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Fall 2000 Part I

Problem 1

Question

A circle of rope of total mass M and radius R is spinning at angular velocity ω about an axis through the center of the circle. What is the tension T in the rope?

Answer

An angular segment $\mathrm{d}\phi$ of the spinning rope has mass $\mathrm{d}m = \rho r \mathrm{d}\phi$. Note that $\omega = \dot{\phi}$. The inward radial force $T\mathrm{d}\phi$ on this angular segment obeys

$$T\mathrm{d}\phi = \rho r a \mathrm{d}\phi = \rho r \left(\omega r\right) \mathrm{d}\phi = \rho \omega^2 r^2 \mathrm{d}\phi \implies \boxed{T = \rho \omega^2 r^2}$$

If the rope has a uniform density, then $M = \rho r$, thus $T = M\omega^2 r$.

Problem 2

Question

A point charge Q is placed at a distance D from the center of an uncharged, solid metal sphere of radius R, thereby polarizing it. What is the potential V of the sphere?

Answer

We may use the method of images to solve this problem. Assume that the point charge is placed a distance D from the center of the sphere and is outside the sphere. The point charge causes negative charge to move to the side of the sphere closest to the point charge, thereby polarizing it. The potential is found by removing the sphere and placing a second point charge $-Q$ at a location C from the center of the sphere such that the total potential $V_T = 0$ at infinity and at R.

Given the above, the potential may be written as

$$V_T\left(x, y, z\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

where $q,\ (q')$ is the real (image) charge and $r,\ (r')$ is the distance between the field point and the image charge. Choose an axis which contains both the field and the image charge. Since the potential is supposed to be zero on the surface of the sphere, $V\left(R, 0, 0\right) = 0$. Solving for when this occurs will lead us to

$$\boxed{q' = -\frac{R}{D}q} \qquad \boxed{C = \frac{R^2}{D}}$$

This completes the problem.

Problem 8

Question

Estimate (a) the average speed (in m/s) and (b) the mean free path (in m) of a nitrogen molecule in this room.

Answer

(a) We relate the kinetic energy of an N_2 molecule with the thermal energy by the equipartition theorem. Since there are 3 translational degrees of freedom,

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$
$$v = \sqrt{\frac{3k_B T}{m}}$$

The mass of the molecule is twice that of a single nitrogen atom which is itself about 14 proton masses. Therefore

$$\boxed{v \approx \sqrt{\frac{3k_B T}{28m_p}} \approx 515\,\frac{\mathrm{m}}{\mathrm{s}}} \tag{F2000 I 8.1}$$

(b) Two particles collide if they come within $2r_0$ of each other where r_0 is the typical radius of the particle. For diatomic nitrogen, we assume $r_0 \approx 2a_0$ where a_0 is the Bohr radius. Then in the time τ that the particle is moving at velocity $\langle v \rangle$, the particle can collide with any other particle within the swept-out volume

$$\mathcal{V} = \pi\left(2r_0\right)^2 \cdot \langle v \rangle \tau$$

Since there are n particles per unit volume, there are $\mathcal{N}$ atoms to collide with:

$$\mathcal{N} = n\mathcal{V} = 4\pi n {r_0}^2 \langle v \rangle \tau$$

On average then, there are $\mathcal{N}$ collisions per length $\langle v \rangle \tau$ traversed, or in its reciprocal form, the mean free path λ is

$$\lambda = \frac{1}{4\pi n {r_0}^2}$$

To estimate the particle density, consider the ideal gas law $PV = Nk_BT$. We can assume atmospheric pressure at room temperature, so the density is

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Putting it all together,

$$\boxed{\lambda \approx \frac{k_B T}{4\pi {r_0}^2 P} \approx 2.90 \cdot 10^{-7}\,\mathrm{m}} \tag{F2000 I 8.2}$$

Spring 2000 Part I

Problem 1

Question

A satellite of mass $m = 500\,\mathrm{kg}$ is in a circular orbit at an altitude $h = 150\,\mathrm{km}$ above the Earth's surface. As a result of air friction, the satellite's orbit degrades. Protected by a heat shield, the satellite eventually impacts with a velocity of $2\,\mathrm{km/s}$. How much energy (in Joules) was released as heat in the process?

