Gaussian Measures and Density Function on $\mathcal{L}^2[p,q]$ Space

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1 Introduction

In this article we aim to define Gaussian measures on the $\mathcal{L}_2[p,q]$ space of all functions that are square-integrable on the compact interval $[a,b]$, where, $p<q$ and $p,q \in \mathbb{R}$. This is a necessary step to define the MEM algorithm on the functional data.

We shall revisit some preliminary concepts of Probability Theory in the following section.

2 Some Preliminary Concepts

Let, $\{\Omega, \mathcal{F}, \mathbb{P}\}$ denote any probability space. We can define a random variable on $\{\Omega, \mathcal{F}, \mathbb{P}\}$ and its law as follows.

- **Definition of Random Variable :**
  A random variable X on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ that takes values in a set E is a mapping $X : \Omega \to E$ such that $I \in \mathcal{B}(E) \Rightarrow X^{-1} \in \mathcal{F}$.

- **Definition of law of a Random Variable :**
  The law of a random variable X on the probaility space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, taking values in the set E is the probability measure $X_{\#\mathbb{P}}(I) = \mathbb{P}(X^{-1}(I)) = \mathbb{P}(X \in I)$, where, $I \in \mathcal{B}(E)$.

We shall also have a look at the change of variables formula.

Theorem 1. *[Change of Variables Formula]*

Let X be a random variable in $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with values in E. Also, let $\varphi : E \to \mathbb{R}$ be a bounded Borel mapping. Then we have,

$$\int_\Omega \varphi(X(\omega))\mathbb{P}(d\omega) = \int_E \varphi(x) X_{\#\mathbb{P}}(dx) \tag{1}$$

Proof. It is enough to prove 1 for the special case $\varphi = \mathbb{1}_I$ for $I \in \mathcal{B}(E)$. Here, $\mathbb{1}_I$ denotes the indicator function of the set I, ie.,

$$\mathbb{1}_I(x) = \begin{cases} 1 & \text{if, } x \in I \\ 0 & \text{if, } x \notin I. \end{cases}$$

In this case we have, $\varphi(X(\omega)) = \mathbb{1}_{X^{-1}(I)}(\omega)$, $\forall \omega \in \Omega$. This gives us,

$$\int_\Omega \varphi(X(\omega))\mathbb{P}(d\omega) = \int_\Omega \mathbb{1}_{X^{-1}(I)}(\omega) = \mathbb{P}(X^{-1}(I)) = X_{\#\mathbb{P}}(I) = \int_E \mathbb{1}_I X_{\#\mathbb{P}}(dx) = \int_E \varphi(x) X_{\#\mathbb{P}}(dx)$$

This completes the proof. $\square$

The other necessary preliminary concepts will be discussed in the Appendix. With the goal of defining a Gaussian measure on the $\mathcal{L}_2[p,q]$ space, we shall introduce some notations in the following section.

3 Notations

We shall abbreviate the $\mathcal{L}_2[p,q]$ space of all sqaure-integrable functions on the compact interval $[p,q]$ as $\mathcal{L}_2$. The reader must not confuse it with the standard $\mathcal{L}_2$ space of all square-integrable functions on $\mathbb{R}$.

We aim to define a Gaussian measure on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$. Towards that direction we introduce the following notations.

- **Inner-product :** For all f and g in $\mathcal{L}_2$ the inner product $\langle f, g\rangle$ between f and g is defined as

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt.$$

.

- **Norm :** For all f in $\mathcal{L}_2$, the norm $\|f\|$ of f is defined as

$$\|f\| = \left[\int_a^b f^2(t)dt\right]^{\frac{1}{2}}.$$

- $\mathbf{L}(\mathcal{L}_2)$: It is the set of all continuous linear operators from $\mathcal{L}_2$ to $\mathcal{L}_2$.
- $\mathbf{L}^+(\mathcal{L}_2)$: It is the set of all $T \in L(\mathcal{L}_2)$ which are symmetric and non-negative, ie, $\langle Tx, y\rangle = \langle x, Ty\rangle$ and, $\langle Tx, x\rangle \geq 0$.
- $\mathbf{L}_1^+(\mathcal{L}_2)$: It is the set of all operators $Q \in L^+(\mathcal{L}_2)$ which are of trace class, ie,

$$Tr(Q) := \sum_{k=1}^{\infty} \langle Qe_k, e_k\rangle < \infty$$

for all completely orthonormal systems $(e_k)_k$ in $\mathcal{L}_2$.

Having our notations defined, we can now move on to define Measures on the $\mathcal{L}_2$ space in the following section.

4 Defining Measures on the $\mathcal{L}_2[p,q]$ Space

In this section we shall define a measure on the $\mathcal{L}_2[p,q]$ space and shall also develop notions of the mean, covariance structure and the Characteristic Function (Fourier Transform) of the measures. We shall then move on to define a Gaussian measure on $\mathcal{L}_2$.
We note that $\mathcal{L}_2$ is an infinite dimensional Hilbert space. Let, $(e_k)_k$ be a completely orthonormal system in $\mathcal{L}_2$. We shall require the projection mapping defined by $P_n : \mathcal{L}_2 \to P_n(\mathcal{L}_2)$ defined by $P_n x = \sum_{k=1}^{n} \langle x, e_k\rangle e_k$, where $x \in \mathcal{L}_2$, to define a Gaussian measure on the $\mathcal{L}_2$ space. We have already proved earlier that $\lim_{n\to\infty} P_n x = x$. This mapping will be necessary later in the discussion. In this section we assume the existence of measures on the $\mathcal{L}_2$ space and prove a number of properties of such measures before progressing further. In the next section we shall prove the existence of such measures.

Theorem 2. *Suppose μ and ν are two measures defined on the $\mathcal{L}_2$ space such that*

$$\int_{\mathcal{L}_2} \varphi(x)\mu(dx) = \int_{\mathcal{L}_2} \varphi(x)\nu(dx)$$

for all continuous and bounded $\varphi : \mathcal{L}_2 \to \mathbb{R}$. Then, $\mu = \nu$.

Proof. We shall prove the theorem for a closed subset C of $\mathcal{L}_2$. Since the closed subsets generate the Borel σ-algebra of $\mathcal{L}_2$, proving the result for any closed subset of $\mathcal{L}_2$ proves the result.
We start with a sequence $(\varphi_n)_n$ of continuous and bounded functions in $\mathcal{L}_2$, such that the following conditions hold.

- $\varphi_n(x) \to \mathbb{1}_C(x)$, $\forall x \in \mathcal{L}_2$.
- $\sup_{x\in\mathcal{L}_2} |\varphi_n(x)| \leq 1$

Here, $\mathbb{1}_C$ denotes the characteristic function of C. An example of such a sequence is given by,

$$\varphi_n(x) = \begin{cases} 1 & \text{if, } x \in C; \\ 1 - nd(x,C) & \text{if, } d(x,C) \leq \frac{1}{n}; \\ 0 & \text{if, } d(x,C) \geq \frac{1}{n} \end{cases}$$

Then, by Dominated Convergence Theorem, we obtain,

$$\int_{\mathcal{L}_2} \varphi_n d\mu = \int_{\mathcal{L}_2} \varphi_n d\nu \implies \mu(C) = \nu(C).$$

This completes the proof. □

Theorem 3. *Suppose μ and ν are two probability measures on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$. If $(P_n)_{\#\mu} = (P_n)_{\#\nu}$ for all $n \in \mathbb{N}$, we have $\mu = \nu$.*

Proof. We consider a bounded and continuous function $\varphi : \mathcal{L}_2 \to \mathbb{R}$. By Dominated Convergence Theorem, we have,

$$\int_{\mathcal{L}_2} \varphi(x)\mu(dx) = \lim_{n\to\infty} \int_{\mathcal{L}_2} \varphi(P_n x)\mu(dx).$$

