$ ), with $a_x$ and $a_y$ given by formulas (VEC-1) above, and

$a_z = z_2 - z_1$ (VEC-2)

**Example 1**:
Compute the components of the vector **p** from the point (1, -4, 7) to the point (-3, 6, 9).

**Answer:**
Simply subtract the coordinates of the "from" point from the coordinates of the "to" point:

$p_x = -3 - 1 = -4$, $p_y = 6 - (-4) = 10$, $p_z = 9 - 7 = 2$

or

**p** = (-4, 10, 2).

If we are using a coordinate system similar to the one sketched in Figure 3 to the right, with the z-axis coming out of the page at you, this is a vector oriented generally leftwards (negative x), upwards (positive y) and forwards out of the page (positive z).

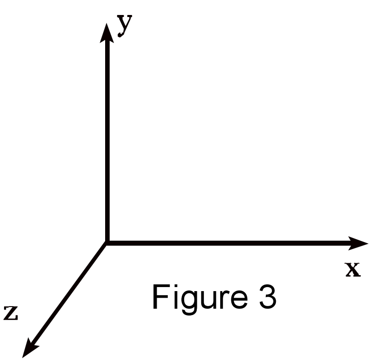

Figure 3

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**Remarks**

**I)** In a coordinate system, the vector from the origin to any point has components equal to the coordinates of the point (since the "from" point has all coordinates equal to zero). Because of this, many authors use the terms "vector" and "point" interchangeably, and use similar notation for the two types of entities.

**ii)** Although vectors often arise as the directed line between a "from" point and a "to" point in a coordinate system, the vector itself contains no specific positional information -- just direction and magnitude. Thus, the vector **q** = (-4, 10, 2), computed as the vector from point (8, 23, 6) to point (12, 33, 8) is considered to the be same vector as **p** obtained in Example 1 above. **p** and **q** both have the same components, and therefore the same length and direction, so they are considered to be identical vectors.

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The following properties and relationships involving vectors and their components are important.

1) the length or magnitude of a vector **a**, denoted |**a**|, can be computed from its components using Pythagoras's formula:

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \text{(VEC-3)}$$

The positive square root is always taken here, so that vectors always have a magnitude of at least zero. A vector will have a magnitude of zero only if all of its components are zeros. If even one component is nonzero (positive or negative), the vector will have a magnitude which is greater than zero.

2) Multiplying each component of a vector by the same constant simply changes the magnitude of the vector by a factor equal to that constant. For example, take **p** = ($p_x$ , $p_y$ , $p_z$ ). Then, this operation of **scalar multiplication** by, say, a factor $\alpha$ gives **q** = $\alpha$**p** = ($q_x$ , $q_y$ , $q_z$ ) = ($\alpha$•$p_x$ , $\alpha$•$p_y$ , $\alpha$•$p_z$ ) and

$$|\mathbf{q}| = \sqrt{q_x^2 + q_y^2 + q_z^2} = \sqrt{(\alpha \cdot p_x)^2 + \left(\alpha \cdot p_y\right)^2 + \left(\alpha \cdot p_z\right)^2} = \sqrt{\alpha^2 \left(p_x^2 + p_y^2 + p_z^2\right)} = \sqrt{\alpha^2} \bullet \sqrt{\left(p_x^2 + p_y^2 + p_z^2\right)} = \alpha \cdot |\mathbf{p}|$$

or, in short,

$$|\alpha \cdot \mathbf{p}| = \alpha \cdot |\mathbf{p}| \qquad \text{(VEC-4)}$$

3) The process of starting with a particular vector and converting it into a vector in the same direction but with magnitude equal to 1 is called **normalization**, and the result is called a **unit vector**. Equation (VEC-4) above indicates that any vector can be normalized by simply dividing each component by the magnitude of the vector:

$$\mathbf{u}_a = \left(\frac{a_x}{|\mathbf{a}|}, \frac{a_y}{|\mathbf{a}|}, \frac{a_z}{|\mathbf{a}|}\right) \qquad \text{(VEC-5)}$$

Here, $\mathbf{u}_a$ is a conventional way of indicating a unit vector in the same direction as vector **a**.

