Quantifying Knowledge Spillovers Using Firm and Product Dynamics

Mohamad Adhami*

January 26, 2026

Link To Latest Version

Abstract

Knowledge spillovers are a common rationale for government support of innovation, yet evidence on their magnitude remains limited. In this paper, I quantify the wedge that spillovers create between social and private rates of return to innovation. To do so, I build a novel semi-endogenous growth model featuring multiproduct firms and endogenous exit of products. In equilibrium, product exit exhibits negative selection and is preceded by a gradual decline in market share, consistent with facts I document using barcode-level data. Through the lens of the model, these dynamics of product exit are informative about spillovers: by accelerating growth across successive generations of new products, stronger spillovers increase the rate at which incumbent products lose market share and exit. Since comprehensive datasets track firms rather than products, I leverage the model to infer the wedge created by spillovers from data on firm exit by age. Across U.S. private nonfarm employer businesses, I infer spillovers that drive a 16 percentage point wedge between the social and private rates of return to innovation.

*Stanford University. Email: adhami@stanford.edu. I am grateful to my advisors Chad Jones and Pete Klenow for their continuous guidance and to Chris Tonetti and Nick Bloom for their generous feedback. The paper also benefited from comments from Shifrah Aron-Dine, Adrien Auclert, Adrien Bilal, Luigi Bocola, Gideon Bornstein, Jean-Félix Brouillette, Benny Kleinman, Alessandra Peter, Monika Piazzesi, Steve Redding, and Martin Schneider. This research was supported by the Leonard W. Ely and Shirly R. Ely Graduate Student Fellowship through a grant to the Stanford Institute for Economic Policy Research. Disclaimer: some results in the paper are calculated based in part on data from Nielsen Consumer LLC and marketing databases provided through the NielsenIQ Datasets at the Kilts Center for Marketing Data Center at The University of Chicago Booth School of Business. The conclusions drawn from the NielsenIQ data are those of the researcher(s) and do not reflect the views of NielsenIQ. NielsenIQ is not responsible for, had no role in, and was not involved in analyzing and preparing the results reported herein.

1 Introduction

Knowledge spillovers have long been recognized as a rationale for government support of innovation (Arrow, 1962). This conventional wisdom recognizes the aggregate stock of knowledge as an input into the production of new ideas, so an individual inventor does not fully capture the social value of their innovation. Quantifying how large these spillovers are is therefore a key ingredient for informed innovation policy (Atkeson and Burstein, 2019; Akcigit, Hanley and Stantcheva, 2022).

Yet, evidence on the magnitude of spillovers "is quite thin" (Bryan and Williams, 2021, p. 290). The challenge is that knowledge flows are inherently difficult to measure. As Griliches notes, "in this desert of data, patent statistics loom up as a mirage of wonderful plenitude and objectivity" (Griliches, 1990, p. 287). Consequently, existing approaches to estimate spillovers rely on patents. However, many innovations are not patented, including the polio vaccine, the World Wide Web, and Linux. Patent data also skew systematically toward manufacturing: the sector accounts for 10% of GDP (BEA, 2025) and TFP growth (BLS, 2025) but 64% of patenting (NCSES, 2023).

This paper uses a novel and complementary model-based approach to quantify knowledge spillovers. Instead of patents or reported R&D, I use data on firm exit by age covering the universe of U.S. private nonfarm employer businesses. To infer the magnitude of knowledge spillovers from these exit patterns, I develop a new semi-endogenous growth model with multiproduct firms and selection into product exit. I find substantial spillovers, driving a 16 percentage point wedge between the social and private rates of return to innovation.

To appreciate why exit rates are informative about spillovers, consider today's inventors creating new products. Compared to their predecessors, they have access to a stock of knowledge made larger by recent innovations. How much does this larger knowledge stock improve the quality of the products they invent? In search for the answer, I turn to dynamics in product markets. The rationale is simple: if higher quality products render lower quality ones obsolete, then, all else equal, stronger spillovers—by accelerating growth in quality across generations of new products—accelerate the rate at which incumbent products lose market share and exit.

Building on this intuition to quantify spillovers requires a model for three reasons. First, the thought experiment above takes as given how much knowledge is available to inventors, but this stock is accumulated endogenously through innovation. Accordingly, I need a model that endogenizes this accumulation process and formalizes how the resulting spillovers lead to a wedge between social and private rates of return to innovation. Second, the thought experiment abstracts from other drivers of exit, but high exit rates could also reflect high volatility of idiosyncratic shocks rather than strong spillovers. The model must therefore allow separately identifying these competing forces. Third, the thought experiment presumes access to product data, which are only available for a few sectors. As a result, the model must also bridge this data gap, showing how to infer the wedge of interest from more widely available firm data.

To meet these three requirements, I develop a new model of growth and firm dynamics. Both new and incumbent firms endogenously invest in R&D to increase the likelihood of creating a new product (variety). The quality of each such new product is drawn from an entry distribution and then evolves as a geometric Brownian motion. While these innovations allow new firms to enter and incumbent firms to expand, the endogenous shutdown of existing products, à la Hopenhayn (1992), acts as a countervailing force, culminating in a firm's exit upon the shutdown of its final product.

In this setting, knowledge spillovers are the source of improvement in the entry distribution over time. Specifically, product innovations by new and incumbent firms increase the aggregate stock of knowledge, and a higher stock of knowledge leads to a first-order stochastic dominance improvement in the entry distribution (Kortum, 1997).

The model delivers on all three requirements. First, it provides an intuitive characterization of the wedge between social and private returns to innovation created by knowledge spillovers. Along the balanced growth path, the social rate of return exceeds the private one, as knowledge spillovers constitute a positive externality that is not internalized in the laissez-faire equilibrium. The resulting wedge is the product of two terms: (i) the spillover elasticity, governing the extent to which a higher stock of knowledge improves the entry distribution, and (ii) the pace of knowledge accumulation, which, in my semi-endogenous growth model, is tied to the rate of population growth.

Second, the model formalizes how the dynamics of product exit are informative about this wedge, even in the presence of idiosyncratic Brownian shocks as another driver of product exit. The key is that a firm shuts down a product when its market share becomes too small to justify incurring the labor-denominated overhead cost. Stronger spillovers accelerate this process: by accelerating quality growth across successive generations of products, they quicken the substitution toward newer ones, accelerating the gradual erosion of an incumbent product's market share. It is this gradual component of product exit, due to downward drift toward the exit threshold, that is informative about the wedge of interest.

Third, the model overcomes the scarcity of product data by providing a way to infer this wedge from widely available data on firm exit by age. This is possible because product dynamics aggregate to determine firm dynamics: a firm starts with a single product, attempts to grow its portfolio via R&D, and exits upon shutdown of its final product. As a result, the model's profile of firm exit by age is governed by three sufficient statistics: (i) the extent of product exit due to downward drift (the component informative about the wedge), (ii) the extent of product exit due to shocks, and (iii) the endogenous rate at which an incumbent firm adds a product to its portfolio per existing product. These are separately identified because they leave distinct signatures on the firm exit hazard over its life cycle: (i) a higher product exit due to downward drift raises firm exit at all ages, (ii) a higher product exit due to shocks increases exit among young firms but, due to selection, decreases it among older ones, and (iii) a higher rate of product addition has little effect on young single-product firms but lowers the exit for mature firms by increasing their average number of products.

Before applying this approach, I provide direct evidence from the consumer packaged goods sector that corroborates the key model features underpinning the sufficient statistic result just described. Specifically, I document five facts using the NielsenIQ Retail Scanner dataset, which provides high-frequency data at the barcode (UPC) level. First, products with higher sales are less likely to exit. Second, the product exit rate is lower among firms with more products. Taken together, these first two facts provide support for negative selection into product exit. Third, the rate of new product introductions per existing product shows no systematic variation with the firm's number of products—consistent with each incumbent product giving birth to a new product at a constant Poisson rate, as in the equilibrium of my model. Fourth, in the lead up to exit, a product's sales decline gradually. Fifth, this gradual decline is driven by a collapse in quantity sold while relative price falls only modestly, a pattern consistent with a negative residual demand shock to the product. Taken together, these last two facts provide support for gradual erosion of a product's appeal preceding exit, which my approach leverages to quantify spillovers.

I apply my novel approach to quantifying spillovers using data covering the universe of U.S. private nonfarm employer businesses. Targeting the profile of *firm* exit at ages 1 through 19, I estimate that the extent of *product* exit due to downward drift toward the exit threshold is 16 percentage points. The model needs such a large estimate for this statistic to match the non-trivial exit rate among older firms: while idiosyncratic shocks are another driver of product exit, firms that survive to old age have positively selected products, so that the sequence of shocks needed to get them to the exit threshold is unlikely.

The wedge between social and private rates of return to innovation implied by these estimation results depends on the drift of the geometric Brownian motion governing the evolution of an incumbent product's quality. As a baseline, I make the conservative assumption that this drift is zero. In this case, the wedge of interest is exactly the extent of product exit due to downward drift, yielding my headline estimate of 16 percentage points. Any positive drift, reflecting for example innovation on incumbent products or learning by doing, would increase this wedge. Intuitively, if incumbent products are themselves improving over time, rationalizing the same rate of product exit due to gradual creative destruction by newer products requires the entry distribution to be improving even more rapidly. That said, I show that for a plausible range of positive values for this drift, the estimated wedge only rises to at most 25 percentage points, confirming that the baseline estimate is not an overly loose lower bound.

A first potential concern is a negative value for this drift, which would bias my estimate of the wedge upward. While at first this might seem difficult to defend, it could reflect taste for novelty: if consumers intrinsically value newness, a product's appeal would drift downward over time as it ages. To assuage this concern, I redo the estimation separately by sector and show that the estimated wedge tends to be largest in sectors where narrative evidence points to an important role for knowledge spillovers.

A second potential concern is that I am attributing all improvements in the entry distribution to the accumulation of knowledge. It is important to clarify that knowledge here is broadly defined, so that the spillovers my approach identifies need not originate within the same sector: they could arise from innovations in other industries or basic research conducted at universities.[1] That

[1] In Section 5, I discuss this multi-sector extension, with the formal model presented in Appendix F.1.

said, I also entertain the possibility that human capital growth could lead to improvements in the entry distribution over time. I show that in an extension allowing for such (exogenous) human capital growth, the wedge of interest equals my baseline estimate scaled by the share of measured productivity growth *not* attributable to human capital. Across three different approaches, I find that this share ranges from 64% to 81%, so that even allowing for such human capital growth, the wedge of interest remains substantial and exceeds 10 percentage points.

To bolster confidence in the quantitative aspects of my results, I show that, despite its parsimony, the quantified model captures several salient features of the data. Specifically, the model closely fits the profile of firm exit by age, the firm entry rate, the share of aggregate growth attributable to incumbent firms, the cross-firm distribution of employment, and average firm employment by firm age. The model also generates plausible rates of aggregate productivity growth. This suggests that, beyond quantifying spillovers, the model provides a useful framework for firm dynamics and growth more broadly.

The main takeaway from the paper is that despite pursuing an alternative approach that allows me to quantify knowledge spillovers across a broader range of firms and sectors, I still find evidence consistent with spillovers being sizable. While the 16 percentage point wedge I estimate is smaller than the 20-45 percentage point range found by Bloom, Schankerman and Van Reenen (2013) using patent and reported R&D data, it remains substantial. Through the lens of my model, it implies that the laissez-faire equilibrium underprovides innovation by a factor of 2 to 3 relative to the first best, suggesting potentially large welfare gains from government support for innovation.

The reason my model requires sizable spillovers to rationalize the data is intuitive. As a result of selection, idiosyncratic shocks alone are not sufficient to match the exit rate of older firms. Instead of introducing an exogenous Poisson death shock as the quantitative firm dynamics literature often does, I model firms as multiproduct and show that I can match the profile of firm exit by age with a downward drift at the product level. In principle, such a downward drift could reflect taste for novelty. However, leveraging the cross-section of sectors, I show evidence favoring a technological obsolescence interpretation. The need for spillovers—in the form of improvements in the entry distribution over time—then follows from the observation that, despite these improvements among the pool of incumbent firms, there has been sustained entry of new firms with a stable average size of entering firms.

After highlighting my contribution to the literature, the rest of the paper proceeds as follows. Section 2 lays out the model, derives the wedge between social and private rates of return to R&D, and formalizes how the dynamics of product exit are informative about this wedge. Section 3 then characterizes the model's *firm* dynamics, a necessary step to quantify this wedge when only firm data are available. Section 4 provides empirical evidence corroborating my treatment of product exit and product addition by incumbent firms. Section 5 puts this apparatus to work using data on the U.S. private nonfarm employer businesses, presenting the headline result as well as those from a series of robustness and validation exercises. Finally, Section 6 concludes.

Contribution to the literature. The paper's theoretical contribution is a new quantitative growth model with multiproduct firms and selection into exit at the product level. Creative destruction plays a prominent role in this model, which relates my work to the large literature on quality ladder models of Schumpeterian growth, building on Aghion and Howitt (1992) and Grossman and Helpman (1991), and reviewed in Aghion, Akcigit and Howitt (2014). More recent contributions include Acemoglu, Akcigit, Alp, Bloom and Kerr (2018), Akcigit and Kerr (2018), Peters (2020), and Cavenaile, Celik and Tian (2025). In contrast to this literature, obsolescence unfolds gradually in my model because new and incumbent products are imperfect substitutes. This choice is motivated by the evidence I document using barcode level data, as well as the findings of Foster, Haltiwanger and Syverson (2008) and Argente, Lee and Moreira (2024). As a consequence, knowledge spillovers are the only source of inefficiency in my model. This is because, in addition to the usual negative business stealing externality, there is a positive consumer surplus externality due to love of variety (Acemoglu, 2009); with CES demand, the two externalities exactly offset, as in Melitz (2003). While this is a special property of the CES aggregator (Dhingra and Morrow, 2019), it is convenient for the purposes of isolating and quantifying knowledge spillovers.

In modeling product dynamics, I build on Luttmer (2007). A first notable difference is that I allow for multiproduct firms. In addition to the evidence on the ubiquity of multiproduct firms (Bernard, Redding and Schott, 2010; Broda and Weinstein, 2010; Arkolakis, Ganapati and Muendler, 2021; Argente, Lee and Moreira, 2018), this enables lower volatility of firm growth among larger firms (Sutton, 2002; Arkolakis, 2016). The second is that Luttmer (2007) models entrants drawing from the incumbent distribution while I follow Kortum (1997) in having them draw from an endogenously shifting distribution. As a result, our models yield diverging comparative statics: in my model, higher volatility of shocks does not increase the growth rate, product entry and exit *rates* are not affected by the cost of entry, and the resulting growth rate is semi-endogenous.

My modeling of multiproduct firms builds on Klette and Kortum (2004) and Luttmer (2011): a firm is a collection of products, each of which, gives "birth" to a new product at a constant Poisson rate in equilibrium. The key difference is that, because product *exit* is not a Poisson process in my model, the firm's number of products evolves as a non-Markovian branching process. Consequently, the stationary firm size distribution is the solution to an infinite system of coupled partial differential equations. The payoff is that this modeling of product exit, and its consequences through the lens of my model, are corroborated by facts I document as well as empirical evidence from Bernard, Redding and Schott (2010) and Hottman, Redding and Weinstein (2016).

As my model features semi-endogenous growth (Jones, 1995a), it is consistent with the aggregate evidence on weak scale effects (Jones, 1995b; Peters, 2022), the firm-level evidence on declining research productivity (Bloom, Jones, Van Reenen and Webb, 2020), and with the literature emphasizing the role of demographics in explaining the slowdown of business dynamism (Karahan, Pugsley and Şahin, 2024; Hopenhayn, Neira and Singhania, 2022).

The link between spillovers and dynamics in product markets is not a peculiar feature of my model. Akcigit and Ates (2023) argue declining knowledge diffusion from frontier to laggard firms can quantitatively explain a large share of the recent slowdown in business dynamism in the U.S. (Decker, Haltiwanger, Jarmin and Miranda, 2016; Akcigit and Ates, 2021). Their model is quite different than mine, as it is a step-by-step innovation model—à la Aghion, Harris and Vickers (1997), Aghion, Harris, Howitt and Vickers (2001), Aghion, Bloom, Blundell, Griffith and Howitt (2005), and Acemoglu and Akcigit (2012)—with Bertrand duopoly in each sector.

The paper's empirical contribution is the quantification of knowledge spillovers using this new theoretical framework. This relates my work to a large body of innovation research, reviewed in Bryan and Williams (2021), for which a central challenge is that knowledge flows are invisible. Apart from papers focusing on specific industries like Griliches (1958) and Irwin and Klenow (1994), the literature has relied on patent data to quantify spillovers. Specifically, following Jaffe, Trajtenberg and Henderson (1993), many have used patent citations as a paper trail for knowledge flows and hence a proxy for spillovers. Compared to my approach, an advantage of using patent citations is the ability to recover the network of spillovers (Acemoglu, Akcigit and Kerr, 2016), which is necessary to characterize optimal sector-specific (Liu and Ma, 2021) or firm-specific (König, Liu and Zenou, 2019) R&D subsidies. However, as Jaffe, Trajtenberg and Fogarty (2000) put it, patent citations are at best a noisy signal of spillovers; a conclusion they reach based on survey evidence on the familiarity of inventors with patents they cite. In fact, among U.S. patents granted between 2001 and 2003, Alcácer, Gittelman and Sampat (2009) find that examiners added, on average, 63% of citations and all citations in 40% of the cases. , although Bryan, Ozcan and Sampat (2020) caveat that this is less of a concern for in-text (as opposed to front-page) citations.

Another common approach in this literature to "detect the path of spillovers in the sands of data" (Griliches, 1992, p. S36) is to leverage variation in proximity between firms. Specifically, Jaffe (1986) uses the patent classification system to define a technological proximity metric between firms, and then looks at the effect of a firm's R&D on R&D by its "technological neighbors". Bloom, Schankerman and Van Reenen (2013) enrich this approach by using line of business data from Compustat to complement technological proximity with proximity in product markets, which allows them to separately identify knowledge spillovers and business stealing. Lucking, Bloom and Van Reenen (2019) extend the analysis to more recent years and Lychagin, Pinkse, Slade and Van Reenen (2016) enrich this approach with a third dimension of proximity tied to geographic location, as in Adams and Jaffe (1996). Zacchia (2019) uses an alternative metric of technological proximity, based on the share of the two firms' inventors who have previously co-patented across firms, and similarly finds a social rate of return twice as large as the private one. Arqué-Castells and Spulber (2022) show that accounting for voluntary technology transfers between firms reduces the gap between the social and private rates of return from 40 to 30 percentage points.

My approach is complementary to these and motivated by the literature acknowledging the limitations of patent and R&D data. First, the propensity to patent an invention varies across industries (Levin, Klevorick, Nelson and Winter, 1987): manufacturing accounts for 64% of patents issued to US companies and 57% of reported R&D (NCSES, 2023). Second, even within sectors where patenting is common, larger firms are more likely to patent an innovation (Cohen, Nelson and

Walsh, 2000; Mezzanotti and Simcoe, 2023; Argente, Baslandze, Hanley and Moreira, 2023). Third, patents capture a small share of the knowledge spilling out of research conducted in universities (Agrawal and Henderson, 2002). Fourth, firms may strategically relabel expenses as R&D (Chen, Liu, Suárez Serrato and Xu, 2021), with these incentives varying over time as the tax code changes (Cowx, Lester and Nessa, 2024). My approach also complements Jones and Summers (2021), who measure the *average* social rate of return to R&D as the growth rate of TFP divided by the share of GDP spent on R&D. In contrast, I focus on the gap between *marginal* rates of return due to spillovers and do not need to take a stance on the share of resources dedicated to innovation.

2 Model

This section lays out the model and derives two key results. The first characterizes the wedge between social and private returns to R&D created by knowledge spillovers. The second formalizes how the dynamics of product exit are informative about the magnitude of this wedge. Appendix A summarizes all the symbols I introduce when setting up and solving the model.

2.1 Economic Environment

Preferences and Technology Time is continuous and population N_t grows at rate $\eta > 0$:

$$\dot{N}_t = \eta N_t \ . \tag{1}$$

Each individual inelastically supplies a unit of labor and household preferences are given by

$$U_0 = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \ ,$$

where consumption c_t is a CES aggregate over a continuum of imperfectly substitutable products

$$c_t = \left[\int_{p \in \Omega_t} \left(Q_{pt} c_{pt} \right)^{\frac{\sigma-1}{\sigma}} dp \right]^{\frac{\sigma}{\sigma-1}} \ .$$

Here, Q_{pt} is the quality of product p at t, c_{pt} its quantity consumed, $\sigma > 1$ the elasticity of substitution, and Ω_t the set of products supplied at t. This set evolves over time as a result of the creation of new products and the endogenous exit of existing ones.

Each of these products is supplied by one, potentially multiproduct, firm. The production technology features a per-product fixed overhead cost $\mathcal{F}$, denoted in labor units. If unpaid, production of that product is irreversibly shut down. The marginal cost of production is otherwise constant, so that quantity produced Y_{pt} is linear in production labor employed L_{pt}:

$$Y_{pt} = AL_{pt} \ .$$

Innovation For a product to be supplied, the underlying blueprint must have already been developed. Such innovations are carried out by entering as well as incumbent firms.

All firms begin as single product firms. Motivated by the findings of Klenow and Li (2025), there is a labor-denominated firm entry cost. Specifically, an individual attempting to set up a new firm—by developing the blueprint for a new product—succeeds at Poisson rate ε. Subsequently, this single product firm can expand its portfolio through R&D. Hiring I_p R&D workers leads to the development of a new product at Poisson rate:

$$\iota\left(I_p\right) = \vartheta \frac{I_p^{1-\delta}}{1-\delta} \quad \text{where} \quad \delta \in (0,1) \ ; \vartheta > 0. \tag{2}$$

R&D investments by incumbents thus run into diminishing returns.

Once the firm becomes multiproduct, it operates as a collection of product lines. As with production, innovation is organized around existing blueprints: for each product p in its portfolio, a firm chooses how many R&D workers I_p to hire. Using the superposition property of the Poisson process, a firm f currently producing the set of products $\{p_1, \ldots, p_n\}$ succeeds in adding a product to its portfolio with Poisson arrival rate

$$\sum_{p \in \{p_1, \ldots, p_n\}} \iota\left(I_p\right) \ .$$

Equivalently, one could write down, as in Klette and Kortum (2004), a firm-level idea production function with constant returns to scale in firm-wide R&D labor and number of products. The product-level formulation allows me to defer the introduction of firm level variables until Section 3.

Knowledge spillovers, which I seek to quantify, stem from the cumulative stock of innovation being an input into the production of new ideas. With innovation taking the form of product development, the cumulative stock of innovation is the mass of product blueprints already developed, which I denote by K_t. Its law of motion is:

$$\dot{K}_t = \varepsilon S_t + \int_{p \in \Omega_t} \iota\left(I_{pt}\right) dp \ . \tag{3}$$

The first term reflects the flow of new blueprints developed by entrants, as S_t is the mass of individuals working as startup entrepreneurs and ε is the Poisson rate an entrepreneur succeeds. The second term reflects the flow of new blueprints developed by incumbents.

Product Quality Whether developed by an entrant or incumbent, the quality of a new product at t is drawn from an entry distribution with complementary cumulative distribution function (CCDF):

$$\overline{F}_t^E(Q) = \Pr\left(\text{Draw}_t > Q\right) = \begin{cases} K_t^{\theta} Q^{-\alpha} & \text{if} \quad Q > K_t^{\frac{\theta}{\alpha}} \\ 1 & \text{otherwise} \end{cases} \quad \text{with} \quad \alpha > 0 \;\text{ and } \theta \geq 0.$$

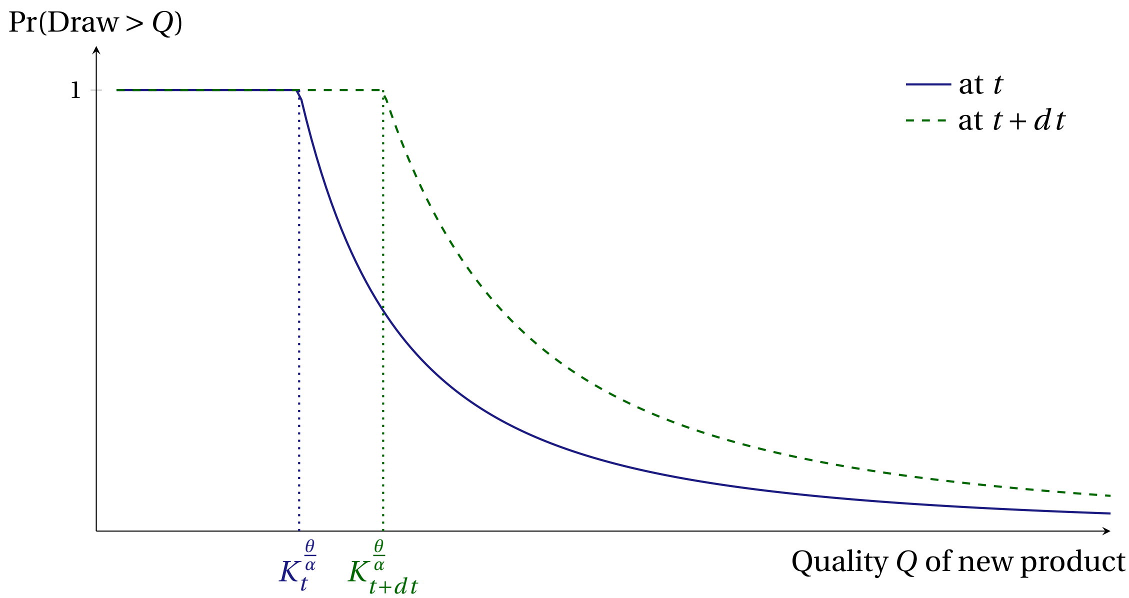


Figure 1: Improvement in entry distribution due to spillovers when $\theta > 0$

In this specification, due to Kortum (1997), α is the time-invariant Pareto shape of the entry distribution, with a higher value for α corresponding to a thinner tail. θ is the spillover elasticity: it governs how strongly a higher cumulative stock of innovation improves—in a first-order stochastic dominance sense—the distribution of quality for new products. For example, when there are no spillovers, $\theta = 0$ and the entry distribution is time-invariant. In contrast, when $\theta > 0$, the flow of new blueprints today increases K_t, which in turn improves the entry distribution inventors draw from in the future, as shown in Figure 1.

After this initial draw, the product's quality Q_{pt} evolves as a geometric Brownian motion with increments that are independent across the continuum of products:

$$d \ln Q_{pt} = \beta dt + \nu dB_{pt} \ .$$

Resource Constraints For each product p, aggregate consumption cannot exceed production:

$$N_t c_{pt} \leq Y_{pt} \ .$$

Labor is the only input in the economy. Each individual inelastically supplies a unit of labor and can work for an incumbent as a production, overhead, or R&D worker. Alternatively, the individual can choose to be a startup entrepreneur and attempt to create a new firm. The resource constraint on labor is thus given by:

$$\int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt}\right) dp + S_t = N_t \ . \tag{4}$$

Table 1: Economic Environment

Population	$\dot{N}_t = \eta N_t$	$\eta > 0$
Preferences	$U_0 = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma}-1}{1-\gamma} dt$	$\rho > \eta$
Consumption	$c_t = \left[\int_{p\in\Omega_t} \left(Q_{pt} c_{pt}\right)^{\frac{\sigma-1}{\sigma}} dp\right]^{\frac{\sigma}{\sigma-1}}$	$\sigma > 1$
Production	Overhead $\mathcal{F}$ (exit if unpaid)	$\mathcal{F} > 0$
	$Y_{pt} = AL_{pt}$	
Flow of new blueprints	$\dot{K}_t = \varepsilon S_t + \int_{p\in\Omega_t} \frac{\vartheta}{1-\delta} I_{pt}^{1-\delta} dp$	$\vartheta > 0 \, ; 0 < \delta < 1$
Entry distribution	$\overline{F}_t^E(Q) = K_t^\theta Q^{-\alpha}$	$\theta \geq 0 \, ; \alpha > 0$
Set of products	Ω_t evolves through entry & exit	
Quality evolution	$d \ln Q_{pt} = \beta dt + \nu dB_{pt}$	$\nu \geq 0$
Resource constraints	$N_t c_{pt} = Y_{pt}$	
	$S_t + \int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt}\right) dp = N_t$	

Notes: The scarce resource to allocate in this economy is labor. The decision of how to divide this resource among its competing uses in production (L), overhead ($\mathcal{F}$), incumbent innovation (I), and entry (S) governs how the the set of products Ω_t consumed evolves over time.

Table 1 summarizes the economic environment of the model. The allocation decision in this economy is how to divide labor among its competing uses, which governs how the set of products Ω_t consumed by households evolves over time. To appreciate the economic trade-offs involved, consider the benefit of a marginal unit of labor in each of its uses. Allocating more labor to production increases the output of products currently supplied. Allocating more labor to overhead expands the range of products that remain active. Because of the love of variety effect, both of these margins enhance current consumption. In contrast, allocating more labor to R&D or entry expands the stock of blueprints, thereby increasing the number and quality of products available for consumption in the future.

2.2 Decision Problems

To pin down how labor is allocated across its competing uses, I consider the following market structure. Each differentiated product is supplied by a single, potentially multiproduct, monopolistically competitive firm. There are no barriers to firm entry and the frictionless labor market is perfectly competitive. The consumption bundle serves as the numeraire so its price is normalized to unity. In terms of assets, in addition to the blueprints which entitle their owners to a stream of dividends, there is a risk-free bond in zero-net supply.

I characterize equilibrium labor allocations and the evolution of the product set Ω_t without explicitly tracking firms. This is possible because a firm's problem is separable across its products. On the technology side, there are no production or innovation synergies across a firm's portfolio. On the demand side, firms are atomistic and the elasticity of substitution is constant across products, so there are no cannibalization effects. The distribution of firms matters only for outcomes such as firm entry and exit rates, which I return to in Section 3.

Household's problem Given a path of interest rates r_t, wages w_t, and product prices P_{pt}, the household chooses consumption of different products p to maximize lifetime utility subject to an intertemporal budget constraint:

$$\max_{\{c_{pt}\}} \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \qquad \text{subject to} \qquad \begin{aligned} c_t &= \left[\int_{p\in\Omega_t} \left(Q_{pt} c_{pt}\right)^{\frac{\sigma-1}{\sigma}} dp \right]^{\frac{\sigma}{\sigma-1}} \\ \dot{a}_t &= \left(r_t - \eta\right) a_t + w_t - \int_{p\in\Omega_t} P_{pt} c_{pt} dp \end{aligned} \tag{5}$$

where a_t is the individual's asset holding. Summing c_{pt} across individuals yields the following demand schedule for product p:

$$Y_{pt} = Q_{pt}^{\sigma-1} P_{pt}^{-\sigma} N_t c_t \ .$$

Incumbent's problem For each product in its portfolio, a firm chooses pricing, production, and R&D to maximize the expected present discounted value of dividends from that product. These consist of profits from operating the product and an option value of expanding the firm's portfolio through R&D. They accrue until the firm optimally chooses to irreversibly shut down production with this product. With $V_t(Q_{pt})$ the value at time t of a product with quality Q_{pt}, the firm solves the following optimal stopping time problem (Dixit and Pindyck, 1994; Stokey, 2009):

$$V_t(Q_{pt}) = \max_{\substack{T \\ \{P_{pt}\} \\ \{I_{pt}, L_{pt}\}}} \mathbb{E}_t \left\{ \int_t^T e^{-\int_t^\tau r_s ds} \left[\underbrace{P_{p\tau} Y_{p\tau} - w_\tau \left(\mathcal{F} + L_{p\tau}\right)}_{\text{operating profit}} + \overbrace{\vartheta \underbrace{\frac{I_{p\tau}^{1-\delta}}{1-\delta}}_{\substack{\text{arrival rate of} \\ \text{new product}}} \underbrace{\int V_\tau(Q)\, dF_t^E(Q)}_{\substack{\text{expected value} \\ \text{of new product}}} - w_\tau I_{p\tau}}^{\text{option value}} \right] d\tau \right\}$$

$$\text{subject to, for all } \ t \le \tau < T, \ \begin{cases} Y_{p\tau} = A L_{p\tau} \\ Y_{p\tau} = Q_{p\tau}^{\sigma-1} P_{p\tau}^{-\sigma} N_\tau c_\tau \\ d \ln Q_{p\tau} = \beta d\tau + \nu dB_{p\tau} \end{cases} \tag{6}$$

The value of a firm is obtained by summing the value of products in its portfolio.

For endogenous exit to occur, the option value of R&D expansion must not exceed the overhead cost. In essence, the effective fixed cost is the overhead net of the option value, and it must be strictly positive for some products to be optimally shut down. Assumption 1 below provides a condition on parameters that guarantees this holds along the equilibrium path. As in Hopenhayn (1992), endogenous exit is then characterized by a quality threshold, denoted $\underline{Q}_t$, below which firms choose to stop supplying a product.

Firm Entry Since there are no barriers to entry, individuals must be indifferent between employment and entrepreneurship in equilibrium. With the wage as the outside option, this requires:

$$S_t \left(w_t - \varepsilon \int V_t(Q) dF_t^E(Q) \right) = 0 \ . \tag{7}$$

Along an interior equilibrium ($S_t > 0$), the expected payoff from attempting entry equals the wage.

2.3 Equilibrium

The model's dynamics center on the evolution of the product set Ω_t. Since quality is the only relevant source of heterogeneity, this set can be represented by a distribution over product qualities. With stationarity in mind, instead of absolute quality it is helpful to work with a product's log-quality relative to the exit threshold as its state variable:

$$q_{pt} = \ln \left(\frac{Q_{pt}}{\underline{Q}_t} \right) \ .$$

I denote the corresponding cross-sectional measure by $m(q,t)$. Using Ito's lemma,

$$dq_{pt} = \left(\beta - g_{Q_t}\right) dt + \nu dB_{pt} \ , \tag{8}$$

where g_{Q_t} is the instantaneous growth rate of $\underline{Q}_t$ at t. It follows that $m(0,t) = 0$ and for $q > 0$ the law of motion for $m(q,t)$ is given by a Kolmogorov Forward Equation (KFE):

$$\dot{m}(q,t) = \underbrace{-\left(\beta - g_{Q_t}\right) \frac{\partial m(q,t)}{\partial q}}_{\text{Drift}} + \underbrace{\frac{\nu^2}{2} \frac{\partial^2 m(q,t)}{\partial q^2}}_{\text{Diffusion}} + \underbrace{\dot{K}_t K_t^{\theta} \underline{Q}_t^{-\alpha} \alpha e^{-\alpha q} \ \mathbb{1}_{\left\{q \geq \frac{\theta}{\alpha} \ln K_t - \ln \underline{Q}_t\right\}}}_{\text{Entry}} \ . \tag{9}$$

Here, the measure of products entering with log-relative quality q is simply the flow of new blueprints $\dot{K}_t$ multiplied by the density of draws at that relative quality.

Definition 1. Given initial population N_0, stock of existing blueprints K_0, and distribution of product log-relative qualities $m(q,0)$, an equilibrium consists of time paths for N_t, K_t, $m(q,t)$, $\underline{Q}_t$, prices $\{r_t, w_t, P_{pt}, V_t(Q_{pt})\}$ and allocations $\{c_t, a_t, c_{pt}, Y_{pt}, L_{pt}, I_{pt}, S_t\}$ such that for all t:

1. c_t, a_t, and c_{pt} solve the household's problem (5)
2. L_{pt}, I_{pt}, P_{pt}, $V_t\left(Q_{pt}\right)$, and $\underline{Q}_t$ solve the incumbent's problem (6)
3. S_t satisfies the free entry condition (7)
4. Y_{pt} satisfies the market clearing condition for product p, $Y_{pt} = N_t \ c_{pt}$
5. w_t clears the labor market (4)
6. r_t clears the asset market, $N_t a_t = \int V_t \left(\underline{Q}_t e^q\right) m(q,t) dq$
7. N_t, K_t, and $m(q,t)$ evolve respectively according to equations (1), (3), and (9).

