Matrix differentiation using limits

let us take

$$\textbf{Input Matrix, } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}$$

$$\textbf{Weight Matrix, } W = \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{j1} \end{bmatrix}$$

$$\textbf{Output Matrix } Y = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{i1} \end{bmatrix}$$

$$\textbf{Y Predict Matrix, } \hat{Y} = \begin{bmatrix} \hat{y}_{11} \\ \hat{y}_{21} \\ \vdots \\ \hat{y}_{i1} \end{bmatrix}$$

Loss [MSE]

$loss = \frac{1}{N}\sum(Y - \hat{Y})^2 \ldots\ldots$ here N is the number of rows in input matrix A

$loss = \frac{1}{N}\sum(Y - A.W)^2$

$$loss = \frac{1}{N}\sum\left\{\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{i1} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix} \cdot \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{j1} \end{bmatrix}\right\}^2$$

$$loss = \frac{1}{N}\sum\left\{\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{i1} \end{bmatrix} - \begin{bmatrix} a_{11}w_{11} + a_{12}w_{21} + \dots + a_{1j}w_{j1} \\ a_{21}w_{11} + a_{22}w_{21} + \dots + a_{2j}w_{j1} \\ \vdots \\ a_{i1}w_{11} + a_{i2}w_{21} + \dots + a_{ij}w_{j1} \end{bmatrix}\right\}^2$$

$$loss = \frac{1}{N}\sum\left\{\begin{bmatrix} y_{11} - a_{11}w_{11} + a_{12}w_{21} + \cdots + a_{1j}w_{j1} \\ y_{21} - a_{21}w_{11} + a_{22}w_{21} + \cdots + a_{2j}w_{j1} \\ \vdots \\ y_{i1} - a_{i1}w_{11} + a_{i2}w_{21} + \cdots + a_{ij}w_{j1} \end{bmatrix}\right\}^2$$

$$loss = \frac{1}{N}\sum\left\{\begin{bmatrix} (y_{11} - a_{11}w_{11} + a_{12}w_{21} + \cdots + a_{1j}w_{j1})^2 \\ (y_{21} - a_{21}w_{11} + a_{22}w_{21} + \cdots + a_{2j}w_{j1})^2 \\ \vdots \\ (y_{i1} - a_{i1}w_{11} + a_{i2}w_{21} + \cdots + a_{ij}w_{j1})^2 \end{bmatrix}\right\}$$

$$loss = \frac{1}{N}\left\{\begin{matrix} (y_{11} - a_{11}w_{11} + a_{12}w_{21} + \cdots + a_{1j}w_{j1})^2 \\ +(y_{21} - a_{21}w_{11} + a_{22}w_{21} + \cdots + a_{2j}w_{j1})^2 \\ \vdots \\ +(y_{i1} - a_{i1}w_{11} + a_{i2}w_{21} + \cdots + a_{ij}w_{j1})^2 \end{matrix}\right\}$$

Grad

$$loss(W) = \frac{1}{N}\left\{\begin{matrix} (y_{11} - a_{11}w_{11} + a_{12}w_{21} + \cdots + a_{1j}w_{j1})^2 \\ +(y_{21} - a_{21}w_{11} + a_{22}w_{21} + \cdots + a_{2j}w_{j1})^2 \\ \vdots \\ +(y_{i1} - a_{i1}w_{11} + a_{i2}w_{21} + \cdots + a_{ij}w_{j1})^2 \end{matrix}\right\}$$

