Quantifying Knowledge Spillovers Using Firm and Product Dynamics

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Abstract

Knowledge spillovers are a common rationale for government support of innovation, yet evidence on their magnitude remains limited. In this paper, I quantify the wedge spillovers create between social and private rates of return to R&D. To do so, I build a novel semi-endogenous growth model featuring multiproduct firms and endogenous exit of products. In equilibrium, product exit exhibits negative selection and is preceded by a gradual decline in sales, consistent with facts I document using barcode-level data. Through the lens of the model, these dynamics of product exit are informative about spillovers: by accelerating growth in the quality of new products, stronger spillovers increase the rate at which incumbent products lose market share and exit. Since comprehensive datasets track firms rather than products, I leverage the model to infer the wedge created by spillovers from data on firm exit by age. Across U.S. private nonfarm businesses, I estimate spillovers that drive a 16 percentage point wedge between the social and private rates of return to R&D.

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1 Introduction

Knowledge spillovers have long been recognized as a rationale for government support of innovation (Arrow, 1962). This conventional wisdom recognizes the aggregate stock of knowledge as an input into the production of new ideas, so an individual inventor does not fully capture the social value of their innovation. Quantifying how large these spillovers are is therefore a key ingredient for informed innovation policy (Atkeson and Burstein, 2019; Akcigit, Hanley and Stantcheva, 2022).

Yet, evidence on the magnitude of spillovers "is quite thin" (Bryan and Williams, 2021, p. 290). Moreover, the limited evidence available relies on patents, thereby not only missing unpatented innovations like the polio vaccine, the World Wide Web, and Linux, but also systematically skewing toward manufacturing: the sector accounts for 10% of GDP (BEA, 2025) and TFP growth (BLS, 2025) but 64% of patenting (NCSES, 2023).

This paper leverages the dynamics of firm exit to quantify the gap between social and private rates of return to R&D created by knowledge spillovers. Instead of patents or reported R&D, I use data on firm exit by age covering the universe of U.S. private nonfarm businesses. To infer the magnitude of spillovers from these exit patterns, I develop a new semi-endogenous growth model with multiproduct firms and selection into product exit. I find substantial spillovers, driving a 16 percentage point wedge between the social and private rates of return to R&D.

To appreciate why exit rates are informative about spillovers, consider today's inventors creating new products. Compared to their predecessors, they have access to a stock of knowledge made larger by recent innovations. How much does this larger knowledge stock improve the quality of the products they invent? In search for the answer, I turn to dynamics in product markets. The rationale is simple: if higher quality products render lower quality ones obsolete, then stronger spillovers—by accelerating growth in the quality of new products—accelerate the rate at which incumbent products lose market share and exit.

Building on this intuition to quantify spillovers requires a model for two reasons. First, the thought experiment above took as given how much knowledge is available to inventors, but this stock is accumulated endogenously through innovation. Accordingly, I need a model that endogenizes this accumulation process and formalizes how the resulting spillovers translate into a higher product exit rate. Second, comprehensive datasets track the exit of firms rather than products, and firms exit for a variety of reasons. Therefore, I need a model to guide the identification of this spillover-driven product exit from the observable dynamics of firm exit.

To meet these requirements, I develop a new model of long run growth and firm dynamics. It features product innovation, where the quality of each new product (variety) is drawn from an entry distribution. Such innovations increase the aggregate stock of knowledge, and spillovers are modeled as a higher knowledge stock leading to a first-order stochastic dominance improvement in the entry distribution (Kortum, 1997). While these innovations allow firms to expand, the endogenous shutdown of existing products, à la Hopenhayn (1992), acts as a countervailing force, culminating in a firm's exit upon the shutdown of its final product. These product exits are driven both by the accumulation of idiosyncratic shocks to quality and competition from newer—and on average better—products that are imperfect substitutes for existing ones.

The model delivers an intuitive characterization of the wedge between social and private returns to R&D created by knowledge spillovers. Along the balanced growth path of my model, the social rate of return to R&D exceeds the private one. The resulting wedge is the product of two terms: (i) the spillover elasticity, governing the extent to which a higher stock of knowledge improves the entry distribution, and (ii) the pace of knowledge accumulation, which, in my semi-endogenous growth model, is tied to the rate of population growth.

Dynamics of product exit are informative about this wedge. The key is that, because of a labor denominated overhead cost, a product exits when its *relative* quality falls below an endogenous threshold. By accelerating quality growth among new products, stronger spillovers accelerate the downward drift of an incumbent product's relative quality. It is precisely this component of product exit, due to drift down to the exit threshold, that is informative about the wedge of interest.

In the absence of comprehensive data on product exit, the profile of firm exit by age is informative about this wedge. The reason is that product dynamics aggregate to determine firm dynamics: a firm starts with a single product, attempts to grow its portfolio via R&D, and exits upon shutdown of its final product. As a result, the model's profile of firm exit by age is governed by three sufficient statistics: (i) the extent of product exit due to downward drift (the component informative about the wedge), (ii) the extent of product exit due to shocks, and (iii) the endogenous rate at which an incumbent firm adds a product to its portfolio. These three statistics are separately identified because they leave distinct signatures on the firm exit hazard over its life cycle: (i) a higher product exit due to downward drift raises firm exit at all ages, (ii) a higher product exit due to shocks increases exit among young firms but, due to selection, decreases it among older ones, and (iii) a higher rate of product addition has little effect on young single product firms but lowers the exit for mature firms by increasing their average number of products.

As this identification approach hinges on the model's mapping between product and firm dynamics, I provide empirical evidence bolstering confidence in my modeling choices. I document four facts using the NielsenIQ Retail Scanner dataset, which provides high-frequency data at the barcode (UPC) level for consumer packaged goods. First, products with higher sales are less likely to exit. Second, the product exit rate is lower among firms with more products. Third, in the lead up to exit, a product's sales decline gradually. Finally, this gradual decline is driven by a collapse in quantity sold while relative price falls only modestly, a pattern consistent with a negative residual demand shock to the product. Taken together, these facts provide support for the model's treatment of product exit: it features negative selection and unfolds gradually. As product exit is at the heart of the mapping between firm and product dynamics, this evidence lends crucial credibility to the identification strategy.

I apply this novel approach to quantifying spillovers using data covering the universe of U.S. private nonfarm businesses. Targeting the profile of firm exit at ages 1 through 19, I find that knowledge spillovers create a 16 percentage point wedge between the social and private rates of return to R&D. Framing the result as a social-private wedge in the rate of return is advantageous for two main reasons. First, it bypasses the need to take a stance on the relevant rate of population growth and makes my finding more robust to alternative specifications of the law of motion for the aggregate stock of knowledge. Second, it facilitates comparison with the canonical estimate due to Bloom, Schankerman and Van Reenen (2013). While the gap I estimate is more conservative than their 20-45 percentage point range, it is still substantial. In canonical models, such a gap leads to vast underprovision of innovation (Jones and Williams, 1998) and means there are potentially large welfare benefits from government support for R&D (Atkeson and Burstein, 2019).

I substantiate this headline finding through a series of validation and robustness exercises. First, to assuage the concern that a taste for novelty drives my results, I show that the estimated wedge tends to be largest in sectors where narrative evidence points to an important role for knowledge spillovers. Second, to bolster confidence in the quantitative aspects of my results, I check the model's quantitative fit to untargeted moments. Finally, I verify that for food manufacturing, where both product and firm level data are available, I obtain similar estimates of the wedge when doing the quantification with product versus firm level data.

After highlighting my contribution to the literature, the rest of the paper proceeds as follows. Section 2 lays out the model, derives the wedge between social and privates rates of return to R&D, and formalizes how the dynamics of product exit are informative about this wedge. Section 3 then characterizes the model's firm dynamics, a necessary step to quantify this wedge when only firm level data are available. To bolster confidence in the model's mapping between firm and product dynamics, Section 4 provides empirical evidence corroborating my treatment of product exit. Section 5 puts this apparatus to work using data on the U.S. private nonfarm businesses, presents the headline result as well as those from a series of robustness and validation exercises. Finally, Section 6 concludes.

Contribution to the literature. The paper's theoretical contribution is a new quantitative growth model with multiproduct firms and selection into exit at the product level. Creative destruction plays a prominent role in this model, which relates my work to the large literature on quality ladder models of Schumpeterian growth, building on Aghion and Howitt (1992) and Grossman and Helpman (1991), and reviewed in Aghion, Akcigit and Howitt (2014). In contrast to this literature, obsolescence unfolds gradually in my model because new and incumbent products are imperfect substitutes. This choice is motivated by the evidence I document using barcode level data, as well as the findings of Foster, Haltiwanger and Syverson (2008) and Argente, Lee and Moreira (2024). As a consequence, knowledge spillovers are the only source of inefficiency in my model. This is because, in addition to the usual negative business stealing externality, there is a positive consumer surplus externality due to love of variety (Acemoglu, 2009); with CES demand, the two externalities exactly offset, as in Melitz (2003). While this is a special property of the CES aggregator (Dhingra and Morrow, 2019), it is convenient for the purposes of isolating and quantifying knowledge spillovers.

In modeling product dynamics, I build on Luttmer (2007), who in turn builds on Hopenhayn (1992) and Melitz (2003). A first notable difference is that I allow for multiproduct firms. In addition to the evidence on the ubiquity of multiproduct firms (Bernard, Redding and Schott, 2010; Broda and Weinstein, 2010; Arkolakis, Ganapati and Muendler, 2021; Argente, Lee and Moreira, 2018; Berlingieri, De Ridder, Lashkari and Rigo, 2025), this enables lower volatility of firm growth among larger firms (Sutton, 2002; Arkolakis, 2016). The second is that Luttmer (2007) models entrants drawing from the incumbent distribution while I follow Kortum (1997) in having them draw from an endogenously shifting distribution. As a result, our models yield diverging comparative statics: in my model, higher volatility of shocks does not increase the growth rate, product entry and exit *rates* are not affected by the cost of entry, and the resulting growth rate is semi-endogenous.

My modeling of multiproduct firms builds on Klette and Kortum (2004) and Luttmer (2011): a firm is a collection of products, each of which, in equilibrium, gives "birth" to a new product at a constant Poisson rate. The key difference is that, because product exit is not a Poisson process in my model, the firm's number of products evolves as a non-Markovian branching process. Consequently, characterizing the stationary firm size distribution involves solving a challenging system of coupled partial differential equations. The payoff is that this modeling of product exit, and its consequences through the lens of my model, are corroborated by facts I document as well as empirical evidence from Bernard, Redding and Schott (2010) and Hottman, Redding and Weinstein (2016).

As my model features semi-endogenous growth (Jones, 1995a), it is consistent with the aggregate evidence on weak scale effects (Jones, 1995b; Peters, 2022), the firm level evidence on declining research productivity (Bloom, Jones, Van Reenen and Webb, 2020), and with the literature emphasizing the role of demographics in explaining the slowdown of business dynamism (Karahan, Pugsley and Şahin, 2024; Hopenhayn, Neira and Singhania, 2022). Unlike the second generation of endogenous growth models (Dinopoulos and Thompson, 1998; Peretto, 1998; Young, 1998; Howitt, 1999; Peters and Walsh, 2024; Aghion, Bergeaud, Boppart and Brouillette, 2025), my model features semi-endogenous growth even along the quality (vertical) dimension.

That the extent of spillovers has a bearing on dynamics in product markets is not a peculiar feature of my model. Akcigit and Ates (2023) argue declining spillovers can quantitatively explain a large share of the recent slowdown in business dynamism in the U.S., specifically the trends documented in Akcigit and Ates (2021). The model they use to reach this conclusion is quite different than mine, as it is a step-by-step innovation model, à la Aghion, Harris and Vickers (1997), Aghion, Harris, Howitt and Vickers (2001), and Aghion, Bloom, Blundell, Griffith and Howitt (2005), with Bertrand duopoly in each sector.

The paper's empirical contribution is the quantification of knowledge spillovers using this new theoretical framework. This relates my work to a large body of innovation research, reviewed in Bryan and Williams (2021), for which a central challenge is that knowledge flows are invisible. Apart from papers focusing on specific industries like Griliches (1958) and Irwin and Klenow (1994), the literature has relied on patent data to quantify spillovers. Specifically, following Jaffe, Trajtenberg and Henderson (1993), many have used patent citations as a paper trail for knowledge flows and hence a proxy for spillovers. Compared to my approach, an advantage of using patent citations is the ability to recover the network of spillovers (Acemoglu, Akcigit and Kerr, 2016), which is necessary to characterize optimal sector-specific (Liu and Ma, 2021) or firm-specific (König, Liu and Zenou, 2019) R&D subsidies. However, as Jaffe, Trajtenberg and Fogarty (2000) put it, patent citations are at best a noisy signal of spillovers; a conclusion they reach based on survey evidence on the familiarity of inventors with patents they cite. In fact, among U.S. patents granted between 2001 and 2003, Alcácer, Gittelman and Sampat (2009) find that examiners added, on average, 63% of citations and all citations in 40% of the cases, although Bryan, Ozcan and Sampat (2020) caveat that this is less of a concern for in-text (as opposed to front-page) citations.

Another common approach in this literature to "detect the path of spillovers in the sands of data" (Griliches, 1992, p. S36) is to leverage variation in proximity between firms. Specifically, Jaffe (1986) uses the patent classification system to define a technological proximity metric between firms, and then looks at the effect of a firm's R&D on R&D by its "technological neighbors". Bloom, Schankerman and Van Reenen (2013) enrich this approach by using line of business data from Compustat to complement technological proximity with proximity in product markets, which allows them to separately identify knowledge spillovers and business stealing. Lucking, Bloom and Van Reenen (2019) extend the analysis to more recent years and Lychagin, Pinkse, Slade and Van Reenen (2016) enrich this approach with a third dimension of proximity tied to geographic location. Zacchia (2019) uses an alternative definition of technological proximity between two firms, based on the share of the two firms' inventors who have previously co-patented across firms, and similarly finds a social rate of return twice as large as the private one. Arqué-Castells and Spulber (2022) show that accounting for voluntary technology transfers between firms reduces the gap between the social and private rates of return from 40 to 30 percentage points.

My approach is complementary to these and motivated by the literature acknowledging the limitations of patent and R&D data. First, the propensity to patent an invention varies across industries (Levin, Klevorick, Nelson and Winter, 1987): manufacturing accounts for 64% of patents issued to US companies and 57% of reported R&D (NCSES, 2023). Second, even within sectors where patenting is common, larger firms are more likely to patent an innovation (Cohen, Nelson and Walsh, 2000; Mezzanotti and Simcoe, 2023; Argente, Baslandze, Hanley and Moreira, 2023). Third, firms may strategically relabel expenses as R&D (Chen, Liu, Suárez Serrato and Xu, 2021), with these incentives varying over time as the tax code changes (Cowx, Lester and Nessa, 2024). My approach also complements Jones and Summers (2021), who show that the long run growth rate divided by the share of GDP spent on R&D is an estimate of the social rate of return to R&D.

2 Model

This section lays out the model and derives two key results. The first characterizes the wedge between social and private returns to R&D created by knowledge spillovers. The second formalizes how the dynamics of product exit are informative about the magnitude of this wedge. Appendix A summarizes all the symbols I introduce when setting up and solving the model.

2.1 Economic Environment

Preferences and Technology Time is continuous and population N_t grows at rate $\eta > 0$:

$$\dot{N}_t = \eta N_t \ . \tag{1}$$

Each individual inelastically supplies a unit of labor and household preferences are given by

$$U_0 = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \ ,$$

where consumption c_t is a CES aggregate over a continuum of imperfectly substitutable products

$$c_t = \left[\int_{p \in \Omega_t} \left(Q_{pt} c_{pt} \right)^{\frac{\sigma-1}{\sigma}} dp \right]^{\frac{\sigma}{\sigma-1}} \ .$$

Here, Q_{pt} is the quality of product p at t, c_{pt} its quantity consumed, $\sigma > 1$ the elasticity of substitution, and Ω_t the set of products supplied at t. This set evolves over time as a result of the creation of new products and the endogenous exit of existing ones.

Each of these products is supplied by one, potentially multiproduct, firm. The production technology features a per-product fixed overhead cost $\mathcal{F}$, denoted in labor units. If unpaid, production of that product is irreversibly shut down. The marginal cost of production is otherwise constant, so that quantity produced Y_{pt} is linear in production labor employed L_{pt}:

$$Y_{pt} = A L_{pt} \ .$$

Innovation For a product to be supplied, the underlying blueprint must have already been developed. Such innovations are carried out by entering as well as incumbent firms.

All firms begin as single product firms. There is a labor denominated firm entry cost, as an individual attempting to set up a new firm, by developing the blueprint for a new product, succeeds at Poisson rate ε. Subsequently, this single product firm can expand its portfolio through R&D. Hiring I_p R&D workers leads to the development of a new product at Poisson rate:

$$\iota \left(I_p \right) = \vartheta \frac{I_p^{1-\delta}}{1-\delta} \quad \text{where} \quad \delta \in (0,1) \ ; \vartheta > 0. \tag{2}$$

R&D investments by incumbents thus run into diminishing returns.

Once the firm becomes multiproduct, it operates as a collection of product lines. As with production, innovation is organized around existing blueprints: for each product p in its portfolio, a firm chooses how many R&D workers I_p to hire. So that a firm f currently producing the set of products $\{p_1, \ldots, p_n\}$ succeeds in adding a product to its portfolio with Poisson arrival rate

$$\sum_{p \in \{p_1, \ldots, p_n\}} \iota\left(I_p\right) \ .$$

Equivalently, one could write down, as in Klette and Kortum (2004), a firm level idea production function with constant returns to scale in total R&D labor and number of products. The product level formulation allows me to defer the introduction of firm level variables until Section 3.

Knowledge spillovers, which I seek to quantify, stem from the cumulative stock of innovation being an input into the production of new ideas. As such, it is going to be helpful to keep track of this stock. With innovation taking the form of product development, this is simply the cumulative mass of product blueprints already developed, which I denote by K_t. This stock is accumulated through the development of new blueprints by entrants and incumbents, so that:

$$\dot{K}_t = \varepsilon S_t + \int_{p \in \Omega_t} \iota\left(I_{pt}\right) dp \ . \tag{3}$$

The first summand reflects the flow of new blueprints developed by entrants, as S_t is the mass of individuals working as startup entrepreneurs and ε is the Poisson rate an entrepreneur succeeds. The second summand reflects the flow of new blueprints developed by incumbents.

Product Quality Whether developed by an entrant or incumbent, the quality of a new product at t is drawn from an entry distribution with countercumulative distribution function (CCDF):

$$\overline{F}_t^E(Q) = \Pr\left(\text{Draw}_t > Q\right) = \begin{cases} K_t^{\theta} Q^{-\alpha} & \text{if} \quad Q > K_t^{\frac{\theta}{\alpha}} \\ 1 & \text{otherwise} \end{cases} \quad \text{with} \quad \alpha > 0 \ \text{ and } \theta \geq 0.$$

In this specification, due to Kortum (1997), α is the time-invariant Pareto shape of the entry distribution, with a higher value for α corresponding to a thinner tail. θ is the spillover elasticity: it governs how strongly a higher cumulative stock of innovation improves – in a first-order stochastic dominance sense – the distribution of quality of new products. For example, when there are no spillovers, $\theta = 0$ and the entry distribution is time-invariant. In contrast, when $\theta > 0$, the flow of new blueprints today increases K_t which in turn improves the entry distribution inventors draw from in the future, as shown in Figure 1.

After this initial draw, the product's quality Q_{pt} evolves as a geometric Brownian motion

$$d \ln Q_{pt} = \beta dt + \nu dB_{pt} \ ,$$

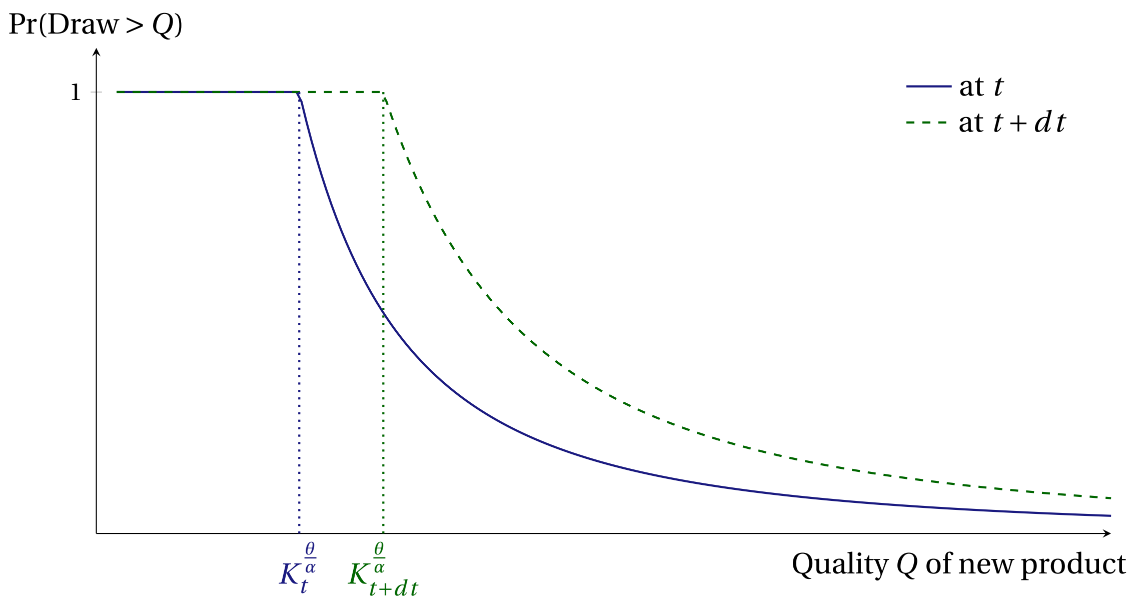


Figure 1: Improvement in entry distribution due to spillovers when $\theta > 0$

with Brownian increments that are independent across the continuum of products.

Resource Constraints For each product p, aggregate consumption cannot exceed production:

$$N_t c_{pt} \leq Y_{pt} \ .$$

Labor is the only productive input in the economy. Each individual inelastically supplies a unit of labor and can work for an incumbent as a production, overhead, or R&D worker. Alternatively, the individual can choose to be a startup entrepreneur and attempt to create a new firm. The resource constraint on labor is thus given by:

$$\int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt}\right) dp + S_t = N_t \ . \tag{4}$$

Table 1 summarizes the economic environment of the model. The allocation decision in this economy is how to divide labor among its competing uses, which governs how the set of products Ω_t consumed by households evolves over time. To appreciate the economic trade-offs involved, it is helpful to consider the benefit of a marginal unit of labor in each of its uses. On the one hand, allocating more labor to production increases the output of products currently supplied, while allocating more labor to overhead expands the range of products that remain active. Because of the love of variety effect, both margins enhance current consumption. On the other hand, allocating more labor to R&D or entry expands the stock of blueprints, thereby increasing the number and improving the quality of products available for consumption in the future.

Table 1: Economic Environment

Population	$\dot{N}_t = \eta N_t$	$\eta > 0$
Preferences	$U_0 = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma}-1}{1-\gamma} dt$	$\rho > \eta$
Consumption	$c_t = \left[\int_{p\in\Omega_t} (Q_{pt} c_{pt})^{\frac{\sigma-1}{\sigma}} dp\right]^{\frac{\sigma}{\sigma-1}}$	$\sigma > 1$
Production	Overhead $\mathcal{F}$ (exit if unpaid)	$\mathcal{F} > 0$
	$Y_{pt} = A L_{pt}$	
Flow of new blueprints	$\dot{K}_t = \varepsilon S_t + \int_{p\in\Omega_t} \frac{\vartheta}{1-\delta} I_{pt}^{1-\delta} dp$	$\vartheta > 0$; $0 < \delta < 1$
Entry distribution	$\overline{F}_t^E(Q) = K_t^\theta Q^{-\alpha}$	$\theta \geq 0$; $\alpha > 0$
Set of products	Ω_t evolves through entry & exit	
Quality evolution	$d \ln Q_{pt} = \beta dt + \nu dB_{pt}$	$\nu \geq 0$
Resource constraints	$N_t c_{pt} = Y_{pt}$	
	$S_t + \int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt}\right) dp = N_t$	

Notes: The scarce resource to allocate in this economy is labor. The decision of how to divide this resource among its competing uses in production (L), overhead ($\mathcal{F}$), incumbent innovation (I), and entry (S) governs how the the set of products Ω_t consumed evolves over time.

2.2 Decision Problems

To pin down how labor is allocated across its competing uses, I consider the following market structure. Each differentiated product is supplied by a single, potentially multiproduct, monopolistically competitive firm. There are no barriers to firm entry and the frictionless labor market is perfectly competitive. The consumption bundle serves as the numeraire so its price is normalized to unity. In terms of assets, in addition to the blueprints which entitle their owners to a stream of dividends, there is a risk-free bond in zero-net supply.

I characterize equilibrium labor allocations and the evolution of the product set Ω_t without explicitly tracking firms. This is possible because a firm's problem is separable across its products. On the technology side, there are no production or innovation synergies across a firm's portfolio. On the demand side, firms are atomistic and the elasticity of substitution is constant across products, so there are no cannibalization effects. The distribution of firms matters only for outcomes such as firm entry and exit rates, which I return to in Section 3.

