Numerical Methods

CACS - 252

Dr. Puskar R. Pokhrel

Associate Professor

Tribhuvan University

Kathmandu, Nepal

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Solution of Non-linear Equations

♦ Polynomial equations:

An equation in the form $a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0 = 0$, where $a_n,\ a_{n-1},\ \cdots\ a_0$ are constants, with $a_n \neq 0$, is *polynomial equation in x of degree n*. The equation has n real or complex roots. The followings are some examples of polynomial equations:

- $3x^2 + x - 1 = 0,$
- $2x^3 + 2x^2 + x - 4 = 0,$
- $x^7 - 3x^2 - x - 2 = 0.$

♦ Transcendental equations:

A *transcendental equation* is an algebraic equation which includes an exponential, or log- arithmic or trigonometric functions or all.

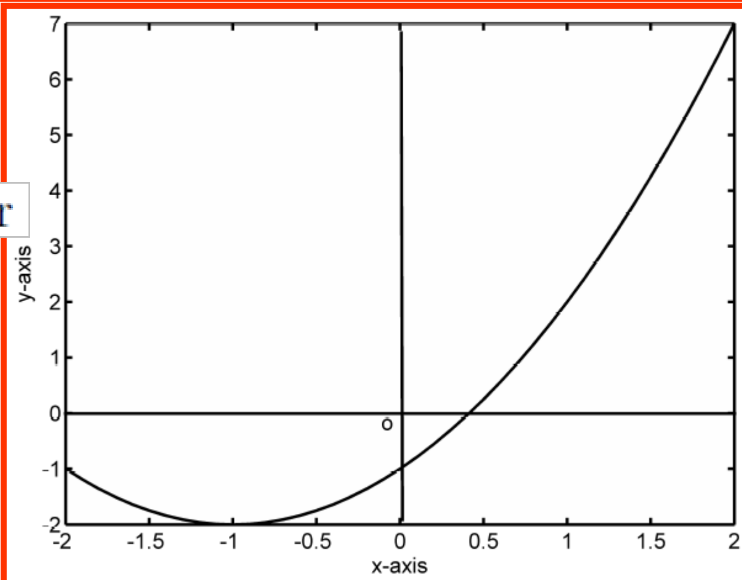


Figure 1.1: $x = \alpha$ is a root of the equation $f(x) = 0$.

♦ Examples:

- $x \sin x - 1 = 0,$
- $e^x + x \tan x = 0,$
- $x^2 \ln x - 1.9 = 0.$

Solution of Non-linear Equations

- Solution of Non-linear Equations:

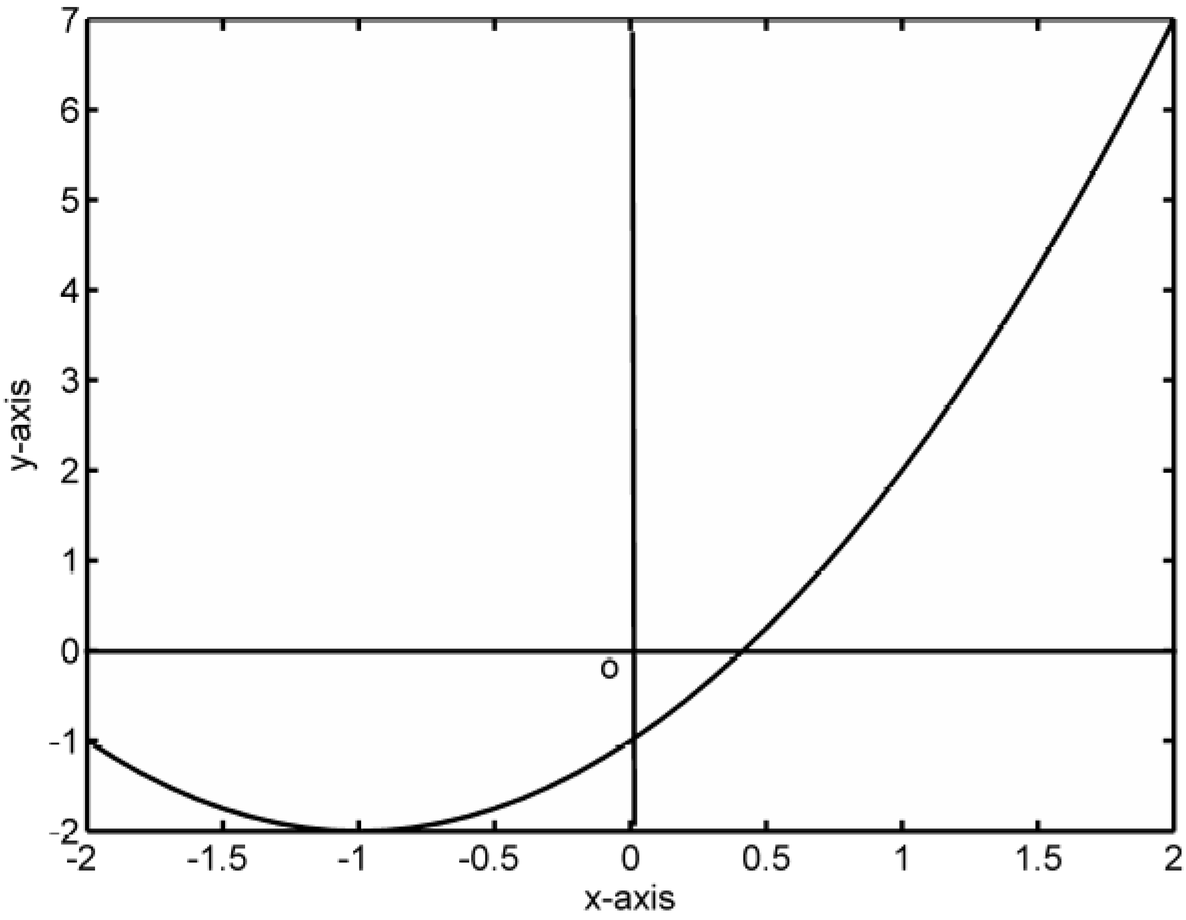


Figure 1.1: $x = \alpha$ is a root of the equation $f(x) = 0$.

- Errors in Computing:

(i) exact numbers and

$3,\ 5,\ 9,\ \sqrt{2},\ \pi,\ e$ are exact numbers

$10/3 = 3.3333333\cdots,\ \pi = 3.14159\cdots$

(ii) approximate numbers.

the numbers 3.3334 and 3.1412

are approximate numbers,

possesses certain degree of accuracy

- Significant figures (digits) :

1, 2, 3, 4, 5, 6, 7, 8, 9

Zero: 0

Non-linear Equations

- Significant figures:

  - 20500 — **Three** significant digits 2, 0, 5
  - 2.0500 — **Five** significant digits 2, 0, 5, 0, 0
  - 0.00025 — **Two** significant digits 2, 5

- Accuracy:

  - 23.65 — **Accurate** to two decimal places,
  - 32. 675 — **Accurate** to three decimal places,

- ## **Errors :**

The difference between the **exact value** and the **approximate value** of quantity is called ***error*.**

- ## **Approximation Errors :**

- can occur the measurement of the data that is not **precise** due to **instruments.**

- actual reading of a piece of paper is **4.5 cm**,

- it can be rounded to **5 cm**, since the ruler does not use decimals, **Approximate error = 0 . 5 cm**

- Actual value **3.14159….** It is rounded to **3.14**, **Approximate error = 0 . 00159**

Non-linear Equations

- **Inherent Errors :**

The quantity which are already present in the statement of the problem before its solution are called *Inherent errors*.