Solving. In equilibrium, consumption per capita and the wage rate are then given by

$$c_t = A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t \frac{L_t}{N_t} \qquad \text{and} \qquad w_t = \frac{\sigma - 1}{\sigma} A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t \ , \tag{10}$$

where L_t denotes aggregate production labor, M_t the measure of products supplied and $\overline{Q}_t$ the power mean of their qualities:

$$M_t \equiv \int_0^{\infty} m(q,t) dq \ \ \text{and} \ \ \overline{Q}_t \equiv \left(\frac{1}{M_t} \int_{p \in \Omega_t} Q_{pt}^{\sigma-1} dp\right)^{\frac{1}{\sigma-1}} = \underline{Q}_t \left(\frac{1}{M_t} \int_0^{\infty} e^{(\sigma-1)q} m(q,t) dq\right)^{\frac{1}{\sigma-1}} \ .$$

Equation 10 highlights the two potential sources of productivity growth: expanding varieties, which contributes to growth due to love of variety, and increasing average quality of products consumed.

Moving on to the model's dynamics, note that the innovation technology in (2) satisfies an Inada condition at 0, so that an incumbent always finds it optimal to do some R&D. Along an interior equilibrium ($S_t > 0$), this level of incumbent innovation per product is constant:

$$\left.\begin{array}{rl}\text{Incumbent's FOC:} & w_t = \vartheta I_{pt}^{-\delta} \displaystyle\int V_t(Q)\,\mathrm{d}F_t^E(Q) \\ \text{Free entry condition:} & w_t = \quad \varepsilon \quad \displaystyle\int V_t(Q)\,\mathrm{d}F_t^E(Q)\end{array}\right\} \implies I_{pt} = I \equiv \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} . \tag{11}$$

Thus the flow of new blueprints at t is simply

$$\dot{K}_t = \varepsilon S_t + \frac{\varepsilon}{1-\delta} I M_t . \tag{12}$$

The scaling by $(1-\delta)^{-1} > 1$ reflects that, while the marginal R&D worker has to be as productive as a startup entrepreneur in equilibrium, the infra-marginal ones will be more productive. Plugging the equilibrium level of incumbent R&D back into the firm's objective function yields an option value from R&D expansion equal to $O w_t$ where $O \equiv \frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$.

Assumption 1. Endogenous exit requires $\mathcal{F} > O$, so the parameters need to satisfy:

$$\mathcal{F} > \frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} .$$

Finally, denoting by $V(q,t)$ the value at t of a product with log-relative quality q, this value function satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:[2]

$$\begin{aligned} V(0,t) = \frac{\partial V(0,t)}{\partial q} = 0 \text{ and } \forall q > 0,\ \ r_t V(q,t) = w_t &\left[\frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{\overline{Q}}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O)\right] \\ &+ \dot{V}(q,t) + \left(\beta - g_{Q_t}\right)\frac{\partial V(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V(q,t)}{\partial q^2} . \end{aligned} \tag{13}$$

2.4 Balanced Growth Path

Definition 2. A balanced growth path (BGP) is an allocation with

1. stationary labor allocations: $\frac{\dot{L}_t}{L_t} = \frac{\dot{S}_t}{S_t} = \frac{\dot{M}_t}{M_t} = \eta$;
2. growth in the quality threshold $\underline{Q}_t$ at constant rate $g_Q \geq 0$;

[2]Using the definition of a product's value from (6), $V(q,t) \equiv V_t\left(\underline{Q}_t e^q\right)$.

3. a stationary distribution of relative qualities:

$$m(q,t) = f_p(q) M_t \ , \ \text{with } f \text{ a probability density function on } (0,\infty) \ .$$

Proposition 1. Along a BGP,

$$\frac{\dot{K}_t}{K_t} = \eta \quad ; \quad g_Q = \frac{\theta}{\alpha}\eta \quad \text{and} \quad g \equiv \frac{\dot{c}_t}{c_t} = \underbrace{\frac{\eta}{\sigma - 1}}_{\text{Variety}} + \underbrace{\frac{\theta}{\alpha}\eta}_{\text{Quality}}$$

Proof. Using the KFE, a time invariant $f_p(q)$ requires $K_t^\theta \underline{Q}_t^{-\alpha}$ to be constant. But $\underline{Q}_t$ grows at constant rate g_Q, so K_t grows at constant rate. That this constant growth rate is η is obtained from (12) combined with stationarity of labor allocations. It follows that $\underline{Q}_t$ grows at rate $\frac{\theta}{\alpha}\eta$. The expression for g is then obtained from (10). □

Assumption 2. Finite lifetime utility and firm value require

$$\rho > \eta + (1-\gamma) g \quad ; \quad (\sigma - 1)\left(\beta - g_Q\right) + \frac{1}{2}(\sigma - 1)^2 \nu^2 < \eta \quad \text{and} \quad \alpha > \sigma - 1 \ .$$

To build intuition for the rate of quality growth g_Q, notice first that it does not depend on the drift β and volatility ν. While these forces affect growth in sales for an incumbent product, they do not contribute to aggregate growth along a path with a stationary distribution of relative quality. Intuitively, the reason is that the quality gains experienced by a product are "lost" once it exits.

It is also noteworthy that, when $\theta = 0$, there is no quality growth along the BGP. This case corresponds to a time-invariant entry distribution. Along the BGP, the (un-normalized) distribution of incumbent quality is then itself time-invariant, as in the stationary equilibrium of Hopenhayn (1992) and Melitz (2003). The sole driver of aggregate productivity growth in this case is the expanding measure of products consumed.

This underscores that the driver of quality growth in this economy is the improvement in the entry distribution. When $\theta > 0$, this improvement happens as a result of the endogenous accumulation of knowledge (blueprints). In this case, as shown in Figure 1, exponential growth in the stock of blueprints K_t at rate η shifts the entry distribution to the right at rate $\frac{\theta}{\alpha}\eta$. The resulting quality growth rate is increasing in η (knowledge accumulated faster) and θ (stronger spillovers), but decreasing in α. Intuitively, when the tail of the entry distribution is thinner (higher α), good ideas are harder to find, so that a given spillover strength θ shifts out the average draw less.

Sustained quality growth thus requires a growing population and positive spillovers. The first is common in semi-endogenous growth models (Jones, 2022): if new ideas drive growth and individuals come up with ideas, then the economy's growth rate is tied to the rate of population growth. That $\theta > 0$ is necessary to sustain quality growth in this model stands in contrast to Kortum (1997). Key to understanding the difference is that, while these two semi-endogenous growth models share the same entry distribution, the mass of products in Kortum (1997) is fixed as new products perfectly substitute for older ones. As a result, what matters for the rate of quality growth in that setup is the maximum draw. Even if the distribution is time invariant, Kortum (1997) shows that exponential growth in the number of draws (due to population growth) from a Pareto distribution leads to exponential growth in the maximum draw. In contrast, what matters for the rate of quality growth in my setting is the average draw from the entry distribution, and for that to grow over time, the entry distribution has to shift out, which requires $\theta > 0$.

While indicative about the *presence* of spillovers ($\theta > 0$), quality growth alone is not informative about their *magnitude*. Rapid quality growth can reflect strong spillovers (high θ) and/or a thick-tailed entry distribution (low α). In addition to this identification challenge, quality growth is notoriously challenging to measure, see for example Bils and Klenow (2001), Bils (2009), Aghion, Bergeaud, Boppart, Klenow and Li (2019), and Atalay, Hortaçsu, Kimmel and Syverson (2025). This is why, to quantify spillovers, I leverage the model's product and firm dynamics. I now turn to characterizing product dynamics along the BGP, and move to firm dynamics in Section 3.

Assumption 3. The parameters are such that, along the BGP,

$$\underline{Q}_t \geq K_t^{\theta/\alpha} \,.$$

In words, this corresponds to the case where, the lower bound of the distribution entrants draw from ($K_t^{\theta/\alpha}$) is no greater than the threshold to supply a product ($\underline{Q}_t$). Proposition 1 guarantees that these grow at the same rate, and in Appendix B.1.3, I show that this ordering obtains when the present discounted value of the effective fixed cost of operation is not too small relative to the entry cost (Equation 24 in Appendix B.1.3 provides the condition on parameters).

Proposition 2. The stationary distribution of product log-relative quality is

$$f_P(q) = \frac{\alpha\zeta}{\zeta - \alpha}\left(e^{-\alpha q} - e^{-\zeta q}\right) \quad \text{where} \quad \zeta \;\equiv\; \frac{g_Q - \beta + \sqrt{(g_Q - \beta)^2 + 2\eta\nu^2}}{\nu^2} \,,$$

and the stationary product entry and exit rates are respectively

$$\frac{E_t}{M_t} = \eta + \frac{\nu^2}{2}\alpha\zeta \quad ; \quad \frac{D_t}{M_t} = \frac{\nu^2}{2}\alpha\zeta \,.$$

Proof. *Follows from solving the KFE along the BGP. Details in Appendix B.1.1* □

It follows that the incumbent quality distribution (in levels) has a Pareto tail with index $\min\{\alpha, \zeta\}$. While α is inherited from the entry distribution, ζ is the Luttmer (2007) tail that arises endogenously as a result of the geometric Brownian motion. This Pareto tail reflects the positive selection of products that have accumulated favorable Brownian shocks over time (as those that accumulate negative shocks get shut down). As such, it is unsurprising that higher volatility ν makes this tail thicker (smaller ζ). In contrast, faster population growth (η) or higher growth in entrant's quality relative to incumbents ($g_Q - \beta$) make this endogenous tail thinner (larger ζ). Intuitively, both of these forces increase the share of products that are young and which haven't had enough time to accumulate as many favorable Brownian shocks.

A distinctive feature of the model—in particular relative to the model with endogenous growth in Luttmer (2007)—is that stationary product entry and exit rates do not depend on the cost of entry (ε, ϑ, and δ). The reason is that, along the BGP, the cost of entry has a proportional effect on the *flow* of entry E_t and the measure of products supplied M_t, leaving the stationary entry rate unaffected. Put differently, "cheaper" entry makes innovation more appealing at all times, equally raising the flow of entry and the measure of products, as the latter reflects cumulative past entry. This is an attractive feature of the model as it makes my quantification robust to unmodeled forces that simply affect the cost of entry.

Proposition 3. Along the BGP, the value of a product with log-relative quality q is

$$V(q,t) = w_t \mathcal{V}(q) \quad \text{where} \quad \mathcal{V}(q) \equiv \frac{\mathcal{F} - O}{r-g}\left[\frac{\xi}{\xi+\sigma-1}e^{(\sigma-1)q} + \frac{\sigma-1}{\xi+\sigma-1}e^{-\xi q} - 1\right]$$
$$\xi \equiv \frac{\beta - g_Q + \sqrt{(\beta-g_Q)^2 + 2\nu^2(r-g)}}{\nu^2},$$

and average production labor per product is

$$\frac{L_t}{M_t} = (\mathcal{F} - O)\,\frac{\alpha\zeta(\sigma-1)}{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}\,\frac{\xi}{\xi+\sigma-1}\,\frac{r - [g + (\sigma-1)(\beta-g_Q) + \frac{\nu^2}{2}(\sigma-1)^2]}{r-g}$$

Proof. Follows from solving the HJB along the BGP. Details in Appendix B.1.2 □

With both L_t/M_t and I_t pinned down, one more equation is needed to fully characterize the stationary labor allocations, and it will follow from the free entry condition:

$$\varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^\infty V(q,t)\alpha e^{-\alpha q} dq = w_t \overset{\text{Proposition 3}}{\Longrightarrow} \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^\infty \mathcal{V}(q)\alpha e^{-\alpha q} dq = 1 . \tag{14}$$

Intuitively, ε is the arrival rate of a new blueprint, $K_t^{\theta}\underline{Q}_t^{-\alpha}$ is the probability that a new blueprint leads to entry (quality draw is above threshold), and the integral gives the expected value of a new product conditional on entry. As I show in Appendix B.1.3, this can be rewritten as

$$\frac{S_t}{M_t} + \frac{I}{1-\delta} = \left(\eta + \frac{\nu^2}{2}\alpha\zeta\right)\int_0^\infty \mathcal{V}(q)\alpha e^{-\alpha q} dq \,;$$

which makes clear that the free entry condition pins down the level of innovation along the BGP.

2.5 Efficiency

Having characterized the stationary allocations arising in equilibrium, I now turn to assessing their efficiency. To do so, I solve for the first best.

I delegate the detailed setup and solution of the planner's problem to Appendix B.2, and instead focus here on comparing the resulting BGP to the equilibrium BGP. Given the semi-endogenous nature of the model, the growth rates are as in Proposition 1 and the stationary distribution of product qualities as in Proposition 2. Contrasting the conditions pinning down stationary labor allocations in the first best (FB) versus equilibrium (DE) reveals the following:

Proposition 4. If $\theta = 0$, the equilibrium BGP is efficient. In contrast, if $\theta > 0$, then the equilibrium BGP features too little innovation with

$$\left(\frac{L_t}{M_t}\right)^{\mathbf{DE}} = \left(\frac{L_t}{M_t}\right)^{\mathbf{FB}} \quad \text{and} \quad I^{\mathbf{DE}} = I^{\mathbf{FB}} \quad \text{but} \quad S_t^{\mathbf{DE}} < S_t^{\mathbf{FB}}\,,\ M_t^{\mathbf{DE}} > M_t^{\mathbf{FB}}\,,\ \text{and}\ L_t^{\mathbf{DE}} > L_t^{\mathbf{FB}}\,.$$

The divergence is due to differing free entry conditions, which I reproduce below:

$$\textbf{Equilibrium}: \quad w_t = \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^\infty w_t \mathcal{V}(q) \alpha e^{-\alpha q} dq$$

$$\textbf{First Best}: \quad \omega_t = \left(1 + \frac{\theta\eta}{\rho + (\gamma - 1)g}\right) \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^\infty \omega_t \mathcal{V}(q) \alpha e^{-\alpha q} dq$$

Here ω_t is the Lagrange multiplier on the labor resource constraint in the planner's problem, whereas w_t is the wage; both of which cancel. So the only difference is the term upfront on the right hand side. Given $\eta > 0$, these conditions coincide if and only $\theta = 0$. Otherwise, when $\theta > 0$, for a given K_t and $\underline{Q}_t$, the planner's expected value from entry is higher. This is because the planner internalizes the positive externality arising from knowledge spillovers. This additional term has a Pigouvian interpretation. The externality consists of any individual failing to internalize that their innovation raises the aggregate stock of knowledge K_t. Along the BGP, this stock grows at rate η, which improves the distribution future entrants draw from at rate $\theta\eta$. The denominator corresponds to taking the present discounted value (it is simply $r - g$), as the benefits of a higher stock of knowledge last forever. As a result of this externality, there is underprovision of innovation in equilibrium.[3]

Contrasting these free entry conditions sheds light on how the planner can decentralize the first best. The policy instrument needs to incentivize the marginal individual to be a startup entrepreneur rather than working for an incumbent firm. One such policy is a profit subsidy at rate $\frac{\theta\eta}{r-g}$, which aligns the social and individual valuation of entry. Since labor is inelastically supplied, an isomorphic tool is a tax on labor income.

[3] Along an interior BGP, this underprovision shows up entirely along the entry of new firm margin, and the per-product incumbent innovation is efficient. I discuss the underlying intuition in detail at end of Appendix B.2.

In practice, governments subside innovation through a wide array of policies (Bloom, Van Reenen and Williams, 2019). With that in mind, to interpret the magnitude of spillovers I estimate, I report the corresponding gap between the social and private rates of return to R&D along the equilibrium BGP. The no-arbitrage condition implies that the private rate of return is r. In contrast, the social rate of return is higher and given by:

$$\boxed{\text{Social Rate of Return to R\&D along Equilibrium BGP} = r + \theta\eta}$$

That the gap should be $\theta\eta$ can be intuitively seen from comparing the above free entry conditions (after converting from present discount value units to annual rates of return). The formal derivation is in Appendix B.3. I follow Jones and Williams (1998) and define the social rate of return to R&D as the rate of return on a variation around the BGP. The variation consists of sacrificing some consumption at t in order to do more R&D, and then eating the proceeds at $t + dt$ by doing sufficiently less R&D to be back to the initial BGP by $t + 2dt$. The rate of return on this variation, in terms of higher consumption at $t + dt$, is precisely the social rate of return to R&D.

That the only source of inefficiency is knowledge spillovers stands in contrast to quality ladder models of Schumpeterian growth (Aghion and Howitt, 1992; Grossman and Helpman, 1991). While my model does feature the negative business stealing externality, it also features a positive consumer surplus externality due to love of variety, as new products are imperfect substitutes for older ones. Given the CES aggregator, the two externalities exactly offset, as in Melitz (2003). While this is a special property of the CES aggregator (Dhingra and Morrow, 2019), it is convenient for the purposes of this paper as the goal is to isolate and quantify knowledge spillovers.

2.6 From product dynamics to knowledge spillovers

My approach to quantifying spillovers is to estimate the wedge $\theta\eta$. I now shed light on why, through the lens of the model, product dynamics are informative about this wedge. To do so, note that the product exit rate from Proposition 2 can be equivalently rewritten as:

$$\frac{D_t}{M_t} = \frac{1}{2}\left(\theta\eta - \alpha\beta\right) + \frac{1}{2}\sqrt{\left(\theta\eta - \alpha\beta\right)^2 \ + \ 2\eta\alpha^2\nu^2} \ . \tag{15}$$

Special case with no drift or volatility ($\beta = \nu = 0$). In this case, after being developed, a product's quality is constant. Over time, if $\theta > 0$ this incumbent competes with an increasingly higher quality pool of products. This gradually erodes the incumbent product's market share and eventually drives it out of business once its relative quality falls below the threshold needed to cover overhead costs. The resulting product exit and entry rates are:

$$\text{Product Exit Rate}\Bigg|_{\beta=\nu=0} = \theta\eta \qquad = \text{gap between social and private rates of return}$$

$$\text{Product Entry Rate}\Bigg|_{\beta=\nu=0} = (1+\theta)\eta$$

A higher wedge $\theta\eta$ raises both the entry and exit rates. Intuitively, when spillovers are stronger (higher θ) or when knowledge is accumulated at a faster pace (higher η), entrants' quality improves more quickly so that incumbents lose market share at a faster pace and exit at higher rate.

With this intuition in mind, it might seem puzzling that a thicker tail (lower α) does not increase the exit rate, since it similarly increases the rate at which entrant's quality grows. To reconcile these two observations, it is important to recognize that the exit rate is the product of two terms: the growth rate of quality and the density of the incumbent distribution near (in this special case, at) the lower bound. As shown above, the former is $g_Q = \frac{\theta}{\alpha}\eta$, and with $\beta = \nu = 0$, the latter is simply α, so that α cancels out.[4] The intuition is that the incumbent distribution inherits the shape of the entry distribution, so when the entry distribution has a thicker tail, the faster increase in quality is exactly offset by a lower density of incumbents near the exit threshold.

In this special case with no Brownian motion, one can recover the wedge $\theta\eta$ from the stationary product exit rate alone. The use of *rates* rather than *levels* is key for identification. The reason is that the measure of products (denominator in this rate) reflects cumulative past entry. So, while a host of factors in the model make entry more appealing, they raise the flow of entry and exit as well as the measure of products proportionally, leaving the entry and exit rates unchanged.

General case with volatility and drift ($\beta \neq 0$ and $\nu > 0$). The reason I entertain volatility in the model is precisely to introduce product exit not due to spillovers. From Equation 15, higher volatility increases the product exit rate; this effect is stronger when population growth is faster (larger η) and when the entry distribution is thinner (larger α)—as both of these increase the density of products near the exit threshold (Proposition 2). With the product exit rate "contaminated" by exit due to volatility, I leverage the profile of product exit by age for identification.

Proposition 5. The hazard rate of a product exiting at age a is

$$d_p(a) = \frac{\ell(a)}{1 - \int_0^a \ell(\tau)d\tau},$$

[4]When $\beta = \nu = 0$, $f_p(q)$ solves a first-order ODE and is given by $f_p(q) = \alpha e^{-\alpha q}$.

where $\ell(a)$ is the density of a product's lifespan (i.e. age at exit) and is given by:

$$\ell(a) = \exp\left(a\left(\frac{(\alpha\nu)^2}{2} - \alpha\left(g_Q - \beta\right)\right)\right)\left(\frac{\alpha\nu}{\sqrt{a}}\phi\left(z_a\right) - \left((\alpha\nu)^2 - \alpha\left(g_Q - \beta\right)\right)\Phi\left(-z_a\right)\right)$$

$$\text{with} \quad z_a \equiv \frac{\sqrt{a}}{\alpha\nu}\left((\alpha\nu)^2 - \alpha\left(g_Q - \beta\right)\right)\ , \ \phi(.) \text{ and } \Phi(.) \text{ standard normal PDF and CDF.}$$

Proof. See Appendix B.4 for proof of Proposition 5 and Proposition 6. □

This shows that $\alpha\left(g_Q - \beta\right) = \theta\eta - \alpha\beta$ and $\alpha\nu$ are sufficient statistics for the profile of product exit rate by age.[5] Intuitively, the former captures the component of the product exit rate resulting from the deterministic downward drift toward the exit threshold, while the latter is a measure of relative volatility and governs the extent of product exit due to idiosyncratic shocks. It is the first of these two statistics that is informative about the wedge $\theta\eta$. By targeting the profile of product exit by age, this component of the product exit rate can be separately identified from exit due to shocks. The reason is that a higher $\theta\eta - \alpha\beta$ increases product exit rate at all ages, whereas a higher $\alpha\nu$ increases the product exit rate at young ages but decreases it at older ones. This non-monotonic effect of volatility on the exit rate by age is a consequence of selection: when volatility is higher, products that survive to old age are even more positively selected.

The challenge this approach runs into is that product data are not widely available, so that my ultimate quantification exercise has to rely on firm data. This requires characterizing the model's firm dynamics, which I now turn to.

3 Model's Firm Dynamics

Through the lens of the model in Section 2, dynamics of product exit are informative about the magnitude of knowledge spillovers. The challenge, however, is that comprehensive datasets track the exit of firms rather than products. This section provides the necessary bridge, formalizing how, within the model, underlying product dynamics aggregate to observable firm dynamics.

3.1 A firm's life cycle

Because I did not track firms when characterizing the equilibrium in Section 2, it is helpful to start with a recap of a firm's life cycle *along the balanced growth path*. I continue to use a product's log-relative quality q_{pt} as its state variable.

[5] Such dimension reductions are common in duration models where the state evolves as a Brownian motion with drift—see Alvarez, Borovičková and Shimer (2023) for another example.

A new firm f starts with a single product p. Since the logarithm of a Pareto-distributed random variable is exponentially distributed, the log-relative quality of the product q_{pt} is drawn from $\text{Exp}(\alpha)$. Thereafter, q_{pt} evolves according to Equation 8 until it hits zero, at which point production of p is irreversibly shut down. While still produced, p has I R&D workers associated with it, as in Equation 11. This R&D generates a new blueprint at Poisson rate $\iota(I)$. This new blueprint leads to product entry when the quality draw is above the threshold $\underline{Q}_t$, so with probability $\overline{F}_t^E(\underline{Q}_t) = K_t^\theta \underline{Q}_t^{-\alpha}$, which is constant along the BGP (by Proposition 1). As a result, while still produced, p "gives birth" to a new product at rate x:

$$x = \iota(I)\,\overline{F}_t^E\left(\underline{Q}_t\right) = \frac{\varepsilon}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} K_t^\theta \underline{Q}_t^{-\alpha} = \frac{1}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} \frac{(r-g)(\alpha+\xi)(\alpha-(\sigma-1))}{(\mathcal{F}-O)(\sigma-1)} \,. \tag{16}$$

The log-relative quality of this new product is again drawn from $\text{Exp}(\alpha)$, then evolves according to Equation 8 with independent Brownian increments across products; while active, this product in turn gives birth to a new one at Poisson rate x.

Accordingly, the number of products firm f produces at ages $a \geq 0$, $\left\{n_f(a)\right\}_{a\geq 0}$, is a branching process. At age 0, f starts with a single product, so $n_f(0) = 1$. Firm f exits once all its products have become obsolete, so its age at exit, or lifespan, is the smallest a such that $n_f(a) = 0$:

$$a_f = \inf\left\{a \geq 0 \mid n_f(a) = 0\right\} \,.$$

This is a random variable for two reasons: product births arrive as a Poisson process, and each product's lifespan is itself random. Denoting by Γ the cumulative distribution function (CDF) of a_f, the hazard rate of firm f exiting at an age a is:

$$d_f(a) = \frac{\Gamma'(a)}{1-\Gamma(a)} \,.$$

Assumption 4. To guarantee that products and firms exit in finite time almost surely, I assume:

$$\beta \leq g_Q \quad \text{and} \quad 0 \leq x \leq \alpha\left(g_Q - \beta\right) \,.$$

Proposition 6. The CDF of a firm's lifespan satisfies $\Gamma(0) = 0$ and, for all $a > 0$,

$$\Gamma(a) = \int_0^a \ell(\tau)\exp\left(\int_0^\tau x\left[\Gamma(a-s)-1\right]ds\right)d\tau \,,$$

where $\ell(.)$ is the density of a product's lifespan, given in Proposition 5.

In conjunction with Proposition 5, this highlights that, x, $\theta\eta - \alpha\beta$ and $\alpha\nu$ are sufficient statistics for the profile of firm exit by age. Intuitively, x controls the rate at which an incumbent firms adds a product to its portfolio, while $\alpha\left(g_Q - \beta\right)$ and $\alpha\nu$ the rates at which it loses a product (as discussed above). Note that, computationally, solving for Γ only requires a single forward iteration on a fine age grid, as $\Gamma(a)$ only depends on $\{\Gamma(\tau) : \tau < a\}$ and $\Gamma(0) = 0$.

3.2 Evolution of the size distribution of firms

Having shed light on the life cycle of a single firm, I now turn to characterizing the evolution of the distribution of firms along the equilibrium BGP. Unlike Klette and Kortum (2004) and Luttmer (2011), the evolution of a firm's number of products is non-Markovian here. A firm with n products becomes a firm with $n-1$ products when it endogenously chooses to shutdown production of one of its products. Since this happens when a product's log-relative quality hits zero, I need to keep track of a firm's portfolio of products. In this vein, define:

$$\mu_{nt}\left(q_1, \ldots, q_n\right) \equiv \text{ measure at } t \text{ of } n\text{-product firms with portfolio } q_1, \ldots, q_n \ .$$

Since μ_{nt} is the measure of n-product firms and a product exits once its relative quality hits zero,

$$\forall i \in \{1, \ldots, n\}, q_i = 0 \implies \mu_{nt}\left(q_1, \ldots, q_n\right) = 0 \ .$$

Put differently, the support of μ_{nt} is the positive orthant $(0, \infty)^n$, and μ_{nt} vanishes on the boundary of this support. To avoid cumbersome notation, I will denote by $\mathbf{q}$ the vector $(q_1, \ldots, q_n)$, and its dimension is to be inferred from the function it is an argument for. It follows that the total measure of products can be written as:

$$M_t = \sum_{n=1}^{\infty} n \underbrace{\int_{\mathbf{q}\in(0,\infty)^n} \mu_{nt}\left(\mathbf{q}\right)\ d\mathbf{q}}_{\text{Measure of } n\text{-product firms}} \ . \tag{17}$$

The evolution of the firm size distribution is then characterized by a system of coupled partial differential equations. Starting with single product firms, for $q > 0$:

$$\begin{aligned}
\dot{\mu}_{1t}(q) \ &= \left(g_Q - \beta\right) \mu'_{1t}(q) \ + \ \frac{\nu^2}{2}\mu''_{1t}(q) && \text{Drift and Diffusion} \\
&- \ x \ \mu_{1t}(q) && \text{Single product firm adds a product} \\
&+ E^f_t \ \alpha \ e^{-\alpha q} && \text{Entry of new firm} \\
&+ \frac{\nu^2}{2}\partial_1\mu_{2t}(0, q) + \frac{\nu^2}{2}\partial_2\mu_{2t}(q, 0) && \text{2-product firm shuts down a product}
\end{aligned}$$

The drift and diffusion terms on the first line echo those from from the product-level KFE in Equation 9. The second line captures that, while still produced ($q > 0$), an incumbent product gives birth to a new one at constant Poisson rate x. This shows up as an outflow because in this case the single product firm becomes a 2-product firm.

The last two lines in the law of motion correspond to inflows. The first source is the entry of new firms, with the corresponding inflow at q given by the total flow of entering firms E_t^f multiplied by the density of draws at that relative quality. The second inflow source is a 2-product firm shutting down the production of either of its products. Since a product's log-relative quality evolves following Equation 8 and exit is the absorbing boundary at 0, the instantaneous flow of 2-product firms who shut down production of their first product *and* whose second product has quality q is given by $\frac{\nu^2}{2}\partial_1\mu_{2t}(0,q)$. To clarify the notation, the second term is the partial derivative of μ_{2t} with respect to its first argument, evaluated at $(0,q)$. Similarly, there is inflow of 2-product firms who shut down production of their second product *and* whose first product has quality q (second summand in last line of the law of motion).[6]

The law of motion for μ_{nt} for $n > 1$ has a similar structure. The key difference is that the sources of inflow now are $(n-1)$ and $(n+1)$-product firms. As such, for $\mathbf{q} \in (0,\infty)^n$, denote by $\mathbf{q}^{\setminus i} \in (0,\infty)^{n-1}$ the vector $\mathbf{q}$ with its i^{th} entry removed and by $\mathbf{q}^{i\leftarrow 0}$ the $(n+1)$ dimensional vector obtained by inserting a 0 into $\mathbf{q}$ at the i^{th} index. It follows that, for $\mathbf{q} \in (0,\infty)^n$,

$$
\begin{aligned}
\dot{\mu}_{nt}(\mathbf{q}) \;=\;& (g_Q - \beta)\sum_{i=1}^{n}\partial_i\mu_{nt}(\mathbf{q}) + \frac{\nu^2}{2}\sum_{i=1}^{n}\partial_i^2\mu_{nt}(\mathbf{q}) && \text{Drift and Diffusion}\\
&-\; n\,x\,\mu_{nt}(\mathbf{q}) && n\text{-product firm adds product}\\
&+\; (n-1)\,x\,\frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\alpha q_i}\mu_{(n-1)t}\left(\mathbf{q}^{\setminus i}\right) && (n-1)\text{-product firm adds a product}\\
&+\; \frac{\nu^2}{2}\sum_{i=1}^{n+1}\partial_i\mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right) && (n+1)\text{-product firm loses a product}
\end{aligned}
$$

The diffusion term on the first line leverages the independence of Brownian increments across products within a firm. The outflow term on the second line reflects that, since each product gives birth to a new one at rate x, a n-product firms expands to $n+1$ products at Poisson rate nx.

The inflow from $n-1$ to n on the third line merits some clarification. A firm with $n-1$ products expands to a n-product firm at Poisson rate $(n-1)x$. What shows up in the law of motion is the inflow at a given $\mathbf{q}$, and any of the n products could be the new one.[7] Suppose for a second that the new product is $i = 1$. The measure of $(n-1)$-product firms with product qualities $(q_2,\ldots,q_n)$ is precisely $\mu_{(n-1)t}(\mathbf{q}^{\setminus 1})$. Such a firm adds a product to its portfolio at rate $(n-1)x$, and if successful, the relative quality of this new product is independent of the firm's portfolio and drawn from the density $\alpha e^{-\alpha q_1}$. The term on the third line of the law of motion is simply obtained by averaging such contributions over $i = 1,\ldots,n$ as each of these is equally likely to be the new product.

Finally, a firm with $n+1$ products becomes a n-product firm when it shut downs production of one of its products. The ultimate line of the law of motion is the resulting inflow at a given $\mathbf{q}$,

[6] Throughout, partial derivatives evaluated at a point on a hyperplane delimiting the positive orthant are taken as limits approaching the hyperplane from inside the support. For $n = 1$, this corresponds to the right derivative at 0.

[7] The firm's state variable is its *set* of products, as reflected by the PDE system satisfying permutation symmetry.

with its structure mirroring that of the $n = 1$ case that I discussed above. For the sake of clarity, $\partial_i \mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right)$ is the partial derivatives of $\mu_{(n+1)t}$ with respect to its i^{th} argument evaluated at $(q_1, \ldots, q_{i-1}, 0, q_i, \ldots, q_n)$.

3.3 Defining a stationary firm size distribution

Along the BGP, the measure of products M_t grows at rate η. For the distribution of products per firm to be stationary, the number of firms needs to grow at rate η as well. So to define a stationary firm size distribution, I start by introducing the following normalized variables. Let Ψ_{nt} be the share of products at t held by firms with n products, so that

$$\Psi_{nt} = \frac{1}{M_t}\, n \int_{\mathbf{q}\in(0,\infty)^n} \mu_{nt}\left(\mathbf{q}\right)\; d\mathbf{q} \quad \text{and} \quad \sum_{n=1}^{\infty} \Psi_{nt} = 1\ .$$

Since the measure of n-product firms grows at rate η along the BGP, also define:

$$f_{nt}(\mathbf{q}) = \frac{n\ \mu_{nt}(\mathbf{q})}{\Psi_{nt}\ M_t}\ .$$

Note that $f_{nt}(\mathbf{q})$ is the probability density function on $(0, \infty)^n$ of product qualities among n-product firms. A stationary firm size distribution amounts to Ψ_n and $f_n(\mathbf{q})$ being time invariant:

$$\forall t, \Psi_{nt} = \Psi_n \;\text{ and }\; f_{nt}(\mathbf{q}) = f_n(\mathbf{q})\ .$$

Plugging these back into the laws of motion outlined above yields the following definition.

Definition 3. A stationary firm size distribution consists of

- a sequence of non-negative numbers $\{\Psi_n\}_{n\geq 1}$ with $\sum_{n=1}^{\infty} \Psi_n = 1$,
- a sequence of functions $\{f_n\}_{n\geq 1}$, with f_n a probability density function on $(0, \infty)^n$,

such that $\{\Psi_n\}$ and $\{f_n\}$ jointly satisfy the following system of coupled PDEs:

$$(\eta{+}x) f_1(q) = \left(g_Q - \beta\right) f_1'(q) + \frac{\nu^2}{2} f_1''(q) + \frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1} \alpha e^{-\alpha q} + \frac{\nu^2}{2} \frac{\Psi_2}{2\,\Psi_1} \left[\partial_1 f_2(0, q) + \partial_2 f_2(q, 0)\right]\ ;$$

with the boundary condition $f_1(0) = 0$; and for $n > 1$,

$$\begin{aligned}\left(\frac{\eta}{n} + x\right) f_n(\mathbf{q}) \;&=\; \frac{1}{n}\sum_{i=1}^{n}\left(\left(g_Q - \beta\right)\partial_i f_n(\mathbf{q}) + \frac{\nu^2}{2}\partial_i^2 f_n(\mathbf{q})\right) \\ &\quad + \;x\,\frac{\Psi_{n-1}}{\Psi_n}\frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\alpha q_i} f_{n-1}\left(\mathbf{q}^{\setminus i}\right) + \frac{\nu^2}{2}\frac{\Psi_{n+1}}{\Psi_n}\frac{1}{n+1}\sum_{i=1}^{n+1}\partial_i f_{n+1}\left(\mathbf{q}^{i\to 0}\right)\end{aligned}$$

with the boundary condition: $\forall\ 1 \le i \le n,\ q_i = 0 \implies f_n(\mathbf{q}) = 0$.