Household's problem Given a path of interest rates r_t, wages w_t, and product prices P_{pt}, the household chooses consumption of different products p to maximize lifetime utility subject to an intertemporal budget constraint:

$$\max_{\{c_{pt}\}} \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma}-1}{1-\gamma} dt \qquad \text{subject to} \qquad \begin{aligned} c_t &= \left[\int_{p\in\Omega_t} \left(Q_{pt}c_{pt}\right)^{\frac{\sigma-1}{\sigma}} dp\right]^{\frac{\sigma}{\sigma-1}} \\ \dot{a}_t &= (r_t-\eta)\, a_t + w_t - \int_{p\in\Omega_t} P_{pt}c_{pt}dp \end{aligned} \tag{5}$$

where a_t is the individual's asset holding. Summing c_{pt} across individuals yields the following demand schedule for product p:

$$Y_{pt} = Q_{pt}^{\sigma-1} P_{pt}^{-\sigma} N_t c_t \ .$$

Incumbent's problem For each product in its portfolio, a firm chooses pricing, production, and R&D to maximize the expected present discounted value of dividends from that product. These dividends consist of net profits from operating the product and the option value associated with expanding the firm's portfolio through R&D. They accrue until the firm optimally chooses to permanently shut down production with this product. With $V_t(Q_{pt})$ the value at time t of a product with quality Q_{pt}, the firm solves:

$$V_t(Q_{pt}) = \max_{\substack{T \\ \{P_{pt}\} \\ \{I_{pt},L_{pt}\}}} \mathbb{E}_t \left\{ \int_t^T e^{-\int_t^\tau r_s ds} \left[\underbrace{P_{p\tau}Y_{p\tau} - w_\tau \left(\mathcal{F} + L_{p\tau}\right)}_{\text{operating profit}} + \overbrace{\underbrace{\vartheta \frac{I_{p\tau}^{1-\delta}}{1-\delta}}_{\substack{\text{arrival rate of} \\ \text{new product}}} \underbrace{\int V_\tau(Q)\, dF_t^E(Q)}_{\substack{\text{expected value} \\ \text{of new product}}} - w_\tau I_{p\tau}}^{\text{option value}} \right] d\tau \right\}$$

$$\text{subject to, for all } \ t \leq \tau < T, \quad \begin{cases} Y_{p\tau} = AL_{p\tau} \\ Y_{p\tau} = Q_{p\tau}^{\sigma-1} P_{p\tau}^{-\sigma} N_\tau c_\tau \\ d\ln Q_{p\tau} = \beta d\tau + \nu dB_{p\tau} \end{cases} \tag{6}$$

The value of a firm is obtained by summing the value of products in its portfolio.

For endogenous exit to occur, the option value of R&D expansion must not exceed the overhead cost. In essence, the effective fixed cost is the overhead net of the option value, and it must be strictly positive for some products to be optimally shut down. Assumption 1 below provides a condition on parameters that guarantees this holds along the equilibrium path. As in Hopenhayn (1992), endogenous exit is then characterized by a quality threshold, denoted $\underline{Q}_t$, below which firms choose to stop production.

Firm Entry Since there are no barriers to entry, individuals must be indifferent between employment and entrepreneurship in equilibrium. With the wage as the outside option, this requires:

$$S_t \left(w_t - \varepsilon \int V_t(Q) dF_t^E(Q) \right) = 0 \,. \tag{7}$$

Along an interior equilibrium ($S_t > 0$), the expected payoff from attempting entry equals the wage.

2.3 Equilibrium

The model's dynamics center on the evolution of the product set Ω_t. Since quality is the only relevant source of heterogeneity, this set can be represented by a distribution over product qualities. With stationarity in mind though, instead of absolute quality, it is helpful to work with a product's quality relative to the exit threshold in log as its state variable:

$$q_{pt} = \ln \left(\frac{Q_{pt}}{\underline{Q}_t} \right) \,.$$

I denote the corresponding cross-sectional measure by $m(q, t)$. Using Ito's lemma,

$$dq_{pt} = \left(\beta - g_{Q_t} \right) dt + \nu dB_{pt} \,, \tag{8}$$

where g_{Q_t} the instantaneous growth rate of $\underline{Q}_t$ at t. It follows that $m(0, t) = 0$ and for $q > 0$ the law of motion for $m(q, t)$ is given by the following Kolmogorov Forward Equation (KFE):

$$\dot{m}(q, t) = \underbrace{- \left(\beta - g_{Q_t} \right) \frac{\partial m(q, t)}{\partial q}}_{\text{Drift}} + \underbrace{\frac{\nu^2}{2} \frac{\partial^2 m(q, t)}{\partial q^2}}_{\text{Diffusion}} + \underbrace{\dot{K}_t K_t^\theta \underline{Q}_t^{-\alpha} \alpha e^{-\alpha q} \, \mathbb{1}_{\left\{ q \geq \frac{\theta}{\alpha} \ln K_t - \ln \underline{Q}_t \right\}}}_{\text{Entry}} \,. \tag{9}$$

Here, the measure of products entering with log-relative quality q is simply the flow of new blueprints $\dot{K}_t$ multiplied by the density of draws at that relative quality.

Definition 1. Given initial population N_0, stock of existing blueprints K_0, and distribution of product log-relative qualities $m(q, 0)$, an equilibrium consists of time paths for N_t, K_t, $m(q, t)$, $\underline{Q}_t$, prices $\{r_t, w_t, P_{pt}, V_t(Q_{pt})\}$ and allocations $\{c_t, a_t, c_{pt}, Y_{pt}, L_{pt}, I_{pt}, S_t\}$ such that for all t:

1. c_t, a_t, and c_{pt} solve the household's problem (5);
2. L_{pt}, I_{pt}, P_{pt}, $V_t\left(Q_{pt}\right)$, and $\underline{Q}_t$ solve the incumbent's problem (6);
3. S_t satisfies the free entry condition (7) ;
4. Y_{pt} satisfies the market clearing condition for product p, $Y_{pt} = N_t \, c_{pt}$;

5. w_t clears the labor market (Equation 4);
6. r_t clears the asset market, $N_t a_t = \int V_t \left(\underline{Q}_t e^q \right) m(q,t) dq$;
7. N_t, K_t, and $m(q,t)$ evolve respectively according to Equation 1, Equation 3, and Equation 9.

Solving. In equilibrium, consumption per capita and the wage rate are then given by

$$c_t = A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t \frac{L_t}{N_t} \qquad \text{and} \qquad w_t = \frac{\sigma-1}{\sigma} A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t \,, \tag{10}$$

where L_t denotes aggregate production labor, M_t the measure of products supplied and $\overline{Q}_t$ the power mean of their qualities:

$$M_t \equiv \int_0^\infty m(q,t) dq \;\text{ and }\; \overline{Q}_t \equiv \left(\frac{1}{M_t} \int_{p \in \Omega_t} Q_{pt}^{\sigma-1} dp \right)^{\frac{1}{\sigma-1}} = \underline{Q}_t \left(\frac{1}{M_t} \int_0^\infty e^{(\sigma-1)q} m(q,t) dq \right)^{\frac{1}{\sigma-1}} .$$

This highlights the two potential sources of productivity growth: expanding varieties, which contributes to growth due to love of variety, and increasing average quality of products consumed.

Moving on to the model's dynamics, note that the innovation technology in Equation 2 satisfies an Inada condition at 0, so that an incumbent always finds it optimal to do some R&D. Along an interior equilibrium ($S_t > 0$), this level of incumbent innovation is constant:

$$\left. \begin{aligned} \text{Incumbent's FOC:} \quad w_t &= \vartheta I_{pt}^{-\delta} \int V_t(Q) \, \mathrm{d}F_t^E(Q) \\ \text{Free entry condition:} \quad w_t &= \;\;\varepsilon \;\; \int V_t(Q) \, \mathrm{d}F_t^E(Q) \end{aligned} \right\} \implies I_{pt} = I \;\equiv\; \left(\frac{\vartheta}{\varepsilon} \right)^{\frac{1}{\delta}} . \tag{11}$$

So that the flow of new blueprints at t is simply

$$\dot{K}_t = \varepsilon S_t + \frac{\varepsilon}{1-\delta} I M_t \,. \tag{12}$$

The scaling by $(1-\delta)^{-1} > 1$ reflects that while the marginal R&D worker has to be as productive as a startup entrepreneur in equilibrium, the infra-marginal ones will be more productive. By plugging the equilibrium level of incumbent R&D back into the firm's objective function yields an option value from R&D expansion equal to $O w_t$ where $O \equiv \frac{\delta}{1-\delta} \left(\frac{\vartheta}{\varepsilon} \right)^{\frac{1}{\delta}}$.

Assumption 1. Endogenous exit requires $\mathcal{F} > O$, so the parameters need to satisfy:

$$\mathcal{F} > \frac{\delta}{1-\delta} \left(\frac{\vartheta}{\varepsilon} \right)^{\frac{1}{\delta}} .$$

Finally, denoting by $V(q,t)$ the value at t of a product with log-relative quality q, this value function satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:[1]

$$V(0,t) = \frac{\partial V(0,t)}{\partial q} = 0 \text{ and } \forall q > 0, \;\; r_t V(q,t) = w_t \left[\frac{1}{\sigma - 1} \frac{L_t}{M_t} \left(\frac{\underline{Q}_t}{\overline{\overline{Q}}_t} \right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O) \right] + \dot{V}(q,t) + \left(\beta - g_{Q_t}\right) \frac{\partial V(q,t)}{\partial q} + \frac{\nu^2}{2} \frac{\partial^2 V(q,t)}{\partial q^2} \,. \tag{13}$$

2.4 Balanced Growth Path (BGP)

Definition 2. A balanced growth path (BGP) is an allocation with

1. stationary labor allocations: $\frac{\dot{L}_t}{L_t} = \frac{\dot{S}_t}{S_t} = \frac{\dot{M}_t}{M_t} = \eta$;
2. growth in the quality threshold $\underline{Q}_t$ at constant rate $g_Q \geq 0$;
3. a stationary distribution of relative qualities:

$$m(q,t) = f_p(q) M_t \;, \text{ with } f \text{ a probability density function on } (0, \infty) \,.$$

Proposition 1. Along a BGP,

$$\frac{\dot{K}_t}{K_t} = \eta \quad ; \quad g_Q = \frac{\theta}{\alpha}\eta \quad \text{and} \quad g \equiv \frac{\dot{c}_t}{c_t} = \underbrace{\frac{\eta}{\sigma - 1}}_{\text{Variety}} + \underbrace{\frac{\theta}{\alpha}\eta}_{\text{Quality}}$$

Proof. Using the KFE, a time invariant $f_p(q)$ requires $K_t^{\theta} \underline{Q}_t^{-\alpha}$ to be constant. But $\underline{Q}_t$ grows at constant rate g_Q, so K_t grows at constant rate. That this constant growth rate is η is obtained from Equation 12 combined with stationarity of labor allocations. It follows that $\underline{Q}_t$ grows at rate $\frac{\theta}{\alpha}\eta$. The expression for g is then obtained from Equation 10. □

Assumption 2. Finite lifetime utility and firm value require

$$\rho > \eta + (1-\gamma) g \quad ; \quad (\sigma - 1)\left(\beta - g_Q\right) + \frac{1}{2}(\sigma-1)^2 \nu^2 < \eta \quad \text{and} \quad \alpha > \sigma - 1 \,.$$

To build intuition for the rate of quality growth g_Q, notice first that it does not depend on the drift β and volatility ν. While these forces affect growth in sales for an incumbent product, they do not contribute to aggregate growth along a path with a stationary distribution of relative quality. Intuitively, the reason is that the gains experienced by a product are "lost" once the product exits.

[1] Using the definition of a product's value from Equation 6, $V(q,t) \equiv V_t\left(\underline{Q}_t e^q\right)$.

It is also noteworthy that when $\theta = 0$, there is no quality growth along the BGP. This case corresponds to a time-invariant entry distribution. Along the BGP, the (un-normalized) distribution of incumbent quality is then itself time-invariant, as in the stationary equilibrium of Hopenhayn (1992) and Melitz (2003). The sole driver of aggregate productivity growth in this case is the expanding measure of products consumed.

This highlights that the driver of quality growth in this economy is the improvement in the entry distribution. When $\theta > 0$, this improvement happens as a result of the endogenous accumulation of knowledge (blueprints). In this case, as shown in Figure 1, exponential growth in the stock of blueprints K_t at rate η shifts the entry distribution to the right at rate $\frac{\theta}{\alpha}\eta$. The resulting quality growth rate is increasing in η (knowledge accumulated faster) and θ (stronger spillovers), but decreasing in α. Intuitively, when the tail of the entry distribution is thinner (higher α), good ideas are harder to find, so that a given spillover strength θ shifts out the distribution less.

Sustained quality growth thus requires a growing population and positive spillovers. The first is common in semi-endogenous growth models (Jones, 2022): if new ideas drive growth and individuals come up with ideas, then the economy's growth rate is tied to the rate of population growth. That $\theta > 0$ is necessary to sustain quality growth in this model stands in contrast to Kortum (1997). Key to understanding the difference is that, while these two semi-endogenous growth models share the same entry distribution, the mass of products in Kortum (1997) is fixed as new products perfectly substitute for older ones. As a result, what matters for the rate of quality growth in that setup is the maximum draw. Even if the distribution is time invariant, Kortum (1997) shows that exponential growth in the number of draws (due to population growth) from a Pareto distribution leads to exponential growth in the maximum draw. In contrast, what matters for the rate of quality growth in my setting is the average draw from the entry distribution, and for that to grow over time, the entry distribution has to shift out, which requires $\theta > 0$.

While indicative about the *presence* of spillovers ($\theta > 0$), quality growth alone is not informative about their *magnitude*. Rapid quality growth can reflect strong spillovers (high θ) and/or a thick-tailed entry distribution (low α). In addition to this identification challenge, quality growth is notoriously challenging to measure (see for example Bils and Klenow (2001), Bils (2009), and Aghion, Bergeaud, Boppart, Klenow and Li (2019)). This is why, to quantify the magnitude of knowledge spillovers as encoded by the elasticity θ, I rely on the model's predictions about product and firm dynamics. I now turn to characterizing product dynamics along the BGP, and return to firm dynamics in Section 3.

Assumption 3. The parameters are such that, along the BGP,

$$\underline{Q}_t \geq K_t^{\theta/\alpha} .$$

In words, this corresponds to the case where, at any point in time along the BGP, the threshold to supply a product ($\underline{Q}_t$) is in the support of the distribution entrants draw from (which has lower bound $K_t^{\theta/\alpha}$). Proposition 1 guarantees that these grow at the same rate, and in Appendix B.1.3, I show that this ordering obtains when the present discounted value of the effective fixed cost of operation is not too small relative to the entry cost.[2]

Proposition 2. The stationary distribution of product log-relative quality is

$$f_P(q) = \frac{\alpha\zeta}{\zeta - \alpha}\left(e^{-\alpha q} - e^{-\zeta q}\right) \quad \text{where} \quad \zeta \equiv \frac{g_Q - \beta + \sqrt{(g_Q - \beta)^2 + 2\eta\nu^2}}{\nu^2},$$

and the stationary product entry and exit rates are respectively

$$\frac{E_t}{M_t} = \eta + \frac{\nu^2}{2}\alpha\zeta \quad ; \quad \frac{D_t}{M_t} = \frac{\nu^2}{2}\alpha\zeta\ .$$

Proof. *Follows from solving the KFE along the BGP. Details in Appendix B.1.1* □

It follows that the incumbent quality distribution (in levels) has a Pareto tail with index $\min\{\alpha, \zeta\}$. While α is inherited from the entry distribution, ζ is the Luttmer (2007) tail that arises endogenously as a result of the geometric Brownian motion. This Pareto tail reflects the positive selection of products that have accumulated favorable Brownian shocks over time (as those that accumulate negative shocks get shut down). As such, it is unsurprising that higher volatility ν makes this tail thicker (smaller ζ). In contrast, faster population growth (η) or higher growth in entrant's quality relative to incumbents ($g_Q - \beta$) make this endogenous tail thinner (larger ζ). Intuitively, both of these forces increase the share of products that are young and which thus haven't had enough time to accumulate as many favorable Brownian shocks.

A distinctive feature of the model – in particular relative to the model with endogenous growth in Luttmer (2007) – is that stationary product entry and exit rates do not depend on the cost of entry (ε, ϑ, and δ). The reason is that, along the BGP, the cost of entry has a proportional effect on the *flow* of entry E_t and the measure of products supplied M_t, leaving the stationary entry rate unaffected. Put differently, "cheaper" entry makes innovation more appealing at all times, equally raising the flow of entry and the measure of products, as the latter reflects cumulative past entry.

Proposition 3. Along the BGP, the value of a product with log-relative quality q is

$$V(q,t) = w_t\mathcal{V}(q) \quad \text{where} \quad \mathcal{V}(q) \equiv \frac{\mathcal{F} - O}{r - g}\left[\frac{\xi}{\xi + \sigma - 1}e^{(\sigma-1)q} + \frac{\sigma - 1}{\xi + \sigma - 1}e^{-\xi q} - 1\right]$$

$$\xi \equiv \frac{\beta - g_Q + \sqrt{(\beta - g_Q)^2 + 2\nu^2(r - g)}}{\nu^2},$$

and average production labor per product is

$$\frac{L_t}{M_t} = (\mathcal{F} - O)\frac{\alpha\zeta(\sigma - 1)}{(\alpha - (\sigma - 1))(\zeta - (\sigma - 1))}\frac{\xi}{\xi + \sigma - 1}\frac{r - [g + (\sigma - 1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma - 1)^2]}{r - g}$$

[2]The condition on parameters is given in Equation 22.

Proof. *Follows from solving the HJB along the BGP. Details in Appendix B.1.2* □

With both L_t/M_t and I_t pinned down, one more equation is needed to fully characterize the stationary labor allocations, and it will follow from the free entry condition:

$$\varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^{\infty} V(q,t)\alpha e^{-\alpha q} dq = w_t \overset{\text{Proposition 3}}{\Longrightarrow} \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^{\infty} \mathcal{V}(q)\alpha e^{-\alpha q} dq = 1 \tag{14}$$

Intuitively, ε is the arrival rate of a new blueprint, $K_t^{\theta} \underline{Q}_t^{-\alpha}$ is the probability that a new blueprint leads to entry (quality draw is above threshold), and the integral gives the expected value of a new product conditional on entry. As I show in Appendix B.1.3, this can be rewritten as

$$\frac{S_t}{M_t} + \frac{I}{1-\delta} = \left(\eta + \frac{\nu^2}{2}\alpha\zeta\right) \int_0^{\infty} \mathcal{V}(q)\alpha e^{-\alpha q} dq \ ;$$

which makes clear that the free entry conditions pins down the level of innovation along the BGP.

2.5 Efficiency

Having characterized the stationary allocations arising in equilibrium, I now turn to assessing their efficiency. To do so, I solve for the first best.

I delegate the detailed setup and solution of the planner's problem to Appendix B.2, and instead focus here on comparing the resulting BGP to the equilibrium BGP. Given the semi-endogenous nature of the model, the growth rates are as in Proposition 1 and the stationary distribution of product qualities as in Proposition 2. Contrasting the conditions pinning down stationary labor allocations in the first best (FB) vs equilibrium (DE) reveals the following.

Proposition 4. If $\theta = 0$, the equilibrium BGP is efficient. In contrast, if $\theta > 0$, then the equilibrium BGP features too little innovation with

$$\left(\frac{L_t}{M_t}\right)^{\mathbf{DE}} = \left(\frac{L_t}{M_t}\right)^{\mathbf{FB}} \quad \text{and} \quad S_t^{\mathbf{DE}} < S_t^{\mathbf{FB}}\ ,\ M_t^{\mathbf{DE}} > M_t^{\mathbf{FB}}\ , L_t^{\mathbf{DE}} > L_t^{\mathbf{FB}}\ .$$

The divergence is due to differing free entry conditions, which I reproduce below:

$$\textbf{Equilibrium:} \quad w_t = \qquad \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^{\infty} w_t \mathcal{V}(q)\alpha e^{-\alpha q} dq$$

$$\textbf{First Best:} \quad \omega_t = \left(1 + \underbrace{\frac{\theta\eta}{\rho + (\gamma-1)g}}_{\text{wedge in PDV units}}\right) \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} \int_0^{\infty} \omega_t \mathcal{V}(q)\alpha e^{-\alpha q} dq$$

Here ω_t is the Lagrange multiplier on the labor resource constraint in the planner's problem, whereas w_t is the wage; both of which cancel. So the only difference is the term upfront on the right hand side. Given $\eta > 0$, these conditions coincide if and only $\theta = 0$. Otherwise, when $\theta > 0$, for a given K_t and $\underline{Q}_t$, the planner's expected value from entry is higher. This is because the planner internalizes the positive externality arising from knowledge spillovers. In fact, the additional term has a very natural Pigouvian interpretation. The externality consists of any individual failing to internalize that their innovation raises the aggregate stock of knowledge K_t. Along the BGP, this stock grows at rate η, which improves the distribution future entrants draw from at rate $\theta\eta$. The denominator corresponds to taking the present discounted value (it is simply $r - g$), as the benefits of a higher stock of knowledge last forever. As a result of this externality, there is underprovision of innovation in equilibrium.[3]

Contrasting these free entry conditions sheds light on how the planner can decentralize the first best. The policy instrument needs to incentivize the marginal individual to be a startup entrepreneur rather than working for an incumbent firm. One such policy is a profit subsidy at rate $\frac{\theta\eta}{r-g}$, which aligns the social and individual valuation of entry. Since labor is inelastically supplied, an isomorphic tool is a tax on labor income.

In practice, governments subside innovation through a wide array of policies (Bloom, Van Reenen and Williams, 2019). With that in mind, to interpret the magnitude of spillovers I estimate, I report the corresponding gap between the social and private rates of return to R&D along the equilibrium BGP. The no-arbitrage condition implies that the private rate of return is r. In contrast, the social rate of return is higher and given by:

$$\boxed{\text{Social Rate of Return to R\&D along Equilibrium BGP} = r + \theta\eta}$$

That the gap should be $\theta\eta$ can be intuitively seen from comparing the above free entry conditions (after converting from present discount value units to annual rates of return). The formal derivation is in Appendix B.3. I follow Jones and Williams (1998) and define the social rate of return to R&D as the rate of return on a variational argument around the BGP. The variational argument consists of, first, momentarily increasing resources dedicated to innovation, then, "eating the proceeds" by sufficiently decreasing resources dedicated to innovation to go back to the initial BGP.

That the only source of inefficiency is knowledge spillovers stands in contrast to quality ladder models of Schumpeterian growth (Aghion and Howitt, 1992; Grossman and Helpman, 1991). While my model does feature the negative business stealing externality, it also features a positive consumer surplus externality due to love of variety, as new products are imperfect substitutes for older ones. Given the CES aggregator, the two externalities exactly offset, as in Melitz (2003). While this is a special property of the CES aggregator (Dhingra and Morrow, 2019), it is convenient for the purposes of this paper as the goal is to isolate and quantify knowledge spillovers.

[3] Along an interior BGP, this underprovision shows up entirely along the entry of new firm margin, and the per-product incumbent innovation is efficient. I discuss the underlying intuition in detail in the appendix.

2.6 From product dynamics to knowledge spillovers

My approach to quantifying spillovers is to estimate the wedge $\theta\eta$, and I now shed light on why, through the lens of the model, product dynamics are informative about this wedge. To do so, note that the product exit rate from Proposition 2 can be equivalently rewritten as:

$$\frac{D_t}{M_t} = \frac{1}{2}(\theta\eta - \alpha\beta) + \frac{1}{2}\sqrt{(\theta\eta - \alpha\beta)^2 \ + \ 2\eta\alpha^2\nu^2} \ . \tag{15}$$

Special case with no drift and volatility ($\beta = \nu = 0$). In this case, after being developed, a product's quality is constant. Over time though, if $\theta > 0$, this incumbent competes with an increasingly higher quality pool of (younger) products. This gradually erodes the incumbent product's market share and eventually drives it out of business once its relative quality falls below the threshold needed to cover overhead costs. The resulting product exit and entry rates are:

$$\text{Product Exit Rate}\Big|_{\beta=\nu=0} \ = \ \theta\eta \qquad = \text{gap between private and social rates of return}$$

$$\text{Product Entry Rate}\Big|_{\beta=\nu=0} \ = \ (1+\theta)\eta$$

A higher elasticity θ raises both the entry and exit rates. Intuitively, when spillovers are stronger, entrants' quality improves more quickly so that incumbents lose market share at a faster pace and exit at higher rate.

With this intuition in mind, it might seem puzzling that a thicker tail (lower α) does not increase the exit rate, since it similarly increases the rate at which entrant's quality grows. To reconcile these two observations, it is important to recognize that the exit rate is the product of two terms: the growth rate of quality and the density of the incumbent distribution near (in this special case, at) the lower bound. As shown above, the former is $g_Q = \frac{\theta}{\alpha}\eta$, and with $\beta = \nu = 0$, the latter is simply α, so that α cancels out. The intuition is that the incumbent distribution inherits the shape of the entry distribution, so when the entry distribution has a thicker tail, the faster increase in quality is exactly offset by a lower density of incumbents near the exit threshold.

In this special case, one can recover the wedge $\theta\eta$ from the stationary product exit rate alone. The use of *rates* rather than *levels* is key for identification. The reason is that the measure of products (denominator in this rate) reflects cumulative past entry. So, while a host of factors in the model make entry more appealing, they raise the flow of entry and exit as well as the measure of products proportionally, leaving the entry and exit rates unchanged.

General case with volatility and drift ($\beta \neq 0$ and $\nu > 0$). The reason I entertain volatility in the model is precisely to introduce product exit not due to spillovers. From Equation 15, higher volatility increases the product exit rate; this effect is stronger when population growth is faster (larger η) and when the entry distribution is thinner (larger α)—as both of these increase the density of products near the exit threshold (Proposition 2). With the product exit rate "contaminated" by exit due to volatility, I leverage the profile of product exit by age for identification.

Proposition 5. The hazard rate of a product exiting at age a is

$$d_p(a) = \frac{\ell(a)}{1 - \int_0^a \ell(\tau) d\tau} \ ,$$

where $\ell(a)$ is the density of a product's lifespan (i.e. age at death) and is given by:

$$\ell(a) = \exp\left(a\left(\frac{(\alpha\nu)^2}{2} - \alpha\left(g_Q - \beta\right)\right)\right)\left(\frac{\alpha\nu}{\sqrt{a}}\phi\left(z_a\right) - \left((\alpha\nu)^2 - \alpha\left(g_Q - \beta\right)\right)\Phi\left(-z_a\right)\right)$$

with $\quad z_a \equiv \dfrac{\sqrt{a}}{\alpha\nu}\left((\alpha\nu)^2 - \alpha\left(g_Q - \beta\right)\right)$, $\phi(.)$ and $\Phi(.)$ standard normal PDF and CDF.

Proof. See Appendix B.4 for proof of Proposition 5 and Proposition 6. □

This shows that, in the model, $\alpha\left(g_Q - \beta\right) = \theta\eta - \alpha\beta$ and $\alpha\nu$ are sufficient statistics for the profile of product exit rate by age. Intuitively, the former captures the component of the product exit rate resulting from the deterministic downward drift toward the exit threshold, while the latter is a measure of relative volatility and governs the extent of product exit due to idiosyncratic shocks. It is the first of these two statistics that is informative about $\theta\eta$. By targeting the profile of product exit by age, this component of the product exit rate can be separately identified from exit due to shocks. The reason is that a higher $\theta\eta - \alpha\beta$ increases product exit rate at all ages, whereas a higher $\alpha\nu$ increases the product exit rate at young ages but decreases it at older ones. This non-monotonic effect of volatility on the exit rate by age is a consequence of selection: when volatility is higher, products that survive to old age are even more positively selected.

