Page 114-115 textbook

Method of Direct Proof (Important)

→ Prove Universal Conditional Statements:

$\forall x \in D,\ P(x) \rightarrow Q(x)$ ← Remember, the negation of "$\forall x \in D,\ P(x) \rightarrow Q(x)$" is "$\exists x \in D \mid P(x) \wedge \sim Q(x)$"

D: a set with infinite elements

$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$	
T	T	T	the interesting cases.
T	F	F	
F	T	T	true by default
F	F	T	

→ The hard cases are when $P(x)$ is true. When $P(x)$ is true, $\boxed{P(x) \rightarrow Q(x)}$'s ~~[illegible]~~ truth is decided entirely by $Q(x)$.

Method of direct proof!

UCS
universal conditional statement

(1) Assume $P(x)$ is true

(2) Show whether $Q(x)$ is true

(3) If $Q(x)$ is true, the UCS is true.
If $Q(x)$ is false, the UCS is false.

Q) Use the method of direct proof to prove that the sum of any two even integers is even.

A) ~~…~~

$\forall x$ and y, if x and y are even integers, ← $P(x)$
then $x+y$ is even. ← $Q(x)$.

① Assume x and y are particular but arbitrarily-chosen even integers → pbac

② Show whether $x+y$ is even

Let $x = 2 \cdot k$ — k is some integer, by dfn of "even"
Let $y = 2 \cdot j$ — j is some integer, by dfn of "even"

Then $x+y = 2k + 2j$

$x+y = 2\underbrace{(k+j)}$ — sum of integers is another integer

$x+y = 2(\text{int})$ ← dfn of even.

∴ ③ $Q(x)$ is true, $x+y$ is even //

Q] Prove the sum of any two odd integers is even.

"if x and y are pbac odd int, then $x+y$ is even"

A] <u>Assume</u> that x and y are pbac odd integers.

We must show $x+y$ is even.

Let $x = 2k+1$ k is some int, by dfn of "odd"

Let $y = 2j+1$ j is some " " " " "

$$
\begin{aligned}
\text{Then } x+y &= 2k+1 + 2j+1 \\
&= 2k + 2j + 2 \\
&= 2\,(\underbrace{k+j+1}) \quad \text{sum of ints = int} \\
&= 2\,(\text{int}) \rightarrow \text{dfn of even.}
\end{aligned}
$$

$\therefore$ odd + odd = even.

Q) Use the method of direct proof
to show that for all integers n,

if n is odd $\longleftarrow P(x)$
then n^2 is odd. $\longleftarrow Q(x)$

A) Assume x is a pbac odd integer

Show x^2 is odd.

Let $x = 2k+1$ by dfn of odd
k ← some int

Then $x^2 = x \cdot x = (2k+1)(2k+1)$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$

product of integers is another integer
sum of int is int
product of ints is int

$= 2(\text{int}) + 1$ = dfn of odd.

∴ if n odd
then n^2 odd. //

Three men go to a motel.

The manager says the room costs $30.

Each man pays $10.

Later the manager realized room only costs $25. So he sent the bellhop to the room with $5.

The bellhop could not figure out how to split $5 between 3 men.

So he gave the men $1 each and kept the other $2.

Therefore the men paid $9 each for a total of $27.

Add the $2 the bellhop kept = $29.

Where did the other dollar go?

Suppose $a + b = c$

$$4a-3a \qquad 4b-3b \qquad 4c-3c$$

$$4a + 4b - 4c = 3a + 3b - 3c$$

$$4(a+b-c) = 3(a+b-c)$$

$$4 = 3$$

//

Sequences

2 4 6 8 10 12 ....

2 3 3 5 10 13 39 43 17...

2 4 6 30 32 34 36 40 ... no e

2 12 1112 3112 132112 1113122112

The Code Book
Simon Singh

Sequences 2, 4, 6, 8, 10...

value v	2	4	6	8	10
index (subscript) k	1	2	3	4	5

← 5 terms

6th term is 12

$V_6 = 6 \cdot 12 = 12$

$V_6 = k \cdot 2 = 12$

Q) Let $a_k = \frac{k}{k+1}$ $k \in \mathbb{Z}^+$

List the initial 5 terms of this sequence.