These coupled partial differential equations are the analogue of the difference equations characterizing the stationary firm size distribution in Klette and Kortum (2004) and Luttmer (2011). They are PDEs instead of difference equations because the non-Markovian evolution of a firm's number of products requires keeping track of product qualities. To make the parallel even clearer, the following definition will prove helpful.

Definition 4. The average *product* exit rate among n-product firms is λ_n, with:

$$\lambda_1 = \frac{\nu^2}{2} f_1'(0) \quad \text{and for } n > 1,\ \lambda_n = \frac{\nu^2}{2}\frac{1}{n}\sum_{i=1}^{n} \int_{\mathbf{q}\in(0,\infty)^{n-1}} \partial_i f_n\left(\mathbf{q}^{i\to 0}\right)\, d\mathbf{q}\,.$$

Proposition 7. Given a stationary firm size distribution $\{\Psi_n, f_n\}$ and the corresponding $\{\lambda_n\}$,

$$\begin{cases} \eta\Psi_1 = -\left(x+\lambda_1\right)\Psi_1 + \lambda_2\Psi_2 + \eta + \frac{\nu^2}{2}\alpha\zeta - x & \\ \frac{\eta}{n}\Psi_n = -\left(x+\lambda_n\right)\Psi_n + \lambda_{n+1}\Psi_{n+1} + x\Psi_{n-1} & \text{for } n > 1 \end{cases}$$

Proof. The n^{th} recurrence relation follows from integrating both sides of the n^{th} PDE over $(0,\infty)^n$ then applying the divergence theorem in $\mathbb{R}^n$. Details in Appendix B.7 $\square$

The recurrence relation in Proposition 7 is a balance condition similar to the one in Luttmer (2011). The difference is that, in his setup, the product exit rate λ does not depend on n, as product exit is a Poisson process. In contrast, here, the average product exit rate among n-product firms depends on the endogenous shape of f_n.

It is perhaps clearer to appreciate the balance interpretation when the recurrence is rewritten in terms of Φ_n, the share of *firms* with n products:

$$\Phi_n \equiv \frac{\frac{\Psi_n}{n}}{\sum_{k=1}^{\infty}\frac{\Psi_k}{k}} \implies \overbrace{-\underbrace{nx\Phi_n}_{n\,\to\,n+1} - \underbrace{n\lambda_n\Phi_n}_{n\,\to\,n-1}}^{\text{Outflow}} + \overbrace{\underbrace{(n-1)x\Phi_{n-1}}_{n-1\,\to\,n} + \underbrace{(n+1)\lambda_{n+1}\Phi_{n+1}}_{n+1\,\to\,n}}^{\text{Inflow}} - \overbrace{\eta\Phi_n}^{\text{Dilution}} = 0$$

Outflows from n correspond to a n-product firm adding or shutting down a product. The former happens with Poisson arrival rate nx. While the latter is not the result of a Poisson shock, the total mass flowing out is $n\lambda_n\Phi_n$ since λ_n is the average product exit rate among n-product firms. Inflows from $n-1$ to n correspond to a $(n-1)$-product firm adding a product, which happens at Poisson rate $(n-1)x$. Inflows from $n+1$ to n correspond to $(n+1)$-product firm shutting down production of one of its products, since the average product exit rate among such firms is λ_{n+1}, the total mass flowing in is $(n+1)\lambda_{n+1}\Phi_{n+1}$. Finally, growth in the total measure of firms at rate η (due to population growth) dilutes these shares.

3.4 Solving for the stationary firm size distribution

Special case with no population growth ($\eta = 0$). Given the semi-endogenous nature of the model, the BGP is now simply a stationary equilibrium. Assumption 2 requires $\beta < 0$, and active firm entry requires $x < -\alpha\beta$. One can guess and verify that a solution to the system is:

$$\forall n \;\; f_n(\mathbf{q}) = \prod_{i=1}^{n} f_P(q_i) \quad ; \quad \lambda_n = \lambda \equiv -\alpha\beta > x > 0 \quad ; \quad \Psi_n = \frac{\lambda - x}{x}\left(\frac{x}{\lambda}\right)^n , \tag{18}$$

where $f_P(q)$ is the unconditional density of product qualities derived in Proposition 2 (plugging $\eta = g_Q = 0$ into the definition of ζ). In this case, conditioning on the number of products a firm has does not alter the distribution of product qualities. As a result, the average product exit rate does not vary with firm size. The resulting distribution of products per firm, given by Ψ_n, matches that of Klette and Kortum (2004).[8] When $\eta = 0$, whether one models product exit as resulting from a Poisson death shock or negative selection has no bearing on the stationary firm size distribution.

General case ($\eta > 0$). When $\eta > 0$, the sequences in (18) no longer constitute a solution to the system in Definition 3. The culprit is the $\frac{\eta}{n}$ term on the left hand side of each PDE.

Two points are worth emphasizing to highlight the intuition behind this result. First, as a consequence of volatility and positive selection into surviving, the distribution of quality among incumbent products first order stochastically dominates that among new products. Second, when $\eta > 0$, the size of firm cohorts grows over time. Taken together, since firms that just entered are necessarily single product, it is unsurprising that the distribution of product quality among single product firms looks different from the distribution of product quality among multiproduct firms.

My strategy to solve the general system is motivated by the fact that $\frac{\eta}{n}$, the term preventing the sequences in (18) from being a solution, vanishes as n grows large. Accordingly, for n large enough, I use the solution obtained under $\eta = 0$ as an approximation for the true solution:

$$\forall n > n_0, \;\; f_n(\mathbf{q}) = \prod_{i=1}^{n} f_P(q_i) \quad ; \quad \lambda_n = \alpha\left(g_Q - \beta\right) \quad ; \quad \frac{\Psi_{n+1}}{\Psi_n} = \frac{x}{\alpha(g_Q - \beta)} .$$

The next step then consists of solving for $\{f_n, \Psi_n, \lambda_n\}_{n \le n_0}$. While finite, this system suffers from a severe curse of dimensionality as each f_n is a pdf on $(0, \infty)^n$.

To address this curse of dimensionality, note that f_n is the joint density of n random variables that one expects to be independent. The reason is twofold: (i) the qualities of a firm's current products have no bearing on the quality of a new product it adds to its portfolio; (ii) the Brownian increments are independent across products. In addition, the system inherently satisfies permutation

[8] See equation (17) of Klette and Kortum (2004). Their nM_n maps to my Ψ_n and their θ to $\frac{\lambda - x}{x}$ here.

symmetry: a firm's state variable is its *set* of products, so their indexing should be irrelevant. So,

$$f_n(\mathbf{q}) \;=\; \prod_{i=1}^{n} \varphi_n(q_j), \quad \text{with } \varphi_n \text{ pdf on } (0,\infty) \text{ satisfying } \varphi_n(0) = 0 \;. \tag{19}$$

Just like in (18), each joint density f_n is the product of n evaluations of the same marginal density. However, the difference is that now the marginal density potentially depends on the firm's number of products. So, for example, while for a 2-product firm the product with index 1 is not systematically any different than the product with index 2, the typical product held by a 2-product firm is allowed to systematically differ from the typical product of a single product firm.

This observation simplifies the stationary firm size distribution to sequences Ψ_n and $\varphi_n(q)$ satisfying a system of coupled ODEs (given in Appendix B.6). This gets rid of the curse of dimensionality because each $\varphi_n(q)$ is a density on $(0,\infty)$. As I explain in Appendix C, I then solve for $\{\Psi_n, \lambda_n, \varphi_n(q)\}_{n\leq n_0}$ using state of the art ODE solvers.

In addition to standard convergence metrics, the theory provides a transparent way of verifying how accurate my solution is. By the law of total probability, aggregating the conditional distributions should yield back the unconditional distribution:

$$f_p(q) = \sum_{n=1}^{\infty} \Psi_n \varphi_n(q) \qquad \text{and} \qquad \frac{D_t}{M_t} = \sum_{n=1}^{\infty} \Psi_n \lambda_n \;.$$

Here $f_p(q)$ is the unconditional distribution of product quality and $\frac{D_t}{M_t}$ is the product exit rate, both characterized in closed form in Proposition 2. Crucially, neither of these identities was used as part of the solution strategy.

A payoff of characterizing the stationary firm size distribution is the ability to pin down the firm exit rate, which is given by:

$$\text{Firm Exit Rate } = \lambda_1 \Phi_1 = \lambda_1 \frac{\Psi_1}{\sum_{n=1}^{\infty} \frac{\Psi_n}{n}} \;.$$

Intuitively, given the continuous time setup of the model, exiting firms are necessarily single product. Such firms exit at rate λ_1 and their share among the total number of firms is Φ_1.

4 Empirical Evidence

My ultimate goal is to use the model to quantify knowledge spillovers. As the preceding sections revealed, I infer spillovers from the gradual component of product exit due to downward drift toward the exit threshold. Since product level data are only available for a few sectors, I leverage Proposition 6 to identify this statistic from the profile of firm exit by age. Therefore, before applying my approach, I provide direct evidence from the consumer packaged goods sector for the key features of the model underpinning this sufficient statistic result: that product exit features negative selection and unfolds gradually, and that among incumbent firms the product addition rate per existing product is constant.

4.1 Data

Motivating and validating the model's treatment of product dynamics requires product-level data. For this purpose, I use the NielsenIQ retail scanner dataset, which provides high-frequency data at the UPC (universal product code, or barcode) level. Its coverage spans consumer packaged goods such as groceries, cosmetics, and cleaning supplies. Different UPCs are partitioned into 104 product categories (groups), including coffee, vitamins, laundry supplies, and pet food.

The analysis draws on data from 2006 to 2019, covering a balanced panel of 25,400 retail stores. Each retailer provides NielsenIQ with weekly data on sales and prices at the UPC level. The results below are obtained by aggregating these data across retailers to the yearly level. This yields 6 million UPC-year observations and 1.2 million unique UPCs. Using the GS1 database, I identify which UPCs are manufactured by the same firm, resulting in a total of 40,600 firms in my sample. The Data Appendix provides further details on my sample construction and data cleaning steps.

Table 2: Summary Statistics – NielsenIQ Retail Scanner Sample

	Mean	Median	P75	P90	P95
Firm sales (thousands of $)	9,171	87	1,335	34,382	66,022
Firm's market share	10%	8%	17%	23%	27%
Number of UPCs per firm	12	3	8	37	62
UPC sales (thousands of $)	502	14	181	1,026	2,377
UPC's market share	1.2%	0.7%	1.3%	2.4%	3.5%

Notes: NielsenIQ Retail Scanner data with GS1 database used to identify the firm manufacturing a UPC. Each statistic is based on annual data from 2006 to 2019, and computed first at the group-year level, then aggregated to the yearly level weighting a group by its share of sales that year, and finally averaged across the 14 years. So, first three rows correspond to firm-level statistics *within a group*.

To validate my model's treatment of product exit, I want to compare UPCs within the same group (sector). Because reporting results for each of the 104 product groups would be impractical, I instead compute different moments at the group–year level, then aggregate them to the yearly level weighting a group by its share of sales that year, and finally average across years.

Table 3: UPC Entry and Exit Dynamics

Entry Rate	Exit Rate	UPC Age At Exit (in years) Mean	UPC Age At Exit (in years) Median
13.6%	12.3%	3.1	2.9

Notes: All statistics are group–year moments, sales-weighted to the year and averaged across years. Last two columns based on exiting UPCs for which age is not left censored.

Table 2 provides summary statistics. Within a product group, the distribution of sales across firms is highly skewed. Consistent with the findings of Hottman, Redding and Weinstein (2016), this reflects a skewed distribution of the number of UPCs per firm, as well as sales per UPC.

Table 3 points to substantial turnover at the UPC level: the annual UPC entry and exit rates are 13.6% and 12.3% respectively. Notice that the average age of an exiting UPC is lower than the one that would prevail if the hazard rate of exit were flat (1/0.123 = 8.1).

4.2 Four facts corroborating the paper's treatment of product exit

Fact 1: UPCs with more sales are less likely to exit

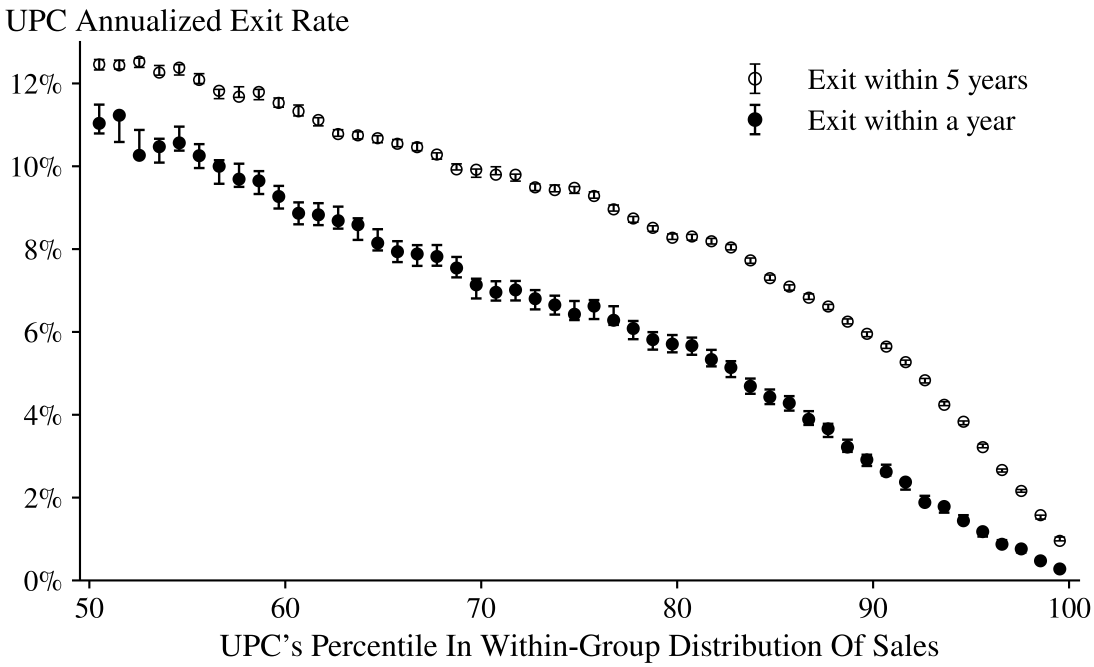
Table 4: UPC Sales and Hazard Rate of Exit

	Exit_{pt}			
$\log(\text{sales}_{pt-1})$	-1.69	-1.56	-2.70	-2.60
	(0.10)	(0.11)	(0.24)	(0.24)
UPC age FE		✓		✓
Group x Year FE	✓	✓		
Firm x Group x Year FE			✓	✓
Observations	2.58M	2.58M	2.50M	2.50M

Notes: Each column reports results from the OLS regression of a dummy for UPC p's exit at t (multiplied by 100) on p's log annual sales in year $t-1$. Sample consists of UPCs with above-median sales in a group-year. Average of the dependent variable is 7.4%, and distributional moments from UPC sales reported in Table 2. Standard error in parentheses, clustered at the group level. UPC age FE includes a separate dummy for left censored UPCs.

The first way to provide empirical support for selection into exit at the UPC level is by looking at how the exit rate varies with sales. Figure 2 shows the annualized hazard rate of UPC exit as a function of a UPC's percentile in the sales distribution for the corresponding group-year. I focus on

Figure 2: UPC Sales and Hazard Rate of Exit



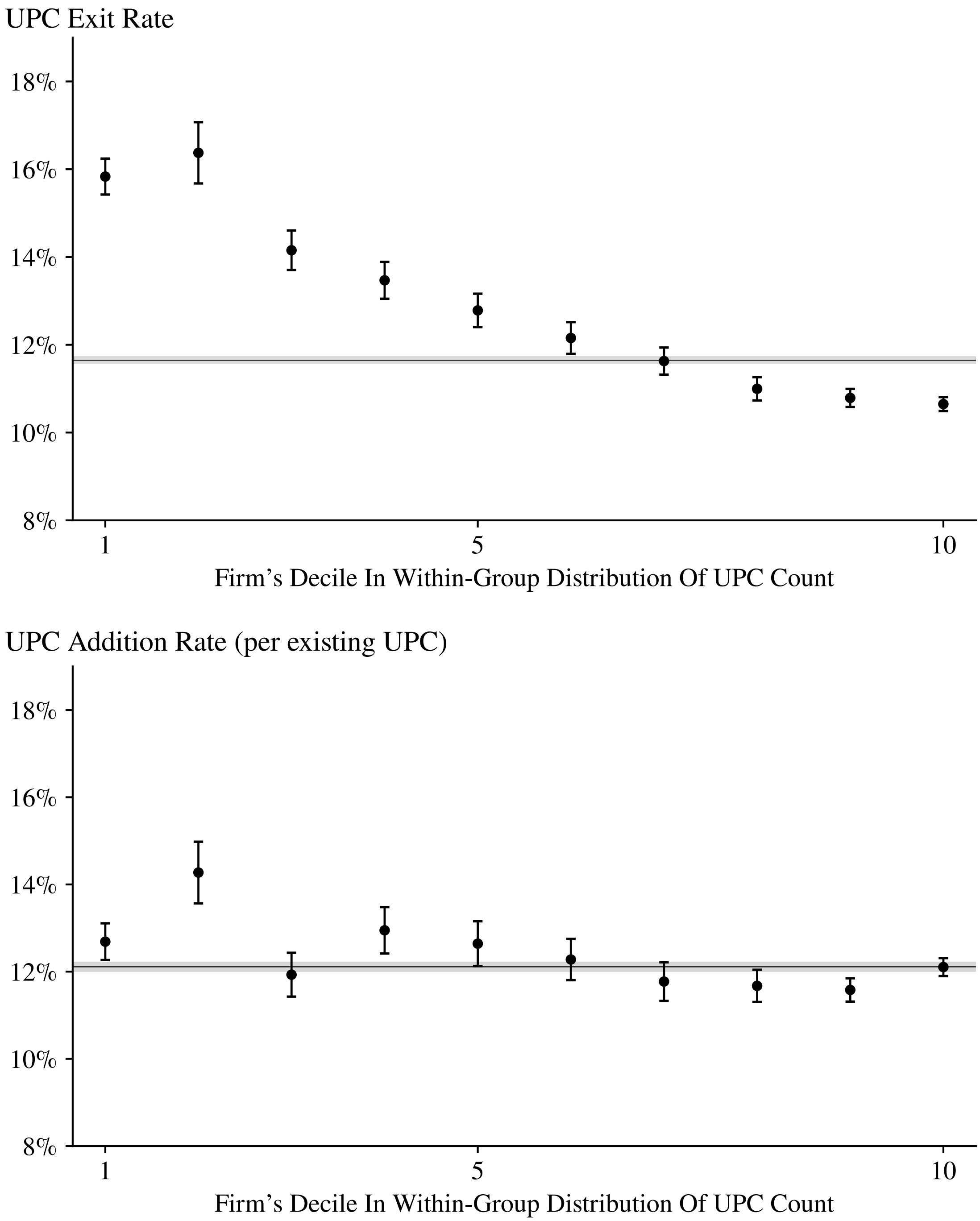
Notes: Cattaneo, Crump, Farrell and Feng (2024) binscatter with 50 bins, corresponding to the 50th through 99th percentile of the distribution of sales within a group in year t. "Exit within a year" refers to year $t+1$ being the UPC's last year of sales. "Exit within 5 years" refers to UPC's last year of sales being $t+6$ or earlier. Vertical bars are 95% pointwise confidence intervals. Underlying number of UPC-year observations is 2.73M (so roughly 54,600 per bin).

UPCs with above median sales (in a group-year) to minimize concerns about measurement error (those with the less sales are sold at very few retailers and account for only 2% of sales).

Figure 2 clearly illustrates that, within a group, UPC exit declines with UPC sales. This is true even when looking at the hazard rate of exiting within 5 years, and when zooming in on the right tail of the sales distribution within a group. Table 4 presents the corresponding regression results. Among products with above-median sales in a group–year, a doubling of sales (≈ 0.693 log units) reduces the exit hazard by 1.17 percentage points. Subsequent columns show that this negative relationship is robust to the inclusion of UPC-age fixed effects and firm-group-year fixed effects. This pattern corroborates the UPC-level evidence in Broda and Weinstein (2010), who find higher exit among smaller and younger UPCs using the NielsenIQ Homescan panel. Showing the same relationship in retailer scanner data, where exit reflects disappearance from store shelves rather than zero purchases by a household sample, lends further credibility to this relationship.[9] Bernard, Redding and Schott (2010) reach similar conclusions at a coarser product definition (5-digit SIC codes in U.S. manufacturing).

[9]Consistent with entry and exit being measured with non-negligible error in the Homescan panel, the entry and exit rates reported in Table 3 are substantially lower than those reported in Broda and Weinstein (2010).

Figure 3: UPC Exit and Addition Rates Across Multiproduct Firms



Notes: Cattaneo et al. (2024) binscatter with 10 bins, corresponding to deciles of distribution of UPC count among multiproduct firms within a group-year. Exiting (entering) UPCs are those for which the current year is the last (first) year of sales. Rates obtained through division by firm's UPC count in corresponding group, averaged between previous and current year. Vertical bars are 95% pointwise confidence intervals. Horizontal line is inverse-weighted mean, with 95% confidence interval around it. Underlying number of firm-year observations is 264,809. Single product firms are not shown because they account for 42% of firms—among them, UPC exit rate is 16.6% and add rate is 5.2%.

Fact 2: UPC exit rate is lower among firms with more UPCs

The top panel of Figure 3 displays another facet of selection into product exit. It shows a firm's UPC exit rate as a function of its number of products. The former is defined as the number of exiting UPCs divided by the number of UPCs in the firm's portfolio (averaged between last and current year). So this is the empirical counterpart of the relationship between λ_n and n in the model. The figure illustrates that, across firms in a group-year, the hazard rate of UPC exit is lower for firms with more UPCs (λ_n decreasing in n). This pattern is clearly inconsistent with models where product exit is a Poisson process. In Section 5, I show that my model can generate this fact as a consequence of negative selection into product exit.

Fact 3: Similar rates of UPC addition per existing UPC across multiproduct firms

The bottom panel of Figure 3 shows that, in contrast to the UPC exit rate, the UPC addition rate does not seem to be systematically related to the firm's UPC count. The addition rate is defined as the firm's number of entering UPCs divided by the number of UPCs in the firm's portfolio (averaged between last and current year). So this is the empirical counterpart of the relationship between x and n in the model. The lack of a systematic relationship provides empirical support for the rate of new product introductions per existing product being constant.

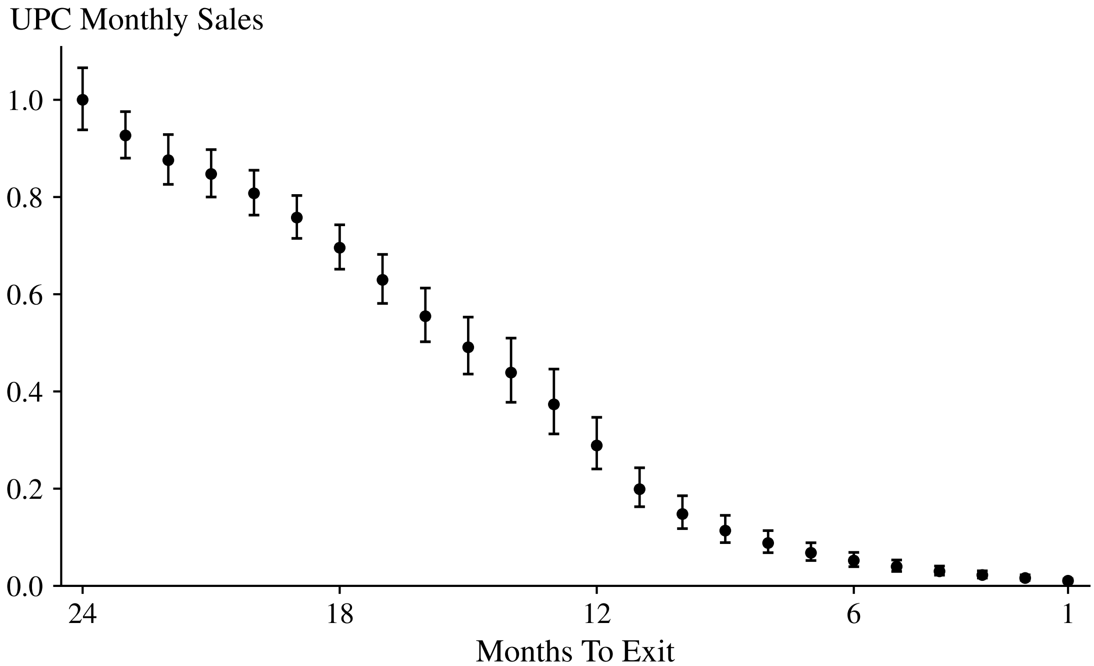
Fact 4: Prior to exit, UPC's sales decline gradually

Besides the granularity, another advantage of the NielsenIQ Retail Scanner dataset is its high frequency, which allows me to track dynamics prior to exit. Specifically, I estimate:

$$\log \text{Sales}_{pt} = \gamma_p + \sum_{m=1}^{24} \beta_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ ; \tag{20}$$

where p indexes a UPC, g its group (product category), and t a month, with γ_p a UPC fixed effect, γ_{gt} a group-month fixed effect, and D_{pt}^m a dummy variable equal to 1 m months prior to the UPC's exit. The path of $\exp(\beta_m)$ then represents the evolution of sales in the two years leading up to exit. Given selection concerns, I only include UPCs that were at least two years old when they exited. Figure 4 plots the resulting path for $\exp(\beta_m)$, normalizing to 1 sales two years prior to exit. The figure reveals that, in the lead up to exit, a UPC experiences a gradual decline in sales. Figure E1 in the Appendix reveals that this gradual decline happens along the extensive margin (number of retailers selling the UPC) as well as the intensive margin (sales of the UPC per store).

Figure 4: Evolution of UPC Sales prior to Exit



Notes: Path of $\exp(\beta_m)$ from Equation 20, normalized such that $\exp(\beta_{24}) = 1$. Number of observations in the underlying regression is 54M with $R^2 = 0.66$. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

Fact 5: Pre-exit price–quantity patterns are consistent with a negative residual demand shock

A final advantage of the NielsenIQ Retail Scanner dataset is the ability to break down UPC sales into price times quantity. This allows me to separately track the evolution of price and quantity prior to exit. Using the same notation as in Equation 20, I estimate:

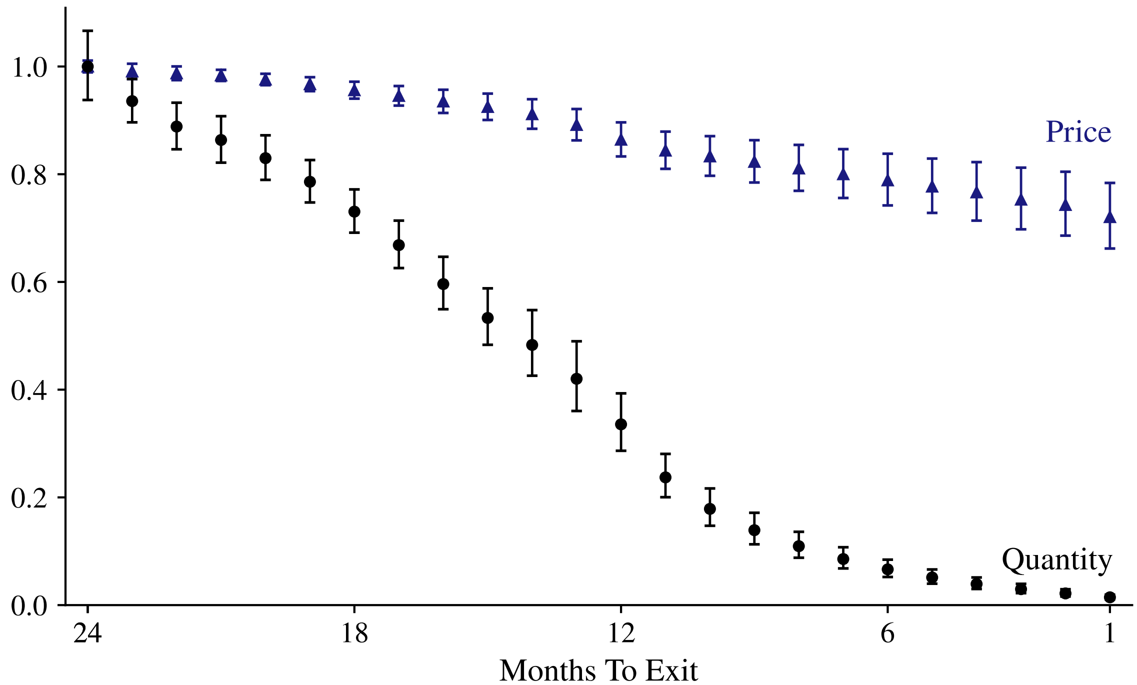
$$\log \text{Price}_{pt} = \gamma_p + \sum_{m=1}^{24} \pi_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ , \tag{21}$$

$$\log \text{Quantity}_{pt} = \gamma_p + \sum_{m=1}^{24} \kappa_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ . \tag{22}$$

The paths of $\exp(\pi_m)$ and $\exp(\kappa_m)$ capture the evolution of price and quantity in the two years prior to exit. Figure 5 shows that, while the product's relative price falls modestly, the quantity sold collapses. These patterns are in line with a gradual negative demand shock in the lead up to exit: despite the UPC becoming relatively cheaper, its consumption is decreasing.

Taken together, these five facts corroborate the paper's treatment of product dynamics. Fact 3 supports each incumbent product "giving birth" to a new product at a constant Poisson rate, a feature the equilibrium of my model inherits from Klette and Kortum (2004) and Luttmer (2011).

Figure 5: Evolution of UPC Price and Quantity prior to Exit



Notes: Price curve corresponds to path of $\exp(\pi_m)$ from Equation 21, normalized such that $\exp(\pi_{24}) = 1$; underlying regression has 54M observations with $R^2 = 0.86$. Quantity curve corresponds to path of $\exp(\kappa_m)$ from Equation 22, normalized such that $\exp(\kappa_{24}) = 1$; underlying regression has 54M observations with $R^2 = 0.68$. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

At the same time, the remaining facts ground the key departure I take relative to these models, as a Poisson exit process at the product level does not generate the observed patterns of negative selection and gradual exit.

5 Quantitative Results

The preceding sections developed the apparatus underlying my novel approach to quantifying the wedge knowledge spillovers create between social and private rates of return to innovation. This section applies the approach across U.S. private nonfarm employer businesses.

5.1 Estimation strategy

My estimation targets the profile of firm exit by age. Specifically, I use the hazard rate of firm exit for ages 1 through 19, as reported in Sterk, Sedláček and Pugsley (2021) using data from the

U.S. Census Longitudinal Business Database (LBD) for the years 1979 to 2012. The coverage of the LBD is the universe of U.S. nonfarm private employer firms; Jarmin and Miranda (2002) and Chow, Fort, Goetz, Goldschlag, Lawrence, Perlman, Stinson and White (2021) provide details on the construction of this administrative dataset.

As emphasized by Hopenhayn, Neira and Singhania (2022) and Karahan, Pugsley and Şahin (2024), these age-specific firm exit rates have been remarkably stable in the U.S. in recent decades. This stability makes them ideal targets for the estimation of a stationary model like mine.

As established in Section 3, the model counterpart of this profile of firm exit by age is governed by three statistics. The first, $\theta\eta - \alpha\beta$, captures the component of the product exit rate driven by the downward drift toward the exit threshold. The second, $\alpha\nu$, is a measure of relative volatility and governs product exit due to idiosyncratic shocks. The third, x, is the endogenous rate at which incumbent firms add a product to their portfolio (per existing product).

Table 5: GMM Estimation Results

	Symbol	Point Estimate
Product exit due to drift	$\theta\eta - \alpha\beta$	0.158
Relative volatility	$\alpha\nu$	0.296
Product addition rate	x	0.127

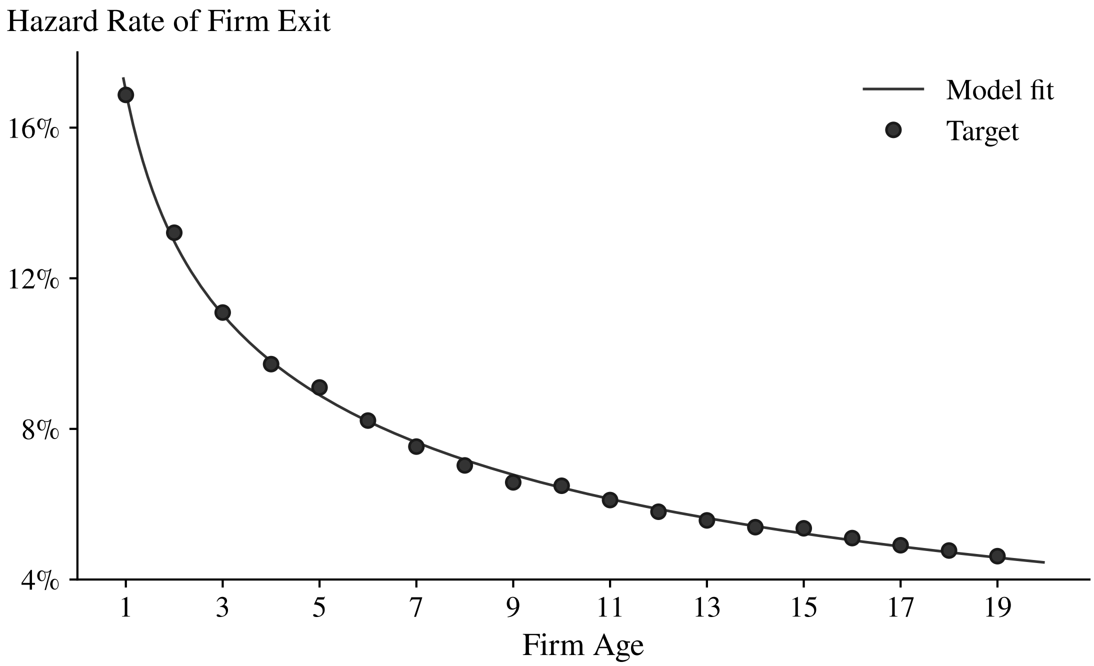
Notes: θ is the spillover elasticity, η the population growth rate, α the thinness of the entry distribution, β and ν the drift and volatility of product quality, and x the rate at which an incumbent firm adds a product to its portfolio (birth rate). GMM objective is an equally-weighted least squares.

I use a Generalized Method of Moments (GMM) procedure to estimate these three statistics, minimizing the equally weighted sum of squared deviations between the 19 empirical and model-implied hazard rates. The latter are obtained by numerically solving the integral equation in Proposition 6 (so no simulation is needed). The resulting estimates are reported in Table 5.

Figure 6 shows the model's excellent fit to the 19 targeted moments. Despite its parsimony, the model tracks the sharp decline in hazard rates at young ages as well as the gradual flattening at older ages, with mean and median absolute deviations below 0.1 percentage points.