 

**Example 2:**
Compute the magnitude of vector **p** = (-4, 10, 2) in example 1 above, and then normalize the vector.

**Answer:**
Using formula (VEC-3),

$$|\mathbf{p}| = \sqrt{(-4)^2 + 10^2 + 2^2} = \sqrt{16 + 100 + 4} = \sqrt{120} \cong 10.954$$

to three decimal places. Then

$$\mathbf{u}_p = \left(\frac{-4}{\sqrt{120}}, \frac{10}{\sqrt{120}}, \frac{2}{\sqrt{120}}\right) \cong (-0.3651, \quad 0.9129, \quad 0.1826)$$

You can easily verify that the magnitude of this last vector is 1 to four decimal places.

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4) Sums of two or more vectors and the difference of two vectors are determined by doing the corresponding sums and/or differences component by component. Thus:
$\mathbf{a + b} = (a_z + b_z\ ,\ a_y + b_y\ ,\ a_z + b_z\ )$
and so on.

5) Item (2) above described multiplication of a vector by a single number, an operation called **scalar multiplication**. A second operation which looks like the multiplication of two vectors is called the **dot product**. It takes two vectors and results in a single value:

$$\mathbf{a \bullet b} = a_x b_x + a_y b_y + a_z b_z \qquad \text{(VEC-6)}$$

That is, multiply corresponding components together, and add up the products. The dot product measures how close in direction two vectors are. In particular, the angle, θ, between two vectors **a** and **b** can be computed from the formula:

$$\cos\theta = \frac{\mathbf{a \bullet b}}{|\mathbf{a}| \cdot |\mathbf{b}|} \quad \Leftrightarrow \quad \mathbf{a \bullet b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta \qquad \text{(VEC-7)}$$

Note that if **a** and **b** are unit vectors, the denominator of the first formula is just 1, and so cos θ is equal to the dot product of the two vectors.

6) Two vectors are perpendicular (make an angle of 90 degrees with each other) if their dot product is equal to zero (since cos 90 = 0). Thus, to check if two vectors are perpendicular, simply check whether or not their dot product is zero.

7) Notice that the magnitude of a vector can be written in terms of its dot product with itself:

$$|a| = \sqrt{a \bullet a} = \sqrt{a_x \cdot a_x + a_y \cdot a_y + a_z \cdot a_z} \qquad \text{(VEC-8)}$$

Thus, to check whether a vector is a unit vector (its magnitude is 1), check whether its dot product with itself is equal to 1.

8) The **vector cross product c = a × b** is an operation that combines two vectors, **a** and **b**, to produce a third vector, **c**, which is perpendicular to the two starting vectors. Here we just give the formulas for the components of **c** (mainly because the procedure for calculating **c** from its basic definition is hard to set up in a wordprocessor document):

$$\begin{aligned} c_x &= a_y b_z - a_z b_y \\ c_y &= a_z b_x - a_x b_z \\ c_z &= a_x b_y - a_y b_x \end{aligned} \qquad \text{(VEC-9)}$$

These are rather tedious formulas, but they are not difficult to apply. Since two distinct vectors determine a plane, and the vector cross product produces a third vector perpendicular to the first two, vector cross products cannot be defined in less than three dimensional coordinate systems, but they are very useful in working with three-dimensional coordinate systems.

**Example 3:**
Find the angle between the two vectors **p** = (2, -3, 5) and **q** = (7, 4, -6). Also, compute the vector cross product, **w**, of these two vectors. Demonstrate explicitly that the vector **w** is perpendicular to both **p** and **q**.