Answer

The solution method will be a simple energy balance equation. In its initial state, the satellite had gravitation potential energy that contributed to its energy equal to

$$V_0 = \frac{GM_E m}{R_E + h}$$

where M_E and R_E are the mass and radius of the Earth. The kinetic energy can be determined by making use of simple circle relations. The centripetal force must be provided by the gravitation force, so

$$\begin{aligned} \frac{{v_0}^2}{R_E + h} &= \frac{GM_E}{(R_E + h)^2} \\ v_0 &= \sqrt{\frac{GM_E}{R_E + h}} \end{aligned}$$

making the initial kinetic energy

$$T_0 = \frac{GM_E m}{2\left(R_E + h\right)}$$

In it's final state, the satellite consists of it's final given velocity's kinetic energy and more gravitational potential energy (with respect to the center of the Earth). They are given by

$$\begin{aligned} T_f &= \frac{1}{2} m {v_f}^2 \\ V_f &= \frac{GM_E m}{R_E} \end{aligned}$$

By conservation of energy, any energy different must result from the loss of energy between the initial and final states, so the lost energy E_{loss} is

$$\begin{aligned} E_{\text{loss}} &= T_0 + V_0 - T_f - V_f \\ &= \frac{GM_E m}{R_E\left(R_E + h\right)}\left(R_E - 2h\right) - \frac{1}{2} m {v_f}^2 \end{aligned}$$

Plugging in all given quantities and constants, we have that

$$\boxed{E_{\text{loss}} = 2.816 \cdot 10^{10}\,\mathrm{J} = 28.16\,\mathrm{GJ}} \tag{S2000 I 1.1}$$

Problem 2

Question

What is the velocity of recoil of an ^{57}Fe nucleus that emits a 100 keV photon, both in units of the speed of light in vacuum and in meters per second.

Answer

In the initial state observed from the iron atom's rest frame, the momentum is zero. After the emission, the photon has a momentum of $p_\gamma = E_\gamma/c$ and consequently, the nucleus must recoil with momentum $p_{Fe} = -p_\gamma$. Knowing that the energy is non-relativistic, then

$$\begin{aligned} m_{Fe} v_{Fe} &= \frac{E_\gamma}{c} \\ v_{Fe} &= \frac{E_\gamma}{57 m_p c} \end{aligned}$$

where we've approximate the mass of the iron atom by a multiple of the proton mass. Plugging in the numbers

$$\boxed{v_{Fe} = 561.4\,\frac{\mathrm{m}}{\mathrm{s}}} \tag{S2000 I 2.1}$$

or in units of c

$$\boxed{v_{Fe} = 1.87 \cdot 10^{-6}\,\mathrm{c}} \tag{S2000 I 2.2}$$

Problem 3

Question

A resistance R and an inductance L are connected in series, and an alternating voltage $V_0 \cos \omega t$ is impressed across the combination. The resulting steady state voltage across the resistance can be written as $V_R \cos(\omega t + \beta)$. Find V_R and β, assuming both V_0 and V_R to be positive.

Answer

The problem is simplified by constucting the solution using complex voltages and currents and recovering the correct component at the end. Since the source voltage is a cosine, the real component will be kept at the end.

The complex impedance for a resistor is simply the resitance itself, so $Z_R = R$. For the inductor, it is $Z_L = i\omega L$. We then use the complex impedances together with Ohm's Law and the first Kirchoff rule to solve for the complex current I.

$$
\begin{aligned}
V_0 e^{i\omega t} &= IR + i\omega LI \\
I &= \frac{V_0}{R + i\omega L} e^{i\omega t}
\end{aligned}
$$

Then by inserting this solution into the voltage law for inductors, we can solve for the complex voltage across the inductor.

$$
\begin{aligned}
V_L &= L\frac{dI}{dt} \\
V_L &= V_0 \frac{i\omega L}{R + i\omega L} e^{i\omega t}
\end{aligned}
$$

The voltage drops across the resistor and inductor must equal the impressed voltage, so we can solve for the unknown voltage across the resistor.

$$
\begin{aligned}
V_0 e^{i\omega t} &= V + V_L \\
V &= V_0 e^{i\omega t} - V_0 \frac{i\omega L}{R + i\omega L} e^{i\omega t} \\
V &= V_0 \left(\frac{R^2 - i\omega RL}{R^2 + \omega^2 L^2} \right) e^{i\omega t}
\end{aligned}
$$

In order to make taking the real component simpler, we put the term in parentheses in complex exponential form according to the relation $z = |z| e^{i \arg z}$.