Thus, by Theorem 1 we see that,

$$\begin{aligned} \int_{\mathcal{L}_2} \varphi(x)\mu(dx) &= \lim_{n\to\infty} \int_{\mathcal{L}_2} \varphi(P_n x)\mu(dx) \\ &= \lim_{n\to\infty} \int_{P_n(\mathcal{L}_2)} \varphi(\rho)(P_n)_{\#\mu}(d\rho) \\ &= \lim_{n\to\infty} \int_{P_n(\mathcal{L}_2)} \varphi(\rho)(P_n)_{\#\nu}(d\rho) \\ &= \lim_{n\to\infty} \int_{\mathcal{L}_2} \varphi(P_n x)\nu(dx) \\ &= \int_{\mathcal{L}_2} \varphi(x)\nu(dx). \end{aligned}$$

Since, φ is arbitrary, by theorem 2, we have $\mu = \nu$. This completes the proof. □

Now, we consider the characteristic function (Fourier transform) of the measure μ. The characteristic function is defined as,

$$\hat{\mu}(h) := \int_{\mathcal{L}_2} e^{i\langle x,h\rangle}\mu(dx), \quad \forall h \in \mathcal{L}_2.$$

This gives us the following result.

Theorem 4. *Suppose μ and ν are two probability measures on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$. If $\hat{\mu}(h) = \hat{\nu}(h)$ for all $h \in \mathcal{L}_2$, we have $\mu = \nu$.*

Proof. We note that $\forall n \in \mathbb{N}$, by equation 1,

$$\hat{\nu}(P_n h) = \int_{\mathcal{L}_2} e^{i\langle x, P_n h\rangle} \nu(dx) = \int_{P_n(\mathcal{L}_2)} e^{i\langle P_n \rho, P_n h\rangle} (P_n)_{\#\mu}(d\rho),$$

and

$$\hat{\mu}(P_n h) = \int_{\mathcal{L}_2} e^{i\langle x, P_n h\rangle} \mu(dx) = \int_{P_n(\mathcal{L}_2)} e^{i\langle P_n \rho, P_n h\rangle} (P_n)_{\#\nu}(d\rho).$$

Now, the measures $(P_n)_{\#\mu}$ and $(P_n)_{\#\nu}$ have the same Characteristic Function, as $\hat{\mu}(P_n h) = \hat{\nu}(P_n h)$, and hence they coincide. Thus, by theorem 3, we have $\mu = \nu$. This completes the proof. □

Now, for a fixed probability measure μ on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$, let us define the mean and covariance of the measure.

4.1 Defining the Mean and the Covariance of a Probability Measure

In this subsection define the mean and the covariance structure of the probability measure μ on the probability space $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2), \mu)$.

4.1.1 Defining Mean of μ

Let us assume that $\int_{\mathcal{L}_2} \|x\| \mu(dx) < \infty$. We define a linear functional $F : \mathcal{L}_2 \to \mathbb{R}$ as,

$$F(h) = \int_{\mathcal{L}_2} \langle x, h\rangle \mu(dx), \quad \forall h \in \mathcal{L}_2. \tag{2}$$

We note that $\forall h \in \mathcal{L}_2$,

$$|F(h)| \leq \int_{\mathcal{L}_2} \|x\| \mu(dx) \|h\| < \infty$$

Recalling that a linear functional is continuous if and only if it is bounded, we see that F defined above is a continuous linear functional. Thus, by Riesz Representation Theorem there exists $m \in \mathcal{L}_2$ such that,

$$\langle m, h\rangle = \mathcal{L}_2 \langle x, h\rangle \mu(dx), \quad \forall h \in \mathcal{L}_2. \tag{3}$$

We shall refer to the m defined by equation 3 as the mean of the probability measure μ, and represent it as,

$$\int_{\mathcal{L}_2} x \mu(dx) = m \tag{4}$$

We shall now be defining the covariance structure of μ in the following subsection.

4.1.2 Defining Covariance of μ

We shall follow a similar technique to define the covariance of the probability measure μ. We begin by assuming that $\int_{\mathcal{L}_2} \|x\|^2 \mu(dx) < \infty$. We define a bilinear form $G : (\mathcal{L}_2 \times \mathcal{L}_2) \to \mathbb{R}$ as,

$$G(h, k) = \int_{\mathcal{L}_2} \langle h, x - m\rangle \langle k, x - m\rangle \mu(dx) \quad h, k \in \mathcal{L}_2. \tag{5}$$

We observe that,

$$|G(h, k)| \leq \int_{\mathcal{L}_2} \|x - m\|^2 \mu(dx) \|h\| \|k\| < \infty.$$

Hence, $G(h,k)$ is continuous. Then, by the Riesz Representation Theorem there exists a unique linear bounded operator $Q \in L(\mathcal{L}_2)$, such that,

$$\langle Qh, k\rangle = \int_{\mathcal{L}_2} \langle h, x-m\rangle\langle k, x-m\rangle \mu(dx), \quad h,k \in \mathcal{L}_2. \tag{6}$$

Q defined by equation 6 can be regarded as the covariance of μ. However, for obvious reasons it needs to be made sure that Q is symmetric, positive and of trace class. The following result ensures that.

Theorem 5. *Let μ be a probability measure on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ having m and Q respectively as the mean and the covariance operator. Then $Q \in L_1^+(\mathcal{L}_2)$, ie, Q is symmetric, positive and of trace class.*

Proof. We shall prove positivity, symmetry and the trace class property one after the other.

- **Proving Positivity :**
  Let $h \in \mathcal{L}_2$. Then, by construction, we have,

  $$\langle Qh, h\rangle = G(h,h) = \int_{\mathcal{L}_2} \langle h, x-m\rangle^2 \mu(dx) \geq 0$$

  Hence, Q is positive

- **Proving Symmetry :**
  Let, $h,k \in \mathcal{L}_2$ be arbitrary. First we note that

  $$G(h,k) = \int_{\mathcal{L}_2} \langle h, x-m\rangle\langle k, x-m\rangle \mu(dx) = \int_{\mathcal{L}_2} \langle k, x-m\rangle\langle h, x-m\rangle \mu(dx) = G(k,h).$$

  Now, by the construction of Q, we have,

  $$\langle Qh, k\rangle = G(h,k) = G(k,h) = \langle Qk, h\rangle = \int_{\mathcal{L}_2} \langle Qk, x-m\rangle\langle h, x-m\rangle \mu(dx) = \int_{\mathcal{L}_2} \langle h, x-m\rangle\langle Qk, x-m\rangle \mu(dx) = \langle h, Qk\rangle$$

  Thus, Q is symmetric.

- **Proving Q is of Trace Class :**
  We start with an orthonormal basis $(e_k)_k$ of $\mathcal{L}_2$. This allows us to write,

  $$\langle Qe_k, e_k\rangle = \int_{\mathcal{L}_2} \langle x-m, e_k\rangle^2 \mu(dx).$$

  By the Monotone Convergence Theorem and the Parseval Identity, we get,

  $$Tr(Q) = \sum_{k=1}^{\infty} \int_{\mathcal{L}_2} \langle x-m, e_k\rangle^2 \mu(dx) = \int_{\mathcal{L}_2} \|x-m\|^2 \mu(dx) < \infty$$

  Hence, Q is of trace class.

  This completes the proof.

$\square$

Theorem 5 enables us to define Q as the covariance operator of the probability measure μ defined on the space $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$.
In the following section, we shall define and and show the existence of a Gaussian measure on the $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ space.