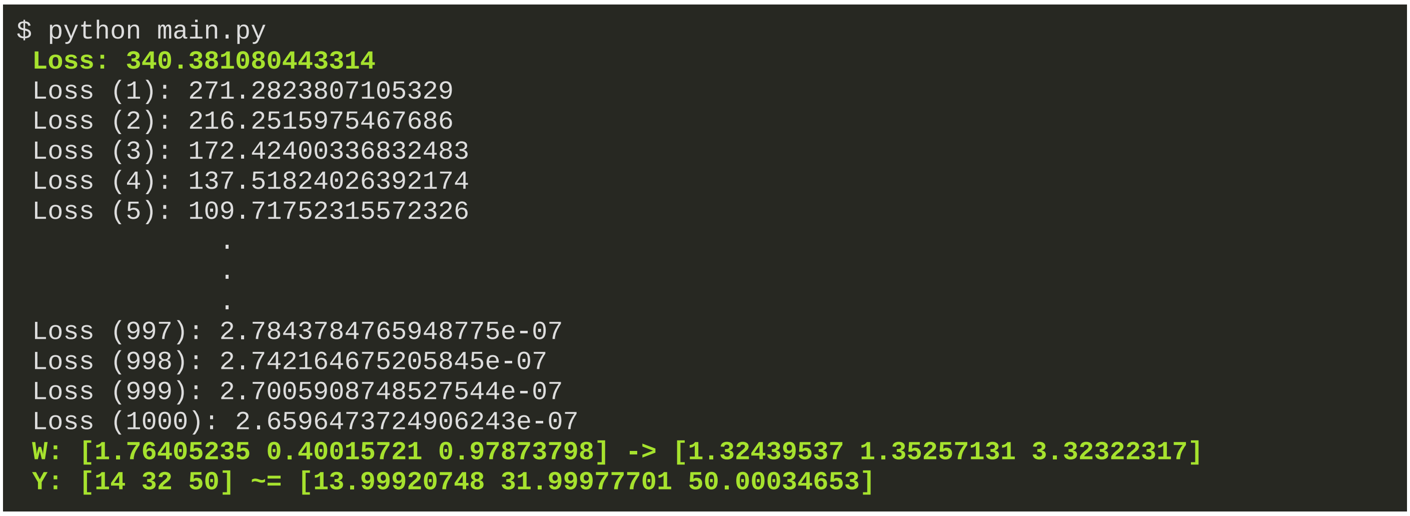
$$\delta w_{j1} = \lim_{h_{j1}\to 0}\frac{loss(w_{j1}+h) - loss(w_{j1}-h)}{2h}$$

$$\delta W = \begin{bmatrix} \lim\limits_{h\to 0}\dfrac{loss(w_{11}+h) - loss(w_{11}-h)}{2h} \\ \lim\limits_{h\to 0}\dfrac{loss(w_{21}+h) - loss(w_{21}-h)}{2h} \\ \vdots \\ \lim\limits_{h\to 0}\dfrac{loss(w_{j1}+h) - loss(w_{j1}-h)}{2h} \end{bmatrix}$$