$$
\begin{aligned}
\left| \frac{R^2 - i\omega RL}{R^2 + \omega^2 L^2} \right| &= \frac{R\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2} \\
\arg \left(\frac{R^2 - i\omega RL}{R^2 + \omega^2 L^2} \right) &= \arctan\left(-\frac{\omega L}{R} \right)
\end{aligned}
$$

Therefore the voltage across the resistor has the form

$$
\begin{aligned}
V &= V_0 \frac{R\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2} e^{i\omega t + \arctan(-\omega L/R)} \\
V &= V_R e^{i\omega t + i\beta} \\
\mathrm{Re}\{V\} &= V_R \cos(\omega t + \beta)
\end{aligned}
$$

where

$$
\boxed{V_R = V_0 \frac{R\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2}} \qquad \boxed{\beta = \arctan\left(-\frac{\omega L}{R} \right)} \tag{S2000 I 3.1}
$$

Problem 4

Question

Find the magnetic flux through a square loop of side a due to current I in a long straight wire. The geometry is as follows: the wire is coplanar with the loop and runs parallel to the loop's closest side, at a distance b away. Write your result as a formula in SI units.

Answer

By Ampère's law in integral form, the magnetic field at a radial distance r away from the wire is given by

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

The flux is then the total magnetic field through the loop. Integrating by lines of length a,

$$\phi = a\int_b^{a+b} \frac{\mu_0 I}{2\pi r} dr$$

$$\boxed{\phi = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{a+b}{a}\right)}$$

Problem 6

Question

By actually evaluating the integral, show that

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e}$$

Answer

Note that the integrand, so start by changing the limits of integration

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\,dx$$

Then expand the cosine into is complex exponential definition

$$\begin{aligned}
&= \frac{1}{4}\left(\int_{-\infty}^\infty \frac{e^{ix}}{1+x^2}\,dx + \int_{-\infty}^\infty \frac{e^{-ix}}{1+x^2}\,dx\right) \\
&= \frac{1}{4}\left(\int_{-\infty}^\infty \frac{e^{ix}}{(x+i)\,(x-i)}\,dx + \int_{-\infty}^\infty \frac{e^{-ix}}{(x+i)\,(x-i)}\,dx\right)
\end{aligned}$$

Complex contour integration lets us evaluate the integrals as limits of the coefficient of a pole as it approaches the pole, so

$$= \frac{1}{4}\left(2\pi i\cdot\lim_{z\to i}\left(\frac{e^{iz}}{z+i}\right) - 2\pi i\cdot\lim_{z\to -i}\left(\frac{e^{-iz}}{z-i}\right)\right)$$

After simplifying,

$$\boxed{\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e}} \tag{S2000 I 6.1}$$

Problem 7

Question

The drag force on a very high speed object of area A, passing through a gas of density ρ at a velocity v is expected to be of the form

$$\text{Force} \sim A^r \rho^s v^t$$

Determine the value of the exponents r, s, and t.

Answer

Force needs to have a unit of inverse time squared, therefore v as the only variable with a time unit sets $t = 2$. Similarly, ρ is the only one with a mass term, so we also immediately know that $s = 1$. That leaves

$$\left[\frac{\text{mass} \cdot \text{distance}}{\text{time}^2}\right] = \left[\frac{\text{mass}}{\text{distance} \cdot \text{time}^2}\right] [\text{distance}]^r$$

Therefore to have the two side have compatible units, $r = 2$.

$$\boxed{r = 2, \quad s = 1, \quad t = 2}$$

Problem 9

Question

For waves in shallow water, the relation between frequency ν and wavelength λ is

$$\nu = \left(\frac{2\pi T}{\rho\lambda^3}\right)^{1/2}$$

where ρ and T are the density and surface tension of water. What is the group velocity of these waves?

Answer

Transforming relation given to use the angular frequency $\omega = 2\pi\nu$ and wave number $k = 2\pi/\lambda$,

$$\omega = \sqrt{\frac{k^3 T}{\rho}}$$

Then the group velocity is simply the partial derivative with respect to k:

$$v_g = \frac{\partial\omega}{\partial k} = \frac{3}{2}\sqrt{\frac{kT}{\rho}}$$

Putting this back into the form which involves only ν and λ, we arrive at the answer

$$\boxed{v_g = \frac{3}{2}\sqrt{\frac{2\pi T}{\rho\lambda}}} \tag{S2000 I 9.1}$$

Problem 10

Question

Find the eigenvalues and corresponding eigenvectors (which need *not* be normalized) of the following matrix:

$$M = \begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & -1 \end{bmatrix}$$

