6.4 Counting Subsets of a Set: Combinations

"But 'glory' doesn't mean 'a nice knock-down argument,'" Alice objected. "When I use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean—neither more nor less." — Lewis Carroll, *Through the Looking Glass,* 1872

Consider the following question:

> Suppose five members of a group of twelve are to be chosen to work as a team on a special project. How many distinct five-person teams can be selected?

This question is answered in Example 6.4.5. It is a special case of the following more general question:

> Given a set S with n elements, how many subsets of size r can be chosen from S?

The number of subsets of size r that can be chosen from S equals the number of subsets of size r that S has. Each individual subset of size r is called an *r-combination* of the set.

• Definition

Let n and r be nonnegative integers with $r \leq n$. An **r-combination** of a set of n elements is a subset of r of the n elements. The symbol $\binom{n}{r}$, which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Note that on a calculator the symbol $C(n, r)$, ${}_nC_r$, $C_{n,r}$, or ${}^nC_r$ is sometimes used instead of $\binom{n}{r}$.

Example 6.4.1 3-Combinations

Let $S = \{\text{Ann, Bob, Cyd, Dan}\}$. Each committee consisting of three of the four people in S is a 3-combination of S.

a. List all such 3-combinations of S. b. What is $\binom{4}{3}$?

Solution

a. Each 3-combination of S is a subset of S of size 3. But each subset of size 3 can be obtained by leaving out one of the elements of S. The 3-combinations are

{Bob, Cyd, Dan}	leave out Ann
{Ann, Cyd, Dan}	leave out Bob
{Ann, Bob, Dan}	leave out Cyd
{Ann, Bob, Cyd}	leave out Dan.

b. Because $\binom{4}{3}$ is the number of 3-combinations of a set with four elements, by part (a), $\binom{4}{3} = 4$. ■

There are two distinct methods that can be used to select r objects from a set of n elements. In an **ordered selection,** it is not only what elements are chosen but also the order in which they are chosen that matters. Two ordered selections are said to be the same if the elements chosen are the same and also if the elements are chosen in the same order. An ordered selection of r elements from a set of n elements is an r-permutation of the set.

In an **unordered selection,** on the other hand, it is only the identity of the chosen elements that matters. Two unordered selections are said to be the same if they consist of the same elements, regardless of the order in which the elements are chosen. An unordered selection of r elements from a set of n elements is the same as a subset of size r or an r-combination of the set.

Example 6.4.2 Unordered Selections

How many unordered selections of two elements can be made from the set $\{0, 1, 2, 3\}$?

Solution An unordered selection of two elements from $\{0, 1, 2, 3\}$ is the same as a 2-combination, or subset of size 2, taken from the set. These can be listed systematically as follows:

$\{0, 1\}, \{0, 2\}, \{0, 3\}$	subsets containing 0
$\{1, 2\}, \{1, 3\}$	subsets containing 1 but not already listed
$\{2, 3\}$	subsets containing 2 but not already listed.

Since this listing exhausts all possibilities, there are six subsets in all. Thus $\binom{4}{2} = 6$, which is the number of unordered selections of two elements from a set of four. ■

When the values of n and r are small, it is reasonable to calculate values of $\binom{n}{r}$ using the method of **complete enumeration** (listing all possibilities) illustrated in Examples 6.4.1 and 6.4.2. But when n and r are large, it is not feasible to compute these numbers by listing and counting all possibilities.

The general values of $\binom{n}{r}$ can be found by a somewhat indirect but simple method. An equation is derived that contains $\binom{n}{r}$ as a factor. Then this equation is solved to obtain a formula for $\binom{n}{r}$. The method is illustrated by Example 6.4.3.

Example 6.4.3 Relation between Permutations and Combinations

Write all 2-permutations of the set $\{0, 1, 2, 3\}$. Find an equation relating the number of 2-permutations, $P(4, 2)$, and the number of 2-combinations, $\binom{4}{2}$, and solve this equation for $\binom{4}{2}$.

Solution According to Theorem 6.2.3, the number of 2-permutations of the set $\{0, 1, 2, 3\}$ is $P(4, 2)$, which equals

$$\frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 12.$$

Now the act of constructing a 2-permutation of $\{0, 1, 2, 3\}$ can be thought of as a two-step process:

Step 1: Choose a subset of two elements from $\{0, 1, 2, 3\}$.

Step 2: Choose an ordering for the two-element subset. ← irrelevant for $\binom{n}{r}$

This process can be illustrated by the possibility tree shown in Figure 6.4.1.

Not just 12·11·10·9·8 bec that counts abcde 5! times

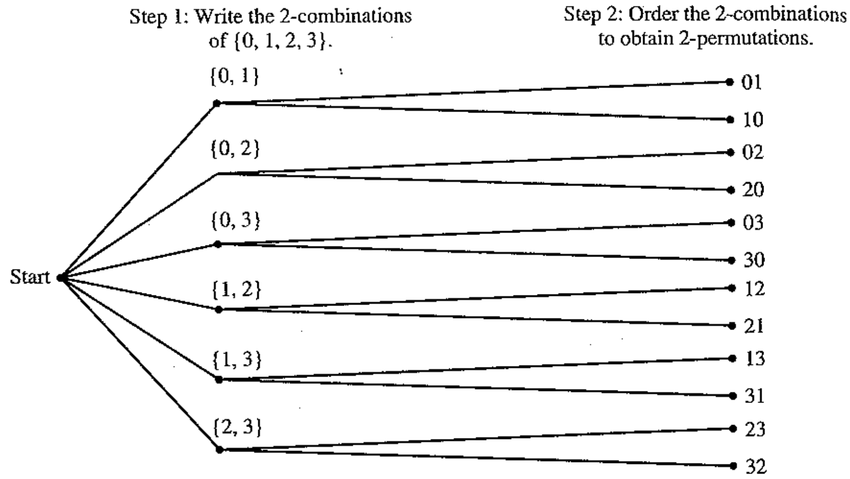

Figure 6.4.1 Relation between Permutations and Combinations

The number of ways to perform step 1 is $\binom{4}{2}$, the same as the number of subsets of si 2 that can be chosen from $\{0, 1, 2, 3\}$. The number of ways to perform step 2 is $2!$, t number of ways to order the elements in a subset of size 2. Because the number of wa of performing the whole process is the number of 2-permutations of the set $\{0, 1, 2,$ which equals $P(4, 2)$, it follows from the product rule that

$$P(4, 2) = \binom{4}{2} \cdot 2!.$$ This is an equation that relates $P(4, 2)$ and $\binom{4}{2}$.

permutations

Solving the equation for $\binom{4}{2}$ gives

$$\binom{4}{2} = \frac{P(4, 2)}{2!}$$

Recall that $P(4, 2) = \dfrac{4!}{(4-2)!}$. Hence, substituting yields

$$\binom{4}{2} = \frac{\frac{4!}{(4-2)!}}{2!} = \frac{4!}{2!(4-2)!} = 6.$$

The reasoning used in Example 6.4.3 applies in the general case as well. To form r-permutation of a set of n elements, first choose a subset of r of the n elements (there $\binom{n}{r}$ ways to perform this step), and then choose an ordering for the r elements (there $r!$ ways to perform this step). Thus the number of r-permutations is

$$P(n, r) = \binom{n}{r} \cdot r!.$$

Now solve for $\binom{n}{r}$ to obtain the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!}.$$

Since $P(n, r) = \dfrac{n!}{(n-r)!}$, substitution gives

$$\binom{n}{r} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}.$$

The result of this discussion is summarized and extended in Theorem 6.4.1.

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> **Theorem 6.4.1**
>
> The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula
>
> $$\binom{n}{r} = \frac{P(n, r)}{r!} \quad \text{first version}$$
>
> or, equivalently,
>
> $$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{second version}$$
>
> where n and r are nonnegative integers with $r \leq n$.