- **Round off Errors :**

The difference between the calculated approximation of a number and its exact mathematical value is called *round-off error* (or rounding error). Round-off errors occur when

Exact number	Round off number	Exact number	Round off number
23.765836	23.7658	23.765836	23.7658$\underline{4}$

Round off Error = 0.000036, **Round off Error** = 0.000004,

Non-linear Equations

- **Truncation Errors :**

  - $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = X.$$

  - $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} = X'.$$

  Error = | X - X ′ |

  - Consider another irrational number 2.71828182845905.

  - The number is approximated by chopping the some digits as 2.71;

Absolute, relative and percentage errors

The *absolute error* is the magnitude of the difference between the exact and approximation value; let X be the true and X' be approximate value of a quantity, then the error

$$E_a = |X - X'| = \delta x$$

$$E_r = \frac{|X - X'|}{|X|} \quad \text{and} \quad E_p = E_r \times 100.$$

Here, $X = 2.718281$, $X' = 2.718$, then

$$\frac{|X - X'|}{|X|} = \frac{|2.718281 - 2.718|}{|2.718281|} = 0.0001033741 < \frac{10^{-3}}{2}.$$

Methods for solving non-linear equations

- Analytical method,
- Graphical method,
- Trial and error method,
- Iterative method.

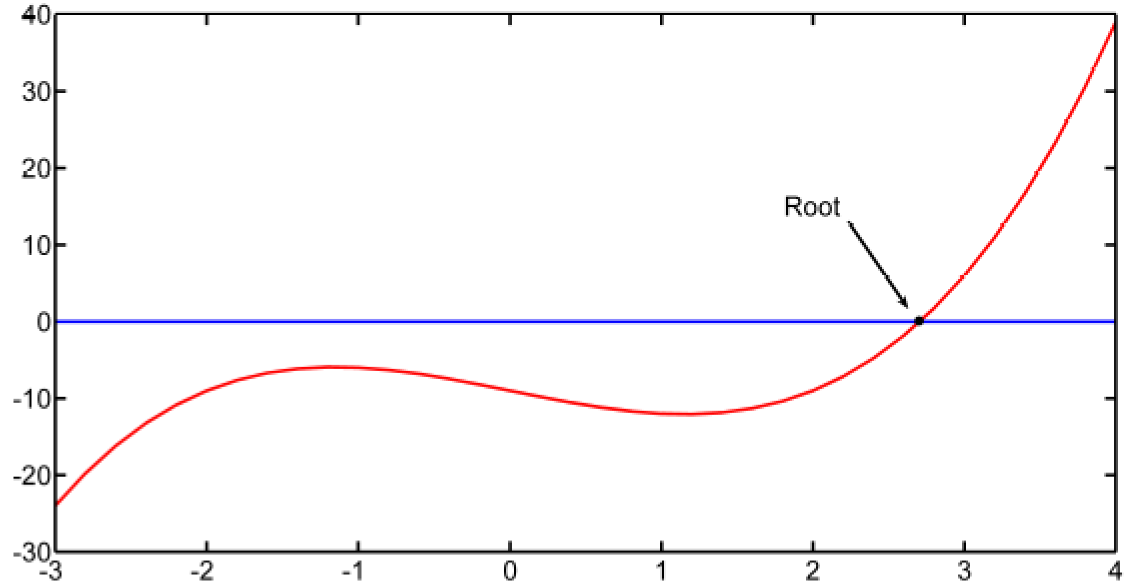
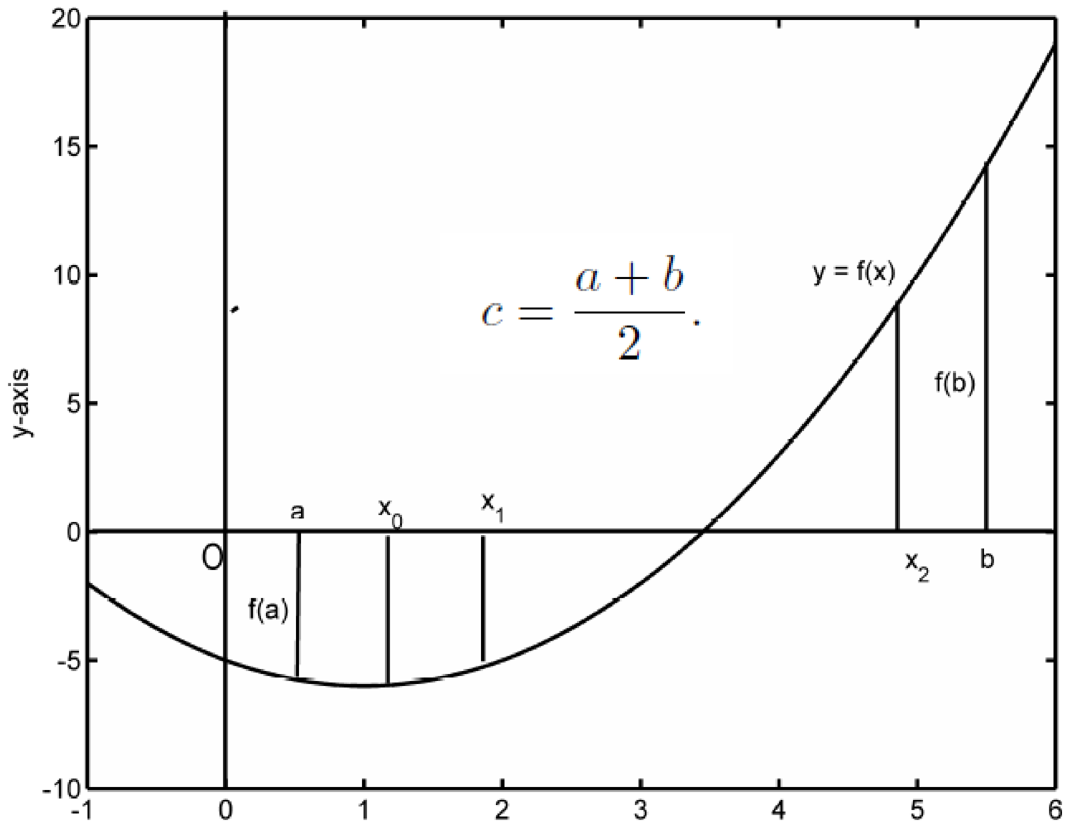