A) $a_1 = \frac{1}{1+1} = \frac{1}{2}$ (substitute 1st element from domain $\mathbb{Z}^+$ for k's)

$a_2 = \frac{2}{2+1} = \frac{2}{3}$

$a_3 = \frac{3}{4}$

$a_4 = \frac{4}{5}$

$a_5 = \frac{5}{6}$

Rule: even powers of $(-1) = +1$

eg $(-1)^4 = +1$

odd powers of $(-1) = -1$

eg $(-1)^5 = -1$

Useful for sequences such as $1, -2, 3, -4, 5, -6 \ldots$

or $-1, 2, -3, 4, -5, 6 \ldots$

Notation:

$$\underset{\substack{\downarrow \\ \text{lowest/first/} \\ \text{initial term}}}{a_m}, a_{m+1}, a_{m+2}, \ldots, a_{n-1}, \underset{\substack{\downarrow \\ \text{final} \\ \text{term}}}{a_n} \qquad m < n$$

---

dfn sequence: a set of elements written in a row.

Q) List the first 7 terms of this sequence:

$$C_k = (-1)^k \qquad k \in \mathbb{Z}^{\text{nonneg}}$$

A)
$$C_0 = (-1)^0 = 1$$
$$C_1 = (-1)^1 = -1$$
$$C_2 = 1$$
$$C_3 = -1$$
$$C_4 = 1$$
$$C_5 = -1$$
$$C_6 = 1$$

Q] Find a formula for a sequence that yields these initial terms (also specify k's domain)

$1, \ 0.25, \ 0.11111\ldots, \ 0.0625, \ \ldots$

A] $a_k = \underline{\qquad}?$

$\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \ldots$

$\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}$

$$\boxed{a_k = \frac{1}{k^2} \qquad k \in \mathbb{Z}^+}$$

Bring a calculator to midterm if you want.

Q] Find a formula that yields this sequence: (specify domain too!)

$$1-\frac{1}{2},\quad \frac{1}{2}-\frac{1}{3},\quad \frac{1}{3}-\frac{1}{4},\quad \frac{1}{4}-\frac{1}{5},\quad \ldots$$

A]

$$a_3 = \frac{1}{3} - \frac{1}{3+1}$$

$$a_k = \frac{1}{k} - \frac{1}{k+1}$$

Common denominator (cross multiply)

$$a_k = \frac{(k+1)-k}{k(k+1)}$$

$$\boxed{a_k = \frac{1}{k^2+k} \qquad k \in \mathbb{Z}^+}$$

Shorthand Summation Notation

4 terms with a sum of 30:

$$2 + 4 + 8 + 16 = 30$$

$$\boxed{\sum_{k=1}^{4} 2^k = 30}$$ } addition of 4 numbers! $2^1+2^2+2^3+2^4 = 30$

	2	4	8	16	
k	1	2	3	4	k
?	2^1	2^2	2^3	2^4	2^k

"the sum, from $k=1$ to $k=4$ of 2 to the k."

lower limit of the sum

capital sigma "sum".

upper limit of the sum

k is the index of the sum

Q Let $a_1 = 3$
$a_2 = -1$
$a_3 = 0$
$a_4 = 2$
$a_5 = 3$

(a) Find $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = 7$

(b) Find $\sum_{k=3}^{4} a_k = a_3 + a_4 = 0 + 2 = 2$

(c) Find $\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 2 = 1$

Q $\sum_{k=3}^{6} k^2$?

A $3^2 + 4^2 + 5^2 + 6^2 = 86$

-13-

Q| Express the following sequence using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \dots + \frac{n+1}{2n}$$

A| Hint: solve the numerator and the denominator independently.

How to Solve it George Polya

denominator only

value	$n+0$	$n+1$	$n+2$	$n+3$	$\dots$ $2n$	$n+k$
index k	0	1	2	3	$\dots$ n	

numerator only:

value	1	2	3	4	$n+1$	$k+1$
k	0	1	2	3	$\dots$ n	

$$\boxed{\sum_{k=0}^{n} \frac{k+1}{k+n}}$$

☆ Read 4.1 — MI Thursday

1.1, 1.2, 1.3 next ~~Tue Monday~~ Thursday July X

CHAPTER 4

SEQUENCES AND MATHEMATICAL INDUCTION

One of the most important tasks of mathematics is to discover and characterize regular patterns, such as those associated with processes that are repeated. The main mathematical structure used to study repeated processes is the *sequence,* and the main mathematical tool used to verify conjectures about patterns governing the arrangement of terms in sequences is *mathematical induction.* In this chapter we introduce the notation and terminology of sequences, show how to use both the ordinary and the strong forms of mathematical induction, and give an application showing how to prove the correctness of computer algorithms.

4.1 Sequences

A mathematician, like a painter or poet, is a maker of patterns.
— G. H. Hardy, *A Mathematician's Apology,* 1940

Imagine that a person decides to count his ancestors. He has two parents, four grandparents, eight great-grandparents, and so forth, These numbers can be written in a row as

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

The symbol "..." is called an *ellipsis.* It is shorthand for "and so forth."

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row.

Position in the row	1	2	3	4	5	6	7...
Number of ancestors	2	4	8	16	32	64	128...