Note that the analysis presented before the theorem proves the theorem in all cases where n and r are positive. If r is zero and n is any nonnegative integer, then $\binom{n}{0}$ is the number of subsets of size zero of a set with n elements. But you know from Section 5.3 that there is only one set that does not have any elements. Consequently, $\binom{n}{0} = 1$. Also

$$\frac{n!}{0!(n-0)!} = \frac{\cancel{n!}}{1 \cdot \cancel{n!}} = 1$$

since $0! = 1$ by definition. (Remember we said that definition would turn out to be convenient!) Hence the formula

$$\binom{n}{0} = \frac{n!}{0!(n-0)!}$$

holds for all integers $n \geq 0$, and so the theorem is true for all nonnegative integers n and r with $r \leq n$.

Many electronic calculators have keys for computing values of $\binom{n}{r}$. Theorem 6.4.1 enables you to compute these by hand as well.

Example 6.4.4 Computing $\binom{n}{r}$ by Hand

n.b. $\binom{8}{5} = \binom{8}{3}$ why?

Compute $\binom{8}{5}$.

Solution By Theorem 6.4.1,

$$\begin{aligned}\binom{8}{5} &= \frac{8!}{5!(8-5)!} \\ &= \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1) \cdot (\cancel{3 \cdot 2 \cdot 1})} \quad \text{always cancel common factors before multiplying} \\ &= 56.\end{aligned}$$

Example 6.4.5 Calculating the Number of Teams

order irrelevant

Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Solution The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$. By Theorem 6.4.1,

See pg 33

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{\cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9 \cdot 8 \cdot \cancel{7!}}{(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1) \cdot \cancel{7!}} = 11 \cdot 9 \cdot 8 = 792.$$

Thus there are 792 distinct five-person teams.

The formula for the number of r-combinations of a set can be applied in a wide variety of situations. Some of these are illustrated in the following examples.

Example 6.4.6 Teams That Contain Both or Neither

plus

Suppose two members ~~of~~ the group of twelve insist on working as a pair—any team must contain either both or neither. How many five-person teams can be formed?

Solution Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B. The set of all possible teams can be partitioned into two subsets as shown in Figure 6.4.2 below.

Because a team that contains both A and B contains exactly three other people from the remaining ten in the group, there are as many such teams as there are subsets of three people that can be chosen from the remaining ten. By Theorem 6.4.1, this number is

$$\binom{10}{3} = \frac{10!}{3!\cdot 7!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot \cancel{7!}}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{7!}} = 120.$$

Because a team that contains neither A nor B contains exactly five people from the remaining ten, there are as many such teams as there are subsets of five people that can be chosen from the remaining ten. By Theorem 6.4.1, this number is

$$\binom{10}{5} = \frac{10!}{5!\cdot 5!} = \frac{\overset{2}{\cancel{10}}\cdot 9\cdot \overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3\cdot 2}\cdot 1\cdot \cancel{5!}} = 252.$$

Because the set of teams that contain both A and B is disjoint from the set of teams that contain neither A nor B, by the addition rule,

$$\begin{bmatrix}\text{number of teams containing}\\ \text{both } A \text{ and } B \text{ or}\\ \text{neither } A \text{ nor } B\end{bmatrix} = \begin{bmatrix}\text{number of teams}\\ \text{containing}\\ \text{both } A \text{ and } B\end{bmatrix} + \begin{bmatrix}\text{number of teams}\\ \text{containing}\\ \text{neither } A \text{ nor } B\end{bmatrix}$$
$$= 120 + 252 = 372.$$

This reasoning is summarized in Figure 6.4.2.

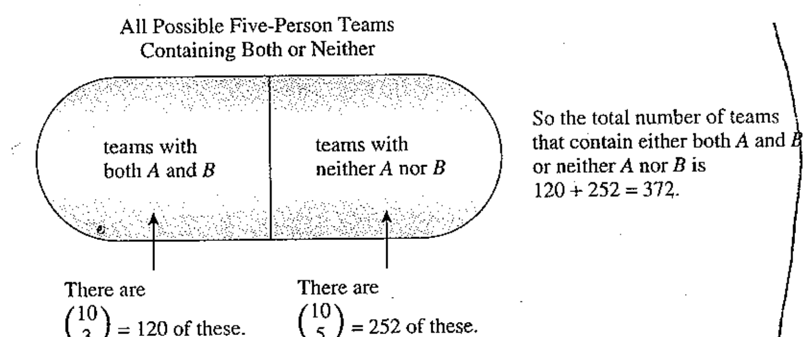

Figure 6.4.2

■

Example 6.4.7 Teams That Do Not Contain Both

group of 12

Suppose two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?

Solution Call the two people who refuse to work together C and D. There are two different ways to answer the given question: One uses the addition rule and the other uses the difference rule.

1st // not a choice

To use the addition rule, partition the set of all teams that don't contain both C and D into three subsets as shown in Figure 6.4.3 below.

Because any team that contains C but not D contains exactly four other people from the remaining ten in the group, by Theorem 6.4.1 the number of such teams is

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{6!}} = 210.$$

Similarly, there are $\binom{10}{4} = 210$ teams that contain D but not C. Finally, by the same reasoning as in Example 6.4.6, there are 252 teams that contain neither C nor D. Thus, by the addition rule,

$$\begin{bmatrix}\text{number of teams that do} \\ \text{not contain both } C \text{ and } D\end{bmatrix} = 210 + 210 + 252 = 672.$$

(+)

This reasoning is summarized in Figure 6.4.3.

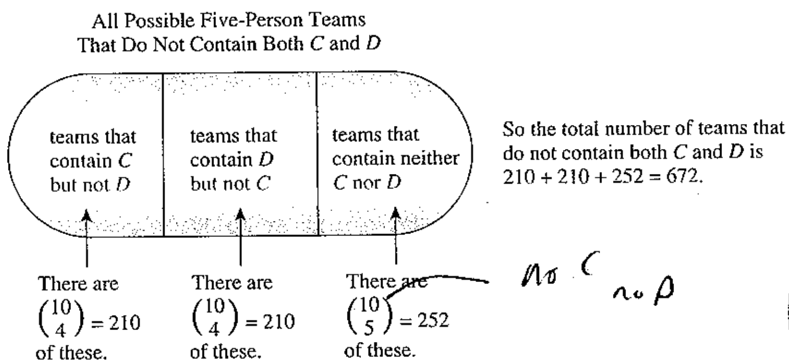

Figure 6.4.3

The alternative solution by the difference rule is based on the following observation: The set of all five-person teams that don't contain both C and D equals the set difference between the set of all five-person teams and the set of all five-person teams that contain both C and D. By Example 6.4.5, the total number of five-person teams is $\binom{12}{5} = 792$. Thus, by the difference rule,

$$\begin{aligned}\begin{bmatrix}\text{number of teams that don't} \\ \text{contain both } C \text{ and } D\end{bmatrix} &= \begin{bmatrix}\text{total number of} \\ \text{teams of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams that} \\ \text{contain both } C \text{ and } D\end{bmatrix} \\ &= \binom{12}{5} - \binom{10}{3} = 792 - 120 = 672.\end{aligned}$$

(−)

This reasoning is summarized in Figure 6.4.4.