To build intuition for the local sensitivity of the moments with respect to the three statistics, Figure 7 shows the effect on the profile of firm exit by age of perturbing each of these statistics while holding the other two fixed. The top panel shows that more *product* exit due to downward drift toward the exit threshold (higher $\theta\eta - \alpha\beta$) raises the *firm* exit rate at all ages. The bottom left panel shows that, in contrast, raising relative volatility (higher $\alpha\nu$) raises the exit rate among young firms but lowers it among older firms. The latter reflects that surviving firms have even more positively selected products when volatility is higher. Finally, the bottom right panel shows that a

Figure 6: Fit of Targeted Moments



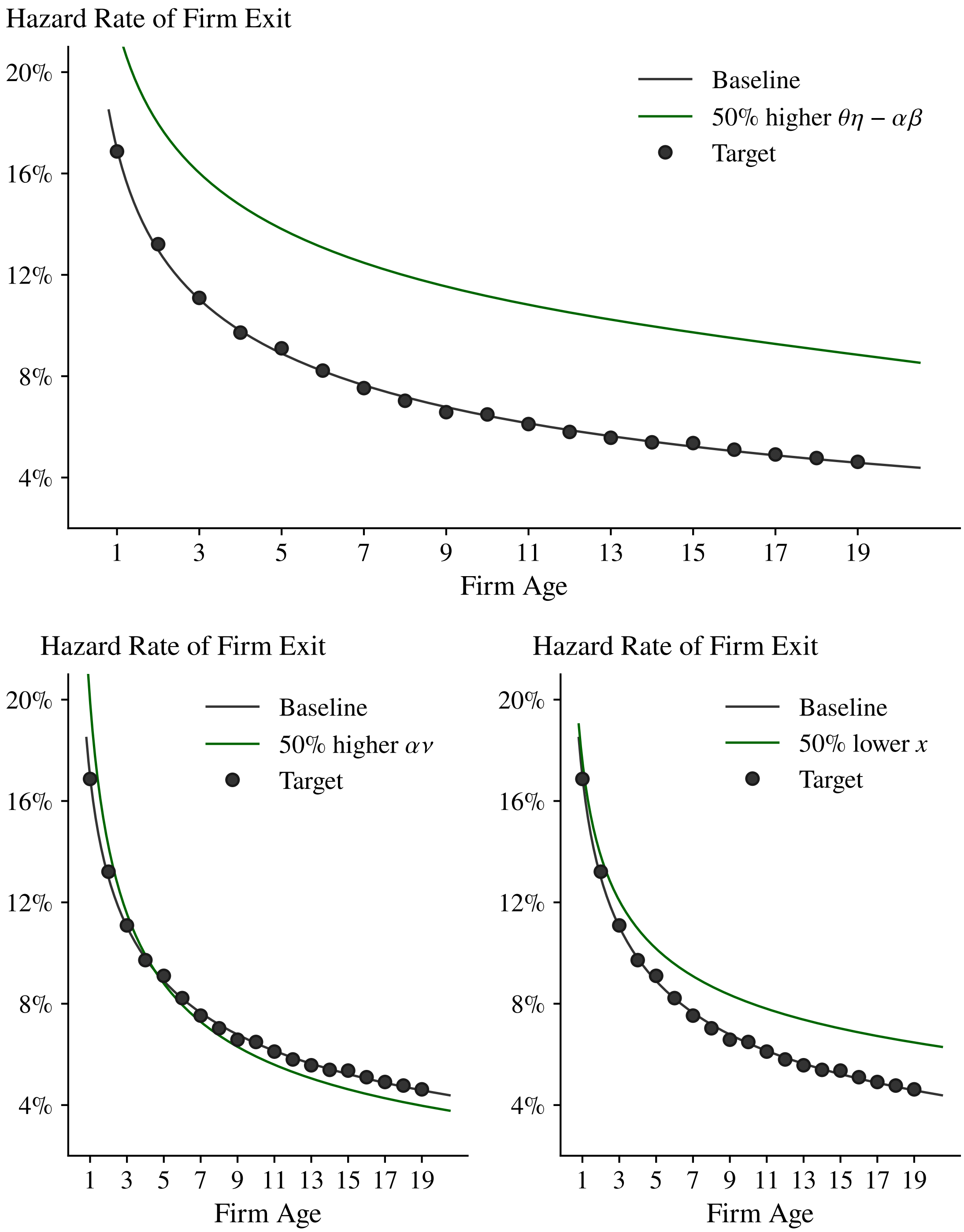
Notes: The 19 target moments are the firm exit rate at ages 1 through 19, reported in Sterk, Sedláček and Pugsley (2021) and calculated from LBD. Across these 19 moments, the absolute difference between model-based and targeted has mean 0.089 p.p. and median 0.063 p.p. and the root mean squared error (RMSE) is 0.11 p.p.

lower product addition rate (x) primarily increases the exit rate for mature firms: while new firms are still single product, this comparative static decreases the number of products older firms have.

Figure 8 is an alternative way to present the results from this same type of exercise. The difference relative to Figure 7 is that the y-axis shows the deviation in percentage points from the baseline hazard rate. The point is to make visually clear that each statistic has a substantial impact on the exit rate at some age. Table E4 in the Appendix makes the same point by showing the percentage point change in the model's hazard rate of firm exit at each age resulting from a perturbation to each of the three statistics. For transparency, I also report—in Table E5 of the Appendix—the sensitivity of each of the three estimated statistics with respect to each of the 19 empirical moments (Andrews, Gentzkow and Shapiro, 2017).

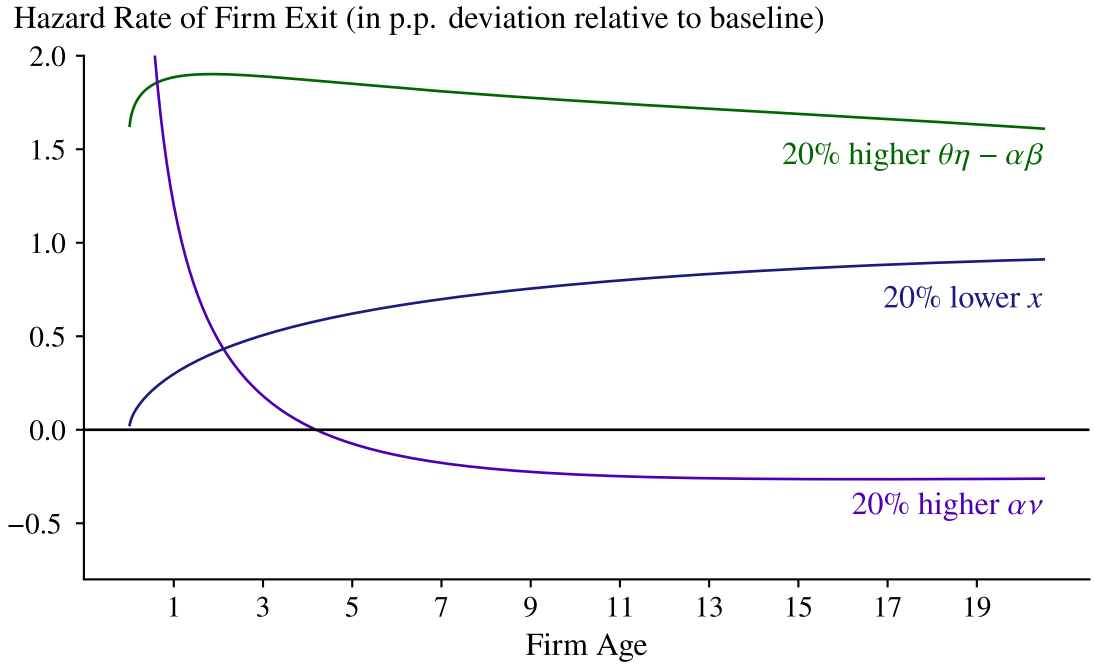
To demonstrate how strongly the moments constrain the estimate of the product exit due to drift, $\theta\eta - \alpha\beta$, I re-estimate restricted versions of the model and assess their fit. Specifically, I exogenously set $\theta\eta - \alpha\beta$ to a value different than 15.8% and estimate the remaining two statistics (relative volatility $\alpha\nu$ and product addition rate x) through a GMM procedure targeting the same firm exit profile. Figure 9 plots the results when $\theta\eta - \alpha\beta$ is exogenously set to 6% (top left panel), 11% (top right panel), 21% (bottom left panel) and 26% (bottom right panel).

Figure 7: Effect of different statistics on firm exit by age



Notes: Each panel shows the effect of changing one of the three estimated statistics, while holding the other two fixed. $\theta\eta - \alpha\beta$ is the product exit rate resulting from downward drift toward the exit threshold, $\alpha\nu$ measures relative volatility of shocks, and x is the product addition rate.

Figure 8: Sensitivity of firm exit at different ages with respect to each statistic

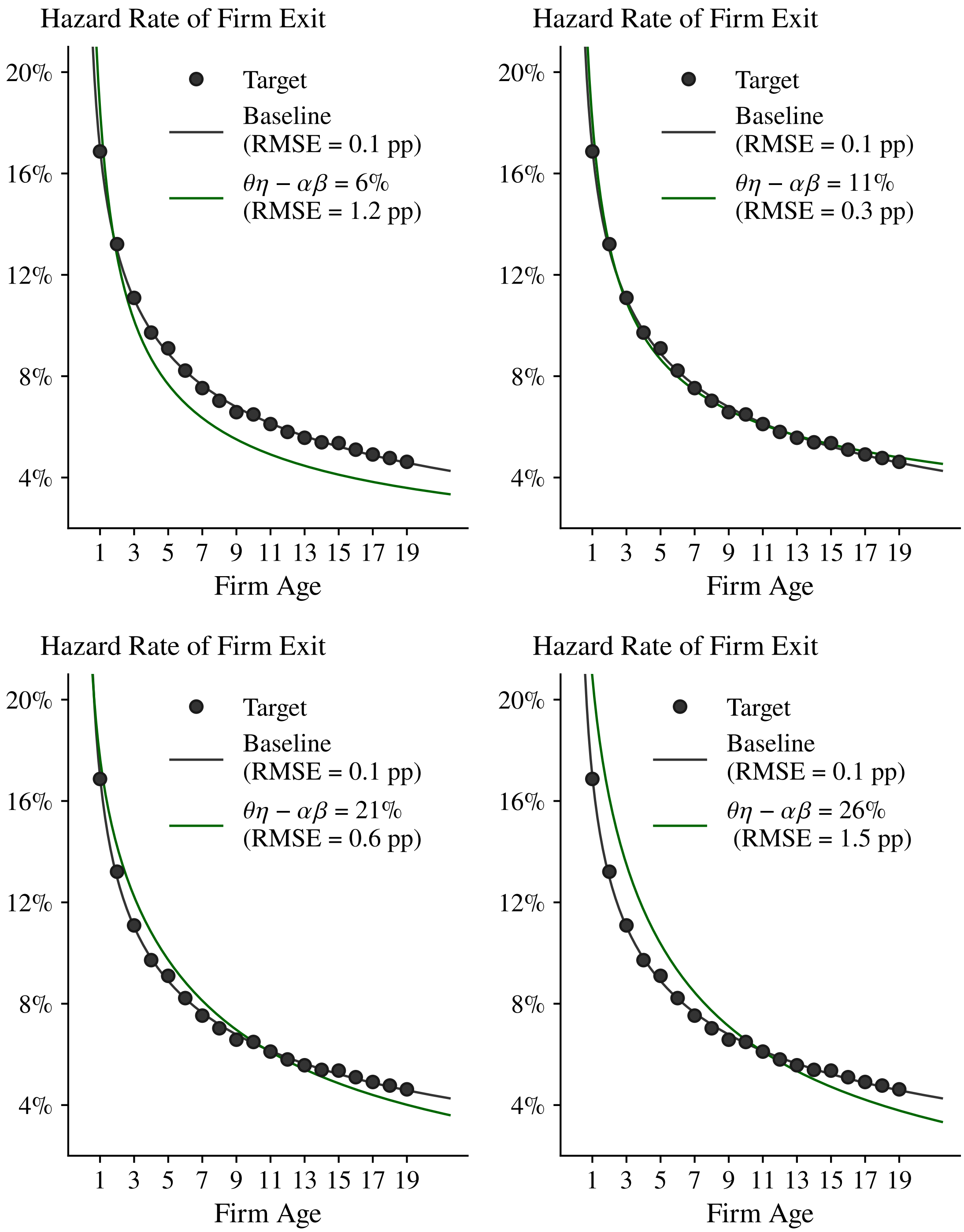


Notes: Each curve shows the effect of changing one of the three estimated statistics, while holding the other two fixed. $\theta\eta - \alpha\beta$ is the product exit rate resulting from downward drift toward the exit threshold; $\alpha\nu$ is a measure of the relative volatility of the Brownian shocks; and x is the rate at which incumbent firms add a product to their portfolio.

Unsurprisingly, all restricted models have a worse statistical fit, with a larger root mean squared error (RMSE) than the baseline unrestricted model. However, it is worth noting that the fit is still good when $\theta\eta - \alpha\beta$ is exogenously set to 11% (top right panel). Given Figure 7 and Figure 8, this is not surprising: one way to "compensate" for the lower $\theta\eta - \alpha\beta$ is with a higher relative volatility $\alpha\nu$ (lifts firm exit among younger firms) and a lower product addition rate x (lifts firm exit among older firms). Consistent with this intuition, the values the GMM procedure yields in this restricted case are $\alpha\nu = 0.41$ (higher than baseline value of 0.296) and $x = 4.1\%$ (lower than baseline value of 12.7%). This highlights the superiority of the baseline estimate: while this restricted model can still achieve a good statistical fit to the targeted moments, it does so by implying a counterfactually low product addition rate of 4.1%, far lower than the 12% observed in my NielsenIQ sample (see Figure 3), which is line with the baseline estimate for x (12.7%).

The reason the fit to targeted moments gets much worse in the $\theta\eta - \alpha\beta = 6\%$ case (top left panel) is that the product addition rate x is bounded below by 0, so that the above "compensation" argument can only go so far. In fact, in this case, the GMM yields $\alpha\nu = 0.51$ and $x = 0$. The top left panel shows that with such little product exit due to drift, the model has a hard time matching the non-trivial exit rate among older firms: while idiosyncratic shocks are another driver of product exit, firms that survived to become old have positively selected products and the sequence of shocks

Figure 9: Fit of restricted models with $\theta\eta - \alpha\beta$ exogenously set to different values



Notes: Each panel plots the 19 target moments, the baseline fit, and the fit of a restricted model with $\theta\eta - \alpha\beta$ exogenously set but $\alpha\nu$ and x re-estimated. Top left: $\alpha\nu = 0.51$, $x = 0$. Top right: $\alpha\nu = 0.41$, $x = 4.1\%$. Bottom left: $\alpha\nu = 0.21$, $x = 20.5\%$. Bottom right: $\alpha\nu = 0.28$, $x = 25.9\%$.

needed to get them to the exit threshold is unlikely.

The bottom two panels complete this sensitivity analysis, showing that setting $\theta\eta - \alpha\beta$ above the baseline estimate (to 21% or 26%) also leads to a substantial deterioration in the model's fit to targeted moments, with the RMSE rising to 0.6 pp and 1.5 pp, respectively. Taken together, these exercises demonstrate that the 15.8% estimate is tightly pinned down, as lower and higher values lead to worse fit to targeted and/or untargeted moments.

5.2 Magnitude of wedge due to knowledge spillovers

With the GMM estimates in hand, I now turn to their implications for the magnitude of knowledge spillovers. The estimation identifies that the product exit rate resulting from the deterministic downward drift toward the exit threshold, $\theta\eta - \alpha\beta$, is 15.8%. Rearranging this expression, the wedge between social and private rates of return to R&D created by knowledge spillovers is:

$$\boxed{\theta\eta = 0.158 + \alpha\beta}$$

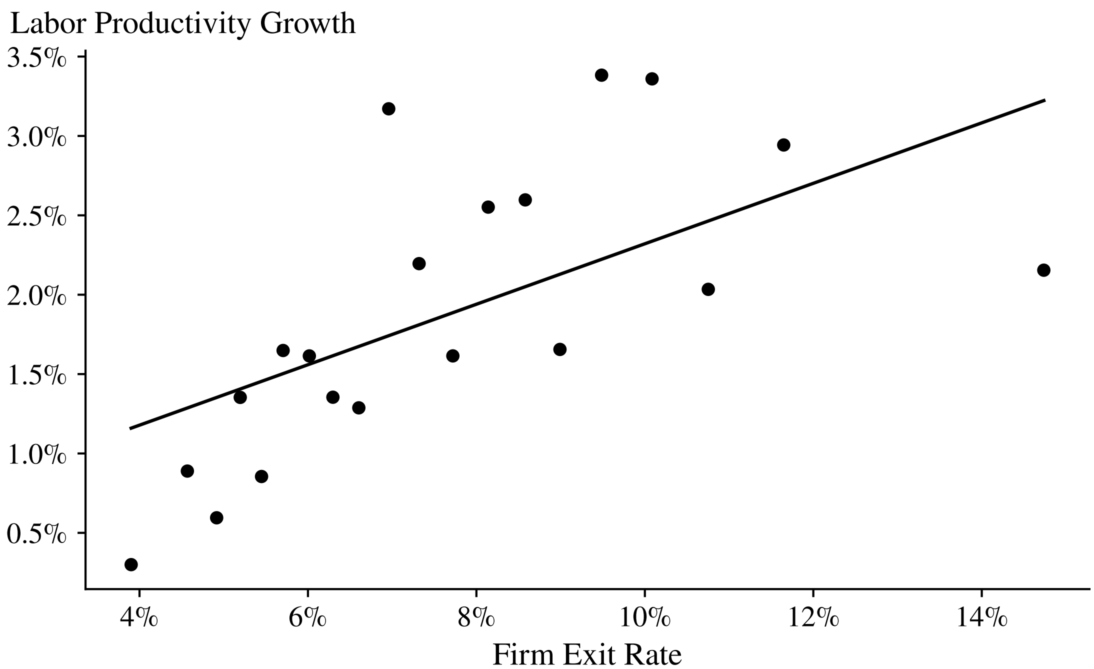
To obtain a benchmark estimate for the spillover wedge, I make a conservative assumption and set the exogenous incumbent quality growth, β, to zero (and later discuss the robustness of my results to relaxing this assumption). This choice is conservative because it leads to a lower bound on the wedge, as any positive $\beta > 0$ would imply an even larger wedge. In addition to being a conservative benchmark, this assumption is consistent with the evidence about the consumer packaged goods sector from Argente, Lee and Moreira (2024): they document that, on average, sales of incumbent products decline and that growth in firm sales is entirely driven by the addition of new products to the firm's portfolio.

Under this assumption, the analysis delivers the paper's main result: knowledge spillovers create a wedge of 15.8 percentage points between the social and private rates of return to R&D.

Taste for novelty as a confounding factor. A potential threat to my approach of backing out the wedge is that the estimated exit due to downward drift might reflect unmodeled forces that are observationally equivalent to a negative drift ($\beta < 0$). Chief among these concerns is a consumer taste for novelty. While preference shocks are accounted for by the Brownian motion, if consumers intrinsically value newness, demand for incumbent products would drift down over time as the product ages. This would lead me to overstate the wedge of interest.

To address this concern, I exploit sectoral heterogeneity. Figure 10 shows that, across industries, there is a positive association between labor productivity growth and the rate of firm exit. If a taste for novelty were the primary driver of my results, there should not be such a systematic relationship between firm exit rates and rates of measured labor productivity growth—as in that case firm exit would be solely driven by idiosyncratic shocks and preference for novelty, neither of which is

Figure 10: Firm exit rate and labor productivity growth across industries



Notes: Cattaneo, Crump, Farrell and Feng (2024) binscatter with 20 bins plotting 5-year average labor productivity growth against the 5-year average firm exit rate at the 4-digit industry level. Underlying number of observations is 1120 with 160 4-digit NAICS code industries observed over seven 5-year periods (from 1988 to 2022). Data on labor productivity growth are from the BLS while data on firm exit from the Census Business Dynamics Statistics.

related to productivity growth.

As a second exercise to address this concern, I re-estimate the model for each 2-digit NAICS sector. If my results were driven by taste for novelty, the estimated product exit due to downward drift should be largest in consumer-facing sectors where fashion and fads play a more important role. Instead, if my analysis plausibly identifies knowledge spillovers, this estimated statistic should be largest in sectors where narrative evidence points to important spillovers.

Before showing these sectoral estimates, it is helpful to clarify what they measure. Instead of a single sector, as in the baseline model, suppose the economy consists of S sectors, with a Cobb-Douglas aggregator across sectors. Innovation is directed toward a sector s, and the quality of a new product in sector s is drawn from an entry distribution with CCDF:[10]

$$\overline{F}^E_{st}(Q) = \prod_{j=1}^{S} K_{jt}^{\theta_{j\to s}} Q^{-\alpha_s} ,$$

[10] Appendix F.1 provides further details about the setup of this multi-sector extension of the model.

where K_{jt} is the cumulative stock of innovation in sector j, and $\theta_{j \to s}$ is the spillover elasticity from sector j to sector s. Along a BGP, the gradual component of product exit due to downward drift toward the exit threshold in sector s is $\eta \sum_{j=1}^{S} \theta_{j \to s} - \alpha_s \beta_s$. Therefore, it is informative about spillovers *received*, rather than *generated*. Accordingly, this metric is not informative about the design of sector-specific R&D policies. That said, it is still useful for two reasons. First, for the purposes of uniform policy across sectors (common for many innovation policies), averaging spillovers received yields spillovers generated. Second, this allows me to assess whether sectors with a larger estimate are plausibly benefiting from spillovers (regardless of their source).

Table 6: Estimated Product Exit due to Downward Drift Across 2-digit Sectors

	Sector	Estimated $\theta\eta - \alpha\beta$	Share of firms	Firm Exit Rate
First 4	Arts, Entertainment, and Recreation	25.6%	1.7%	8.6%
	Mining, Quarrying, and Oil and Gas Extraction	21.6%	0.4%	8.2%
	Transportation and Warehousing	18.9%	2.8%	10.8%
	Information	18.5%	1.2%	10.2%
Last 4	Finance and Insurance	10.5%	4.1%	7.7%
	Health Care and Social Assistance	7.0%	10.8%	6.2%
	Utilities	5.5%	0.1%	4.2%
	Management of Companies and Enterprises	3.3%	0.5%	3.6%

Notes: Estimated $\theta\eta - \alpha\beta$ obtained through GMM targeting sector's firm exit rate at ages 1, 2, 3, 4, 5, 8, 13, and 18. Underlying data are from the Business Dynamics Statistics for the years 1996-2019.

For each of the nineteen 2-digit NAICS sectors, I use the same GMM strategy as above to estimate the three statistics by targeting the sector's profile of firm exit by age. Table 6 presents the estimated product exit due to downward drift ($\theta\eta - \alpha\beta$) for eight sectors: the four in which the procedure yields the highest estimated wedge, and the four in which it leads to the lowest one.

The fact that the highest estimate is for Arts, Entertainment, and Recreation is consistent with a taste for novelty being a potential concern. However, several factors suggest that this confounder does not drive my results. First, this is a relatively small sector, accounting for less than 2% of both firms and employment in the U.S. economy. Second, and more importantly, the pattern among the other sectors with a high estimate provides strong evidence supporting my interpretation of the results. As I discuss next, these are technology-intensive industries where narrative evidence points to a central role for exactly the kind of spillovers my model is designed to capture.

The Mining, Quarrying, and Oil and Gas Extraction sector provides a prime example, as knowledge spillovers played a central role in enabling the shale gas boom. Within the sector, the common narrative credits George Mitchell's company with developing a breakthrough formula for combining horizontal drilling with slickwater fracturing, a process the rest of the industry then "adapted with awesome rapidity" (Golden and Wiseman, 2015, p. 960). But cross-industry spillovers also played a crucial role, as this breakthrough formula itself built on a "web of technological developments that helped spur the shale gas boom" (Golden and Wiseman, 2015, p. 973), including "3D seismic imaging techniques [...] that have benefited from advances in computing and that draw on technology originally developed to track submarines" (Golden and Wiseman, 2015, p. 973).

This example also clearly illustrates how my approach is complementary to those relying on patents to quantify spillovers. In fact, "although it is somewhat surprising and counterintuitive, during the late 1990s and early 2000s, neither Mitchell nor Devon pursued patent protection for their respective innovations in slickwater hydraulic fracturing and horizontal drilling" (Cahoy et al., 2013, p. 291). However, crucially for my approach, by lowering the cost of natural gas, these innovations reduced demand for coal and left a detectable trace in product markets: they drove (old) firms whose businesses relied on coal out of business (Linn and McCormack, 2019).

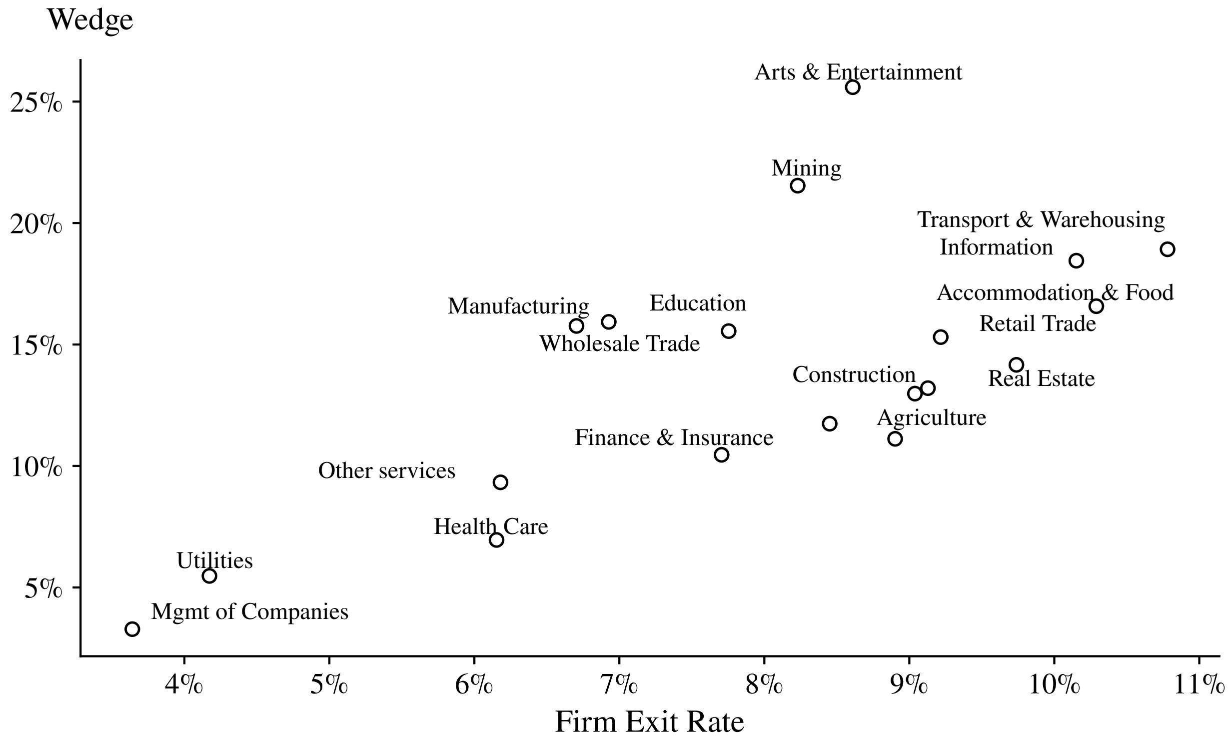
Turning to Transportation and Warehousing, the major developments the sector has experienced over this period also built on new technologies developed in other sectors. Arguably the most pivotal of these was the Global Positioning System (GPS). Originally developed for military purposes, GPS became widely available for commercial use in the U.S. in the mid 1990s. Its high precision capabilities were critical "to unlocking most of the benefits of telematics", the "field of technology that uses in-vehicle equipment to remotely monitor vehicles" (O'Connor et al., 2019, p. 13-1). Advanced on-board computers (OBCs) are an example of such equipment adopted by trucking companies and Hubbard (2003) estimates that they increased capacity utilization among adopting trucks by 13%. These gains in fleet management stem from dynamic route optimization as well as improved monitoring of driver behavior (Hubbard, 2000).

A more recent transformation in the transportation sector is the rise of ride-sharing platforms like Uber and Lyft. This new business model also leverages a confluence of technologies developed in other sectors, combining the ubiquity of GPS-enabled smartphones and mobile data networks with sophisticated matching algorithms. Cramer and Krueger (2016) document a higher utilization rate among UberX drivers compared to traditional taxi drivers, and argue that the more efficient technology for matching drivers and passengers is a leading contender in explaining this finding.

The case of the Information sector is perhaps the least surprising, as it encompasses software and digital industries, home to the Open Source Software (OSS) paradigm. In fact, OSS offers a tangible illustration of my model's aggregate stock of knowledge: it is a public stock that firms simultaneously contribute to and benefit from (Gortmaker, 2025).

The final column of Table 6 reveals a clear pattern: the four sectors with the highest estimated product exit due to downward drift ($\theta\eta - \alpha\beta$) also exhibit substantially higher firm exit rates than the four with the lowest estimate. However, the relationship is not monotonic. For instance, Transportation and Warehousing has a higher exit rate than Mining (10.8% vs. 8.2%), yet its

Figure 11: Estimated Wedge and Firm Exit Rate Across Sectors



Notes: Estimated wedge is the point estimate of $\theta\eta - \alpha\beta$ from a GMM procedure targeting the sector's firm exit at ages 1, 2, 3, 4, 5, 8, 13, and 18. Pearson correlation coefficient between firm exit rate and estimated wedge is 0.68. Underlying data are Business Dynamics Statistics for 1996-2019.

estimated $\theta\eta - \alpha\beta$ is lower. The reason is that, in addition to differences in downward drift, exit rates also reflect differences in volatility, incumbent innovation rates, and compositional differences in the age distribution of firms.

This underscores the necessity of using the model to learn about spillovers from firm exit rates. By leveraging the entire profile of firm exit by age (shown for these eight sectors in Figure E2), the GMM procedure identifies the component due to downward drift. This being said, Figure 11, which shows the results across all sectors, confirms that the correlation between firm exit rate and my estimated statistic is reasonably strong (0.68).

5.3 Quantitative Validation

With the cross-sectoral evidence providing *qualitative* support for my interpretation of the results, I now turn to bolstering confidence in their *quantitative* aspect.

Direct validation of the magnitude of spillovers is notoriously difficult, as knowledge flows themselves are inherently unobservable. However, this challenge highlights a key advantage of the structural approach taken in this paper. Because the estimation is embedded within a general equilibrium model, it generates a rich set of targeted as well as untargeted predictions about firm dynamics and growth. The goal of this subsection is to show that these quantitative predictions closely align with the corresponding patterns observed in the data.

In this vein, I solve for the stationary firm size distribution characterized in Section 3 and then simulate the resulting distribution of employment across firms. As I explain in Appendix C.2, the statistics estimated by the GMM (Table 5) are not sufficient for these purposes. Indeed, four additional (combination of) parameters need to be calibrated: the population growth rate (η), the thinness of the entry distribution relative to the elasticity of substitution ($\alpha/(\sigma-1)$), the constant component of product-level employment which consists of overhead and R&D labor ($\mathcal{F}+I$), and the production employment for a product approaching the exit threshold ($\underline{L}$).[11]

Table 7: Values for calibrated parameters

	Symbol	Value
Population growth rate	η	0.01
Index of Pareto tail	$\frac{\alpha}{\sigma-1}$	1.06
Fixed employment per product	$\mathcal{F}+I$	0.89
Production employment at exit threshold	$\underline{L}$	0.37

Notes: α is the thinness of the entry distribution, $\mathcal{F}$ and I are respectively the per-product overhead R&D employment, and $\underline{L}$ is the limit of product-level production employment as a product's q approaches 0 (See Appendix C.3 for definition in terms of model's structural parameters).

Table 7 reports values for these four calibrated parameters. Since the population growth η is the net firm entry rate along the BGP, I set $\eta = 1\%$ to match the average annual growth rate in the number of private nonfarm employer businesses between 1978 and 2019. To calibrate $\alpha/(\sigma-1)$, I take advantage of the fact that this is the Pareto tail index of the distribution of employment across firms. The reason is that the number of products per firm has a geometric-like thin tail, so that the Pareto tail in the cross-firm employment distribution arises from the thick tailed distribution of product quality. From Proposition 2, the Pareto tail index of this distribution is $\min\{\alpha,\zeta\}$. The parameters from Table 5 along with my calibration $\eta = 1\%$ imply:

$$\frac{\zeta}{\alpha} = \frac{\theta\eta + \sqrt{(\theta\eta)^2 + 2\eta(\alpha\nu)^2}}{(\alpha\nu)^2} \approx 3.7 \quad \Longrightarrow \quad \min\{\alpha,\zeta\} = \alpha.$$

Since product-level employment is proportional to quality raised to the power $\sigma-1$, it follows that the Pareto tail of the distribution of employment is $\alpha/(\sigma-1)$. As a result, to match the tail of 1.06

[11]The relevant details and definitions in terms of model's structural parameters are provided in Appendix C.3.

in the data (Luttmer, 2007), I set $\alpha/(\sigma-1) = 1.06$. Finally, to pin down $\mathcal{F} + I$ and $\underline{L}$, I target an average of 22 workers per firm and 58% of firms having 1 to 4 employees (both targets computed as averages for private nonfarm businesses between 1978 and 2019).

Product and Firm Dynamics. Table 8 displays entry and exit rates of firms and products. Despite only targeting the profile of firm exit until age 19, the model closely fits the overall exit rate of firms. With η calibrated to the growth rate in the number of firms, it is then unsurprising that the model also closely fits the firm entry rate.

Table 8: Product and Firm Entry and Exit Rates

	Model	Data
Product Entry Rate	17.1%	-
Product Exit Rate	16.1%	-
Firm Entry Rate	9.8%	9.9%
Firm Exit Rate	8.8%	8.9%

Notes: Data on firm entry and exit rates are from the Business Dynamics Statistics for 1978-2019 and cover the universe of U.S. private nonfarm employer businesses. No available data on product entry and exit rates with such coverage.

While empirical counterparts for the product entry and exit rates are not available for such broad coverage of the economy, I include them in the table to emphasize the high churn at the product level. This implies high turnover within the firm, in line with the findings of Broda and Weinstein (2010), Argente, Lee and Moreira (2018), and Argente, Lee and Moreira (2024) for the consumer packaged goods sector, Bernard, Redding and Schott (2010) for the U.S. manufacturing sector, and Berlingieri, De Ridder, Lashkari and Rigo (2025) for the French manufacturing sector.

Incumbents' Contribution to Growth. My quantitative strategy did not target the share of growth due to incumbent firms. Based largely on the employment growth of surviving incumbent firms, Garcia-Macia, Hsieh and Klenow (2019) estimate this share to be 75.2% across U.S. private nonfarm employer businesses.

Given that (i) entry of new products is the engine of productivity growth in my model and (ii) at any point in time entering and innovating incumbent firms draw the quality of their new products from the same distribution, the incumbent's contribution to growth in my model is simply the share of product entry accounted for by incumbent firms. From Table 8, the product entry rate is 17.1%, and from Table 5, each product in the portfolio of an incumbent firm leads to entry of new product at rate 12.7%. Therefore, the share of growth due to incumbents in the model is 74.3% (12.7/17.1), remarkably close to the estimate of 75.2% from Garcia-Macia, Hsieh and Klenow (2019).

Firm Size Distribution. Table 9 compares the model-implied firm size distribution (employees per firm) to its empirical counterpart. It is worth reiterating that the calibration only disciplined three features of this distribution: the Pareto tail index (1.06), the average firm size (22), and the share of firms in the 1-4 employee bin (58%). Yet, as the table conveys, the model is still capable to track the empirical share of firms and the share of employment across different firm size bins.