Figure 1: The graph of $f(x) = 0$

Bisection method

Theorem 2 *(Bisection method) To derive a formula of bisection method for finding a real root of non- linear equation* $f(x) = 0$



$$c = \frac{a+b}{2}.$$

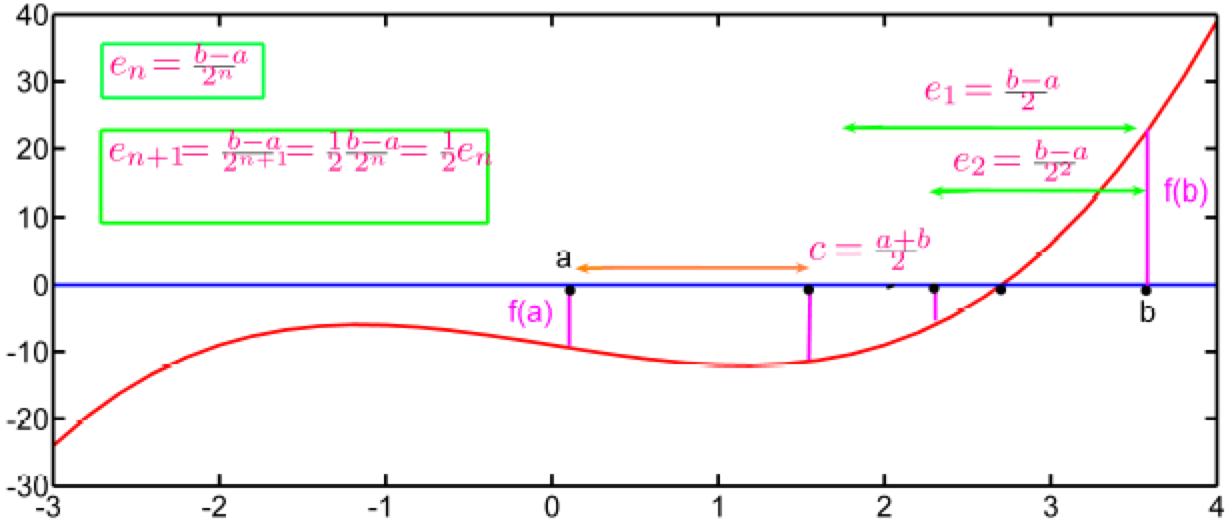


Figure 4: The nth iteration of the approximation root of $f(x) = 0$.

Definition (**Rate of convergence**)

An iterative method has the rate of convergence $p > 0$, for which $\exists$ a constant $C \neq 0$ such that $|e_{n+1}| \leq C|e_n|^p$, where $e_n = c - \xi$.

- Further $e_{n+1} = (1/2)e_n$, bisection method *converges linearly*.

Convergence

- Tolerance ϵ, $\dfrac{b-a}{2^n} \le \epsilon$, i.e., $n \ge \dfrac{\log(b-a) - \log\epsilon}{\log 2}$.

Error, ϵ:	0.01	0.001	0.0001	0.00001
No. iter., n:	6	9	13	16

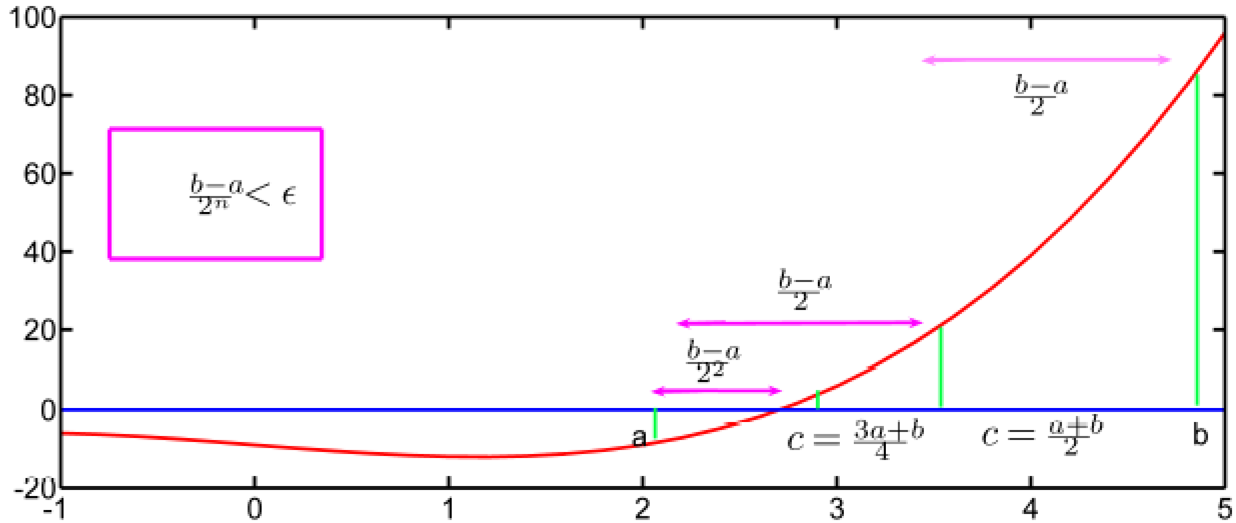


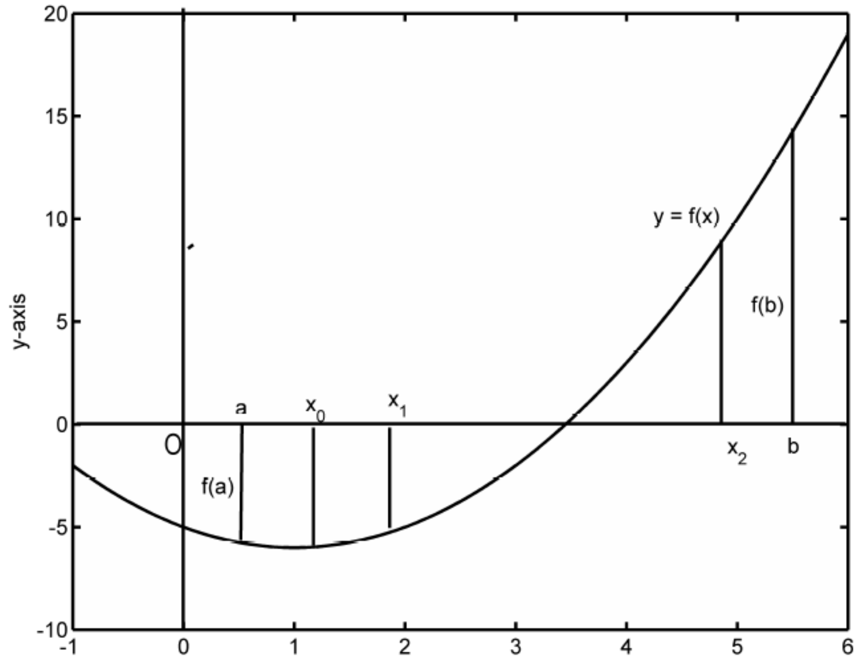
Figure 5: Error in the root of of $f(x) = 0$.

Solution of Non-linear Equations

Example 1 Find an approximation real root of the non-linear equation $x^3 - 4x - 9 = 0$ correct to three decimal places with initial guesses $a = 2$ and $b = 3$ using bisection method

Solution: Here $f(x) = x^3 - 4x - 9 = 0$

- Let the two initial guesses are $a = 2$, $b = 3$,
- $f(2) = 8 - 8 - 9 = -9 < 0$,
- $f(3) = 27 - 12 - 9 = 6 > 0$,
- $f(a) \times f(b) = f(2) \times f(3) = (-9) \times 6 = -54 < 0$.