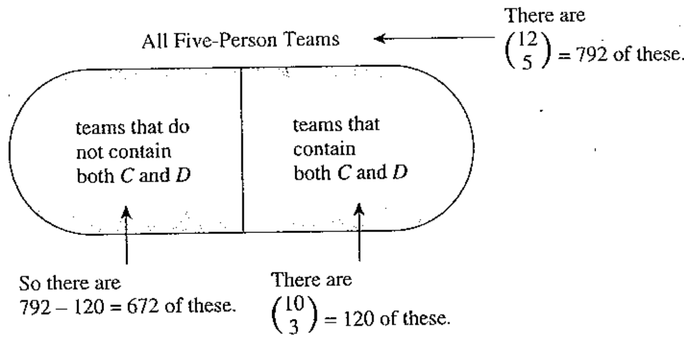

Figure 6.4.4 ■

Before we begin the next example, a remark on the phrases *at least* and *at most* is in order:

> The phrase **at least n** means "n or more."
> The phrase **at most n** means "n or fewer."

For instance, if a set consists of three elements and you are to choose at least two, you will choose two or three; if you are to choose at most two, you will choose none, or one, or two.

Example 6.4.8 Teams with Members of Two Types

Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Solution

a. To answer this question, think of forming a team as a two-step process:

Step 1: Choose the men.

Step 2: Choose the women.

There are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven. Hence, by the product rule,

$$\begin{aligned}\begin{bmatrix}\text{number of teams of five that}\\ \text{contain three men and two women}\end{bmatrix} &= \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!}\cdot\frac{7!}{2!5!}\\ &= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{2}\cdot 1\cdot\cancel{2}\cdot 1}\\ &= 210.\end{aligned}$$

b. This question can also be answered either by the addition rule or by the difference rule. The solution by the difference rule is shorter and is shown first.

Observe that the set of five-person teams containing at least one man equals the set difference between the set of all five-person teams and the set of five-person teams that do not contain any men. See Figure 6.4.5 below.

Now a team with no men consists entirely of five women chosen from the seven women in the group, so there are $\binom{7}{5}$ such teams. Also, by Example 6.4.5, the total number of five-person teams is $\binom{12}{5} = 792$. Hence, by the difference rule,

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5! \cdot 2!}$$

$$= 792 - \frac{7 \cdot \overset{3}{\cancel{6}} \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{2} \cdot 1} = 792 - 21 = 771.$$

This reasoning is summarized in Figure 6.4.5.

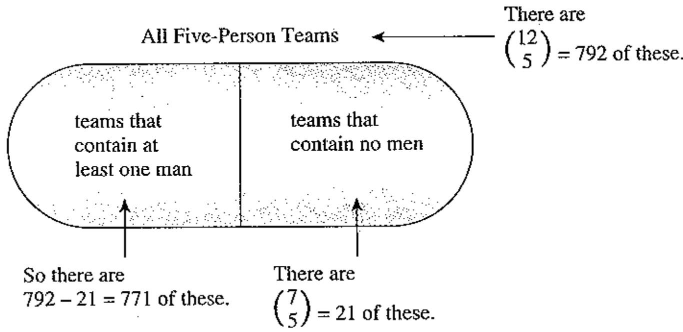

Figure 6.4.5

Alternatively, to use the addition rule, observe that the set of teams containing at least one man can be partitioned as shown in Figure 6.4.6 on page 342. The number of teams in each subset of the partition is calculated using the method illustrated in part (a). There are

$\binom{5}{1}\binom{7}{4}$ teams with one man and four women

$\binom{5}{2}\binom{7}{3}$ teams with two men and three women

$\binom{5}{3}\binom{7}{2}$ teams with three men and two women

$\binom{5}{4}\binom{7}{1}$ teams with four men and one woman

$\binom{5}{5}\binom{7}{0}$ teams with five men and no women.

Hence, by the addition rule,

$$
\begin{aligned}
&\begin{bmatrix}\text{number of teams with}\\ \text{at least one man}\end{bmatrix}\\
&= \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3} + \binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}\\
&= \frac{5!}{1!4!}\cdot\frac{7!}{4!3!} + \frac{5!}{2!3!}\cdot\frac{7!}{3!4!} + \frac{5!}{3!2!}\cdot\frac{7!}{2!5!} + \frac{5!}{4!1!}\cdot\frac{7!}{1!6!} + \frac{5!}{5!0!}\cdot\frac{7!}{0!7!}\\
&= \frac{5\cdot\cancel{4!}\cdot 7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\cancel{4!}\cdot\cancel{3\cdot 2}\cdot\cancel{4!}} + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}\cdot 7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\cancel{3!}\cdot\cancel{2}\cdot\cancel{4!}\cdot\cancel{3\cdot 2}} + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}\cdot 7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5!}}{\cancel{2}\cdot\cancel{3!}\cdot\cancel{5!}\cdot\cancel{2}}\\
&\quad + \frac{5\cdot\cancel{4!}\cdot 7\cdot\cancel{6!}}{\cancel{4!}\cdot\cancel{6!}} + \frac{\cancel{5!}\cdot\cancel{7!}}{\cancel{5!}\cdot\cancel{7!}}\\
&= 175 + 350 + 210 + 35 + 1 = 771.
\end{aligned}
$$

This reasoning is summarized in Figure 6.4.6.

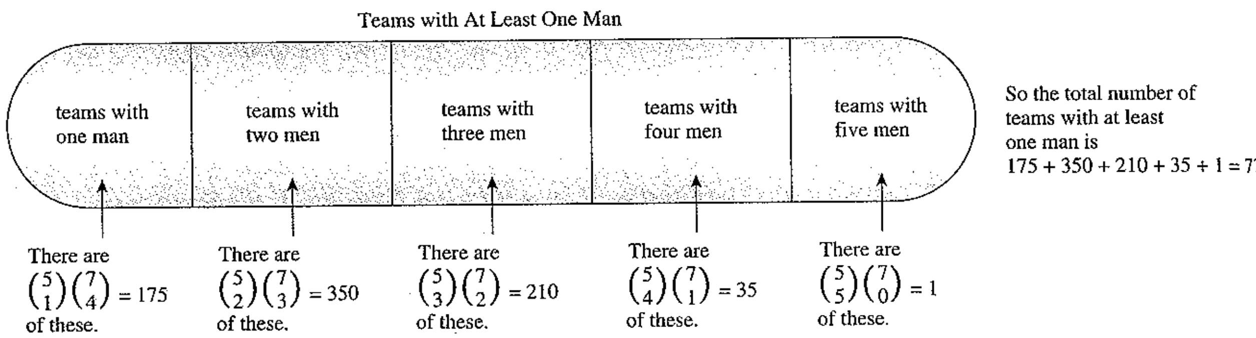

Figure 6.4.6

c. As shown in Figure 6.4.7 below, the set of teams containing at most one man can b partitioned into the set that does not contain any men and the set that contains exactl one man. Hence, by the addition rule,

$$
\begin{aligned}
\begin{bmatrix}\text{number of teams}\\ \text{with at}\\ \text{most one man}\end{bmatrix} &= \begin{bmatrix}\text{number of}\\ \text{teams without}\\ \text{any men}\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{teams with}\\ \text{one man}\end{bmatrix}\\
&= \binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4} = 21 + 175 = 196.
\end{aligned}
$$

This reasoning is summarized in Figure 6.4.7.

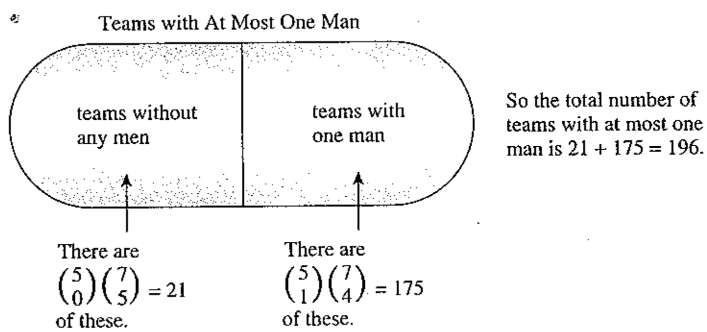

Figure 6.4.7

Example 6.4.9 Poker Hand Problems

The game of poker is played with an ordinary deck of cards (see Example 6.1.1). Various five-card holdings are given special names, and certain holdings beat certain other holdings. The named holdings are listed from highest to lowest below.