**Answer:**
To get the angle between the two vectors, we need to calculate their magnitudes and their dot product:

$$|\mathbf{p}| = \sqrt{4+9+25} = \sqrt{38} \cong 6.1644$$
$$|\mathbf{q}| = \sqrt{49+16+36} = \sqrt{101} \cong 10.050$$
$$\mathbf{p} \bullet \mathbf{q} = (2)(7) + (-3)(4) + (5)(-6) = 14 - 12 - 30 = -28$$

So, if $\theta$ is the angle between **p** and **q**, then by formula (VEC-7) above,

$$\cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}| \cdot |\mathbf{q}|} = \frac{-28}{\sqrt{38}\sqrt{101}} \cong -0.451966$$

and so

$$\theta \cong \cos^{-1}(-0.451966) \cong 116.87^{o} \text{ or } 2.0398 \text{ radians}$$

Thus, if the vectors **p** and **q** were placed tail-to-tail, then the angle between the two in the plane containing the two vectors would be approximately $116.87^{o}$ or 2.0398 radians. You can perhaps picture this mentally if you note that **p** is pointing downwards, forward, and to the right, whereas **q** is pointing more to the right, upwards and backwards.

To get the components of **w**, just apply the formulas (VEC-9) above:

$$w_x = p_y\, q_z - p_z\, q_y = (-3)(-6) - (5)(4) = 18 - 20 = -2$$
$$w_y = p_z\, q_x - p_x\, q_z = (5)(7) - (2)(-6) = 35 + 12 = 47$$
$$w_z = p_x\, q_y - p_y\, q_x = (2)(4) - (-3)(7) = 8 + 21 = 29$$

Thus, **w** = (-2, 47, 29), a vector directed steeply upwards and forwards in the coordinate system sketched in Figure 3 earlier.

To confirm that **w** really is perpendicular to both **p** and **q**, we just need to verify that the dot products **w•p** and **w•q** are each zero. We get:

$$\mathbf{w}\bullet\mathbf{p} = (-2)(2) + (47)(-3) + (29)(5) = -4 - 141 + 145 = 0$$
$$\mathbf{w}\bullet\mathbf{q} = (-2)(7) + (47)(4) + (29)(-6) = -14 + 188 - 174 = 0$$

completing the required demonstration.

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## Projections

This topic is somewhat advanced, but the ideas and formulas are required in computer graphics in, among other things, setting up the so-called synthetic camera coordinate system for simulating the viewing of a scene from abitrarily specified locations and directions. The notion of a **projection** is a generalization of the simply idea of resolving a vector into components with respect to coordinate axis directions that was described at the beginning of this document.

The idea is sketched out in Figure 4 to the right. We start with a vector **a** represented by a dotted arrow in the figure, and a second vector **b** represented by the heavy solid arrow in the figure. The goal is to find two new vectors, $\mathbf{b}_{a\|}$ (parallel to **a**, and called the *"projection of **b** along **a**"*), and a second vector $\mathbf{b}_{a\perp}$ (perpendicular to **a**), so that the resultant of $\mathbf{b}_{a\|}$ and $\mathbf{b}_{a\perp}$ is **b** itself. We then say that we have **resolved b** into a component $\mathbf{b}_{a\|}$ parallel to **a**, and a component $\mathbf{b}_{a\perp}$ perpendicular to **a**.

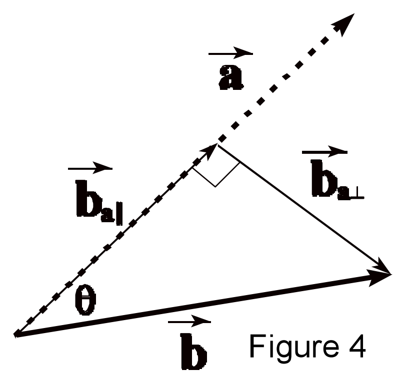

Figure 4

The details of the following derivation of the formulas should not be too difficult to follow given the material presented so far. First, define a unit vector, $\mathbf{u}_a$, in the direction of vector **a**:

$$\mathbf{u}_a = \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{a_x}{|\mathbf{a}|}, \frac{a_y}{|\mathbf{a}|}, \frac{a_z}{|\mathbf{a}|}\right)$$

as we've done in the past. Then, by definition of what we mean by $|\mathbf{b}_{a\|}|$, we can write that