- A root lies between 2 and 3.

$c = (a + b)/2 = (2 + 3)/2 = 2.5$,

- $f(2.5) = (2.5)^3 - 4 \times 2.5 - 9 = -3.3750 < 0$.

Solution of Non-linear Equations

- Here $f(x) = x^3 - 4x - 9 = 0$, $a = 2$, $b = 3$, $c = 2.5$,

  $f(2) = 8 - 8 - 9 = -9$, $f(3) = 27 - 12 - 9 = 6 > 0$, $f(2.5) = (2.5)^3 - 4 \times 2.5 - 9 = -3.3750$

Step	a	b	c	f (c)	f (a)	f (b)
1.	2.0000	3.0000	2.5000	-3.3750	-9.0000	6.0000
2.	2.5000	3.0000	2.7500	0.7969	-3.3750	6.0000
3.	2.5000	2.7500	2.6250	-1.4121	-3.3750	0.7969
4.	2.6250	2.7500	2.6875	-0.3391	-1.4121	0.7969
5.	2.6875	2.7500	2.7188	0.2209	-0.3391	0.7969
6.	2.6875	2.7188	2.7031	-0.0611	-0.3391	0.2209
7.	2.7031	2.7188	2.7109	0.0794	-0.0611	0.2209
8.	2.7031	2.7109	2.7070	0.0090	-0.0611	0.0794
9.	2.7031	2.7070	2.7051	-0.0260	-0.0611	0.0090
10.	2.7051	2.7070	2.7061	-0.0085	-0.0260	0.0090
11.	2.7061	2.7070	2.7065	0.0003	-0.0085	0.0090

root = 2.706, error = 0.0005.

Solution of Non-linear Equation - Bisection method

Example 2 Find at least two possible roots of the non-linear equation $x^2 - \sin x - 0.5 = 0$ correct to two decimal places (i) a = 0, b = 2 (ii) a = -1, b = 0 using bisection method

Solution: Here $f(x) = x^2 - \sin x - 0.5 = 0$, $a = 0, b = 2$, $f(0) = 0$, $f(2) = 2.5907$

Steps	a	b	c	f(c)	f(a)	f(b)
1	0.0000	2.0000	1.0000	-0.3415	-0.5000	2.5907
2	1.0000	2.0000	1.5000	0.7525	-0.3415	2.5907
3	1.0000	1.5000	1.2500	0.1135	-0.3415	0.7525
4	1.0000	1.2500	1.1250	-0.1366	-0.3415	0.1135
5	1.1250	1.2500	1.1875	-0.0173	-0.1366	0.1135
6	1.1875	1.2500	1.2188	0.0467	-0.0173	0.1135
7	1.1875	1.2188	1.2031	0.0143	-0.0173	0.0467
8	1.1875	1.2031	1.1953	-0.0016	-0.0173	0.0143
9	1.1953	1.2031	1.1992	0.0064	-0.0016	0.0143

The approximation roots in 8^{th} and 9^{th} iterations are same accurate to two decimal places,

root = 1.19, error = 0.0039

Example 2 Find at least two possible roots of the non-linear equation $x^2 - \sin x - 0.5 = 0$ correct to two decimal places (i) a = 0, b =2 (ii) a = -1, b = 0 using bisection method

Steps	a	b	c	f(c)	f(a)	f(b)
1	-1.0000	0.0000	-0.5000	0.2294	1.3415	-0.5000
2	-0.5000	0.0000	-0.2500	-0.1901	0.2294	-0.5000
3	-0.5000	-0.2500	-0.3750	0.0069	0.2294	-0.1901
4	-0.3750	-0.2500	-0.3125	-0.0949	0.0069	-0.1901
5	-0.3750	-0.3125	-0.3438	-0.0448	0.0069	-0.0949
6	-0.3750	-0.3438	-0.3594	-0.0192	0.0069	-0.0448
7	-0.3750	-0.3594	-0.3672	-0.0062	0.0069	-0.0192
8	-0.3750	-0.3672	-0.3711	0.0003	0.0069	-0.0062

The approximation roots in 7^{th} and 8^{th} iterations are same accurate to two decimal places,

