Numerical Methods

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Unit 2: Interpolations and Approximation

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Interpolation and Approximation

Interpolation:

If $y_0, y_1, y_2, \cdots, y_n$ are values of y at points $a = x_0, x_1, x_2, \cdots, x_n = b$, and the process of determining the values of y within the interval (x_0, x_n) is called *interpolation* and the values of y outside the interval (x_0, x_n) is called *extrapolation*.

Interpolation and Approximation

- Lagrange's Linear Interpolation: $(x_0, y_0), (x_1, y_1)$

$$y(x) = a_0(x - x_1) + a_1(x - x_0).$$

$$y(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

- Lagrange's Quadratic Interpolation: $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$y(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2) + a_2(x - x_0)(x - x_1).$$

$$y(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

- Lagrange's Cubic Interpolation: $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$\begin{aligned} y(x) = {} & a_0(x - x_1)(x - x_2)(x - x_3) + a_1(x - x_0)(x - x_2)(x - x_3) \\ & + a_2(x - x_0)(x - x_1)(x - x_3) + a_3(x - x_0)(x - x_1)(x - x_2). \end{aligned}$$

$$\begin{aligned} y(x) = {} & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 \end{aligned}$$

Interpolation and Approximation

Theorem 8 ***(Linear polynomial)*** *To derive Lagrange's linear polynomial from the given two samples* (x_0, y_0) *and* (x_1, y_1)

Proof: Let $y_0, y_1, \cdots, y_n$ be the values at $x_0, x_1, \cdots, x_n$, then a polynomial is constructed from two samples (x_0, y_0) and (x_1, y_1). Let a linear polynomial is determined by

♦ Lagrange's Linear Interpolation: $(x_0, y_0), (x_1, y_1)$

$$y(x) = a_0(x - x_1) + a_1(x - x_0).$$

where a_0 and a_1 are constants. Equation 2.1 passes through the points (x_0, y_0) and (x_1, y_1), so using $y = y_0$ when $x = x_0$ in (2.1), to get

$$y(x_0) = y_0 = a_0(x_0 - x_1) + 0, \qquad \therefore \quad a_0 = \frac{y_0}{x_0 - x_1},$$

and using $y = y_1$ when $x = x_1$ in (2.1), to get

$$y(x_1) = y_1 = 0 + a_0(x_1 - x_0), \qquad \therefore \quad a_1 = \frac{y_1}{x_1 - x_0}.$$

$$y(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

Interpolation and Approximation

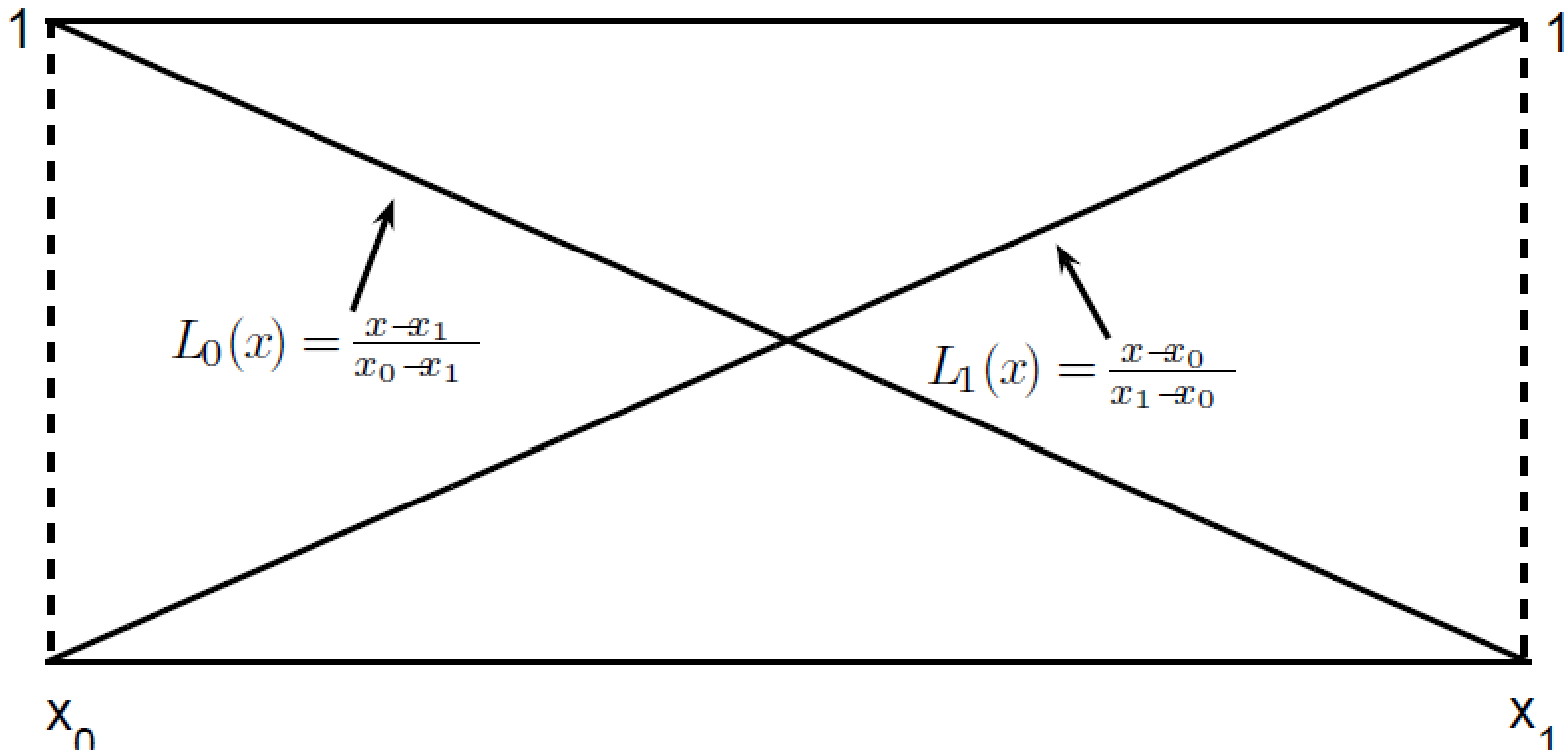


Figure 2.1: Linear shape function as $L_1(x_0) = 1$, and $L_2(x_1) = 1$.