The challenge this approach runs into is that product level data are not widely available, so that my ultimate quantification exercise has to rely on firm level data. This requires characterizing the model's firm dynamics, which I now turn to.

3 Model's Firm Dynamics

Through the lens of the model laid out in Section 2, dynamics of product exit are informative about the magnitude of knowledge spillovers. The challenge, however, is that comprehensive datasets track the exit of firms rather than products. This section provides the necessary bridge, formalizing how, within the model, underlying product dynamics aggregate to observable firm dynamics.

3.1 A firm's life cycle

Because I did not track firms when characterizing the equilibrium in Section 2, it is helpful to start with a recap of a firm's life cycle *along the balanced growth path*. I continue to use a product's log-relative quality q_{pt} as its state variable.

A new firm f starts with a single product p with $q_{pt} \sim \text{Exp}(\alpha)$. Thereafter, q_{pt} evolves according to Equation 8 until it hits zero, at which point production of p is irreversibly shut down. While still produced, p has I R&D workers associated with it (see Equation 11). They generate a new blueprint at Poisson rate $\iota(I)$. This new blueprint leads to product entry when the quality draw is above the threshold $\underline{Q}_t$, so with probability $\overline{F}_t^E(\underline{Q}_t) = K_t^\theta \underline{Q}_t^{-\alpha}$, which is constant along the BGP (by Proposition 1). As a result, while still produced, p "gives birth" to a new product at rate x:

$$x = \iota(I)\,\overline{F}_t^E\left(\underline{Q}_t\right) = \frac{\varepsilon}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} K_t^\theta \underline{Q}_t^{-\alpha} = \frac{1}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} \frac{(r-g)(\alpha+\xi)(\alpha-(\sigma-1))}{(\mathcal{F}-O)(\sigma-1)} \ .$$

The log-relative quality of this new product is again drawn from $\text{Exp}(\alpha)$, then evolves according to Equation 8 (with Brownian increments independent across products); while active, it in turn gives birth to a new product at Poisson rate x.

Accordingly, the number of products firm f produces at ages $a \geq 0$, $\left\{n_f(a)\right\}_{a\geq 0}$, is a branching process. At age 0, f starts with a single product, so $n_f(0) = 1$. Firm f exits once all its products have become obsolete, so its age at exit, or lifespan, is the smallest a such that $n_f(a) = 0$:

$$a_f = \inf\left\{a \geq 0 \mid n_f(a) = 0\right\} \ .$$

This is a random variable for two reasons: product births arrive as a Poisson process, and each product's lifespan is itself random. Denoting by Γ the cumulative distribution function (CDF) of a_f, the hazard rate of firm f exiting at an age a is:

$$d_f(a) = \frac{\Gamma'(a)}{1-\Gamma(a)} \ .$$

Assumption 4. To guarantee that products and firms exit in finite time almost surely, I assume:

$$\beta \leq g_Q \quad \text{and} \quad 0 \leq x \leq \alpha\left(g_Q - \beta\right) \ .$$

Proposition 6. The CDF of a firm's lifespan satisfies $\Gamma(0) = 0$ and, for all $a > 0$,

$$\Gamma(a) = \int_0^a \ell(\tau)\exp\left(\int_0^\tau x\left[\Gamma(a-s)-1\right]ds\right)d\tau \ ,$$

where $\ell(.)$ is the density of a product's lifespan, given in Proposition 5.

In conjunction with Proposition 5, this highlights that, in the model, x, $\theta\eta - \alpha\beta$ and $\alpha\nu$ are sufficient statistics for the profile of firm exit by age. Intuitively, x controls the rate at which an incumbent firms adds a product to its portfolio, while $\alpha\left(g_Q-\beta\right)$ and $\alpha\nu$ the rates at which it loses a product (as discussed above). Note also that, computationally, solving for Γ only requires a single forward iteration on a fine age grid, as $\Gamma(a)$ only depends on $\{\Gamma(\tau):\tau<a\}$ and $\Gamma(0)=0$.

3.2 Evolution of the size distribution of firms

Having shed light on the life cycle of a single firm, I now turn to characterizing the evolution of the distribution of firms along the equilibrium BGP. Unlike Klette and Kortum (2004) and Luttmer (2011), the evolution of a firm's number of products is non-Markovian here. A firm with n products becomes a firm with $n-1$ products when it endogenously chooses to shutdown production of one of its products. Since this happens when a product's log-relative quality hits zero, I need to keep track of a firm's portfolio of products. In this vein, define:

$$\mu_{nt}\left(q_1,\ldots,q_n\right)\equiv\text{ measure at } t \text{ of } n\text{-product firms with portfolio } q_1,\ldots,q_n\ .$$

Since μ_{nt} is the measure of n-product firms and a product exits once its relative quality hits zero,

$$\forall i\in\{1,\ldots,n\},q_i=0\implies\mu_{nt}\left(q_1,\ldots,q_n\right)=0\ .$$

Put differently, the support of μ_{nt} is the positive orthant $(0,\infty)^n$, and μ_{nt} vanishes on the boundary of this support. To avoid cumbersome notation, I will denote by $\mathbf{q}$ the vector $(q_1,\ldots,q_n)$, and its dimension is to be inferred from the function it is an argument for. It follows that the total measure of products can be written as:

$$M_t=\sum_{n=1}^{\infty}n\underbrace{\int_{\mathbf{q}\in(0,\infty)^n}\mu_{nt}\left(\mathbf{q}\right)\ d\mathbf{q}}_{\text{Measure of } n\text{-product firms}}\ . \tag{16}$$

The evolution of the firm size distribution is then characterized by a system of coupled partial differential equations. Starting with single product firms, for $q>0$:

$$\begin{aligned}
\dot{\mu}_{1t}(q)\ &=\left(g_Q-\beta\right)\mu_{1t}'(q)\ +\ \frac{\nu^2}{2}\mu_{1t}''(q) && \text{Drift and Diffusion}\\
&-\ x\ \mu_{1t}(q) && \text{Single product firm adds a product}\\
&+E_t^f\ \alpha\ e^{-\alpha q} && \text{Entry of new firm}\\
&+\frac{\nu^2}{2}\partial_1\mu_{2t}(0,q)+\frac{\nu^2}{2}\partial_2\mu_{2t}(q,0) && \text{2-product firm shuts down a product}
\end{aligned}$$

The drift and diffusion terms on the first line echo those from from the product level KFE in Equation 9. The second line capture that, while still produced ($q>0$), an incumbent product gives birth to a new one at constant Poisson rate x. This shows up as an outflow because in this case the single product firm becomes a 2-product firm.

The last two lines in the law of motion correspond to inflows. The first source is the entry of new firms, with the corresponding inflow at q given by the total flow of entering firms E^f_t multiplied by the density of draws at that relative quality. The second inflow source is a 2-product firm shutting down the production of either of its products. Since a product's log-relative quality evolves following Equation 8 and exit is the absorbing boundary at 0, the instantaneous flow of 2-product firms who shut down production of their first product *and* whose second product has quality q is given by $\frac{\nu^2}{2}\partial_1\mu_{2t}(0,q)$. To clarify the notation, the second term is the partial derivative of μ_{2t} with respect to its first argument, evaluated at $(0,q)$. Similarly, there is inflow of 2-product firms who shut down production of their second product *and* whose first product has quality q (second summand in last line of the law of motion).[4]

The law of motion for μ_{nt} for $n > 1$ has a similar structure. The key difference is that the sources of inflow now are $(n-1)$ and $(n+1)$-product firms. As such, for $\mathbf{q} \in (0,\infty)^n$, denote by $\mathbf{q}^{\setminus i} \in (0,\infty)^{n-1}$ the vector $\mathbf{q}$ with its i^{th} entry removed and by $\mathbf{q}^{i\leftarrow 0}$ the $(n+1)$ dimensional vector obtained by inserting a 0 into $\mathbf{q}$ at the i^{th} index. It follows that, for $\mathbf{q} \in (0,\infty)^n$,

$$
\begin{aligned}
\dot{\mu}_{nt}(\mathbf{q}) \;&=\; (g_Q-\beta)\sum_{i=1}^{n}\partial_i\mu_{nt}(\mathbf{q}) + \frac{\nu^2}{2}\sum_{i=1}^{n}\partial_i^2\mu_{nt}(\mathbf{q}) && \text{Drift and Diffusion}\\
&-\; n\,x\,\mu_{nt}(\mathbf{q}) && n\text{-product firm adds product}\\
&+\; (n-1)\,x\,\frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\alpha q_i}\mu_{(n-1)t}\left(\mathbf{q}^{\setminus i}\right) && (n-1)\text{-product firm adds a product}\\
&+\; \frac{\nu^2}{2}\sum_{i=1}^{n+1}\partial_i\mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right) && (n+1)\text{-product firm loses a product}
\end{aligned}
$$

The diffusion term on the first line leverages the independence of Brownian increments across products within a firm. The outflow term on the second line reflects that, since each product gives birth to a new one at rate x, a n-product firms expands to $n+1$ products at Poisson rate nx.

The inflow from $n-1$ to n on the third line merits some clarification. A firm with $n-1$ products expands to a n-product firm at Poisson rate $(n-1)x$. What shows up in the law of motion is the inflow at a given $\mathbf{q}$, and any of the n products could be the new one.[5] Suppose for a second that the new product is $i=1$. The measure of $(n-1)$-product firms with product qualities $(q_2,\dots,q_n)$ is precisely $\mu_{(n-1)t}(\mathbf{q}^{\setminus 1})$. Such a firm adds a product to its portfolio at rate $(n-1)x$, and if successful, the relative quality of this new product is independent of the firm's portfolio and drawn from the density $\alpha e^{-\alpha q_1}$. The term on the third line of the law of motion is simply obtained by averaging

[4]Throughout, partial derivatives evaluated at a point on a hyperplane delimiting the positive orthant are taken as limits approaching the hyperplane from inside the support. For $n=1$, this corresponds to the right derivative at 0.

[5]Intuitively, the firm's state variable is the set of product qualities rather than the vector of product qualities. This is why the system should satisfy permutation symmetry.

such contributions over $i = 1, \ldots, n$ – as each of these is equally likely to be the new product.

Finally, a firm with $n+1$ products becomes a n-product firm when it shut downs production of one of its products. The ultimate line of the law of motion is the resulting inflow at a given $\mathbf{q}$, with its structure mirroring that of the $n = 1$ case that I discussed above. For the sake of clarity, $\partial_i \mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right)$ is the partial derivatives of $\mu_{(n+1)t}$ with respect to its i^{th} argument evaluated at $(q_1, \ldots, q_{i-1}, 0, q_i, \ldots, q_n)$.

3.3 Defining a stationary firm size distribution

Along the BGP, the measure of products M_t grows at rate η. For the distribution of products per firm to be stationary, the number of firms needs to grow at rate η as well. So to define a stationary firm size distribution, I start by introducing the following normalized variables. Let Ψ_{nt} be the share of products at t held by firms with n products, so that

$$\Psi_{nt} = \frac{1}{M_t}\, n \int_{\mathbf{q}\in(0,\infty)^n} \mu_{nt}(\mathbf{q})\; d\mathbf{q} \quad \text{and} \quad \sum_{n=1}^{\infty} \Psi_{nt} = 1 \;.$$

Since the measure of n-product firms grows at rate η along the BGP, also define:

$$f_{nt}(\mathbf{q}) = \frac{n\; \mu_{nt}(\mathbf{q})}{\Psi_{nt}\; M_t} \;.$$

Note that $f_{nt}(\mathbf{q})$ is the probability density function on $(0,\infty)^n$ of product qualities among n-product firms. A stationary firm size distribution amounts to Ψ_n and $f_n(\mathbf{q})$ being time invariant:

$$\forall t, \Psi_{nt} = \Psi_n \;\text{ and }\; f_{nt}(\mathbf{q}) = f_n(\mathbf{q}) \;.$$

Plugging these back into the laws of motion outlined above yields the following definition.

Definition 3. A stationary firm size distribution consists of

- a sequence of non-negative numbers $\{\Psi_n\}_{n\geq 1}$ with $\sum_{n=1}^{\infty} \Psi_n = 1$,
- a sequence of functions $\{f_n\}_{n\geq 1}$, with f_n a probability density function on $(0,\infty)^n$,

such that $\{\Psi_n\}$ and $\{f_n\}$ jointly satisfy the following system of coupled PDEs:

$$(\eta+x) f_1(q) = \left(g_Q - \beta\right) f_1'(q) + \frac{\nu^2}{2} f_1''(q) + \frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}\alpha e^{-\alpha q} + \frac{\nu^2}{2}\frac{\Psi_2}{2\,\Psi_1}\left[\partial_1 f_2(0,q) + \partial_2 f_2(q,0)\right] \;;$$

with the boundary condition $f_1(0) = 0$; and for $n > 1$,

$$\left(\frac{\eta}{n} + x\right) f_n(\mathbf{q}) \;=\; \frac{1}{n}\sum_{i=1}^{n}\left(\left(g_Q - \beta\right)\partial_i f_n(\mathbf{q}) + \frac{\nu^2}{2}\partial_i^2 f_n(\mathbf{q})\right)$$
$$+\; x\,\frac{\Psi_{n-1}}{\Psi_n}\frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\alpha q_i} f_{n-1}\left(\mathbf{q}^{\setminus i}\right) + \frac{\nu^2}{2}\frac{\Psi_{n+1}}{\Psi_n}\frac{1}{n+1}\sum_{i=1}^{n+1}\partial_i f_{n+1}\left(\mathbf{q}^{i\to 0}\right)$$

with the boundary condition: $\forall\ 1 \leq i \leq n,\ q_i = 0 \implies f_n(\mathbf{q}) \;=\; 0.$

These coupled partial differential equations are the analogue of the difference equations characterizing the stationary firm size distribution in Klette and Kortum (2004) and Luttmer (2011). They are PDEs instead of difference equations because the non-Markovian evolution of a firm's number of products requires keeping track of product qualities. To make the parallel even clearer, the following definition will prove helpful.

Definition 4. The average *product* exit rate among n-product firms is λ_n, with:

$$\lambda_1 = \frac{\nu^2}{2} f_1'(0) \quad \text{and for } n > 1,\ \lambda_n = \frac{\nu^2}{2}\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbf{q}\in(0,\infty)^{n-1}} \partial_i f_n\left(\mathbf{q}^{i\to 0}\right)\, d\mathbf{q}\,.$$

Proposition 7. Given a stationary firm size distribution $\{\Psi_n, f_n\}$ and the corresponding $\{\lambda_n\}$,

$$\begin{cases} \eta\Psi_1 = -\left(x + \lambda_1\right)\Psi_1 + \lambda_2\Psi_2 \qquad + \eta + \frac{\nu^2}{2}\alpha\zeta - x & \\ \frac{\eta}{n}\Psi_n = -\left(x + \lambda_n\right)\Psi_n + \lambda_{n+1}\Psi_{n+1} + x\Psi_{n-1} & \text{for } n > 1 \end{cases}$$

Proof. The n^{th} recurrence relation follows from integrating both sides of the n^{th} PDE over $(0,\infty)^n$ then applying the divergence theorem in $\mathbb{R}^n$. Details in Appendix B.7 □

The recurrence relation in Proposition 7 is a balance condition similar to the one in Luttmer (2011). The difference is that in his setup, the product exit rate λ does not depend on n, as product exit is modeled as a Poisson process. In contrast, here, the average product exit rate among n-product firms depends on the endogenous shape of f_n.

It is perhaps clearer to appreciate the balance interpretation when the recurrence is equivalently rewritten in terms of Φ_n, the share of *firms* with n products:

$$\Phi_n \equiv \frac{\frac{\Psi_n}{n}}{\sum_{k=1}^{\infty}\frac{\Psi_k}{k}} \implies -\overbrace{\underbrace{nx\Phi_n}_{n\,\to\,n+1} - \underbrace{n\lambda_n\Phi_n}_{n\,\to\,n-1}}^{\text{Outflow}} + \overbrace{\underbrace{(n-1)x\Phi_{n-1}}_{n-1\,\to\,n} + \underbrace{(n+1)\lambda_{n+1}\Phi_{n+1}}_{n+1\,\to\,n}}^{\text{Inflow}} - \overbrace{\eta\Phi_n}^{\text{Dilution}} \;=\; 0$$

Outflows from n correspond to a n-product firm adding or shutting down a product. The former happens with Poisson arrival rate nx. While the latter is not the result of a Poisson shock, the total mass flowing out is $n\lambda_n\Phi_n$ since λ_n is the average product exit rate among n-product firms.

Inflows from $n-1$ to n correspond to a $(n-1)$-product firm adding a product, which happens at Poisson rate $(n-1)x$. Inflows from $n+1$ to n correspond to $(n+1)$-product firm shutting down production of one of its products, since the average product exit rate among such firms is λ_{n+1}, the total mass flowing in is $(n+1)\lambda_{n+1}\Phi_{n+1}$. Finally, growth in the total measure of firms at rate η (due to population growth) dilutes these shares.

3.4 Solving for the stationary firm size distribution

To solve the system from Definition 3, I start with a special case that then informs the general solution strategy.

Special case with no population growth ($\eta = 0$). Given the semi-endogenous nature of the model, the BGP is now simply a stationary equilibrium. Assumption 2 requires $\beta < 0$, and active firm entry requires $x < -\alpha\beta$. One can guess and verify that a solution to the system is:

$$\forall n \;\; f_n(\mathbf{q}) = \prod_{i=1}^{n} f_p(q_i) \quad ; \quad \lambda_n = \lambda \equiv -\alpha\beta > x > 0 \quad ; \quad \Psi_n = \frac{\lambda - x}{x}\left(\frac{x}{\lambda}\right)^n , \tag{17}$$

where $f_p(q)$ is the unconditional density of product qualities derived in Proposition 2 (plugging $\eta = g_Q = 0$ into the definition of ζ). In this case, conditioning on the number of products a firm has does not alter the distribution of product qualities. As a result, the average product exit rate does not vary with firm size. The resulting distribution of products per firm, given by Ψ_n, matches that of Klette and Kortum (2004).[6] When $\eta = 0$, whether one models product exit as resulting from a Poisson death shock or negative selection has no bearing on the stationary firm size distribution.

General case ($\eta > 0$). When $\eta > 0$, the sequences in Equation 17 no longer constitute a solution to the system in Definition 3. The culprit is the $\frac{\eta}{n}$ term on the left hand side of each PDE.

Two points are worth emphasizing to highlight the intuition behind this result. First, as a consequence of volatility and positive selection into surviving, the distribution of quality among incumbent products first order stochastically dominates that among new products. Second, when $\eta > 0$, the size of firm cohorts grows over time. Taken together, since firms that just entered are necessarily single product, it is unsurprising that the distribution of product quality among single product firms looks different from the distribution of product quality among multiproduct firms.

My strategy to solve the general system is the following and motivated by the fact that $\frac{\eta}{n}$, the term preventing the sequences in Equation 17 from being a solution, vanishes as n grows large.

[6] See equation (17) of Klette and Kortum (2004). Their nM_n maps to my Ψ_n and their θ to $\frac{\lambda - x}{x}$ here.

Accordingly, I guess that for some large n_0,

$$\forall n > n_0, \quad f_n(\mathbf{q}) = \prod_{i=1}^{n} f_p(q_i) \quad ; \quad \lambda_n = \alpha\left(g_Q - \beta\right) \quad ; \quad \frac{\Psi_{n+1}}{\Psi_n} = \frac{x}{\alpha(g_Q - \beta)} \ .$$

The next step then consists of solving for $\{f_n, \Psi_n, \lambda_n\}_{n \le n_0}$. While finite, this system suffers from a severe curse of dimensionality as each f_n is a pdf on $(0, \infty)^n$.

To address this curse of dimensionality, note that f_n is the joint density of n random variables that one expects to be independent. The reason is twofold: (i) the qualities of a firm's current products have no bearing on the quality of a new product it adds to its portfolio; (ii) the Brownian increments are independent across products. In addition, the system inherently satisfies permutation symmetry: a firm's state variable is its *set* of products, so their indexing should be irrelevant. So,

$$f_n(\mathbf{q}) \ = \ \prod_{i=1}^{n} \varphi_n(q_j), \quad \text{with } \varphi_n \text{ pdf on } (0,\infty) \text{ satisfying } \varphi_n(0) = 0 \ . \tag{18}$$

Just like in Equation 17, each joint density f_n is the product of n evaluations of the same marginal density. However, the difference is that now the marginal density potentially depends on the firm's number of products. So, for example, while for a 2-product firm the product with index 1 is not systematically any different than the product with index 2, the typical product held by a 2-product firm is allowed to systematically differ from the typical product of a single product firm.

This observation simplifies the stationary firm size distribution to sequences Ψ_n and $\varphi_n(q)$ satisfying a system of coupled ordinary differential equations (given in Appendix B.6). This gets rid of the curse of dimensionality because each $\varphi_n(q)$ is a density on $(0, \infty)$. As I explain in Appendix C, this allows me to solve for $\{\Psi_n, \lambda_n, \varphi_n(q)\}_{n \le n_0}$ using state of the art ODE solvers.

In addition to standard convergence metrics, the theory provides a transparent way of verifying my solution. By the law of total probability, aggregating the conditional distributions should yield back the unconditional distribution:

$$f_p(q) = \sum_{n=1}^{\infty} \Psi_n \varphi_n(q) \qquad \text{and} \qquad \frac{D_t}{M_t} = \sum_{n=1}^{\infty} \Psi_n \lambda_n \ .$$

Here $f_p(q)$ is the unconditional distribution of product quality and $\frac{D_t}{M_t}$ is the product exit rate, both characterized in closed form in Proposition 2. Crucially, neither of these identities was used as part of the solution strategy.

A payoff of characterizing the stationary firm size distribution is the ability to pin down the firm exit rate, which is given by:

$$\text{Firm Exit Rate } = \lambda_1 \Phi_1 = \lambda_1 \frac{\Psi_1}{\sum_{n=1}^{\infty} \frac{\Psi_n}{n}} \ .$$

Intuitively, given the continuous time setup of the model, exiting firms are necessarily single product. Such firms exit at rate λ_1 and their share among the total number of firms is Φ_1.

4 Empirical Evidence

My ultimate goal is to use the model to quantify knowledge spillovers. As discussed earlier, a central challenge is that while the model links spillovers to the dynamics of product exit, comprehensive datasets track exit of firms rather than products. My quantitative strategy therefore hinges on the model's mapping between firm and product dynamics. At the heart of this mapping is the model's novel treatment of product exit. So before proceeding to the quantitative exercise, in this section I provide empirical evidence to corroborate my modeling of product exit.

4.1 Data

Motivating and validating the model's treatment of product exit requires product level data. For this purpose, I use the NielsenIQ retail scanner dataset, which provides high-frequency data at the UPC (universal product code, or barcode) level. Its coverage spans consumer packaged goods such as groceries, cosmetics, and cleaning supplies. Different UPCs are partitioned into 104 product categories (groups), including coffee, vitamins, laundry supplies, and pet food.

The analysis draws on data from 2006 to 2019, covering a balanced panel of 25,400 retail stores. Each retailer provides Nielsen with weekly data on sales and quantities sold at the UPC level. The results below are obtained by aggregating these data across retailers to the yearly level. This yields 6 million UPC-year observations and 1.2 million unique UPCs. Using the GS1 database, I identify which UPCs are manufactured by the same firm, resulting in a total of 40,600 firms in my sample. The Data Appendix provides further details on my sample construction and data cleaning steps.

To validate my model's treatment of product exit, I want to compare UPCs within the same group. Because reporting results for each of the 104 product groups would be impractical, I instead compute different moments at the group–year level, then aggregate them to the yearly level weighting a group by its share of sales that year, and finally average across years.

Table 2 provides summary statistics. Within a product group, the distribution of sales across firms is highly skewed. Consistent with the findings of Hottman, Redding and Weinstein (2016), this reflects a skewed distribution of the number of UPCs per firm, as well as sales per UPC.

Table 3 points to substantial turnover at the UPC level: the annual UPC entry and exit rates are 13.6% and 12.3% respectively. Notice that the average age of an exiting UPC is lower than the one that would prevail if the hazard rate of exit were flat ($1/0.123 = 8.1$).

Table 2: Summary Statistics – NielsenIQ Retail Scanner Sample

	Mean	Median	P75	P90	P95
Firm sales (thousands of $)	9,171	87	1,335	34,382	66,022
Firm's market share	10%	8%	17%	23%	27%
Number of UPCs per firm	12	3	8	37	62
UPC sales (thousands of $)	502	14	181	1,026	2,377
UPC's market share	1.2%	0.7%	1.3%	2.4%	3.5%

Notes: NielsenIQ Retail Scanner data with GS1 database used to identify the firm manufacturing a UPC. Each statistic is based on annual data from 2006 to 2019, and computed first at the group-year level, then aggregated to the yearly level weighting a group by its share of sales that year, and finally averaged across the 14 years. So, first three rows correspond to firm level statistics *within a group*.

Table 3: UPC Entry and Exit Dynamics

Entry Rate	Exit Rate	UPC Age At Exit (in years) Mean	Median
13.6%	12.3%	3.1	2.9

Notes: All statistics are group–year moments, sales-weighted to the year and averaged across years. Last two columns based on exiting UPCs for which age is not left censored.