The number corresponding to position 1 is 2, which equals 2^1. The number corresponding to position 2 is 4, which equals 2^2. For positions 3, 4, 5, 6, and 7, the corresponding numbers are 8, 16, 32, 64, and 128, which equal $2^3, 2^4, 2^5, 2^6$, and 2^7, respectively. For a general value of k, let A_k be the number of ancestors in the kth generation back. The pattern of computed values strongly suggests the following for each k:

$$A_k = 2^k.*$$

*Strictly speaking, the true value of A_k is probably less than 2^k when k is large, because ancestors from one branch of the family tree may also appear on other branches of the tree.

In this section we define the term **sequence** informally as a set of elements written in a row. (We give a more formal definition of sequence in terms of functions in Section 7.1.) In the sequence denoted

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n,$$

each individual element a_k (read "a sub k") is called a **term.** The k in a_k is called a **subscript** or **index,** m (which may be any integer) is the subscript of the **initial term,** and n (which must be greater than or equal to m) is the subscript of the **final term.** The notation

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

denotes an **infinite sequence.** An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

The following example shows that it is possible for two different formulas to give sequences with the same terms.

Example 4.1.1 Finding Terms of Sequences Given by Explicit Formulas

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for all integers } k \geq 1,$$

$$b_i = \frac{i-1}{i} \quad \text{for all integers } i \geq 2.$$

Compute the first five terms of both sequences.

Solution

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

As you can see, the first terms of both sequences are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$; in fact, it can be shown that all terms of both sequences are identical. ■

The next example shows that an infinite sequence may have only a finite number of values.

Example 4.1.2 An Alternating Sequence

Compute the first six terms of the sequence $c_0, c_1, c_2, \ldots$ defined as follows:

$$c_j = (-1)^j \quad \text{for all integers } j \geq 0.$$

Solution

$$
\begin{aligned}
c_0 &= (-1)^0 = 1\\
c_1 &= (-1)^1 = -1\\
c_2 &= (-1)^2 = 1\\
c_3 &= (-1)^3 = -1\\
c_4 &= (-1)^4 = 1\\
c_5 &= (-1)^5 = -1
\end{aligned}
$$

Thus the first six terms are $1, -1, 1, -1, 1, -1$. By exercises 29 and 30 of Section 3.1, even powers of -1 equal 1 and odd powers of -1 equal -1. It follows that the sequence oscillates endlessly between 1 and -1. ■

In Examples 4.1.1 and 4.1.2 the task was to compute initial values of a sequence given by an explicit formula. The next example treats the question of how to find an explicit formula for a sequence with given initial terms. Any such formula is a guess, but it is very useful to be able to make such guesses.

Example 4.1.3 Finding an Explicit Formula to Fit Given Initial Terms

Find an explicit formula for a sequence that has the following initial terms:

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \ldots$$

Solution Denote the general term of the sequence by a_k and suppose the first term is a_1. Then observe that the denominator of each term is a perfect square. Thus the terms can be rewritten as

$$
\begin{array}{cccccc}
\dfrac{1}{1^2}, & \dfrac{(-1)}{2^2}, & \dfrac{1}{3^2}, & \dfrac{(-1)}{4^2}, & \dfrac{1}{5^2}, & \dfrac{(-1)}{6^2}.\\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow\\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6
\end{array}
$$

Note that the denominator of each term equals the square of the subscript of that term, and that the numerator equals ± 1. Hence

$$a_k = \frac{\pm 1}{k^2}.$$

Also the numerator oscillates back and forth between $+1$ and -1; it is $+1$ when k is odd and -1 when k is even. To achieve this oscillation, insert a factor of $(-1)^{k+1}$ (or $(-1)^{k-1}$) into the formula for a_k. [*For when k is odd, $k+1$ is even and thus* $(-1)^{k+1} = +1$; *and when k is even, $k+1$ is odd and thus* $(-1)^{k+1} = -1$.] Consequently, an explicit formula that gives the correct first six terms is

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1.$$

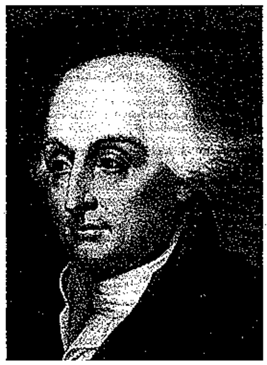
Note that making the first term a_0 would have led to the alternative formula

$$a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for all integers } k \geq 0.$$

You should check that this formula also gives the correct first six terms. ■

> **Caution!** Two sequences may start off with the same initial values but diverge later on. See exercise 7 at the end of this section.