♦ Lagrange's Quadratic Interpolation:

Let $y_0, y_1, \cdots, y_n$ be the values at $x_0, x_1, \cdots, x_n$, then a polynomial is constructed from three samples (x_0, y_0), (x_1, y_1) and (x_2, y_2). Let a quadratic polynomial is

$$y(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2) + a_2(x - x_0)(x - x_1). \qquad (1)$$

Three samples: $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$y(x_0) = y_0 = a_0(x_0 - x_1)(x_0 - x_2) + 0 + 0, \quad \therefore \quad a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)},$$

using $y = y_1$ when $x = x_1$ in (1)

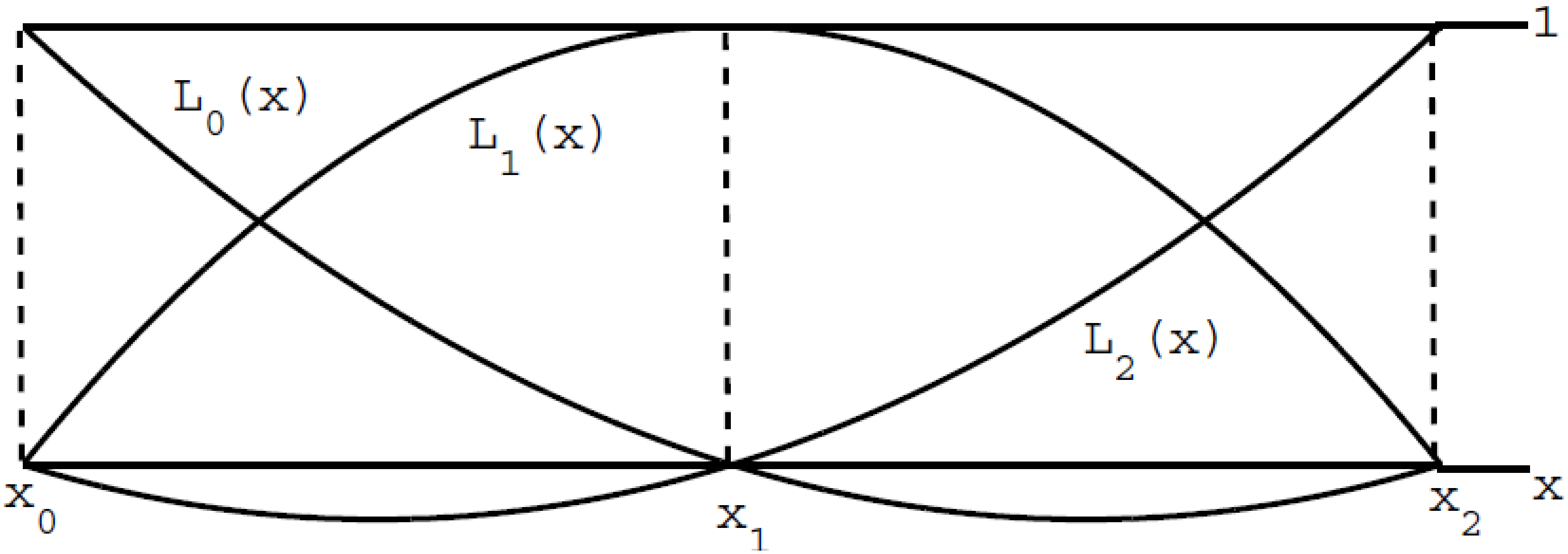
$$y(x_1) = y_1 = 0 + a_1(x_1 - x_0)(x_1 - x_2) + 0, \quad \therefore \quad a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)},$$

and using $y = y_2$ when $x = x_2$

$$y(x_2) = y_2 = 0 + 0 + a_2(x_2 - x_0)(x_2 - x_1) + 0, \quad \therefore \quad a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}.$$

$$y(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2.$$

Interpolation and Approximation



Interpolation and Approximation

Ex.1: Applying Lagrange's interpolation formula to evaluate $f(9)$ from the following data:

x	3	5	11
$f(x)$	120	290	960

Let $x_0 = 3$, $x_1 = 5$, $x_2 = 11$; $y_0 = 120$, $y_1 = 290$, $y_2 = 960$. Lagrange's interpolation polynomial is

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-5)(x-11)}{(3-5)(3-11)}(120) + \frac{(x-3)(x-11)}{(5-3)(5-11)}(290) + \frac{(x-3)(x-5)}{(11-3)(11-5)}(960)$$

$$= 7.5x^2 - 120x + 412.5 - 24.1667x^2 + 338.33x - 797.50 + 20x^2 - 160x + 300$$

$$y(x) = 3.3333x^2 + 58.33x - 85.$$

Interpolation and Approximation

- Lagrange's Cubic Interpolation:

Let $y_0, y_1, \cdots, y_n$ be the values at $x_0, x_1, \cdots, x_n$, then a polynomial is constructed from four samples (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3). Let a cubic polynomial is

$$y(x) = a_0(x-x_1)(x-x_2)(x-x_3) + a_1(x-x_0)(x-x_2)(x-x_3)$$
$$+ a_2(x-x_0)(x-x_1)(x-x_3) + a_3(x-x_0)(x-x_1)(x-x_2). \quad \textbf{(1)}$$

Four samples: $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$

Using $x = x_0$, $y = y_0$ in (1),

$$y(x_0) = y_0 = a_0(x_0-x_1)(x_0-x_2)(x_0-x_3) + 0, \quad \therefore \quad a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)},$$

Using $x = x_1$, $y = y_1$ in (1),

$$y(x_1) = y_1 = 0 + a_1(x_1-x_0)(x_1-x_2)(x_1-x_3), \quad \therefore \quad a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)},$$

Using $x = x_2$, $y = y_2$ in (1),

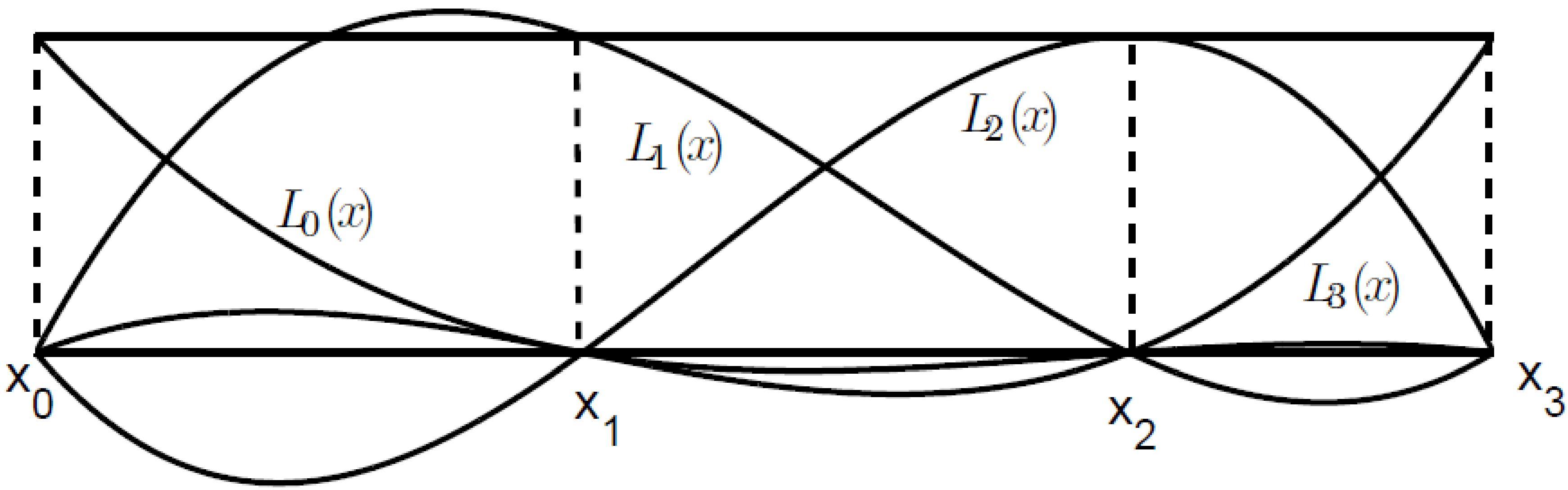
$$y(x_2) = y_2 = 0 + a_2(x_2-x_0)(x_2-x_1)(x_2-x_3), \quad \therefore \quad a_2 = \frac{y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)},$$