Royal flush: 10, J, Q, K, A of the same suit

Straight flush: five adjacent denominations of the same suit but not a royal flush—aces can be high or low, so A, 2, 3, 4, 5 of the same suit is a straight flush.

Four of a kind: four cards of one denomination—the fifth card can be any other in the deck

Full house: three cards of one denomination, two cards of another denomination

Flush: five cards of the same suit but not a straight or a royal flush

Straight: five cards of adjacent denominations but not all of the same suit—aces can be high or low

Three of a kind: three cards of the same denomination and two other cards of different denominations

Two pairs: two cards of one denomination, two cards of a second denomination, and a fifth card of a third denomination

One pair: two cards of one denomination and three other cards all of different denominations

No pairs: all cards of different denominations but not a straight or straight flush or flush

a. How many five-card poker hands contain two pairs?

b. If a five-card hand is dealt at random from an ordinary deck of cards, what is the probability that the hand contains two pairs?

Solution

a. Consider forming a hand with two pairs as a four-step process:

Step 1: Choose the two denominations for the pairs. choices

Step 2: Choose two cards from the smaller denomination.

Step 3: Choose two cards from the larger denomination.

Step 4: Choose one card from those remaining.

The number of ways to perform step 1 is $\binom{13}{2}$ because there are 13 denominations in all. The number of ways to perform steps 2 and 3 is $\binom{4}{2}$ because there are four cards of each denomination, one in each suit. The number of ways to perform step 4 is $\binom{44}{1}$ because removing the eight cards in the two chosen denominations from the 52 in the deck leaves 44 from which to choose the fifth card. Thus

$$\begin{aligned}
\begin{bmatrix}\text{the total number of}\\ \text{hands with two pairs}\end{bmatrix} &= \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}\\
&= \frac{13!}{2!(13-2)!}\cdot\frac{4!}{2!(4-2)!}\cdot\frac{4!}{2!(4-2)!}\cdot\frac{44!}{1!(44-1)!}\\
&= \frac{13\cdot 12\cdot 11!}{(2\cdot 1)\cdot 11!}\cdot\frac{4\cdot 3\cdot 2!}{(2\cdot 1)\cdot 2!}\cdot\frac{4\cdot 3\cdot 2!}{(2\cdot 1)\cdot 2!}\cdot\frac{44\cdot 43!}{1\cdot 43!}\\
&= 78\cdot 6\cdot 6\cdot 44 \;=\; 123{,}552.
\end{aligned}$$

b. The total number of five-card hands from an ordinary deck of cards is $\binom{52}{5} = 2{,}598{,}960$. Thus if all hands are equally likely, the probability of obtaining a hand with two pairs is $\dfrac{123{,}552}{2{,}598{,}960} \cong 4.75\%$. ■

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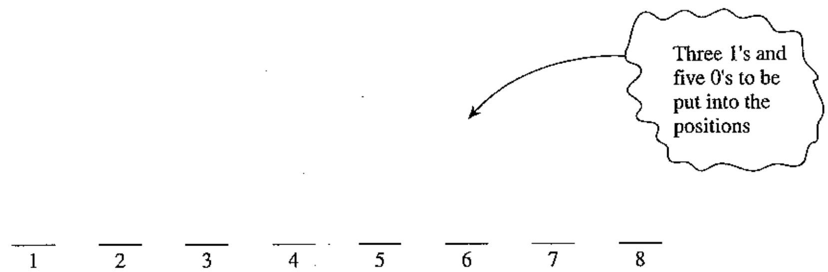
Example 6.4.10 Number of Bit Strings with Fixed Number of 1's

How many eight-bit strings have exactly three 1's?

Solution To solve this problem, imagine eight empty positions into which the 0's and 1's of the bit string will be placed.



Once a subset of three positions has been chosen from the eight to contain 1's, then the remaining five positions must all contain 0's (since the string is to have exactly three 1's). It follows that the number of ways to construct an eight-bit string with exactly three 1's is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1's. By Theorem 6.4.1, this equals

$$\binom{8}{3} = \frac{8!}{3!\cdot 5!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{5!}} = 56.$$ ■

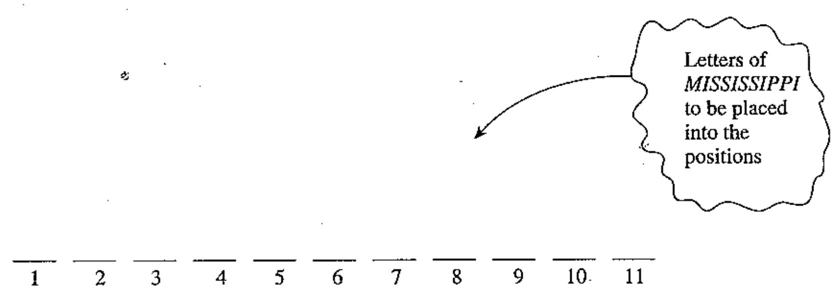
Example 6.4.11 Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*:

IIMSSPISSIP, ISSSPMIIPIS, PIMISSSSIIP, and so on.

How many distinguishable orderings are there?

Solution This example generalizes Example 6.4.10. Imagine placing the 11 letters of *MISSISSIPPI* one after another into 11 positions.



Because copies of the same letter cannot be distinguished from one another, once the positions for a certain letter are known, then all copies of the letter can go into the positions in any order. It follows that constructing an ordering for the letters can be thought of as a four-step process:

Step 1: Choose a subset of four positions for the S's.

Step 2: Choose a subset of four positions for the I's.

Step 3: Choose a subset of two positions for the P's.

Step 4: Choose a subset of one position for the M.

Since there are 11 positions in all, there are $\binom{11}{4}$ subsets of four positions for the S's. Once the four S's are in place, there are seven positions that remain empty, so there are $\binom{7}{4}$ subsets of four positions for the I's. After the I's are in place, there are three positions left empty, so there are $\binom{3}{2}$ subsets of two positions for the P's. That leaves just one position for the M. But $1 = \binom{1}{1}$. Hence by the multiplication rule,

$$\begin{aligned}
\begin{bmatrix}\text{number of ways to}\\ \text{position all the letters}\end{bmatrix} &= \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}\\
&= \frac{11!}{4!\cancel{7!}}\cdot\frac{\cancel{7!}}{4!\cancel{3!}}\cdot\frac{\cancel{3!}}{2!\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\cancel{0!}}\\
&= \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!} = 34{,}650.
\end{aligned}$$

■

In exercise 18 at the end of the section, you are asked to show that changing the order in which the letters are placed into the positions does not change the answer to this example.

The same reasoning used in this example can be used to derive the following general theorem.

Theorem 6.4.2

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinct permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

Some Advice about Counting

Students learning counting techniques often ask, "How do I know what to multiply and what to add? When do I use the multiplication rule and when do I use the addition rule?" Unfortunately, these questions have no easy answers. You need to imagine, as vividly as possible, the objects you are to count. You might even start to make an actual list of the items you are trying to count to get a sense for how to obtain them in a systematic way.

You should then construct a model that would allow you to continue counting the objects one by one if you had enough time. If you can imagine the elements to be counted as being obtained through a multistep process (in which each step is performed in a fixed number of ways regardless of how preceding steps were performed), then you can use the multiplication rule. The total number of elements will be the product of the number of ways to perform each step. If, however, you can imagine the set of elements to be counted as being broken up into disjoint subsets, then you can use the addition rule. The total number of elements in the set will be the sum of the number of elements in each subset.