5 Defining Gaussian Measure on $\mathcal{L}_2$ Space

In this section we shall define a Gaussian measure on the $\mathcal{L}_2$ space. Let, $a \in \mathcal{L}_2$ and $Q \in L_1^+(\mathcal{L}_2)$. We define a measure $\mu := N_{a,Q}$ on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ as a measure having mean a, covariance operator Q and characteristic function,

$$\hat{N}_{a,Q}(h) = \exp\{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle\}, \quad h \in \mathcal{L}_2. \tag{7}$$

We shall say the measure $N_{a,Q}$ is non-degenerate if $Ker(Q) := \{x \in \mathcal{L}_2 : Qx = 0\} = \{0\}$.
We are going to establish that for any arbitrary $a \in \mathcal{L}_2$ and any $Q \in L_1^+(\mathcal{L}_2)$ there exists a unique Gaussian measure $N_{a,Q}$ defined on the space $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$.
We notice that, since $Q \in L_1^+(\mathcal{L}_2)$, there exists a complete orthonormal basis $(e_k)_k$ of $\mathcal{L}_2$ and a sequence $(\lambda_k)_k$ of non-negative real numbers, such that

$$Qe_k = \lambda_k e_k; \quad k \in \mathbb{N}.$$

Let us define, for all $k \in \mathbb{N}$, x_k to be the coefficient of e_k in the basis expansion of x with respect to the completely orthonormal basis $(e_k)_k$, ie., $x = \sum_{k=1}^{\infty} x_k e_k$. This can also be mathematically represented as,

$$x_k = \langle x, e_k\rangle; \quad k \in \mathbb{N}$$

We consider the natural isomorphism $\gamma : \mathcal{L}_2 \to l_2$ where, l_2 denotes the Hilbert space of all real sequences $(x_k)_k$ such that, $\sum_{k=1}^{\infty} x_k^2 < \infty$, defined by,

$$\mathcal{L}_2 \to l_2, x \to \Gamma(x) = (x_1, x_2, \dots)$$

We then consider the product measure, $\mu := \bigtimes_{k=1}^{\infty} N_{a_k,\lambda_k}$ where, $a_k = \langle a, e_k\rangle$. We must note that μ is defined on $\mathbb{R}^\infty := \bigtimes_{k=1}^{\infty} \mathbb{R}$ and not l_2. We shall show later that μ is concentrated on l_2 in the sense that $\mu(l_2) = 1$. Finally, we shall move on to show that μ as defined above is a Gaussian measure on the l_2 space. However, before proceeding, we need to revisit some concepts on countable products of measures, as they will be essential for further discussion.

5.1 Revisiting Countable Products of Measures

Suppose, $(\zeta_1, \zeta_2, \dots)$ is a sequence of probability measures defined on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$. We aim to define a product measure on the space, $\mathbb{R}^\infty = \bigtimes_{k=1}^{\infty} \mathbb{R}$, consisting of all sequences $p = (p_1, p_2, \dots)$ of real numbers. We endow $\mathbb{R}$ with the following metric.

$$d(p,q) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\max_{1 \le k \le n}\{|p_k - q_k|\}}{1 + \max_{1 \le k \le n}\{|p_k - q_k|\}}, \tag{8}$$

where $p = (p_1, p_2, \dots)$ and $q = (q_1, q_2, \dots)$. We can verify that $\mathbb{R}^\infty$ endowed with the above metric d is complete, ie., all Cauchy sequences in $\mathbb{R}^\infty$ are convergent in $\mathbb{R}^\infty$. In addition, the above metric induces the product topology.

Let, $\mathcal{C} = \{I_{n,A} : n \in \mathbb{N} \text{ and } A \in \mathcal{B}(\mathbb{R}^n)\}$, where,

$$I_{n,A} = \{x = (x_1, x_2, \dots) \in \mathbb{R}^\infty : (x_1, x_2, \dots, x_n) \in A\}. \tag{9}$$

It is easy to see from equation 9 that,

$$I_{n,A} = I_{(n+k),(A \times X_{n+1} \times X_{n+2} \times \cdots \times X_{n+k})}; \quad n, k \in \mathbb{N}. \tag{10}$$

Let, $I_{n,A}$ and $I_{m,B}$ be two arbitrary cylindrical subsets. From equation 10 we can see that,

$$\begin{aligned} I_{n,A} \bigcup I_{m,B} &= I_{(m+n),(A \times X_{n+1} \times X_{n+2} \times \cdots \times X_{m+n})} \bigcup I_{(m+n),(B \times X_{m+1} \times X_{m+2} \times \cdots \times X_{m+n})} \\ &= I_{(m+n),[(A \times X_{n+1} \times X_{n+2} \times \cdots \times X_{m+n}) \bigcup (B \times X_{m+1} \times X_{m+2} \times \cdots \times X_{m+n})]}. \end{aligned} \tag{11}$$

Also, we can check that,

$$I_{n,A}^c = I_{n,A^c}. \tag{12}$$

Thus, we can see that $\mathcal{C}$ is an algebra on $\mathbb{R}^\infty$. In addition, the σ-algebra induced by $\mathcal{C}$ is the same as $\mathcal{B}(\mathbb{R}^\infty)$, because any ball (with respect to the metric d defined by equation 8) is a countable intersection of the cylindrical sets defined by 9.

We shall now define the product measure on $\mathcal{C}$ as,

$$\mu(I_{n,A}) = (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(A) \tag{13}$$

Equations 10 and 11 show that μ is additive. The following result shows that μ is also σ-additive on $\mathcal{C}$. This would imply by the Caretheodory Extension Theorem that μ can be uniquely extended to probability measure on the product σ-algebra $\mathcal{B}(\mathbb{R}^\infty)$.

Theorem 6. *μ is σ-additive on $\mathcal{C}$, and hence, it possessses an unique extension to a probability measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$.*

Proof. To show that μ is σ-additive on $\mathcal{C}$, it is enough to prove that μ is continuous at ϕ. Towards that direction, let $(E_j)_j$ be a decreasing sequence on $\mathcal{C}$, such that for some fixed $\epsilon > 0$, we have $\mu(E_j) \geq \epsilon$ for all $j \in \mathbb{N}$. We shall show that $\bigcap_{j=1}^\infty E_j \neq \phi$.

Let us define, $\forall p \in \mathbb{N}$, $\mathbb{R}_p^\infty = \bigtimes_{n=p+1}^\infty \mathbb{R}$ and $\mu^{(p)} = \bigtimes_{n=p+1}^\infty \mu_n$. Also suppose,

$$E_j(\alpha) = \{x \in \mathbb{R}_1^\infty : (\alpha, x) \in E_j\}; \quad \alpha \in \mathbb{R}$$

and,

$$F_j^{(1)} = \{\alpha \in \mathbb{R} : \mu^{(1)}(E_j(\alpha)) \geq \frac{\epsilon}{2}\}; \quad j \in \mathbb{N}.$$

Then, by Fubini's Theorem we have,

$$\begin{aligned}
\mu(E_j) &= \int_{\mathbb{R}} \mu^{(1)}(E_j(\alpha))\mu_1(d\alpha) \\
&= \int_{F_j^{(1)}} \mu^{(1)}(E_j(\alpha))\mu_1(d\alpha) + \int_{[F_j^{(1)}]^c} \mu^{(1)}(E_j(\alpha))\mu_1(d\alpha) \\
&\leq \mu_1(F_j^{(1)}) + \frac{\epsilon}{2}
\end{aligned} \tag{14}$$