4.2 Four facts corroborating the paper's treatment of product exit

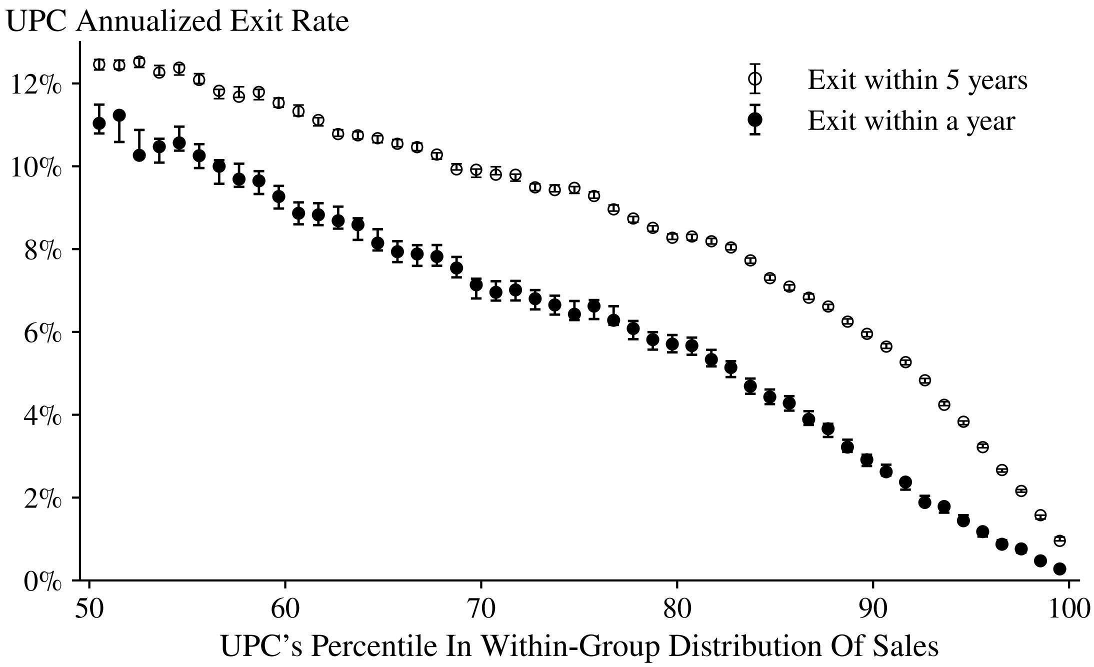
Fact 1: UPCs with more sales are less likely to exit

The first way to provide empirical support for selection into exit at the UPC level is by looking at how the exit rate varies with sales. Figure 2 shows the annualized hazard rate of UPC exit as a function of a UPC's percentile in the sales distribution for the corresponding group-year. I focus on UPCs with above median sales (in a group-year) to minimize concerns about measurement error (those with the less sales are sold at very few retailers and account for only 2% of sales).

Figure 2 clearly illustrates that within a group, even at the UPC level, exit declines with sales. This is true even when looking at the hazard rate of exiting within 5 years, and when zooming in on the right tail of the sales distribution within a group.

Table 4 presents the corresponding regression results. Among products with above-median sales in a group–year, a doubling of sales (≈ 0.693 log units) reduces the exit hazard by 1.17 percentage

Figure 2: UPC Sales and Hazard Rate of Exit



Notes: Cattaneo, Crump, Farrell and Feng (2024) binscatter with 50 bins, corresponding to the 50th through 99th percentile of the distribution of sales within a group in year t. "Exit within a year" refers to year $t+1$ being the UPC's last year of sales. "Exit within 5 years" refers to UPC's last year of sales being $t+6$ or earlier. Vertical bars are 95% pointwise confidence intervals. Underlying number of UPC-year observations is 2.73M (so roughly 54,600 per bin).

Table 4: UPC Sales and Hazard Rate of Exit

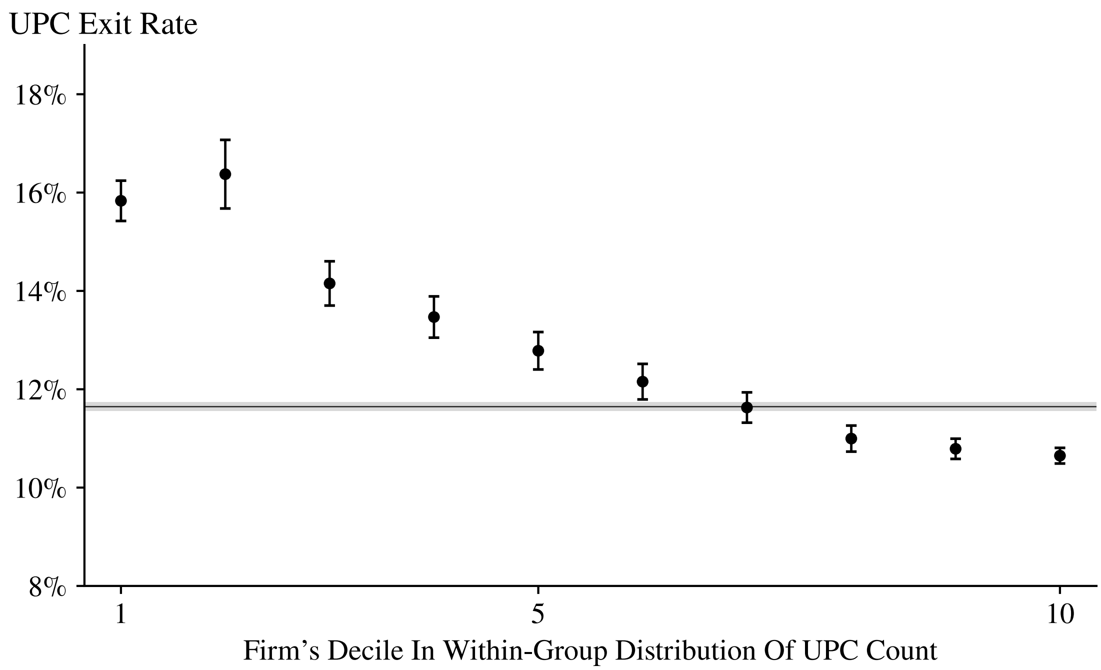
	Exit_{pt}			
$\log(\text{sales}_{pt-1})$	-1.69	-1.56	-2.70	-2.60
	(0.10)	(0.11)	(0.24)	(0.24)
UPC age FE		✓		✓
Group x Year FE	✓	✓		
Firm x Group x Year FE			✓	✓
Observations	2.58M	2.58M	2.50M	2.50M

Notes: Each column reports results from the OLS regression of a dummy for UPC p's exit at t on p's log annual sales in year $t-1$. Sample consists of UPCs with above-median sales in a group-year. Average of the dependent variable is 7.4%, and distributional moments from UPC sales reported in Table 2. Standard error in parentheses, clustered at the group level. UPC age FE includes a separate dummy for left censored UPCs.

points. Subsequent columns show that this negative relationship is robust to the inclusion of UPC-age fixed effects and firm-group-year fixed effects. This pattern corroborates the UPC-level evidence in Broda and Weinstein (2010), who find higher exit among smaller and younger UPCs using the Nielsen Homescan panel. Showing the same relationship in retailer scanner data, where exit reflects disappearance from store shelves rather than zero purchases by a household sample, lends further credibility to this relationship.[7] In addition, Bernard, Redding and Schott (2010) reach similar conclusions at a coarser product definition (5-digit SIC codes in U.S. manufacturing).

Fact 2: UPC exit rate is lower among firms with more UPCs

Figure 3: UPC Exit Rate Across Multiproduct Firms



Notes: Cattaneo et al. (2024) binscatter with 10 bins, corresponding to the deciles of the distribution of UPC count among multiproduct firms within a group in year t. Exiting UPCs are those for which the current year is the last year of sales; rates are obtained through division by firm's UPC count in corresponding group, averaged between previous and current year. Vertical bars are 95% pointwise confidence intervals. Horizontal line is inverse-weighted mean, with 95% confidence interval around it. Underlying number of firm-year observations is 264,809. Single product firms are not shown because they account for 42% of firms – among them, UPC exit rate is 16.6%.

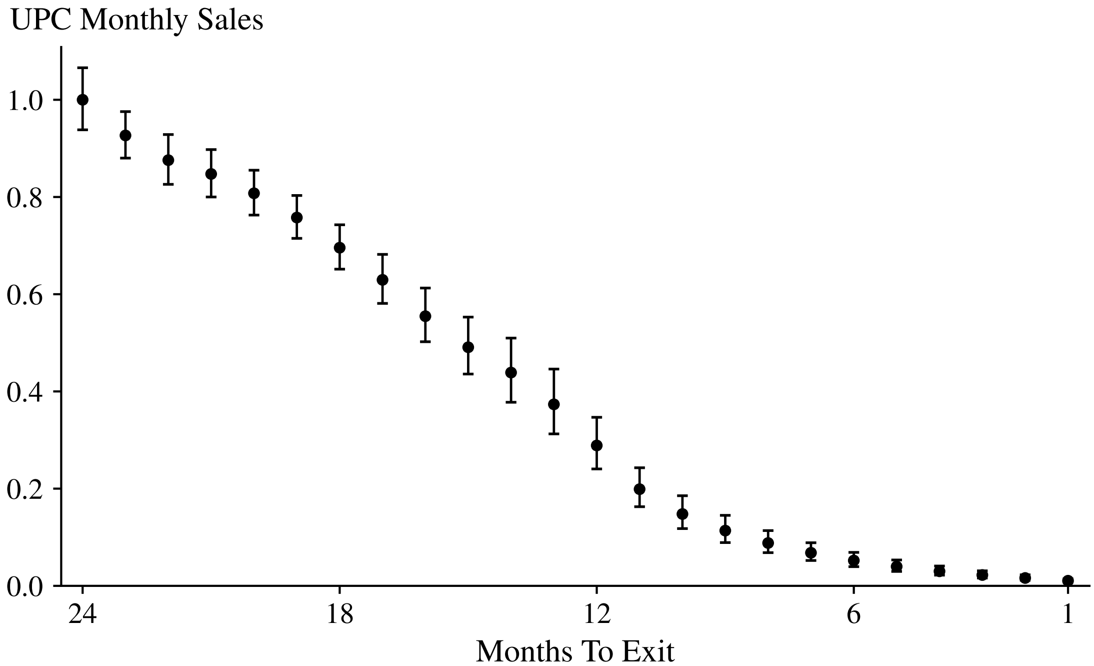
Figure 3 displays another facade of selection into product exit. It shows a firm's UPC exit rate as a function of its number of products. The latter is defined as the number of exiting UPCs divided

[7] Consistent with entry and exit being measured with non-negligible error in the Homescan panel, the entry and exit rates reported in Table 3 are substantially lower than those reported in Broda and Weinstein (2010).

by the number of UPCs in the firm's portfolio (averaged between last and current year). So this is the empirical counterpart of the relationship between λ_n and n in the model. The figure illustrates that, across firms in a group-year, the hazard rate of UPC exit is lower for firms with more UPCs. This pattern is clearly inconsistent with models where product exit is modeled as a Poisson process. In Section 5, I show that my model can generate this fact as a consequence of selection. In contrast, Figure E2 shows no clear tend between the UPC addition rate and the firm's number of UPCs, providing empirical support for the constant rate of product addition in my model.

Fact 3: Prior to exit, UPC's sales decline gradually

Figure 4: Evolution of UPC Sales prior to Exit



Notes: Path of $\exp(\beta_m)$ from Equation 19, normalized such that $\exp(\beta_{24}) = 1$. Number of observations in the underlying regression is 54M with $R^2 = 0.66$. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

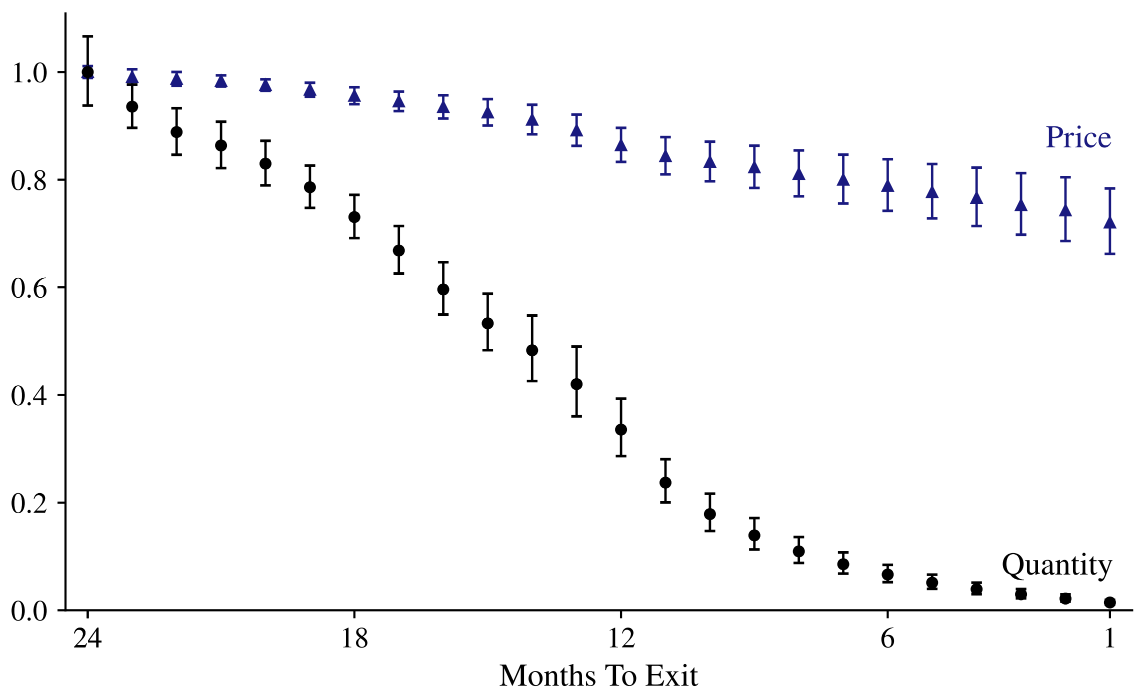
Besides the granularity, another advantage of the NielsenIQ Retail Scanner dataset is its high frequency, which allows me to track dynamics prior to exit. Specifically, I estimate:

$$\log \text{Sales}_{pt} = \gamma_p + \sum_{m=1}^{24} \beta_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \; ; \tag{19}$$

where p indexes a UPC, g its group (product category), and t a month, with γ_p a UPC fixed effect, γ_{gt} a group-month fixed effect, and D_{pt}^m a dummy variable equal to 1 m months prior to the UPC's exit. The path of $\exp(\beta_m)$ then represents the evolution of sales in the two years leading up to exit. Given selection concerns, I only include UPCs that were at least two years old when they exited. Figure 4 plots the resulting path for $\exp(\beta_m)$, normalizing to 1 sales two years prior to exit. The figure reveals that, in the lead up to exit, a UPC experiences a gradual decline in sales. Figure E1 in the appendix reveals that this gradual decline happens along the extensive margin (number of retailers selling the UPC) as well as the intensive margin (sales of the UPC per store).

Fact 4: Pre-exit price–quantity patterns are consistent with a negative residual demand shock

Figure 5: Evolution of UPC Price and Quantity prior to Exit



Notes: Price curve corresponds to path of $\exp(\pi_m)$ from Equation 20, normalized such that $\exp(\pi_{24}) = 1$; underlying regression has 54M observations with $R^2 = 0.86$. Quantity curve corresponds to path of $\exp(\kappa_m)$ from Equation 21, normalized such that $\exp(\kappa_{24}) = 1$; underlying regression has 54M observations with $R^2 = 0.68$. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

A final advantage of the NielsenIQ Retail Scanner dataset is the ability to break down UPC sales into price times quantity. This allows me to separately track the evolution of price and quantity prior to exit. Specifically, using the same notation as in Equation 19, I estimate:

$$\log \text{Price}_{pt} = \gamma_p + \sum_{m=1}^{24} \pi_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ , \tag{20}$$

$$\log \text{Quantity}_{pt} = \gamma_p + \sum_{m=1}^{24} \kappa_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ . \tag{21}$$

The paths of $\exp(\pi_m)$ and $\exp(\kappa_m)$ respectively capture the evolution of price and quantity in the two years prior to exit. Figure 5 shows that, while the product's relative price falls modestly, the quantity sold collapses. These patterns are in line with a negative demand shock in the lead up to exit: despite the UPC becoming relatively cheaper, its consumption is decreasing.

Taken together, these four facts provide strong empirical corroboration for the paper's novel treatment of product exit. They show that, consistent with the model's predictions, there is negative selection into product exit. With confidence established in this central and most novel component of the model, I now turn to the full quantitative exercise: using the model to infer the magnitude of knowledge spillovers from firm level dynamics.

5 Quantitative Results

The preceding sections provide the apparatus needed to quantify knowledge spillovers using data on firm exit by age. This section puts this apparatus to work, delivering the paper's main quantitative finding: across U.S. private nonfarm businesses, I estimate that knowledge spillovers create a 16 percentage point wedge between the social and private rates of return to R&D.

5.1 Estimation strategy

My estimation targets the profile of firm exit rates by age. Specifically, I use the hazard rate of firm exit for ages 1 through 19, as reported in Sterk, Sedláček and Pugsley (2021) using data from the U.S. Census Longitudinal Business Database (LBD) which covers the universe of nonfarm private employer firms in the U.S. As emphasized by recent work (Hopenhayn, Neira and Singhania, 2022; Karahan, Pugsley and Şahin, 2024), these age-specific firm exit rates have been remarkably stable over recent decades. This stability makes them ideal targets for the estimation of a stationary model.

As established in Section 3, the model-counterpart of this profile is governed by three sufficient statistics. The first, $\theta\eta - \alpha\beta$, captures the component of the product exit rate driven by the downward drift toward the exit threshold. The second, $\alpha\nu$, is a measure of relative volatility and governs product exit due to idiosyncratic shocks. The third, x, is the endogenous rate at which incumbent firms add a product to their portfolio.

I use a Generalized Method of Moments (GMM) procedure to estimate these three statistics,

Table 5: GMM Estimation Results

	Symbol	Point Estimate
Quality drift (scaled)	$\theta\eta - \alpha\beta$	0.158
Quality volatility (scaled)	$\alpha\nu$	0.296
Product addition rate	x	0.127

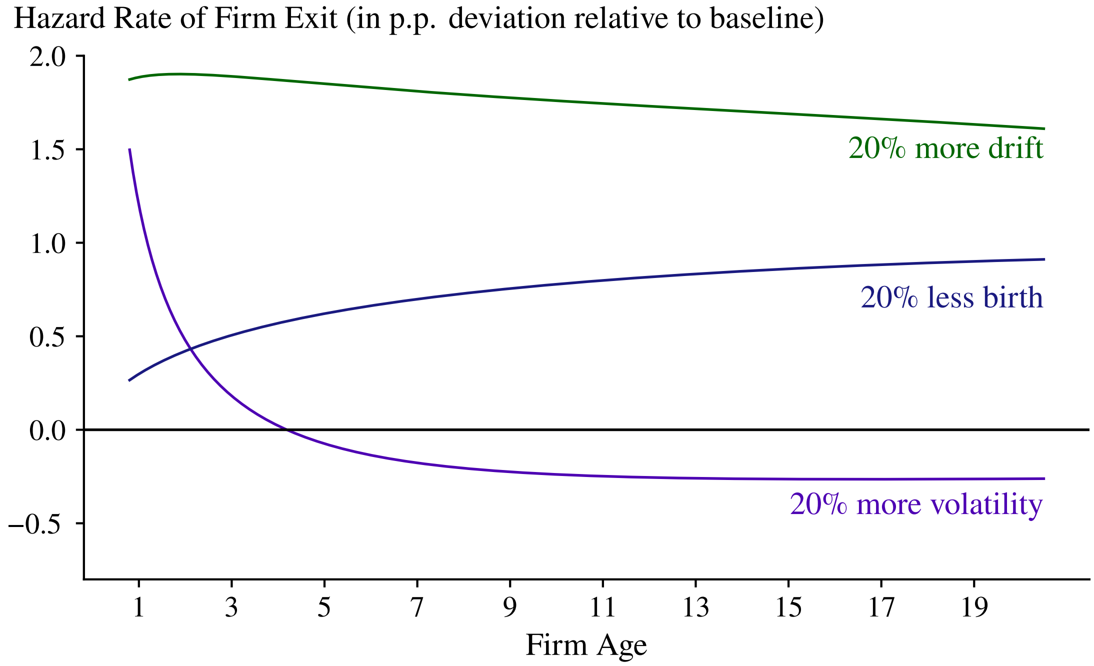
Notes: θ is the spillover elasticity, η the population growth rate, α the thinness of the entry distribution, β and ν the drift and volatility of product quality, and x the rate at which an incumbent firm adds a product to its portfolio (birth rate). GMM objective is an equally-weighted least squares. No standard errors reported because empirical targets are calculated from population-wide data.

minimizing the equally weighted sum of squared deviations between the 19 empirical and model-implied hazard rates. The latter are obtained by numerically solving the integral equation in Proposition 6. The resulting estimates are reported in Table 5. Because the empirical targets are constructed from population data covering the universe of U.S. nonfarm private employer firms, there is no sampling uncertainty, and I therefore do not report standard errors.

Figure 6: Fit of Targeted Moments

Notes: The 19 target moments are the firm exit rate at ages 1 through 19, reported in Sterk, Sedláček and Pugsley (2021) and calculated from LBD. Across these 19 moments, the absolute difference between model-based and targeted has mean 0.089 p.p. and median 0.063 p.p.

Figure 7: Intuition for identification



Notes: Each curve shows the effect of changing one of the three estimated statistics, while holding the other two fixed.

Figure 6 shows the fit of the model to the 19 targeted moments. Despite its parsimony, the model tracks the sharp decline in hazard rates at young ages as well as the gradual flattening at older ages, with mean and median absolute deviations below 0.1 percentage points.

To build intuition for identification around these point estimates, Figure 7 plots the change in the hazard rate resulting from a 20% change in each of the three statistics, holding the others fixed. The y-axis shows the deviation in percentage points from the baseline hazard rate. The figure makes visually clear that each statistic has both a substantial impact and a distinct signature across firm ages. A more negative drift raises the exit rate across all firm ages. In contrast, higher volatility raises the exit rate among young firms but lowers it among older firms. The latter reflects that surviving firms have even more positively selected products when volatility is higher. Finally, a lower product addition rate primarily increases the exit rate for mature firms: while very young firms are still single product, this comparative static decreases the number of products older firms have. This demonstrates how different parts of the exit-age curve are informative about different underlying economic forces, allowing the GMM procedure to separately identify them.

5.2 Magnitude of knowledge spillovers

With the GMM estimates in hand, I now turn to their implications for the magnitude of knowledge spillovers. The estimation identifies that the product exit rate resulting from the deterministic downward drift toward the exit threshold, $\theta\eta - \alpha\beta$, is 15.8%. Rearranging this expression, the wedge between social and private rates of return to R&D created by knowledge spillovers is:

$$\theta\eta = 0.158 + \alpha\beta \; .$$

To obtain a benchmark estimate for the spillover wedge, I make a conservative assumption and set the exogenous incumbent quality growth, β, to zero (and later discuss the robustness of my results to relaxing this assumption). This choice is conservative because it leads to a lower bound on the wedge, as any positive $\beta > 0$ would imply an even larger wedge. Intuitively, ignoring a positive drift when backing out spillovers biases the estimate downward: I would rationalize a low exit rate as resulting from the entry distribution improving slowly, while the actual reason is that incumbents improve over their life cycle.

Under this assumption, the analysis delivers the paper's main result: knowledge spillovers create a wedge of 15.8 percentage points between the social and private rates of return to R&D.

A feature of my approach is that it identifies the wedge itself, the product of the spillover elasticity θ and the long-run growth rate of the knowledge stock g_K, which equals η in my model. This is a strength of the identification strategy. It makes the estimated wedge robust to alternative specifications of the law of motion for K_t. For instance, one could easily envision efficiency units of labor as the input into the innovation process. With growth in human capital at rate g_h, the knowledge stock would then grow along the BGP at rate $\eta + g_h$. My approach would identify the product $\theta(\eta + g_h)$, which remains the economically relevant object for quantifying the social-private R&D return gap.

Taste for novelty as a confounding factor. A threat to my approach of backing out the wedge is that the estimated exit due to downward drift might reflect unmodeled forces that are observationally equivalent to a negative drift ($\beta < 0$). Chief among these concerns is a consumer taste for novelty. While preference shocks are accounted for by the Brownian motion, if consumers intrinsically value newness, demand for incumbent products would drift down over time as the product ages. This would lead my estimation to attribute the resulting exit to spillovers, even in their absence.

To address this concern and bolster the credibility of my findings, I exploit sectoral heterogeneity and re-estimate the model for each 2-digit NAICS sector. If my results were driven by taste for novelty, the estimated wedge should be largest in consumer facing sectors where fashion and fads play a more important role. Instead, if my analysis plausibly identifies knowledge spillovers, the estimated wedge should be largest in sectors where narrative evidence points to important spillovers.

Before showing these sectoral estimates, it is crucial to clarify what they measure. Instead of a single sector, as in the baseline model, suppose the economy consists of S sectors, with a Cobb-Douglas aggregator across sectors. Innovation is directed toward a sector s, and the quality of a new product in sector s is drawn from an entry distribution with CCDF:

$$\overline{F}^{E}_{st}(Q) = \prod_{j=1}^{S} K_{jt}^{\theta_{j\to s}} Q^{-\alpha_s} ,$$

where K_{jt} is the cumulative stock of innovation in sector j, and $\theta_{j\to s}$ is the spillover elasticity from sector j to sector s. Along a BGP, the wedge I recover for sector s is $\eta \sum_{j=1}^{S} \theta_{j\to s}$. Therefore, it should be interpreted as a measure of spillovers *received*, rather than spillovers *generated*. Accordingly, this metric is not informative about the design of sector-specific R&D policies. This being said, it is still useful for two reasons. First, for the purposes of uniform policy across sectors (common for many innovation policies), averaging spillovers received yields spillovers generated. Second, this allows me to assess whether sectors where I identify a large wedge are plausibly benefiting from spillovers, the source of which can be the sector itself or other sectors in the economy.

For each 2-digit NAICS sector, I use the same GMM strategy as above to estimate the three statistics by targeting the sector's profile of firm exit by age. Table 6 presents the estimated wedge for eight sectors: the four in which the procedure yields the highest estimated wedge, and the four in which it leads to the lowest one.

Table 6: Estimated Spillover Wedge Across 2-digit Sectors

	Sector	Estimated Wedge	Share of firms	Firm Exit Rate
First 4	Arts, Entertainment, and Recreation	25.6%	1.7%	8.6%
	Mining, Quarrying, and Oil and Gas Extraction	21.6%	0.4%	8.2%
	Transportation and Warehousing	18.9%	2.8%	10.8%
	Information	18.5%	1.2%	10.2%
Last 4	Finance and Insurance	10.5%	4.1%	7.7%
	Health Care and Social Assistance	7.0%	10.8%	6.2%
	Utilities	5.5%	0.1%	4.2%
	Management of Companies and Enterprises	3.3%	0.5%	3.6%

Notes: The "Estimated Wedge" is the point estimate of $\theta\eta - \alpha\beta$ from a GMM procedure targeting the firm exit rate at ages 1, 2, 3, 4, 5, 8, 13, and 18. Underlying data are from the Business Dynamics Statistics for the years 1996-2019.