One of the most common mistakes students make is to count certain possibilities more than once.

Example 6.4.12 Double Counting

Consider again the problem of Example 6.4.8(b). A group consists of five men and seven women. How many teams of five contain at least one man?

Caution! The following is a ***false solution.*** Imagine constructing the team as a two-step process:

Step 1: Choose a subset of one man from the five men.

Step 2: Choose a subset of four others from the remaining eleven people.

Hence, by the multiplication rule, there are $\binom{5}{1} \cdot \binom{11}{4} = 1{,}650$ five-person teams that contain at least one man.

Analysis of the False Solution: The problem with the solution above is that some teams are counted more than once. Suppose the men are Anwar, Ben, Carlos, Dwayne, and Ed and the women are Fumiko, Gail, Hui-Fan, Inez, Jill, Kim, and Laura. According to the method described above, one possible outcome of the two-step process is as follows:

Outcome of step 1: Anwar

Outcome of step 2: Ben, Gail, Inez, and Jill.

In this case the team would be {Anwar, Ben, Gail, Inez, Jill}. But another possible outcome is

Outcome of step 1: Ben

Outcome of step 2: Anwar, Gail, Inez, and Jill,

which also gives the team {Anwar, Ben, Gail, Inez, Jill}. Thus this one team is given b two different branches of the possibility tree, and so it is counted twice. ∎

The best way to avoid mistakes such as the one described above is to visualize th possibility tree that corresponds to any use of the multiplication rule and the set partitio that corresponds to a use of the addition rule. Check how your division into steps work by applying it to some actual data—as was done in the analysis above—and try to pic data that are as typical or generic as possible.

It often helps to ask yourself (1) "Am I counting everything?" and (2) "Am I countin anything twice?" When using the multiplication rule, these questions become (1) "Do every outcome appear as some branch of the tree?" and (2) "Does any outcome appear c more than one branch of the tree?" When using the addition rule, the questions becon (1) "Does every outcome appear in some subset of the diagram?" and (2) "Do any tw subsets in the diagram share common elements?"

Exercise Set 6.4

1. a. List all 2-combinations for the set $\{x_1, x_2, x_3\}$. Deduce the value of $\binom{3}{2}$.
   b. List all unordered selections of four elements from the set $\{a, b, c, d, e\}$. Deduce the value of $\binom{5}{4}$.

2. a. List all 3-combinations for the set $\{x_1, x_2, x_3, x_4, x_5\}$. Deduce the value of $\binom{5}{3}$.
   b. List all unordered selections of two elements from the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. Deduce the value of $\binom{6}{2}$.

3. Write an equation relating $P(7, 2)$ and $\binom{7}{2}$.

4. Write an equation relating $P(8, 3)$ and $\binom{8}{3}$.

5. Compute each of the following.
   a. $\binom{5}{0}$ b. $\binom{5}{1}$ c. $\binom{5}{2}$
   d. $\binom{5}{3}$ e. $\binom{5}{4}$ f. $\binom{5}{5}$

6. A student council consists of 15 students.
   a. In how many ways can a committee of six be selected from the membership of the council?
   b. Two council members have the same major and are not permitted to serve together on a committee. How many ways can a committee of six be selected from the membership of the council?
   c. Two council members always insist on serving on committees together. If they can't serve together, they won't serve at all. How many ways can a committee of six be selected from the council membership?
   d. Suppose the council contains eight men and seven women.
      (i) How many committees of six contain three men and three women?
      (ii) How many committees of six contain at least one woman?
   e. Suppose the council consists of three freshmen, four sophomores, three juniors, and five seniors. How many committees of eight contain two representatives from each class?

7. A computer programming team has 13 members.
   a. How many ways can a group of seven be chosen to work on a project?
   b. Suppose seven team members are women and six are men.
      (i) How many groups of seven can be chosen that contain four women and three men?
      (ii) How many groups of seven can be chosen that contain at least one man?
      (iii) How many groups of seven can be chosen that contain at most three women?
   c. Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?
   d. Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

H 8. An instructor gives an exam with twelve questions. Students are allowed to choose any ten to answer.
   a. How many different choices of ten questions are there?
   b. Suppose five questions require proof and seven do not.
      (i) How many groups of ten questions contain four that require proof and six that do not?
      (ii) How many groups of ten questions contain at least one that requires proof?
      (iii) How many groups of ten questions contain at most three that require proof?
   c. Suppose the exam instructions specify that at most one of questions 1 and 2 may be included among the ten. How many different choices of ten questions are there?
   d. Suppose the exam instructions specify that either both questions 1 and 2 are to be included among the ten or neither is to be included. How many different choices of ten questions are there?

9. A club is considering changing its by-laws. In an initial straw vote on the issue, 24 of the 40 members of the club favored the change and 16 did not. A committee of six is to be chosen from the 40 club members to devote further study to the issue.
   a. How many committees of six can be formed from the club membership?
   b. How many of the committees will contain at least three club members who, in the preliminary survey, favored the change in the by-laws?

   (If you do not have a calculator that computes values of $\binom{n}{r}$, write your answers as numeric expressions using the symbol $\binom{n}{r}$ for some particular values of n and r.)

10. Two new drugs are to be tested using a group of 60 laboratory mice, each tagged with a number for identification purposes. Drug A is to be given to 22 mice, drug B is to be given to another 22 mice, and the remaining 16 mice are to be used as controls. How many ways can the assignment of treatments to mice be made? (A single assignment involves specifying the treatment for each mouse—whether drug A, drug B, or no drug.)

✱ 11. Refer to Example 6.4.9. For each poker holding below, (1) find the number of five-card poker hands with that holding; (2) find the probability that a randomly chosen set of five cards has that holding.
   a. royal flush b. straight flush c. four of a kind
   d. full house e. flush f. straight
   g. three of a kind h. one pair
   i. no repeated denomination and not of five adjacent denominations

12. How many pairs of two distinct integers chosen from the set $\{1, 2, 3, \ldots, 101\}$ have a sum that is even?

13. A coin is tossed ten times. In each case the outcome H (for heads) or T (for tails) is recorded. (One possible outcome of the ten tossings is denoted $THHTTTHTTH$.)
  a. What is the total number of possible outcomes of the coin-tossing experiment?
  b. In how many of the possible outcomes are exactly five heads obtained?
  c. In how many of the possible outcomes are at least eight heads obtained?
  d. In how many of the possible outcomes is at least one head obtained?
  e. In how many of the possible outcomes is at most one head obtained?

14. a. How many 16-bit strings contain exactly seven 1's?
  b. How many 16-bit strings contain at least thirteen 1's?
  c. How many 16-bit strings contain at least one 1?
  d. How many 16-bit strings contain at most one 1?

15. a. How many even integers are in the set

$$\{1, 2, 3, \ldots, 100\}?$$

  b. How many odd integers are in the set

$$\{1, 2, 3, \ldots, 100\}?$$

  c. How many ways can two integers be selected from the set $\{1, 2, 3, \ldots, 100\}$ so that their sum is even?
  d. How many ways can two integers be selected from the set $\{1, 2, 3, \ldots, 100\}$ so that their sum is odd?

16. Suppose that three computer boards in a production run of forty are defective. A sample of five is to be selected to be checked for defects.
  a. How many different samples can be chosen?
  b. How many samples will contain at least one defective board?
  c. What is the probability that a randomly chosen sample of five contains at least one defective board?

17. Ten points labeled $A, B, C, D, E, F, G, H, I, J$ are arranged in a plane in such a way that no three lie on the same straight line.
  a. How many straight lines are determined by the ten points?
  b. How many of these straight lines do not pass through point A?
  c. How many triangles have three of the ten points as vertices?
  d. How many of these triangles do not have A as a vertex?