Using $x = x_3$, $y = y_3$ in (1),

$$y(x_3) = y_3 = 0 + a_2(x_3-x_0)(x_3-x_1)(x_3-x_2), \quad \therefore \quad a_3 = \frac{y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.$$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3.$$

Interpolation and Approximation



Interpolation and Approximation

- **Lagrange's General Interpolation:**

N samples: $(x_0, y_0), (x_1, y_1), \ldots\ldots (x_n, y_n)$,

$$
\begin{aligned}
y(x) = & \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 \\
& + \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)} y_2 + \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_3-x_0)(x_3-x_1)\cdots(x_3-x_n)} y_3 \cdots \\
& + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n .
\end{aligned}
$$

Example:

Applying Lagrange's interpolation formula to find the interpolated value for $x = 3.0$ from the following data:

x	3.2	2.7	1.0	4.8
y	22.0	17.8	14.2	38.3

Example: Applying Lagrange's interpolation formula to find the interpolated value for $x = 3.0$ from the following data:

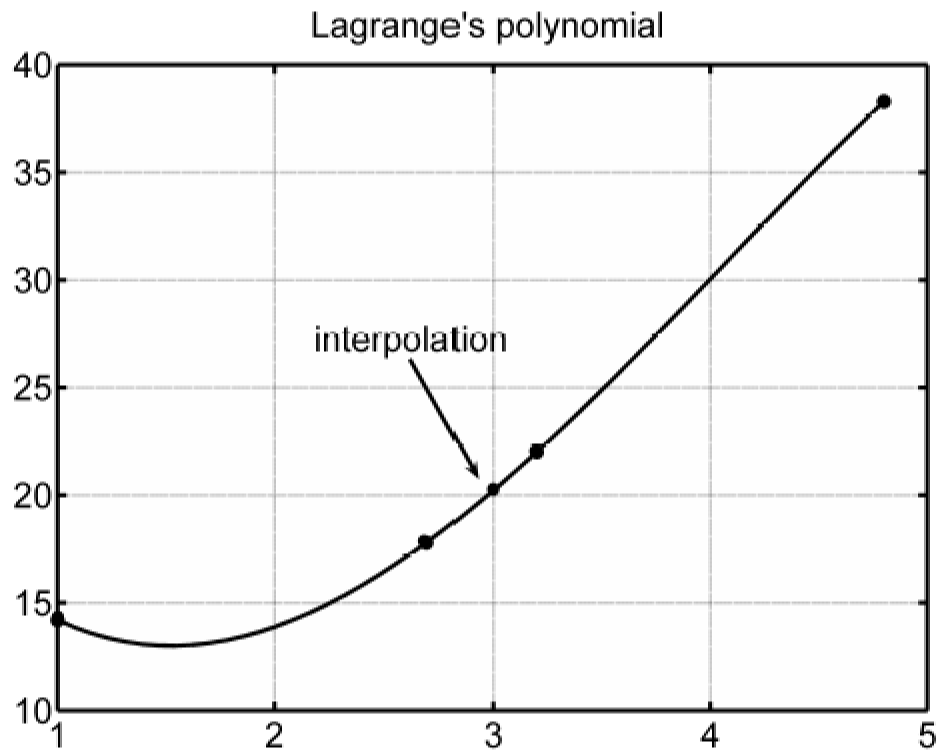
x	3.2	2.7	1.0	4.8
y	22.0	17.8	14.2	38.3

Let $x_0 = 3.2$, $x_1 = 2.7$, $x_2 = 1.0$, $x_3 = 4.8$, $y_0 = 22.0$, $y_1 = 17.8$, $y_2 = 14.2$, $y_3 = 38.3$.

Lagrange's interpolation polynomial is

$$\begin{aligned} y(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\ &\quad +\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \\ &= \frac{(x-2.7)(x-1)(x-4.8)(22)}{(3.2-2.7)(3.2-1)(3.2-4.8)}+\frac{(x-3.2)(x-1)(x-4.8)(17.8)}{(2.7-3.2)(2.7-1)(2.7-4.8)} \end{aligned}$$

$$\begin{aligned} y(3) \quad = \quad & -\frac{22}{1.76}(3-2.7)(3-1.0)(3-4.8)+\frac{17.8}{1.785}(3-3.2)(3-1.0)(3-4.8) \\ & -\frac{14.2}{14.212}(3-3.2)(3-2.7)(3-4.8)+\frac{38.3}{12.768}(3-3.2)(x-2.7)(3-1.0)=20.212. \end{aligned}$$



Example: A curve passes through the points (2, 1), (3, 5), (4, 7), (8, 9). Interpolate at $x = 6$ applying Lagrange's interpolation formula

Interpolation and Approximation

A curve passes through the points (2, 1), (3, 5), (4, 7), (8, 9). Interpolate at $x = 6$ applying Lagrange's interpolation formula

Lagrange's interpolation polynomial is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Lab 5: To implement **Lagrange's Interpolation Formula** for estimating data from n discrete points in C - programming

4 data points : (2,1), (3,5), (4,7), (8,9), Interpolate at x = 6

$$
\begin{aligned}
&= \frac{(x-3)(x-4)(x-8)}{(2-3)(2-4)(3-8)}(1) + \frac{(x-2)(x-4)(x-8)}{(3-2)(3-4)(3-8)}(5) \\
&\quad + \frac{(x-2)(x-3)(x-8)}{(4-2)(4-3)(4-8)}(7) + \frac{(x-2)(x-3)(x-4)}{(8-2)(8-3)(8-4)}(9).
\end{aligned}
$$

$$
\begin{aligned}
y(6) &= \frac{(6-3)(6-4)(6-8)}{(2-3)(2-4)(3-8)}(1) + \frac{(6-2)(6-4)(6-8)}{(3-2)(3-4)(3-8)}(5) \\
&\quad + \frac{(6-2)(6-3)(6-8)}{(4-2)(4-3)(4-8)}(7) + \frac{(6-2)(6-3)(6-4)}{(8-2)(8-3)(8-4)}(9) = 7.8.
\end{aligned}
$$