Table 9: Firm Size Distribution

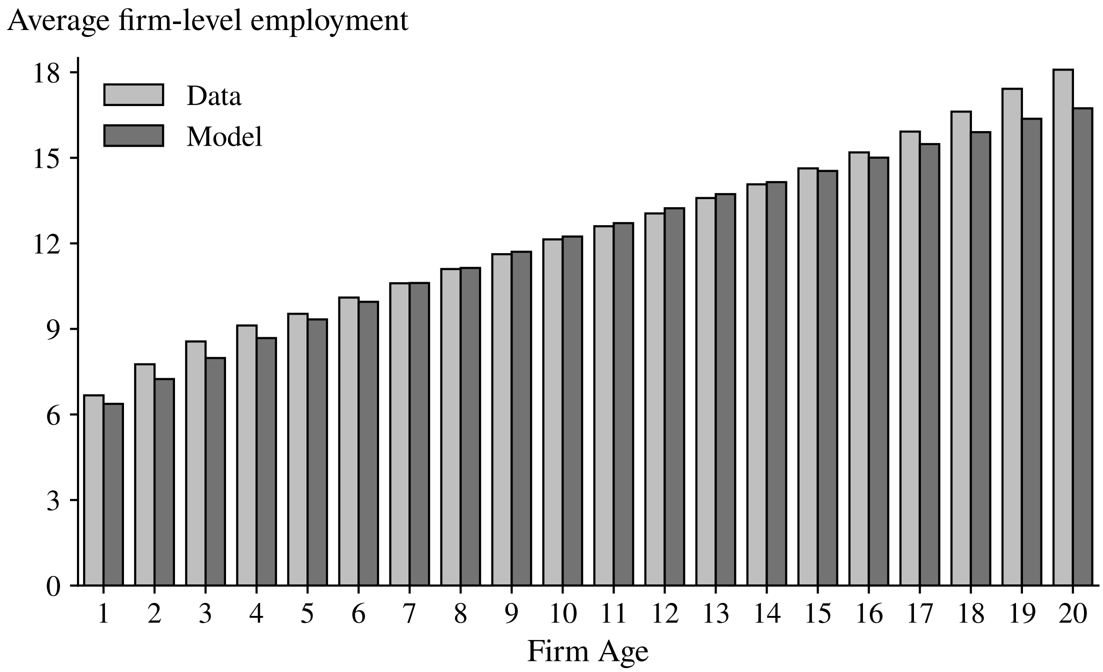
Firm size bin	Share of firms		Share of employment	
	Model	Data	Model	Data
1 to 4 employees	58.1%	58.1%	8.1%	6.2%
5 to 9 employees	20.5%	19.0%	8.1%	6.2%
10 to 19 employees	11.9%	11.4%	9.6%	7.5%
20 to 99 employees	8.5%	9.5%	17.2%	17.5%
100 to 499 employees	0.8%	1.6%	9.0%	13.8%
500+ employees	0.2%	0.4%	48.0%	48.8%

Notes: Data are from the Business Dynamics Statistics for 1978-2019 and cover the universe of U.S. private nonfarm employer businesses.

Firm Size by Firm Age. Despite average firm size by firm age not being targeted, Figure 12 shows that the model closely fits the average number of employees per firm at ages 1 through 20.

Differences in firm level employment stem from differences in (i) product count, as well as (ii) production employment per product (which, unlike R&D and overhead labor, depends on the product's relative quality). Figure E3 shows the average of these two statistics as a function of the firm's age in the quantified model. It conveys that older firms tend to employ more workers because they tend to have *more* and endogenously *better* products. The figure shows that, quantitatively, it is the former of these two mechanisms that matters more for generating the gradient of average firm size with respect to firm age. In contrast, Table E7 shows that employment per product plays a much more important role in accounting for cross-sectional heterogeneity in the firm size distribution: compared to firms with 1-4 employees, firms with 20+ employees have five times more products but employ 54 times more production workers per product. This feature of the quantified model is appealing for two reasons. First, it is consistent with the findings of Hottman, Redding and

Figure 12: Average Firm Size by Firm Age



Notes: Data on firm average employment by firm age as reported in Sterk, Sedláček and Pugsley (2021) and calculated from the LBD. To compute the model counterpart, I simulate a cohort of 5 million firms up until age 20, compute average employment by age, repeat this simulation 1000 times, and report the median of average employment by age across the simulations.

Weinstein (2016) that the quality (rather than the count) of a firm's products explains most of the variation in sales across firms. Second, since the arrival rate of new products scales linearly with the firm's number of products (rather than its workers), the model generates a declining relationship between arrival rate of a new product per worker employed and firm size—in line with the evidence from Akcigit and Kerr (2018).

Productivity growth. My quantitative strategy does not target a specific growth rate, a choice motivated by the well-acknowledged challenge of measuring quality improvements in the data (Bils and Klenow, 2001; Bils, 2009; Aghion, Bergeaud, Boppart, Klenow and Li, 2019; Atalay, Hortaçsu, Kimmel and Syverson, 2025). Nevertheless, it is instructive to examine the aggregate growth rates implied by the quantified model. Recall that:

$$g_Q = \frac{\theta}{\alpha}\eta \implies g_Q = \frac{\theta\eta}{\frac{\alpha}{\sigma-1}(\sigma-1)} \quad \text{and} \quad g = \frac{\eta}{\sigma-1} + g_Q\,. \tag{23}$$

Therefore, to calculate these rates, in addition to $\theta\eta$, η and $\frac{\alpha}{\sigma-1}$ (for which values have already been assigned), I need to take a stance on the elasticity of substitution across products (σ). I consider values for σ in the range of 5 to 10, in line with estimates for product-level elasticities from Hottman, Redding and Weinstein (2016).

Table 10: Growth Implications of Quantified Model

Elasticity of substitution (σ)	Quality growth (g_Q)	TFP growth (g)
5	3.73%	3.98%
7.5	2.29%	2.45%
10	1.66%	1.77%

Notes: TFP growth is the sum of quality growth and variety growth. Calculated using Equation 23 with $\eta = 0.01$, $\theta\eta = 0.158$ (so $\beta = 0$), and $\alpha/(\sigma - 1) = 1.06$.

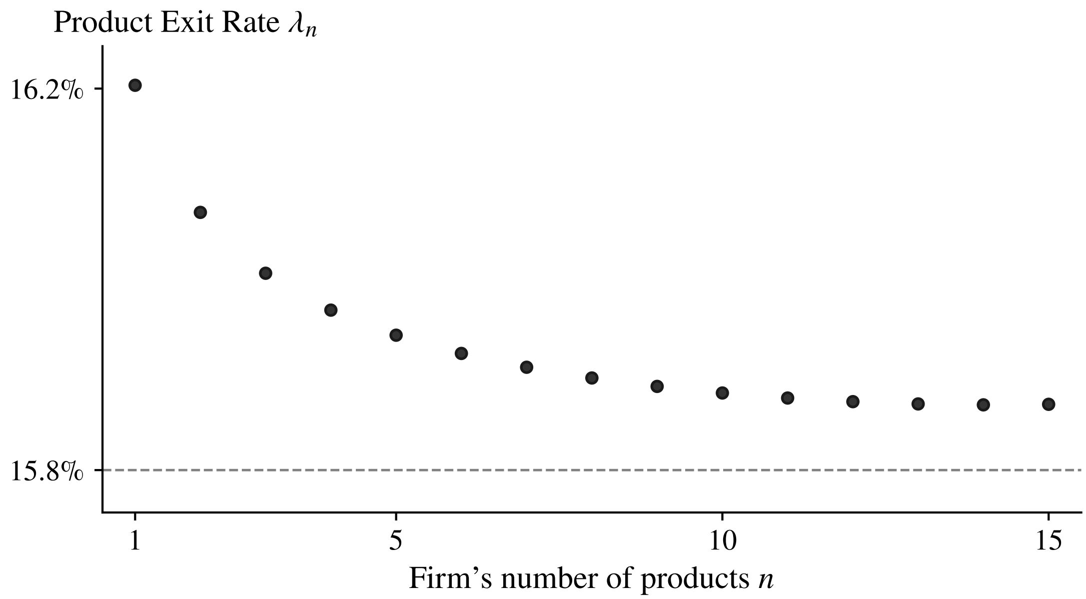
Table 10 reports the results. Since standard price deflators do not fully capture gains from product variety (Feenstra, 1994; Broda and Weinstein, 2006), the model's quality growth rate, g_Q, is the appropriate counterpart to measured productivity growth. Over the 1978-2019 period, average labor productivity growth in the U.S. private nonfarm business sector was 1.84%. This figure aligns remarkably well with the model's implied growth rate of 1.66% obtained with $\sigma = 10$ (value of the elasticity of substitution used in Atkeson and Burstein (2008)). Moreover, the higher growth rates implied by lower elasticities are also plausible, as a large body of research argues that measured productivity growth is understated. These estimates of "missing growth"—often attributed to how statistical agencies handle product exit and entry—range from 0.5 percentage points for the private nonfarm sector (Aghion, Bergeaud, Boppart, Klenow and Li, 2019) to as high as 2 percentage points for consumer durables (Bils, 2009). The model's growth implications are thus plausible and sit comfortably within the range of empirical estimates.

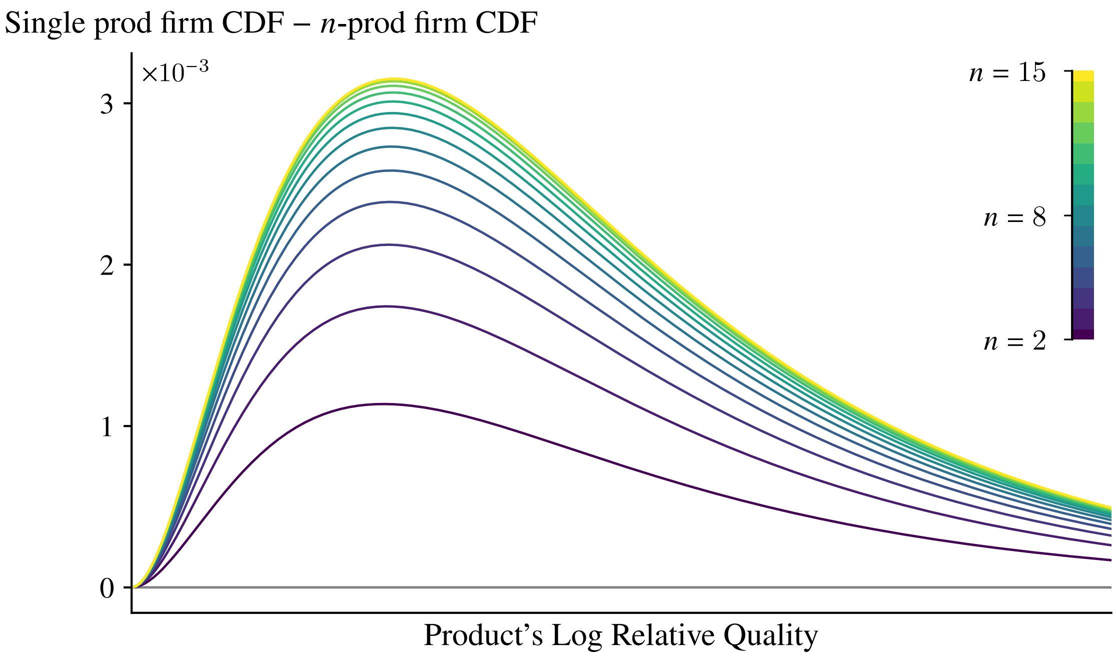
Product exit rate across firms. In Section 4, I showed that the average product exit rate is lower among firms with more products (Fact 2). There, I interpreted the fact as evidence in support of moving away from modeling product exit as a Poisson process. Here, I show how the quantified model endogenously generates this feature as a consequence of negative selection into product exit.

The top panel of Figure 13 illustrates this finding. It plots the model's relationship between a firm's average product exit rate (λ_n) against its number of products (n). The rate is highest for single product firms at 16.2% and declines monotonically as firms grow, slowly approaching a limit of 15.8%. While the range is narrower than the one in Figure 3, it is worth keeping in mind that the underlying samples are different (consumer packaged goods versus the universe of private nonfarm employer businesses).

The bottom panel of Figure 13 shows how the model generates this prediction: firms with *more*

Figure 13: Consequences of Selection on Model's Firm Size Distribution





Notes: The top panel shows λ_n, the average product exit rate among n-product firms from Definition 4, as a function of n. The bottom panel shows the CDF of product log-relative quality (q) among single product firms minus the CDF of product product log-relative quality (q) among n-product firms ($n > 1$). A positive gap means the latter ($n > 1$) first order stochastically dominates the former ($n = 1$). Refer to Appendix C for details about computational solution.

products have *better* products. The panel plots the difference between the cumulative distribution function (CDF) of product log-relative quality among single-product firms and that among n-product firms for $n > 1$. The fact that this difference is positive for the different values of $n > 1$ indicates first order stochastic dominance: the product quality distribution for multiproduct firms is better than for single product firms. Since the difference from the single product CDF widens with n, the figure also reveals a ranking of product quality by firm's number of products: the product quality distribution among n-product firms first-order stochastically dominates that among n'-product firms whenever $n > n'$. Finally, the vertical gap between the curves for consecutive values of n shrinks as n increases, showing that the distribution of product quality converges to its large n limit.

This feature of the model's stationary equilibrium is a consequence of positive selection into survival at the product level. Since a firm starts with a single product and gradually expands it portfolio through R&D, ending up with many products requires good draws and/or positive shocks.

5.4 Robustness Checks and Discussion

This final subsection explores the robustness and broader implications of the paper's main quantitative result. First, I assess the sensitivity of the estimated gap between social and private rates of return to entertaining positive drift in the quality of incumbent products—reflecting learning by doing or incumbent innovation. Second, I examine how taking into account growth due to human capital alters my quantitative results. Third, I explore the impact of doing the quantification with product-level instead of firm-level data. Finally, as an alternative metric for the strength of spillovers I am estimating, I quantify the extent to which innovation is under-provided relative to the first best.

Robustness to positive drift in quality of incumbent product ($\beta > 0$). My baseline estimate of 15.8 percentage points for the gap between social and private rates of return to innovation follows from the conservative assumption that β, the exogenous drift in an incumbent's product quality, is 0. While a literal interpretation of $\beta < 0$ seems implausible, I explained above (in section 5.2) that it is isomorphic to a taste for novelty and showed evidence that such explanations do not seem to be driving my results. I now discuss the sensitivity of my results to entertaining $\beta > 0$. This will unambiguously make the gap I estimate even larger, as $\theta\eta = \alpha\beta + 0.158$. This is also fairly intuitive: my estimation identifies the rate of product exit resulting from the gradual creative destruction of older products by newer ones. If incumbent products are themselves improving over time ($\beta > 0$), they become a moving target. To rationalize the same rate of product exit due to gradual creative destruction, the quality distribution of new entrants must therefore be advancing even more rapidly to overcome this incumbent quality growth (larger $\theta\eta$). So rather than assessing the direction of the bias, the point of this exercise is to assess how loose of a lower bound the 15.8 pp is when $\beta > 0$.

Table 11 presents the results from this robustness exercise. It displays the estimate of $\theta\eta$, the gap between social and private rates of return to R&D, for different calibrations of $\beta > 0$ and σ (the dependence on the latter is due the calibration pinning down $\alpha/(\sigma - 1) = 1.06$). The key

Table 11: Sensitivity of estimated wedge ($\theta\eta$) to positive drift in incumbent product quality

σ \ β	0.25%	0.5%	0.75%	1%
5	16.9%	17.8%	18.8%	19.8%
7.5	17.5%	19.1%	20.7%	22.3%
10	18.2%	20.3%	22.6%	24.8%

Notes: Spillover wedge $\theta\eta = \alpha\beta + 0.158 = 1.06(\sigma - 1)\beta + 0.158$, for different values of $\beta > 0$ and σ. β is the exogenous drift in the quality of an incumbent product, σ is the elasticity of substitution between products. Baseline estimate of 15.8% obtained under the conservative assumption that $\beta = 0$.

takeaway is that the 15.8 percentage points headline number, while conservative, is not an overly loose lower bound: even when allowing for 1% drift in incumbent product *quality* and an elasticity of substitution as high as 10, the estimated wedge only increases to 24.8 percentage points.

Human capital growth. Given that my model abstracts from human capital growth, I now assess how incorporating (exogenous) growth in human capital alters my results. I distinguish between two potential manifestations of human capital growth.

A first possibility is that human capital growth leads to an increase in the efficiency units supplied by each individual per unit of time. For individuals who choose to be entrepreneurs, human capital growth then translates into more draws from the entry distribution. As a result, along the balanced growth path, the cumulative stock of products created (K_t) grows at rate $\eta + g_h$, where $g_h > 0$ is the growth rate of human capital. As I show in Appendix F.2, under the assumption that $\beta = 0$ (no incumbent drift), my approach still identifies θg_K, which is the gap between social and private rates of return to innovation. Put differently, my results are robust to such human capital growth, which only affects how the estimated wedge of 15.8% is split between the spillover elasticity θ and the growth rate of K.

A second possibility is that higher human capital leads to draws from a better entry distribution. In this case, human capital growth then leads to improvements in the entry distribution over time and my baseline estimate is biased upward. In Appendix F.2, I show that in this case, I can recover the gap between social and private rates of return to innovation using:

$$\theta\eta = 0.158 \times (1 - \text{Share of measured productivity growth due to human capital growth}) .$$

In Appendix F.2, I propose three different approaches to calculate the share of measured productivity growth due to human capital growth. These approaches range in scope: the first uses the standard Mincerian return to account for growth resulting from rising educational attainment; the second uses the BLS labor composition index to account for reallocation of hours worked toward workers with more education but also more experience; and the third, most comprehensive method, extends the BLS labor composition index to also account for within worker type human capital growth (resulting for example from higher quality of schooling). These methods imply that human capital growth can explain between 19% and 36% of measured productivity growth. Accordingly, the resulting estimate for $\theta\eta$ falls to a range between 0.10 and 0.128. The takeaway is that even after allowing for such a role for human capital growth, the results point to knowledge spillovers leading to a wedge of at least 10 percentage points between social and private rates of return to innovation.

Quantifying Spillovers with Product Data. As I emphasized on a number of occasions, the key insight the paper leverages to quantify spillovers links dynamics of *product* exit to the wedge created by spillovers. The reason I rely on firm-level data to do the quantification is that comprehensive data at the product level are not available.

As a robustness exercise, I assess the sensitivity of my results to doing the quantification with product data instead of firm data. This requires a setting where both product and firm data are available, so that I can compare the results across the two methods. The food manufacturing sector provides such an opportunity. For *firm* exit by age, I use data from the Business Dynamics Statistics for the 3-digit NAICS sector 311 (food manufacturing).[12] For *product* exit by age, I use my NielsenIQ sample and exclude UPCs classified under "Health & Beauty Care", "Non Food Grocery", and "General Merchandise" so that the sample is comparable to food manufacturing.

The quantification with product-level data leverages Proposition 5. Specifically, by targeting the profile of *product* exit by age, I recover $\theta\eta - \alpha\beta$ and $\alpha\nu$. Identification follows from the former raising product exit at all ages and the latter raising it for young products but lowering it for older ones (due to selection). In contrast, the quantification with firm-level data uses the same GMM strategy as above, and identifies the rate of incumbent innovation x in addition to $\theta\eta - \alpha\beta$ and $\alpha\nu$.

Table 12 presents the results from this exercise. Focusing on the spillover wedge—the statistic of interest—the estimate obtained using product-level data is 16.3%, while that obtained from firm-level data is 15.9%. This alignment lends credibility to the headline finding, suggesting my methodology successfully recovers the gradual component of product exit from firm data.

Extent of innovation underprovision. An alternative approach to assessing how large knowledge spillovers are is to quantify the resulting extent of innovation underprovision in the laissez-faire equilibrium. Specifically, I define:

$$\text{Innovation underprovision } = \frac{S_t^{\mathbf{FB}} + (1-\delta) I_t^{\mathbf{FB}} M_t^{\mathbf{FB}}}{S_t^{\mathbf{DE}} + (1-\delta) I_t^{\mathbf{DE}} M_t^{\mathbf{DE}}} .$$

[12] While I do link NielsenIQ to the GS1 database and can hence observe which products are produced by the same firm, firm age is censored for more than 80% of firms, preventing any meaningful estimation with firm-level data.

Table 12: Results with Product vs Firm Level Data for Food Manufacturing

		Product Data	Firm Data
Product exit due to drift	$\theta\eta - \alpha\beta$	0.170	0.160
		[0.167, 0.173]	
Relative volatility	$\alpha\nu$	0.362	0.484
		[0.342, 0.378]	

Notes: $\theta\eta - \alpha\beta$ is product exit due to downward drift, $\alpha\nu$ governs extent of product exit due to shocks. Product-level results are obtained from an equally weighted GMM targeting the profile of *product* exit at ages 1 through 9, with underlying data from NielsenIQ for sample years 2016, 2017, and 2018. Corresponding 95% confidence intervals obtained using 1000 bootstraps. Firm-level results are obtained from a GMM targeting the profile of *firm* exit at ages 1, 2, 3, 4, 5, 8, 13, and 18 in the food manufacturing sector (NAICS 311), with underlying data from the Business Dynamics Statistics.

In words, this is the level of innovation in the first-best relative to that in the laissez-faire equilibrium. For each of these allocations, the level of innovation reflects innovation by entrants as well as incumbents, with the $(1-\delta)^{-1}$ accounting for the fact that the infra-marginal R&D worker is more productive than a startup entrepreneur (as reflected in Equation 12).

Table 13: Extent of innovation underprovision along stationary laissez-faire equilibrium

σ \ $r-g$	5%	10%	15%	20%
5	2.8	2.1	1.9	1.7
7.5	3.2	2.3	1.9	1.7
10	3.4	2.4	2.0	1.7

Notes: Table reports level of innovation along first best BGP relative to that along laissez-faire BGP. So an entry of 2 means there is twice as much innovation along the first best. The level of innovation used is $S_t + (1-\delta)^{-1} IM_t$, so that it reflects innovation by entrants and incumbents and takes into account that the infra-marginal R&D worker is more productive than a startup entrepreneur.

Unlike the gap between social and private rates of return to innovation, this metric depends on the effective discount rate, $r-g$, and the thinness of the entry distribution $\alpha = 1.06(\sigma - 1)$. In Table 13, I report the extent of underprovision along the BGP for different values of these parameters. I find that the social planner desires between two and three times more innovation than the laissez-faire economy provides along the BGP. This substantial degree of underprovision offers an alternative illustration of how strong the knowledge spillovers that I am backing out seem to be.

6 Conclusion

Knowledge spillovers have long been recognized as a reason the social rate of return to R&D might exceed the private one. Despite this serving as a common rationale for government support for R&D, the existing evidence remains "quite thin" (Bryan and Williams, 2021, p. 290).

This paper attempts to quantify this gap by leveraging data on firm exit by age. To do so, I develop a new semi-endogenous growth model featuring multiproduct firms and negative selection into product exit. Building on Kortum (1997), I model spillovers as the source of endogenous improvement in the entry distribution from which the quality of new products is drawn. Quantifying spillovers then amounts to measuring the extent to which this entry distribution is improving over time. When spillovers are stronger, turnover (or churn) is higher, which translates into higher product entry and exit rates.

In the model, the strength of spillovers is not the only force affecting the extent of turnover at the product level. As in Hopenhayn (1992), high product entry and exit rates could reflect high volatility of idiosyncratic product-level shocks. However, targeting the profile of exit by age allows disentangling these separate drivers of exit. This is a consequence of selection: while higher volatility increases the exit rate, it decreases it among products that survive to be old, as these are even more positively selected when volatility is higher.

Given that product level data are only available for a handful of sectors, the model explicitly features innovating multiproduct firms. In this regard, a key feature of the equilibrium is that firms succeed in adding a product to their portfolio at a rate that scales linearly with their number of products, as in Klette and Kortum (2004). Empirically, this corresponds to the ratio of new to current products being independent of the firm's product count, which is corroborated by evidence I document (Figure 3) as well as the findings of Berlingieri, De Ridder, Lashkari and Rigo (2025). As a result, I prove that the model's profile of firm exit by age is governed by three sufficient statistics: this constant product addition rate, a metric of the relative volatility of idiosyncratic shocks (which governs the extent of product exit due to shocks), and the extent of product exit due to the endogenous gradual obsolescence resulting from consumers reallocating toward the newer and higher quality products. It is the last of these statistics that is tightly linked to the gap between social and private rates of return to R&D created by knowledge spillovers.

In a quantitative application to the universe of U.S. private nonfarm employer businesses, I find that knowledge spillovers create a 16 percentage point wedge between the social and private rates of return to R&D. The importance of the finding is twofold. First, it provides a crucial estimate for a wedge that is a common rationale for government support for innovation, but for which there is limited quantitative evidence (Bryan and Williams, 2021). Second, this estimate is obtained from data on firm exit by age with far broader coverage than traditional innovation proxies like patents or reported R&D. By finding sizable spillovers even in this more comprehensive sample of firms and using a very different methodology, my results strengthen the limited existing evidence that spillovers are sizable and that there are therefore potentially large welfare benefits from government policies that support innovation.

While my analysis attempts to quantify the magnitude of knowledge spillovers, it leaves open an important set of questions that matter for the policy implications of my results. Are these spillovers primarily driven by basic research conducted in universities or by applied research within firms? Which sectors are the source of these spillovers? How important are the different mechanisms through which they occur—such as labor mobility or supply linkages (Lim, 2009)? Given the evidence on large spillovers from publicly funded R&D (Azoulay, Graff Zivin, Li and Sampat, 2018; Myers and Lanahan, 2022; Fieldhouse and Mertens, 2025), a natural question is how much of the wedge I estimate is already internalized by existing innovation policies, including grants, subsidies, and tax credits? I leave these exciting questions to future research.

References

Acemoglu, Daron, *Introduction to Modern Economic Growth*, Princeton University Press, 2009. Hardcover.

_ and Ufuk Akcigit, "Intellectual Property Rights Policy, Competition and Innovation," *Journal of the European Economic Association*, 02 2012, *10* (1), 1–42.

_ , _ , and William R. Kerr, "Innovation network," *Proceedings of the National Academy of Sciences*, 2016, *113* (41), 11483–11488.

_ , _ , Harun Alp, Nicholas Bloom, and William Kerr, "Innovation, Reallocation, and Growth," *American Economic Review*, November 2018, *108* (11), 3450–91.

Adams, James D. and Adam B. Jaffe, "Bounding the Effects of RD: An Investigation Using Matched Establishment-Firm Data," *The RAND Journal of Economics*, 1996, *27* (4), 700–721.

Aghion, Philippe and Peter Howitt, "A Model of Growth Through Creative Destruction," *Econometrica*, 1992, *60* (2), 323–351.

_ , Antonin Bergeaud, Timo Boppart, Peter J. Klenow, and Huiyu Li, "Missing Growth from Creative Destruction," *American Economic Review*, August 2019, *109* (8), 2795–2822.

_ , Christopher Harris, and John Vickers, "Competition and growth with step-by-step innovation: An example," *European Economic Review*, 1997, *41* (3), 771–782. Paper and Proceedings of the Eleventh Annual Congress of the European Economic Association.

_ , _ , Peter Howitt, and John Vickers, "Competition, Imitation and Growth with Step-by-Step Innovation," *The Review of Economic Studies*, 07 2001, *68* (3), 467–492.

— , Nick Bloom, Richard Blundell, Rachel Griffith, and Peter Howitt, "Competition and Innovation: an Inverted-U Relationship," *The Quarterly Journal of Economics*, 05 2005, *120* (2), 701–728.

— , Ufuk Akcigit, and Peter Howitt, "Chapter 1 - What Do We Learn From Schumpeterian Growth Theory?," in Philippe Aghion and Steven N. Durlauf, eds., *Handbook of Economic Growth*, Vol. 2 of *Handbook of Economic Growth*, Elsevier, 2014, pp. 515–563.

Agrawal, Ajay and Rebecca Henderson, "Putting Patents in Context: Exploring Knowledge Transfer from MIT," *Management Science*, 2002, *48* (1), 44–60.

Akcigit, Ufuk and Sina T. Ates, "Ten Facts on Declining Business Dynamism and Lessons from Endogenous Growth Theory," *American Economic Journal: Macroeconomics*, January 2021, *13* (1), 257–98.

— and — , "What Happened to US Business Dynamism?," *Journal of Political Economy*, 2023, *131* (8), 2059–2124.

— and William R. Kerr, "Growth through Heterogeneous Innovations," *Journal of Political Economy*, 2018, *126* (4), 1374–1443.

— , Douglas Hanley, and Stefanie Stantcheva, "Optimal Taxation and R&D Policies," *Econometrica*, 2022, *90* (2), 645–684.

Alcácer, Juan, Michelle Gittelman, and Bhaven Sampat, "Applicant and examiner citations in U.S. patents: An overview and analysis," *Research Policy*, 2009, *38* (2), 415–427.

Alvarez, Fernando, Katarína Borovičková, and Robert Shimer, "Decomposing Duration Dependence in a Stopping Time Model," *The Review of Economic Studies*, 12 2023, *91* (6), 3151–3189.

Andrews, Isaiah, Matthew Gentzkow, and Jesse M. Shapiro, "Measuring the Sensitivity of Parameter Estimates to Estimation Moments*," *The Quarterly Journal of Economics*, 06 2017, *132* (4), 1553–1592.

Argente, David, Munseob Lee, and Sara Moreira, "Innovation and product reallocation in the great recession," *Journal of Monetary Economics*, 2018, *93*, 1–20. Carnegie-Rochester-NYU Conference on Public Policy held at the Stern School of Business at New York University.

— , — , and — , "The Life Cycle of Products: Evidence and Implications," *Journal of Political Economy*, 2024, *132* (2), 337–390.

— , Salomé Baslandze, Douglas Hanley, and Sara Moreira, "Patents to Products: Product Innovation and Firm Dynamics," March 2023. Working paper.

Arkolakis, Costas, "A Unified Theory of Firm Selection and Growth *," *The Quarterly Journal of Economics*, 02 2016, *131* (1), 89–155.

— , Sharat Ganapati, and Marc-Andreas Muendler, "The Extensive Margin of Exporting Products: A Firm-Level Analysis," *American Economic Journal: Macroeconomics*, October 2021, *13* (4), 182–245.

Arqué-Castells, Pere and Daniel F. Spulber, "Measuring the Private and Social Returns to R&D: Unintended Spillovers versus Technology Markets," *Journal of Political Economy*, 2022, *130* (7), 1860–1918.

Arrow, Kenneth J., "Economic Welfare and the Allocation of Resources for Invention," in "The Rate and Direction of Inventive Activity: Economic and Social Factors," Princeton University Press, 1962, pp. 609–626.

Atalay, Enghin, Ali Hortaçsu, Nicole Kimmel, and Chad Syverson, "Why Is Manufacturing Productivity Growth So Low?," NBER Working Paper 34264, National Bureau of Economic Research 2025.

Atkeson, Andrew and Ariel Burstein, "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, December 2008, *98* (5), 1998–2031.

— and — , "Aggregate Implications of Innovation Policy," *Journal of Political Economy*, 2019, *127* (6), 2625–2683.

Azoulay, Pierre, Joshua S Graff Zivin, Danielle Li, and Bhaven N Sampat, "Public R&D Investments and Private-sector Patenting: Evidence from NIH Funding Rules," *The Review of Economic Studies*, 06 2018, *86* (1), 117–152.

BEA, "Value Added by Industry," 2025. Accessed: 2025-09-14.

Berlingieri, Giuseppe, Maarten De Ridder, Danial Lashkari, and Davide Rigo, "Creative Destruction through Innovation Bursts," April 2025. Working paper.

Bernard, Andrew B., Stephen J. Redding, and Peter K. Schott, "Multiple-Product Firms and Product Switching," *American Economic Review*, March 2010, *100* (1), 70–97.

Bils, Mark, "Do Higher Prices for New Goods Reflect Quality Growth or Inflation?*," *The Quarterly Journal of Economics*, 05 2009, *124* (2), 637–675.

— and Peter J. Klenow, "Quantifying Quality Growth," *American Economic Review*, September 2001, *91* (4), 1006–1030.

Bloom, Nicholas, Charles I. Jones, John Van Reenen, and Michael Webb, "Are Ideas Getting Harder to Find?," *American Economic Review*, April 2020, *110* (4), 1104–44.

— , John Van Reenen, and Heidi Williams, "A Toolkit of Policies to Promote Innovation," *Journal of Economic Perspectives*, August 2019, *33* (3), 163–84.

__ , Mark Schankerman, and John Van Reenen, “IDENTIFYING TECHNOLOGY SPILLOVERS AND PRODUCT MARKET RIVALRY,” *Econometrica*, 2013, *81* (4), 1347–1393.

BLS, “Total factor productivity by major industries,” 2025. Accessed: 2025-09-14.

Broda, Christian and David E. Weinstein, “Globalization and the Gains From Variety*,” *The Quarterly Journal of Economics*, 05 2006, *121* (2), 541–585.

__ and __ , “Product Creation and Destruction: Evidence and Price Implications,” *American Economic Review*, June 2010, *100* (3), 691–723.

Bryan, Kevin A. and Heidi L. Williams, “Chapter 13 - Innovation: market failures and public policies,” in Kate Ho, Ali Hortaçsu, and Alessandro Lizzeri, eds., *Handbook of Industrial Organization, Volume 5*, Vol. 5 of *Handbook of Industrial Organization*, Elsevier, 2021, pp. 281–388.

__ , Yasin Ozcan, and Bhaven Sampat, “In-text patent citations: A user’s guide,” *Research Policy*, 2020, *49* (4), 103946.

Cahoy, Daniel R., Joel Gehman, and Zhen Lei, “Fracking Patents: The Emergence of Patents as Information-Containment Tools in Shale Drilling,” *Michigan Telecommunications and Technology Law Review*, 2013, *19* (2), 279.

Cattaneo, Matias D., Richard K. Crump, Max H. Farrell, and Yingjie Feng, “On Binscatter,” *American Economic Review*, May 2024, *114* (5), 1488–1514.

Cavenaile, Laurent, Murat Celik, and Xu Tian, “Are Markups Too High? Competition, Strategic Innovation, and Industry Dynamics,” Working Paper, University of Toronto, Department of Economics 2025. Accessed [insert date you accessed it].

Chen, Zhao, Zhikuo Liu, Juan Carlos Suárez Serrato, and Daniel Yi Xu, “Notching RD Investment with Corporate Income Tax Cuts in China,” *American Economic Review*, July 2021, *111* (7), 2065–2100.

Chow, Melissa C., Teresa C. Fort, Christopher Goetz, Nathan Goldschlag, James Lawrence, Elisabeth Ruth Perlman, Martha Stinson, and T. Kirk White, “Redesigning the Longitudinal Business Database,” NBER Working Paper 28839, National Bureau of Economic Research 2021.

Cohen, Wesley M, Richard R Nelson, and John P Walsh, “Protecting Their Intellectual Assets: Appropriability Conditions and Why U.S. Manufacturing Firms Patent (or Not),” Working Paper 7552, National Bureau of Economic Research February 2000.

Cowx, Mary, Rebecca Lester, and Michelle L. Nessa, “The Consequences of Limiting the Tax Deductibility of R&D,” Research Paper 4998845, Stanford University Graduate School of Business July 2024. Available at SSRN: https://ssrn.com/abstract=4998845 or http://dx.doi.org/10.2139/ssrn.4998845.

Cramer, Judd and Alan B. Krueger, "Disruptive Change in the Taxi Business: The Case of Uber," *American Economic Review*, May 2016, *106* (5), 177–82.

Crump, Kenny S. and Charles J. Mode, "A general age-dependent branching process. I," *Journal of Mathematical Analysis and Applications*, December 1968, *24* (3), 494–508.

_ and _ , "A general age-dependent branching process. II," *Journal of Mathematical Analysis and Applications*, January 1969, *25* (1), 8–17.

Decker, Ryan A., John Haltiwanger, Ron S. Jarmin, and Javier Miranda, "Declining Business Dynamism: What We Know and the Way Forward," *American Economic Review*, May 2016, *106* (5), 203–07.