Thus we have,

$$\mu_1(F_j^{(1)}) \geq \frac{\epsilon}{2}.$$

μ_1 being a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, it is continuous at ϕ. Hence, as the sequence $(F_j^{(1)})$ is decreasing, $\exists \bar{\alpha_1} \in \mathbb{R}$ such that,

$$\mu^1(E_j(\bar{\alpha_1})) \geq \frac{\epsilon}{2}; \quad j \in \mathbb{N},$$

and as a result,

$$E_j(\bar{\alpha}_1) \neq \phi \tag{15}$$

Let us now set,

$$E_j(\bar{\alpha_1}, \alpha_2) = \{x_2 \in \mathbb{R}_2^\infty : (\bar{\alpha_1}, \alpha_2, x) \in E_j\}; \quad j \in \mathbb{N}, \alpha_2 \in \mathbb{R},$$

and,

$$F_j^{(2)} = \{\alpha_2 \in \mathbb{R} : \mu^{(2)}(E_j(\alpha)) \geq \frac{\epsilon}{2}\}; j \in \mathbb{N}.$$

Again by Fubini's Theorem, we have

$$\begin{aligned}\mu^1(E_j(\bar{\alpha_1})) &= \int_{\mathbb{R}} \mu^{(2)}(E_j(\bar{\alpha_1}, \alpha_2))\mu_2(d\alpha_2)\\ &= \int_{F_j^{(2)}} \mu^{(2)}(E_j(\bar{\alpha_1}, \alpha_2))\mu_2(d\alpha_2) + \int_{[F_j^{(2)}]^c} \mu^{(2)}(E_j(\bar{\alpha_1}, \alpha_2))\mu_2(d\alpha_2\\ &\leq \mu_2(F_j^{(2)}) + \frac{\epsilon}{4}.\end{aligned}$$

Therefore, we have,

$$\mu_2(F_j^{(2)}) \geq \frac{\epsilon}{4}.$$

Now, since $(F_j^{(2)})$ is decreasing, there exists $\bar{\alpha_2} \in \mathbb{R}$, such that,

$$\mu^2(E_j(\bar{\alpha_1}, \bar{\alpha_2})) \geq \frac{\epsilon}{4}, \quad j \in \mathbb{N}.$$

Consequently,

$$E_j(\bar{\alpha_1}, \bar{\alpha_2}) \neq \phi. \tag{16}$$

Moving the argument forward in a similar manner, we can construct a sequence $(\bar{\alpha_1}, \bar{\alpha_2}, \dots) \in \mathbb{R}^\infty$ such that,

$$E_j(\bar{\alpha_1}, \bar{\alpha_2}, \dots, \bar{\alpha_n}) \neq \phi, \tag{17}$$

where,

$$E_j(\bar{\alpha_1}, \bar{\alpha_2}, \dots, \bar{\alpha_n}) = \{x \in \mathbb{R}_n^\infty : (\alpha_1, \alpha_2, \dots \alpha_n, x) \in E_j\}, \quad n \in \mathbb{N}.$$

This implies that,

$$(\alpha_1, \alpha_2, \dots) \in \bigcap_{j=1}^{\infty} E_j$$

Thus, $\bigcap_{j=1}^\infty E_j \neq \phi$, and hence, μ is σ-additive on $\mathcal{C}$ and consequently on $\mathcal{B}(\mathbb{R}^\infty)$.
This completes the proof. $\square$

In the following subsection, we shall show the existence of Gaussian measure on the $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ space.

5.2 Definition and Existence of Gaussian Measure on the $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ Space

As defined earlier, let,

$$\mu = \bigtimes_{k=1}^{\infty} N_{a_k, \lambda_k} \tag{18}$$

To show the existence of a Gaussian measure on the $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ space, we first show that l_2 is a Borel subset of $\mathbb{R}^\infty$ and then show that the measure μ is concentrated on the l_2 space, in the sense that $\mu(l_2) = 1$.

Theorem 7. *l_2 is a Borel subset of $\mathbb{R}^\infty$.*

Proof. Let us define a sequence of functions $(\pi_i)_i$ such that, $\forall i \in \mathbb{N}$, $\pi_i : \mathbb{R}^\infty \to \mathbb{R}$ and,

$$\pi_i(x) = x_i; \quad x \in \mathbb{R}^\infty.$$

We can see that the functions π_i are continuous on $\mathbb{R}^\infty$ by the very definition of the product topology, and hence it is Borel. Hence, the function $f : \mathbb{R}^\infty \to [0, \infty]$ defined by $f(x) = \sum_{i=1}^\infty \pi_i(x)^2$ is also Borel, as it is a sum of countably many Borel functions. However, f is just the square of the l_2 norm, and hence, $l_2 = f^{-1}([0, \infty))$ is a Borel subset of $\mathbb{R}^\infty$.
This completes the proof. $\square$

Theorem 8. *We have, $\mu(l_2) = 1$.*

Proof. Using the Monotone Convergence Theorem, we see that,

$$\begin{aligned}\int_{\mathbb{R}^\infty} \sum_{k=1}^{\infty} x_k^2 \mu(dx) &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} x_k^2 N_{a_k,\lambda_k}(dx_k) \\ &= \sum_{k=1}^{\infty} (a_k^2 + \lambda_k) \end{aligned} \tag{19}$$

Now, since $a \in l_2$ and Q is of trace class, we have, $\sum_{k=1}^{\infty} a_k^2 = \|a\|_{l_2}^2 < \infty$ and $\sum_{k=1}^{\infty} \lambda_k < \infty$. Thus, we have, $\int_{\mathbb{R}^\infty} \sum_{k=1}^{\infty} x_k^2 \mu(dx) < \infty$. Therefore,

$$\mu(\{x \in \mathbb{R}^\infty : \|x\|_{l_2}^2 < \infty\}) = 1.$$

This completes the proof. □

The following result proves the existence of a Gaussian measure on the $\mathcal{L}_2$ space.

Theorem 9. *There exists a unique probability measure μ on $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ with mean a, covariance operator Q and characteristic function given by,*

$$\hat{\mu}(h) = e^{\{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle\}} \tag{20}$$

μ can be denoted by $N_{a,Q}$.

Proof. We shall check the restriction of the product measure μ defined by equation 18 to the l_2 space satisfies the necessary properties.
We get from equation 19 that

$$\int_{\mathcal{L}_2} \|x\|^2 \mu(dx) = Tr(Q) + \|a\|_{l_2}^2 \tag{21}$$

For the remainder of the proof, we assume that $Ker(Q) = \{0\}$ and (without any loss of generality) that,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \ldots$$

Suppose, $\{P_1, P_2, \ldots\}$ is a sequence of projection mappings, where $\forall n \in \mathbb{N}, P_n(x) = \sum_{k=1}^{n} \langle x, e_k\rangle e_k$ and let $h \in \mathcal{L}_2$.

- **Proof that the mean of μ is a :**

  We note that, $|\langle x,h\rangle| \leq \|x\| \cdot \|h\|$ and $\int_{\mathcal{L}_2} \|x\| \mu(dx) < \infty$. So, by the Dominated Convergence Theorem,

$$\begin{aligned}\int_{\mathcal{L}_2} \langle x,h\rangle \mu(dx) &= \lim_{n\to\infty} \int_{\mathcal{L}_2} \langle P_n x, h\rangle \mu(dx) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\mathcal{L}_2} x_k h_k \mu(dx) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} h_k \int_{\mathbb{R}} x_k N_{a_k,\lambda_k}(dx_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} h_k a_k \\ &= \lim_{n\to\infty} \langle P_n a, h\rangle \\ &= \langle a,h\rangle \end{aligned}$$

This shows that the mean of the product measure μ is a.