$$\mathbf{b}_{a\|} = |\mathbf{b}_{a\|}|\ \mathbf{u}_a$$

That is, $\mathbf{b}_{a\|}$ is a vector of length $|\mathbf{b}_{a\|}|$ in the direction of $\mathbf{u}_a$. (Remember, multiplying the unit length vector, $\mathbf{u}_a$, by the constant $|\mathbf{b}_{a\|}|$ gives us a new vector in the direction of $\mathbf{u}_a$ but with length multiplied by this value $|\mathbf{b}_{a\|}|$. Since the length of $\mathbf{u}_a$ is 1 by construction, the length of $\mathbf{b}_{a\|}$ calculated in this way will be just 1 times $|\mathbf{b}_{a\|}|$ or just $|\mathbf{b}_{a\|}|$.)

From Figure 4, and simple trigonometry

$$|\mathbf{b}_{a\|}| = |\mathbf{b}| \cos\theta$$

where, by formula (VEC-7) above

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|}$$

Thus, in detail,

$$\mathbf{b}_{a\|} = |\mathbf{b}| \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} \mathbf{u}_a = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|} \mathbf{u}_a = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = (\mathbf{u}_a \bullet \mathbf{b}) \mathbf{u}_a \qquad \text{(VEC-10)}$$

The last three expressions in this formula are all equivalent -- use whichever is more convenient. This is a vector formula, and so really represents a set of formulas, one for each component of the vectors:

$$(\mathbf{b}_{a\|})_x = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|} (\mathbf{u}_a)_x = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2} (\mathbf{a})_x = (\mathbf{u}_a \bullet \mathbf{b})(\mathbf{u}_a)_x$$

and similarly for y- and z-components.

To get a formula for the component of **b** perpendicular to **a**, we just note that the goal is to have

$$\mathbf{b}_{a\|} + \mathbf{b}_{a\perp} = \mathbf{b}$$

as a vector sum. Thus,

$$\mathbf{b}_{a\perp} = \mathbf{b} - \mathbf{b}_{a\|} = \mathbf{b} - \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \mathbf{b} - (\mathbf{u}_a \bullet \mathbf{b}) \mathbf{u}_a \qquad \text{(VEC-11)}$$

or, component-wise:

$$(\mathbf{b}_{a\perp})_x = b_x - \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2} a_x = b_x - (\mathbf{u}_a \bullet \mathbf{b})(\mathbf{u}_a)_x$$

and similarly for the y- and z-components. Notice that it is easy to confirm that $\mathbf{b}_{\mathbf{a}\perp}$ is perpendicular to $\mathbf{b}_{a\|}$: we just confirm that $\mathbf{b}_{a\|} \bullet \mathbf{b}_{\mathbf{a}\perp} = 0$. But,

$$\begin{aligned}
\mathbf{b}_{a\|} \bullet \mathbf{b}_{\mathbf{a}\perp} &= [\,(\mathbf{u}_a \bullet \mathbf{b})\, \mathbf{u}_a\,] \bullet [\,\mathbf{b} - (\mathbf{u}_a \bullet \mathbf{b})\, \mathbf{u}_a\,] \\
&= (\mathbf{u}_a \bullet \mathbf{b})\, \mathbf{u}_a \bullet [\,\mathbf{b} - (\mathbf{u}_a \bullet \mathbf{b})\, \mathbf{u}_a\,] \\
&= (\mathbf{u}_a \bullet \mathbf{b})\, [\mathbf{u}_a \bullet \mathbf{b} - (\mathbf{u}_a \bullet \mathbf{b})\, \mathbf{u}_a \bullet \mathbf{u}_a\,] \\
&= (\mathbf{u}_a \bullet \mathbf{b})\, [\mathbf{u}_a \bullet \mathbf{b} - \mathbf{u}_a \bullet \mathbf{b}] \\
&= (\mathbf{u}_a \bullet \mathbf{b})\, [0] = 0
\end{aligned}$$

To get the third line, we kept in mind that the dot product operation occurs only between vectors, and to get the fourth line, we used the fact that $\mathbf{u}_a \bullet \mathbf{u}_a = 1$ since $\mathbf{u}_a$ is constructed to be a unit vector.