Dhingra, Swati and John Morrow, "Monopolistic Competition and Optimum Product Diversity under Firm Heterogeneity," *Journal of Political Economy*, 2019, *127* (1), 196–232.

Dixit, Avinash K. and Robert S. Pindyck, *Investment under Uncertainty*, Princeton, NJ: Princeton University Press, 1994.

Feenstra, Robert C., "New Product Varieties and the Measurement of International Prices," *The American Economic Review*, 1994, *84* (1), 157–177.

Fieldhouse, Andrew J and Karel Mertens, "The Social Returns to Public RD," Working Paper 33780, National Bureau of Economic Research May 2025.

Foster, Lucia, John Haltiwanger, and Chad Syverson, "Reallocation, Firm Turnover, and Efficiency: Selection on Productivity or Profitability?," *American Economic Review*, March 2008, *98* (1), 394–425.

Garcia-Macia, Daniel, Chang-Tai Hsieh, and Peter J. Klenow, "How Destructive Is Innovation?," *Econometrica*, 2019, *87* (5), 1507–1541.

Golden, John M. and Hannah J. Wiseman, "The Fracking Revolution: Shale Gas as a Case Study in Innovation Policy," *Emory Law Journal*, 2015, *64* (4), 955.

Gortmaker, Jeff, "Open Source Software Policy in Industry Equilibrium," January 2025. Working Paper.

Griliches, Zvi, "Research Costs and Social Returns: Hybrid Corn and Related Innovations," *Journal of Political Economy*, 1958, *66* (5), 419–431.

_ , "Patent Statistics as Economic Indicators: A Survey," Working Paper 3301, National Bureau of Economic Research March 1990.

_ , "The Search for RD Spillovers," *The Scandinavian Journal of Economics*, 1992, *94*, S29–S47.

Grossman, Gene M. and Elhanan Helpman, "Quality Ladders in the Theory of Growth," *The Review of Economic Studies*, 01 1991, *58* (1), 43–61.

Heckman, James J, Lance Lochner, and Christopher Taber, "Explaining Rising Wage Inequality: Explorations with a Dynamic General Equilibrium Model of Labor Earnings with Heterogeneous Agents," *Review of Economic Dynamics*, 1998, *1* (1), 1–58.

Hopenhayn, Hugo A., "Entry, Exit, and firm Dynamics in Long Run Equilibrium," *Econometrica*, 1992, *60* (5), 1127–1150.

Hopenhayn, Hugo, Julian Neira, and Rish Singhania, "From Population Growth to Firm Demographics: Implications for Concentration, Entrepreneurship and the Labor Share," *Econometrica*, 2022, *90* (4), 1879–1914.

Hottman, Colin J., Stephen J. Redding, and David E. Weinstein, "Quantifying the Sources of Firm Heterogeneity *," *The Quarterly Journal of Economics*, 03 2016, *131* (3), 1291–1364.

Hubbard, Thomas N., "The Demand for Monitoring Technologies: The Case of Trucking," *The Quarterly Journal of Economics*, 2000, *115* (2), 533–560.

_ , "Information, Decisions, and Productivity: On-Board Computers and Capacity Utilization in Trucking," *American Economic Review*, September 2003, *93* (4), 1328–1353.

Irwin, Douglas A. and Peter J. Klenow, "Learning-by-Doing Spillovers in the Semiconductor Industry," *Journal of Political Economy*, 1994, *102* (6), 1200–1227.

Jaffe, Adam B., "Technological Opportunity and Spillovers of R D: Evidence from Firms' Patents, Profits, and Market Value," *The American Economic Review*, 1986, *76* (5), 984–1001.

_ , Manuel Trajtenberg, and Michael S. Fogarty, "Knowledge Spillovers and Patent Citations: Evidence from a Survey of Inventors," *The American Economic Review*, 2000, *90* (2), 215–218.

_ , _ , and Rebecca Henderson, "Geographic Localization of Knowledge Spillovers as Evidenced by Patent Citations," *The Quarterly Journal of Economics*, 1993, *108* (3), 577–598.

Jagers, Peter, *Branching Processes with Biological Applications* Wiley Series in Probability and Mathematical Statistics, London; New York; Sydney; Toronto: John Wiley & Sons, 1975.

Jarmin, Ron S. and Javier Miranda, "The Longitudinal Business Database," CES Working Paper CES-WP-02-17, U.S. Census Bureau, Center for Economic Studies 2002.

Jones, Benjamin F. and Lawrence H. Summers, "A Calculation of the Social Returns to Innovation," in Austan Goolsbee and Benjamin F. Jones, eds., *Innovation and Public Policy*, Chicago: University of Chicago Press, 2021, pp. 13–60.

Jones, Charles I., “R&D-Based Models of Economic Growth,” *Journal of Political Economy*, 1995, *103* (4), 759–784.

— , “Time Series Tests of Endogenous Growth Models,” *The Quarterly Journal of Economics*, 1995, *110* (2), 495–525.

— , “The Past and Future of Economic Growth: A Semi-Endogenous Perspective,” *Annual Review of Economics*, 2022, *14* (Volume 14, 2022), 125–152.

— and John C. Williams, “Measuring the Social Return to R&D*,” *The Quarterly Journal of Economics*, 11 1998, *113* (4), 1119–1135.

Karahan, Fatih, Benjamin Pugsley, and Ayşegül Şahin, “Demographic Origins of the Start-up Deficit,” *American Economic Review*, July 2024, *114* (7), 1986–2023.

Klenow, Peter J. and Huiyu Li, “Entry Costs Rise with Growth,” *Journal of Political Economy Macroeconomics*, 2025, *3* (1), 43–74.

Klette, Tor Jakob and Samuel Kortum, “Innovating Firms and Aggregate Innovation,” *Journal of Political Economy*, 2004, *112* (5), 986–1018.

Kortum, Samuel S., “Research, Patenting, and Technological Change,” *Econometrica*, 1997, *65* (6), 1389–1419.

König, Michael D., Xiaodong Liu, and Yves Zenou, “R&D Networks: Theory, Empirics, and Policy Implications,” *The Review of Economics and Statistics*, 07 2019, *101* (3), 476–491.

Lagakos, David, Benjamin Moll, Tommaso Porzio, Nancy Qian, and Todd Schoellman, “Life Cycle Wage Growth across Countries,” *Journal of Political Economy*, 2018, *126* (2), 797–849.

Levin, Richard C., Alvin K. Klevorick, Richard Nelson, and Sidney Winter, “Appropriating the Returns from Industrial Research and Development,” *Brookings Papers on Economic Activity*, 1987, *18* (3, Special Issue on Microeconomics), 783–832.

Lim, Kwanghui, “The many faces of absorptive capacity: spillovers of copper interconnect technology for semiconductor chips,” *Industrial and Corporate Change*, 11 2009, *18* (6), 1249–1284.

Linn, Joshua and Kristen McCormack, “The roles of energy markets and environmental regulation in reducing coal-fired plant profits and electricity sector emissions,” *The RAND Journal of Economics*, 2019, *50* (4), 733–767.

Liu, Ernest and Song Ma, “Innovation Networks and RD Allocation,” Working Paper 29607, National Bureau of Economic Research December 2021.

Lucking, Brian, Nicholas Bloom, and John Van Reenen, "Have R&D Spillovers Declined in the 21st Century?," *Fiscal Studies*, 2019, *40* (4), 561–590.

Luttmer, Erzo G. J., "Selection, Growth, and the Size Distribution of Firms," *The Quarterly Journal of Economics*, 08 2007, *122* (3), 1103–1144.

_ , "On the Mechanics of Firm Growth," *The Review of Economic Studies*, 02 2011, *78* (3), 1042–1068.

Lychagin, Sergey, Joris Pinkse, Margaret E. Slade, and John Van Reenen, "Spillovers in Space: Does Geography Matter?," *The Journal of Industrial Economics*, 2016, *64* (2), 295–335.

Melitz, Marc J., "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 2003, *71* (6), 1695–1725.

Mezzanotti, Filippo and Timothy Simcoe, "Innovation and Appropriability: Revisiting the Role of Intellectual Property," Working Paper 31428, National Bureau of Economic Research July 2023.

Myers, Kyle R. and Lauren Lanahan, "Estimating Spillovers from Publicly Funded RD: Evidence from the US Department of Energy," *American Economic Review*, July 2022, *112* (7), 2393–2423.

National Center for Science and Engineering Statistics (NCSES), "Business Enterprise Research and Development: 2020," NSF 23-314. Alexandria, VA: National Science Foundation 2023. Available at https://ncses.nsf.gov/pubs/nsf23314/.

Nuño, Galo and Benjamin Moll, "Social optima in economies with heterogeneous agents," *Review of Economic Dynamics*, 2018, *28*, 150–180.

O'Connor, A. C., M. P. Gallaher, K. Clark-Sutton, D. Lapidus, Z. T. Oliver, T. J. Scott, D. W. Wood, M. A. Gonzalez, E. G. Brown, and J. Fletcher, "Economic Benefits of the Global Positioning System (GPS)," Technical Report RTI Report Number 0215471, RTI International, Research Triangle Park, NC June 2019. Sponsored by the National Institute of Standards and Technology.

Peters, Michael, "Heterogeneous Markups, Growth, and Endogenous Misallocation," *Econometrica*, 2020, *88* (5), 2037–2073.

_ , "Market Size and Spatial Growth—Evidence From Germany's Post-War Population Expulsions," *Econometrica*, 2022, *90* (5), 2357–2396.

Redner, Sidney, *A Guide to First-Passage Processes*, Cambridge University Press, 2001.

Sterk, Vincent, Petr Sedláček, and Benjamin Pugsley, "The Nature of Firm Growth," *American Economic Review*, February 2021, *111* (2), 547–79.

Stokey, Nancy L., *The Economics of Inaction: Stochastic Control Models with Fixed Costs*, Princeton, NJ: Princeton University Press, 2009.

Sutton, John, “The variance of firm growth rates: the ‘scaling’ puzzle,” *Physica A: Statistical Mechanics and its Applications*, 2002, *312* (3), 577–590.

Zacchia, Paolo, “Knowledge Spillovers through Networks of Scientists,” *The Review of Economic Studies*, 05 2019, *87* (4), 1989–2018.

APPENDIX

A Summary of Notation and Symbols

Table A1: Summary of notation and symbols

η	Population growth rate
ρ	Rate of time preference
γ	Coefficient of relative risk aversion
σ	Elasticity of substitution
β	Drift in incumbent's log quality
ν	Diffusion in incumbent's log quality
θ	Spillover elasticity
α	Thinness of entry distribution
ε	Inverse of firm entry cost
$\mathcal{F}$	Overhead cost
δ	Per-product diminishing returns in incumbent innovation
ϑ	Scale parameter in incumbent innovation
A	Process efficiency
ζ	Luttmer tail index (defined in Proposition 2)
ξ	Parameter in HJB solution (defined in Proposition 3)
I	Equilibrium level of R&D labor per incumbent product (defined in Equation 11)
x	Product addition rate per existing product along BGP (defined in Equation 16)
$\underline{L}$	Production employment at exit threshold (defined in Table C3)
$\theta\eta - \alpha\beta$	Product Exit due to downward drift
$\alpha\nu$	Relative volatility of shocks

Table A2: Summary of notation and symbols (continued)

N_t	Population	K_t	Stock of products created
Q_{pt}	Product's quality	$\underline{Q}_t$	Endogenous exit threshold
q_{pt}	Product's log quality relative to $\underline{Q}_t$	$m(q,t)$	Cross-sectional measure of q_{pt}
r_t	Interest rate	w_t	Wage
P_{pt}	Price of p	$V_t(Q_{pt})$	Product's value, with $V(q,t) = V_t\left(\underline{Q}_t e^q\right)$
c_t	Per capita consumption	a_t	Individual's asset holding
c_{pt}	Per capita consumption of p	Y_{pt}	Aggregate supply of p
L_{pt}	p's production labor	I_{pt}	p's R&D labor (= I in equilibrium)
S_t	Startup entrepreneurs (entry labor)	$\overline{F}_t^E(.)$	CCDF of entry distribution
g_Q	Quality growth rate	g	Consumption per capita growth rate
$\mathcal{V}(q)$	Stationary product value	Ow_t	Option value of product addition
L_t	Aggregate production labor	Ω_t	Set of products supplied at t
M_t	Measure of products	$\overline{Q}_t$	Average quality supplied
$f_p(q)$	Stationary PDF of q_{pt}	E_t^f	Flow of entering firms
E_t/M_t	Product entry rate	D_t/M_t	Product exit rate
$\ell(a)$	PDF of product's lifespan	$d_p(a)$	Product's exit hazard at age a
$\Gamma(a)$	CDF of firm's lifespan	$d_f(a)$	Firm's exit hazard at age a
$\mu_{nt}(\mathbf{q})$	Measure of $n-$prod firms with portfolio $\mathbf{q}$	$f_n(\mathbf{q})$	$= \prod_{i=1}^{n} \varphi_n(q_i)$ PDF among $n-$prod firms
Ψ_n	Share of products held by $n-$prod firms	Φ_n	Share of firms with n products
λ_n	Mean prod exit rate for $n-$prod firm		

B Proofs and derivations of results in main text

B.1 Product characteristics along balanced growth path

B.1.1 Proof of Proposition 2

Along the BGP, $m(q,t) = M_t f_p(q)$ and M_t grows at rate η, so:

$$\frac{\partial m(q,t)}{\partial t} = \eta\, M_t\, f_p(q) \quad ; \quad \frac{\partial m(q,t)}{\partial q} = M_t\, f_p'(q) \quad ; \quad \frac{\partial^2 m(q,t)}{\partial q^2} = M_t\, f_p''(q)$$

Plugging back into the KFE (Equation 9) yields a second order ODE in $f_p(q)$:

$$\frac{\nu^2}{2}\, f_p''(q) \;+\; \big(g_Q - \beta\big)\, f_p'(q) \;-\; \eta\, f_p(q) \;=\; -\frac{E_t}{M_t}\, \alpha\, e^{-\alpha q}\,.$$

The stationary distribution solves this ODE subject to:

$$\int_0^\infty f_p(q)\, dq \;=\; 1 \quad ; \quad f_p(q) \geq 0 \quad ; \quad f_p(0) \;=\; 0$$

The first requirement leads to a zero coefficient on the positive homogeneous root, so:

$$f_p(q) = C_2 e^{-\zeta q} + C_3 e^{-\alpha q} \quad \text{with} \;\; C_3 = \frac{-\alpha \frac{E_t}{M_t}}{\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta}$$

and ζ as defined in Proposition 2. The boundary condition $f_p(0) = 0$ yields $C_2 = -C_3$. Hence

$$1 = \int_0^\infty f_p(q)dq = \int_0^\infty C_3\left(e^{-\alpha q} - e^{-\zeta q}\right)dq \implies C_3 = \frac{\alpha\zeta}{\zeta - \alpha} \implies f_p(q) \;=\; \frac{\alpha\,\zeta}{\zeta - \alpha}\left(e^{-\alpha q} - e^{-\zeta q}\right)$$

Equating the two expressions for C_3 and simplifying yields an expression for the entry rate:

$$\begin{aligned}
\frac{\alpha\zeta}{\zeta - \alpha} \;=\; \frac{-\frac{E_t}{M_t}\,\alpha}{\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta} \quad &\implies \frac{E_t}{M_t} \;=\; \frac{\zeta}{\alpha - \zeta}\left(\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta\right) \\
&\overset{\star}{\implies} \frac{E_t}{M_t} \;=\; \frac{\zeta}{\alpha - \zeta}\left(\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \left(\frac{\nu^2}{2}\zeta^2 - (g_Q - \beta)\zeta\right)\right) \\
&\implies \frac{E_t}{M_t} \;=\; \frac{\nu^2}{2}\zeta\,\alpha \;+\; \frac{\nu^2}{2}\zeta^2 - (g_Q - \beta)\zeta \\
&\overset{\star}{\implies} \boxed{\frac{E_t}{M_t} \;=\; \frac{\nu^2}{2}\zeta\,\alpha \;+\; \eta}
\end{aligned}$$

where $\star$ follows from $-\zeta$ being a root of the characteristic polynomial. Notice the intuitive expression I got for the product entry rate:

$$\frac{E_t}{M_t} = \eta + \frac{\nu^2}{2} f'(0) .$$

Since the measure of products grows at rate η, the product entry rate exceeds the product exit rate by η. The second summand is precisely the exit rate, as it gives the instantaneous rate at which, in the stationary distribution, q – which evolves according to the SDE in Equation 8 – hits the absorbing boundary condition at 0.

B.1.2 Proof of Proposition 3

Plugging the stationary distribution into the definition of $\overline{Q}_t$ yields

$$\overline{Q}_t \equiv \left(\frac{1}{M_t}\int_{p\in\Omega_t} Q_{pt}^{\sigma-1}dp\right)^{\frac{1}{\sigma-1}} = \underline{Q}_t\left(\int_0^\infty e^{(\sigma-1)q} f_p(q)dq\right)^{\frac{1}{\sigma-1}}$$

Assumption 2 guarantees that $\alpha > \sigma - 1$ and $\zeta > \sigma - 1$, so that:

$$\left(\frac{\overline{Q}_t}{\underline{Q}_t}\right)^{\sigma-1} = \frac{\alpha}{\alpha-(\sigma-1)}\,\frac{\zeta}{\zeta-(\sigma-1)} .$$

Plugging back into the HJB yields

$$\begin{aligned} r_t V(q,t) = w_t &\left[\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{(\sigma-1)\alpha\zeta}\frac{L_t}{M_t}e^{(\sigma-1)q} - (\mathcal{F}-O)\right] \\ &+ \dot{V}(q,t) + \left(\beta - g_{Q_t}\right)\frac{\partial V(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V(q,t)}{\partial q^2} . \end{aligned}$$

Guess that $V(q,t) = w_t\mathcal{V}(q)$, then $\mathcal{V}(q)$ solves the second order ODE with constant coefficients:[13]

$$\frac{\nu^2}{2}\mathcal{V}''(q) + (\beta - g_Q)\mathcal{V}'(q) - (r-g)\mathcal{V}(q) = -\left[\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)}\,\frac{L_t}{M_t}\,e^{(\sigma-1)q} - (\mathcal{F}-O)\right]$$

subject to:

$$\mathcal{V}(q) \geq 0 \;;\; \mathcal{V}(0) = 0 \;;\; \mathcal{V}'(0) = 0 \text{ and } \mathcal{V}(q) < \frac{\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)}\,\frac{L_t}{M_t}\,e^{(\sigma-1)q}}{r - \left[g + (\sigma-1)(\beta-g_Q) + \frac{\nu^2}{2}(\sigma-1)^2\right]} .$$

[13] L_t and M_t both grow at rate η, so their ratio is constant.

The first constraint follows from the fact that the firm can always choose to shutdown production. The second and third constraints are respectively the value matching and smooth pasting conditions. To understand the fourth condition, note that the flow of dividends, gross of overhead and option value, is the numerator of the right hand side scaled by w_t. Using Ito's lemma, this flow grows at rate $g + (\sigma - 1)(\beta - g_Q) \; + \; \frac{\nu^2}{2}(\sigma - 1)^2$. So the right hand side of the fourth constraint is the PDV of flow of dividends gross of the overhead and option value. The inequality then follows from $\mathcal{F} > O$. The solution to this ODE is:

$$\mathcal{V}(q) \;=\; C_1\, e^{zq} \;+\; C_2\, e^{-\xi q} \;+\; \frac{\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)}}{r - \left[g + (\sigma-1)(\beta - g_Q) \;+\; \frac{\nu^2}{2}(\sigma-1)^2\right]}\,\frac{L_t}{M_t}\, e^{(\sigma-1)q} \;-\; \frac{\mathcal{F} - O}{r - g}$$

$$\text{with} \quad z \equiv \frac{-(\beta - g_Q) + \sqrt{(g_Q - \beta)^2 + 2\nu^2(r-g)}}{\nu^2} \;\text{ and }\; \xi \equiv \frac{\beta - g_Q + \sqrt{(\beta - g_Q)^2 + 2\nu^2(r-g)}}{\nu^2}$$

Since $z > 0$ (while $-\xi < 0$), satisfying the inequality constraints on $\mathcal{V}(q)$ requires $C_1 = 0$. The value matching and smooth pasting conditions are two equations in two unknowns, $\frac{L_t}{M_t}$ and C_2. Solving yields:

$$\mathcal{V}(q) = \frac{\mathcal{F} - O}{r - g}\left[\frac{\xi}{\xi + \sigma - 1}e^{(\sigma-1)q} + \frac{\sigma - 1}{\xi + \sigma - 1}e^{-\xi q} - 1\right]$$

$$\frac{L_t}{M_t} = (\mathcal{F} - O)\,\frac{\alpha\zeta(\sigma-1)}{(\alpha - (\sigma-1))(\zeta - (\sigma-1))}\frac{\xi}{\xi + \sigma - 1}\frac{r - [g + (\sigma-1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma-1)^2]}{r - g}\,.$$

B.1.3 Labor allocations along balanced growth path

The expression for $\frac{L_t}{M_t}$ was obtained above when solving the HJB. The remaining equation follows from the free entry condition along the BGP. Using a change of variables, relative quality of new products is drawn from the density

$$\frac{K_t^\theta}{\underline{Q}_t^\alpha}\alpha e^{-\alpha q} \quad \text{for} \quad q \;\geq\; \ln\left(\frac{K_t^\theta}{\underline{Q}_t^\alpha}\right)$$

where this pdf is time-invariant along the BGP since $\underline{Q}_t^\alpha$ and K_t^θ both grow at rate $\theta\eta$. Using Assumption 3, the free entry condition reads

$$\varepsilon\int_0^\infty w_t\mathcal{V}(q)\frac{K_t^\theta}{\underline{Q}_t^\alpha}\alpha e^{-\alpha q}\, dq \;=\; w_t$$

The integration starts at 0 because for $q \leq 0$, $\mathcal{V}(q) = 0$. Plugging in $\mathcal{V}$ yields:

$$\frac{\underline{Q}_t^\alpha}{K_t^\theta} \;=\; \varepsilon \, \frac{\mathcal{F} - O}{r - g} \, \frac{(\sigma - 1)\xi}{(\alpha + \xi)(\alpha - (\sigma - 1))} \, .$$

So Assumption 3 places a lower bound on the PDV of the effective fixed cost of operation relative to the entry cost:

$$\frac{\frac{\mathcal{F} - O}{r - g}}{\frac{1}{\varepsilon}} \;\geq\; \left(1 + \frac{\alpha}{\xi}\right)\left(\frac{\alpha}{\sigma - 1} - 1\right) \, . \tag{24}$$

To see how the expression for $\frac{\underline{Q}_t^\alpha}{K_t^\theta}$ helps us pin down the labor allocations, note that

$$\eta K_t = \dot{K}_t \;=\; \varepsilon S_t \;+\; \frac{\vartheta}{1 - \delta} I^{1-\delta} M_t \;\Longrightarrow\; \frac{S_t}{M_t} \;=\; \frac{\eta}{\varepsilon} \frac{K_t}{M_t} \;-\; \frac{I}{1 - \delta}$$

where I used the fact that $I = \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$. Now,

$$E_t = \overline{F}_t^E \left(\underline{Q}_t\right) \dot{K}_t = \frac{K_t^\theta}{\underline{Q}_t^\alpha} \eta K_t \;\Longrightarrow\; \frac{K_t}{M_t} = \frac{1}{\eta} \frac{E_t}{M_t} \frac{\underline{Q}_t^\alpha}{K_t^\theta}$$

Plugging the expression for $\frac{K_t}{M_t}$ back into $\frac{S_t}{M_t}$, I get:

$$\frac{S_t}{M_t} \;=\; \frac{1}{\varepsilon} \, \frac{E_t}{M_t} \, \frac{\underline{Q}_t^\alpha}{K_t^\theta} \;-\; \frac{I}{1 - \delta} \, .$$

Plugging in the expressions for the entry rate and $\frac{\underline{Q}_t^\alpha}{K_t^\theta}$ yields:

$$\frac{S_t}{M_t} \;=\; \left(\eta \;+\; \frac{\nu^2}{2}\zeta\alpha\right) \frac{\mathcal{F} - O}{r - g} \, \frac{(\sigma - 1)\xi}{(\alpha + \xi)(\alpha - (\sigma - 1))} \;-\; \frac{I}{1 - \delta}$$

Adding $\mathcal{F} + I$ on both sides and using $O = \frac{\delta}{1-\delta} I$ yields:

$$\frac{S_t}{M_t} \;+\; I \;+\; \mathcal{F} \;=\; (\mathcal{F} - O) \left[\frac{\eta \;+\; \frac{\nu^2}{2}\zeta\alpha}{r - g} \, \frac{(\sigma - 1)\xi}{(\alpha + \xi)(\alpha - (\sigma - 1))} \;+\; 1\right] .$$

I now have all I need to solve for $\frac{M_t}{N_t}$ using the labor resource constraint.

B.2 Planner's problem

Define $z_{pt} = \ln Q_{pt}$ and denote by $\mu(z,t)$ the measure of products with log-quality z at t. The entry distribution in terms of z then has density

$$\widetilde{f}_t^E(z) = \alpha K_t^\theta e^{-\alpha z} \text{ for } z \geq \frac{\theta}{\alpha} \ln K_t$$

By symmetry, the planner picks $I_{pt} = I_t$ and strict positivity follows from the Inada condition at 0. Moreover, once total production L_t is chosen, standard CES allocations yield

$$L_t = \int_{z>\underline{z}_t} L_t(z)\mu(z,t)dz \implies c_t = \frac{AL_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}.$$

So the planner's problem features an infinite dimensional state variable (Nuño and Moll, 2018):

$$\max_{\substack{S_t,L_t,I_t\\ d_t}} \int_0^\infty e^{-(\rho-\eta)t}\,\frac{\left(A\frac{L_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}\right)^{1-\gamma}-1}{1-\gamma}\,dt$$

subject to $\mu\left(\underline{z}_t, t\right) = 0$

$$\forall z > \underline{z}_t, \quad \dot{\mu}(z,t) = -\beta\,\frac{\partial\mu(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\mu(z,t)}{\partial z^2} + \dot{K}_t\,K_t^\theta \alpha e^{-\alpha z}\mathbb{1}_{\{z>\frac{\theta}{\alpha}\ln K_t\}}$$

$$\dot{\underline{z}}_t = d_t \geq 0$$

$$\dot{K}_t = \varepsilon S_t + \frac{\vartheta}{1-\delta} I_t^{1-\delta} \int_{z>\underline{z}_t} \mu(z,t)dz$$

$$N_t = S_t + L_t + \int_{z\geq\underline{z}_t} (\mathcal{F} + I_t)\;\mu(z,t)\,dz$$

This is an optimal control problem with S_t, L_t, I_t, and d_t (how much to lift the threshold) as controls. The states are $\underline{z}_t$ (lowest quality still "alive"), K_t and $\{\mu(z,t)\}$. Let ω_t be the Lagrange multiplier on the labor resource constraint, $\Upsilon(z,t)$ be the costate associated with $\mu(z,t)$, χ_t the costate associated with K_t, and Ξ_t the costate associated with $\underline{z}_t$. Then the current-value Hamiltonian is:

$$\mathcal{H} = \frac{c_t^{1-\gamma}-1}{1-\gamma} + \chi_t\left(\varepsilon S_t + \frac{\vartheta}{1-\delta} I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right) + \int \Upsilon(z,t)\dot{\mu}(z,t)dz$$
$$+\,\omega_t\left(N_t - S_t - L_t - \int_{z\geq\underline{z}_t}(\mathcal{F}+I_t)\;\mu(z,t)\,dz\right) + \Xi_t d_t$$

It is going to be helpful to plug the KFE into $\int \Upsilon(z,t)\dot{\mu}(z,t)dz$ and then integrate by parts to move the derivatives to Υ. To deal with the boundary terms,

Assumption 5. I suppose (and verify later) that:

$$\lim_{z\to\infty} \Upsilon(z,t)\mu(z,t) = 0 \quad \text{and} \quad \lim_{z\to\infty} \Upsilon(z,t)\frac{\partial \mu(z,t)}{\partial z} = 0\ .$$

Using these assumptions, along with $\mu(\underline{z}_t, t) = 0$:

$$\int_{\underline{z}_t}^{\infty} \Upsilon(z,t)\dot{\mu}(z,t)dz = -\frac{\nu^2}{2}\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z}\Upsilon\left(\underline{z}_t,t\right) + \int_{\underline{z}_t}^{\infty} \mu(z,t)\left(\beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2 \Upsilon(z,t)}{\partial z^2}\right)dz$$
$$+ \dot{K}_t K_t^{\theta}\int_{\underline{z}_t}^{\infty} \Upsilon(z,t)\alpha e^{-\alpha z}\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz$$

$$\implies \mathcal{H} = \frac{\left(A\frac{L_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\ \mu(z,t)\ dz\right)^{\frac{1}{\sigma-1}}\right)^{1-\gamma} - 1}{1-\gamma} + \chi_t\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)$$
$$-\frac{\nu^2}{2}\Upsilon\left(\underline{z}_t,t\right)\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z} + \int_{\underline{z}_t}^{\infty}\mu(z,t)\left(\beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2 \Upsilon(z,t)}{\partial z^2}\right)dz$$
$$+\left(\varepsilon S_t\ +\ \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz$$
$$+\omega_t\left(N_t - S_t - L_t - \int_{z\geq\underline{z}_t}(\mathcal{F}+I_t)\ \mu(z,t)\ dz\right) + \Xi_t d_t\ .$$

The optimality conditions for the controls L_t, S_t, I_t and d_t are (respectively):

$$\mathcal{H}_{L_t} = 0 \implies \omega_t = \frac{c_t^{1-\gamma}}{L_t} \tag{25}$$

$$\mathcal{H}_{S_t} = 0 \implies \omega_t = \varepsilon\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz\right) \tag{26}$$

$$\mathcal{H}_{I_t} = 0 \implies \omega_t = \vartheta I_t^{-\delta}\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz\right) \tag{27}$$

$$\Xi_t d_t = 0 \implies d_t = 0\ \text{ or }\ \Xi_t = 0$$

$$\underline{z}_t \text{ adjoint } \implies (\rho-\eta)\Xi_t = \dot{\Xi}_t - \frac{\nu^2}{2}\left(\frac{\partial \Upsilon\left(\underline{z}_t,t\right)}{\partial z}\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z} + \Upsilon\left(\underline{z}_t,t\right)\frac{\partial^2 \mu\left(\underline{z}_t,t\right)}{\partial z^2}\right)$$
$$-\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)K_t^\theta \alpha e^{-\alpha \underline{z}_t}\Upsilon\left(\underline{z}_t,t\right)\mathbb{1}_{\{\underline{z}_t>\frac{\theta}{\alpha}\ln K_t\}}dz \tag{28}$$

$$K_t \text{ adjoint } \implies (\rho-\eta)\chi_t = \dot{\chi}_t + \theta\frac{\dot{K}_t}{K_t}\left(K_t^\theta\int_{\underline{z}_t}^\infty \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\{z>\frac{\theta}{\alpha}\ln K_t\}}dz - \Upsilon\left(\frac{\theta}{\alpha}\ln K_t,t\right)\mathbb{1}_{\{\underline{z}_t\le\frac{\theta}{\alpha}\ln K_t\}}\right) \tag{29}$$

Finally, adjoint corresponding to $\mu(z,t)$ yields:

$$(\rho-\eta)\Upsilon(z,t) = \dot{\Upsilon}(z,t) + \beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2} + \frac{1}{\sigma-1}\frac{c_t^{1-\gamma}}{M_t}\left(\frac{e^z}{\overline{Q}_t}\right)^{\sigma-1} - \omega_t(\mathcal{F}+I_t)$$
$$+\frac{\vartheta}{1-\delta}I_t^{1-\delta}\left(\chi_t + K_t^\theta\int_{\underline{z}_t}^\infty \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{z>\frac{\theta}{\alpha}\ln K_t}dz\right) \tag{30}$$

where I used M_t and $\overline{Q}_t$ as defined in the main text so that

$$\int_{z\ge\underline{z}_t}\mu(z,t)dz = M_t \quad \text{and} \quad \int_{z\ge\underline{z}_t}e^{(\sigma-1)z}\mu(z,t)dz = M_t\overline{Q}_t^{\sigma-1}$$

Finally, the transversality conditions are

$$0 = \lim_{t\to\infty}e^{-(\rho-\eta)t}\Xi_t\underline{z}_t = \lim_{t\to\infty}e^{-(\rho-\eta)t}\chi_t K_t = \lim_{t\to\infty}e^{-(\rho-\eta)t}\Upsilon(z,t)\mu(z,t)$$

$$\text{Combining Equation 26 and Equation 27} \implies I_t = \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} \implies \text{as in equilibrium!}$$

Plugging Equation 27 and Equation 25 into Equation 30 and using $O \equiv \frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$ (as in equilibrium):

$$(\rho-\eta)\Upsilon(z,t) = \dot{\Upsilon}(z,t) + \beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2} + \omega_t\left(\frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{e^z}{\overline{Q}_t}\right)^{\sigma-1} - (\mathcal{F}-O)\right)$$

Change of coordinates

With stationarity in mind, it is going to be helpful to work with relative coordinates: $q = z - \underline{z}_t$. For $q \geq 0$, define the value function $V^{\mathrm{SP}}(q,t)$ and cross-sectional distribution $m(q,t)$ by

$$V^{\mathrm{SP}}(q,t) = \Upsilon(q + \underline{z}_t, t) \quad \text{and} \quad m(q,t) = \mu(q + \underline{z}_t, t)$$

Using the chain rule and $\dot{\underline{z}}_t = d_t$, the planner's HJB becomes:

$$(\rho - \eta)V^{\mathrm{SP}}(q,t) = \dot{V}^{\mathrm{SP}}(q,t) + (\beta - d_t)\frac{\partial V^{\mathrm{SP}}(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V^{\mathrm{SP}}(q,t)}{\partial q^2}$$
$$+ \omega_t \left[\frac{1}{\sigma - 1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{\overline{Q}}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O) \right]$$

The KFE is as in Equation 9. Equation 26, Equation 28, and Equation 29 become:

$$\omega_t = \varepsilon\left(\chi_t + K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}V^{\mathrm{SP}}(q,t)\mathbb{1}_{\left\{q > \ln \frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}\right\}}dq\right), \tag{31}$$

$$(\rho - \eta)\Xi_t = \dot{\Xi}_t - \frac{\nu^2}{2}\left[\frac{\partial V^{\mathrm{SP}}(0,t)}{\partial q}\frac{\partial m(0,t)}{\partial q} + V^{\mathrm{SP}}(0,t)\frac{\partial^2 m(0,t)}{\partial q^2}\right]$$
$$- \left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}M_t\right)\alpha K_t^{\theta}\underline{Q}_t^{-\alpha}V^{\mathrm{SP}}(0,t)\mathbb{1}_{\left\{\frac{\theta}{\alpha}\ln K_t < 0\right\}}, \tag{32}$$

$$(\rho - \eta)\chi_t = \dot{\chi}_t + \theta\frac{\dot{K}_t}{K_t}\left(K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}V^{\mathrm{SP}}(q,t)\mathbb{1}_{\left\{q > \ln \frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}\right\}}dq - V^{\mathrm{SP}}\left(\ln\frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}, t\right)\mathbb{1}_{\left\{\ln \frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t} > 0\right\}}\right) \tag{33}$$

Finally, the transversality conditions become:

$$0 = \lim_{t\to\infty} e^{-(\rho-\eta)t}\Xi_t\underline{z}_t = \lim_{t\to\infty} e^{-(\rho-\eta)t}\chi_t K_t = \lim_{t\to\infty} e^{-(\rho-\eta)t}V^{\mathrm{SP}}(q,t)m(q,t)$$

And Assumption 5 becomes

$$\lim_{q\to\infty} V^{\mathrm{SP}}(q,t)m(q,t) = 0 \quad \text{and} \quad \lim_{q\to\infty} V^{\mathrm{SP}}(q,t)\frac{\partial m(q,t)}{\partial q} = 0\,.$$

Balanced Growth Path

The definition of BGP was given in Definition 2. Proposition 1 still applies. Here again, I focus on the BGP satisfying Assumption 3 – as the below will make clear, Equation 24 is sufficient since:

$$\left(\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}\right)^{\textbf{FB}} \geq \left(\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}\right)^{\textbf{DE}} .$$

Since the KFE is unchanged, $f_p(q)$ is as in equilibrium. For the HJB, guess and verify $V^{\text{SP}}(q,t) = \omega_t \mathcal{V}(q)$. As I will show, this is not an abuse of notation, as $\mathcal{V}(q)$ will coincide with the function defined in Proposition 3. To get there, note that

$$\text{Equation 25} \implies \frac{\dot{\omega}_t}{\omega_t} = (1-\gamma)g - \eta ,$$

so that by plugging back into the HJB, we know $\mathcal{V}^{\text{SP}}(q)$ satisfies the following ODE:

$$(\rho + (\gamma-1)g)\mathcal{V}(q) = (\beta - g_Q)\mathcal{V}'(q) + \frac{\nu^2}{2}\mathcal{V}''(q) + \frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{\overline{Q}}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O)$$

This matches the ODE I solve in Appendix B.1.2 to prove Proposition 3. What remains to be shown is that the boundary conditions are the same. The ones at 0 follow from Equation 32 combined with $\Xi_t = 0$ along the BGP. The latter follows directly from $g_Q > 0$ if $\theta > 0$ and from $\underline{Q}_t > K_t^{\frac{\theta}{\alpha}}$ otherwise. As such, the TVC for Ξ_t is trivially satisfied and:

$$-\frac{\nu^2}{2}\omega_t\left[\mathcal{V}'(0)f_p'(0) + \mathcal{V}(0)f_p''(0)\right] = \eta K_t \alpha K_t^{\theta}\underline{Q}_t^{-\alpha}\mathcal{V}(0)$$

Suppose $\mathcal{V}(0) \neq 0$, then the growth rate of the LHS is $(1-\gamma)g - \eta$ while that of the RHS is η, which is a contradiction. Hence $\mathcal{V}(0) = 0$. Since $f_p''(0) > 0$, it follows that $\mathcal{V}'(0) = 0$. The last boundary condition will follow from Assumption 5, which requires

$$\lim_{q\to\infty}\mathcal{V}(q)f_p(q) = 0 \quad \text{and} \quad \lim_{q\to\infty}\mathcal{V}(q)f_p'(q) = 0$$

In addition to the requirements $\alpha > \sigma - 1$ and $\zeta > \sigma - 1$, these lead to the inequality constraint on $\mathcal{V}(q)$ (see Appendix B.1.2). As a result, we end up with the same ODE and boundary condition so that $\mathcal{V}(q)$ is indeed the same.