Newton's divided interpolation formula $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$+ \cdots + a_n(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}).$$

Example: Applying Newton's divided difference method to find a polynomial, and hence evaluate y at $x = 7.5$

x	2	4	9	10
y	10	14	15	17

Example: Applying Newton's divided difference method to find a polynomial, and hence find y at $x = 0.3$

x	0	1	3	4	7
y	1	3	49	129	813

Newton's Forward difference formula

If $x_0,\ x_1,\ \cdots,\ x_n$ *are given set of observations with common difference* h,
$y_0,\ y_1,\ \cdots,\ y_n$ *are their corresponding values, then approximation polynomial function*

$$
\begin{aligned}
y_p(x) \quad &= \quad y_0 + p\triangle y_0 + \frac{p(p-1)}{2!}\triangle^2 y_0 + \frac{p(p-1)(p-2)}{3!}\triangle^3 y_0 + \cdots \\
&+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\triangle^n y_0, \ \ where\ \ p = \frac{x-x_0}{h}
\end{aligned}
$$

Example: Find a polynomial and estimate $y(6)$ using Newton's forward interpolation technique from the following data:

x	0	5	10	15
y	7	11	14	18

Newton's backward difference formula

If $x_0, x_1, \cdots, x_n$ *are given set of observations with common difference* h, $y_0, y_1, \cdots, y_n$ *are their corresponding values, then approximation polynomial fun*

$$y_p(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$+ \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n, \quad \textit{where } p = \frac{x - x_n}{h}$$

Example Using Newton's backward interpolation technique, construct an interpolating polynomial of degree 3 for the data: $f(-0.75) = -0.0718125$, $f(-0.5) = -0.02475$, $f(-0.25) = 0.3349375$, $f(0) = 1.10100$. Hence find $f(-1/3)$

Solution: Here $x_3 = 0$, $h = 0.25$, the difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	-0.0718125			
		0.0470625		
-0.5	-0.02475		0.312625	
		0.3596875		0.09375
-0.25	0.3349375		0.400375	
		0.7660625		
0	1.10100			

$x = x_3 + ph$ gives $p = (x - 0)/0.25 = 4x$, Newton's interpolation formula is

- Newton's backward interpolation formula is

$$y_p(x) = y_6 + p\nabla y_6 + \frac{p(p+1)}{2!}\nabla^2 y_6 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_6.$$

$$y(x) = 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2} \times (0.400375) + \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$\therefore \quad y(x) = x^3 + 4.001x^2 + 4.002x + 1.101$$

$$y(-1/3) = (-1/3)^3 + 4.001\,(-1/3)^2 + 4.002\,(-1/3) + 1.101 = 0.1745.$$

Example: Approximate $y(2)$ and $y(10)$ from following the data:

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Newton's forward interpolation formula is

$$y_p(x) = y_0 + p\triangle y_0 + \frac{p(p-1)}{2!}\triangle^2 y_0 + \frac{p(p-1)(p-2)}{3!}\triangle^3 y_0.$$

where $x = x_0 + p\,h, \quad \therefore \quad p = \frac{x - x_0}{h}$

- Newton's backward interpolation formula is

$$y_p(x) = y_6 + p\nabla y_6 + \frac{p(p+1)}{2!}\nabla^2 y_6 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_6.$$

where $x = x_3 + p\,h, \quad \therefore \quad p = \frac{x - x_3}{h}$

x	y	$\triangle y$	$\triangle^2 y$	$\triangle^3 y$	$\triangle^4 y$
3	$\underline{4.8}$				
		$\underline{3.6}$			
4	8.4		$\underline{2.5}$		
		6.1		$\underline{0.5}$	
5	14.5		3.0		$\underline{0}$
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		$\underline{0}$
		16.6		$\underline{0.5}$	
8	52.8		$\underline{4.5}$		
		$\underline{21.1}$			
9	$\underline{73.9}$				

Newton's forward interpolation formula is

$$y_p(x) = y_0 + p\triangle y_0 + \frac{p(p-1)}{2!}\triangle^2 y_0 + \frac{p(p-1)(p-2)}{3!}\triangle^3 y_0.$$

- Newton's forward interpolation formula is

$$y_p(x) = y_0 + p\triangle y_0 + \frac{p(p-1)}{2!}\triangle^2 y_0 + \frac{p(p-1)(p-2)}{3!}\triangle^3 y_0.$$

- Taking $x = x_0 + ph$ gives $p = x - 3$. When $x = 2$, $p = -1$.

$$y(2) = 4.8 + (-1) \times (3.6) + \frac{(-1)(-1-1)}{2!}(2.5) + \frac{(-1)(-1-1)(-1-2)}{3!}(0.5)$$

$$= 4.8 - 3.6 + 2.5 - 0.5 = 3.2$$

- Newton's backward interpolation formula is

$$y_p(x) = y_6 + p\nabla y_6 + \frac{p(p+1)}{2!}\nabla^2 y_6 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_6.$$

- Here $x_6 = 9$, $y_6 = 73.9$, $h = 1$, $x = x_6 + ph$ gives $p = x - 9$. When $x = 10$, $p = 1$

$$y(10) = 73.9 + 1(21.1) + \frac{1(1+1)}{2!}(4.5) + \frac{1(1+1)(1+2)}{3!}(0.5)$$