Summation Notation



CORBIS

Joseph Louis Lagrange (1736–1813)

Consider again the example in which $A_k = 2^k$ represented the number of ancestors a person has in the kth generation back. What is the total number of ancestors for the past six generations? The answer is

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 126.$$

It is convenient to use a shorthand notation to write such sums. In 1772 the French mathematician Joseph Louis Lagrange introduced the capital Greek letter sigma, Σ, to denote the word *sum* (or *summation*), and the notation

$$\sum_{k=1}^{n} a_k$$

to represent the sum given in **expanded form** by

$$a_1 + a_2 + a_3 + \cdots + a_n.$$

More generally, if m and n are integers and $m \leq n$, then the **summation from k equals m to n of a_k** is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

and call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Example 4.1.4 Computing Summations

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$. Compute the following:

a. $\displaystyle\sum_{k=1}^{5} a_k$ b. $\displaystyle\sum_{k=2}^{2} a_k$ c. $\displaystyle\sum_{k=1}^{2} a_{2k}$

Solution

a. $\displaystyle\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\displaystyle\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\displaystyle\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$ ■

Oftentimes, the terms of a summation are expressed using an explicit formula. For instance, it is common to see summations such as

$$\sum_{k=1}^{5} k^2 \quad \text{or} \quad \sum_{i=0}^{8} \frac{(-1)^i}{i+1}.$$

Example 4.1.5 When the Terms of a Summation Are Given by a Formula

Compute the following summation:

$$\sum_{k=1}^{5} k^2.$$

Solution

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

■

When the upper limit of a summation is a variable, an ellipsis is used to write the summation in expanded form.

Example 4.1.6 Changing from Summation Notation to Expanded Form

Write the following summation in expanded form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}.$$

Solution

$$\begin{aligned}
\sum_{i=0}^{n} \frac{(-1)^i}{i+1} &= \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1} \\
&= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1} \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}
\end{aligned}$$

■

Example 4.1.7 Changing from Expanded Form to Summation Notation

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}.$$

Solution The general term of this summation can be expressed as $\frac{k+1}{n+k}$ for integers k from 0 to n. Hence

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}.$$

■

> ***Caution!*** The expanded form of a sum may appear ambiguous for small values of n. For instance, consider
>
> $$1^2 + 2^2 + 3^2 + \cdots + n^2.$$
>
> This expression is intended to represent the sum of squares of consecutive integers starting with 1^2 and ending with n^2. Thus, if $n = 1$ the sum is just 1^2, if $n = 2$ the sum is $1^2 + 2^2$, and if $n = 3$ the sum is $1^2 + 2^2 + 3^2$.

Example 4.1.8 Evaluating $a_1, a_2, a_3, \ldots, a_n$ for Small n

What is the value of the expression $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$ when $n = 1$? $n = 2$? $n = 3$?

Solution

When $n = 1$, the expression equals $\frac{1}{1 \cdot 2} = \frac{1}{2}$.

When $n = 2$, it equals $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$.

When $n = 3$, it is $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{6}{12} + \frac{2}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$. ■

A more mathematically precise definition of summation, called a *recursive definition*, is the following:* If m and n are any integers with $m < n$, then

$$\sum_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n \quad \text{for all integers } n > m.$$

When solving problems, it is often useful to rewrite a summation using the recursive form of the definition, either by separating off the final term of a summation or by adding a final term to a summation.

Example 4.1.9 Separating Off a Final Term and Adding On a Final Term

a. Rewrite $\sum_{i=1}^{n} \frac{1}{k^2}$ by separating off the final term.

b. Write $\sum_{i=0}^{n-1} 2^k + 2^n$ as a single summation.

Solution

a. $\sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{n-1} \frac{1}{i^2} + \frac{1}{n^2}$ b. $\sum_{k=0}^{n-1} 2^k + 2^n = \sum_{k=0}^{n} 2^k$ ■

In certain sums each term is a difference of two quantities. When you write such sums in expanded form, you sometimes see that all the terms cancel except the first and the last. Successive cancellation of terms collapses the sum like a telescope.

*Recursive definitions are discussed in Section 8.4.

Example 4.1.10 A Telescoping Sum

Some sums can be transformed into telescoping sums, which then can be rewritten as a simple expression. For instance, observe that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

Use this identity to find a simple expression for $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Solution

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{k(k+1)} \\
&\quad = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&\quad = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\
&\quad = 1 - \frac{1}{n+1}.
\end{aligned}$$

■

Product Notation

The notation for the product of a sequence of numbers is analogous to the notation for their sum. The Greek capital letter pi, Π, denotes a product. For example,

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5.$$

More generally, the **product from k equals m to n of a_k** is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. That is,

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

A recursive definition for the product notation is the following: If m and n are any integers with $m < n$, then

$$\prod_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

Example 4.1.11 Computing Products

Compute the following products:

a. $\displaystyle\prod_{k=1}^{5} k$ b. $\displaystyle\prod_{k=1}^{1} \frac{k}{k+1}$

Solution

a. $\displaystyle\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ b. $\displaystyle\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

■

Factorial Notation

The product of all consecutive integers up to a given integer occurs so often in mathematics that it is given a special notation—*factorial* notation.