The fact that the highest wedge is in Arts, Entertainment, and Recreation is consistent with a "taste for novelty" being a potential concern. However, this confounder does not drive my results. First, this is a relatively small sector, accounting for less than 2% of both firms and employment in the U.S. economy. Second, and more importantly, the pattern among the other high-wedge sectors provides strong evidence supporting my interpretation of the wedge. As I discuss next, these are technology-intensive industries where narrative evidence points to a central role for exactly the kind of spillovers my model is designed to capture.

The Mining, Quarrying, and Oil and Gas Extraction sector provides a prime example, as knowledge spillovers played a central role in enabling the shale gas boom. Within the sector, the common narrative credits George Mitchell's company with developing a breakthrough formula for combining horizontal drilling with slickwater fracturing, a process the rest of the industry then "adapted with awesome rapidity" (Golden and Wiseman, 2015, p. 960). But cross-industry spillovers also played a crucial role, as this breakthrough formula itself built on a "web of technological developments that helped spur the shale gas boom" (Golden and Wiseman, 2015, p. 973), including "3D seismic imaging techniques [...] that have benefited from advances in computing and that draw on technology originally developed to track submarines" (Golden and Wiseman, 2015, p. 973).

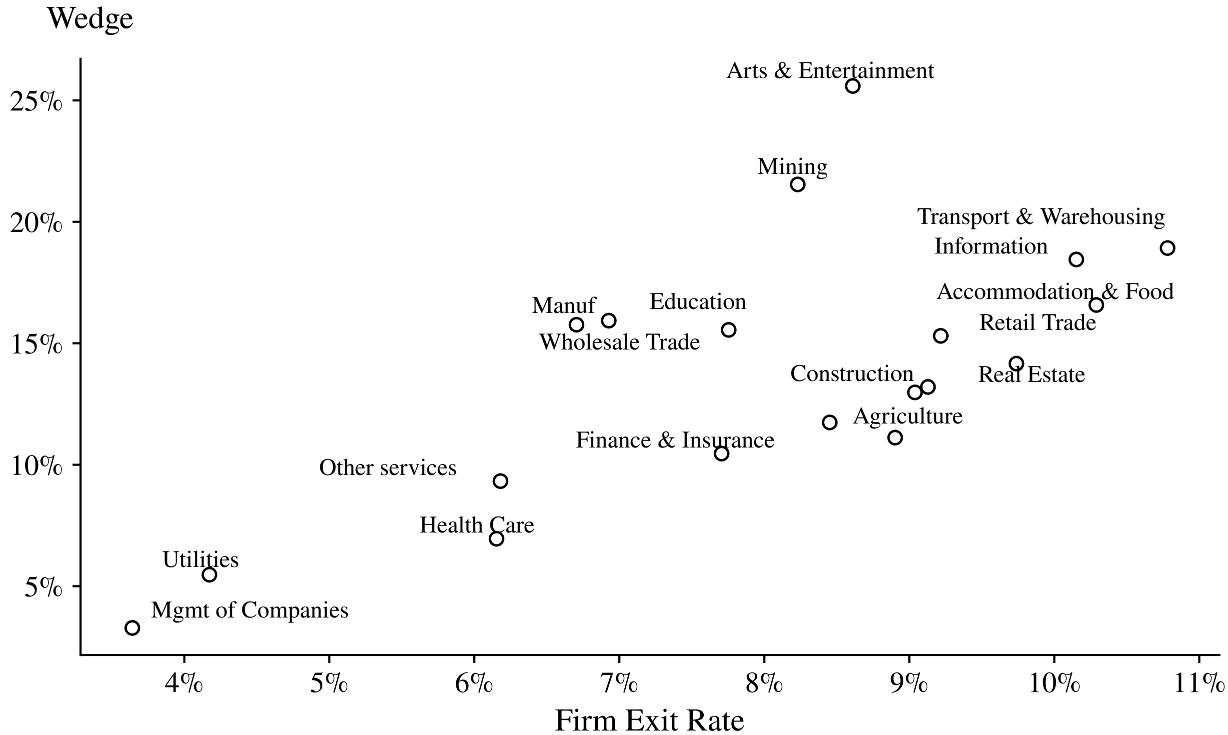
This example also clearly illustrates how my approach is complementary to those relying on patents to quantify spillovers. In fact, "although it is somewhat surprising and counterintuitive, during the late 1990s and early 2000s, neither Mitchell nor Devon pursued patent protection for their respective innovations in slickwater hydraulic fracturing and horizontal drilling" (Cahoy et al., 2013, p. 291). But by lowering the cost of natural gas, these innovations reduced demand for coal and left a detectable trace in product markets: they drove (old) firms whose businesses relied on coal out of business (Linn and McCormack, 2019).

Turning to Transportation and Warehousing, the major developments the sector has experienced over this period also built on new technologies developed in other sectors. Arguably the most pivotal of these was the Global Positioning System (GPS). Originally developed for military purposes, GPS became widely available for commercial use in the U.S. in the mid 1990s. Its high precision capabilities were critical "to unlocking most of the benefits of telematics", the "field of technology that uses in-vehicle equipment to remotely monitor vehicles" (O'Connor et al., 2019, p. 13-1). Advanced on-board computers (OBCs) are an example of such equipment adopted by trucking companies and Hubbard (2003) estimates that they increased capacity utilization among adopting trucks by 13%. These gains in fleet management stem from dynamic route optimization as well as improved monitoring of driver behavior (Hubbard, 2000).

A more recent transformation in the transportation sector is the rise of ride-sharing platforms like Uber and Lyft. This new business model also leverages a confluence of technologies developed in other sectors, combining the ubiquity of GPS-enabled smartphones and mobile data networks with sophisticated matching algorithms. Cramer and Krueger (2016) document a higher utilization rate among UberX drivers compared to traditional taxi drivers, and argue that the more efficient technology for matching drivers and passengers is a leading contender in explaining this finding.

The case of the Information sector is perhaps the least surprising, as it encompasses software and digital industries, home to the Open Source Software (OSS) paradigm. In fact, OSS offers a tangible illustration of my model's aggregate stock of knowledge: it is a public stock that firms simultaneously contribute to and benefit from Gortmaker (2025).

Figure 8: Estimated Wedge and Firm Exit Rate Across Sectors



Notes: The wedge is the point estimate of $\theta\eta - \alpha\beta$ from a GMM procedure targeting the sector's firm exit at ages 1, 2, 3, 4, 5, 8, 13, and 18. Pearson correlation coefficient between firm exit rate and estimated wedge is 0.68. Underlying data are Business Dynamics Statistics for 1996-2019.

The final column of Table 6 reveals a clear pattern: the four sectors with the highest estimated wedge also exhibit substantially higher firm exit rates than the four with the lowest wedge. However, the relationship is not monotonic. For instance, Transportation and Warehousing has a higher exit rate than Mining (10.8% vs. 8.2%), yet its estimated wedge is lower. The reason is that, in addition to differences in downward drift, exit rates could differ across sectors because of differences in volatility, incumbent innovation rate, and compositional differences in the age distribution of firms.

This underscores the necessity of using the model to learn about spillovers from firm exit rates. By leveraging the entire profile of firm exit by age (shown for these eight sectors in Figure E4), it identifies the component due to downward drift. This being said, Figure 8, which shows the results across all sectors, confirms that the correlation between firm exit rate and my estimated wedge is reasonably strong (0.68). The figure also highlights that the magnitude of the estimated wedge is typically larger than the firm exit rate. The reason is that the wedge is tied to the product exit rate, which will be larger than the firm exit rate when firms are multiproduct.

5.3 Quantitative Validation

With the cross-sectoral evidence providing *qualitative* support for my interpretation of the results, I now turn to bolstering confidence in their *quantitative* aspect.

Direct validation of the magnitude of spillovers is notoriously difficult, as knowledge flows themselves are inherently unobservable. However, this challenge highlights a key advantage of the structural approach taken in this paper. Because the estimation is embedded within a general equilibrium model, it generates a rich set of untargeted predictions about firm and product dynamics. By demonstrating that these untargeted predictions align with established facts from the literature, I substantiate the quantitative plausibility of the headline estimate.

In this vein, I solve for the stationary firm size distribution following the approach outlined in Section 3. In addition to the values in Table 5, this requires calibrating the population growth rate η. Since η is the net firm entry rate along the BGP, I set $\eta = 1\%$ to match the average annual growth rate in the number of private nonfarm businesses between 1978 and 2019.

Table 7: Product and Firm Entry and Exit Rates

	Model	Data
Product Entry Rate	17.1%	-
Product Exit Rate	16.1%	-
Firm Entry Rate	9.8%	9.9%
Firm Exit Rate	8.8%	8.9%

Notes: Model outcomes obtained with $\alpha(g_Q - \beta) = 0.158$, $\alpha\nu = 0.296$, $x = 0.127$, and $\eta = 1\%$. Data on firm entry and exit rates are from the Business Dynamics Statistics for 1978-2019 and cover the universe of U.S. private nonfarm businesses. No available data on product entry and exit rates with such coverage.

Product and firm dynamics. Table 7 displays entry and exit rates of firms and products. As I target the profile of firm exit by age and pick η to match the growth in the number of firms, the model's close fit for the firm entry and exit rates is unsurprising. What I want to emphasize instead is the high churn at the product level, which implies high churn within the firm. This is consistent with the findings of Broda and Weinstein (2010), Argente, Lee and Moreira (2018), and Argente, Lee and Moreira (2024) for the consumer packaged goods sector and Bernard, Redding and Schott (2010) for the manufacturing sector.

Incumbents' Contribution to Growth. The estimation did not target the share of growth due to incumbent firms, which Garcia-Macia, Hsieh and Klenow (2019) estimate to be 75.2% across U.S. private nonfarm businesses. Given that entry of new products is the engine of aggregate growth in my model, the corresponding metric here is the share of product entry accounted for by incumbent firms. From Table 7, the product entry rate is 17.1%, and from Table 5, an incumbent firm adds a product to its portfolio at rate 12.7%. Therefore, the share of growth due to incumbents in the model is 74.3%, remarkably close to the estimate from Garcia-Macia, Hsieh and Klenow (2019).

Consequences of selection. In Section 4, I showed that the average product exit rate is lower among firms with more products (Fact 2). There, I interpreted the fact as evidence in support of moving away from modeling product exit as a Poisson process. Here, I show how the estimated model endogenously generates this feature as a consequence of selection.

The top panel of Figure 9 illustrates this finding. It plots a firm's average product exit rate (λ_n) against its number of products (n). The rate is highest for single product firms at 16.2% and declines monotonically as firms grow, slowly approaching a limit of 15.8%. While the range is narrower than the one in Figure 3, it is worth keeping in mind that the underlying samples are different.

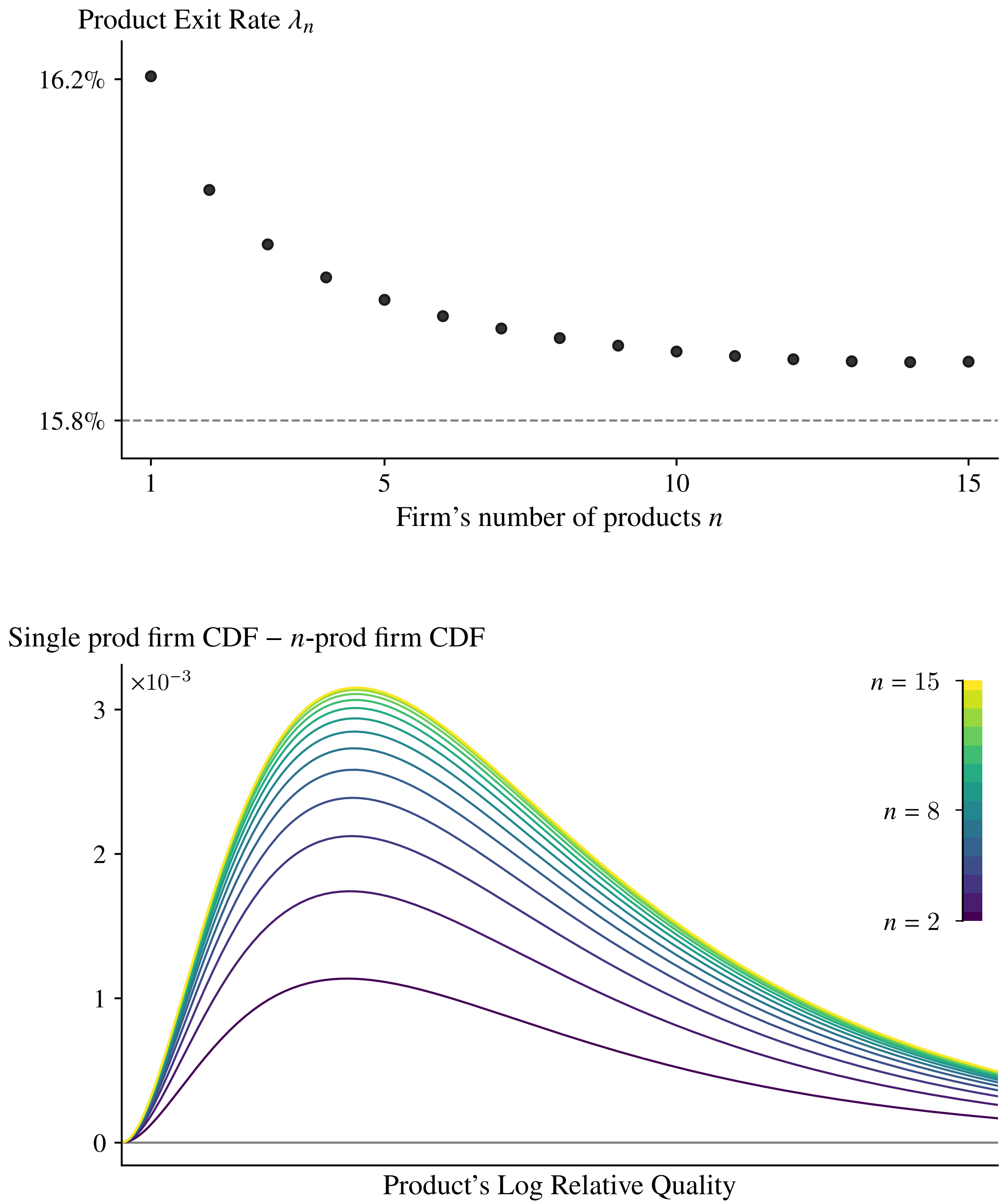
The bottom panel of Figure 9 shows that this reflects firms with *more* products endogenously having *better* products. It plots the difference between the cumulative distribution function (CDF) of product log-relative quality among single-product firms and that among n-product firms for $n > 1$. The fact that this difference is positive for the different values of $n > 1$ indicates first order stochastic dominance: the product quality distribution for multiproduct firms is better than for single product firms. Since the difference from the single product CDF widens with n, the figure also reveals a ranking of product quality by firm's number of products: the product quality distribution among n-product firms first-order stochastically dominates that among n'-product firms whenever $n > n'$. Finally, the vertical gap between the curves for consecutive values of n shrinks as n increases, showing that the distribution of product quality converges to its large n limit.

These features of the stationary equilibrium are consequences of positive selection into survival at the product level. Since a firm starts with a single product and gradually expands it portfolio through R&D, ending up with many products requires good draws and/or positive shocks.

5.4 Robustness Checks

Robustness to $\beta > 0$. The headline estimate follows from the conservative assumption that β, the exogenous drift in an incumbent's product quality, is 0. While a literal interpretation of $\beta < 0$ is hard to defend, I explained above that it is isomorphic to a taste for novelty and showed evidence that such explanations do not seem to be driving my results. I now discuss the sensitivity to my results to entertaining $\beta > 0$. This will unambiguously make the gap I estimate even larger, but the point is to show that the 15.8% gap is not a very loose lower bound.

Figure 9: Consequences of Selection on Firm Size Distribution



Notes: The top panel shows λ_n, the average product exit rate among n-product firms from Definition 4, as a function of n. The bottom panel shows the CDF of product log-relative quality (q) among single product firms minus the CDF of product product log-relative quality (q) among n-product firms ($n > 1$). A positive gap means the latter ($n > 1$) first order stochastically dominates the former ($n = 1$). Refer to Appendix C for details about computational solution.

To do so, I need to calibrate the value of α as $\theta\eta = 15.8\% + \alpha\beta$. A transparent calibration strategy follows from realizing that $\frac{\alpha}{\sigma-1}$ is the Pareto tail of the firm size distribution. The reason is that the number of products per firm has a geometric-like thin tail, so that the Pareto tail in the distribution of sales across firms is inherited from the distribution of sales per product. From Proposition 2, the Pareto tail index of the distribution of quality across products is $\min\{\alpha, \zeta\}$. The parameters from Table 5 along with my calibration of $\eta = 1\%$ imply that $\alpha < \zeta$ because

$$\frac{\zeta}{\alpha} = \frac{\theta\eta + \sqrt{(\theta\eta)^2 + 2\eta(\alpha\nu)^2}}{(\alpha\nu)^2} \approx 3.7 \ .$$

Since product sales are proportional to quality raised to the power $\sigma - 1$, it follows that the Pareto tail of the distribution of sales is $\frac{\alpha}{\sigma-1}$. As a result, to match the tail of 1.06 in the data (Luttmer, 2007), I set $\frac{\alpha}{\sigma-1} = 1.06$. For validation, I check that the resulting standard deviation in the annual growth rate of sales across firms (0.41) is consistent with the empirical analogue for U.S. private nonfarm businesses (0.45, Sterk, Sedláček and Pugsley (2021)).[8] To get the corresponding α, I consider a range of values for the elasticity of substitution σ.

Table 8: Robustness to positive drift in incumbent product quality

		β			
		0.25%	0.5%	0.75%	1%
σ	4	16.6%	17.3%	18.1%	18.8%
	6	17.1%	18.3%	19.6%	20.8%
	8	17.6%	19.3%	21.1%	22.8%
	10	18.1%	20.3%	22.6%	24.8%

Notes: Spillover wedge $\theta\eta$ for different values of $\beta > 0$ and σ. β is the exogenous drift in the quality of an incumbent product, σ is the elasticity of substitution between products. $\frac{\alpha}{\sigma-1}$ calibrated to 1.06.

Table 8 presents the results from this robustness exercise. It displays the estimate of $\theta\eta$, the gap between social and private rates of return to R&D, for different calibrations of $\beta > 0$ and σ (the dependence on the latter is due the calibration pinning down $\frac{\alpha}{\sigma-1}$). The key takeaway is that the 15.8% headline number, while conservative, is not a loose lower bound: even when allowing for 1% drift in incumbent product *quality* and an elasticity of substitution of 10 (which combined imply that if it weren't for substitution toward newer products, incumbent product sales would grow by 10% annually), the estimated gap is 24.8%.

[8] Given values in Table 5 and $\eta = 1\%$, $\frac{\alpha}{\sigma-1}$ pins down this standard deviation because it governs the dispersion in sales of new products firms add to their portfolios.

Quantifying Spillovers with Product Data. As I emphasized on a number of occasions, the key insight the paper leverages to quantify spillovers links dynamics of *product* exit to the magnitude of spillovers. The reason I rely on firm level data to do the quantification is that comprehensive data at the product level are not available.

As a robustness exercise, I assess the sensitivity of my results to doing the quantification with product instead of firm level data. This requires a setting where both product and firm level data are available, so that I can compare the results across the two methods. The food manufacturing sector provides such an opportunity. For *firm* exit by age, I use data from the Business Dynamics Statistics for the 3-digit NAICS sector 311 (food manufacturing).[9] For *product* exit by age, I use my Nielsen sample and exclude UPCs classified under "Health & Beauty Care", "Non Food Grocery", and "General Merchandise" (so that the sample is comparable to food manufacturing).

The quantification with product level data leverages Proposition 5. Specifically, by targeting the profile of *product* exit by age, I recover $\theta\eta - \alpha\beta$ and $\alpha\nu$. Identification follows from the former raising product exit at all ages and the latter raising it for young products but lowering it for older ones (due to selection). In contrast, the quantification with firm level data uses the same GMM strategy as above, and identifies the rate of incumbent innovation x in addition to $\theta\eta - \alpha\beta$ and $\alpha\nu$.

Table 9: Results with Product vs Firm Level Data for Food Manufacturing

	Product Level Data	Firm Level Data
$\theta\eta - \alpha\beta$	0.163 (0.001)	0.159
$\alpha\nu$	0.277 (0.006)	0.484
x	-	0.081

Notes: $\theta\eta - \alpha\beta$ is product exit due to downward drift, $\alpha\nu$ governs extent of product exit due to shocks, and x is the rate at which incumbent firms add a product to their portfolio. Product level results are obtained from an optimally weighted GMM strategy targeting the profile of *product* exit at ages 1 through 11, with underlying data from NielsenIQ. Corresponding standard errors are in parentheses. Firm level results are obtained from a GMM strategy targeting the profile of *firm* exit at ages 1, 2, 3, 4, 5, 8, 13, and 18 in the food manufacturing sector (NAICS 311), with underlying data from the Business Dynamics Statistics. No standard errors reported because underlying Census data covers the entire population.

Table 9 presents the results from this exercise. Focusing on the spillover wedge—the statistic of interest—the estimate obtained using product level data is 16.3%, while that obtained from firm level data is 15.9%. This alignment lends credibility to the headline finding, suggesting my methodology successfully recovers the gradual component of product exit from firm level data.

[9] While I do link NielsenIQ to the GS1 database and can hence observe which products are produced by the same firm, firm age is censored for more than 80% of firms, preventing any meaningful estimation with firm level data.

6 Conclusion

Knowledge spillovers have long been recognized as a reason the social rate of return to R&D might exceed the private one. Despite this serving as a common rationale for government support for R&D, existing evidence on this gap "is quite thin" (Bryan and Williams, 2021, p. 290).

This paper introduces a new approach to quantify this gap by leveraging data on firm exit by age. The core intuition is that stronger knowledge spillovers accelerate growth among entrants, which in turn hastens the obsolescence of incumbent products. The advantage of this approach, relative to existing ones, is its reliance on data that are far more comprehensive and widely available than patents or reported R&D.

To carry out the quantification, I develop a new semi-endogenous growth model featuring multiproduct firms and negative selection into product exit. By parsimoniously matching key features of firm and product dynamics, the model provides a tractable building block for future research exploring questions beyond the scope of this paper.

Applying this framework to U.S. private nonfarm businesses, my headline finding is that knowledge spillovers create a 16 percentage point wedge between the social and private rates of return to R&D. This result strengthens the existing empirical evidence that spillovers are sizable and that there are potentially large welfare benefits from government policies that support innovation.