- **Proof that the Covariance Operator of μ is Q :**
  To prove that we proceed in a similar fashion. We fix any arbitrary $y, z \in \mathcal{L}_2$. Then we have,

$$
\begin{aligned}
\int_{\mathcal{L}_2} \langle (x-a), y\rangle \langle (x-a), z\rangle \mu(dx) &= \lim_{n\to\infty} \int_{\mathcal{L}_2} \langle P_n(x-a), y\rangle_n (x-a), z\rangle \mu(dx) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\mathcal{L}_2} (x_k - a_k)^2 y_k z_k \mu(dx) \\
&= \lim_{n\to\infty} y_k z_k \int_{\mathbb{R}} (x_k - a_k)^2 N_{a_k, \lambda_k}(dx_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} y_k z_k \lambda_k \\
&= \lim_{n\to\infty} \langle P_n Q y, z\rangle \\
&= \langle Q y, z\rangle
\end{aligned} \tag{22}
$$

  This shows that Q is the covariance operator of μ.

- **Proof that $\hat{\mu}(h) = e^{\{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle\}}$ is the Characteristic Function of μ :**
  Let $h \in \mathcal{L}_2$ be arbitrary. Then we have,

$$
\begin{aligned}
\int_{\mathcal{L}_2} e^{i\langle x,h\rangle} \mu(dx) &= \lim_{n\to\infty} \int_{\mathcal{L}_2} e^{i\langle P_n x, h\rangle} \mu(dx) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \int_{\mathbb{R}} e^{i x_k h_k} N_{a_k, \lambda_k}(dx_k) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} e^{i a_k h_k - \frac{1}{2}\lambda_k h_k^2} \\
&= \lim_{n\to\infty} e^{i\langle P_n a, h\rangle} e^{-\frac{1}{2}\langle P_n Q h, h\rangle} \\
&= e^{i\langle a,h\rangle} e^{-\frac{1}{2}\langle Qh,h\rangle} \\
&= e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}
\end{aligned} \tag{23}
$$

  This shows that the characteristic function of the product measure μ is given by $\hat{\mu}(h) = e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}$.

Also, by theorem 4, we conclude that the measure μ defined in equation 18 is the unique Gaussian measure defined on the $(\mathcal{L}_2, \mathcal{B}(\mathcal{L}_2))$ space. $\square$

Now, equipped with the Fourier Transform (or, characteristic function), we attempt to obtain the density function by taking the inverse Fourier Transform of the characteristic function.

6 Theorem

The following work is inspired by the following two theorems.

Theorem 10. *Suppose m is the Lebesgue measure on $\mathbb{R}$. If $f \in \mathcal{L}_2$ and its Fourier Transform $\hat{f} \in \mathcal{L}_1$, then,*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt} dm(t),$$

almost everywhere.

Theorem 11. *If $\hat{f}$ is the characteristic function of a real valued random variable X, then its inverse Fourier Transform gives us the density function f, provided, $\hat{f} \in \mathcal{L}_1$.*

7 Assumptions :

We shall need to verify the following assumptions. They will be necessary for obtaining the density.

- 1. The Fourier Transform (Characteristic Function) of the Gaussian Distribution on the $\mathcal{L}_2$ space is an $\mathcal{L}_1$ function.
- 2. A result similar to Theorem 10 holds for Gaussian Measure on $\mathcal{L}_2$ space, ie., we have $\forall x \in \mathcal{L}_2$,

$$f(x) = \int_{\mathcal{L}_2} e^{i\langle x,t\rangle} \hat{f}(t) \mu(dt) = \int_{\mathcal{L}_2} e^{i\langle x,t\rangle} e^{i\langle a,t\rangle - \frac{1}{2}\langle Qt,t\rangle} \mu(dt),$$

  where, μ is the Gaussian measure on the $\mathcal{L}_2$ space, with mean $a \in \mathcal{L}_2$ and covariance operator $Q \in L_1^+(\mathcal{L}_2)$.

- 3. The Dominated Convergence Theorem allows us to interchange product and integrals.

- 4. The Inverse Fourier Transform on the $\mathcal{L}_2$ space with respect to the Gaussian measure gives us the density function of Gaussian measure on the $\mathcal{L}_2$ space. (Even if this assumption is not true, it should not be much of a problem for our final goal of clustering of functional data. We shall then refer to it simply as the Inverse Fourier Transform).

8 Calculation

By Assumptions 1, 2 and 4, we have the density function f of Gaussian Measure on $\mathcal{L}_2$ space, given by,

$$f(x) = \int_{\mathcal{L}_2} e^{i\langle x,t\rangle} \hat{f}(t) \mu(dt) = \int_{\mathcal{L}_2} e^{i\langle x,t\rangle} e^{i\langle a,t\rangle - \frac{1}{2}\langle Qt,t\rangle} \mu(dt), \tag{24}$$

Let, $\{e_1, e_2, \dots\}$ denote a completely orthonormal basis of the $\mathcal{L}_2$ space. Then, we can write,

$$a = \sum_{k=1}^{\infty} a_k e_k; \quad t = \sum_{k=1}^{\infty} t_k e_k; \quad x = \sum_{k=1}^{\infty} x_k e_k,$$

where, $\{a_1, a_2, \dots\}$, $\{t_1, t_2, \dots\}$ and $\{x_1, x_2, \dots\}$ are three sequences of real numbers in the l_2 space.
Also, since $Q \in L_1^+(\mathcal{L}_2)$, we have a sequence of positive real numbers $\{\lambda_1, \lambda_2, \dots\}$ such that $\forall k \in \mathbb{N}$, we have,

$$Qe_k = \lambda_k e_k$$

We recall that the Gaussian measure on the $\mathcal{L}_2$ space is defined by the product measure,

$$\mu = \bigtimes_{k=1}^{\infty} N_{a_k,\lambda_k},$$

where, N_{a_k,λ_k} denotes the Gaussian measure on $\mathbb{R}$ with mean $a_k \in \mathbb{R}$ and variance $\lambda_k \in \mathbb{R}_+$.
Being equipped with the above representations of a, t, x and Q, it will be helpful to compute the necessary inner products.

$$\langle a+x, t\rangle = \langle \sum_{k=1}^{\infty}(a_k + x_k)e_k, \sum_{k=1}^{\infty} t_k e_k\rangle = \sum_{k=1}^{\infty}(a_k + x_k)t_k; \tag{25}$$

$$\langle Qt, t\rangle = \langle \sum_{k=1}^{\infty} \lambda_k t_k e_k, \sum_{k=1}^{\infty} t_k e_k\rangle = \sum_{k=1}^{\infty} \lambda_k t_k^2. \tag{26}$$

Thus, from equation 24 we obtain,

$$\begin{aligned}
f(x) &= \int_{\mathcal{L}_2} e^{\{i\sum_{k=1}^{\infty}(a_k+x_k)t_k - \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k t_k^2\}}\mu(dt) \\
&= \int_{\mathcal{L}_2} \prod_{k=1}^{\infty} e^{\{i(a_k+x_k)t_k - \frac{1}{2}\lambda_k t_k^2\}}\mu(dt) && (27) \\
&= \prod_{k=1}^{\infty} \int_{\mathbb{R}} e^{\{i(a_k+x_k)t_k - \frac{1}{2}\lambda_k t_k^2\}} N_{a_k,\lambda_k}(dt_k) \quad \text{(by Assumption 3 and assuming independence of the Normal densities)} \\
&= \prod_{k=1}^{\infty} \int_{\mathbb{R}} e^{ix_k t_k} e^{\{ia_k t_k - \frac{1}{2}\lambda_k t_k^2\}} N_{a_k,\lambda_k}(dt_k) && (28)
\end{aligned}$$

From theorem 10, we can see that equation 27 gives the inverse Fourier Transform of $\hat{f}(t_k)$ with respect to the Gaussian Measure on $\mathbb{R}$, and hence by theorem 11 $\int_{\mathbb{R}} e^{ix_k t_k} e^{\{ia_k t_k - \frac{1}{2}\lambda_k t_k^2\}} N_{a_k,\lambda_k}(dt_k)$ is the density function of Gaussian measure on $\mathbb{R}$. Denoting this density by $\phi(x_k; a_k, \lambda_k)$ we see that the Gaussian density (by Assumption-4) function on the $\mathcal{L}_2$ space is given by,

$$\begin{aligned}
f(x) &= \prod_{k=1}^{\infty} \phi(x_k; a_k, \lambda_k) \\
&= \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi\lambda_k}} e^{-\frac{1}{2}\frac{(x_k-a_k)^2}{\lambda_k}}. && (29)
\end{aligned}$$

9 Justification of Assumption-1

We shall need the following lemma for justifying Assumption 1.