18. Suppose that you placed the letters in Example 6.4.11 into positions in the following order: first the M, then the I's, then the S's, and then the P's. Show that you would obtain the same answer for the number of distinguishable orderings.

19. a. How many distinguishable ways can the letters of the word $HULLABALOO$ be arranged?
  b. How many distinguishable arrangements of the letters of $HULLABALOO$ begin with U and end with L?
  c. How many distinguishable arrangements of the letters of $HULLABALOO$ contain the two letters HU next to each other in order?

20. a. How many distinguishable ways can the letters of the word $MILLIMICRON$ be arranged?
  b. How many distinguishable arrangements of the letters of $MILLIMICRON$ begin with M and end with N?
  c. How many distinguishable arrangements of the letters of $MILLIMICRON$ contain the letters CR next to each other in order and also the letters ON next to each other in order?

21. When the expression $(a + b)^4$ is multiplied out, terms of the form *aaaa, abaa, baba, bbba,* and so on are obtained. Consider the set S of all strings of length 4 over $\{a, b\}$.
  a. What is $N(S)$? In other words, how many strings of length 4 can be constructed using a's and b's?
  b. How many strings of length 4 over $\{a, b\}$ have three a's and one b?
  c. How many strings of length 4 over $\{a, b\}$ have two a's and two b's?

A 22. In Morse code, symbols are represented by variable-length sequences of dots and dashes. (For example, $A = \cdot -$, $1 = \cdot - - - -$, $? = \cdot \cdot - - \cdot \cdot$.) How many different symbols can be represented by sequences of seven or fewer dots and dashes?

23. Each symbol in the Braille code is represented by a rectangular arrangement of six dots, each of which may be raised or flat against a smooth background. For instance, when the word Braille is spelled out, it looks like this:

Given that at least one of the six dots must be raised, how many symbols can be represented in the Braille code?

24. On an 8×8 chessboard, a rook is allowed to move any number of squares either horizontally or vertically. How many different paths can a rook follow from the bottom-left square of the board to the top-right square of the board if all moves are to the right or upward?

25. The number 42 has the prime factorization $2 \cdot 3 \cdot 7$. Thus 42 can be written in four ways as a product of two positive integer factors: $1 \cdot 42$, $6 \cdot 7$, $14 \cdot 3$, and $2 \cdot 21$.
  a. List the distinct ways the number 210 can be written as a product of two positive integer factors.
  b. If $n = p_1 p_2 p_3 p_4$, where the p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?
  c. If $n = p_1 p_2 p_3 p_4 p_5$, where the p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?
  d. If $n = p_1 p_2 \cdots p_k$, where the p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?

H ∗26. A student council consists of three freshmen, four sophomores, four juniors, and five seniors. How many committees of eight members of the council contain at least one member from each class?

∗27. An alternative way to derive Theorem 6.4.1 uses the following *division rule:* Let n and k be integers so that k divides n. If a set consisting of n elements is divided into subsets that each contain k elements, then the number of such subsets is n/k. Explain how Theorem 6.4.1 can be derived using the division rule.

28. Find the error in the following reasoning: "Consider forming a poker hand with two pairs as a five-step process.

Step 1: Choose the denomination of one of the pairs.

Step 2: Choose the two cards of that denomination.

Step 3: Choose the denomination of the other of the pairs.

Step 4: Choose the two cards of that second denomination.

Step 5: Choose the fifth card from the remaining denominations.

There are $\binom{13}{1}$ ways to perform step 1, $\binom{4}{2}$ ways to perform step 2, $\binom{12}{1}$ ways to perform step 3, $\binom{4}{2}$ ways to perform step 4, and $\binom{44}{1}$ ways to perform step 5. Therefore, the total number of five-card poker hands with two pairs is $13 \cdot 6 \cdot 12 \cdot 6 \cdot 44 = 247{,}104$."

6.5 *r*-Combinations with Repetition Allowed

The value of mathematics in any science lies more in disciplined analysis and abstract thinking than in particular theories and techniques. —Alan Tucker, 1982

In Section 6.4 we showed that there are $\binom{n}{r}$ r-combinations, or subsets of size r, of a set of n elements. In other words, there are $\binom{n}{r}$ ways to choose r distinct elements without regard to order from a set of n elements. For instance, there are $\binom{4}{3} = 4$ ways to choose three elements out of a set of four: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$.

In this section we ask: How many ways are there to choose r elements without regard to order from a set of n elements *if repetition is allowed?* A good way to imagine this is to visualize the n elements as categories of objects from which multiple selections may be made. For instance, if the categories are labeled 1, 2, 3, and 4 and three elements are chosen, it is possible to choose two elements of type 3 and one of type 1, or all three of type 2, or one each of types 1, 2 and 4. We denote such choices by $[3, 3, 1]$, $[2, 2, 2]$, and $[1, 2, 4]$, respectively. Note that because order does not matter, $[3, 3, 1] = [3, 1, 3] = [1, 3, 3]$, for example.

• Definition

An **r-combination with repetition allowed,** or **multiset of size r,** chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

Example 6.5.1 *r*-Combinations with Repetition Allowed

Write a complete list to find the number of 3-combinations with repetition allowed, or multisets of size 3, that can be selected from $\{1, 2, 3, 4\}$. Observe that because the order in which the elements are chosen does not matter, the elements of each selection may be written in increasing order, and writing the elements in increasing order will ensure that no combinations are overlooked.

26. a. The number of students who checked at least one of the magazines is $N(T \cup N \cup U) = N(T) + N(N) + N(U) - N(T \cap N) - N(T \cap U) - N(N \cap U) + N(T \cap N \cap U) = 28 + 26 + 14 - 8 - 4 - 3 + 2 = 55$.

b. By the difference rule, the number of students who checked none of the magazines is the total number of students minus the number who checked at least one magazine. This is $100 - 55 = 45$.

d. The number of students who read *Time* and *Newsweek* but not *U.S. News* is

$$N((T \cap N) - N(T \cap N \cap U)) = 8 - 2 = 6.$$

28. Let

$M =$ the set of married people in the sample,

$Y =$ the set of people between 20 and 30 in the sample, and

$F =$ the set of females in the sample.

Then the number of people in the set $M \cup Y \cup F$ is less than or equal to the size of the sample. And so

$$\begin{aligned}
1{,}200 &\geq N(M \cup Y \cup F) \\
&= N(M) + N(Y) + N(F) - N(M \cap Y) \\
&\quad - N(M \cap F) - N(Y \cap F) + N(M \cap Y \cap F) \\
&= 675 + 682 + 684 - 195 - 467 - 318 + 165 \\
&= 1{,}226.
\end{aligned}$$

This is impossible since $1{,}200 < 1{,}226$, so the polltaker's figures are inconsistent. They could not have occurred as a result of an actual sample survey.

30. Let A be the set of all positive integers less than 1,000 that are not multiples of 2, and let B be the set of all positive integers less than 1,000 that are not multiples of 5. Since the only prime factors of 1,000 are 2 and 5, the number of positive integers that have no common factors with 1,000 is $N(A \cap B)$. Let the universe U be the set of all positive integers less than 1,000. Then A^c is the set of positive integers less than 1,000 that are multiples of 2, B^c is the set of positive integers less than 1,000 that are multiples of 5, and $A^c \cap B^c$ is the set of positive integers less than 1,000 that are multiples of 10. By one of the procedures discussed in Section 6.1 or 6.2, it is easily found that $N(A^c) = 499$, $N(B^c) = 199$, and $N(A^c \cap B^c) = 99$. Thus, by the inclusion/exclusion rule,

$$\begin{aligned}
N(A^c \cup B^c) &= N(A^c) + N(B^c) - N(A^c \cap B^c) \\
&= 499 + 199 - 99 = 599.
\end{aligned}$$

But by De Morgan's law, $N(A^c \cup B^c) = N((A \cap B)^c)$, and so

$$N((A \cap B)^c) = 599. \qquad (*)$$

Now since $(A \cap B)^c = U - (A \cap B)$, by the difference rule we have

$$N((A \cap B)^c) = N(U) - N(A \cap B). \qquad (**)$$

Equating the right-hand sides of $(*)$ and $(**)$ gives $N(U) - N(A \cap B) = 599$. And because $N(U) = 999$, we conclude that $999 - N(A \cap B) = 599$, or, equivalently, $N(A \cap B) = 999 - 599 = 400$. So there are 400 positive integers less than 1,000 that have no common factor with 1,000.

