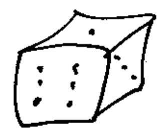
Counting and probability

Similar - sounding problems with ~ 6 different solutions.

dfn: Random: A process is random iff one outcome is certain to occur (from a set of outcomes) but it is not possible to predict with certainty which outcome that will be.

eg Roll a die  assume perfectly weighted.

Possible outcomes: $\{1,2,3,4,5,6\}$: Sample space for rolling one die

the set of all possible outcomes of a random process.

dfn: event: a subset of a sample space.

$SS = \{1,2,3,4,5,6\}$

Event $= \{4\}$ or $\{4,5\}$ → the event of rolling a 4 or 5.

<u>dfn</u>: equally-likely probability.

If S is a finite space in which all outcomes are equally likely, and E is an event in S, the probability of E, $P(E)$, is

$$\boxed{P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes in } S}}$$

$S = \{1, 2, 3, 4, 5, 6\}$

$P(4) = \frac{1}{6} = 16.7\%$

$P(4 \text{ or } 5) = \frac{1+1}{6} = 33.3\%$

Notation: For any finite set, $N(A)$ denotes the number of elements in A.

$$\boxed{P(E) = \frac{N(E)}{N(S)}}$$

The equally-likely-probability formula

A deck of playing cards:

52 cards $\left.A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\right\}$ 13

suits :- hearts } red
- diamonds

- clubs } black
- spades

3 face cards : J Q K

Q) What is the sample space?

A) $\left.\begin{matrix} \{A\heartsuit, & 2\heartsuit, & 3\heartsuit \ldots & Q\heartsuit, & K\heartsuit, \\ A\diamondsuit, & 2\diamondsuit, & \ldots & & K\diamondsuit, \\ A\clubsuit, & \ldots & & & K\clubsuit, \\ A\spadesuit, & & & & K\spadesuit\} \end{matrix}\right\}$ 52 elements

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Q) What is the event that a chosen card is a red face card?

A) $\{J\heartsuit, Q\heartsuit, K\heartsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit\}$ 6 elements.

Q) What is P(red face card)?

A) $P(\text{red face}) = \dfrac{N(\text{red face})}{N(\text{all cards})} = \dfrac{6}{52} = 11.5\%$

Q) What is P(2, 3, 4, or 5 and black)?

A) $\dfrac{8}{52} = 15.4\%$

Roll a pair of dice.

Q) Sample space?

{11, 12, 13, 14, 15, 16
21,
31
41
51
61, 62, 63, 64, 65, 66}

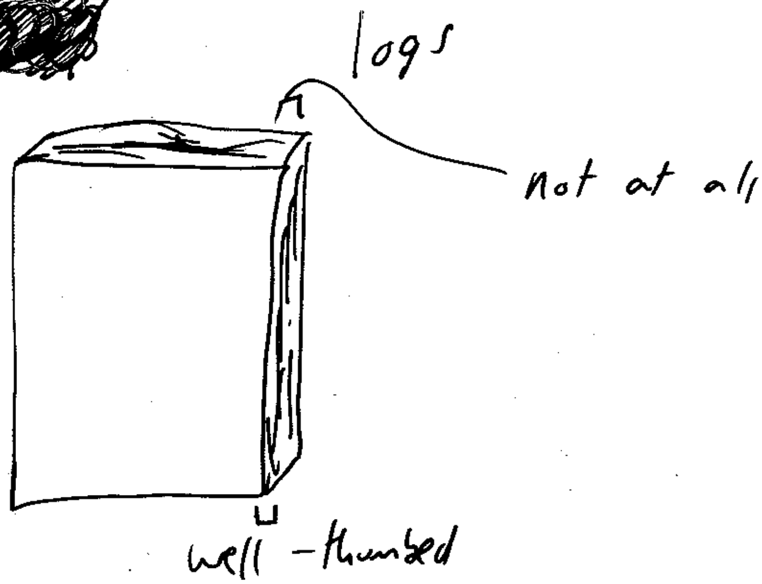
$N(SS) = 36$

A) P(Sum of 8)?