> **• Definition**
>
> For each positive integer n, the quantity **n factorial** denoted $n!$, is defined to be the product of all the integers from 1 to n:
>
> $$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$
>
> **Zero factorial,** denoted $0!$, is defined to be 1:
>
> $$0! = 1.$$

The definition of zero factorial as 1 may seem odd, but, as you will see when you read Chapter 6, it is convenient for many mathematical formulas.

Example 4.1.12 The First Ten Factorials

$$\begin{array}{ll} 0! = 1 & 1! = 1 \\ 2! = 2 \cdot 1 = 2 & 3! = 3 \cdot 2 \cdot 1 = 6 \\ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 & 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 & 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040 \\ 8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 & 9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ \quad = 40{,}320 & \quad = 362{,}880 \end{array}$$

■

As you can see from the example above, the values of $n!$ grow very rapidly: For instance, $40! \cong 8.16 \times 10^{47}$, which is a number that is too large to be computed exactly using the standard integer arithmetic of the machine-specific implementations of many computer languages. (The symbol $\cong$ means "is approximately equal to.")

A recursive definition for factorial is the following: Given any nonnegative integer n,

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}.$$

Example 4.1.13 illustrates the usefulness of the recursive definition for making computations.

Example 4.1.13 Computing with Factorials

Simplify the following expressions:

a. $\dfrac{8!}{7!}$ b. $\dfrac{5!}{2! \cdot 3!}$ c. $\dfrac{1}{2! \cdot 4!} + \dfrac{1}{3! \cdot 3!}$ d. $\dfrac{(n+1)!}{n!}$ e. $\dfrac{n!}{(n-3)!}$

Solution

a. $\dfrac{8!}{7!} = \dfrac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$

b. $\dfrac{5!}{2! \cdot 3!} = \dfrac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$

c. $$\frac{1}{2!\cdot 4!}+\frac{1}{3!\cdot 3!}=\frac{1}{2!\cdot 4!}\cdot\frac{3}{3}+\frac{1}{3!\cdot 3!}\cdot\frac{4}{4}$$ by multiplying each numerator and denominator by just what is necessary to obtain a common denominator

$$=\frac{3}{3\cdot 2!\cdot 4!}+\frac{4}{3!\cdot 4\cdot 3!}$$ by rearranging factors

$$=\frac{3}{3!\cdot 4!}+\frac{4}{3!\cdot 4!}$$ because $3\cdot 2!=3!$ and $4\cdot 3!=4!$

$$=\frac{7}{3!\cdot 4!}$$ by the rule for adding fractions with a common denominator

$$=\frac{7}{144}$$

d. $$\frac{(n+1)!}{n!}=\frac{(n+1)\cdot\cancel{n!}}{\cancel{n!}}=n+1$$

e. $$\frac{n!}{(n-3)!}=\frac{n\cdot(n-1)\cdot(n-2)\cdot\cancel{(n-3)!}}{\cancel{(n-3)!}}=n\cdot(n-1)\cdot(n-2)$$
$$=n^3-3n^2+2n$$ ■

Properties of Summations and Products

The following theorem states general properties of summations and products. The proof of the theorem is discussed in Section 8.4.

Theorem 4.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k+b_k)$$

2. $$c\cdot\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c\cdot a_k$$ generalized distributive law

3. $$\left(\prod_{k=m}^{n} a_k\right)\cdot\left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n}(a_k\cdot b_k).$$

Example 4.1.14 Using Properties of Summation and Product

Let $a_k = k+1$ and $b_k = k-1$ for all integers k. Write each of the following expressions as a single summation or product:

a. $\displaystyle\sum_{k=m}^{n} a_k + 2\cdot\sum_{k=m}^{n} b_k$ b. $\displaystyle\left(\prod_{k=m}^{n} a_k\right)\cdot\left(\prod_{k=m}^{n} b_k\right)$