Lemma 9.1. *Suppose* $\chi \sim N(0, \sigma^2)$ *distribution. Then,* $\mathbb{E}(\cos(b\chi)) = e^{-\frac{1}{2}b^2\sigma^2}$ *and* $\mathbb{E}(\sin(b\chi)) = 0$, $\forall b \in \mathbb{R}$.

Proof. We note that $\chi \sim N(0, \sigma^2) \Rightarrow b\chi \sim N(0, b^2\sigma^2)$. We also recall that if $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}(\sin(x)) = \sin(\mu)e^{-\frac{\sigma^2}{2}}$. This gives us,

$$\mathbb{E}(\sin(b\chi)) = \sin(0)e^{-\frac{b^2\sigma^2}{2}} = 0$$

.

Also,

$$\mathbb{E}(\cos(b\chi)) = \int_{\mathbb{R}} \cos(b\chi)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\frac{\chi^2}{\sigma^2}}d\chi$$

Substituting, $\frac{\chi}{\sqrt{2\sigma^2}} = t$, we obtain,

$$\begin{aligned}
\mathbb{E}(\cos(b\chi)) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}} \cos(b\sqrt{2\sigma^2}t)e^{-t^2}\sqrt{2\sigma^2}dt \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \cos(b\sqrt{2\sigma^2}t)e^{-t^2}dt \\
&= \frac{1}{\sqrt{\pi}}\sqrt{\pi}e^{-\frac{1}{4}b^2\times 2\sigma^2} \\
&= e^{-\frac{1}{2}b^2\sigma^2}
\end{aligned}$$

This completes the proof. □

$$
\begin{aligned}
\int_{\mathcal{L}_2} e^{i\langle a,h\rangle-\frac{1}{2}\langle Qh,h\rangle}\mu(dh) &= \int_{\mathcal{L}_2} e^{i\sum_{k=1}^{\infty} a_kh_k-\frac{1}{2}\sum_{k=1}^{\infty}\lambda_kh_k^2}\mu(dh)\\
&= \int_{\mathcal{L}_2}\prod_{k=1}^{\infty} e^{ia_kh_k-\frac{1}{2}\lambda_kh_k^2}\mu(dh)\\
&= \int_{\mathcal{L}_2}\lim_{n\to\infty}\prod_{k=1}^{n} e^{ia_kh_k-\frac{1}{2}\lambda_kh_k^2}\mu(dh)\\
&= \lim_{n\to\infty}\prod_{k=1}^{n}\int_{\mathbb{R}} e^{ia_kh_k-\frac{1}{2}\lambda_kh_k^2}N_{a_k,\lambda_k}(dh_k)[\text{By DCT and Fubini's Theorem, since }\ \mu=\bigtimes_{k=1}^{\infty}N_{a_k,\lambda_k}]\\
&= \prod_{k=1}^{\infty}\int_{\mathbb{R}} e^{ia_kh_k-\frac{1}{2}\lambda_kh_k^2}N_{a_k,\lambda_k}(dh_k)\\
&= \prod_{k=1}^{\infty}\int_{\mathbb{R}} e^{ia_kh_k-\frac{1}{2}\lambda_kh_k^2}\phi(h_k;a_k,\lambda_k)dh_k\ [\text{By Radon-Nikodym Theorem as the mesure }\ N_{a_k,\lambda_k}\ \text{is}\ \ \sigma-\text{finite}]\\
&= \prod_{k=1}^{\infty}\int_{\mathbb{R}} e^{ia_kh_k-\frac{1}{2}\lambda_kh_k^2}\frac{1}{\sqrt{2\pi\lambda_k}}e^{-\frac{1}{2}\frac{(h_k-a_k)^2}{\lambda_k}}dh_k\\
&= \prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}} e^{ia_kh_k-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k\\
&= \prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}[\cos(a_kh_k)+i\sin(a_kh_k)]e^{-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k \qquad [\text{Using Euler's Identity}]\\
&= \prod_{k=1}^{\infty}\left[\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k + i\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\sin(a_kh_k)e^{-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k\right]
\end{aligned} \tag{30}
$$

Let us define $\forall k\in\mathbb{N}$, $Re_k=\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k$ and $Im_k=\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\sin(a_kh_k)e^{-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k$.

Re_k and Im_k respectively denote the real and the imaginary parts of $\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}e^{ia_kh_k-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k$. We shall now determine Re_k and Im_k individually.

$$
\begin{aligned}
Re_k &= \frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}[\lambda_kh_k^2+\frac{(h_k-a_k)^2}{\lambda_k}]}dh_k\\
&= \frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}\frac{[(\lambda_k^2+1)h_k^2+a_k^2-2a_kh_k]}{\lambda_k}}dh_k
\end{aligned}
$$

Plugging in $\lambda_k^2+1=\beta_k^2$, we see get,

$$
\begin{aligned}
Re_k &= \frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}\frac{[\beta_k^2h_k^2+a_k^2-2a_kh_k]}{\lambda_k}}dh_k\\
&= \frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}\frac{[\beta_k^2h_k^2+\frac{a_k^2}{\beta_k^2}-2\beta_kh_k\cdot\frac{a_k}{\beta_k}]}{\lambda_k}}e^{-\frac{1}{2}\frac{(a_k^2-\frac{a_k^2}{\beta_k^2})}{\lambda_k}}dh_k\\
&= \frac{1}{\sqrt{2\pi\lambda_k}}e^{-\frac{1}{2}a_k^2(1-\frac{1}{\beta_k^2})}\int_{\mathbb{R}}\cos(a_kh_k)e^{-\frac{1}{2}\frac{(\beta_kh_k-\frac{a_k}{\beta_k})^2}{\lambda_k}}dh_k
\end{aligned} \tag{31}
$$

We substitute, $\beta_k h_k - \frac{a_k}{\beta_k} = \Gamma_k$. Hence, $dh_k = \frac{d\Gamma_k}{h_k}$. This gives us,

$$
\begin{aligned}
Re_k &= \frac{1}{\sqrt{2\pi\lambda_k}} e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}} \int_{\mathbb{R}} \cos\left(\frac{a_k}{\beta_k}\Gamma_k + \frac{a_k^2}{\beta_k^2}\right) e^{-\frac{1}{2}\frac{\Gamma_k^2}{\lambda_k}} \frac{d\Gamma_k}{\beta_k} \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k} \int_{\mathbb{R}} \cos\left(\frac{a_k}{\beta_k}\Gamma_k + \frac{a_k^2}{\beta_k^2}\right) \frac{1}{\sqrt{2\pi\lambda_k}} e^{-\frac{1}{2}\frac{\Gamma_k^2}{\lambda_k}} d\Gamma_k
\end{aligned}
\tag{32}
$$