$$= 73.9 + 21.1 + 4.5 + 0.5 = 100.$$

Least Square Method

◆ **Fitting the data** $x: x_1 \quad x_2 \quad \dots x_n$

$y: y_1 \quad y_2 \quad \dots y_n$ **in** $y = f(x) = a x + b$

◆ **Fitting in a straight line:**

$$y = a x + b$$

◆ **Normal equations are:**

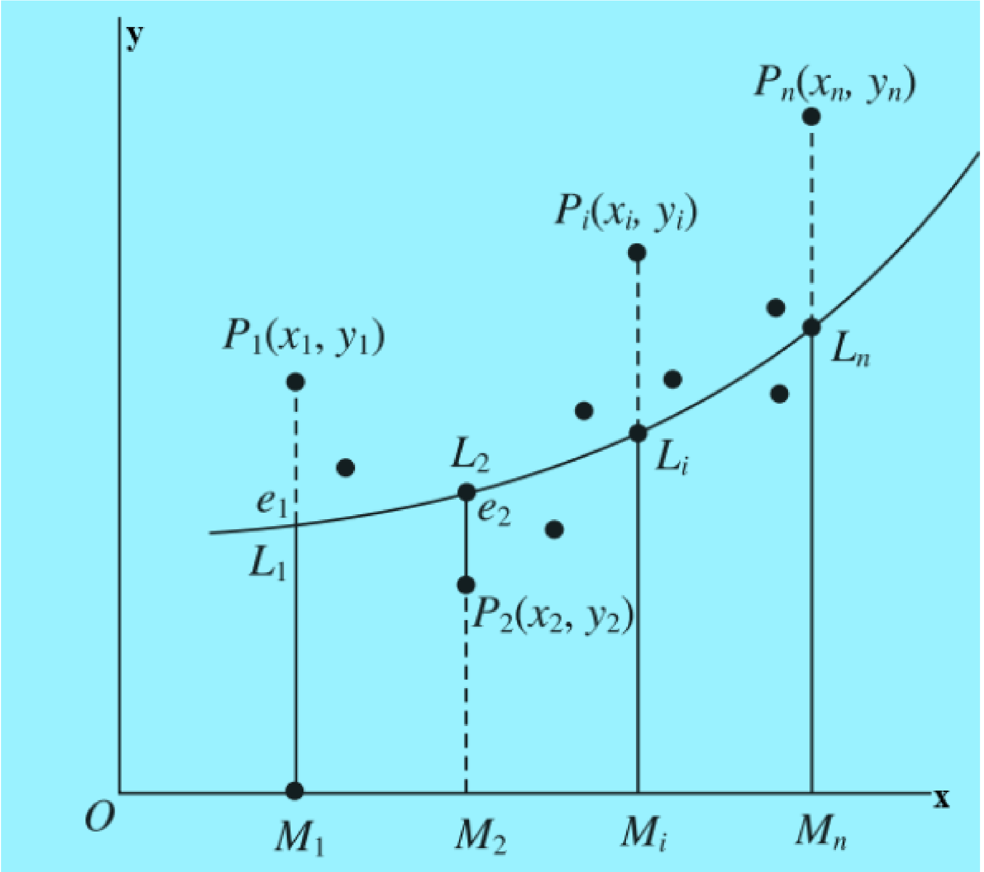
$$\sum y = a \sum x + b n$$

$$\sum x y = a \sum x^2 + b \sum x$$

◆ **Solving these equations:**

$$a = \frac{\sum x \sum y - n \sum x y}{(\sum x)^2 - n \sum x^2},$$

$$b = \frac{\sum x \sum x y - \sum x^2 \sum y}{(\sum x)^2 - n \sum x^2}$$



Least Square Method

Example Fit the following set of data into the straight line $y = a\,x + b$

x	0	1	2	3	4
y	4	9	8	12	10

Also find the value of *y* at $x = 3.5$

Solution: Here, the number of data, $n = 5$

x	y	x^2	$x\,y$
0	4	0	0
1	9	1	9
2	8	4	16
3	12	9	36
4	10	16	40
$\sum x = 10$	$\sum y = 43$	$\sum x^2 = 30$	$\sum xy = 101$

Least Square Method

Example Fit the following set of data into the straight line $y = a\,x + b$

x	0	1	2	3	4
y	4	9	8	12	10

Also find the value of y at $x = 3.5$

- **Normal equations are:**

$$\sum y = a \sum x + b\, n$$

$$\sum x\, y = a \sum x^2 + b \sum x$$

- **So, the normal equations give:**

$$43 = 10a + 5b,$$

$$101 = 30a + 10b.$$

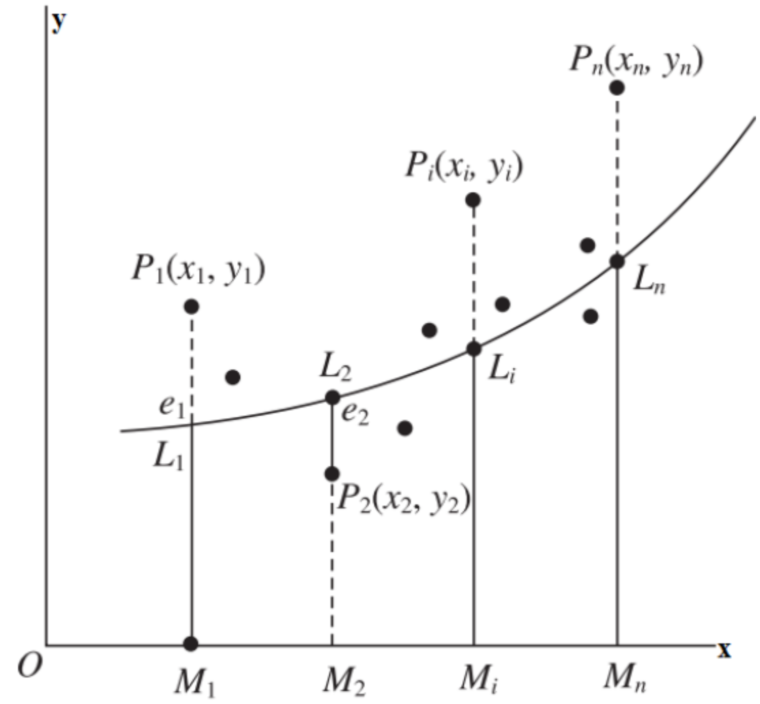
- **Solving these normal equations:** $a = 1.5, \quad b = 5.6$

- **Straight line:** $y = 1.5\, x + 5.6$

$$y\,(3.5) = 1.5 \times 3.5 + 5.6 = 10.85$$

Least Square Method

Fitting the data in $y = a + b x^2$

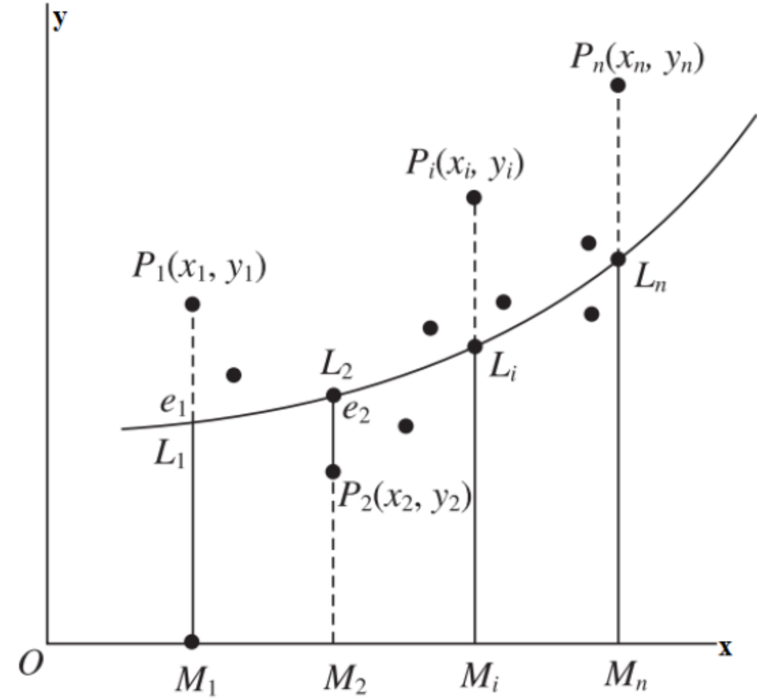


Fit the parabola $y = a + bx^2$ from the following set of data:

x	-1	0	1	2
y	2	5	3	0

Least Square Method

1. **Fitting the data in exponential function $f(x) = a\,e^{b\,x}$**



Least Square Method

Example Fit the following set of data into a curve in the form $y = a\,e^{bx}$

x	3	4	5	6	7	8
y	16.8	11.5	8.4	6.3	4.2	2.1

and hence find y when $x = 3.5$

Solution: where $Y = \log_e y$, and the constants $\log_e a = A$. Number of data, $n = 6$, and various summations are obtained as

x	y	x^2	$Y = \log_e y$	xY
3	16.8	9	2.82137	8.46411
4	11.5	16	2.44234	7.32702
5	8.4	25	2.12823	10.64115
6	6.3	36	1.84054	11.04324
7	4.2	49	1.43508	10.04556
8	2.1	64	0.74193	5.93544
$\sum x = 33$		$\sum x^2 = 199$	$\sum Y = 11.40949$	$\sum xY = 53.45652$

The constants $A = \log_e a$ and b are determined by solving the normal equations as

Least Square Method

- **Normal equations are:** $\sum Y = A\,n + b\sum x$

$$\sum xY = A\sum x + b\sum x^2$$

- **So, the normal equations give:**

$$11.40949 = 6A + 33b,$$

$$53.45652 = 33A + 199b.$$

- **Solving these normal equations:**

$A = 4.82307,\ b = -0.53118.$ So that

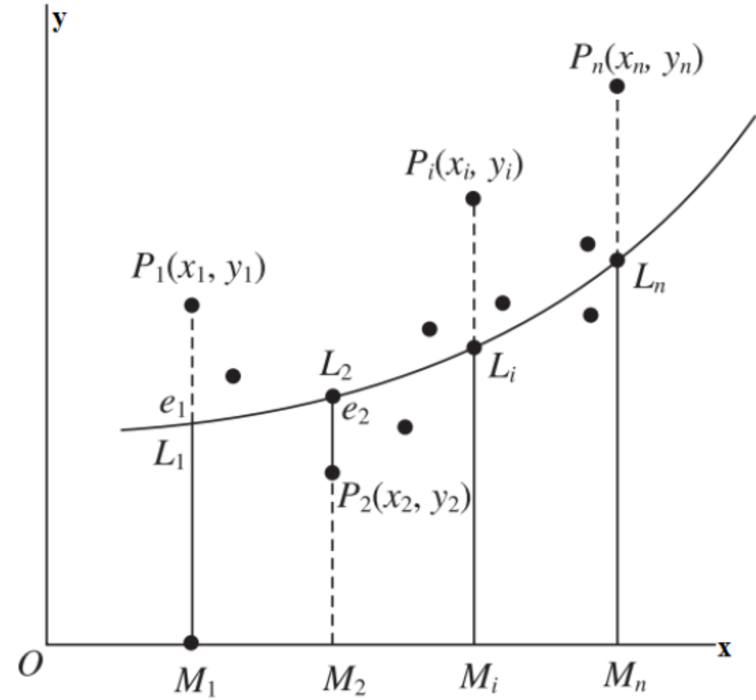
$$a = e^A = 124.34624.$$

- **Exponential curve:**

$$y = 124.34624\, e^{-0.531180\,x}.$$

Least Square Method

Fitting the data in exponential function $y = a\,b^{x}$



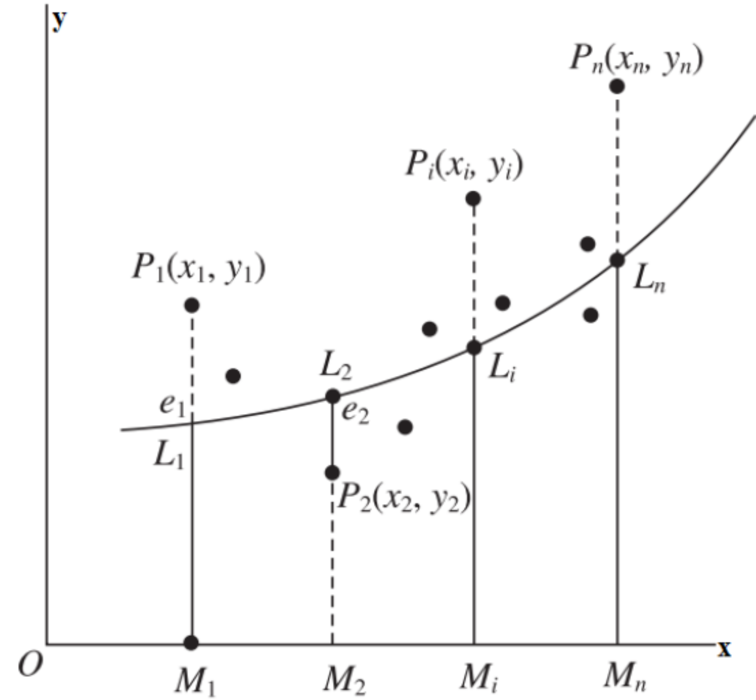
Least Square Method

Example: **Fitting in exponential function $y = a\,b^{x}$ from the following data:**

x	2	4	6	8	10	12
y	16	11.1	8.7	6.4	4.7	2.6

and hence evaluate $y(5.8)$

Fitting the data in quadratic function $y = ax^2 + bx + c$



Least Square Method

Example: **Fitting in quadratic function $y = a x^2 + b x + c$ from the following data:**