Solution

a.
$$\begin{aligned}
\sum_{k=m}^{n} a_k + 2\cdot\sum_{k=m}^{n} b_k &= \sum_{k=m}^{n}(k+1) + 2\cdot\sum_{k=m}^{n}(k-1) && \text{by substitution}\\
&= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2\cdot(k-1) && \text{by Theorem 4.1.1 (2)}\\
&= \sum_{k=m}^{n}((k+1) + 2\cdot(k-1)) && \text{by Theorem 4.1.1 (1)}\\
&= \sum_{k=m}^{n}(3k-1) && \text{by algebraic simplification}
\end{aligned}$$

b.
$$\begin{aligned}
\left(\prod_{k=m}^{n} a_k\right)\cdot\left(\prod_{k=m}^{n} b_k\right) &= \left(\prod_{k=m}^{n}(k+1)\right)\cdot\left(\prod_{k=m}^{n}(k-1)\right) && \text{by substitution}\\
&= \prod_{k=m}^{n}(k+1)\cdot(k-1) && \text{by Theorem 4.1.1 (3)}\\
&= \prod_{k=m}^{n}(k^2-1) && \text{by algebraic simplification}
\end{aligned}$$

■

Change of Variable

Observe that

$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$

and also that

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$$

Hence

$$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$$

This equation illustrates the fact that the symbol used to represent the index of a summation can be replaced by any other symbol as long as the replacement is made in each location where the symbol occurs. As a consequence, the index of a summation is called a dummy variable. A dummy variable is a symbol that derives its entire meaning from its local context. Outside of that context (both before and after), the symbol may have another meaning entirely.

The appearance of a summation can be altered by more complicated changes of variable as well. For example, observe that

$$\begin{aligned}\sum_{j=2}^{4}(j-1)^2 &= (2-1)^2+(3-1)^2+(4-1)^2\\ &= 1^2+2^2+3^2\\ &= \sum_{k=1}^{3}k^2.\end{aligned}$$

A general procedure to transform the first summation into the second is illustrated in Example 4.1.15.

Example 4.1.15 Transforming a Sum by a Change of Variable

Transform the following summation by making the specified change of variable.

$$\textit{summation: } \sum_{k=0}^{6}\frac{1}{k+1} \qquad \textit{change of variable: } j=k+1$$

Solution First calculate the lower and upper limits of the new summation:

$$\text{When } k=0, \quad j=k+1=0+1=1.$$
$$\text{When } k=6, \quad j=k+1=6+1=7.$$

Thus the new sum goes from $j=1$ to $j=7$.

Next calculate the general term of the new summation. You will need to replace each occurrence of k by an expression in j:

$$\text{Since } j=k+1, \text{ then } k=j-1.$$
$$\text{Hence } \frac{1}{k+1}=\frac{1}{(j-1)+1}=\frac{1}{j}.$$

Finally, put the steps together to obtain

$$\sum_{k=0}^{6}\frac{1}{k+1}=\sum_{j=1}^{7}\frac{1}{j}. \qquad 4.1.1$$

■

Equation (4.1.1) can be given an additional twist by noting that because the j in the right-hand summation is a dummy variable, it may be replaced by any other variable name, as long as the substitution is made in every location where j occurs. In particular, it is legal to substitute k in place of j to obtain

$$\sum_{j=1}^{7}\frac{1}{j}=\sum_{k=1}^{7}\frac{1}{k}. \qquad 4.1.2$$

Putting equations (4.1.1) and (4.1.2) together gives

$$\sum_{k=0}^{6}\frac{1}{k+1}=\sum_{k=1}^{7}\frac{1}{k}.$$

Sometimes it is necessary to shift the limits of one summation in order to add it to another. An example is the algebraic proof of the binomial theorem, given in Section 6.7. A general procedure for making such a shift when the upper limit is part of the summand is illustrated in Example 4.1.16.

Example 4.1.16 When the Upper Limit Appears in the Expression to Be Summed

a. Transform the following summation by making the specified change of variable.

$$\textit{summation: } \sum_{k=1}^{n+1}\left(\frac{k}{n+k}\right) \qquad \textit{change of variable: } j = k-1$$

b. Transform the summation obtained in part (a) by changing all j's to k's.

Solution

a. When $k = 1$, then $j = k - 1 = 1 - 1 = 0$. (So the new lower limit is 0.) When $k = n + 1$, then $j = k - 1 = (n + 1) - 1 = n$. (So the new upper limit is n.)