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APPENDIX

A Summary of Notation and Symbols **54**

B Proofs and derivations of results in main text **56**

B.1 Product characteristics along balanced growth path 56

B.1.1 Proof of Proposition 2 56

B.1.2 Proof of Proposition 3 57

B.1.3 Labor allocations along balanced growth path 58

B.2 Planner's problem 60

B.3 Derivation of the social rate of return to R&D 65

B.4 Proof of Proposition 5 and Proposition 6 68

B.5 Proof of Proposition 7 69

B.6 Dimension Reduction 71

C Computational Appendix **71**

C.1 Solving integral equation from Proposition 6 71

C.2 Solving for stationary firm size distribution 72

D Data Appendix **75**

D.1 NielsenIQ Retail Scanner Dataset 75

D.2 Publicly Available U.S. Census Data 76

E Additional Figures and Tables **76**

E.1 Empirical Results 76

E.2 Quantitative Results from the Model 77

A Summary of Notation and Symbols

Table A1: Symbols of endogenous objects featuring in definition of equilibrium (Definition 1)

N_t	Population
K_t	Stock of products created
$m(q,t)$	Cross-sectional measure of incumbent product log-relative qualities
$\underline{Q}_t$	Endogenous exit threshold
r_t	Interest rate
w_t	Wage
P_{pt}	Price of p
Q_{pt}	Product's quality
q_{pt}	Product's log quality relative to exit threshold
$V_t(Q_{pt})$	Product's value, with $V(q,t) = V_t\left(\underline{Q}_t e^q\right)$
c_t	Per capita consumption
a_t	Individual's asset holding
c_{pt}	Per capita consumption of p
Y_{pt}	Aggregate supply of p
L_{pt}	p's production labor
I_{pt}	p's R&D labor
S_t	Startup entrepreneurs (entry labor)

Table A2: Symbols of parameters and auxiliary objects

η	Population growth rate	ρ	Rate of time preference
γ	Coefficient of relative risk aversion	σ	Elasticity of substitution
β	Drift in incumbent's log quality	ν	Diffusion in incumbent's log quality
θ	Spillover elasticity	α	Thinness of entry distribution
ε	Inverse of firm entry cost	$\mathcal{F}$	Overhead cost
δ	Diminishing returns in incumbent innovation	ϑ	Scale param. in incumbent innovation
ζ	Luttmer tail index (Proposition 2)	ξ	param. in HJB solution (Proposition 3)
A	Process efficiency	$\overline{F}_t^E(.)$	CCDF of entry distribution
g_Q	Quality growth rate	g	Consumption per capita growth rate
$\mathcal{V}(q)$	Stationary product value	Ow_t	Option value of product addition
I	R&D labor per incumbent product	L_t	Aggregate production labor
M_t	Measure of products	$\overline{Q}_t$	Average quality supplied
$f_p(q)$	Stationary PDF of incumbent prod	E_t^f	Flow of entering firms
E_t/M_t	Product entry rate	D_t/M_t	Product exit rate
$\ell(a)$	PDF of product's lifespan	$d_p(a)$	Product's exit hazard at age a
$\Gamma(a)$	CDF of firm's lifespan	$d_f(a)$	Firm's exit hazard at age a
x	Firm's product addition rate	λ_n	Mean prod exit rate for $n-$prod firm
$\mu_{nt}(\mathbf{q})$	Measure of $n-$prod firms with portfolio $\mathbf{q}$	$f_n(\mathbf{q})$	$= \prod_{i=1}^{n} \varphi_n(q_i)$ PDF among $n-$prod firms
Ψ_n	Share of products held by $n-$prod firms	Φ_n	Share of firms with n products
$\theta\eta - \alpha\beta$	Product Exit due to downward drift	$\alpha\nu$	Relative volatility of shocks

B Proofs and derivations of results in main text

B.1 Product characteristics along balanced growth path

B.1.1 Proof of Proposition 2

Along the BGP, $m(q,t) = M_t f_p(q)$ and M_t grows at rate η, so:

$$\frac{\partial m(q,t)}{\partial t} = \eta\, M_t\, f_p(q) \quad ; \quad \frac{\partial m(q,t)}{\partial q} = M_t\, f_p'(q) \quad ; \quad \frac{\partial^2 m(q,t)}{\partial q^2} = M_t\, f_p''(q)$$

Plugging back into the KFE (Equation 9) yields a second order ODE in $f_p(q)$:

$$\frac{\nu^2}{2}\, f_p''(q) \;+\; \big(g_Q - \beta\big)\, f_p'(q) \;-\; \eta\, f_p(q) \;=\; -\frac{E_t}{M_t}\,\alpha\, e^{-\alpha q}\,.$$

The stationary distribution solves this ODE subject to:

$$\int_0^\infty f_p(q)\, dq \;=\; 1 \quad ; \quad f_p(q) \ge 0 \quad ; \quad f_p(0) \;=\; 0$$

The first requirement leads to a zero coefficient on the positive homogeneous root, so:

$$f_p(q) = C_2 e^{-\zeta q} + C_3 e^{-\alpha q} \quad \text{with} \;\; C_3 = \frac{-\alpha \frac{E_t}{M_t}}{\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta}$$

and ζ as defined in Proposition 2. The boundary condition $f_p(0) = 0$ yields $C_2 = -C_3$. Hence

$$1 = \int_0^\infty f_p(q) dq = \int_0^\infty C_3 \left(e^{-\alpha q} - e^{-\zeta q}\right) dq \implies C_3 = \frac{\alpha\zeta}{\zeta - \alpha} \implies f_p(q) \;=\; \frac{\alpha\,\zeta}{\zeta - \alpha}\left(e^{-\alpha q} - e^{-\zeta q}\right)$$

Equating the two expressions for C_3 and simplifying yields an expression for the entry rate:

$$\begin{aligned}
\frac{\alpha\zeta}{\zeta - \alpha} \;=\; \frac{-\frac{E_t}{M_t}\,\alpha}{\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta} \quad &\implies \frac{E_t}{M_t} \;=\; \frac{\zeta}{\alpha - \zeta}\left(\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta\right) \\
&\overset{\star}{\implies} \frac{E_t}{M_t} \;=\; \frac{\zeta}{\alpha - \zeta}\left(\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \left(\frac{\nu^2}{2}\zeta^2 - (g_Q - \beta)\zeta\right)\right) \\
&\implies \frac{E_t}{M_t} \;=\; \frac{\nu^2}{2}\zeta\,\alpha \;+\; \frac{\nu^2}{2}\zeta^2 - (g_Q - \beta)\zeta \\
&\overset{\star}{\implies} \boxed{\frac{E_t}{M_t} \;=\; \frac{\nu^2}{2}\zeta\,\alpha \;+\; \eta}
\end{aligned}$$

where $\star$ follows from $-\zeta$ being a root of the characteristic polynomial. Notice the intuitive expression I got for the product entry rate:

$$\frac{E_t}{M_t} = \eta + \frac{\nu^2}{2} f'(0) .$$

Since the measure of products grows at rate η, the product entry rate exceeds the product exit rate by η. The second summand is precisely the exit rate, as it gives the instantaneous rate at which, in the stationary distribution, q – which evolves according to the SDE in Equation 8 – hits the absorbing boundary condition at 0.

B.1.2 Proof of Proposition 3

Plugging the stationary distribution into the definition of $\overline{Q}_t$ yields

$$\overline{Q}_t \equiv \left(\frac{1}{M_t}\int_{p\in\Omega_t} Q_{pt}^{\sigma-1} dp\right)^{\frac{1}{\sigma-1}} = \underline{Q}_t \left(\int_0^\infty e^{(\sigma-1)q} f_p(q) dq\right)^{\frac{1}{\sigma-1}}$$

Assumption 2 guarantees that $\alpha > \sigma - 1$ and $\zeta > \sigma - 1$, so that:

$$\left(\frac{\overline{Q}_t}{\underline{Q}_t}\right)^{\sigma-1} = \frac{\alpha}{\alpha - (\sigma - 1)} \frac{\zeta}{\zeta - (\sigma - 1)} .$$

Plugging back into the HJB yields

$$\begin{aligned} r_t V(q,t) = w_t &\left[\frac{(\alpha - (\sigma - 1))(\zeta - (\sigma - 1))}{(\sigma - 1)\alpha\zeta} \frac{L_t}{M_t} e^{(\sigma-1)q} - (\mathcal{F} - O)\right] \\ &+ \dot{V}(q,t) + \left(\beta - g_{Q_t}\right) \frac{\partial V(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V(q,t)}{\partial q^2} . \end{aligned}$$

Guess that $V(q,t) = w_t \mathcal{V}(q)$, then $\mathcal{V}(q)$ solves the second order ODE with constant coefficients:[10]

$$\frac{\nu^2}{2}\mathcal{V}''(q) + (\beta - g_Q)\mathcal{V}'(q) - (r - g)\mathcal{V}(q) = -\left[\frac{(\alpha - (\sigma - 1))(\zeta - (\sigma - 1))}{\alpha\zeta(\sigma - 1)} \frac{L_t}{M_t} e^{(\sigma-1)q} - (\mathcal{F} - O)\right]$$

subject to:

$$\mathcal{V}(q) \geq 0 \; ; \; \mathcal{V}(0) = 0 \; ; \; \mathcal{V}'(0) = 0 \text{ and } \mathcal{V}(q) < \frac{\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)} \frac{L_t}{M_t} e^{(\sigma-1)q}}{r - \left[g + (\sigma - 1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma - 1)^2\right]} .$$

[10] L_t and M_t both grow at rate η, so their ratio is constant.

The first constraint follows from the fact that the firm can always choose to shutdown production. The second and third constraints are respectively the value matching and smooth pasting conditions. To understand the fourth condition, note that the flow of dividends, gross of overhead and option value, is the numerator of the right hand side scaled by w_t. Using Ito's lemma, this flow grows at rate $g + (\sigma - 1)(\beta - g_Q) \; + \; \frac{\nu^2}{2}(\sigma - 1)^2$. So the right hand side of the fourth constraint is the PDV of flow of dividends gross of the overhead and option value. The inequality then follows from $\mathcal{F} > O$. The solution to this ODE is:

$$\mathcal{V}(q) \; = \; C_1 \, e^{zq} \; + \; C_2 \, e^{-\xi q} \; + \; \frac{\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)}}{r - \left[g + (\sigma - 1)(\beta - g_Q) \; + \; \frac{\nu^2}{2}(\sigma - 1)^2\right]} \frac{L_t}{M_t} \, e^{(\sigma-1)q} \; - \; \frac{\mathcal{F} - O}{r - g}$$

$$\text{with} \quad z \equiv \frac{-(\beta - g_Q) + \sqrt{(g_Q - \beta)^2 + 2\nu^2(r - g)}}{\nu^2} \;\; \text{and} \;\; \xi \equiv \frac{\beta - g_Q + \sqrt{(\beta - g_Q)^2 + 2\nu^2(r - g)}}{\nu^2}$$

Since $z > 0$ (while $-\xi < 0$), satisfying the inequality constraints on $\mathcal{V}(q)$ requires $C_1 = 0$. The value matching and smooth pasting conditions are two equations in two unknowns, $\frac{L_t}{M_t}$ and C_2. Solving yields:

$$\mathcal{V}(q) = \frac{\mathcal{F} - O}{r - g} \left[\frac{\xi}{\xi + \sigma - 1} e^{(\sigma-1)q} + \frac{\sigma - 1}{\xi + \sigma - 1} e^{-\xi q} - 1 \right]$$

$$\frac{L_t}{M_t} = (\mathcal{F} - O) \, \frac{\alpha\zeta(\sigma - 1)}{(\alpha - (\sigma - 1))(\zeta - (\sigma - 1))} \frac{\xi}{\xi + \sigma - 1} \frac{r - [g + (\sigma - 1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma - 1)^2]}{r - g} \, .$$

B.1.3 Labor allocations along balanced growth path

The expression for $\frac{L_t}{M_t}$ was obtained above when solving the HJB. The remaining equation follows from the free entry condition along the BGP. Using a change of variables, relative quality of new products is drawn from the density

$$\frac{K_t^\theta}{\underline{Q}_t^\alpha} \alpha e^{-\alpha q} \quad \text{for} \quad q \; \geq \; \ln\left(\frac{K_t^\theta}{\underline{Q}_t^\alpha}\right)$$

where this pdf is time-invariant along the BGP since $\underline{Q}_t^\alpha$ and K_t^θ both grow at rate $\theta\eta$. Using Assumption 3, the free entry condition reads

$$\varepsilon \int_0^\infty w_t \mathcal{V}(q) \frac{K_t^\theta}{\underline{Q}_t^\alpha} \alpha e^{-\alpha q} \, dq \; = \; w_t$$

The integration starts at 0 because for $q \le 0$, $\mathcal{V}(q) = 0$. Plugging in $\mathcal{V}$ yields:

$$\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}} = \varepsilon \, \frac{\mathcal{F} - O}{r - g} \, \frac{(\sigma - 1)\xi}{(\alpha + \xi)(\alpha - (\sigma - 1))} .$$

So Assumption 3 places a lower bound on the PDV of the effective fixed cost of operation relative to the entry cost:

$$\frac{\frac{\mathcal{F} - O}{r - g}}{\frac{1}{\varepsilon}} \ge \left(1 + \frac{\alpha}{\xi}\right)\left(\frac{\alpha}{\sigma - 1} - 1\right) . \tag{22}$$

To see how the expression for $\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}$ helps us pin down the labor allocations, note that

$$\eta K_t = \dot{K}_t = \varepsilon S_t + \frac{\vartheta}{1 - \delta} I^{1-\delta} M_t \implies \frac{S_t}{M_t} = \frac{\eta}{\varepsilon} \frac{K_t}{M_t} - \frac{I}{1 - \delta}$$

where I used the fact that $I = \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$. Now,

$$E_t = \overline{F}_t^E \left(\underline{Q}_t\right) \dot{K}_t = \frac{K_t^{\theta}}{\underline{Q}_t^{\alpha}} \eta K_t \implies \frac{K_t}{M_t} = \frac{1}{\eta} \frac{E_t}{M_t} \frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}$$

Plugging the expression for $\frac{K_t}{M_t}$ back into $\frac{S_t}{M_t}$, I get:

$$\frac{S_t}{M_t} = \frac{1}{\varepsilon} \frac{E_t}{M_t} \frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}} - \frac{I}{1 - \delta} .$$

Plugging in the expressions for the entry rate and $\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}$ yields:

$$\frac{S_t}{M_t} = \left(\eta + \frac{\nu^2}{2} \zeta \alpha\right) \frac{\mathcal{F} - O}{r - g} \frac{(\sigma - 1)\xi}{(\alpha + \xi)(\alpha - (\sigma - 1))} - \frac{I}{1 - \delta}$$

Adding $\mathcal{F} + I$ on both sides and using $O = \frac{\delta}{1-\delta} I$ yields:

$$\frac{S_t}{M_t} + I + \mathcal{F} = (\mathcal{F} - O) \left[\frac{\eta + \frac{\nu^2}{2} \zeta \alpha}{r - g} \frac{(\sigma - 1)\xi}{(\alpha + \xi)(\alpha - (\sigma - 1))} + 1\right] .$$

I now have all I need to solve for $\frac{M_t}{N_t}$ using the labor resource constraint.

B.2 Planner's problem

Define $z_{pt} = \ln Q_{pt}$ and denote by $\mu(z,t)$ the measure of products with log-quality z at t. The entry distribution in terms of z then has density

$$\widetilde{f}_t^E(z) = \alpha K_t^\theta e^{-\alpha z} \text{ for } z \geq \frac{\theta}{\alpha} \ln K_t$$

By symmetry, the planner picks $I_{pt} = I_t$ and strict positivity follows from the Inada condition at 0. Moreover, once total production L_t is chosen, standard CES allocations yield

$$L_t = \int_{z>\underline{z}_t} L_t(z)\mu(z,t)dz \implies c_t = \frac{AL_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}.$$

So the planner's problem is:

$$\max_{\substack{S_t,L_t,I_t\\ d_t}} \int_0^\infty e^{-(\rho-\eta)t}\,\frac{\left(A\frac{L_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}\right)^{1-\gamma}-1}{1-\gamma}\,dt$$

$$\text{subject to}\quad \mu\left(\underline{z}_t, t\right) = 0$$

$$\forall z > \underline{z}_t,\quad \dot{\mu}(z,t) = -\beta\,\frac{\partial\mu(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\mu(z,t)}{\partial z^2} + \dot{K}_t\,K_t^\theta\alpha e^{-\alpha z}\mathbb{1}_{\{z>\frac{\theta}{\alpha}\ln K_t\}}$$

$$\dot{\underline{z}}_t = d_t \geq 0$$

$$\dot{K}_t = \varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz$$

$$N_t = S_t + L_t + \int_{z\geq\underline{z}_t}(\mathcal{F}+I_t)\;\mu(z,t)\;dz$$

This is an optimal control problem with S_t, L_t, I_t, and d_t (how much to lift the threshold) as controls. The states are $\underline{z}_t$ (lowest quality still "alive"), K_t and $\{\mu(z,t)\}$. Let ω_t be the Lagrange multiplier on the labor resource constraint, $\Upsilon(z,t)$ be the costate associated with $\mu(z,t)$, χ_t the costate associated with K_t, and Ξ_t the costate associated with $\underline{z}_t$. Then the current-value Hamiltonian is:

$$\mathcal{H} = \frac{c_t^{1-\gamma}-1}{1-\gamma} + \chi_t\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right) + \int \Upsilon(z,t)\dot{\mu}(z,t)dz$$
$$+\,\omega_t\left(N_t - S_t - L_t - \int_{z\geq\underline{z}_t}(\mathcal{F}+I_t)\;\mu(z,t)\;dz\right) + \Xi_t d_t$$

It is going to be helpful to plug the KFE into $\int \Upsilon(z,t)\dot{\mu}(z,t)dz$ and then integrate by parts to move the derivatives to Υ. To deal with the boundary terms,

Assumption 5. I suppose (and verify later) that:

$$\lim_{z\to\infty} \Upsilon(z,t)\mu(z,t) = 0 \quad \text{and} \quad \lim_{z\to\infty} \Upsilon(z,t)\frac{\partial \mu(z,t)}{\partial z} = 0\ .$$

Using these assumptions, along with $\mu(\underline{z}_t, t) = 0$:

$$\int_{\underline{z}_t}^{\infty} \Upsilon(z,t)\dot{\mu}(z,t)dz = -\frac{\nu^2}{2}\frac{\partial \mu\left(\underline{z}_t, t\right)}{\partial z}\Upsilon\left(\underline{z}_t, t\right) + \int_{\underline{z}_t}^{\infty} \mu(z,t)\left(\beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2 \Upsilon(z,t)}{\partial z^2}\right)dz$$
$$+ \dot{K}_t K_t^{\theta}\int_{\underline{z}_t}^{\infty} \Upsilon(z,t)\alpha e^{-\alpha z}\mathbb{1}_{\left\{z > \frac{\theta}{\alpha}\ln K_t\right\}}dz$$

$$\begin{aligned}
\implies \mathcal{H} = {} & \frac{\left(A\frac{L_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\ \mu(z,t)\ dz\right)^{\frac{1}{\sigma-1}}\right)^{1-\gamma} - 1}{1-\gamma} + \chi_t\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t} \mu(z,t)dz\right) \\
& - \frac{\nu^2}{2}\Upsilon\left(\underline{z}_t, t\right)\frac{\partial \mu\left(\underline{z}_t, t\right)}{\partial z} + \int_{\underline{z}_t}^{\infty} \mu(z,t)\left(\beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2 \Upsilon(z,t)}{\partial z^2}\right)dz \\
& + \left(\varepsilon S_t\ +\ \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t} \mu(z,t)dz\right)K_t^{\theta}\int_{\underline{z}_t}^{\infty} \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z > \frac{\theta}{\alpha}\ln K_t\right\}}dz \\
& + \omega_t\left(N_t - S_t - L_t - \int_{z\geq\underline{z}_t} (\mathcal{F} + I_t)\ \mu(z,t)\ dz\right) + \Xi_t d_t\ .
\end{aligned}$$

The optimality conditions for the controls L_t, S_t, I_t and d_t are (respectively):

$$\mathcal{H}_{L_t} = 0 \implies \omega_t = \frac{c_t^{1-\gamma}}{L_t} \tag{23}$$

$$\mathcal{H}_{S_t} = 0 \implies \omega_t = \varepsilon\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty} \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z > \frac{\theta}{\alpha}\ln K_t\right\}}dz\right) \tag{24}$$

$$\mathcal{H}_{I_t} = 0 \implies \omega_t = \vartheta I_t^{-\delta}\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty} \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z > \frac{\theta}{\alpha}\ln K_t\right\}}dz\right) \tag{25}$$

$$\Xi_t d_t = 0 \implies d_t = 0 \text{ or } \Xi_t = 0$$

$$\underline{z}_t \text{ adjoint} \implies (\rho-\eta)\Xi_t = \dot{\Xi}_t - \frac{\nu^2}{2}\left(\frac{\partial \Upsilon\left(\underline{z}_t,t\right)}{\partial z}\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z} + \Upsilon\left(\underline{z}_t,t\right)\frac{\partial^2 \mu\left(\underline{z}_t,t\right)}{\partial z^2}\right)$$
$$-\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)K_t^{\theta}\alpha e^{-\alpha \underline{z}_t}\Upsilon\left(\underline{z}_t,t\right)\mathbb{1}_{\{\underline{z}_t>\frac{\theta}{\alpha}\ln K_t\}}dz \tag{26}$$

$$K_t \text{ adjoint} \implies (\rho-\eta)\chi_t = \dot{\chi}_t+\theta\frac{\dot{K}_t}{K_t}\left(K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\{z>\frac{\theta}{\alpha}\ln K_t\}}dz - \Upsilon\left(\frac{\theta}{\alpha}\ln K_t,t\right)\mathbb{1}_{\{\underline{z}_t\le\frac{\theta}{\alpha}\ln K_t\}}\right) \tag{27}$$

Finally, adjoint corresponding to $\mu(z,t)$ yields:

$$(\rho-\eta)\Upsilon(z,t) = \dot{\Upsilon}(z,t) + \beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2} + \frac{1}{\sigma-1}\frac{c_t^{1-\gamma}}{M_t}\left(\frac{e^z}{\overline{Q}_t}\right)^{\sigma-1} - \omega_t(\mathcal{F}+I_t)$$
$$+\frac{\vartheta}{1-\delta}I_t^{1-\delta}\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{z>\frac{\theta}{\alpha}\ln K_t}dz\right) \tag{28}$$

where I used M_t and $\overline{Q}_t$ as defined in the main text so that

$$\int_{z\ge\underline{z}_t}\mu(z,t)dz = M_t \quad \text{and} \quad \int_{z\ge\underline{z}_t}e^{(\sigma-1)z}\mu(z,t)dz = M_t\overline{Q}_t^{\sigma-1}$$

Finally, the transversality conditions are

$$0 = \lim_{t\to\infty}e^{-(\rho-\eta)t}\Xi_t\underline{z}_t = \lim_{t\to\infty}e^{-(\rho-\eta)t}\chi_t K_t = \lim_{t\to\infty}e^{-(\rho-\eta)t}\Upsilon(z,t)\mu(z,t)$$

$$\text{Combining Equation 24 and Equation 25} \implies I_t = \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} \implies \text{as in equilibrium!}$$

Plugging Equation 25 and Equation 23 into Equation 28 and using $O \equiv \frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$ (as in equilibrium):

$$(\rho-\eta)\Upsilon(z,t) = \dot{\Upsilon}(z,t) + \beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2} + \omega_t\left(\frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{e^z}{\overline{Q}_t}\right)^{\sigma-1} - (\mathcal{F}-O)\right)$$

Change of coordinates

With stationarity in mind, it is going to be helpful to work with relative coordinates: $q = z - \underline{z}_t$. For $q \geq 0$, define the value function $V^{\mathrm{SP}}(q,t)$ and cross-sectional distribution $m(q,t)$ by

$$V^{\mathrm{SP}}(q,t) = \Upsilon(q + \underline{z}_t, t) \quad \text{and} \quad m(q,t) = \mu(q + \underline{z}_t, t)$$

Using the chain rule and $\dot{\underline{z}}_t = d_t$, the planner's HJB becomes:

$$\begin{aligned}(\rho - \eta)V^{\mathrm{SP}}(q,t) = \dot{V}^{\mathrm{SP}}(q,t) + (\beta - d_t)\frac{\partial V^{\mathrm{SP}}(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V^{\mathrm{SP}}(q,t)}{\partial q^2} \\ + \omega_t\left[\frac{1}{\sigma - 1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{\overline{Q}}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O)\right]\end{aligned}$$

The KFE is as in Equation 9. Equation 24, Equation 26, and Equation 27 become:

$$\omega_t = \varepsilon\left(\chi_t + K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}V^{\mathrm{SP}}(q,t)\mathbb{1}_{\left\{q > \ln\frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}\right\}}dq\right), \tag{29}$$

$$\begin{aligned}(\rho - \eta)\Xi_t = \dot{\Xi}_t - \frac{\nu^2}{2}\left[\frac{\partial V^{\mathrm{SP}}(0,t)}{\partial q}\frac{\partial m(0,t)}{\partial q} + V^{\mathrm{SP}}(0,t)\frac{\partial^2 m(0,t)}{\partial q^2}\right] \\ - \left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}M_t\right)\alpha K_t^{\theta}\underline{Q}_t^{-\alpha}V^{\mathrm{SP}}(0,t)\mathbb{1}_{\left\{\frac{\theta}{\alpha}\ln K_t < 0\right\}},\end{aligned} \tag{30}$$

$$(\rho - \eta)\chi_t = \dot{\chi}_t + \theta\frac{\dot{K}_t}{K_t}\left(K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}V^{\mathrm{SP}}(q,t)\mathbb{1}_{\left\{q > \ln\frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}\right\}}dq - V^{\mathrm{SP}}\left(\ln\frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}, t\right)\mathbb{1}_{\left\{\ln\frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t} > 0\right\}}\right) \tag{31}$$

Finally, the transversality conditions become:

$$0 = \lim_{t\to\infty} e^{-(\rho-\eta)t}\Xi_t\underline{z}_t = \lim_{t\to\infty} e^{-(\rho-\eta)t}\chi_t K_t = \lim_{t\to\infty} e^{-(\rho-\eta)t}V^{\mathrm{SP}}(q,t)m(q,t)$$

And Assumption 5 becomes

$$\lim_{q\to\infty} V^{\mathrm{SP}}(q,t)m(q,t) = 0 \quad \text{and} \quad \lim_{q\to\infty} V^{\mathrm{SP}}(q,t)\frac{\partial m(q,t)}{\partial q} = 0\,.$$

Balanced Growth Path

The definition of BGP was given in Definition 2. Proposition 1 still applies. Here again, I focus on the BGP satisfying Assumption 3 – as the below will make clear, Equation 22 is sufficient since:

$$\left(\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}\right)^{\mathbf{FB}} \geq \left(\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}\right)^{\mathbf{DE}} .$$

Since the KFE is unchanged, $f_p(q)$ is as in equilibrium. For the HJB, guess and verify $V^{\text{SP}}(q,t) = \omega_t \mathcal{V}(q)$. As I will show, this is not an abuse of notation, as $\mathcal{V}(q)$ will coincide with the function defined in Proposition 3. To get there, note that

$$\text{Equation 23} \implies \frac{\dot{\omega}_t}{\omega_t} = (1-\gamma)g - \eta \, ,$$

so that by plugging back into the HJB, we know $\mathcal{V}^{\text{SP}}(q)$ satisfies the following ODE:

$$(\rho + (\gamma-1)g)\mathcal{V}(q) = (\beta - g_Q)\mathcal{V}'(q) + \frac{\nu^2}{2}\mathcal{V}''(q) + \frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{\overline{Q}}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O)$$

This matches the ODE I solve in Appendix B.1.2 to prove Proposition 3. What remains to be shown is that the boundary conditions are the same. The ones at 0 follow from Equation 30 combined with $\Xi_t = 0$ along the BGP. The latter follows directly from $g_Q > 0$ if $\theta > 0$ and from $\underline{Q}_t > K_t^{\frac{\theta}{\alpha}}$ otherwise. As such, the TVC for Ξ_t is trivially satisfied and:

$$-\frac{\nu^2}{2}\omega_t\left[\mathcal{V}'(0)f_p'(0) + \mathcal{V}(0)f_p''(0)\right] = \eta K_t \alpha K_t^{\theta}\underline{Q}_t^{-\alpha}\mathcal{V}(0)$$

Suppose $\mathcal{V}(0) \neq 0$, then the growth rate of the LHS is $(1-\gamma)g - \eta$ while that of the RHS is η, which is a contradiction. Hence $\mathcal{V}(0) = 0$. Since $f_p''(0) > 0$, it follows that $\mathcal{V}'(0) = 0$. The last boundary condition will follow from Assumption 5, which requires

$$\lim_{q\to\infty} \mathcal{V}(q)f_p(q) = 0 \quad \text{and} \quad \lim_{q\to\infty} \mathcal{V}(q)f_p'(q) = 0$$

In addition to the requirements $\alpha > \sigma - 1$ and $\zeta > \sigma - 1$, these lead to the inequality constraint on $\mathcal{V}(q)$ (see Appendix B.1.2). As a result, we end up with the same ODE and boundary condition so that $\mathcal{V}(q)$ is indeed the same.