root = - 0.37, error = | - 0.3711 + 0.3672| = 0.0039

Solution of Non-linear Equation- Bisection method

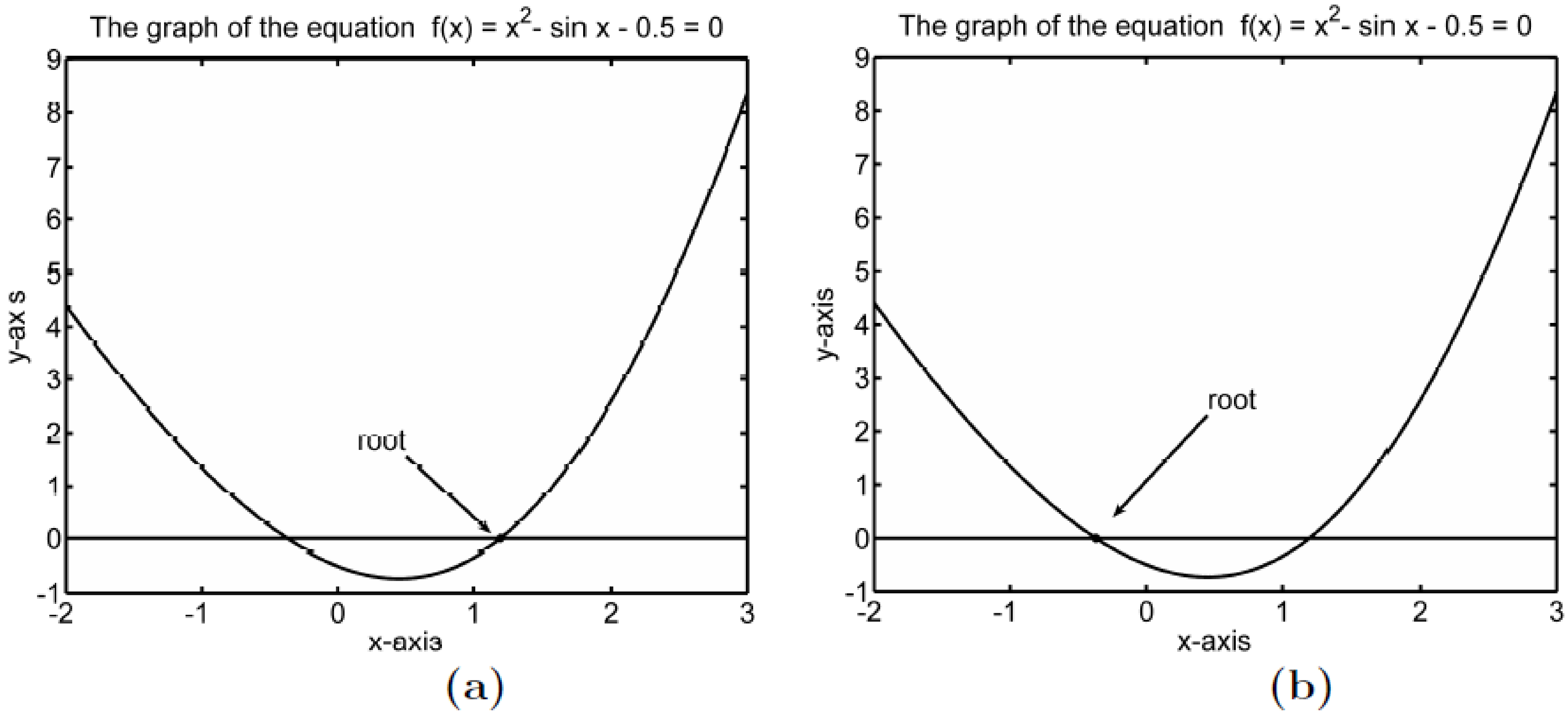
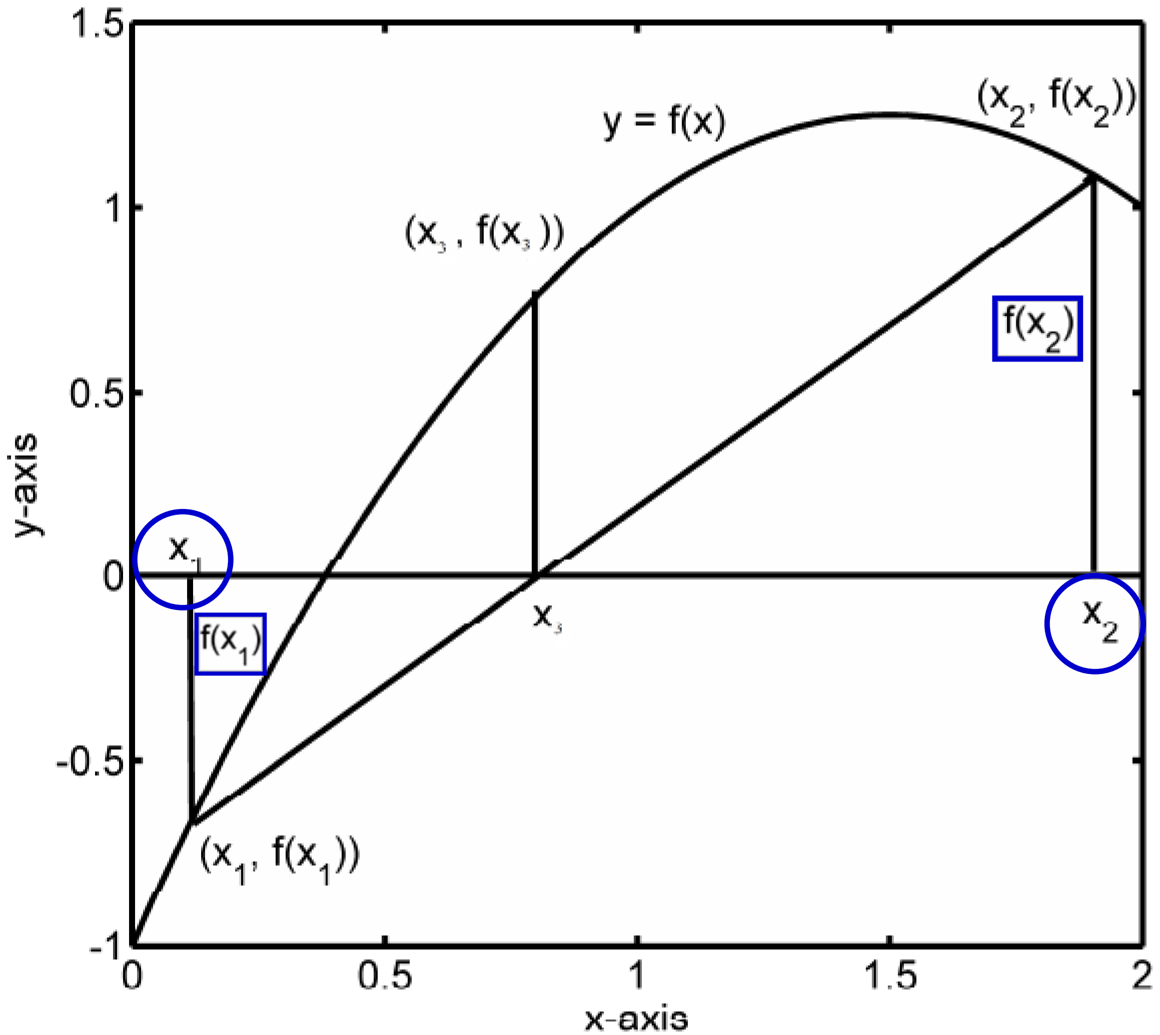


Figure 1.5: The root = 1.19 in (a) and the root = - 0.37 in (b).

Solution of Non-linear Equation- False position method

Derive a formula for False-Position to find a real root of equation $f(x) = 0$



Solution of Non-linear Equation- False-position method

Soln: equation of a line joining

two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is

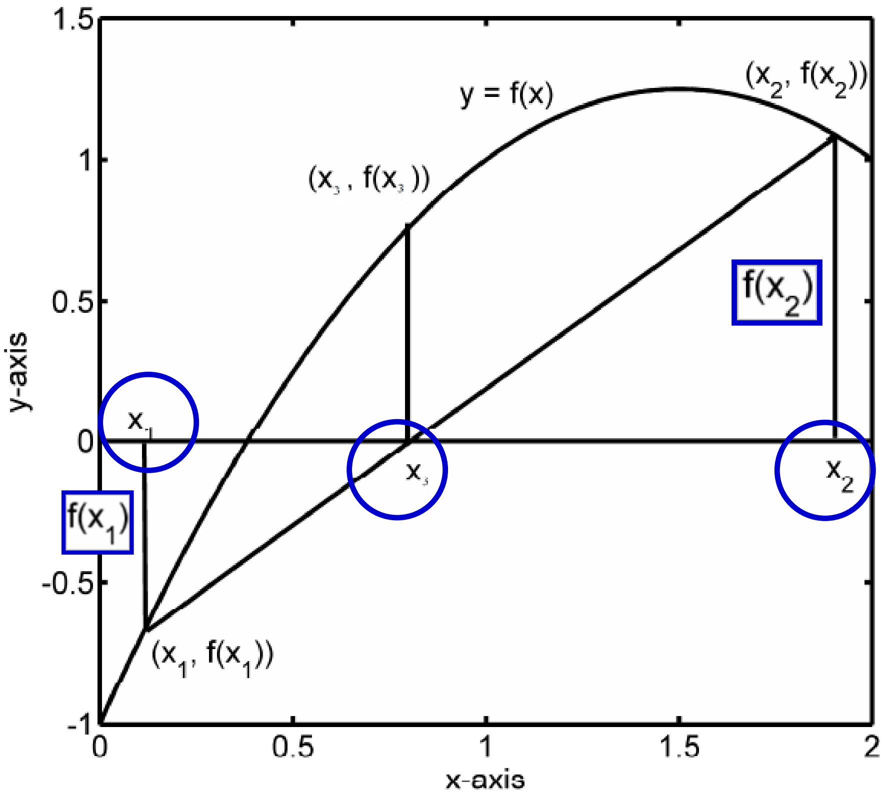
$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