Since $j = k - 1$, then $k = j + 1$. Also note that n is a constant as far as the terms of the sum are concerned. It follows that

$$\frac{k}{n+k} = \frac{j+1}{n+(j+1)}$$

and so the general term of the new summation is

$$\frac{j+1}{n+(j+1)}.$$

Therefore,

$$\sum_{k=1}^{n+1}\frac{k}{n+k} = \sum_{j=0}^{n}\frac{j+1}{n+(j+1)}. \qquad 4.1.3$$

b. Changing all the j's to k's in the right-hand side of equation (4.1.3) gives

$$\sum_{j=0}^{n}\frac{j+1}{n+(j+1)} = \sum_{k=0}^{n}\frac{k+1}{n+(k+1)} \qquad 4.1.4$$

Combining equations (4.1.3) and (4.1.4) results in

$$\sum_{k=1}^{n+1}\frac{k}{n+k} = \sum_{k=0}^{n}\frac{k+1}{n+(k+1)}.$$

■

Sequences In Computer Programming

An important data type in computer programming consists of finite sequences. In computer programming contexts, these are usually referred to as *one-dimensional arrays*. For example, consider a program that analyzes the wages paid to a sample of 50 workers. Such a program might compute the average wage and the difference between each individual wage and the average. This would require that each wage be stored in memory for retrieval later in the calculation. To avoid the use of entirely separate variable names for all of the 50 wages, each is written as a term of a one-dimensional array:

$$W[1], W[2], W[3], \ldots, W[50].$$

Note that the subscript labels are written inside square brackets. The reason is that until relatively recently, it was impossible to type actual dropped subscripts on most computer keyboards.

The main difficulty programmers have when using one-dimensional arrays is keeping the labels straight.

Example 4.1.17 Dummy Variable in a Loop

The index variable for a **for-next** loop is a dummy variable. For example, the following three algorithm segments all produce the same output:

1. **for** $i := 1$ **to** n	2. **for** $j := 0$ **to** $n - 1$	3. **for** $k := 2$ **to** $n + 1$
print $a[i]$	**print** $a[j + 1]$	**print** $a[k - 1]$
next i	**next** j	**next** k ■

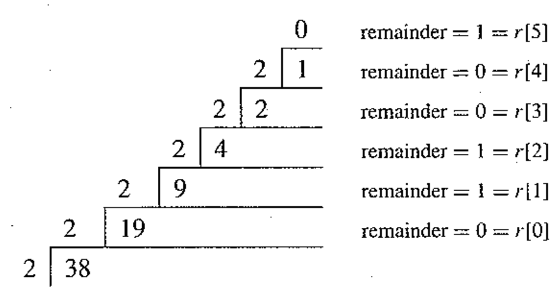
The recursive definitions for summation, product, and factorial lead naturally to computational algorithms. For instance, here are two sets of pseudocode to find the sum of $a[1], a[2], \ldots, a[n]$. The one on the left exactly mimics the recursive definition by initializing the sum to equal $a[1]$; the one on the right initializes the sum to equal 0. In both cases the output is $\sum_{k=1}^{n} a[k]$.

$s := a[1]$	$s := 0$
for $k := 2$ **to** n	**for** $k := 1$ **to** n
$s := s + a[k]$	$s := s + a[k]$
next k	**next** k

Application: Algorithm to Convert from Base 10 to Base 2 Using Repeated Division by 2

Section 1.5 contains some examples of converting integers from decimal to binary notation. The method shown there, however, is only convenient to use with small numbers. A systematic algorithm to convert any nonnegative integer to binary notation uses repeated division by 2.

Suppose a is a nonnegative integer. Divide a by 2 using the quotient-remainder theorem to obtain a quotient $q[0]$ and a remainder $r[0]$. If the quotient is nonzero, divide by 2 again to obtain a quotient $q[1]$ and a remainder $r[1]$. Continue this process until a quotient of 0 is obtained. At each stage, the remainder must be less than the divisor, which is 2. Thus each remainder is either 0 or 1. The process is illustrated below for $a = 38$. (Read the divisions from the bottom up.)



The results of all these divisions can be written as a sequence of equations:

$$\begin{aligned} 38 &= 19 \cdot 2 + 0, \\ 19 &= 9 \cdot 2 + 1, \\ 9 &= 4 \cdot 2 + 1, \\ 4 &= 2 \cdot 2 + 0, \\ 2 &= 1 \cdot 2 + 0, \\ 1 &= 0 \cdot 2 + 1. \end{aligned}$$

By repeated substitution, then,

$$\begin{aligned} 38 &= 19{\cdot}2 + 0 \\ &= (9{\cdot}2 + 1){\cdot}2 + 0 \;=\; 9{\cdot}2^2 + 1{\cdot}2 + 0 \\ &= (4{\cdot}2 + 1){\cdot}2^2 + 1{\cdot}2 + 0 \;=\; 4{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2 + 0 \\ &= (2{\cdot}2 + 0){\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2 + 0 \\ &= 2{\cdot}2^4 + 0{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2 + 0 \\ &= (1{\cdot}2 + 0){\cdot}2^4 + 0{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2 + 0 \\ &= 1{\cdot}2^5 + 0{\cdot}2^4 + 0{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2 + 0. \end{aligned}$$