36. *Hint:* Use the generalized distributive law for sets from exercise 35, Section 5.2.

Section 6.4

1. a. 2-combinations: $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$.

Hence, $\binom{3}{2} = 3$.

b. Unordered selections: $\{a, b, c, d\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$, $\{a, c, d, e\}$, $\{b, c, d, e\}$.

Hence, $\binom{5}{4} = 5$.

3. $P(7, 2) = \binom{7}{2} \cdot 2!$

5. a. $\binom{5}{0} = \dfrac{5!}{0!(5-0)!} = \dfrac{\cancel{5!}}{1 \cdot \cancel{5!}} = 1$

b. $\binom{5}{1} = \dfrac{5!}{1!(5-1)!} = \dfrac{5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{1 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}} = 5$

6. a. number of committees of 6

$$\begin{aligned}
&= \binom{15}{6} = \frac{15!}{(15-6)!6!} \\
&= \frac{\cancel{15} \cdot \overset{7}{\cancel{14}} \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot \cancel{9!}}{\cancel{9!} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 5{,}005
\end{aligned}$$

b.

$$\begin{aligned}
&\left[\begin{array}{l}\text{number of committees}\\ \text{that don't contain } A\\ \text{and } B \text{ together}\end{array}\right] \\
&= \left[\begin{array}{l}\text{number of}\\ \text{committees with } A\\ \text{and five others—}\\ \text{none of them } B\end{array}\right] + \left[\begin{array}{l}\text{number of}\\ \text{committees with } B\\ \text{and five others—}\\ \text{none of them } A\end{array}\right] \\
&\qquad + \left[\begin{array}{l}\text{number of committees}\\ \text{with neither } A \text{ nor } B\end{array}\right] \\
&= \binom{13}{5} + \binom{13}{5} + \binom{13}{6} \\
&= 1{,}287 + 1{,}287 + 1{,}716 = 4{,}290
\end{aligned}$$

Alternative solution:

$$\begin{aligned}
&\left[\begin{array}{l}\text{number of committees}\\ \text{that don't contain } A\\ \text{and } B \text{ together}\end{array}\right] \\
&= \left[\begin{array}{l}\text{total number}\\ \text{of committees}\end{array}\right] - \left[\begin{array}{l}\text{number of committees}\\ \text{that contain both } A \text{ and } B\end{array}\right] \\
&= \binom{15}{6} - \binom{13}{4} \\
&= 5{,}005 - 715 = 4{,}290
\end{aligned}$$

c. $\begin{bmatrix}\text{number of}\\ \text{committees with}\\ \text{both } A \text{ and } B\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{committees with}\\ \text{neither } A \text{ and } B\end{bmatrix}$

$$= \binom{13}{4} + \binom{13}{6} = 715 + 1{,}716 = 2{,}431$$

d. (i) $\begin{bmatrix}\text{number of subsets}\\ \text{of three men}\\ \text{chosen from eight}\end{bmatrix} \cdot \begin{bmatrix}\text{number of subsets}\\ \text{of three women}\\ \text{chosen from seven}\end{bmatrix}$

$$= \binom{8}{3}\binom{7}{3} = 56 \cdot 35 = 1{,}960$$

(ii) $\begin{bmatrix}\text{number of committees}\\ \text{with at least one woman}\end{bmatrix}$

$$= \begin{bmatrix}\text{total number of}\\ \text{committees}\end{bmatrix} - \begin{bmatrix}\text{number of all-male}\\ \text{committees}\end{bmatrix}$$

$$= \binom{15}{6} - \binom{8}{6} = 5{,}005 - 28$$

$$= 4{,}977$$

e. $\begin{bmatrix}\text{number of}\\ \text{ways to choose}\\ \text{two freshmen}\end{bmatrix} \cdot \begin{bmatrix}\text{number of}\\ \text{ways to choose two}\\ \text{sophomores}\end{bmatrix}$

$$\cdot \begin{bmatrix}\text{number of ways}\\ \text{to choose two juniors}\end{bmatrix} \cdot \begin{bmatrix}\text{number of ways}\\ \text{to choose two seniors}\end{bmatrix}$$

$$= \binom{3}{2}\binom{4}{2}\binom{3}{2}\binom{5}{2}$$

$$= 540$$

8. *Hint:* The answers are: **a.** 66 **b.** (i) 35 (ii) 66 (iii) 10 **c.** 21 **d.** 46

9. b. $\binom{24}{3}\binom{16}{3} + \binom{24}{4}\binom{16}{2} + \binom{24}{5}\binom{16}{1} + \binom{24}{6}\binom{16}{0} = 3{,}223{,}220$

11. a. (1) 4 (because there are as many royal flushes as there are suits)

(2) $\frac{4}{\binom{52}{5}} = \frac{4}{2{,}598{,}960} \cong 0.0000015$

c. (1) $13 \cdot \binom{48}{1} = 624$ (because one can first choose the denomination of the four-of-a-kind and then choose one additional card from the 48 remaining)

(2) $\frac{624}{\binom{52}{5}} = \frac{624}{2{,}598{,}960} = 0.00024$

f. (1) Imagine constructing a straight (including a straight flush and a royal flush) as a six-step process: step 1 is to choose the lowest denomination of any card of the five (which can be any one of $A, 2, \ldots, 10$), step 2 is to choose a card of that denomination, step 3 is to choose a card of the next higher denomination, and so forth until all five cards have been selected. By the multiplication rule, the number of ways to perform this process is

$$10 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10 \cdot 4^5 = 10{,}240.$$

By parts (a) and (b), 40 of these numbers represent royal or straight flushes, so there are $10{,}240 - 40 = 10{,}200$ straights in all.

(2) $\frac{10{,}200}{\binom{52}{5}} = \frac{10{,}200}{2{,}598{,}960} \cong 0.0039$

13. a. $2^{10} = 1{,}024$

d. $\begin{bmatrix}\text{number of outcomes}\\ \text{with at least one head}\end{bmatrix}$

$$= \begin{bmatrix}\text{total number}\\ \text{of outcomes}\end{bmatrix} - \begin{bmatrix}\text{number of outcomes}\\ \text{with no heads}\end{bmatrix}$$

$$= 1{,}024 - 1 = 1{,}023$$

15. a. 50 **b.** 50

c. To get an even sum, both numbers must be even or both must be odd. Hence

$$\begin{bmatrix}\text{number of subsets of two integers from}\\ \text{1 to 100 inclusive whose sum is even}\end{bmatrix}$$

$$= \begin{bmatrix}\text{number of subsets}\\ \text{of two even}\\ \text{integers chosen from}\\ \text{the 50 possible}\end{bmatrix} + \begin{bmatrix}\text{number of subsets}\\ \text{of two odd}\\ \text{integers chosen from}\\ \text{the 50 possible}\end{bmatrix}$$

$$= \binom{50}{2} + \binom{50}{2} = 2{,}450.$$

d. To obtain an odd sum, one of the numbers must be even and the other odd. Hence the answer is $\binom{50}{1} \cdot \binom{50}{1} = 2{,}500$. Alternatively, note that the answer equals the total number of subsets of two integers chosen from 1 through 100 minus the number of such subsets for which the sum of the elements is even. Thus the answer is $\binom{100}{2} - 2{,}450 = 2{,}500$.