- It meets x-axis at x= x_3, y = 0

$$0 - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_1).$$

- Solving for x_3, to get

- $$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}.$$



Solution of Non-linear Equation- False-position method

Example Applying False-position method to find a real root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places with two guesses $x_1=2$, and $x_2 = 3$

Soln: Let $f(x) = x^3 - 4x - 9 = 0$, taking two initial guesses $a = 2$, $b = 3$, so that $f(2) = 8 - 8 - 9 = -9 < 0$, $f(3) = 27 - 12 - 9 = 6 > 0$. So, a real root lies between 2 and 3, the formula of False-Position or Regula-Falsi method is

- $$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}.$$

- Putting $x_1 = 2$, $x_2 = 3$, the *first approximation* to the root is

$$x_3 = \frac{2 \times f(3) - 3 \times f(2)}{f(3) - f(2)} = \frac{2 \times 6 - 3 \times (-9)}{6 + 9} = \frac{12 + 27}{15} = 2.6.$$

Now $f(2.6) = (2.6)^3 - 4 \times 2.6 - 9 = -1.824 < 0$, the root lies between 2.6 and 3.

- The other approximation roots are in the following table:

Solution of Non-linear Equation- False-position method

i	x1	x2	x3	f(x1)	f(x2)	f(x3)
1	2.0000	3.0000	2.60000	-9.0000	6.0000	-1.8240
2	2.6000	3.0000	2.69325	-1.8240	6.0000	-0.2372
3	2.6933	3.0000	2.70492	-0.2372	6.0000	-0.0289
4	2.7049	3.0000	2.70633	-0.0289	6.0000	-0.0035
5	2.7063	3.0000	2.70650	-0.0035	6.0000	-0.0004

root, x3 = 2.706,Error = 0.00017

Solution of Non-linear Equation- False-position method

Example Applying false-position method to find a real root of the equation $x e^x - \cos x = 0$ correct to **three decimal places** with $x_1 = 0$ and $x_2 = 1$

Soln: Let $f(x) = xe^x - \cos x = 0$, taking two initial guesses $x_1 = 0$ and $x_2 = 1$, $\epsilon = 10^{-4} = 0.0001$, so that $f(x_1) = f(0) = -1 < 0$, $f(x_2) = f(1) = 1 \times e^1 - \cos(1) = 2.1780 > 0$.

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0f(1) - 1f(0)}{f(1) - f(0)} = \frac{0 - 1(-1)}{2.1780 + 1} = 0.3147.$$

$$\therefore \quad x_3 = 0.3147$$

$$f(x_3) = f(0.3147) = 0.3147 \times e\,0.3147 - \cos(0.3147) = -0.5199.$$

Solution of Non-linear Equation- False-position method

i	x1	x2	x3	f(x1)	f(x2)	f(x3)
1	0.0000	1.0000	0.31467	-1.0000	2.1780	-0.5199
2	0.3147	1.0000	0.44673	-0.5199	2.1780	-0.2035
3	0.4467	1.0000	0.49402	-0.2035	2.1780	-0.0708
4	0.4940	1.0000	0.50995	-0.0708	2.1780	-0.0236
5	0.5099	1.0000	0.51520	-0.0236	2.1780	-0.0078
6	0.5152	1.0000	0.51692	-0.0078	2.1780	-0.0025
7	0.5169	1.0000	0.51748	-0.0025	2.1780	-0.0008
8	0.5175	1.0000	0.51767	-0.0008	2.1780	-0.0003

Root = 0.517

Solution of Non-linear Equation- Newton-Raphson method

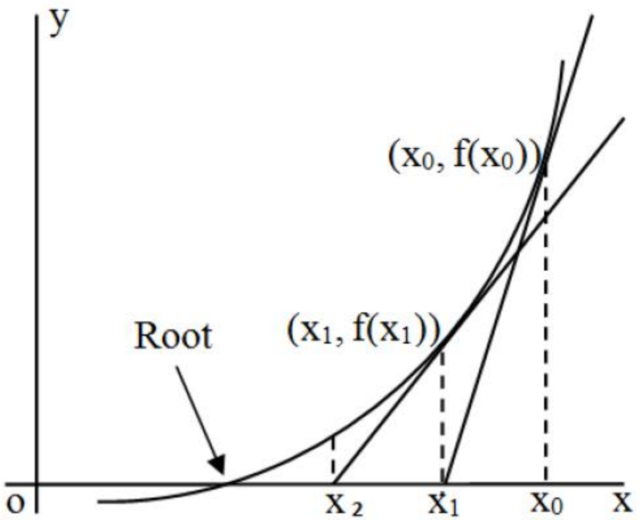
Theorem 5 *Derive a formula for Newton-Raphson method to find a real root of non-linear equation* $f(x) = 0$

Taking initial guess x_0 to non-linear equation $f(x) = 0$, the equation of a tangent line at the point $(x_0, f(x_0))$ to the graph $y = f(x)$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

This tangent line intersects x-axis at the point $(x_1, 0)$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$



- In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3, \cdots .$$

Solution of Non-linear Equation- Newton-Raphson method

Example : Applying Newton-Raphson method to find a real root of the equation $x e^x - \cos x = 0$ correct to **three decimal places** with $x_0 = 1$

Soln:

Let $f(x) = xe^x - \cos x = 0$, $f'(x) = xe^x + e^x + \sin x$. Taking the initial guess $x_0 = 1$.

Applying Newton-Raphson formula, the *first approximation* to the root is

Applying Newton-Raphson formula, the *first approximation* to the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(1) = 1 \times e^1 - \cos(1) = 2.1779$, $f'(1) = 1 \times e^1 + e^1 + \sin(1) = 6.2780$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - (2.1779) / (6.2780) = 0.65309,$$

Solution of Non-linear Equation- Newton-Raphson method

Steps	x0	x1=x0-f(x0)/df(x0)	f(x1)
1	1.00000	0.65308	0.46064
2	0.65308	0.53134	0.04180
3	0.53134	0.51791	0.00046
4	0.51791	0.51776	0.00000

root, x1 = 0.5178.

Solution of Non-linear Equation- Newton-Raphson method

Example : Applying Newton-Raphson method to find a real root of the equation $x \log_{10} x = 12.2$ correct to four decimal places with $x_0 = 10$

Soln : Here $f(x) = x \log_{10} x - 12.2$, $f'(x) = \log_{10} e + \log_{10} x$.

Take initial guess $x_0 = 10$.

The *first approximation root* is

$$x_1 = x_0 - f(x_0) / f'(x_0)$$

The *first approximation root* is

$$x_1 = x_0 - f(x_0) / f'(x_0) = 10 - f(10) / f'(10) = 11.53385$$

The *first approximation root* is

$$x_1 = x_0 - f(x_0) / f'(x_0) = 10 - f(10) / f'(10) = 11.53385$$

Solution of Non-linear Equation- Newton-Raphson method

Example : **Applying Newton-Raphson method to find a real root of the equation $x \log_{10} x = 12.2$ correct to four decimal places with $x_0 = 10$**

Soln : Here $f(x) = x \log_{10} x - 12.2$, $f'(x) = \log_{10} e + \log_{10} x$.