Let us call $c_k = \frac{a_k}{\beta_k}$ and $d_k = \frac{a_k^2}{\beta_k^2} = c_k^2$. Noting that, $\frac{1}{\sqrt{2\pi\lambda_k}} e^{-\frac{1}{2}\frac{\Gamma_k^2}{\lambda_k}}$ is the probability density function of a univariate normal random variable $Z \sim N(0, \lambda_k)$, we obtain,

$$
\begin{aligned}
Re_k &= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k} \mathbb{E}(\cos(c_k Z + d_k)) \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k} \mathbb{E}(\cos(c_k Z)\cos(d_k) - \sin(c_k Z)\sin(d_k)) \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k} \cos(d_k) e^{-\frac{1}{2}c_k^2\lambda_k} \quad \text{[Using Lemma 9.1]} \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\sqrt{1+\lambda_k^2}} \cos\left(\frac{a_k^2}{\beta_k^2}\right) e^{-\frac{1}{2}\frac{a_k^2}{1+\lambda_k^2}\lambda_k} \\
&= \frac{e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\sqrt{1+\lambda_k^2}} \cos\left(\frac{a_k^2}{1+\lambda_k^2}\right)
\end{aligned}
\tag{33}
$$

Similarly, we can show that,

$$
\begin{aligned}
Im_k &= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k}\mathbb{E}(\sin(c_k Z + d_k)) \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k}\mathbb{E}(\sin(c_k Z)\cos(d_k) + \cos(c_k Z)\sin(d_k)) \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k}[\cos(d_k)\mathbb{E}(\sin(c_k Z)) + \sin(d_k)\mathbb{E}(\cos(c_k Z))] \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k}\sin(d_k)e^{-\frac{1}{2}c_k^2\lambda_k} \qquad \text{[Using Lemma 9.1]} \\
&= \frac{e^{-\frac{1}{2}\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k}\sin\left(\frac{a_k^2}{1+\lambda_k^2}\right)e^{-\frac{1}{2}\frac{a_k^2}{1+\lambda_k^2}\lambda_k} \\
&= \frac{e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\beta_k}\sin\left(\frac{a_k^2}{1+\lambda_k^2}\right)
\end{aligned} \tag{34}
$$

Thus, we have obtained,

$$
Re_k = \frac{e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\sqrt{1+\lambda_k^2}}\cos\left(\frac{a_k^2}{1+\lambda_k^2}\right), \quad \text{and,}
$$

$$
Im_k = \frac{e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\sqrt{1+\lambda_k^2}}\sin\left(\frac{a_k^2}{1+\lambda_k^2}\right),
$$

and hence from equation 30 we get,

$$
\begin{aligned}
\left|\int_{\mathcal{L}_2} e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}\mu(dh)\right| &= \left|\prod_{k=1}^{\infty}\left[\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_k h_k)e^{-\frac{1}{2}[\lambda_k h_k^2 + \frac{(h_k-a_k)^2}{\lambda_k}]}dh_k + i\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\sin(a_k h_k)e^{-\frac{1}{2}[\lambda_k h_k^2 + \frac{(h_k-a_k)^2}{\lambda_k}]}dh_k\right]\right| \\
&= \prod_{k=1}^{\infty}\left|\left[\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\cos(a_k h_k)e^{-\frac{1}{2}[\lambda_k h_k^2 + \frac{(h_k-a_k)^2}{\lambda_k}]}dh_k + i\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}}\sin(a_k h_k)e^{-\frac{1}{2}[\lambda_k h_k^2 + \frac{(h_k-a_k)^2}{\lambda_k}]}dh_k\right]\right| \\
&= \prod_{k=1}^{\infty}\left|Re_k + iIm_k\right| \\
&= \prod_{k=1}^{\infty}\frac{e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\sqrt{1+\lambda_k^2}} \leq 1 \qquad \left[\text{Since, } e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}} \leq 1 \leq \sqrt{1+\lambda_k^2} \Rightarrow \frac{e^{-\frac{\lambda_k a_k^2}{1+\lambda_k^2}}}{\sqrt{1+\lambda_k^2}} \leq 1\right]
\end{aligned}
$$

This shows that the Fourier Transform, $e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}$ is integrable on $\mathcal{L}_2$.

10 Justification of Assumption-3

Let us write, $a_k + x_k = b_k, \forall k \in \mathbb{N}$. Then, equation 27 can be rewritten as,

$$f(x) = \int_{\mathcal{L}_2} \prod_{k=1}^{\infty} e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2} \mu(dt) \tag{35}$$

Set, $g_n(t) = \prod_{k=1}^{n} e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2}$. It is easy to see that,

$$|g_n(t)| = \left| \prod_{k=1}^{n} e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2} \right| = \prod_{k=1}^{n} \left| e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2} \right| = \prod_{k=1}^{n} e^{-\frac{1}{2}\lambda_k t_k^2} = e^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k t_k^2}$$

Clearly, $|g_n(t)|$ defines a sequence of decreasing functions for fixed $t \in \mathcal{L}_2$. Therefore,

$$|g_n(t)| \le |g_1(t)| = e^{-\frac{1}{2}\lambda_1 t_1^2},$$

which is integrable. Thus the Dominated Convergence Theorem applies. This gives us,

$$\begin{aligned}
f(x) &= \int_{\mathcal{L}_2} \lim_{n\to\infty} g_n(t) \mu(dt) \\
&= \lim_{n\to\infty} \int_{\mathcal{L}_2} g_n(t)\mu(dt) \\
&= \lim_{n\to\infty} \int_{\mathcal{L}_2} \prod_{k=1}^{n} e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2} \mu(dt) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \int_{\mathbb{R}} e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2} N_{a_k,\lambda_k}(dt_k) \quad \text{[By Fubini's Theorem, since } \mu = \bigtimes_{k=1}^{\infty} N_{a_k,\lambda_k}\text{]} \\
&= \prod_{k=1}^{\infty} \int_{\mathbb{R}} e^{ib_k t_k - \frac{1}{2}\lambda_k t_k^2} N_{a_k,\lambda_k}(dt_k)
\end{aligned} \tag{36}$$

11 Deriving the Fourier Transform from the Density Function

Let, $h \in \mathcal{L}_2$ be arbitrary. Then, we have,

$$\begin{aligned}
\hat{f}(h) &= \int_{\mathcal{L}_2} e^{-i\langle h,t\rangle} f(t)\mu(dt) \quad \text{[By Definition.]} \\
&= \int_{\mathcal{L}_2} e^{-i\sum_{k=1}^{\infty} h_k t_k} \prod_{k=1}^{\infty} \phi(t_k; a_k, \lambda_k)\mu(dt) \\
&= \int_{\mathcal{L}_2} \prod_{k=1}^{\infty} e^{-ih_k t_k} \phi(t_k; a_k, \lambda_k)\mu(dt) \\
&= \prod_{k=1}^{\infty} \int_{\mathbb{R}} e^{-ih_k t_k} \phi(t_k; a_k, \lambda_k) N_{a_k,\lambda_k}(dt_k) \quad \text{[Using Dominated Convergence Theorem and Fubini's Theorem as before.]} \\
&= \prod_{k=1}^{\infty} \hat{f}(h_k) \quad \text{[As } \int_{\mathbb{R}} e^{-ih_k t_k} \phi(t_k; a_k, \lambda_k) N_{a_k,\lambda_k}(dt_k) \text{ is the Fourier Transform of the one-dimensional Gaussian measure.]} \\
&= \prod_{k=1}^{\infty} e^{ia_k h_k - \frac{1}{2}\lambda_k h_k^2} \\
&= e^{i\sum_{k=1}^{\infty} a_k, h_k - \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k h_k^2} \\
&= e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}
\end{aligned} \tag{37}$$

Thus, we see that the density function f defined by equation 27 gives has the same Fourier Transform as the one used to define a Gaussian Measure on the $\mathcal{L}_2$ space. Hence, by the uniqueness of the Fourier Transform, we can say that f defines the Gaussian density function on the $\mathcal{L}_2$ space.