Finally, to obtain the planner's analogue of the free entry condition, divide both sides of Equation 29 by ω_t, which yields

$$\frac{1}{\varepsilon} = \frac{\chi_t}{\omega_t} + K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}\mathcal{V}(q)dq \implies \frac{\dot{\chi}_t}{\chi_t} = \frac{\dot{\omega}_t}{\omega_t} = (1-\gamma)g - \eta \tag{32}$$

Plugging this growth rate of $\frac{\dot{\chi}_t}{\chi_t}$ into Equation 31,

$$\rho + (\gamma - 1)g = \frac{\omega_t}{\chi_t}\theta\eta\frac{K_t^\theta}{\underline{Q}_t^\alpha}\int_0^\infty \alpha e^{-\alpha q}\mathcal{V}(q)dq \implies \frac{\chi_t}{\omega_t} = \frac{\theta\eta}{\rho + (\gamma - 1)g}K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \alpha e^{-\alpha q}\mathcal{V}(q)dq$$

$$\overset{\text{Equation 32}}{\implies} \boxed{1 = \varepsilon\left(1 + \frac{\theta\eta}{\rho + (\gamma - 1)g}\right)K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \alpha e^{-\alpha q}\mathcal{V}(q)dq}$$

This is the planner's analogue of the free entry condition, and it is the only condition that differs across the first best and equilibrium. That the level of incumbent innovation per product (I) is efficient might seem surprising at first, as this activity generates positive knowledge spillovers – just as much as the entry of new firms. The way to think about it is as follows. Due to spillovers, there is too little aggregate innovation in equilibrium. However, the incumbent's innovation technology (Equation 2) satisfies an Inada condition at 0. So, both in equilibrium and the first best, initial units of innovation are carried out by incumbents – until diminishing returns push the marginal product from this technology to ε, at which point the rest of innovation to be done is carried with the linear entry technology (new firms). Along an interior BGP, this point is necessarily reached, so that the underprovision of innovation shows up entirely along the entry of new firms margin.

B.3 Derivation of the social rate of return to R&D

Here I derive the social rate of return to R&D as the return on a variational argument around a BGP (Jones and Williams, 1998). For these purposes, note that the economy is simply given by:

$$\begin{aligned}
Y_t &= L_t \underline{Q}_t A_t^{\frac{1}{\sigma-1}} \qquad \text{where } A_t \equiv A\int_0^\infty e^{(\sigma-1)q}m(q,t)dq \\
\dot{m}(q,t) &= \left(g_Q - \beta\right)\frac{\partial m(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 m(q,t)}{\partial q^2} + \varepsilon R_t K_t^\theta \underline{Q}_t^{-\alpha}\alpha e^{-\alpha q} \\
m(0,t) &= 0 \\
\dot{K}_t &= \varepsilon R_t \\
N_t &= L_t + R_t + (\mathcal{F} - O)M_t \qquad \text{where } R_t \equiv S_t + \frac{1}{1-\delta}IM_t\,.
\end{aligned}$$

Denoting by ∇ deviations from the initial balanced growth path, note that from the law of motion:

$$\forall t, \quad \nabla\dot{m}(q,t) = \mathcal{L}\nabla m(q,t) + \varepsilon\alpha e^{-\alpha q}\left(K_t^\theta \underline{Q}_t^{-\alpha}\nabla R_t + \theta K_t^{\theta-1}\underline{Q}_t^{-\alpha}R_t\nabla K_t - K_t^\theta \alpha \underline{Q}_t^{-\alpha-1}R_t\nabla\underline{Q}_t\right)$$

$$\text{where} \quad \mathcal{L} \equiv -(\beta - g_Q)\partial_q + \frac{\nu^2}{2}\partial_{qq}\,.$$

The specific variational argument of interest is:

1. from t to $t + dt$, the economy does more R&D by reducing L_t and raising R_t;
2. from $t + dt$ to $t + 2dt$, the economy "eats the proceeds" by doing sufficiently less R&D to be back at initial BGP path by $t + 2dt$.

The social rate of return is then defined as the rate of return on this variational argument as $dt \to 0$:

$$\widetilde{r} \equiv \lim_{dt \to 0} \frac{\nabla Y_{t+dt} - \frac{Y_t}{L_t} \nabla R_t}{\frac{Y_t}{L_t} \nabla R_t dt} .$$

Intuitively, this is a rate of return because $\frac{Y_t}{L_t} \nabla R_t$ is the amount of output (and hence consumption) that the variational argument sacrifices at t, while ∇Y_{t+dt} is the resulting increase in output at $t + dt$. Since m is a state variable and the variational argument starts at t, $\nabla m(q,t) = 0$, so

$$m(q, t + dt) = m(q,t) + \dot{m}(q,t)dt \implies \nabla m(q, t + dt) = \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \alpha e^{-\alpha q} \nabla R_t dt$$

In contrast

$$\begin{aligned} \nabla m(q, t + 2dt) &= \nabla m(q, t + dt) + \nabla \dot{m}(q, t + dt) dt \\ &= \nabla m(q, t + dt) + \left(\mathcal{L} \nabla m(q, t + dt) + \varepsilon \underline{Q}_{t+dt}^{-\alpha} \alpha e^{-\alpha q} \left(K_{t+dt}^{\theta} \nabla R_{t+dt} + \theta K_{t+dt}^{\theta-1} \nabla K_{t+dt} R_{t+dt} \right) \right) dt \\ \text{with} \quad & \mathcal{L} \nabla \mu(q, t + dt) = \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \alpha e^{-\alpha q} \left((\beta - g_Q)\alpha + \frac{\nu^2}{2} \alpha^2 \right) \nabla R_t dt \\ & \dot{K}_{t+dt} = K_t + \varepsilon R_t dt \implies \nabla K_{t+dt} = \varepsilon \nabla R_t dt \end{aligned}$$

The variational argument requires $\nabla m(q, t + 2dt) = 0.$ Solving for ∇R_{t+dt} in terms of ∇R_t yields

$$-\nabla R_{t+dt} = \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \nabla R_t + \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \left((\beta - g_Q)\, \alpha + \frac{\nu^2}{2} \alpha^2 \right) + \theta \varepsilon \frac{R_{t+dt}}{K_{t+dt}} \right] \nabla R_t dt .$$

The increase in output at $t + dt$ is due to higher TFP and higher productional labor:

$$Y_{t+dt} = L_{t+dt} \underline{Q}_{t+dt} A_{t+dt}^{\frac{1}{\sigma-1}} \implies \nabla Y_{t+dt} = \frac{Y_{t+dt}}{L_{t+dt}} \nabla L_{t+dt} + \frac{1}{\sigma - 1} \frac{Y_{t+dt}}{A_{t+dt}} \nabla A_{t+dt}$$

Now

$$\nabla L_{t+dt} + (\mathcal{F} - O) \nabla M_{t+dt} + \nabla R_{t+dt} = 0 \quad \text{and} \quad \nabla M_{t+dt} = K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon \nabla R_t dt$$

$$\implies \nabla L_{t+dt} = \frac{\underline{Q}_t^{-\alpha} K_t^{\theta}}{\underline{Q}_{t+dt}^{-\alpha} K_{t+dt}^{\theta}} \nabla R_t + \left[\frac{\underline{Q}_t^{-\alpha} K_t^{\theta}}{\underline{Q}_{t+dt}^{-\alpha} K_{t+dt}^{\theta}} \left((\beta - g_Q)\, \alpha + \frac{\nu^2}{2} \alpha^2 \right) + \theta \varepsilon \frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon (\mathcal{F} - O) \right] \nabla R_t dt$$

$$\Longrightarrow \nabla Y_{t+dt} = \frac{Y_{t+dt}}{L_{t+dt}} \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}}\right)^{\theta} \nabla R_t + \frac{1}{\sigma-1} \frac{Y_{t+dt}}{A_{t+dt}} \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \frac{\alpha}{\alpha-(\sigma-1)} \nabla R_t dt$$
$$+ \frac{Y_{t+dt}}{L_{t+dt}} \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}}\right)^{\theta} \left((\beta - g_Q)\,\alpha + \frac{\nu^2}{2}\alpha^2 \right) + \theta\varepsilon \frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon(\mathcal{F} - O) \right] \nabla R_t dt$$
$$\Longrightarrow \frac{\nabla Y_{t+dt} - \frac{Y_t}{L_t}\nabla R_t}{\frac{Y_t}{L_t}\nabla R_t dt} = \frac{1}{dt}\left[\frac{\frac{Y_{t+dt}}{L_{t+dt}}}{\frac{Y_t}{L_t}} \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}}\right)^{\theta} - 1 \right] + \frac{1}{\sigma-1} \frac{\frac{Y_{t+dt}}{A_{t+dt}}}{\frac{Y_t}{L_t}} \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \frac{\alpha}{\alpha-(\sigma-1)}$$
$$+ \frac{\frac{Y_{t+dt}}{L_{t+dt}}}{\frac{Y_t}{L_t}} \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}}\right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}}\right)^{\theta} \left((\beta - g_Q)\,\alpha + \frac{\nu^2}{2}\alpha^2 \right) + \theta\varepsilon \frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon(\mathcal{F} - O) \right]$$

Define

$$P_{A_t} \equiv \frac{Y_t}{L_t} \underline{Q}_t^{\alpha} K_t^{-\theta}$$

Then taking limits yields

$$\widetilde{r} = \frac{\dot{P}_{A_t}}{P_{A_t}} + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\varepsilon\frac{R_t}{K_t} - \varepsilon(\mathcal{F} - O) K_t^{\theta} \underline{Q}_t^{-\alpha} + \frac{\frac{1}{\sigma-1}\frac{Y_t}{A_t}}{P_{A_t}} \frac{\alpha}{\alpha-(\sigma-1)} \varepsilon$$

$$\text{with} \quad \frac{\dot{P}_{A_t}}{P_{A_t}} = g \ ; \ \dot{K}_t = \varepsilon R_t \ ; \ \frac{\dot{K}_t}{K_t} = \eta \ ; \ A_t = \int_0^{\infty} e^{(\sigma-1)q} M_t \frac{\alpha\zeta}{\zeta-\alpha}(e^{-\alpha q} - e^{-\zeta q}) dq$$
$$\Longrightarrow \widetilde{r} = g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta - \varepsilon(\mathcal{F} - O) K_t^{\theta} \underline{Q}_t^{-\alpha} + \frac{L_t}{M_t} \frac{\zeta - (\sigma-1)}{\zeta(\sigma-1)} \varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha}$$

This gives the social rate of return to R&D as a function of the allocations. Along the decentralization equilibrium, these allocations satisfy

$$\frac{\zeta - (\sigma-1)}{\zeta(\sigma-1)} \frac{L_t}{M_t} = (\mathcal{F} - O) \frac{\alpha}{\alpha - (\sigma-1)} \frac{\xi}{\xi + \sigma - 1} \frac{r - [g + (\sigma-1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma-1)^2]}{r - g}$$
$$\varepsilon K_t^{\theta} \underline{Q}_t^{-\alpha} = \frac{r-g}{\mathcal{F} - O} \frac{\alpha + \xi}{\xi} \frac{\alpha - (\sigma-1)}{\sigma - 1}$$
$$\Longrightarrow \widetilde{r}_{\mathbf{DE}} = g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\left(1 + \frac{\alpha}{\xi}\right)\left(1 - \frac{\alpha}{\sigma-1}\right)$$
$$+ \frac{\alpha(\alpha+\xi)}{(\sigma-1)(\xi+\sigma-1)} \left(r - g - (\sigma-1)(\beta - g_Q) - (\sigma-1)^2 \frac{\nu^2}{2} \right)$$

But from the definition of ξ in Proposition 3 : $\quad \frac{\nu^2}{2}\xi^2 - (\beta - g_Q)\xi = r - g$

Hence

$$
\begin{aligned}
\widetilde{r}_{\mathbf{DE}} &= g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\frac{\alpha+\xi}{\xi}\left(1 - \frac{\alpha}{\sigma-1}\right) + \frac{\alpha(\alpha+\xi)}{\sigma-1}\left(\frac{\nu^2}{2}(\xi - (\sigma-1)) - (\beta - g_Q)\right) \\
&= g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\frac{\alpha+\xi}{\xi}\left(1 - \frac{\alpha}{\sigma-1}\right) + \frac{\alpha(\alpha+\xi)}{\xi(\sigma-1)}\left(r - g - \frac{\nu^2}{2}\xi(\sigma-1)\right) \\
&= \theta\eta + g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + (\alpha+\xi)\left(\frac{\nu^2}{2}\xi - (\beta - g_Q) - \frac{\nu^2}{2}\alpha\right) \\
&= \theta\eta + g + \frac{\nu^2}{2}\xi^2 - (\beta - g_Q)\xi = \theta\eta + g + r - g \implies \boxed{\widetilde{r}_{\mathbf{DE}} = r + \theta\eta}
\end{aligned}
$$

B.4 Proof of Proposition 5 and Proposition 6

The unconditional density $\ell(a)$ is obtained by averaging the conditional density $\ell(a|q)$ over the distribution of initial draws, noting that $\ell(a|q)$ is the density of the first-passage time from above of a drifted Brownian motion (see equation 3.2.13 in Redner (2001)):

$$
\ell(a) = \int_0^\infty \alpha e^{-\alpha q}\ell(a|q)dq \quad \text{with} \quad \ell(a|q) = \frac{q}{\nu\sqrt{2\pi a^3}}\exp\left(-\frac{\left(q + (\beta - g_Q)\,a\right)^2}{2\nu^2 a}\right).
$$

Turning to the integral equation; for the firm to have exited by age A, its initial product must have died at some age $0 \le a \le A$. With $\mathcal{L}$ the initial product's lifespan, by the law of total probability:

$$
\Gamma(A) = \Pr(f\text{'s lifespan} \le A) = \int_0^A \ell(a)\Pr(f\text{'s lifespan} \le A \mid \mathcal{L} = a)\,da\,.
$$

If firms were forever single product, the conditional probability in above expression would be 1. With $x > 0$, and conditional on $\mathcal{L} = a$, the initial product may give birth at ages $s \in (0, a)$. By the recursive structure of the branching process, a birth at age s produces a lineage that is obsolete by age A with probability $\Gamma(A - s)$. Since births on $(0, a)$ form a Poisson point process of rate x, splitting this interval into subintervals of length Δ, indexed by i and with midpoints s_i:

$$
\begin{aligned}
\Pr(f\text{'s lifespan} \le A \mid \mathcal{L} = a) &= \lim_{\Delta\downarrow 0}\prod_i (x\Delta)\Gamma(A - s_i) + (1 - x\Delta)1 = \lim_{\Delta\downarrow 0}\prod_i \exp\left(x\Delta\left[\Gamma(A - s_i) - 1\right]\right) \\
&= \lim_{\Delta\downarrow 0}\exp\left(\sum_i x\Delta\left[\Gamma(A - s_i) - 1\right]\right) = \exp\left(\int_0^a x\left[\Gamma(A - s) - 1\right]ds\right)\,.
\end{aligned}
$$

This leverages that $\{n_f(a)\}_{a\ge 0}$ is a single-type Crump–Mode–Jagers (general age-dependent) branching process; see Crump and Mode (1968, 1969) for foundational theory and Jagers (1975) for applications in population dynamics.

B.5 Proof of Proposition 7

Integrating both sides of the n^{th} PDE over $(0,\infty)^n$ and using linearity of the integral yields:

$$\begin{aligned}\int_{\mathbf{q}\in(0,\infty)^n}(\eta+xn)f_n(\mathbf{q})\,d\mathbf{q} \;=\; &\int_{\mathbf{q}\in(0,\infty)^n}\sum_{j=1}^{n}\left[\left(g_Q-\beta\right)\frac{\partial f_n(\mathbf{q})}{\partial q_j}\;+\;\frac{\nu^2}{2}\frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2}\right]\,d\mathbf{q}\\ &+\int_{\mathbf{q}\in(0,\infty)^n} x\,\frac{\Psi_{n-1}}{\Psi_n}\sum_{j=1}^{n}\alpha e^{-\alpha q_j} f_{n-1}\left(\mathbf{q}^{\setminus j}\right)\,d\mathbf{q}\\ &+\frac{\nu^2}{2}\frac{n\,\Psi_{n+1}}{\Psi_n}\frac{1}{n+1}\sum_{j=1}^{n+1}\int_{\mathbf{q}\in(0,\infty)^n}\frac{\partial f_{n+1}\left(\mathbf{q}^{j\to 0}\right)}{\partial q_j}d\mathbf{q}\end{aligned}$$

The goal is to simplify each of four integrals that appear in the expression. Starting with the LHS,

$$f_n \text{ pdf on } (0,\infty)^n \implies \int_{\mathbf{q}\in(0,\infty)^n}(\eta+xn)f_n(\mathbf{q})\,d\mathbf{q} \;=\; \eta+xn\;.$$

On the RHS, three integrals show up:

- Since f_{n-1} is a pdf on $(0,\infty)^{n-1}$ and $\alpha e^{-\alpha q}$ a pdf on $(0,\infty)$,

$$\begin{aligned}&\int_{\mathbf{q}\in(0,\infty)^i} x\frac{\Psi_{n-1}}{\Psi_n}\sum_{j=1}^{n}\alpha e^{-\alpha q_j} f_{i-1}\left(\mathbf{q}^{\setminus j}\right)\,d\mathbf{q}\\ &=\sum_{j=1}^{n}x\frac{\Psi_{n-1}}{\Psi_n}\left(\int_0^\infty \alpha e^{-\alpha q_j}\,dq_j\right)\left(\int_{\mathbf{q}^{\setminus j}\in(0,\infty)^{n-1}} f_{n-1}\left(\mathbf{q}^{\setminus j}\right)d\mathbf{q}^{\setminus j}\right)=\sum_{j\,=\,1}^{n}x\frac{\Psi_{n-1}}{\Psi_n}\;=\;n\,x\,\frac{\Psi_{n-1}}{\Psi_n}\end{aligned}$$

- Leveraging the definition of λ_n for all n (specifically for $n+1$)

$$\frac{\nu^2}{2}\frac{n\,\Psi_{n+1}}{\Psi_n}\frac{1}{n+1}\sum_{j=1}^{n+1}\int_{\mathbf{q}\in(0,\infty)^n}\frac{\partial f_{n+1}\left(\mathbf{q}^{j\to 0}\right)}{\partial q_j}d\mathbf{q}\;=\;\frac{n\,\Psi_{n+1}}{\Psi_n}\lambda_{n+1}$$

- Finally, to evaluate the integral on the first line of the right hand side, define the continuously differentiable vector field on $(0,\infty)^n$:

$$V(\mathbf{q})\;=\;(V_1(\mathbf{q}),\ldots,V_n(\mathbf{q}))\quad\text{where}\quad V_j(\mathbf{q})\;\equiv\;g_Q\,f_n(\mathbf{q})\;+\;\frac{\nu^2}{2}\frac{\partial f_n(\mathbf{q})}{\partial q_j}$$

  The divergence of this vector field is

$$\text{div}\,V\;=\;\sum_{j=1}^{n}g_Q\frac{\partial f_n(\mathbf{q})}{\partial q_j}\;+\;\frac{\nu^2}{2}\frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2}$$

$$\Longrightarrow \int_{(0,\infty)^n} \left(g_Q \sum_{j=1}^{n} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \; + \; \frac{\nu^2}{2} \sum_{j=1}^{n} \frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2} \right) d\mathbf{q} \; = \; \int_{(0,\infty)^n} \text{div}\, V(\mathbf{q}) \; d\mathbf{q}$$

I can now apply the divergence theorem on $(0, R)^i$ and then take limits as $R \longrightarrow \infty$. This transforms the volume integral of the divergence over the positive orthant into a surface integral on the hyperplanes delimiting the positive orthant:

$$\begin{aligned}
\int\limits_{(0,R)^n} \text{div} V(\mathbf{q}) d\mathbf{q} &= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} V \cdot (-e_j) dS + \int_{\{q_j=R\}} F \cdot (e_j) dS \right) \\
&= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} - \left(g_Q f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS + \int_{\{q_j=R\}} \left(g_Q f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS \right) \\
&= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} -\frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} dS + \int_{\{q_j=R\}} \left(g_Q f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS \right)
\end{aligned}$$

where the last step uses the boundary condition that f_n vanishes on any of the hyperplanes delimiting the positive orthant. Taking limits as $R \longrightarrow \infty$, only the first integral within each sum survives, as both f_n and its partial derivatives vanish when any of its entries grows to infinity. And each of these integrals is a surface integral on one of the n hyperplanes delimiting the positive orthant, so the integration is with respect to all variables other than q_j, where q_j itself is zero. Using the notation defined above:

$$\int_{(0,\infty)^n} \text{div}\, V(\mathbf{q}) \; = \; \sum_{j=1}^{n} -\frac{\nu^2}{2} \int_{\{q_j=0\}} \frac{\partial f_i(\mathbf{q})}{\partial q_j} \; = \; -\, n \, \lambda_n$$

Putting it all together,

$$\eta + xn \; = \; -\, n \, \lambda_n \; + \; n \, x \, \frac{\Psi_{n-1}}{\Psi_n} \; + \; \frac{n\Psi_{n+1}}{\Psi_n} \lambda_{n+1}$$

Multiplying both sides by $\frac{\Psi_n}{n}$, I get:

$$\frac{\eta}{n} \Psi_n \; = \; -\, (x \; + \; \lambda_n) \, \Psi_n \; + x \Psi_{n-1} \; + \; \lambda_{n+1} \Psi_{n+1}$$

B.6 Dimension Reduction

Lemma 1. Given the ansatz from Equation 18, the stationary firm size distribution consists of sequences Ψ_n and φ_n satisfying the following system of coupled ordinary differential equations:

$$\frac{\nu^2}{2}\varphi_1''(q) + (g_Q - \beta)\varphi_1'(q) - (\eta + x)\varphi_1(q) = -\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}\alpha e^{-\alpha q} - \frac{\Psi_2}{\Psi_1}\lambda_2\varphi_2(q) \,,$$

$$\text{and for } n > 1, \quad \frac{\nu^2}{2}\varphi_n''(q) + (g_Q - \beta)\varphi_n'(q) - (\eta + nx + (n-1)\lambda_n)\varphi_n(q)$$
$$= -\frac{x\Psi_{n-1}}{\Psi_n}\left(\alpha e^{-\alpha q} + (n-1)\varphi_{n-1}(q)\right) - n\frac{\Psi_{n+1}}{\Psi_n}\lambda_{n+1}\varphi_{n+1}(q)$$

with $\forall n \geq 1$, $\varphi_n(0) = 0$ and $\lambda_n = \frac{\nu^2}{2}\varphi_n'(0)$.

Proof. For $n > 1$, the n^{th} ODEs is obtained by plugging in the ansatz into the n^{th} PDE then integrating both sides over $(0, \infty)^{n-1}$, so by integrating out all dimensions but one. □

C Computational Appendix

C.1 Solving integral equation from Proposition 6

$$\Gamma(A) = \int_0^A \ell(a) \exp\left(\int_0^a x\,[\Gamma(A-s) - 1]\,ds\right) da$$

As I highlighted in the main text, this can be computed with a marching forward algorithm, as each $\Gamma(A)$ only depends on lower ages and $\Gamma(0) = 0$. To speed up the process (specifically the inner integral), define $R(A) = \exp\left(-x\int_0^A \Gamma(u)du\right)$, so that $R_0 = 1$ and by the fundamental theorem of calculus $R'(A) = -xR(A)\Gamma(A)$. Now to see how this simplifies the integral equation:

$$\begin{aligned}
\Gamma(A) &= \int_0^A \ell(a)\exp\left(\int_0^a x\,[\Gamma(A-s)-1]\,ds\right) da = \int_0^A \ell(a)\exp\left(x\int_0^a \Gamma(A-s)ds - xa\right) da \\
&= \int_0^A \ell(a)\exp\left(x\int_{A-a}^A \Gamma(u)du - xa\right) da = \int_0^A \ell(a)\exp\left(x\int_0^A \Gamma(u)du - x\int_0^{A-a}\Gamma(u)du - xa\right) da \\
&= \int_0^A \ell(a)\exp\left(x\int_0^A \Gamma(u)du\right)\exp\left(-x\int_0^{A-a}\Gamma(u)du\right)\exp(-xa)\,da = \int_0^A \ell(a)\frac{1}{R(A)}R(A-a)e^{-xa}da \\
&\implies \Gamma(A)R(A) = \int_0^A e^{-xa}\ell(a)R(A-a)da
\end{aligned}$$

Evaluation of the RHS does not depend on values at A, since $\ell(0) = 0$. So I get $\Gamma(A)R(A)$ by simply evaluating the integral. I then get $R(A)$ using the ODE $R'(A) = -xR(A)\Gamma(A)$ and previous value for $R(A)$. I then divide $\Gamma(A)R(A)$ by $R(A)$ to get $\Gamma(A)$.

For the hazard rate, I need $\Gamma'(A)$. Denoting $Y(A) = \Gamma(A)R(A)$, it follows that

$$\Gamma'(A) = \frac{Y'(A)}{R(A)} + x\Gamma(A)^2$$

$$\text{where } \; Y'(A) = e^{-xA}\ell(A) - x\int_0^A e^{-xa}\ell(a)Y(A-a)da$$

Because:

$$\begin{aligned}
Y(A) &= \int_0^A e^{-xa}\ell(a)R(A-a)da \\
Y'(A) &= e^{-xA}\ell(A) + \int_0^A e^{-xa}\ell(a)R'(A-a)da \quad \text{ since } \; R(A) = 1 \\
Y'(A) &= e^{-xA}\ell(A) - x\int_0^A e^{-xa}\ell(a)Y(A-a)da \quad \text{ using ODE for } \; R\,.
\end{aligned}$$

To see how this helps getting $\Gamma'(A)$:

$$Y(A) = \Gamma(A)R(A) \implies Y'(A) = \Gamma'(A)R(A) + \Gamma(A)R'(A) \implies \Gamma'(A) = \frac{Y'(A)}{R(A)} + \Gamma(A)\frac{R'(A)}{R(A)}$$

Using the ODE for R, this simplifies to

$$\Gamma'(A) = \frac{Y'(A)}{R(A)} + \Gamma(A)^2 x\,.$$

Details for numerical implementation given a set of parameters. To get the hazard rate of firm exit up to age 20, I work on a discrete age grid with 20500 points between 0 and 20.5. I use the above algorithm to "fill" D in a single forward march, using Numpy's builtin numerical integration with the trapezoid rule.

GMM to identify parameters. My GMM objective is an equally weighted least squares deviations of model vs empirical hazard rate of firm exit at ages 1 through 19. I use the least squares routine provided by Python's SciPy library.