x	2	4	6	8	10	12
y	16	11.1	8.7	6.4	4.7	2.6

and hence evaluate $y(5.8)$

Least Square Method

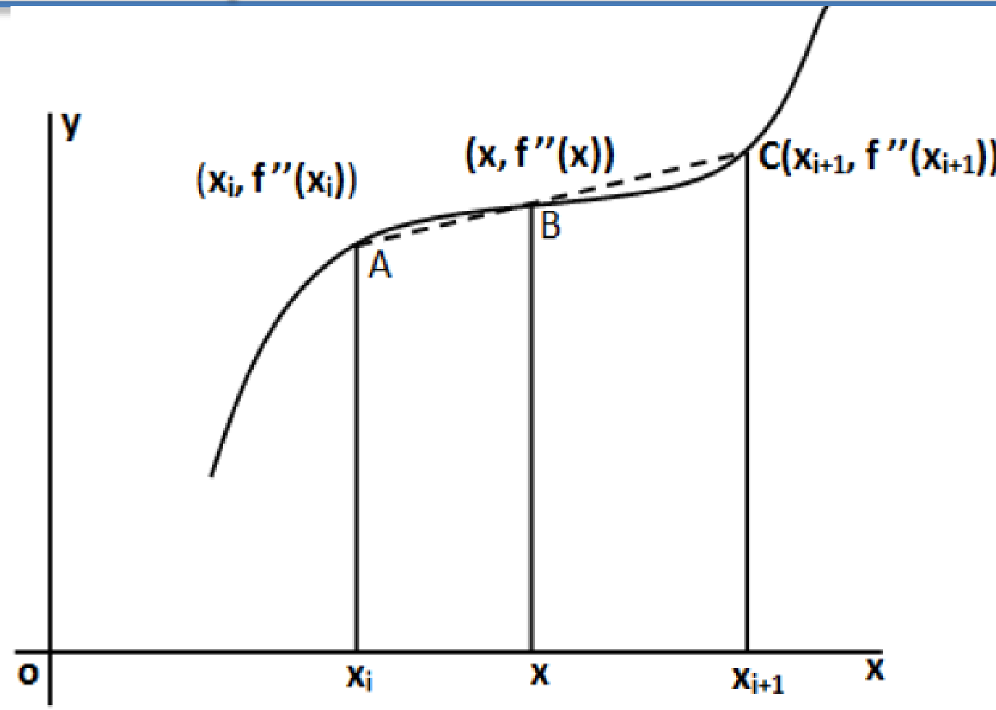
Spline Interpolation

A spline is a polynomial between each pair of tabulated points. A function $S(x)$ is a spline of degree k on $[a,b]$ if $S \in C^{k-1}\,[a,b]$, $a = x_0 < x_1 < \cdots < x_n = b$ and

$$S(x) = \begin{cases} S_0(x), & x_0 \le x \le x_1 \\ S_1(x), & x_1 \le x \le x_2 \\ \vdots & \\ S_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases}$$

where $S_i(x) \in P^k$.

Cubic Spline Interpolation



Number of sub interval = n with equal space = h, cubic spline function in the interval $x_{i-1} \le x \le x_i$ is defined as

$$S_i(x) = \frac{a_{i-1}}{6h}(h^2 u_i - u_i^3) + \frac{a_i}{6h}(u_{i-1}^3 - h^2 u_{i-1}) + \frac{1}{h}(y_i u_{i-1} - y_{i-1} u_i),$$

where the coefficients, $a_i's$ are determined by

$$a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$$ for i =1,2,…,n-1

$where,\ u_i = x - x_i,\ u_{i-1} = x - x_{i-1},$ **with $a_0 = 0$, $a_n = 0$**

Cubic Spline Interpolation

Example: Using cubic spline interpolation technique, estimate $y(9)$ from the following data:

x	4	6	8	10
y	2	5	8	6

Soln:

Number of sub interval , n = 3 with equal space h = 2, cubic spline function in the interval $x_{i-1} \leq x \leq x_i$ is defined as

$$S_i(x) = \frac{a_{i-1}}{6h}(h^2 u_i - u_i^3) + \frac{a_i}{6h}(u_{i-1}^3 - h^2 u_{i-1}) + \frac{1}{h}(y_i u_{i-1} - y_{i-1} u_i), \quad (1)$$

where the coefficients, $a_i's$ are determined by

$$a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right). \quad (2)$$

for i =1, 2, and $a_0 = 0$, $a_3 = 0$

$$where,\ u_i = x - x_i,\ u_{i-1} = x - x_{i-1},$$

Putting i =1 in (2), we get

$$a_0 + 4a_1 + a_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2). \quad (3)$$

Using $a_0 = 0$, and putting $y_0 = 2$, $y_1 = 5$, $y_2 = 8$ in (3), we get

$$4a_1 + a_2 = \frac{6}{4}(2 - 10 + 8) = 0,$$

$$\therefore \qquad 4a_1 + a_2 = 0, \qquad (4)$$

Putting i = 2 in (2), we get

$$a_1 + 4a_2 + a_3 = \frac{6}{h^2}(y_1 - 2y_2 + y_3). \qquad (5)$$

Using $a_3 = 0$, and putting $y_1 = 5,\ y_2 = 8,\ y_3 = 6,$ in (5), we get

$$a_1 + 4a_2 + a_3 = \frac{6}{4}(5 - 16 + 6) = -7.5,$$

$$\therefore \qquad a_1 + 4a_2 = -7.5. \qquad (6)$$

Solving (4) and (6), we get $a_1 = 0.5$, $a_2 = -2$.

Putting i = 3 in (1) we get

$$S_3(x) = \frac{a_2}{6h}\left(h^2 u_3 - u_3^3\right) + \frac{a_3}{6h}\left(u_2^3 - h^2 u_2\right) + \frac{1}{h}\left(y_3 u_2 - y_2 u_3\right).$$

Putting $u_2 = x - x_2 = x - 8$, $u_3 = x - x_3 = x - 10$, $y_2 = 8$, $y_3 = 6$, $h = 2$, $a_1 = 0.5$, $a_2 = -2$ in this equation, to get

$$\begin{aligned} S_3(x) &= -\frac{2}{12}\left[4(x-10) - (x-10)^3\right] + 0 + \frac{1}{2}\left[6(x-8) - 8(x-10)\right] \\ &= -\frac{1}{6}\left(4x - 40 - x^3 + 30x^2 - 300x + 1000\right) + 0.5\left(-2x + 32\right) \\ \therefore \quad S_3(x) &= \frac{1}{6}x^3 - 5x^2 + \frac{145}{3}x - 144. \end{aligned}$$

Putting x = 9 in S_3, we get

$$y(9) = S_3(9) = \frac{1}{6}(9)^3 - 5(9)^2 + \frac{145}{3}(9) - 144 = 7.5.$$

Cubic Spline Interpolation

Example: **Find cubic spline polynomial from the following data:**

x	1	3	5	7
y	12	6	4	1

, and hence evaluate $y(6)$

Soln: Here step size $h = 2$, and number of subinterval $n = 3$, a cubic spline function in the interval $x_{i-1} \leq x \leq x_i$ is defined as

$$S_i(x) = \frac{a_{i-1}}{6h}(h^2 u_i - u_i^3) + \frac{a_i}{6h}(u_{i-1}^3 - h^2 u_{i-1}) + \frac{1}{h}(y_i\, u_{i-1} - y_{i-1}\, u_i), \quad (1)$$

where the coefficients, $a_i's$ are determined by

$$a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right). \quad (2)$$

Putting i =1 in (2), to get

$$a_0 + 4a_1 + a_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2). \quad (3)$$