Finally, to obtain the planner's analogue of the free entry condition, divide both sides of Equation 31 by ω_t, which yields

$$\frac{1}{\varepsilon} = \frac{\chi_t}{\omega_t} + K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}\mathcal{V}(q)dq \implies \frac{\dot{\chi}_t}{\chi_t} = \frac{\dot{\omega}_t}{\omega_t} = (1-\gamma)g - \eta \tag{34}$$

Plugging this growth rate of $\frac{\dot{\chi}_t}{\chi_t}$ into Equation 33,

$$\rho + (\gamma - 1)g = \frac{\omega_t}{\chi_t}\theta\eta\frac{K_t^\theta}{\underline{Q}_t^\alpha}\int_0^\infty \alpha e^{-\alpha q}\mathcal{V}(q)dq \implies \frac{\chi_t}{\omega_t} = \frac{\theta\eta}{\rho + (\gamma - 1)g}K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \alpha e^{-\alpha q}\mathcal{V}(q)dq$$

$$\overset{\text{Equation 34}}{\implies} \boxed{1 = \varepsilon\left(1 + \frac{\theta\eta}{\rho + (\gamma - 1)g}\right)K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \alpha e^{-\alpha q}\mathcal{V}(q)dq}$$

This is the planner's analogue of the free entry condition, and it is the only condition that differs across the first best and equilibrium. That the level of incumbent innovation per product (I) is efficient might seem surprising at first, as this activity generates positive knowledge spillovers – just as much as the entry of new firms. The way to think about it is as follows. Due to spillovers, there is too little aggregate innovation in equilibrium. However, the incumbent's innovation technology (Equation 2) satisfies an Inada condition at 0. So, both in equilibrium and the first best, initial units of innovation are carried out by incumbents – until diminishing returns push the marginal product from this technology to ε, at which point the rest of innovation to be done is carried with the linear entry technology (new firms). Along an interior BGP, this point is necessarily reached, so that the underprovision of innovation shows up entirely along the entry of new firms margin.

B.3 Derivation of the social rate of return to R&D

Here I derive the social rate of return to R&D as the return on a variational argument around a BGP (Jones and Williams, 1998). For these purposes, note that the economy is simply given by:

$$\begin{aligned}
Y_t &= L_t\underline{Q}_t A_t^{\frac{1}{\sigma-1}} \qquad \text{where } A_t \equiv A\int_0^\infty e^{(\sigma-1)q}m(q,t)dq \\
\dot{m}(q,t) &= \left(g_Q - \beta\right)\frac{\partial m(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 m(q,t)}{\partial q^2} + \varepsilon R_t K_t^\theta \underline{Q}_t^{-\alpha}\alpha e^{-\alpha q} \\
m(0,t) &= 0 \\
\dot{K}_t &= \varepsilon R_t \\
N_t &= L_t + R_t + (\mathcal{F} - O)M_t \qquad \text{where } R_t \equiv S_t + \frac{1}{1-\delta}IM_t\ .
\end{aligned}$$

Denoting by ∇ deviations from the initial balanced growth path, note that from the law of motion:

$$\forall t, \quad \nabla\dot{m}(q,t) = \mathcal{L}\nabla m(q,t) + \varepsilon\alpha e^{-\alpha q}\left(K_t^\theta \underline{Q}_t^{-\alpha}\nabla R_t + \theta K_t^{\theta-1}\underline{Q}_t^{-\alpha}R_t\nabla K_t - K_t^\theta \alpha \underline{Q}_t^{-\alpha-1}R_t\nabla \underline{Q}_t\right)$$

$$\text{where} \quad \mathcal{L} \equiv -(\beta - g_Q)\partial_q + \frac{\nu^2}{2}\partial_{qq}\ .$$

The specific variational argument of interest is:

1. from t to $t+dt$, the economy does more R&D by reducing L_t and raising R_t;
2. from $t+dt$ to $t+2dt$, the economy "eats the proceeds" by doing sufficiently less R&D to be back at initial BGP path by $t+2dt$.

The social rate of return is then defined as the rate of return on this variational argument as $dt \to 0$:

$$\widetilde{r} \equiv \lim_{dt\to 0} \frac{\nabla Y_{t+dt} - \frac{Y_t}{L_t}\nabla R_t}{\frac{Y_t}{L_t}\nabla R_t dt} .$$

Intuitively, this is a rate of return because $\frac{Y_t}{L_t}\nabla R_t$ is the amount of output (and hence consumption) that the variational argument sacrifices at t, while ∇Y_{t+dt} is the resulting increase in output at $t+dt$. Since m is a state variable and the variational argument starts at t, $\nabla m(q,t) = 0$, so

$$m(q,t+dt) = m(q,t) + \dot{m}(q,t)dt \implies \nabla m(q,t+dt) = \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \alpha e^{-\alpha q} \nabla R_t dt$$

In contrast

$$\begin{aligned}
\nabla m(q,t+2dt) &= \nabla m(q,t+dt) + \nabla \dot{m}(q,t+dt)dt \\
&= \nabla m(q,t+dt) + \left(\mathcal{L}\nabla m(q,t+dt) + \varepsilon \underline{Q}_{t+dt}^{-\alpha} \alpha e^{-\alpha q}\left(K_{t+dt}^{\theta}\nabla R_{t+dt} + \theta K_{t+dt}^{\theta-1}\nabla K_{t+dt} R_{t+dt}\right)\right) dt \\
\text{with} \quad & \mathcal{L}\nabla\mu(q,t+dt) = \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta}\alpha e^{-\alpha q}\left((\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2\right)\nabla R_t dt \\
& \dot{K}_{t+dt} = K_t + \varepsilon R_t dt \implies \nabla K_{t+dt} = \varepsilon \nabla R_t dt
\end{aligned}$$

The variational argument requires $\nabla m(q,t+2dt) = 0$. Solving for ∇R_{t+dt} in terms of ∇R_t yields

$$-\nabla R_{t+dt} = \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha}\left(\frac{K_t}{K_{t+dt}}\right)^{\theta}\nabla R_t + \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha}\left(\frac{K_t}{K_{t+dt}}\right)^{\theta}\left((\beta - g_Q)\,\alpha + \frac{\nu^2}{2}\alpha^2\right) + \theta\varepsilon\frac{R_{t+dt}}{K_{t+dt}}\right]\nabla R_t dt .$$

The increase in output at $t+dt$ is due to higher TFP and higher productional labor:

$$Y_{t+dt} = L_{t+dt}\underline{Q}_{t+dt} A_{t+dt}^{\frac{1}{\sigma-1}} \implies \nabla Y_{t+dt} = \frac{Y_{t+dt}}{L_{t+dt}}\nabla L_{t+dt} + \frac{1}{\sigma-1}\frac{Y_{t+dt}}{A_{t+dt}}\nabla A_{t+dt}$$

Now

$$\nabla L_{t+dt} + (\mathcal{F} - O)\nabla M_{t+dt} + \nabla R_{t+dt} = 0 \quad \text{and} \quad \nabla M_{t+dt} = K_t^{\theta}\underline{Q}_t^{-\alpha}\varepsilon\nabla R_t dt$$

$$\implies \nabla L_{t+dt} = \frac{\underline{Q}_t^{-\alpha}K_t^{\theta}}{\underline{Q}_{t+dt}^{-\alpha}K_{t+dt}^{\theta}}\nabla R_t + \left[\frac{\underline{Q}_t^{-\alpha}K_t^{\theta}}{\underline{Q}_{t+dt}^{-\alpha}K_{t+dt}^{\theta}}\left((\beta - g_Q)\,\alpha + \frac{\nu^2}{2}\alpha^2\right) + \theta\varepsilon\frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta}\underline{Q}_t^{-\alpha}\varepsilon(\mathcal{F} - O)\right]\nabla R_t dt$$

$$\implies \nabla Y_{t+dt} = \frac{Y_{t+dt}}{L_{t+dt}}\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha}\left(\frac{K_t}{K_{t+dt}}\right)^{\theta}\nabla R_t + \frac{1}{\sigma-1}\frac{Y_{t+dt}}{A_{t+dt}}\varepsilon\underline{Q}_t^{-\alpha}K_t^{\theta}\frac{\alpha}{\alpha-(\sigma-1)}\nabla R_t dt$$
$$+\frac{Y_{t+dt}}{L_{t+dt}}\left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha}\left(\frac{K_t}{K_{t+dt}}\right)^{\theta}\left((\beta-g_Q)\,\alpha+\frac{\nu^2}{2}\alpha^2\right)+\theta\varepsilon\frac{R_{t+dt}}{K_{t+dt}}-K_t^{\theta}\underline{Q}_t^{-\alpha}\varepsilon(\mathcal{F}-O)\right]\nabla R_t dt$$
$$\implies \frac{\nabla Y_{t+dt}-\frac{Y_t}{L_t}\nabla R_t}{\frac{Y_t}{L_t}\nabla R_t dt} = \frac{1}{dt}\left[\frac{\frac{Y_{t+dt}}{L_{t+dt}}}{\frac{Y_t}{L_t}}\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha}\left(\frac{K_t}{K_{t+dt}}\right)^{\theta}-1\right]+\frac{1}{\sigma-1}\frac{\frac{Y_{t+dt}}{A_{t+dt}}}{\frac{Y_t}{L_t}}\varepsilon\underline{Q}_t^{-\alpha}K_t^{\theta}\frac{\alpha}{\alpha-(\sigma-1)}$$
$$+\frac{\frac{Y_{t+dt}}{L_{t+dt}}}{\frac{Y_t}{L_t}}\left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha}\left(\frac{K_t}{K_{t+dt}}\right)^{\theta}\left((\beta-g_Q)\,\alpha+\frac{\nu^2}{2}\alpha^2\right)+\theta\varepsilon\frac{R_{t+dt}}{K_{t+dt}}-K_t^{\theta}\underline{Q}_t^{-\alpha}\varepsilon(\mathcal{F}-O)\right]$$

Define

$$P_{A_t} \equiv \frac{Y_t}{L_t}\underline{Q}_t^{\alpha}K_t^{-\theta}$$

Then taking limits yields

$$\widetilde{r} = \frac{\dot{P}_{A_t}}{P_{A_t}} + (\beta-g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\varepsilon\frac{R_t}{K_t} - \varepsilon(\mathcal{F}-O)K_t^{\theta}\underline{Q}_t^{-\alpha} + \frac{\frac{1}{\sigma-1}\frac{Y_t}{A_t}}{P_{A_t}}\frac{\alpha}{\alpha-(\sigma-1)}\varepsilon$$

$$\text{with}\quad \frac{\dot{P}_{A_t}}{P_{A_t}} = g\ ;\ \dot{K}_t = \varepsilon R_t\ ;\ \frac{\dot{K}_t}{K_t} = \eta\ ;\ A_t = \int_0^{\infty} e^{(\sigma-1)q}M_t\frac{\alpha\zeta}{\zeta-\alpha}(e^{-\alpha q}-e^{-\zeta q})dq$$
$$\implies \widetilde{r} = g + (\beta-g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta - \varepsilon(\mathcal{F}-O)K_t^{\theta}\underline{Q}_t^{-\alpha} + \frac{L_t}{M_t}\frac{\zeta-(\sigma-1)}{\zeta(\sigma-1)}\varepsilon K_t^{\theta}\underline{Q}_t^{-\alpha}$$

This gives the social rate of return to R&D as a function of the allocations. Along the decentralization equilibrium, these allocations satisfy

$$\frac{\zeta-(\sigma-1)}{\zeta(\sigma-1)}\frac{L_t}{M_t} = (\mathcal{F}-O)\frac{\alpha}{\alpha-(\sigma-1)}\frac{\xi}{\xi+\sigma-1}\frac{r-[g+(\sigma-1)(\beta-g_Q)+\frac{\nu^2}{2}(\sigma-1)^2]}{r-g}$$
$$\varepsilon K_t^{\theta}\underline{Q}_t^{-\alpha} = \frac{r-g}{\mathcal{F}-O}\frac{\alpha+\xi}{\xi}\frac{\alpha-(\sigma-1)}{\sigma-1}$$
$$\implies \widetilde{r}_{\mathbf{DE}} = g + (\beta-g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\left(1+\frac{\alpha}{\xi}\right)\left(1-\frac{\alpha}{\sigma-1}\right)$$
$$+\frac{\alpha(\alpha+\xi)}{(\sigma-1)(\xi+\sigma-1)}\left(r-g-(\sigma-1)(\beta-g_Q)-(\sigma-1)^2\frac{\nu^2}{2}\right)$$

But from the definition of ξ in Proposition 3 : $\quad \frac{\nu^2}{2}\xi^2 - (\beta-g_Q)\xi = r-g$

Hence

$$\begin{aligned}
\widetilde{r}_{\mathbf{DE}} &= g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\frac{\alpha+\xi}{\xi}\left(1 - \frac{\alpha}{\sigma-1}\right) + \frac{\alpha(\alpha+\xi)}{\sigma-1}\left(\frac{\nu^2}{2}(\xi - (\sigma-1)) - (\beta - g_Q)\right) \\
&= g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\frac{\alpha+\xi}{\xi}\left(1 - \frac{\alpha}{\sigma-1}\right) + \frac{\alpha(\alpha+\xi)}{\xi(\sigma-1)}\left(r - g - \frac{\nu^2}{2}\xi(\sigma-1)\right) \\
&= \theta\eta + g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + (\alpha+\xi)\left(\frac{\nu^2}{2}\xi - (\beta - g_Q) - \frac{\nu^2}{2}\alpha\right) \\
&= \theta\eta + g + \frac{\nu^2}{2}\xi^2 - (\beta - g_Q)\xi = \theta\eta + g + r - g \implies \boxed{\widetilde{r}_{\mathbf{DE}} = r + \theta\eta}
\end{aligned}$$

B.4 Proof of Proposition 5 and Proposition 6

The unconditional density $\ell(a)$ is obtained by averaging the conditional density $\ell(a|q)$ over the distribution of initial draws, noting that $\ell(a|q)$ is the density of the first-passage time from above of a drifted Brownian motion (see equation 3.2.13 in Redner (2001)):

$$\ell(a) = \int_0^\infty \alpha e^{-\alpha q}\ell(a|q)dq \quad \text{with} \quad \ell(a|q) = \frac{q}{\nu\sqrt{2\pi a^3}}\exp\left(-\frac{\left(q + (\beta - g_Q)\,a\right)^2}{2\nu^2 a}\right).$$

Turning to the integral equation; for the firm to have exited by age A, its initial product must have died at some age $0 \le a \le A$. With $\mathcal{L}$ the initial product's lifespan, by the law of total probability:

$$\Gamma(A) = \Pr(f\text{'s lifespan} \le A) = \int_0^A \ell(a)\Pr(f\text{'s lifespan} \le A \mid \mathcal{L} = a)\,da\;.$$

If firms were forever single product, the conditional probability in above expression would be 1. With $x > 0$, and conditional on $\mathcal{L} = a$, the initial product may give birth at ages $s \in (0, a)$. By the recursive structure of the branching process, a birth at age s produces a lineage that is obsolete by age A with probability $\Gamma(A - s)$. Since births on $(0, a)$ form a Poisson point process of rate x, splitting this interval into subintervals of length Δ, indexed by i and with midpoints s_i:

$$\begin{aligned}
\Pr(f\text{'s lifespan} \le A \mid \mathcal{L} = a) &= \lim_{\Delta\downarrow 0}\prod_i (x\Delta)\Gamma(A - s_i) + (1 - x\Delta)1 = \lim_{\Delta\downarrow 0}\prod_i \exp\left(x\Delta\left[\Gamma(A - s_i) - 1\right]\right) \\
&= \lim_{\Delta\downarrow 0}\exp\left(\sum_i x\Delta\left[\Gamma(A - s_i) - 1\right]\right) = \exp\left(\int_0^a x\left[\Gamma(A - s) - 1\right]ds\right)\;.
\end{aligned}$$

This leverages that $\{n_f(a)\}_{a\ge 0}$ is a single-type Crump–Mode–Jagers (general age-dependent) branching process; see Crump and Mode (1968, 1969) for foundational theory and Jagers (1975) for applications in population dynamics.

B.5 Proof of Proposition 7

Integrating both sides of the n^{th} PDE over $(0,\infty)^n$ and using linearity of the integral yields:

$$\begin{aligned}\int_{\mathbf{q}\in(0,\infty)^n} (\eta + xn) f_n(\mathbf{q})\, d\mathbf{q} \;=\; & \int_{\mathbf{q}\in(0,\infty)^n} \sum_{j=1}^{n} \left[\left(g_Q - \beta\right) \frac{\partial f_n(\mathbf{q})}{\partial q_j} \;+\; \frac{\nu^2}{2} \frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2} \right] d\mathbf{q} \\ & + \int_{\mathbf{q}\in(0,\infty)^n} x\, \frac{\Psi_{n-1}}{\Psi_n} \sum_{j=1}^{n} \alpha e^{-\alpha q_j} f_{n-1}\left(\mathbf{q}^{\setminus j}\right) d\mathbf{q} \\ & + \frac{\nu^2}{2} \frac{n\, \Psi_{n+1}}{\Psi_n} \frac{1}{n+1} \sum_{j=1}^{n+1} \int_{\mathbf{q}\in(0,\infty)^n} \frac{\partial f_{n+1}\left(\mathbf{q}^{j\to 0}\right)}{\partial q_j} d\mathbf{q}\end{aligned}$$

The goal is to simplify each of four integrals that appear in the expression. Starting with the LHS,

$$f_n \text{ pdf on } (0,\infty)^n \implies \int_{\mathbf{q}\in(0,\infty)^n} (\eta + xn) f_n(\mathbf{q})\, d\mathbf{q} \;=\; \eta + xn\,.$$

On the RHS, three integrals show up:

- Since f_{n-1} is a pdf on $(0,\infty)^{n-1}$ and $\alpha e^{-\alpha q}$ a pdf on $(0,\infty)$,

$$\begin{aligned}&\int_{\mathbf{q}\in(0,\infty)^i} x \frac{\Psi_{n-1}}{\Psi_n} \sum_{j=1}^{n} \alpha e^{-\alpha q_j} f_{i-1}\left(\mathbf{q}^{\setminus j}\right) d\mathbf{q} \\ &= \sum_{j=1}^{n} x \frac{\Psi_{n-1}}{\Psi_n} \left(\int_0^\infty \alpha e^{-\alpha q_j}\, dq_j \right) \left(\int_{\mathbf{q}^{\setminus j}\in(0,\infty)^{n-1}} f_{n-1}\left(\mathbf{q}^{\setminus j}\right) d\mathbf{q}^{\setminus j} \right) = \sum_{j\,=\,1}^{n} x \frac{\Psi_{n-1}}{\Psi_n} \;=\; n\, x\, \frac{\Psi_{n-1}}{\Psi_n}\end{aligned}$$

- Leveraging the definition of λ_n for all n (specifically for $n+1$)

$$\frac{\nu^2}{2} \frac{n\, \Psi_{n+1}}{\Psi_n} \frac{1}{n+1} \sum_{j=1}^{n+1} \int_{\mathbf{q}\in(0,\infty)^n} \frac{\partial f_{n+1}\left(\mathbf{q}^{j\to 0}\right)}{\partial q_j} d\mathbf{q} \;=\; \frac{n\, \Psi_{n+1}}{\Psi_n} \lambda_{n+1}$$

- Finally, to evaluate the integral on the first line of the right hand side, define the continuously differentiable vector field on $(0,\infty)^n$:

$$V(\mathbf{q}) \;=\; \left(V_1(\mathbf{q}), \ldots, V_n(\mathbf{q})\right) \quad \text{where} \quad V_j(\mathbf{q}) \;\equiv\; \left(g_Q - \beta\right)\, f_n(\mathbf{q}) \;+\; \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j}$$

  The divergence of this vector field is

$$\operatorname{div} V \;=\; \sum_{j=1}^{n} \left(g_Q - \beta\right) \frac{\partial f_n(\mathbf{q})}{\partial q_j} \;+\; \frac{\nu^2}{2} \frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2}$$

$$\implies \int_{(0,\infty)^n} \left(\left(g_Q - \beta\right) \sum_{j=1}^{n} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \; + \; \frac{\nu^2}{2} \sum_{j=1}^{n} \frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2} \right) d\mathbf{q} \; = \; \int_{(0,\infty)^n} \operatorname{div} V(\mathbf{q}) \; d\mathbf{q}$$

I can now apply the divergence theorem on $(0, R)^i$ and then take limits as $R \to \infty$. This transforms the volume integral of the divergence over the positive orthant into a surface integral on the hyperplanes delimiting the positive orthant:

$$\begin{aligned}
\int\limits_{(0,R)^n} \operatorname{div} V(\mathbf{q}) d\mathbf{q} &= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} V \cdot (-e_j) dS + \int_{\{q_j=R\}} F \cdot (e_j) dS \right) \\
&= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} - \left(\left(g_Q - \beta\right) f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS + \int_{\{q_j=R\}} \left(\left(g_Q - \beta\right) f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS \right) \\
&= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} - \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} dS + \int_{\{q_j=R\}} \left(\left(g_Q - \beta\right) f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS \right)
\end{aligned}$$

where the last step uses the boundary condition that f_n vanishes on any of the hyperplanes delimiting the positive orthant. Taking limits as $R \to \infty$, only the first integral within each sum survives, as both f_n and its partial derivatives vanish when any of its entries grows to infinity. And each of these integrals is a surface integral on one of the n hyperplanes delimiting the positive orthant, so the integration is with respect to all variables other than q_j, where q_j itself is zero. Using the notation defined above:

$$\int_{(0,\infty)^n} \operatorname{div} V(\mathbf{q}) \; = \; \sum_{j=1}^{n} -\frac{\nu^2}{2} \int_{\{q_j=0\}} \frac{\partial f_i(\mathbf{q})}{\partial q_j} \; = \; - \, n \, \lambda_n$$

Putting it all together,

$$\eta + xn \; = \; - \, n \, \lambda_n \; + \; n \, x \, \frac{\Psi_{n-1}}{\Psi_n} \; + \; \frac{n \Psi_{n+1}}{\Psi_n} \lambda_{n+1}$$

Multiplying both sides by $\frac{\Psi_n}{n}$, I get:

$$\frac{\eta}{n} \Psi_n \; = \; - \left(x \; + \; \lambda_n\right) \Psi_n \; + x \Psi_{n-1} \; + \; \lambda_{n+1} \Psi_{n+1}$$

B.6 Dimension Reduction

Lemma 1. Given the ansatz from Equation 19, the stationary firm size distribution consists of sequences Ψ_n and φ_n satisfying the following system of coupled ordinary differential equations:

$$\frac{\nu^2}{2}\varphi_1''(q) + (g_Q - \beta)\varphi_1'(q) - (\eta + x)\varphi_1(q) = -\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}\alpha e^{-\alpha q} - \frac{\Psi_2}{\Psi_1}\lambda_2\varphi_2(q)\ ,$$

$$\text{and for } n > 1, \quad \frac{\nu^2}{2}\varphi_n''(q) + (g_Q - \beta)\varphi_n'(q) - (\eta + nx + (n-1)\lambda_n)\varphi_n(q)$$
$$= -\frac{x\Psi_{n-1}}{\Psi_n}\left(\alpha e^{-\alpha q} + (n-1)\varphi_{n-1}(q)\right) - n\frac{\Psi_{n+1}}{\Psi_n}\lambda_{n+1}\varphi_{n+1}(q)$$

with $\forall n \geq 1$, $\varphi_n(0) = 0$ and $\lambda_n = \frac{\nu^2}{2}\varphi_n'(0)$.

Proof. For $n > 1$, the n^{th} ODEs is obtained by plugging in the ansatz into the n^{th} PDE then integrating both sides over $(0, \infty)^{n-1}$, so by integrating out all dimensions but one. □

C Computational Appendix

C.1 Solving integral equation from Proposition 6

$$\Gamma(A) = \int_0^A \ell(a)\exp\left(\int_0^a x\left[\Gamma(A-s) - 1\right] ds\right) da$$

As I highlighted in the main text, this can be computed with a marching forward algorithm, as each $\Gamma(A)$ only depends on lower ages and $\Gamma(0) = 0$. To speed up the process (specifically the inner integral), define $R(A) = \exp\left(-x\int_0^A \Gamma(u)du\right)$, so that $R_0 = 1$ and by the fundamental theorem of calculus $R'(A) = -xR(A)\Gamma(A)$. Now to see how this simplifies the integral equation:

$$\begin{aligned}
\Gamma(A) &= \int_0^A \ell(a)\exp\left(\int_0^a x\left[\Gamma(A-s) - 1\right] ds\right) da = \int_0^A \ell(a)\exp\left(x\int_0^a \Gamma(A-s)ds - xa\right) da \\
&= \int_0^A \ell(a)\exp\left(x\int_{A-a}^A \Gamma(u)du - xa\right) da = \int_0^A \ell(a)\exp\left(x\int_0^A \Gamma(u)du - x\int_0^{A-a}\Gamma(u)du - xa\right) da \\
&= \int_0^A \ell(a)\exp\left(x\int_0^A \Gamma(u)du\right)\exp\left(-x\int_0^{A-a}\Gamma(u)du\right)\exp\left(-xa\right) da = \int_0^A \ell(a)\frac{1}{R(A)}R(A-a)e^{-xa}da \\
\implies& \Gamma(A)R(A) = \int_0^A e^{-xa}\ell(a)R(A-a)da
\end{aligned}$$

Evaluation of the RHS does not depend on values at A, since $\ell(0) = 0$. So I get $\Gamma(A)R(A)$ by simply evaluating the integral. I then get $R(A)$ using the ODE $R'(A) = -xR(A)\Gamma(A)$ and previous value for $R(A)$. I then divide $\Gamma(A)R(A)$ by $R(A)$ to get $\Gamma(A)$.

For the hazard rate, I need $\Gamma'(A)$. Denoting $Y(A) = \Gamma(A)R(A)$, it follows that

$$\Gamma'(A) = \frac{Y'(A)}{R(A)} + x\Gamma(A)^2$$

$$\text{where } \; Y'(A) = e^{-xA}\ell(A) - x\int_0^A e^{-xa}\ell(a)Y(A-a)da$$

Because:

$$\begin{aligned}
Y(A) &= \int_0^A e^{-xa}\ell(a)R(A-a)da \\
Y'(A) &= e^{-xA}\ell(A) + \int_0^A e^{-xa}\ell(a)R'(A-a)da \quad \text{ since } \; R(A) = 1 \\
Y'(A) &= e^{-xA}\ell(A) - x\int_0^A e^{-xa}\ell(a)Y(A-a)da \quad \text{ using ODE for } \; R\,.
\end{aligned}$$

To see how this helps getting $\Gamma'(A)$:

$$Y(A) = \Gamma(A)R(A) \implies Y'(A) = \Gamma'(A)R(A) + \Gamma(A)R'(A) \implies \Gamma'(A) = \frac{Y'(A)}{R(A)} + \Gamma(A)\frac{R'(A)}{R(A)}$$

Using the ODE for R, this simplifies to

$$\Gamma'(A) = \frac{Y'(A)}{R(A)} + \Gamma(A)^2 x\,.$$

Details for numerical implementation given a set of parameters. To get the hazard rate of firm exit up to age 20, I work on a discrete age grid with 20500 points between 0 and 20.5. I use the above algorithm to "fill" D in a single forward march, using Numpy's builtin numerical integration with the trapezoid rule.

GMM to identify parameters. My GMM objective is an equally weighted least squares deviations of model vs empirical hazard rate of firm exit at ages 1 through 19. I use the least squares routine provided by Python's SciPy library.

C.2 Solving for stationary firm size distribution

A first transformation of the system. Given that my GMM strategy identifies x, $\alpha(g_Q - \beta) = \theta\eta - \alpha\beta$, and $\alpha\nu$, I start by rewriting the system from Lemma 1 in terms of these parameter combinations and η. To do so, I do the following change of variables

$$s \equiv \alpha q \quad \text{and} \quad \delta_n(s) \equiv \frac{\varphi_n\left(\frac{s}{\alpha}\right)}{\alpha} \implies \varphi_n(q) = \alpha\delta_n(\alpha q) \, ,$$

and note that

$$\frac{\nu^2}{2}\alpha\zeta = \frac{1}{2}\alpha\left(g_Q - \beta\right) + \frac{1}{2}\sqrt{\left(\alpha\left(g_Q - \beta\right)\right)^2 + 2\eta\left(\alpha\nu\right)^2}$$

so that this term in the first ODE is taken care of. Plugging into the initial system yields:

$$\forall n \geq 1 \, , \ \ \delta_n(0) = 0 \ \ ; \ \ \int_0^\infty \delta_n(s)ds = 1 \ \ ; \ \ \lambda_n = \frac{(\alpha\nu)^2}{2}\delta_n'(0) \ \ ; \ \ \lim_{s\to\infty}\delta(s) = \lim_{s\to\infty}\delta'(s) = 0$$

$$\frac{(\alpha\nu)^2}{2}\delta_1''(s) + \alpha(g_Q - \beta)\delta_1'(s) - (\eta + x)\delta_1(s) = -\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}e^{-s} - \frac{\Psi_2}{\Psi_1}\lambda_2\delta_2(s)$$

and for $n > 1$

$$\frac{(\alpha\nu)^2}{2}\delta_n''(s) + \alpha(g_Q - \beta)\delta_n'(s) - \left(\eta + nx + (n-1)\lambda_n\right)\delta_n(s) = -x\frac{\Psi_{n-1}}{\Psi_n}\left(e^{-s} + (n-1)\delta_{n-1}(s)\right) - n\frac{\Psi_{n+1}}{\Psi_n}\lambda_{n+1}\delta_{n+1}($$

with the recurrence relation that only depends on identified parameters:

$$\begin{cases} \eta\,\Psi_1 & = \; -\left(x + \lambda_1\right)\Psi_1 \; + \; \lambda_2\Psi_2 \; + \; \eta \; + \; \frac{\nu^2}{2}\alpha\,\zeta \; - \; x \\ \frac{\eta}{n}\,\Psi_n & = \; -\left(x + \lambda_n\right)\Psi_n \; + \; \lambda_{n+1}\Psi_{n+1} \; + \; x\Psi_{n-1} \qquad \text{for } \; n \; > \; 1 \end{cases}$$

The limit (fixed point) of the system becomes

$$\delta_\infty(s) = \frac{\frac{\tau}{\alpha}}{\frac{\tau}{\alpha} - 1}\left(e^{-s} - e^{-\frac{\tau}{\alpha}s}\right) \quad \text{where } \; \frac{\tau}{\alpha} = 2\frac{\alpha(g_Q - \beta)}{(\alpha\nu)^2}$$

along with

$$\lambda_\infty = \alpha(g_Q - \beta) \quad \text{and} \quad \frac{\Psi_{n+1}}{\Psi_n} \; \sim \; \frac{x}{\lambda_\infty} \; < \; 1 \, .$$

A second transformation of the system. While the above system is in principle solvable, to guarantee numerical stability I do a second transformation that drastically improves the system's conditioning. In that vein, let:

$$u_n(s) \equiv e^{\frac{\tau}{\alpha}s}\delta_n(s) \quad \text{and} \quad R_n \equiv \frac{\Psi_n}{\Psi_{n+1}}$$

The motivation for working with ratios is that as n grows large, Ψ_n converges to zero, while the ratio of consecutive terms converges to a strictly positive number. Defining the initial condition

$$R_0 \;=\; \frac{1}{x}\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}\;,$$

The recurrence is then given for all $n \geq 1$ by

$$-\left(\frac{\eta}{n} + x + \lambda_n\right) + \frac{\lambda_{n+1}}{R_n} + xR_{n-1} \;=\; 0\;.$$

Plugging the recurrence into the RHS to avoid having divisions by R_n, the $n = 1$ ODE becomes:

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}u_1''(s) = &-\left[\alpha(g_Q - \beta) - \frac{\tau}{\alpha}(\alpha\nu)^2\right]u_1'(s) + \left[\eta + x + \frac{\tau}{\alpha}\alpha(g_Q - \beta) - \left(\frac{\tau}{\alpha}\right)^2\frac{(\alpha\nu)^2}{2}\right]u_1(s)\\ &- xR_0 e^{-\left(1-\frac{\tau}{\alpha}\right)s} + \left[xR_0 - (\eta + x + \lambda_1)\right]u_2(s)\;.\end{aligned}$$

The $n > 1$ ODE becomes:

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}u_n''(s) = &-\left[\alpha(g_Q - \beta) - \frac{\tau}{\alpha}(\alpha\nu)^2\right]u_n'(s) + \left[\eta + nx + (n-1)\lambda_n + \frac{\tau}{\alpha}\alpha(g_Q - \beta) - \left(\frac{\tau}{\alpha}\right)^2\frac{(\alpha\nu)^2}{2}\right]u_n(s)\\ &- xR_{n-1}\left(e^{-\left(1-\frac{\tau}{\alpha}\right)s} + (n-1)u_{n-1}(s)\right) + n\left[xR_{n-1} - \left(\frac{\eta}{n} + x + \lambda_n\right)\right]u_{n+1}(s)\;.\end{aligned}$$

The boundary conditions are with $u_n(0) = 0$; $\lambda_n = \frac{(\alpha\nu)^2}{2}u_n'(0)$ and, for s_max large enough $\delta_n(s_{max}) = 0$ (exponentially decaying tail).