Note that each coefficient of a power of 2 on the right-hand side above is one of the remainders obtained in the repeated division of 38 by 2. This is true for the left-most 1 as well, because $1 = 0 \cdot 2 + 1$. Thus

$$38_{10} = 100110_2 = (r[5]r[4]r[3]r[2]r[1]r[0])_2.$$

In general, if a nonnegative integer a is repeatedly divided by 2 until a quotient of zero is obtained and the remainders are found to be $r[0], r[1], \ldots, r[k]$, then by the quotient-remainder theorem each $r[i]$ equals 0 or 1, and by repeated substitution from the theorem,

$$a = 2^k{\cdot}r[k] + 2^{k-1}{\cdot}r[k-1] + \cdots + 2^2{\cdot}r[2] + 2^1{\cdot}r[1] + 2^0{\cdot}r[0]. \qquad 4.1.5$$

Thus the binary representation for a can be read from equation (4.1.5):

$$a_{10} = (r[k]r[k-1]\cdots r[2]r[1]r[0])_2.$$

Example 4.1.18 Converting from Decimal to Binary Notation Using Repeated Division by 2

Use repeated division by 2 to write the number 29_{10} in binary notation.

Solution

$$\begin{array}{rrl} & 0 & \text{remainder} = r[4] = 1 \\ 2\,| & 1 & \text{remainder} = r[3] = 1 \\ 2\,| & 3 & \text{remainder} = r[2] = 1 \\ 2\,| & 7 & \text{remainder} = r[1] = 0 \\ 2\,| & 14 & \text{remainder} = r[0] = 1 \\ 2\,| & 29 & \end{array}$$

Hence $29_{10} = (r[4]r[3]r[2]r[1]r[0])_2 = 11101_2$. ■

The procedure we have described for converting from base 10 to base 2 is formalized in the following algorithm:

Algorithm 4.1.1 Decimal to Binary Conversion Using Repeated Division by 2

[*In Algorithm* 4.1.1 *the input is a nonnegative integer a. The aim of the algorithm is to produce a sequence of binary digits* $r[0], r[1], r[2], \ldots, r[k]$ *so that the binary representation of a is*

$$(r[k]r[k-1]\cdots r[2]r[1]r[0])_2.$$

That is,

$$a = 2^k \cdot r[k] + 2^{k-1} \cdot r[k-1] + \cdots + 2^2 \cdot r[2] + 2^1 \cdot r[1] + 2^0 \cdot r[0].]$$

Input: a [*a nonnegative integer*]

Algorithm Body:

$q := a, i := 0$

[*Repeatedly perform the integer division of q by* 2 *until q becomes* 0. *Store successive remainders in a one-dimensional array* $r[0], r[1], r[2], \ldots, r[k]$. *Even if the initial value of q equals 0, the loop should execute one time (so that* $r[0]$ *is computed). Thus the guard condition for the* **while** *loop is* $i = 0$ *or* $q \neq 0$.]

while ($i = 0$ or $q \neq 0$)

$r[i] := q \text{ mod } 2$

$q := q \text{ div } 2$

[$r[i]$ *and q can be obtained by calling the division algorithm.*]

$i := i + 1$

end while

[*After execution of this step, the values of* $r[0], r[1], \ldots, r[i-1]$ *are all* 0*'s and* 1*'s, and* $a = (r[i-1]r[i-2]\cdots r[2]r[1]r[0])_2$.]

Output: $r[0], r[1], r[2], \ldots, r[i-1]$ [*a sequence of integers*]

Exercise Set 4.1*

Write the first four terms of the sequences defined by the formulas in 1–6.

1. $a_k = \dfrac{k}{10+k}$, for all integers $k \geq 1$.

2. $b_j = \dfrac{5-k}{5+k}$, for all integers $j \geq 1$.

3. $c_i = \dfrac{(-1)^i}{3^i}$, for all integers $i \geq 0$.

4. $d_m = 1 + \left(\dfrac{1}{2}\right)^m$ for all integers $m \geq 0$.

5. $e_n = \left\lfloor \dfrac{n}{2} \right\rfloor \cdot 2$, for all integers $n \geq 0$.

6. $f_n = \left\lfloor \dfrac{n}{4} \right\rfloor \cdot 4$, for all integers $n \geq 1$.

7. Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for all integers $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words. (A definition of logarithm is given in Section 7.1.)

8. $g_n = \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

9. $h_n = n \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms given in 10–16.

10. $-1, 1, -1, 1, -1, 1$

11. $0, 1, -2, 3, -4, 5$

12. $\dfrac{1}{4}, \dfrac{2}{9}, \dfrac{3}{16}, \dfrac{4}{25}, \dfrac{5}{36}, \dfrac{6}{49}$

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ∗ signals that an exercise is more challenging than usual.