C.2 Solving for stationary firm size distribution

A first transformation of the system. Given that my GMM strategy identifies x, $\alpha(g_Q - \beta) = \theta\eta - \alpha\beta$, and $\alpha\nu$, I start by rewriting the system from Lemma 1 in terms of these parameter combinations and η. To do so, I do the following change of variables

$$s \equiv \alpha q \quad \text{and} \quad \delta_n(s) \equiv \frac{\varphi_n\left(\frac{s}{\alpha}\right)}{\alpha} \implies \varphi_n(q) = \alpha\delta_n(\alpha q)\ ,$$

and note that

$$\frac{\nu^2}{2}\alpha\zeta = \frac{1}{2}\alpha\left(g_Q - \beta\right) + \frac{1}{2}\sqrt{\left(\alpha\left(g_Q - \beta\right)\right)^2 + 2\eta\left(\alpha\nu\right)^2}$$

so that this term in the first ODE is taken care of. Plugging into the initial system yields:

$$\forall n \geq 1\ ,\ \ \delta_n(0) = 0\ \ ;\ \ \int_0^\infty \delta_n(s)ds = 1\ \ ;\ \ \lambda_n = \frac{(\alpha\nu)^2}{2}\delta_n'(0)\ \ ;\ \ \lim_{s\to\infty}\delta(s) = \lim_{s\to\infty}\delta'(s) = 0$$

$$\frac{(\alpha\nu)^2}{2}\delta_1''(s) + \alpha(g_Q - \beta)\delta_1'(s) - (\eta + x)\delta_1(s) = -\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}e^{-s} - \frac{\Psi_2}{\Psi_1}\lambda_2\delta_2(s)$$

and for $n > 1$

$$\frac{(\alpha\nu)^2}{2}\delta_n''(s) + \alpha(g_Q - \beta)\delta_n'(s) - \left(\eta + nx + (n-1)\lambda_n\right)\delta_n(s) = -x\frac{\Psi_{n-1}}{\Psi_n}\left(e^{-s} + (n-1)\delta_{n-1}(s)\right) - n\frac{\Psi_{n+1}}{\Psi_n}\lambda_{n+1}\delta_{n+1}($$

with the recurrence relation that only depends on identified parameters:

$$\begin{cases} \eta\ \Psi_1 & = -\left(x + \lambda_1\right)\Psi_1\ +\ \lambda_2\Psi_2\ +\ \eta\ +\ \frac{\nu^2}{2}\alpha\ \zeta\ -\ x \\ \frac{\eta}{n}\ \Psi_n & = -\left(x + \lambda_n\right)\Psi_n\ +\ \lambda_{n+1}\Psi_{n+1}\ +\ x\Psi_{n-1} \qquad \text{for}\ \ n\ >\ 1 \end{cases}$$

The limit (fixed point) of the system becomes

$$\delta_\infty(s) = \frac{\frac{\tau}{\alpha}}{\frac{\tau}{\alpha} - 1}\left(e^{-s} - e^{-\frac{\tau}{\alpha}s}\right) \quad \text{where} \quad \frac{\tau}{\alpha} = 2\frac{\alpha(g_Q - \beta)}{(\alpha\nu)^2}$$

along with

$$\lambda_\infty = \alpha(g_Q - \beta) \quad \text{and} \quad \frac{\Psi_{n+1}}{\Psi_n}\ \sim\ \frac{x}{\lambda_\infty}\ <\ 1\ .$$

A second transformation of the system. While the above system is in principle solvable, to guarantee numerical stability I do a second transformation that drastically improves the system's conditioning. In that vein, let:

$$u_n(s) \equiv e^{\frac{\tau}{\alpha}s}\delta_n(s) \quad \text{and} \quad R_n \equiv \frac{\Psi_n}{\Psi_{n+1}}$$

The motivation for working with ratios is that as n grows large, Ψ_n converges to zero, while the ratio of consecutive terms converges to a strictly positive number. Defining the initial condition

$$R_0 \;=\; \frac{1}{x}\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1} \;,$$

The recurrence is then given for all $n \geq 1$ by

$$-\left(\frac{\eta}{n} + x + \lambda_n\right) + \frac{\lambda_{n+1}}{R_n} + xR_{n-1} \;=\; 0 \;.$$

Plugging the recurrence into the RHS to avoid having divisions by R_n, the $n = 1$ ODE becomes:

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}u_1''(s) = &-\left[\alpha(g_Q - \beta) - \frac{\tau}{\alpha}(\alpha\nu)^2\right]u_1'(s) + \left[\eta + x + \frac{\tau}{\alpha}\alpha(g_Q - \beta) - \left(\frac{\tau}{\alpha}\right)^2\frac{(\alpha\nu)^2}{2}\right]u_1(s)\\ &- xR_0e^{-\left(1-\frac{\tau}{\alpha}\right)s} + \left[xR_0 - (\eta + x + \lambda_1)\right]u_2(s) \;.\end{aligned}$$

The $n > 1$ ODE becomes:

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}u_n''(s) = &-\left[\alpha(g_Q - \beta) - \frac{\tau}{\alpha}(\alpha\nu)^2\right]u_n'(s) + \left[\eta + nx + (n-1)\lambda_n + \frac{\tau}{\alpha}\alpha(g_Q - \beta) - \left(\frac{\tau}{\alpha}\right)^2\frac{(\alpha\nu)^2}{2}\right]u_n(s)\\ &- xR_{n-1}\left(e^{-\left(1-\frac{\tau}{\alpha}\right)s} + (n-1)u_{n-1}(s)\right) + n\left[xR_{n-1} - \left(\frac{\eta}{n} + x + \lambda_n\right)\right]u_{n+1}(s) \;.\end{aligned}$$

The boundary conditions are with $u_n(0) = 0$; $\lambda_n = \frac{(\alpha\nu)^2}{2}u_n'(0)$ and, for s_max large enough $\delta_n(s_{max}) = 0$ (exponentially decaying tail).

Numerical solution. I transform the system of 2nd order ODEs into a system of 1st order ODEs by introducing the derivatives u' as auxiliary variables. I solve for u_n, u_n', λ_n, and R_n using solve-bvp from Python's Scipy library, which is designed to solve such systems of differential-algebraic equations. The n_0 I choose for truncation purposes is 25 (and checked robustness to decreasing n_0 to 40), with a tolerance of 10^{-5} and $s_{max} = 8$.

Verification of solution. After solving for $u_n(q)$, I obtain $\delta_n(q)$ using

$$\delta_n(s) = e^{-\frac{\tau}{\alpha}s}u_n(s) \;.$$

As discussed in the main text, the theory provides a transparent way to verify the solution:

$$\sum_{n=1}^{\infty}\Psi_n\varphi_n(q) = \frac{\alpha\;\zeta}{\zeta - \alpha}\left(e^{-\alpha q} \;-\; e^{-\zeta q}\right)$$

In terms of what I solved for numerically, this becomes

$$\sum_{n=1}^{\infty} \Psi_n \delta_n(s) = \frac{\zeta/\alpha}{\zeta/\alpha - 1}\left(e^{-s} - e^{-\frac{\zeta}{\alpha}s}\right) \text{ where } \frac{\zeta}{\alpha} = \frac{\alpha(g_Q - \beta) + \sqrt{(\alpha(g_Q - \beta))^2 + 2\eta(\alpha\nu)^2}}{(\alpha\nu)^2}.$$

The absolute deviation between the LHS (computed numerically) and the RHS (closed form) has mean 0.0003 and maximum 0.0013.

D Data Appendix

This appendix provides detailed information on the data sources, sample construction, and variable definitions used in the empirical analysis of the paper. Section D.1 describes the product level data used to document the facts in Section 4. Section D.2 describes the firm level data used for the main quantitative estimation in Section 5.

D.1 NielsenIQ Retail Scanner Dataset

Sample Construction. The analysis is restricted to a balanced panel of approximately 25,400 retail stores that are continuously present for the entire 14 year period. This restriction ensures that product exit is not mechanically driven by store closures.

Each UPC in the data is already assigned to one of roughly 120 product group codes. In building my sample, I exclude unclassified products, fresh produce, non-scannable ("magnet") products, control brands, and products classified under "seasonal", "prep food-deli", or groups that get discontinued by Nielsen (deferred modules). These exclusions insure that products in my sample can be consistently mapped across different retailers at a point in time as well as over time.

Variable definitions. I obtain a UPC's sales (in \$) and volume of sales (quantity sold) by summing across retailers in a given time period. Dividing the former by the latter yields the average unit price. A UPC is defined as entering in year t if it has zero sales in year $t-1$ and positive sales in year t. A UPC is defined as exiting in year t if it has positive sales in year t and zero sales in year $t+1$. A UPC's age in a given year is defined as the current year minus its first year of appearance in the sample. UPCs present in 2006 are considered left-censored

Data cleaning. To guarantee accurate measurement of entry and exit, I drop any UPC that records more than one entry or exit event over the sample period.

To allow for meaningful comparisons of quantity and price within a product group, I harmonize product size units. First, I convert units to a common standard where possible (e.g., pounds and kilograms are converted to ounces; liters and quarts are converted to milliliters). For a small number of products, Nielsen provides a secondary size. If a product's primary unit does not match the modal unit of its product group, but its secondary unit does, I use the secondary unit for harmonization. If after these harmonization steps a product group's modal unit of measure accounts for less than 95% of sales, then this group is excluded from my analysis. Similarly, if the group has fewer than 100 products, it is excluded from the sample.

D.2 Publicly Available U.S. Census Data

To quantify spillovers I use publicly available tabulations based on the U.S. Census Longitudinal Business Database (LBD). This is an administrative dataset covering the universe of private nonfarm businesses in the U.S. As it tracks establishments, in tabulations based on this dataset, a firm exits is defined as all its establishments closing.

Estimation across all private nonfarm businesses. The firm exit rate at ages 1 through 19 that I use are provided in the replication package of Sterk, Sedláček and Pugsley (2021).

Estimation at the 2-digit sector level. I use the Business Dynamics Statistics, publicly provided by the U.S. Census Bureau. Specifically, I use the State by Firm Age two-way tabulation, which provides, for each 2-digit NAICS sector, the number of firms as well as the number of exiting firms at ages 1, 2, 3, 4, 5, 8, 13, and 18. Since the underlying data source is the LBD (which starts in 1978), I use data from 1996 onward to avoid having any left censored firm in any of my age bins. I define the firm exit rate in a cell as the number of exiting firms divided by the average number of firms between last the previous and current year.

E Additional Figures and Tables

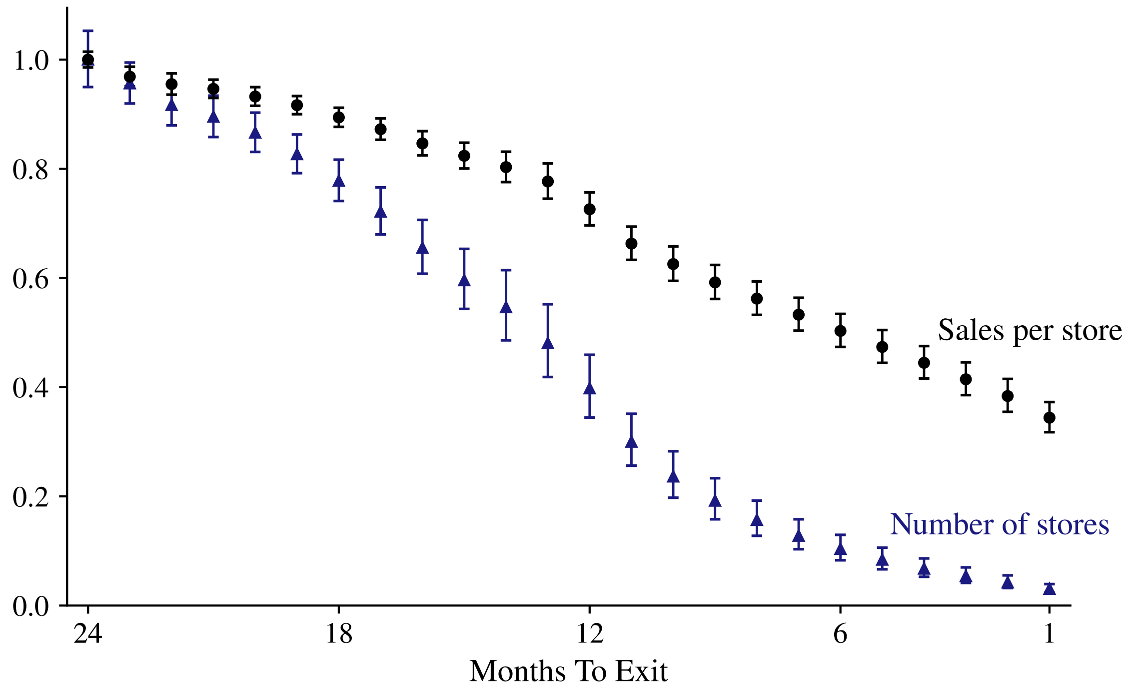
E.1 Empirical Results

To document the gradual exit process along both the extensive (number of stores at which UPC is sold) and intensive (sales per store) margins, I run the following regressions:

$$\log \text{Sales per store}_{pt} = \gamma_p + \sum_{m=1}^{24} \pi_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ , \tag{33}$$

$$\log \text{Number of stores}_{pt} = \gamma_p + \sum_{m=1}^{24} \kappa_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \ ; \tag{34}$$

Figure E1: Extensive and intensive margins in the lead up to exit

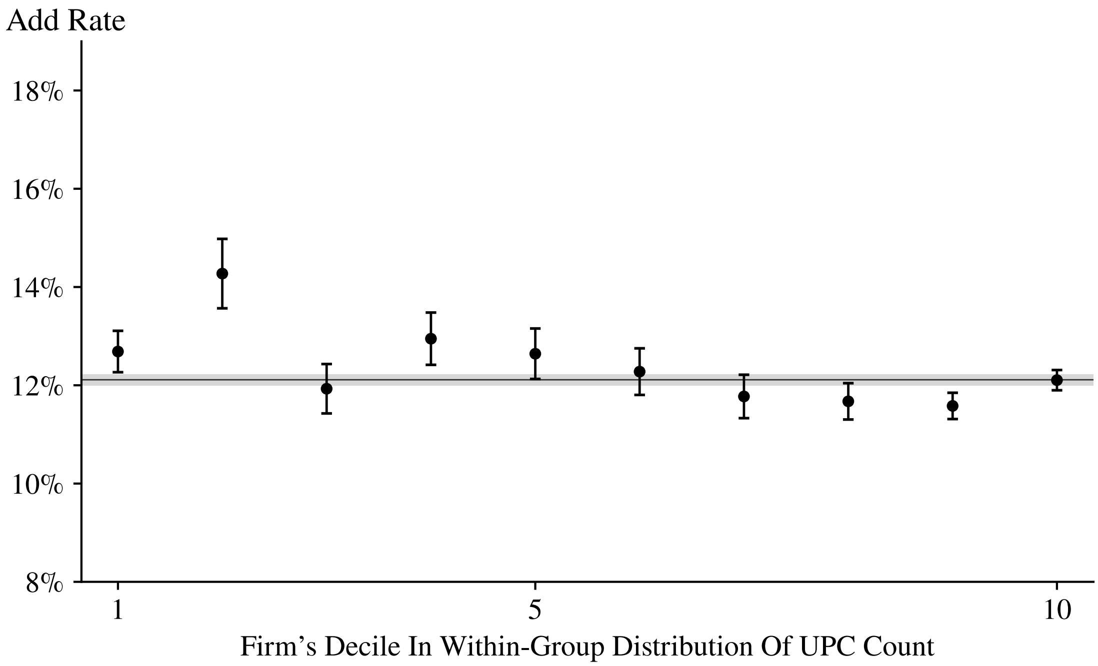


Notes: Sales per store curve corresponds to path of $\exp(\pi_m)$ from Equation 33, normalized such that $\exp(\pi_{24}) = 1$; underlying regression has 54M observations. Number of stores curve corresponds to path of $\exp(\kappa_m)$ from Equation 34, normalized such that $\exp(\kappa_{24}) = 1$; underlying regression has 54M observations. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

where p indexes a UPC, g its group (product category), and t a month, with γ_p a UPC fixed effect, γ_{gt} a group-month fixed effect, and D^m_{pt} a dummy variable equal to 1 m months prior to the UPC's exit. Figure E1 plots $\exp(\pi_m)$ and $\exp(\kappa_m)$, respectively representing the paths of sales per store and number of stores in the two years leading up to the UPC's exit.

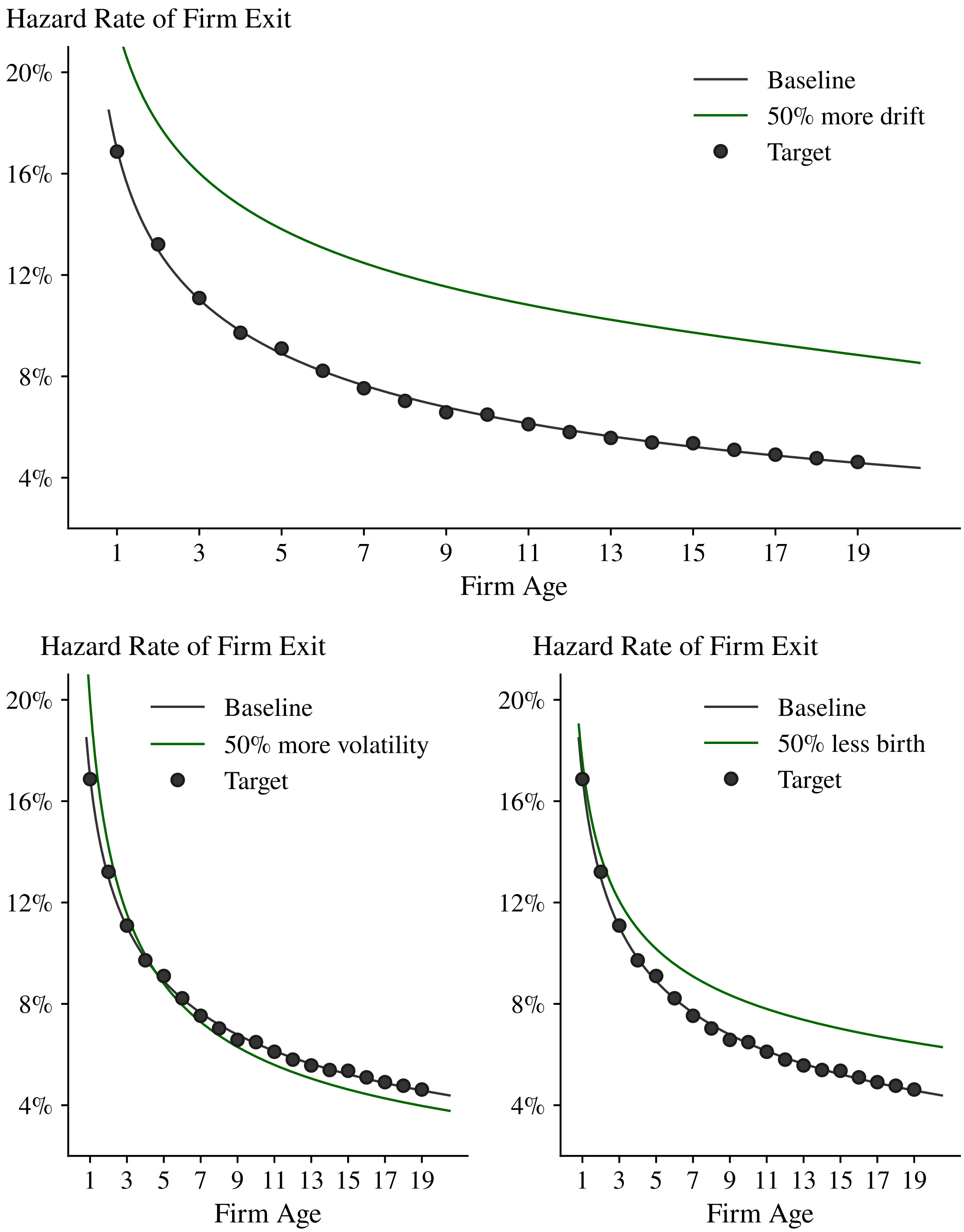
E.2 Quantitative Results from the Model

Figure E2: UPC Addition Rate Across Multiproduct Firms



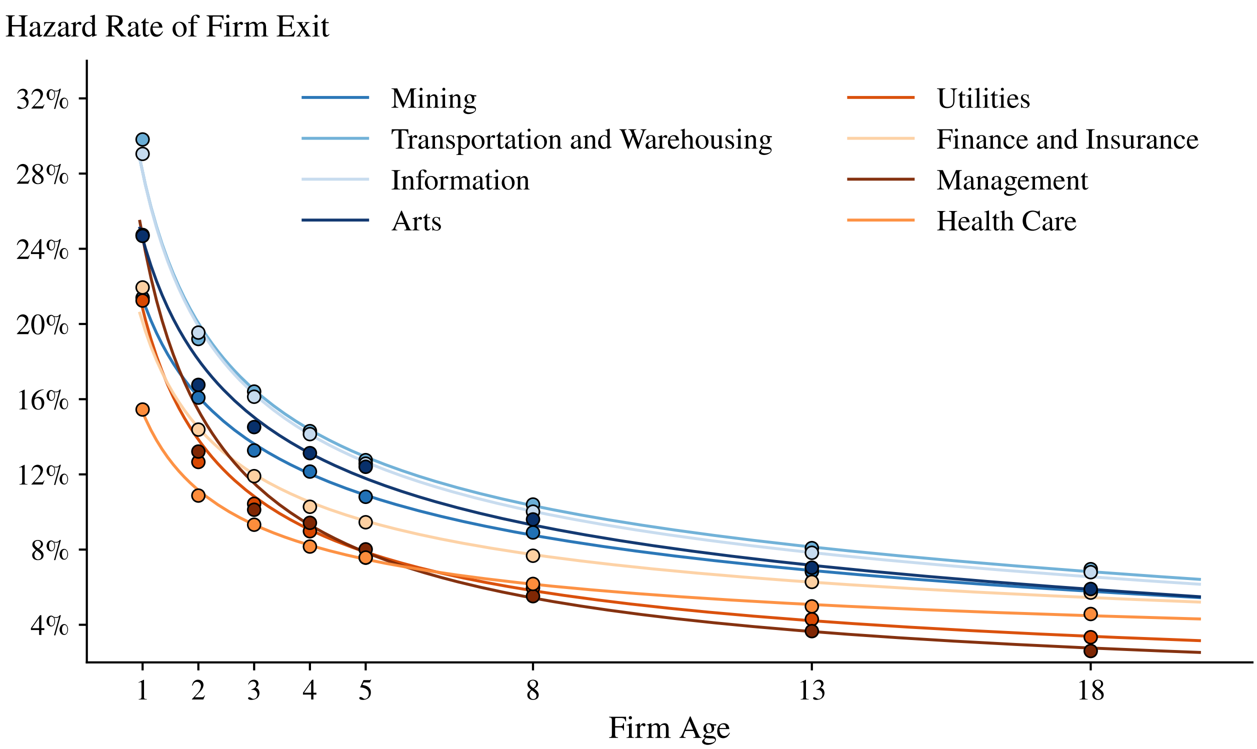
Notes: Cattaneo et al. (2024) binscatter with 10 bins, corresponding to the deciles of the distribution of UPC count among multiproduct firms within a group in year t. New UPCs are those for which the current year is the first year of sales; rates are obtained through division by firm's UPC count in corresponding group, averaged between previous and current year. Vertical bars are 95% pointwise confidence intervals. Horizontal line is inverse-weighted mean, with 95% confidence interval around it. Underlying number of firm-year observations is 264,809. Single product firms are not shown because they account for 42% of firms – among them, UPC add rate is 5.2%.

Figure E3: Intuition for identification



Notes: Each panel shows the effect of changing one of the three estimated statistics, while holding the other two fixed.

Figure E4: Firm Exit by Age in Sectors with Largest and Smallest Estimated Wedges



Notes: Sectors in blue are the four in which I estimate the largest wedge. Sectors in orange are the four in which I estimate the smallest wedge. Dots correspond to empirical moments, each curve is the model's fit from a GMM targeting that sector's profile of firm exit by age. The GMM objective puts equal weights on ages 1 through 5, and five times the weight on ages 8, 14, and 18. Underlying data is from Business Dynamics Statistics for the years 1996 to 2019.

Table E3: GMM Estimation Results Across Sectors

Sector	Firm Exit Rate	Share of firms	Share of emp	Estimation Results $\theta\eta - \alpha\beta$	$\alpha\nu$	x
Agriculture, Forestry, Fishing, and Hunting	8.9%	0.4%	0.1%	11.1%	0.50	0
Mining, Quarrying, and Oil and Gas Extraction	8.2%	0.3%	0.5%	21.5%	0.38	0.16
Utilities	4.2%	0.1%	0.6%	5.5%	0.62	0
Construction	9.0%	11.6%	5.4%	13.0%	0.57	0
Manufacturing	6.7%	5.1%	11.6%	15.8%	0.36	0.14
Wholesale Trade	7.8%	5.7%	5.0%	15.6%	0.44	0.09
Retail Trade	9.2%	12.2%	13.0%	15.3%	0.55	0.04
Transportation and Warehousing	10.8%	2.8%	3.7%	18.9%	0.70	0
Information	10.2%	1.2%	2.9%	18.4%	0.71	0
Finance & Insurance	7.7%	4.1%	5.3%	10.5%	0.48	0
Real Estate & Leasing	9.1%	4.5%	1.7%	13.2%	0.49	0
Professional, Scientific, & Tech Services	8.5%	12.2%	6.5%	11.7%	0.53	0
Management of Companies & Enterprises	3.6%	0.5%	2.7%	3.3%	0.88	0
Administrative & Support & Waste Management & Remediation Services	9.7%	5.3%	7.9%	14.2%	0.59	0
Educational Services	6.9%	1.3%	2.6%	15.9%	0.41	0.10
Health Care and Social Assistance	6.2%	10.8%	14.3%	7.0%	0.36	0
Arts, Entertainment, and Recreation	8.6%	1.7%	1.7%	25.6%	0.47	0.18
Accommodation and Food Services	10.3%	8.0%	9.8%	16.6%	0.55	0.02
Other Services (except Public Administration)	6.2%	12.0%	4.6%	9.3%	0.57	0

Notes: Underlying data are Business Dynamics Statistics for 1996-2019. The estimation targets the sector's profile of firm exit by age, specifically at ages 1, 2, 3, 4, 5, 8, 13, and 18.