$$= \frac{N(\text{Sum of } 8)}{N(SS)} = \boxed{\frac{5}{36} = 13.9\%}$$

35
26
44
62
53

[redacted]



$1/e$

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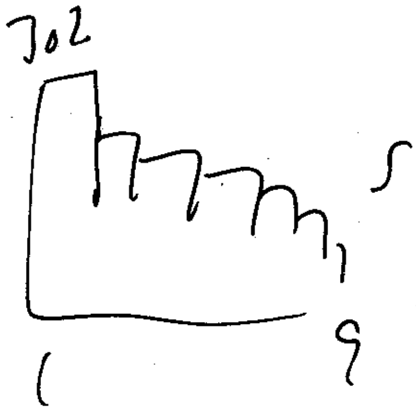
area countries

→ 30% 1 ——

→ 5% 9 ——



area rivers
pop.
cons.
news.
spec.
p...
m.

Benford's Law

Q] If three women each have two children, what is the probability that each woman has one boy and one girl?

A]

W1		W2		W3
↓		↓		↓
BB		$\frac{2}{4}$		$\frac{2}{4}$
$\frac{2}{4}$ [BG, GB				
GG	AND		AND	

"multiply"

$$\frac{2}{4} \times \frac{2}{4} \times \frac{2}{4} = \frac{8}{64} = \frac{1}{8} = 12.5\%$$

Two teams, A and B, play in a tournament until one team either:

- wins 2 games in a row

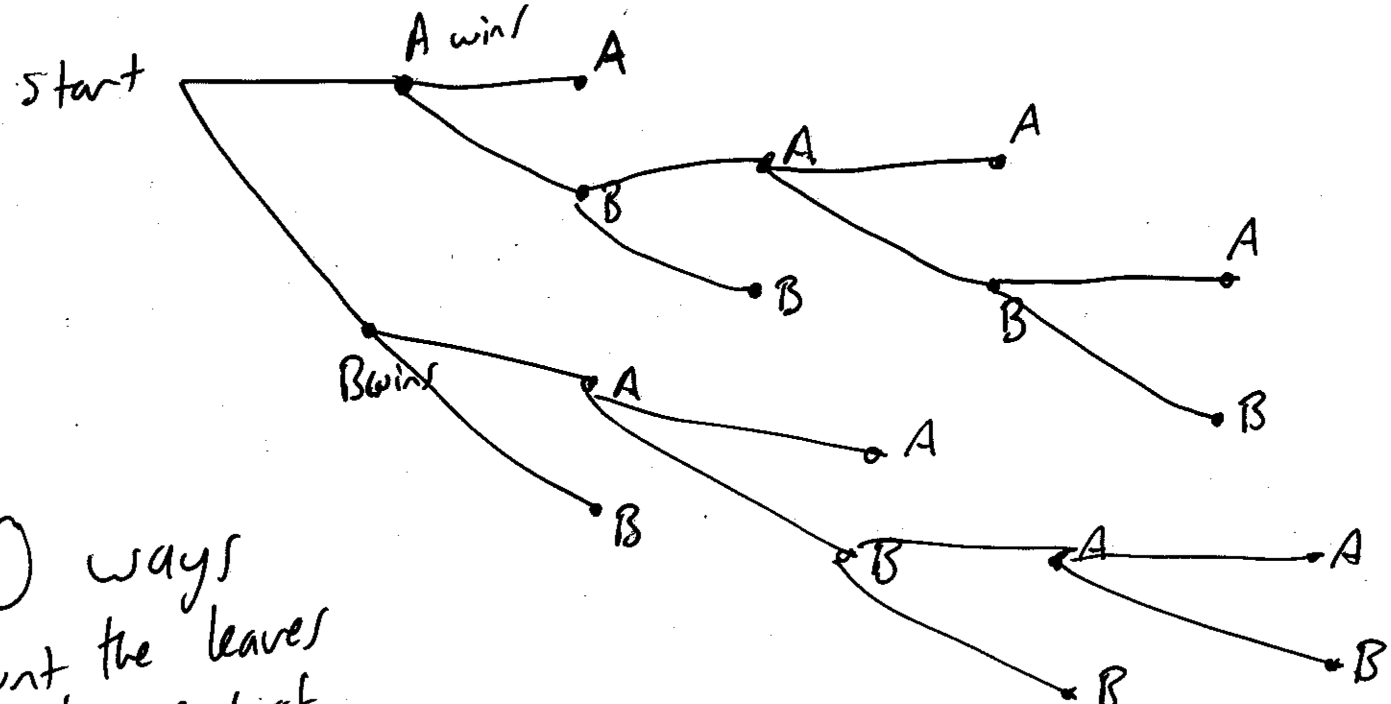
or

- wins a total of 3 games.