Putting $y_0 = 12$, $y_1 = 6$, $y_2 = 4$, $h = 2$, and $a_0 = 0$ in (3), to get

$$4a_1 + a_2 = \frac{6}{4}(12 - 12 + 4) = 6,$$

$$\therefore \quad 4a_1 + a_2 = 6, \quad (4)$$

Cubic Spline Interpolation

Putting i =2 in (2), we get

$$a_1 + 4a_2 + a_3 = \frac{6}{h^2}(y_1 - 2y_2 + y_3). \qquad (5)$$

Using $a_3 = 0$, and putting $y_1 = 6$, $y_2 = 4$, $y_3 = 1$, $h = 2$ in (5), we get

$$a_1 + 4a_2 = \frac{6}{4}(6 - 8 + 1) = -1.5,$$

$$\therefore \qquad a_1 + 4a_2 = -1.5. \qquad (6)$$

Solving (4) and (6), to get $a_1 = 1.7$, and $a_2 = -0.8$.

Putting i = 1 in (1), the cubic spline function in the interval $x_0 \le x \le x_1$ i.e. $1 \le x \le 3$ is

$$S_1(x) = \frac{a_0}{6h}(h^2 u_1 - u_1^3) + \frac{a_1}{6h}(u_0^3 - h^2 u_0) + \frac{1}{h}(y_1 u_0 - y_0 u_1).$$

Putting the values of $u_0 = x - x_0 = x - 1$, $u_1 = x - x_1 = x - 3$, $y_0 = 12$, $y_1 = 6$, $a_0 = 0$, $a_1 = 1.7$ in this equation, to get

$$\begin{aligned} S_1(x) &= 0 + \frac{1.7}{12}\left[(x-1)^3 - 4(x-1)\right] + \frac{1}{2}\left[6(x-1) - 12(x-3)\right] \\ &= 0.1083(x^3 - 3x^2 + 3x - 1 - 4x + 4) + 0.5(6x - 6 - 12x + 36) \end{aligned}$$

$$\therefore \qquad S_1(x) = 0.142x^3 - 0.425x^2 - 3.142x + 15.425.$$

Cubic Spline Interpolation

Putting i = 2 in (1), the cubic spline function in the interval $x_1 \le x \le x_2$ i.e. $3 \le x \le 5$ is

$$S_2(x) = \frac{a_1}{6h}(h^2 u_2 - u_2^3) + \frac{a_2}{6h}(u_1^3 - h^2 u_1) + \frac{1}{h}(y_2 u_1 - y_1 u_2).$$

Putting the values of $u_1 = x - x_1 = x - 3$, $u_2 = x - x_2 = x - 5$, $y_1 = 6$, $y_2 = 4$, $a_1 = 1.3$, $a_2 = 0.8$ in this equation, to get

$$\begin{aligned} S_2(x) &= \frac{1.7}{12}\left[4(x-5) - (x-5)^3\right] - \frac{0.8}{12}\left[(x-3)^3 - 4(x-3)\right] \\ &\quad + \frac{1}{2}\left[4(x-3) - 6(x-5)\right] \\ &= 0.142\left(-x^3 + 15x^2 - 71x + 105\right) - 0.067(x^3 - 9x^2 + 23x - 15) \\ &\quad + 0.5\left(-2x + 18\right) \end{aligned}$$

$$\therefore \quad S_2(x) = -0.209x^3 + 2.734x^2 - 12.623x + 24.9.$$

Putting i = 3 in (1), the cubic spline function in the interval $x_2 \le x \le x_3$ i.e. $5 \le x \le 7$ is

$$S_3(x) = \frac{a_2}{6h}(h^2 u_3 - u_3^3) + \frac{a_3}{6h}(u_2^3 - h^2 u_2) + \frac{1}{h}(y_3 u_2 - y_2 u_3). \qquad (7)$$

Putting the values of $u_2 = x - x_2 = x - 5$, $u_3 = x - x_3 = x - 7$, $y_2 = 4$, $y_3 = 1$, $a_1 = 1.7$, $a_2 = -0.8$ in this equation, to get

$$\begin{aligned} S_3(x) &= -\frac{0.8}{12}\left[4(x-7) - (x-7)^3\right] + 0 + \frac{1}{2}\left[1(x-5) - 4(x-7)\right] \\ &= -0.0667\left(4x - 28 - x^3 + 21x^2 - 147x + 343\right) + 0.5\left(-3x + 23\right) \\ \therefore \quad S_3(x) &= 0.0667x^3 - 1.4x^2 + 8.038x - 9.51. \end{aligned}$$

Hence, the cubic spline function is

$$y(x) = \begin{cases} S_1(x) = 0.142x^3 - 0.425x^2 - 3.142x + 15.425, & 1 \le x \le 3 \\ S_2(x) = -0.209x^3 + 2.734x^2 - 12.623x + 24.9, & 3 \le x \le 5 \\ S_3(x) = 0.0667x^3 - 1.4x^2 + 8.038x - 9.51, & 5 \le x \le 7 \end{cases}.$$

Putting $x = 6$ in $S_3(x)$ in this equation, to get

$$y(6) = 0.0667(6)^3 - 1.4(6)^2 + 8.038(6) - 9.51 = 2.73.$$

Assignment - II

1. Applying Lagrange's linear interpolation method to interpolate at x =2.25 from the following data:

x	2	3	7	10
y	15	16	18	21

2. Applying Lagrange's quadratic interpolation method to interpolate at x =3 from the following data:

x	1	2	5	7	11
y	3	12	147	160	200

3. Applying Lagrange's interpolation method to interpolate at x = 0.7 from the following data

x	0.2	0.5	0.6	0.9	1.1
y	0.6931	1.6094	1.7918	2.1972	2.7193

Assignment - II

4. Obtain a polynomial applying Newton's divided difference method, and hence interpolate at $x = 4.5$ from the following data:

x	0	1	3	4	7
y	1	3	49	129	813

5. Estimate $y(1.8)$ using Newton's divided interpolation from the following data:

x	0	0.5	1	2
y	1	1.8987	3.7183	11.3891

6. Obtain the value of $y(6)$ from the following data, using Newton's forward interpolation formula:

x	0	5	10	15	20
y	9	13	16	20	26

Assignment - II

7. Find $y(2.5)$ using natural cubic spline interpolation formula from the following data:

x	1	2	3
y	1	1/2	1/3

8. Find a cubic spline function from the following data:

x	-1	0	1	2
y	1	4	13	15

, and hence estimate $y(0.5)$