12 Remarks on the Density Obtained

We note the following remarks pertaining to the density function obtained above in 29.

Remark 12.1. *According to theorem 5, Q is of trace class. This means that* $\sum_{k=1}^{\infty} \lambda_k < \infty$*, which in turn implies,* $\lim_{k\to\infty} \lambda_k = 0$*. Thus, the factors* $\frac{1}{\sqrt{2\pi\lambda_k}}$ *in 29 diverge to* ∞ *as k tends to* ∞*. As a result, the resulting density can assume infinite value.*

Remark 12.2. *In fact the density assumes infinite value for uncountably many functions. For example, here is an uncountable set on which for each of the functions, the density assumes infinite value.*

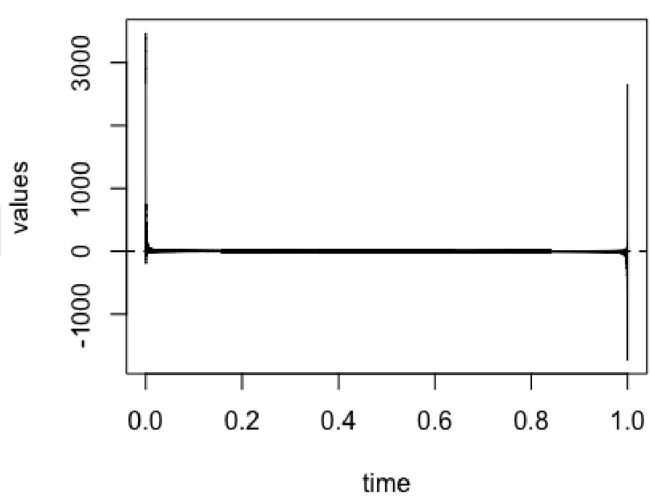
$$S_\infty = \{x = \sum_{k=1}^{\infty} x_k e_k \in \mathcal{L}_2 : x_1 \in \mathbb{R} \text{ and } x_j = a_j, \forall j \geq 2\}$$

Remark 12.3. *The density can assume any value in* $[0, \infty]$*. We provide a some examples of* $\mathcal{L}_2$ *functions for which the density assumes the values* $0, 1, 5$ *and* ∞*.*

For each of the following examples, we have $a_k = 0$ *and* $\lambda_k = \frac{1}{2\pi k^2}, \forall k \in \mathbb{N}$*. We note that these definitions satisfy the conditions that* $\sum_{k=1}^{\infty} \lambda_k < \infty$ *and* $a = \sum_{k=1}^{\infty} a_k e_k = 0 \in \mathcal{L}_2$*. The following functions are considered on the compact interval* $[0, 1]$ *and the Fourier Basis Functions are used to produce the orthonormal basis for the construction of the functions.*

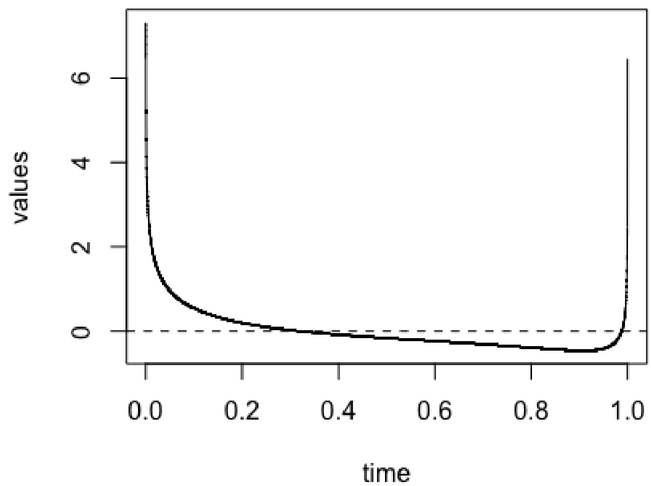
EXAMPLE - 1 : $f(x) = 0$
Suppose, $x = \sum_{k=1}^{\infty} x_k e_k$*, where,* $x_k = \frac{1}{\sqrt{\pi} k^{\frac{3}{4}}}$*.*

The function is plotted below.



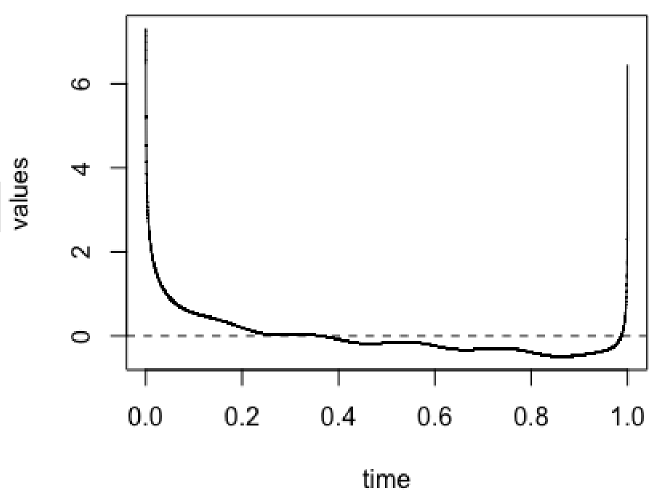
The above function x produces the density $f(x) = 0$.

EXAMPLE - 2 : $f(x) = 1$
Suppose, $x = \sum_{k=1}^{\infty} x_k e_k$, *where,* $x_k = \sqrt{\frac{\ln k}{\pi k^2}}$.
The function is plotted below.



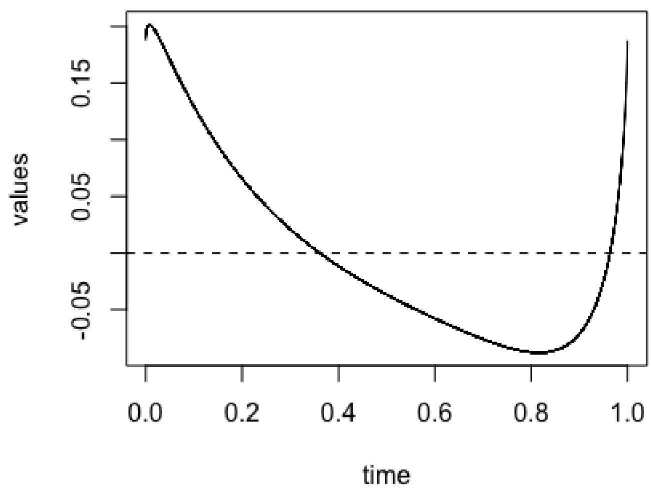
The above function x produces the density $f(x) = 1$.

EXAMPLE - 3 : $f(x) = 5$
Suppose, $x = \sum_{k=1}^{\infty} x_k e_k$, *where,* $x_k = \sqrt{\frac{\ln k}{\pi k^2}}$ *if* $k \neq 10$, *and* $x_{10} = \sqrt{\frac{\ln 2}{100\pi}}$.
The function is plotted below.



The above function x produces the density $f(x) = 5$.

EXAMPLE - 4 : $f(x) = \infty$
Suppose, $x = \sum_{k=1}^{\infty} x_k e_k$*, where,* $x_k = \frac{\ln k}{\pi k^2}$.
The function is plotted below.



The above function x produces the density $f(x) = \infty$.

Remark 12.4. *We can check that the density function so produced integrates to* 1.