Simple Python Example

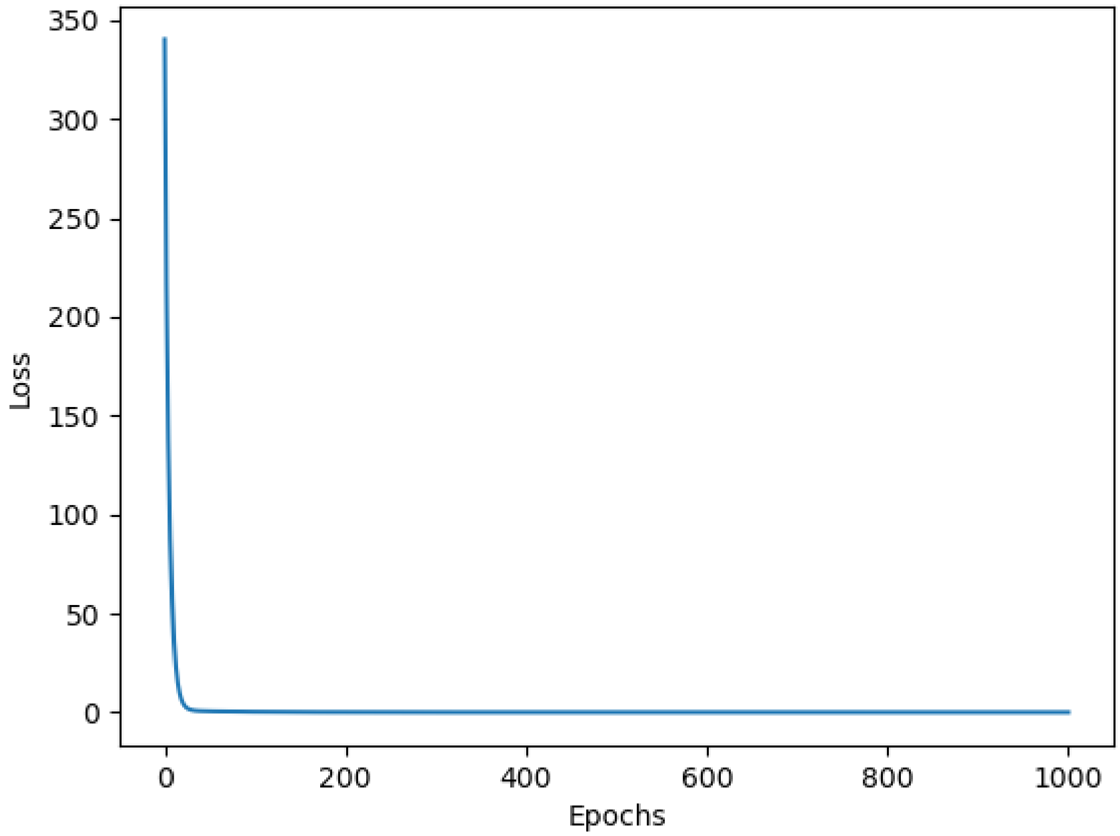
```
import numpy as np
import matplotlib.pyplot as plt
np.random.seed(0)

# Matrices
A = np.array([
    [1, 2, 3],
    [4, 5, 6],
    [7, 8, 9]
], dtype='float32')

W = np.random.randn(3, 1)
previous_W = W.copy()

Y = np.array([[14], [32], [50]])

# Loss function [MSE]
def loss(Y, A, W):
    return ((Y - A @ W)**2).mean()

# Grad
def dW(Y, A, W, h):
    dw = []
    for i in range(len(W)):
        w_plus = W.copy()
        w_minus = W.copy()

        w_plus[i][0] += h
        w_minus[i][0] -= h
        dw.append((loss(Y, A, w_plus) - loss(Y, A, w_minus)) / 2*h)
    return np.array(dw).reshape(-1, 1)


print('Loss:', loss(Y, A, W))
losses = [loss(Y, A, W)]

# iteration
for i in range(1000):

    W = W - dW(Y, A, W, 0.1) # grad

    if i > 995 or i < 5:
        print(f'Loss ({i+1}):', loss(Y, A, W))
    elif i in [6,7,8]:
        print('            .')
    losses.append(loss(Y, A, W))

# Output
print(f'W: {previous_W.flatten()} -> {W.flatten()}')
print(f'Y: {Y.flatten()} ~= {(A @ W).flatten()}')

# Graph
plt.plot(losses)
plt.xlabel('Epochs')
plt.ylabel('Loss')
plt.savefig('graph.png')
plt.close()
```

Output



```
$ python main.py
 Loss: 340.381080443314
 Loss (1): 271.2823807105329
 Loss (2): 216.2515975467686
 Loss (3): 172.42400336832483
 Loss (4): 137.51824026392174
 Loss (5): 109.71752315572326
              .
              .
              .
 Loss (997): 2.7843784765948775e-07
 Loss (998): 2.742164675205845e-07
 Loss (999): 2.7005908748527544e-07
 Loss (1000): 2.6596473724906243e-07
 W: [1.76405235 0.40015721 0.97873798] -> [1.32439537 1.35257131 3.32322317]
 Y: [14 32 50] ~= [13.99920748 31.99977701 50.00034653]
```

Graph


Loss
Epochs