There are no ties.

Q) How many ways can this tournament be played?

Draw a tree:



10 ways

count the leaves (at the end of the branches).

Q) What is the probability the tournament lasts 5 games?

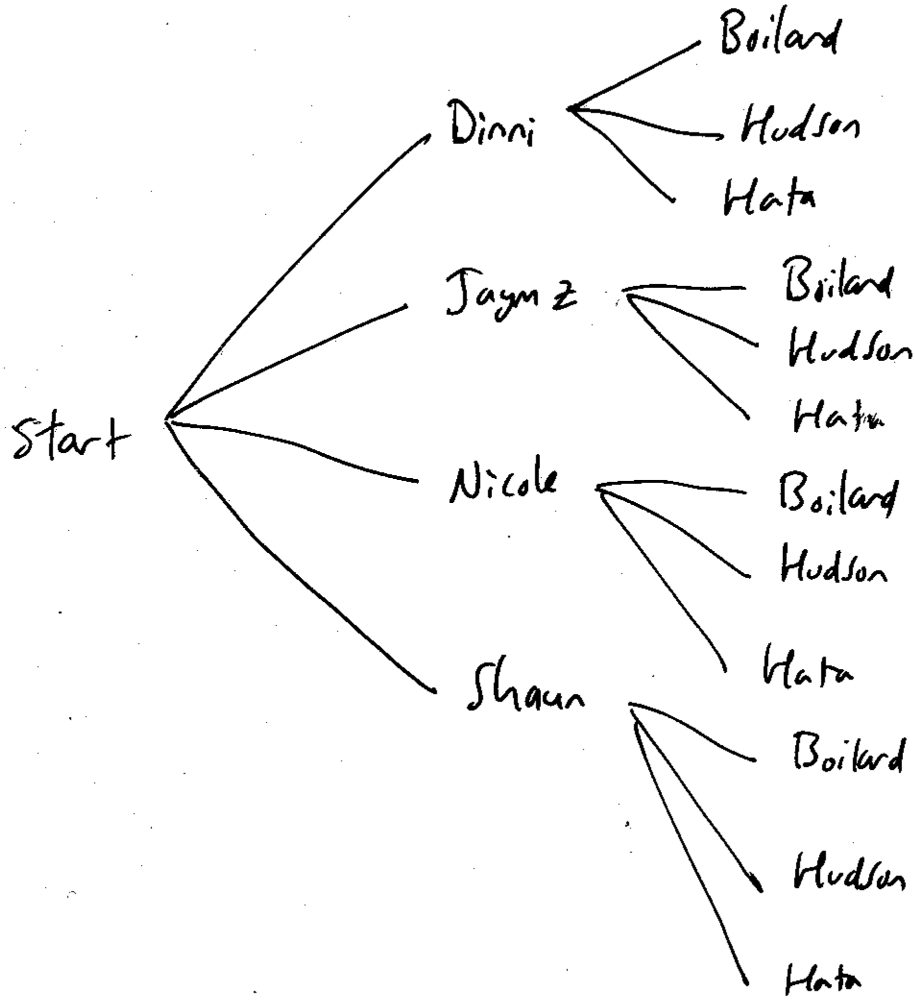
A) $P(5) = \frac{N(5)}{N(SS)} = \frac{4}{10} = 40\%.$

Q) How many different combinations of first name, last name can be made from these sets:

$F = \{$ Dinni, Jaymz, Nicole, Shaun $\}$

$L = \{$ Boilard, Hudson, Hata $\}$

A)



Count the leaves: 12 full names

OR $4 \times 3 = \underline{\underline{12}}$ Multiplication Rule

Mult. Rule:

If an operation consists of k steps and:

Step 1 can be done in n_1 ways and

Step 2 can be done in n_2 ways and

$\vdots$

Step k can be done in n_k ways, then

the ENTIRE operation can be done in

$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$ ways.