Take initial guess, $x_0 = 10$.

The *first approximation root* is

$$x_1 = x_0 - f(x_0) / f'(x_0)$$

$f(10) = 10 \log_{10} 10 - 12.2 = -2.2$, $f'(10) = \log_{10} e + \log_{10} 10 = 1.4343$

By Newton-Rahpson method, the *first approximation root* is

$$x_1 = x_0 - f(x_0) / f'(x_0) = 10 - (-2.2) / 1.4343 = 11.53385$$

Solution of Non-linear Equation- Newton-Raphson method

```
================================
i          x0          x1
================================
```

$x1 = x_0 - f(x_0) / f'(x_0)$

i	x0	x1
1	10.00000	11.53385
2.	11.53385	11.50132
3.	11.50132	11.50133

```
================================
root x1 = 11.5013
```

Solution of Non-linear Equation- Newton-Raphson method

Example. Applying Newton-Raphson method to find a real root of $x^3 - 5x + 3 = 0$ correct to four decimal places with $x_0 = 2$

Soln: Let $f(x) = x^3 - 5x + 3 = 0$, $f'(x) = 3x^2 - 5$

Taking the initial guess $x_0 = 2$.

Applying Newton-Raphson formula, the *first approximation* to the root is

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. $f(2) = 2^3 - 5 \times 2 + 3 = 1$, $f'(2) = 3 \times 2^2 - 5 = 7$,

$x_1 = 2 - (1/7) = 1.8571$,

Steps	x0	x1=x0-f(x0)/df(x0)	f(x1)
1	2.00000	1.85714	0.11953
2	1.85714	1.83479	0.00277
3	1.83479	1.83424	0.00000
4	1.83424	1.83424	0.00000

root, x1 = 1.8342.

Applications - Newton-Raphson method

Problem 1 *If N is a real number, then iterative formula to find $\frac{1}{N}$ is $x_{n+1} = x_n(2 - Nx_n)$*

Solution: Let $x = \frac{1}{N}$, i.e., $\frac{1}{x} - N = 0$, taking $f(x) = \frac{1}{x} - N$, so that $f'(x) = -\frac{1}{x^2}$. Newton's iteration formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - N}{-x_n^{-2}}, \\ &= x_n + x_n - Nx_n^2 = 2x_n - Nx_n^2 = x_n(2 - Nx_n). \end{aligned}$$

Example: Find the value of 1 / 6

Solution of Non-linear Equation- Fixed-point iteration method

Definition 3 *(Fixed Point) If a function g: $\Re \to \Re$ such that $x = g(x)$, then a real number x is called fixed point of a function $g(x)$*

Theorem 6 *To derive a formula for finding a real root of the equation $f(x) = 0$*

Example: Applying fixed point iteration method to find a real root of $x^3 - 3x - 8 = 0$ correct to four decimal places

Solution: Let $f(x) = x^3 - 3x - 8 = 0$,

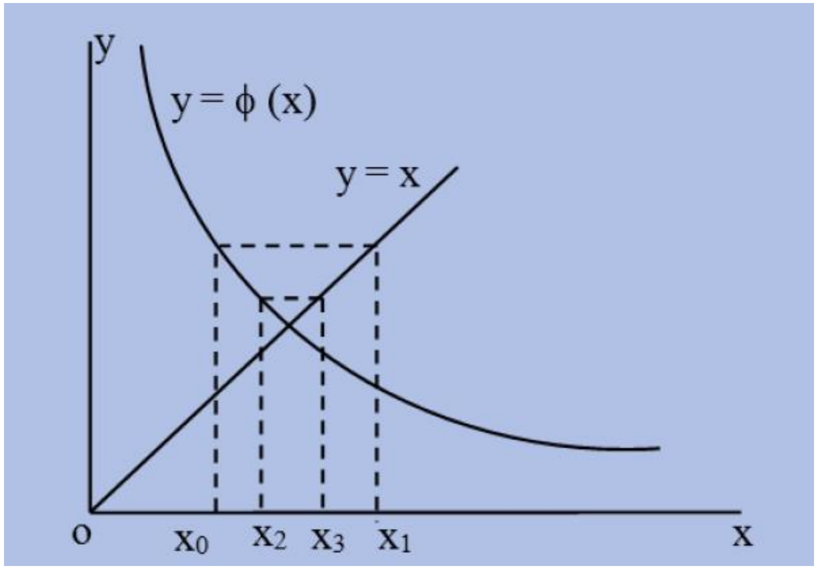
equation can be rearranged as

$$x = (3x + 8)^{1/3}.$$

$f(2) = -6$, $f(3) = 10$, a root lies between 2 and 3,

Initial guess $x_0 = 2$

$$x_1 = g(x_0) = (3x_0 + 8)^{1/3} = (3 \times 2 + 8)^{1/3} = 2.4101,$$



Solution of Non-linear Equation- Fixed-point iteration method

Example: Applying fixed point iteration method to find a real root of $2x - 1 = 2\sin x$ correct to four decimal places

Solution: Let $f(x) = 2x - 1 - 2\sin x = 0$, $f(0) = -1 < 0$, $f(2) = 1.1814 > 0$, a root lies between 0 and 2, then equation can be rearranged as

$$x = \frac{1}{2} + \sin x = g(x).$$

i	x0	x1
1	0.500000	2.097618
2	2.097618	1.638440
3	1.638440	1.738756
4	1.738756	1.715271
5	1.715271	1.720684
6	1.720684	1.719432

root x1 = 1.719432

Solution of Non-linear Equation- Fixed-point iteration method

Example: Applying fixed point iteration method to find a real root of $3x - 2\log_{10} x - 8 = 0$ correct to three decimal places

Solution: Let $f(x) = 3x - 2\log_{10} x - 8 = 0$, $f(2) = 3 \times 2 - 2 \times \log_{10} 2 - 8 = -2.6021 < 0$, $f(3) = 3 \times 3 - 2 \times \log_{10} 3 - 8 = 0.0458 > 0$, a root lies between 2 and 3, then equation

$$x = \frac{2}{3}(\log_{10} x + 4).$$

Solution of System of Non-linear Equations

A non-linear system of n equations with n variables is written in the form as:

$$f_1(x_{1\,,}\; x_2 \quad \ldots,\, x_n) = 0;$$
$$f_2\,(x_{1\,,}\; x_2 \quad \ldots,\, x_n) = 0;$$
$$\ldots$$
$$\ldots$$
$$f_n(\, x_{1\,,}\; x_2 \quad \ldots,\, x_n) = 0.$$

System of Non-linear Equations

A non-linear system of n equations with n variables is written in the form as:

$$\mathbf{X} = (x_1, x_2 \quad \ldots, x_n) \in \Re^n; \quad \mathbf{F} = [\, f_1 \quad f_2 \quad \ldots \quad f_n \,]'.$$

$$\mathbf{F(X) = 0.}$$

System of Non-linear Equations

Newton- Rahpson's method:

$$\mathbf{X} = (x_{1\,,}\ x_2 \quad \ldots,\ x_n) \in \Re^n; \qquad \mathbf{F} = [\, f_1 \quad f_2 \quad \ldots \quad f_n\,]'.$$

$$\mathbf{F(X) = 0.}$$

The first approximation root is

$$\mathbf{X_1 = X_0 - F(X_0) / F'(X_0)}$$

The second approximation root is

$$\mathbf{X_2 = X_1 - F(X_1) / F'(X_1)}$$

The n^{th} approximation root is

$$\mathbf{X_n = X_{n-1} - F(X_{n-1}) / F'(X_{n-1})}$$

System of Non-linear Equations

Newton- Rahpson's method:

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{bmatrix}, \quad \mathbf{F'}(\mathbf{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Non-linear Equations

Example : Applying Newton-Raphson method to solve a system of non-linear equation $x^2 + y - x - 1 = 0,\ x^2 - 2y^2 - y = 0$ with initial guess $x_0 = 1,\ y_0 = 0$ accurate to three decimal places

Solution :

Let $f_1 = x^2 + y - x - 1 = 0$ $f_2 = x^2 - 2y^2 - y = 0$. So

$$\mathbf{F} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 + y - x - 1 \\ x^2 - 2y^2 - y \end{bmatrix},$$

$$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -4y - 1 \end{bmatrix}.$$

Non-linear Equations

Applying Newton-Raphson method to solve a system of non-linear equation $x^2 + y - x - 1 = 0,\ x^2 - 2y^2 - y = 0$

the *first approximation* to the solution is

$$\mathbf{X}_1 = \mathbf{X}_0 - [\mathbf{F}'(\mathbf{X}_0)]^{-1}\mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

Non-linear Equations

Applying Newton-Raphson method to solve a system of non-linear equation $x^2+y-x-1=0,\ x^2-2y^2-y=0$

and the *second approximation*to the solution is

$$\mathbf{X}_2 = \mathbf{X}_1 - [\mathbf{F}'(\mathbf{X}_1)]^{-1}\mathbf{F}(\mathbf{X}_1) = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0 & 1\\2 & -5\end{bmatrix}^{-1}\begin{bmatrix}0\\-2\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{7}\begin{bmatrix}5 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{7}\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1.2857\\0.7143\end{bmatrix},$$

Non-linear Equations

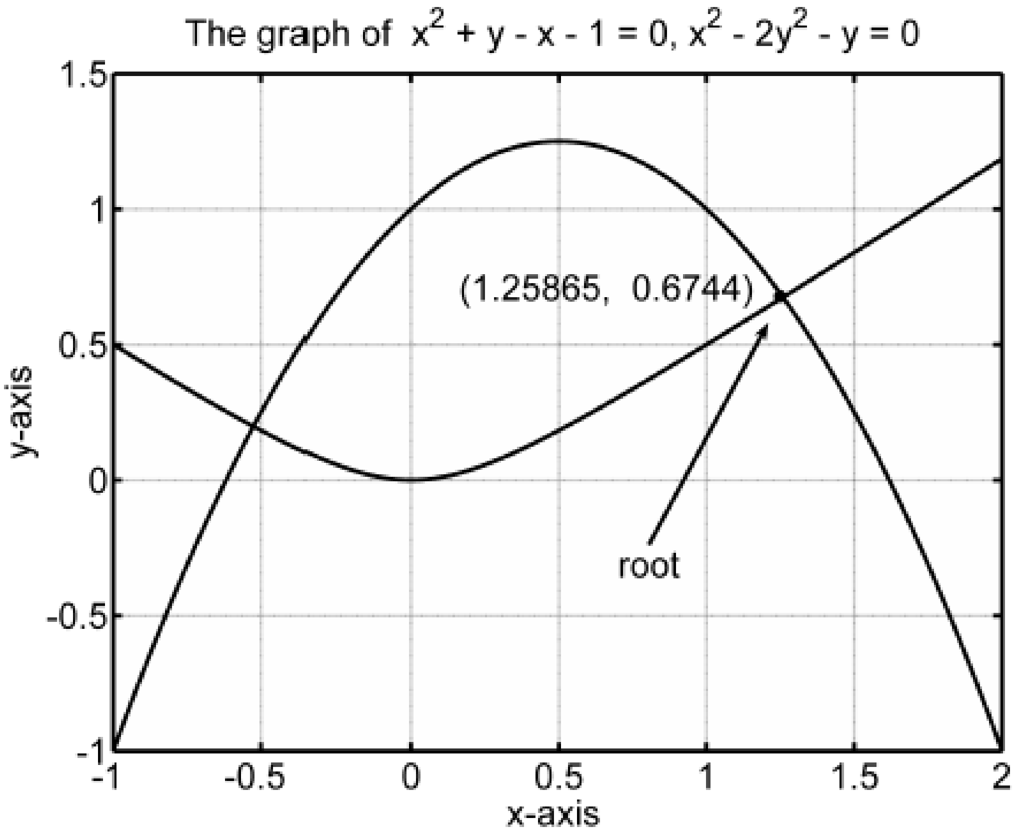
Applying Newton-Raphson method to solve a system of non-linear equation
$x^2 + y - x - 1 = 0$, $x^2 - 2y^2 - y = 0$

Iterations	x	y
0.	1.00000	0.00000
1.	1.00000	1.00000
2.	1.28571	0.71428
3.	1.25869	0.67511
4.	1.25865	0.67444
5.	1.25865	0.67444

roots, x = 1.25865, y = 0.67444.

Non-linear Equations

Applying Newton-Raphson method to solve a system of non-linear equation $x^2 + y - x - 1 = 0,\ x^2 - 2y^2 - y = 0$



Assignment - 1

1. Find an approximate root of the following non-linear equations applying bisection method correct to two decimal places (error $\epsilon = 10^{-3}$) with given initial guesses a and b

   (i) $x^3 + 2x - 2 = 0, \quad a = 0, b = 1$ (ii) $x^3 + x^2 - 1 = 0, \quad a = 0, b = 1$

   (iii) $e^{-x} \sin x - \cos x = 0, \quad a = 1, b = 2$ (iv) $x^4 + 2x - 10 = 0, \quad a = 1, b = 2$

   (v) $\log_{10} x - 1.2 = 0, \quad a = 2, b = 3$ (vi) $x^3 - 4x - 9 = 0, \quad a = 2, b = 3$

2. Find a real root of a non-linear equation obtained by the intersection of $y = x - 2$ and $y = \log_{10} x$ with accuracy 0.001, two initial guesses $a = 2$, $b = 3$ applying bisection method

3. Find an approximate root of the following equations correct to three decimal places correct to three decimal places with given *initial* guesses x_1 and $x_2$applying False-position (Regula-False) method

   (i) $x^3 + 3x - 7 = 0, \; x_1 = 1, x_2 = 2$ (ii) $3x + \sin x = e^x, \; x_1 = 0, x_2 = 1$

   (iii) $x^2 - e^x - 3x + 2 = 0, \; x_1 = 0, x_2 = 1$ (iv) $x^2 - (1 - x)^5 = 0, \; x_1 = 0, x_2 = 1$

   (v) $e^{\cos x} - x \sin x = 0, \; x_1 = 1, x_2 = 2$ (vi) $\sin^2 x - x^2 + 1 = 0, \; x_1 = 1, x_2 = 2$