17. a. Two points determine a line. Hence

$$\begin{bmatrix}\text{number of straight}\\ \text{lines determined}\\ \text{by the ten points}\end{bmatrix} = \begin{bmatrix}\text{number of subsets}\\ \text{of two points}\\ \text{chosen from ten}\end{bmatrix}$$

$$= \binom{10}{2} = 45.$$

19. a. $\frac{10!}{2!1!1!3!2!1!} = 151{,}200$ since there are 2 A's, 1 B, 1 H, 3 L's, 2 O's, and 1 U

b. $\frac{8!}{2!1!1!2!2!} = 5{,}040$ **c.** $\frac{9!}{1!2!1!3!2!} = 15{,}120$

21. a. There are two choices for each of four positions in the string, so the answer is $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.

b. $\begin{bmatrix}\text{number of strings}\\ \text{with three } a\text{'s}\\ \text{and one } b\end{bmatrix} = \begin{bmatrix}\text{number of ways to pick}\\ \text{a subset of three positions}\\ \text{out of four into}\\ \text{which to place the } a\text{'s}\end{bmatrix}$

$$= \binom{4}{3} = 4$$

24. Rook must move seven squares to the right and seven squares up, so

$$\begin{bmatrix}\text{the number of}\\ \text{paths the rook}\\ \text{can take}\end{bmatrix} = \begin{bmatrix}\text{the number}\\ \text{of orderings}\\ \text{of seven R's}\\ \text{and seven U's}\end{bmatrix}$$

where R stands for "right" and U stands for "up"

$$= \frac{14!}{7!7!} = 3{,}432.$$

25. b. *Solution 1:* One factor can be 1, and the other factor can be the product of all the primes. (This gives 1 factorization.) One factor can be one of the primes, and the other factor can be the product of the other three. (This gives $\binom{4}{1} = 4$ factorizations.) One factor can be a product of two of the primes, and the other factor can be a product of the two other primes. The number $\binom{4}{2} = 6$ counts all possible sets of two primes chosen from the four primes, and each set of two primes corresponds to a factorization. Note, however, that the set $\{p_1, p_2\}$ corresponds to the same factorization as the set $\{p_3, p_4\}$, namely, $p_1 p_2 p_3 p_4$ (just written in a different order). In general, each choice of two primes corresponds to the same factorization as one other choice of two primes. Thus the number of factorizations in which each factor is a product of two primes is $\frac{\binom{4}{2}}{2} = 3$. (This gives 3 factorizations.) The foregoing cases account for all the possibilities, so the answer is $4 + 3 + 1 = 8$.

Solution 2: Let $S = \{p_1, p_2, p_3, p_4\}$. Let $p_1 p_2 p_3 p_4 = P$, and let $f_1 f_2$ be any factorization of P. The product of the numbers in any subset $A \subseteq S$ can be used for f_1, with the product of the numbers in A^c being f_2. There are as many ways to write $f_1 f_2$ as there are subsets of S, namely $2^4 = 16$ (by Theorem 5.3.1). But given any factors f_1 and f_2, $f_1 f_2 = f_2 f_1$. Thus counting the number of ways to write $f_1 f_2$ counts each factorization twice, so the answer is $\frac{16}{2} = 8$.

26. *Hint:* Use the difference rule and the generalization of the inclusion/exclusion rule for 4 sets. (See exercise 36 in Section 6.3.)

Section 6.5

1. a. $\binom{5+3-1}{5} = \binom{7}{5} = \frac{7 \cdot 6}{2} = 21$.

b. The three elements of the set are 1, 2 and 3. The 5-combinations are [1, 1, 1, 1, 1], [1, 1, 1, 1, 2], [1, 1, 1, 1, 3], [1, 1, 1, 2, 2], [1, 1, 1, 2, 3], [1, 1, 1, 3, 3], [1, 1, 2, 2, 2], [1, 1, 2, 2, 3], [1, 1, 2, 3, 3], [1, 1, 3, 3, 3], [1, 2, 2, 2, 2], [1, 2, 2, 2, 3], [1, 2, 2, 3, 3], [1, 2, 3, 3, 3], [1, 3, 3, 3, 3], [2, 2, 2, 2, 2], [2, 2, 2, 2, 3], [2, 2, 2, 3, 3], [2, 2, 3, 3, 3], [2, 3, 3, 3, 3], and [3, 3, 3, 3, 3].

2. a. $\binom{4+3-1}{4} = \binom{6}{4} = \frac{6 \cdot 5}{2} = 15$

3. a. $\binom{20+6-1}{20} = \binom{25}{20} = 53{,}130$

b. If at least three are eclairs, then 17 additional pastries are selected from six kinds. The number of selections is $\binom{17+6-1}{17} = \binom{22}{17} = 26{,}334$.

Note: In parts (a) and (b), it is assumed that the selections being counted are unordered.

c. By parts (a) and (b), the probability that at least three eclairs are among the pastries selected is $26334/53130 \cong 0.496 = 49.6\%$.

d. If exactly three of the pastries are eclairs, then 17 additional pastries are selected from five kinds. The number of selections is

$$\binom{17+5-1}{17} = \binom{21}{17} = 5{,}985.$$

Hence the probability that a random selection includes exactly three eclairs is $5985/53130 \cong 0.113 = 11.3\%$.

5. The answer equals the number of 4-combinations with repetition allowed that can be formed from a set of n elements. It is

$$\binom{4+n-1}{4} = \binom{n+3}{4} = \frac{(n+3)(n+2)(n+1)n\cancel{(n-1)!}}{4!\cancel{(n-1)!}} = \frac{n(n+1)(n+2)(n+3)}{24}.$$

8. As in Example 6.5.4, the answer is the same as the number of quadruples of integers (i, j, k, m) for which $1 \le i \le j \le k \le m \le n$. By exercise 5, this number is $\binom{n+3}{4} = \frac{n(n+1)(n+2)(n+3)}{24}$.

10. Think of the number 20 as divided into 20 individual units and the variables x_1, x_2, and x_3 as three categories into which these units are placed. The number of units in category x_i indicates the value of x_i in a solution of the equation. By Theorem 6.5.1, the number of ways to select 20 objects from the three categories is $\binom{20+3-1}{20} = \binom{22}{20} = \frac{22 \cdot 21}{2} = 231$, so there are 231 nonnegative integer solutions to the equation.

11. The analysis for this exercise is the same as for exercise 10 except that since each $x_i \ge 1$, we can imagine taking 3 of the 20 units, placing one in each category x_1, x_2, and x_3, and then distributing the remaining 17 units among the three categories. The number of ways to do this is $\binom{17+3-1}{17} = \binom{19}{17} = \frac{19 \cdot 18}{2} = 171$, so there are 171 positive integer solutions to the equation.

18. a. Because only ten eclairs are available, any selection of 20 pastries contains k eclairs, where $0 \le k \le 10$. Since such a selection includes $20 - k$ of the other five kinds of pastry, the number of such selections is $\binom{(20-k)+5-1}{20-k} = \binom{24-k}{20-k}$ (by Theorem 6.5.1.) Therefore, by the addition rule, the total number of selections is $\sum_{k=0}^{10} \binom{24-k}{20-k}$. The numerical value of this expression is 51,128, which can be obtained using a calculator that automatically computes values of $\binom{n}{r}$ or using a symbolic manipulation computer program such as Derive, Maple, or Mathematica.

b. For each combination of k eclairs and m napoleon slices, choose $20 - (k + m)$ pastries from the remaining four