Numerical solution. I transform the system of 2nd order ODEs into a system of 1st order ODEs by introducing the derivatives u' as auxiliary variables. I solve for u_n, u_n', λ_n, and R_n using solve-bvp from Python's Scipy library, which is designed to solve such systems of differential-algebraic equations. The n_0 I choose for truncation purposes is 25 (and checked robustness to increasing n_0 to 40), with a tolerance of 10^{-5} and $s_{max} = 8$.

Verification of solution. After solving for $u_n(q)$, I obtain $\delta_n(q)$ using

$$\delta_n(s) = e^{-\frac{\tau}{\alpha}s}u_n(s)\;.$$

As discussed in the main text, the theory provides a transparent way to verify the solution:

$$\sum_{n=1}^{\infty}\Psi_n\varphi_n(q) = \frac{\alpha\;\zeta}{\zeta - \alpha}\left(e^{-\alpha q}\;-\;e^{-\zeta q}\right)$$

In terms of what I solved for numerically, this becomes

$$\sum_{n=1}^{\infty} \Psi_n \delta_n(s) = \frac{\zeta/\alpha}{\zeta/\alpha - 1}\left(e^{-s} - e^{-\frac{\zeta}{\alpha}s}\right) \text{ where } \frac{\zeta}{\alpha} = \frac{\alpha(g_Q - \beta) + \sqrt{(\alpha(g_Q - \beta))^2 + 2\eta(\alpha\nu)^2}}{(\alpha\nu)^2} .$$

The absolute deviation between the LHS (computed numerically) and the RHS (closed form) has mean 0.0003 and maximum 0.0013.

C.3 Employment distribution along BGP

In equilibrium, production employment for product p is given by

$$L_{pt} = e^{(\sigma-1)q} \cdot \frac{L_t}{M_t} \cdot \frac{\alpha - (\sigma - 1)}{\alpha} \frac{\zeta - (\sigma - 1)}{\zeta} .$$

Using the change of variable $s \equiv \alpha q$, the expression for L/M along the BGP and the identities that ξ verifies yields

$$L_{pt} = e^{\frac{\sigma-1}{\alpha}s}(\mathcal{F}-O)(\sigma-1)\left[1 - \frac{\frac{1}{2}(\beta - g_Q)(\sigma - 1) + \frac{1}{2}\sqrt{(\beta - g_Q)^2(\sigma - 1)^2 + 2\nu^2(r - g)(\sigma - 1)^2}}{r - g}\right] ;$$

which I rewrite as:

$$L_{pt} = \underline{L} e^{\frac{\sigma-1}{\alpha}s} ,$$

where $\underline{L}$ is production labor employment at the exit threshold ($s = q = 0$).

Employment at a firm with n products with portfolio $(s_1, \ldots, s_n)$ is then given by:

$$(\mathcal{F} + I)n + \sum_{i=1}^{n} \underline{L} e^{\frac{\sigma-1}{\alpha}s} .$$

Appendix C.2 makes clear that, to solve for the stationary distribution of number of products per firm and conditional distribution of s among firms with n products, one only needs a value for the population growth rate η and values for the statistics estimated by the GMM (reported in Table 5). Once these distributions are characterized, to compute employment, one needs: $\frac{\sigma-1}{\alpha}$, $\mathcal{F} + I$, and $\underline{L}$. Calibrated values for these parameters are reported in Table 7.

It is worth noting that between Table 7 and Table 5, I am pinning down values for three statistics that are complicated functions of structural parameters. These are reproduced in Table C3. The goal is to highlight that there are enough degrees of freedom (structural parameters) to jointly satisfy the three requirements: for example, given values for other parameters, one can pick values for $\mathcal{F}$, δ and $\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$ in order to match the estimated value for x and calibrated values for $\underline{L}$ and $\mathcal{F} + I$.

Table C3: Calibrated and estimated statistics in terms of structural parameters

Statistic	As function of structural parameters
x	$\frac{1}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}\frac{\left(\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)\right)(\alpha-(\sigma-1))}{\left(\mathcal{F}-\frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}\right)(\sigma-1)}\left(\alpha+\frac{\beta-\frac{\theta}{\alpha}\eta+\sqrt{(\beta-\frac{\theta}{\eta}\alpha)^2+2\nu^2\left(\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)\right)}}{\nu^2}\right)$
$\mathcal{F}+I$	$\mathcal{F}+\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$
$\underline{L}$	$\left(\mathcal{F}-\frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}\right)(\sigma-1)\left[1-\frac{\frac{1}{2}(\beta-\frac{\theta}{\alpha}\eta)(\sigma-1)+\frac{1}{2}\sqrt{(\beta-\frac{\theta}{\eta}\alpha)^2(\sigma-1)^2+2\nu^2\left(\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)\right)(\sigma-1)^2}}{\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)}\right]$

D Data Appendix

This appendix provides detailed information on the data sources, sample construction, and variable definitions used in the empirical analysis of the paper. Section D.1 describes the product data used to document the facts in Section 4. Section D.2 describes the firm data used for the main quantitative estimation in Section 5.

D.1 NielsenIQ Retail Scanner Dataset

Sample Construction. The analysis is restricted to a balanced panel of approximately 25,400 retail stores that are continuously present for the entire 14 year period. This restriction ensures that product exit is not mechanically driven by store closures.

Each UPC in the data is already assigned to one of roughly 120 product group codes. In building my sample, I exclude unclassified products, fresh produce, non-scannable (“magnet”) products, control brands, and products classified under “seasonal”, “prep food-deli”, or groups that get discontinued by NielsenIQ (deferred modules). These exclusions insure that products in my sample can be consistently mapped across different retailers at a point in time as well as over time.

Variable definitions. I obtain a UPC’s sales (in \$) and volume of sales (quantity sold) by summing across retailers in a given time period. Dividing the former by the latter yields the average unit price. A UPC is defined as entering in year t if it has zero sales in year $t-1$ and positive sales in year t. A UPC is defined as exiting in year t if it has positive sales in year t and zero sales in year $t+1$. A UPC’s age in a given year is defined as the current year minus its first year of appearance in the sample. UPCs present in 2006 are considered left-censored

Data cleaning. To guarantee accurate measurement of entry and exit, I drop any UPC that records more than one entry or exit event over the sample period.

To allow for meaningful comparisons of quantity and price within a product group, I harmonize product size units. First, I convert units to a common standard where possible (e.g., pounds and kilograms are converted to ounces; liters and quarts are converted to milliliters). For a small number of products, NielsenIQ provides a secondary size. If a product's primary unit does not match the modal unit of its product group, but its secondary unit does, I use the secondary unit for harmonization. If after these harmonization steps a product group's modal unit of measure accounts for less than 95% of sales, then this group is excluded from my analysis. Similarly, if the group has fewer than 100 products, it is excluded from the sample.

D.2 Publicly Available U.S. Census Data

To quantify spillovers I use publicly available tabulations based on the U.S. Census Longitudinal Business Database (LBD). This is an administrative dataset covering the universe of private nonfarm employer businesses in the U.S. As it tracks establishments, in tabulations based on this dataset, a firm exits is defined as all its establishments closing.

Estimation across all private nonfarm employer businesses. The firm exit rate at ages 1 through 19 that I use are provided in the replication package of Sterk, Sedláček and Pugsley (2021).

Estimation at the 2-digit sector level. I use the Business Dynamics Statistics, publicly provided by the U.S. Census Bureau. Specifically, I use the State by Firm Age two-way tabulation, which provides, for each 2-digit NAICS sector, the number of firms as well as the number of exiting firms at ages 1, 2, 3, 4, 5, 8, 13, and 18. Since the underlying data source is the LBD (which starts in 1978), I use data from 1996 onward to avoid having any left censored firm in any of my age bins. I define the firm exit rate in a cell as the number of exiting firms divided by the average number of firms between last the previous and current year.

E Additional Figures and Tables

E.1 Empirical Results

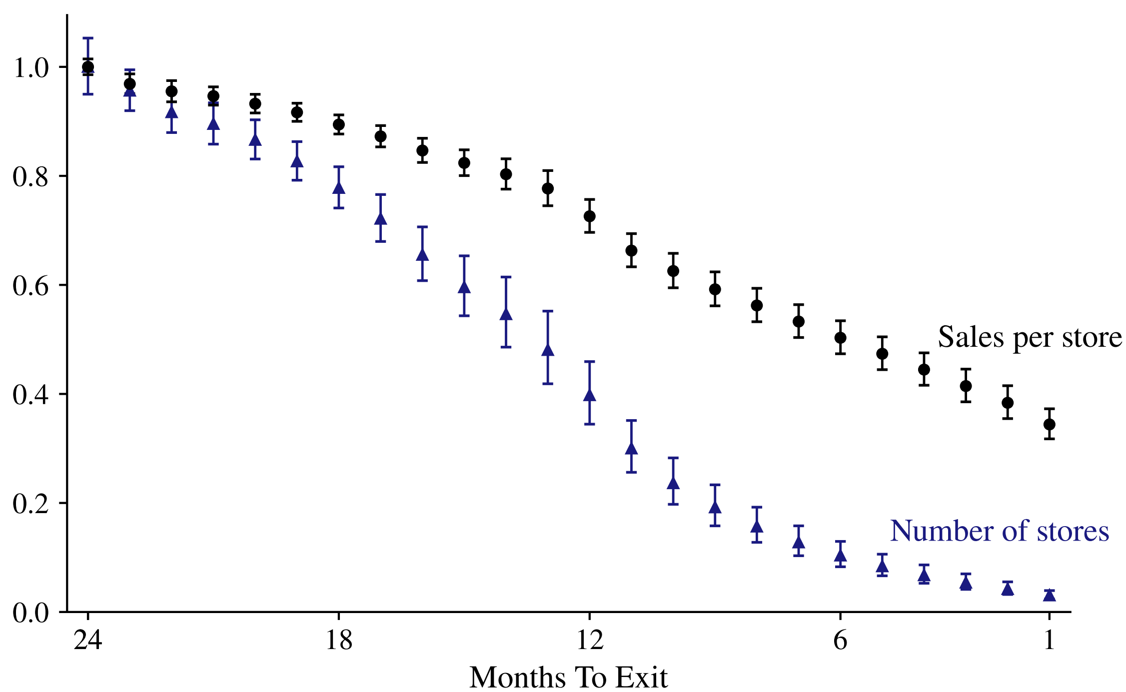
To document the gradual exit process along both the extensive (number of stores at which UPC is sold) and intensive (sales per store) margins, I run the following regressions:

$$\log \text{Sales per store}_{pt} = \gamma_p + \sum_{m=1}^{24} \pi_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} , \tag{35}$$

$$\log \text{Number of stores}_{pt} = \gamma_p + \sum_{m=1}^{24} \kappa_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} ; \tag{36}$$

where p indexes a UPC, g its group (product category), and t a month, with γ_p a UPC fixed effect, γ_{gt} a group-month fixed effect, and D_{pt}^m a dummy variable equal to 1 m months prior to the UPC's exit. Figure E1 plots $\exp(\pi_m)$ and $\exp(\kappa_m)$, respectively representing the paths of sales per store and number of stores in the two years leading up to the UPC's exit.

Figure E1: Extensive and intensive margins in the lead up to exit



Notes: Sales per store curve corresponds to path of $\exp(\pi_m)$ from Equation 35, normalized such that $\exp(\pi_{24}) = 1$; underlying regression has 54M observations. Number of stores curve corresponds to path of $\exp(\kappa_m)$ from Equation 36, normalized such that $\exp(\kappa_{24}) = 1$; underlying regression has 54M observations. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

E.2 Quantitative Results

Table E4: Sensitivity of model-implied hazard rates to the three different statistics

	$\theta\eta - \alpha\beta$	$\alpha\nu$	x
Model's hazard rate of firm exit at age 1	9.16	5.98	-1.78
Model's hazard rate of firm exit at age 2	9.17	2.28	-2.49
Model's hazard rate of firm exit at age 3	9.07	0.75	-2.98
Model's hazard rate of firm exit at age 4	8.95	-0.06	-3.34
Model's hazard rate of firm exit at age 5	8.82	-0.54	-3.63
Model's hazard rate of firm exit at age 6	8.70	-0.84	-3.86
Model's hazard rate of firm exit at age 7	8.58	-1.04	-4.06
Model's hazard rate of firm exit at age 8	8.48	-1.17	-4.22
Model's hazard rate of firm exit at age 9	8.38	-1.25	-4.37
Model's hazard rate of firm exit at age 10	8.29	-1.31	-4.49
Model's hazard rate of firm exit at age 11	8.20	-1.35	-4.60
Model's hazard rate of firm exit at age 12	8.12	-1.37	-4.69
Model's hazard rate of firm exit at age 13	8.05	-1.38	-4.78
Model's hazard rate of firm exit at age 14	7.98	-1.39	-4.86
Model's hazard rate of firm exit at age 15	7.91	-1.39	-4.92
Model's hazard rate of firm exit at age 16	7.84	-1.39	-4.98
Model's hazard rate of firm exit at age 17	7.77	-1.38	-5.04
Model's hazard rate of firm exit at age 18	7.71	-1.37	-5.08
Model's hazard rate of firm exit at age 19	7.65	-1.36	-5.13

Notes: The entry in row i and column j gives the **percentage point** change in the model-implied hazard rate of firm exit at age i resulting from a **100% change** in statistic j. These are local semi-elasticities, evaluated at the estimated values of the three statistics.

Table E5: Sensitivity of estimated statistics to empirical hazard rates

	$\theta\eta - \alpha\beta$	$\alpha\nu$	x
Empirical hazard rate of firm exit at age 1	-5.27	18.26	-14.02
Empirical hazard rate of firm exit at age 2	4.51	0.11	7.76
Empirical hazard rate of firm exit at age 3	6.73	-5.19	12.84
Empirical hazard rate of firm exit at age 4	6.86	-6.77	13.30
Empirical hazard rate of firm exit at age 5	6.23	-6.86	12.05
Empirical hazard rate of firm exit at age 6	5.30	-6.27	10.10
Empirical hazard rate of firm exit at age 7	4.25	-5.37	7.90
Empirical hazard rate of firm exit at age 8	3.18	-4.33	5.63
Empirical hazard rate of firm exit at age 9	2.14	-3.23	3.40
Empirical hazard rate of firm exit at age 10	1.13	-2.14	1.26
Empirical hazard rate of firm exit at age 11	0.18	-1.07	-0.79
Empirical hazard rate of firm exit at age 12	-0.73	-0.03	-2.71
Empirical hazard rate of firm exit at age 13	-1.57	0.95	-4.53
Empirical hazard rate of firm exit at age 14	-2.37	1.89	-6.23
Empirical hazard rate of firm exit at age 15	-3.11	2.77	-7.83
Empirical hazard rate of firm exit at age 16	-3.81	3.61	-9.32
Empirical hazard rate of firm exit at age 17	-4.46	4.39	-10.72
Empirical hazard rate of firm exit at age 18	-5.07	5.13	-12.03
Empirical hazard rate of firm exit at age 19	-5.64	5.82	-13.25

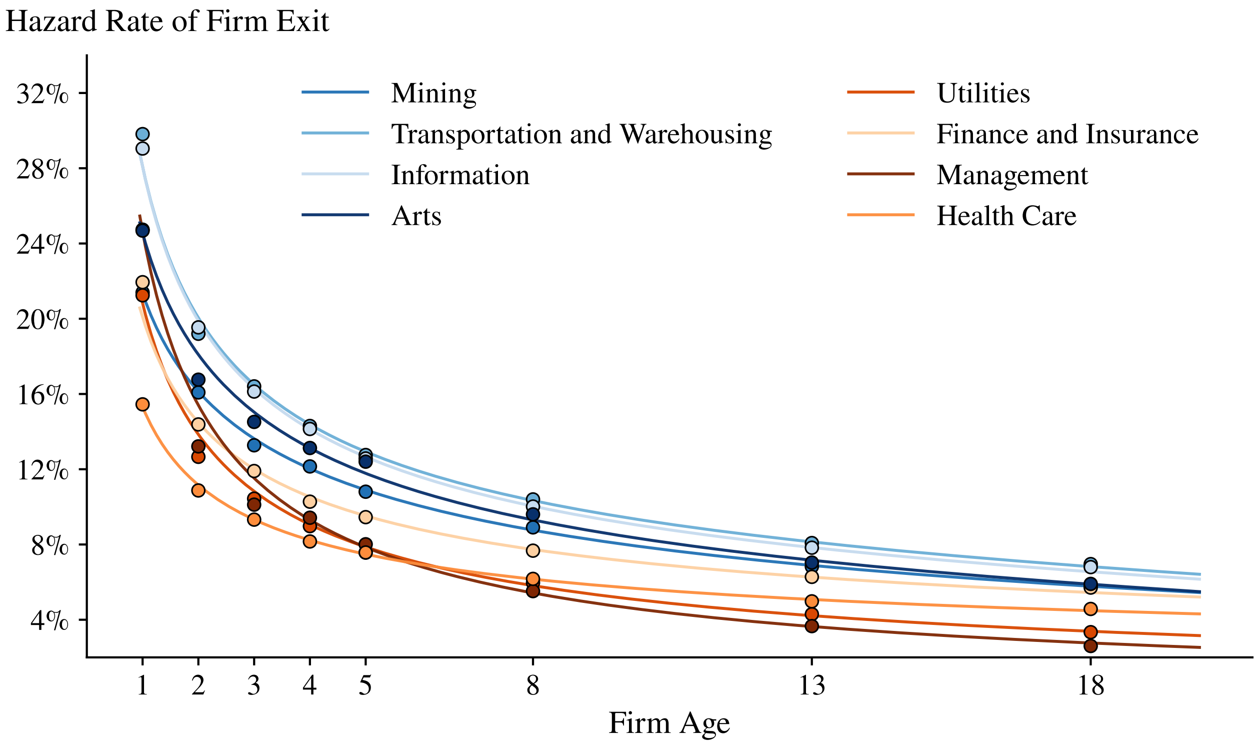
Notes: The entry in row i and column j gives the **percent change** in statistic j resulting from a **1 p.p. change** in the empirical hazard rate of firm exit at age i. These are local semi-elasticities, evaluated at the estimated parameter values.

Table E6: GMM Estimation Results Across Sectors

Sector	Firm Exit Rate	Share of firms	Share of emp	Estimated $\theta\eta - \alpha\beta$
Agriculture, Forestry, Fishing, and Hunting	8.9%	0.4%	0.1%	11.1%
Mining, Quarrying, and Oil and Gas Extraction	8.2%	0.3%	0.5%	21.5%
Utilities	4.2%	0.1%	0.6%	5.5%
Construction	9.0%	11.6%	5.4%	13.0%
Manufacturing	6.7%	5.1%	11.6%	15.8%
Wholesale Trade	7.8%	5.7%	5.0%	15.6%
Retail Trade	9.2%	12.2%	13.0%	15.3%
Transportation and Warehousing	10.8%	2.8%	3.7%	18.9%
Information	10.2%	1.2%	2.9%	18.4%
Finance & Insurance	7.7%	4.1%	5.3%	10.5%
Real Estate & Leasing	9.1%	4.5%	1.7%	13.2%
Professional, Scientific, & Tech Services	8.5%	12.2%	6.5%	11.7%
Management of Companies & Enterprises	3.6%	0.5%	2.7%	3.3%
Administrative & Support & Waste Management & Remediation Services	9.7%	5.3%	7.9%	14.2%
Educational Services	6.9%	1.3%	2.6%	15.9%
Health Care and Social Assistance	6.2%	10.8%	14.3%	7.0%
Arts, Entertainment, and Recreation	8.6%	1.7%	1.7%	25.6%
Accommodation and Food Services	10.3%	8.0%	9.8%	16.6%
Other Services (except Public Administration)	6.2%	12.0%	4.6%	9.3%

Notes: Underlying data are Business Dynamics Statistics for 1996-2019. The estimation targets the sector's profile of firm exit by age, specifically at ages 1, 2, 3, 4, 5, 8, 13, and 18.

Figure E2: Firm Exit by Age in Sectors with Largest and Smallest Estimated Wedges



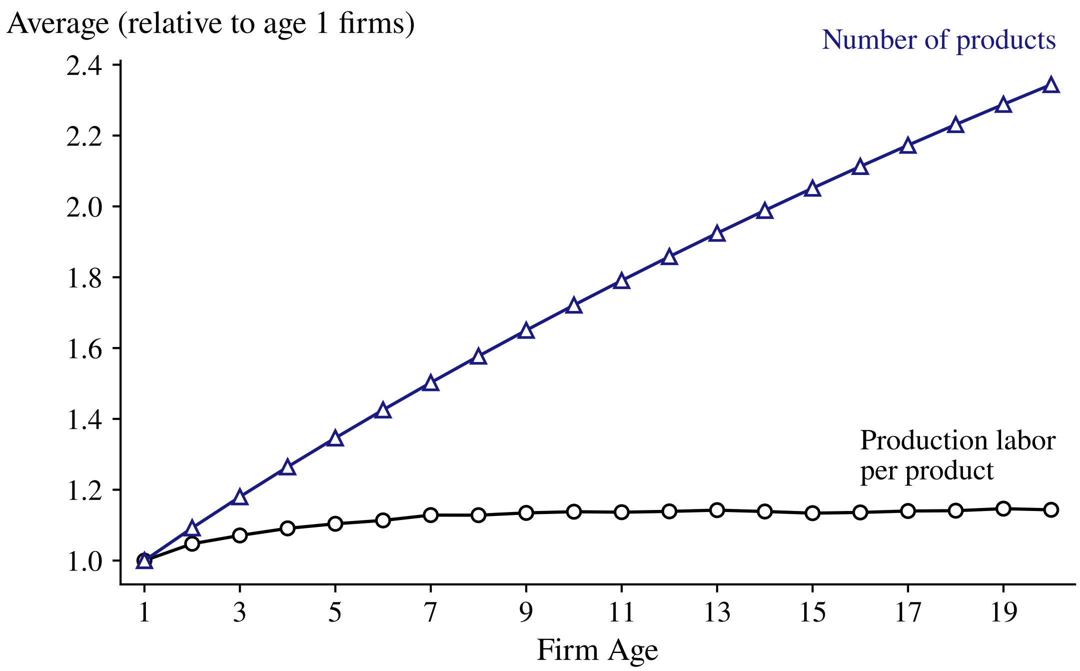
Notes: Sectors in blue are the four in which I estimate the largest wedge. Sectors in orange are the four in which I estimate the smallest wedge. Dots correspond to empirical moments, each curve is the model's fit from a GMM targeting that sector's profile of firm exit by age. The GMM objective puts equal weights on ages 1 through 5, and five times the weight on ages 8, 14, and 18. Underlying data is from Business Dynamics Statistics for the years 1996 to 2019.

Table E7: Sources of Firm Size Heterogeneity in the model

	Firm-level Employment	Number of Products	Per-product Production Employment
1 to 4 employees	2.3	1.2	1.1
5 to 9 employees	6.5	2.5	2.3
10 to 19 employees	13.4	4.1	3.8
20+ employees	151.4	6.1	59.3

Notes: Each statistic reported corresponds to an average across firms within that size bin.

Figure E3: Model's firm size by firm age: product count versus employment per product



Notes: To obtain the results, I simulate a cohort of 5 million firms up until age 20, compute average employment and average product count across firms at each age, repeat this simulation 1000 times, and report medians across the simulations.

F Extensions

F.1 Multi-sector version of the model

The model presented in the main text features a single sector. In this appendix, I present a version of the model with multiple sectors. To avoid cumbersome notation, I shut down volatility ($\nu = 0$) and drift ($\beta = 0$). While these matter for the quantitative aspect of the results, they do not affect the qualitative point I want to make, which is that product turnover or churn in a sector reflects knowledge spillovers *received* by that sector.

Suppose the consumption good is a Cobb-Douglas aggregate of different sectors:

$$c_t = \prod_{s=1}^{S} c_{st}^{\beta_s} \quad \text{where } \beta_s \in (0,1) \text{ and } \sum_{s=1}^{S} \beta_s = 1 \, .$$

As the below will make clear, Cobb Douglas aggregation across sectors is needed to achieve a BGP with constant share of employment in each sector, since different sectors will grow at different rates.

Within each sector, c_{st} is a CES aggregate of a continuum of differentiated varieties, as in the model in the main text:

$$c_{st} = \left[\int_{p \in \Omega_{st}} \left(Q_{pt} c_{pt} \right)^{\frac{\sigma_s - 1}{\sigma_s}} \, dp \right]^{\frac{\sigma_s}{\sigma_s - 1}} .$$

The set Ω_{st} evolves through entry and exit. Exit results from the presence of a sector-specific fixed overhead cost $\mathcal{F}_s$, denoted in labor units. After payment of this fixed cost, quantity is linear in production labor.

The creation of new products is carried out by incumbent firms in that sector as well as entering firms. Entry of new firms is the result of individuals who choose to be entrepreneurs. If individual i chooses to be an entrepreneur, they draw their idiosyncratic and sector specific entrepreneurship ability from a multivariate Fréchet distribution with cumulative distribution function:

$$F\left(\varepsilon_1, \ \ldots \ , \varepsilon_S\right) \ = \ \exp\left[- \left(\sum_{s=1}^{S} \varepsilon_s^{-\frac{\widetilde{\lambda}}{1-\kappa}} \right)^{1-\kappa} \right]$$

where $\widetilde{\lambda}$ captures dispersion in entrepreneurship ability between individuals, with lower $\widetilde{\lambda}$ corresponding to more dispersion, and κ measures the correlation between entrepreneurship ability across the different sectors. After observing these draws, the individual chooses toward which sector to direct their entrepreneurship endeavors. If they choose sector s, they generate a new blueprint with Poisson arrival rate $\eta_s \cdot \varepsilon_{is}$ where η_s captures sector-specific features common to all entrepreneurs (ideas are harder to find in some sectors).[14] Denoting by K_{st} the stock of sector s products ever created, its law of motion is then given by:

$$\dot{K}_{st} \ = \ \int_{i \in \mathcal{R}_{st}} \eta_s \cdot \varepsilon_{is} \quad \text{where} \quad \mathcal{R}_{st} \ \equiv \ \text{Sector } s \text{ entrepreneurs} \ .$$

The quality of a new product in sector s at time t is drawn from a sector specific entry distribution with CCDF $\overline{F}^E_{st}(Q)$ given by:

$$\overline{F}^E_{st}(Q) \ = \ \Pr\left(\text{Draw}_{st} \ > Q\right) \ = \ \begin{cases} 1 & \text{if } Q \ < \ \prod_j \ K_{jt}^{\frac{\theta_{j \to s}}{\alpha_s}} \\ \prod_j \ K_{jt}^{\theta_{j \to s}} Q^{-\alpha_s} & \text{otherwise} \end{cases} ,$$

where $\theta_{j \to s}$ encodes the strength of knowledge spillovers from sector j to sector s.

A balanced growth path features (i) constant allocation of labor across sectors s and occupations (so that production labor and the measure of products grows at rate η in each sector), (ii) constant growth in the sector-specific exit threshold $\underline{Q}_{st}$, and (iii) stationary distribution of relative quality in each sector.

[14] The individual specific component introduces the diminishing returns necessary to ensure an interior solution for the allocation of entrepreneurship across sectors.

Along such a balanced growth path,

$$\frac{\dot{K}_{st}}{K_{st}} = \eta \quad \text{and} \quad g = \sum \beta_s \left(\frac{\eta}{\sigma_S - 1} + g_{Q_s} \right) ,$$

where g_{Q_s}, the growth rate of quality in sector s, is given by:

$$g_{Q_s} = \frac{\Theta_s}{\alpha_s}\eta \quad \text{with} \quad \Theta_s \equiv \sum_{j=1}^{S} \theta_{j\to s} .$$

The resulting stationary product exit and entry rates are:

$$\frac{D_{st}}{M_{st}} = \Theta_s \eta \quad \text{and} \quad \frac{E_{st}}{M_{st}} = (1 + \Theta_s)\, \eta .$$

As I mentioned at the top of this subsection, I have set $\nu = 0$ in deriving these results, that's why all turnover is due to spillovers. The point I want to illustrate is that the entry and exit rates in sector s are function of $\sum_j \theta_{j\to s}$, which is a measure of the spillovers *received* by sector s.

F.2 Model incorporating human capital growth

The goal of this subsection is to assess how incorporating human capital growth alters my results. The answer depends on whether human capital growth leads to more draws from the entry distribution or better draws from the entry distribution.

F.2.1 Higher human capital $\implies$ more draws from entry distribution

Suppose that each individual supplies h_t efficiency units of labor per unit of time. The baseline version of the model corresponds to $h_t \equiv 1$ for all t. Here instead I want to entertain exogenous growth in h_t at rate g_h:

$$\dot{h}_t = g_h h_t .$$

The only change relative to the environment of the baseline model is that the labor resource constraint is in efficiency units now and reads:

$$\int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt} \right) dp + S_t = h_t N_t .$$

Along the balanced growth path of this model, it follows that

$$\frac{\dot{M}_t}{M_t} = \frac{\dot{S}_t}{S_t} = \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} = \eta + g_h \quad \text{and} \quad g_Q = \frac{\theta}{\alpha}\left(\eta + g_h\right) .$$

Proceeding as I did for the baseline model, one can show that along the equilibrium BGP, the social rate of return to R&D is

$$\widetilde{r}_{\mathbf{DE}} = r + \theta(\eta + g_h) \ ,$$

so that the gap between social and private rate of return is $\widetilde{r} - r = \theta(\eta + g_h) = \theta g_K$. The dynamics of product exit are still informative about this gap as the KFE now leads to the following stationary product entry and exit rates:

$$\frac{E_t}{M_t} = \eta + g_h + \frac{\nu^2}{2}\alpha\widetilde{\zeta} \quad \text{where} \quad \widetilde{\zeta} \equiv \frac{g_Q - \beta + \sqrt{(g_Q-\beta)^2 + 2\nu^2(\eta + g_h)}}{\nu^2} \ ,$$
$$\frac{D_t}{M_t} = \frac{1}{2}\alpha(g_Q - \beta) + \frac{1}{2}\sqrt{(\alpha(g_Q-\beta))^2 + 2\nu^2(\eta+g_h)} = \frac{1}{2}(\theta g_K - \alpha\beta) + \frac{1}{2}\sqrt{(\theta g_K - \alpha\beta)^2 + 2\nu^2(\eta + g_h)} \ .$$

Intuitively the reason is that the SDE for a product's relative quality can still be written as:

$$dq_{pt} = \left(\beta - \frac{\theta}{\alpha}g_K\right)dt + \nu dB_{pt} \ ,$$

So that product exit due to downward drift still exactly maps to the wedge of interest when $\beta = 0$.

The takeaway from this extension is that my baseline approach is robust to having human capital growth that leads to more draws from the entry distribution. The only effect such exogenous growth has is how to "split" the estimated wedge between θ and growth in the knowledge stock (K_t).

F.2.2 Higher human capital $\implies$ better draws from entry distribution

Suppose instead that higher human capital shows up as an ability to draw from a better distribution:

$$\overline{F}_t^E(Q) = h_t^\lambda K_t^\theta Q^{-\alpha} \ .$$

With $\dot{h}_t = g_h h_t$, human capital growth then leads to a shift out in the entry distribution over time. It follows that along the balanced growth path,

$$\frac{\dot{M}_t}{M_t} = \frac{\dot{S}_t}{S_t} = \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} = \eta \quad \text{and} \quad g_Q = \frac{\theta}{\alpha}\eta + \frac{\lambda}{\alpha}g_h \ .$$

Proceeding as I did for the baseline model, one can show that along the equilibrium BGP, the social rate of return to R&D is still

$$\widetilde{r}_{\mathbf{DE}} = r + \theta\eta \ .$$

It is also still the case that the extent of product exit due to downward drift to the exit threshold is $\alpha(g_Q - \beta)$, which is now equal to $\theta\eta + \lambda g_h - \alpha\beta$. Therefore

$$\theta\eta + \lambda g_h - \alpha\beta = 15.8\% \ \overset{\beta=0}{\implies} \ \boxed{\theta\eta + \lambda g_h = 15.8\%} \ .$$

To see how to split the 15.8% between $\theta\eta$ and λg_h, note that

$$g_Q = \frac{1}{\alpha}\left(\theta\eta + \lambda g_h\right) \ .$$

Since standard price deflators do not typically capture gains from product variety (Feenstra, 1994; Broda and Weinstein, 2006), the model's quality growth rate, g_Q, is the appropriate counterpart to measured productivity growth. Then the share of the 15.8% due to λg_h is the share of measured productivity growth due to human capital growth.

I here explore three different ways of calculating the share of measured growth due to human capital growth.

1. The first uses a Mincerian return framework to translate growth in average years of schooling to wage growth. Specifically, using CPS Historical Time Series Tables provided by the US Census Bureau, I calculate an increase of 2.19 in the average years of schooling for the Civilian noninstitutionalized population age 25 years and older between 1978 and 2019. With a Mincerian return of 7% for an extra year of schooling, this yields an annual contribution of

   $$(\exp(0.07 * 2.19))^{1/42} - 1 = 0.366 \text{ p.p.}$$

   With measured productivity growth averaging 1.84% per annum (labor productivity growth for private nonfarm businesses from BLS), this corresponds to human capital accounting for 19% of measured productivity growth. In this case

   $$\theta\eta = (1 - 0.19)0.158 \implies \theta\eta = 0.128$$

2. The second uses the labor composition index provided by the BLS to measure the contribution of human capital growth to productivity growth. Relative to the first approach, this also captures the growth in average human capital resulting from reallocation of hours worked towards older individuals, with more experience. For the private nonfarm sector between 1978 and 2019, the average annual growth rate of this labor composition index is 0.43%. This approaches therefore attributes 24% of measured productivity growth to rising human capital. So with this metric,

   $$\theta\eta = (1 - 0.24)0.158 \implies \theta\eta = 0.12 \ .$$

3. A shortcoming of the labor composition index is that it does not capture growth in human capital within a "type" (schooling x years of experience). Such growth might result for example from higher quality of schooling or better investments during childhood. With this in mind, I extend the BLS methodology to allow for within-type growth in human capital. The challenge with identifying this within type growth is that micro data on wages only identify relative changes in human capital. To overcome this identification challenge, I build on Lagakos, Moll, Porzio, Qian and Schoellman (2018) and impose the Heckman, Lochner

and Taber (1998) assumption of no returns to experience at the end of the life cycle. Using data on wage growth of workers in the flat portion of their human capital-experience profile allows identifying the time fixed effect of wage growth (the non human capital component of TFP, which is common to everyone). Once this time fixed effect is recovered, within-type growth is simply this type's wage growth net of the time fixed effect.
I implement this approach using data from the BLS, with 355 worker-types (5 schooling groups $\times$ 71 age categories) and 35 to 40 years of potential experience as the range of experience over which there are no returns to additional experience.
I find an average estimate of within type growth of 0.24%. Combined with the 0.43% resulting from compositional growth (see item 2 above), this yields an average annual growth rate of 0.67% in this human capital index. With this metric, human capital growth accounts for 36% of measured growth. So the resulting estimate for $\theta\eta$ is:

$$\theta\eta = (1 - 0.36)0.158 \implies \theta\eta = 0.10$$