**<u>Example 4</u>:**
Resolve the vector **b** = (7, -3) into components parallel and perpendicular to the vector **a** = (1, 2).

**Answer**
This example involves vectors in a plane to reduce the total amount of arithmetic that needs to be done. Here each vector involves just an x- and a y-component. If you were working in three dimensions, you'd need to include a z-component in all of your calculations.

Before doing the actual calculations, we note the following. Movement in the direction of **a** involves shifting two units in the y-direction for every unit of movement in the x-direction (if you like, lines in the direction of **a** have a slope of 2). We expect the $\mathbf{b}_{\|}$ we get eventually to reflect this ratio in values of x- and y-components (and you'll see that this must be the case, since $\mathbf{b}_{\|}$ is just a scalar multiple of **a**) . Further, from knowledge of the relationship between slopes of perpendicular lines in a plane, we expect lines in the direction of $\mathbf{b}_{\perp}$ to have slopes of -1/2, so that the ratio of the x- to the y-component of $\mathbf{b}_{\perp}$ should be -2 to 1.

Now, we can apply the formulas. We choose to start by calculating $\mathbf{u}_a$ , the unit vector in the direction of **a**. Since

$$|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5} \cong 2.23607$$

we get

$$\mathbf{u}_a = \frac{1}{\sqrt{5}}(1,2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \cong (0.44721, 0.89443)$$

Thus,

$$\mathbf{u}_a \bullet \mathbf{b} \cong (7)(0.44721) + (-3)(0.89443) \cong 0.44721$$

Thus, applying formula (VEC-10) above for the x- and y-components separately, we get

$$(\mathbf{b}_{a_\parallel})_x = (\mathbf{u}_a \bullet \mathbf{b}) \cdot (\mathbf{u}_a)_x \cong (0.44721)(0.44721) = 0.2$$

and

$$(\mathbf{b}_{a_\parallel})_y = (\mathbf{u}_a \bullet \mathbf{b}) \cdot (\mathbf{u}_a)_y \cong (0.44721)(0.89443) = 0.4$$

That is,

$$\mathbf{b}_{a_\parallel} = (0.2, 0.4)$$

You can see from this calculation that the dot product, $\mathbf{u}_a \bullet \mathbf{b}$, is a measure of what part of vector **b** is in the direction of **a**.

We now apply formula (VEC-11) component-by-component to get

$$(\mathbf{b}_{a_\perp})_x = b_x - (\mathbf{u}_a \bullet \mathbf{b}) \cdot (\mathbf{u}_a)_x \cong 7 - (0.44721)(0.44721) = 6.8$$

and

$$(\mathbf{b}_{a_\perp})_y = b_y - (\mathbf{u}_a \bullet \mathbf{b}) \cdot (\mathbf{u}_a)_y \cong -3 - (0.44721)(0.89443) = -3.4$$

so that

$$\mathbf{b}_{a_\perp} = (6.8, -3.4).$$

Notice that both $\mathbf{b}_{a_\parallel}$ and $\mathbf{b}_{a_\perp}$ have components in the expected ratio. Further, we can easily verify that $\mathbf{b}_{a_\perp}$ really is perpendicular to **a** since

$$\mathbf{b}_{a_\perp} \bullet \mathbf{a} = (6.8)(1) + (-3.4)(2) = 6.8 - 6.8 = 0.$$

Notice as well that neither $\mathbf{b}_{a_\parallel}$ and $\mathbf{b}_{a_\perp}$ are unit vectors (and generally it will be the case that this procedure does not result immediately in unit vectors). However, we could easily calculate the corresponding unit vectors using the method previously illustrated.

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Vectors will arise in a variety of topics in computer graphics, and the elementary properties listed and illustrated in this short summary often make the manipulation of vectors and derivation of formulas involving directions, etc., very straightforward. Examples of applications we will consider in this course that make use of vectors and their properties are in the development of a simple lighting model and in the development of the so-called synthetic camera model.