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eg Picking a BCIT password.

Choose 6 characters (from 26 letters and 10 digits).

Q: How many different passwords can be created?

A: Step 1: pick first character : 36 ways

Step 2: second 36

$\vdots$

Step 6: sixth 36 ways

repetitions allowed

$$36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6 \text{ passwords}$$

$$\approx 176\,782\,336$$

Q: Same question but no duplicate characters allowed:

A: step:

1 - 36 ways
2 - 35
3 - 34
4 - 33
5 - 32
6 - 31

1 402 410 240 passwords.

Looks like a factorial:

eg $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$0! = 1$ by definition

Q] If all passwords are equally likely, what is the probability that a password chosen at random contains a repeated character?

A] $P(\text{repeats}) = \dfrac{N(\text{repeats})}{N(SS)} = \dfrac{N(SS) - N(\text{no repeats})}{N(SS)}$

$$= \frac{2176\,782\,336 - 1402\,410\,240}{2176\,782\,336}$$

$$\boxed{= 35.6\%}$$

<u>dfn</u> Permutation: → "ordering"

A permutation of a set of objects is an ordering of the objects in a row.

eg The permutations of abc are

abc
acb
bac
bca
cab
cba

Think of permuting abc as a 3-step process:
- pick the first character: 3
- pick the second character: 2
- pick the third character: 1

∴ the # of ways to pick the entire order is $3 \cdot 2 \cdot 1 = 6$

☆ Permutations are factorial-ish.

☆ For any integer $n \geq 1$ the number of permutations of a set with n elements is $n!$

Consider the word COQUITLAM

Q) How many ways can the letters be arranged in a row?

A) $9! = 362880$ ways/words/orderings/permutations.
$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Q) Same question but QU must always occur in that order "QU".

A) Treat $\boxed{QU}$ as one letter.
$\therefore 8! = 40320$

Q) Same question but $\boxed{QU}$ could also be $\boxed{UQ}$.

A) $2 * 8! = 80640$

A lock requires 3 selections of numbers, each from 1 to 30 (inclusive).

Q) How many diff combinations are possible?

$$30 \cdot 30 \cdot 30 = 27000$$

A)

Q) How many, if no repeated #s allowed.

A) $30 \cdot 29 \cdot 28 = 24360$

Q) Probability (no repeated #s in a combination)?

A) $P\binom{no}{rep} = \dfrac{N\binom{no}{rep}}{N(comb)} = \dfrac{24360}{27000} = 90.2\%$

Q) Probability (repeated #s in a combination)?

A) $100\% - 90.2\% = 9.8\%$

Permutations of <u>selected</u> elements.

<u>dfn</u>: An r-permutation of a set of n elements is an ordered selection of r elements taken from a set of n elements.

no repeats

Notation: the # of r-permutations of a <u>set</u> of n elements is written $P(n,r)$ "n permuter"

Calculator: nPr : 49 nPr 6

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}} \quad 1 \leq r \leq n$$

<u>Q)</u> What is $P(10,6)$?

<u>A)</u> $\frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot\cancel{4!}}{\cancel{4!}} = \boxed{151200 \text{ ways}}$

Q) Consider the 26-letter alphabet.
How many different 10-letter "words" can be made?

A) $P(26,10) = \frac{26!}{(26-10)!} = 26 \cdot 25 \cdot 24 \cdot \ldots \cdot 18 \cdot 17 = $ 19 trillion

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Q) How many 5-permutations are there of a set of 5 objects?

A) $\frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120$

Q) What is $P(49,6)$?

A) $\frac{49!}{(49-6)!} = \frac{49!}{43!} =$ 10 billion

Q) How many ways can 3 letters of the word BYTES be chosen and written in a row?

A) $P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$

Q) Same question but the first letter must be T?

A) BYES } pick 2 : $P(4,2) = \frac{4!}{(4-2)!} = 4 \cdot 3 = 12$