Answer

The eigenvalue equation is found from the standard procedure of adding a parameter λ into the matrix and taking the determinant equal to zero:

$$\det\left\{\begin{bmatrix} 1-\lambda & 0 & -i \\ 0 & 2-\lambda & 0 \\ i & 0 & -1-\lambda \end{bmatrix}\right\} = 0 = (1-\lambda)(2-\lambda)(-1-\lambda) - i(-i)(2-\lambda)$$

$$0 = -(1-\lambda)(1+\lambda)(2-\lambda) - (2-\lambda)$$

$$0 = -(2-\lambda)\left[(1-\lambda)(1+\lambda)+1\right]$$

Solving the two equations $0 = (2-\lambda)$ and $(1-\lambda)(1+\lambda) = -1$ gives the three eigenvalues

$$\boxed{\lambda = \left\{-\sqrt{2}, \sqrt{2}, 2\right\}} \tag{S2000 I 10.1}$$

Starting with the eigenvalue $\lambda = 2$, we solve for its eigenvector using the usual Gaussian elimination approach:

$$\det\left\{\begin{bmatrix} -1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -3 \end{bmatrix}\right\}$$

The second variable is completely unconstrained, so we can set that component of the vector to 1. The first and last rows are incompatible, so that means that both the first and third variables must be zero. This gives us the eigenvector

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In a similar manner for the eigenvalues $\lambda = \pm\sqrt{2}$, we perform Gaussian elimination. When all free parameters have been set arbitrarily to 1 and other constraints considered, we end up with the three eigenvectors:

$$\boxed{\lambda = -\sqrt{2} \quad \rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ -i\left(1+\sqrt{2}\right) \end{bmatrix}} \tag{S2000 I 10.2}$$

$$\boxed{\lambda = \sqrt{2} \quad \rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -i\left(1-\sqrt{2}\right) \end{bmatrix}} \tag{S2000 I 10.3}$$

$$\boxed{\lambda = 2 \quad \rightarrow \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \tag{S2000 I 10.4}$$

Problem 12

Question

Suppose the electron were to have spin $\frac{3}{2}$ instead of spin $\frac{1}{2}$. What would then be the atomic numbers Z of the *three* lowest-mass noble gases, i.e. the equivalents of helium, neon, and argon?

Answer

In a spin $\frac{3}{2}$ particle, there are 4 possible spin configurations corresponding to the spin projections $s_z = \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$. Because of this, each projection of the orbital angular momentum can hold 4 electrons instead of just two. This means we can make use of spectroscopic notation to easily count up to the atoms of interest.

$\ell = 0 \to 1\ \ell_z$ state	$1s^4$
$\ell = 1 \to 3\ \ell_z$ states	$1s^4 2p^{12} 2s^4$
$\ell = 2 \to 5\ \ell_z$ states	$1s^4 2p^{12} 2s^4 3d^{20} 3p^{12} 3s^4$

Since all shells are filled at each level, we just have to sum the number of electrons in each line above to get the Z number of the new noble gases.

$$\boxed{Z = \{4, 20, 56\}} \tag{S2000 I 12.1}$$

Spring 2002 Part I

Problem 1

Question

A 1000 kg automobile has ground clearance of 18 cm but when loaded with an extra 500 kg from its 4 passengers it only clears the ground by 12 cm. The car's shock absorbers are ineffective. At what speed (in miles per hour) will the car bounce in resonance when it travels along a smooth road containing transverse tar patches every 15 m? Assume that the front and rear suspensions have the same bouncing frequency.

Answer

To find the natural resonant frequency of the vehicle, we use the two data points about it's clearance to obtain the spring constant k. We know that a constant force on a spring system does not change the dynamics, so it's only the change in mass and distance which are relevant. Therefore by simple equilibrium requirements,

$$\begin{aligned} k\Delta L &= mg \\ k &= \frac{mg}{\Delta L} \end{aligned}$$

where m and M are the masses of the passengers and empty car respectively and ΔL is the difference in the clearance between the unloaded and loaded car.

Next, we know that the resonant frequency of a simple harmonic oscillator is given by $\omega = \sqrt{k/m_{\text{tot}}}$, so

$$\begin{aligned} \omega &= \sqrt{\frac{mg}{(M+m)\,\Delta L}} \\ f &= \frac{1}{2\pi}\sqrt{\frac{mg}{(M+m)\,\Delta L}} \end{aligned}$$

where the second line has converted from angular to linear frequency. We then simply relate that to how often the car hits a tar patch and solve for the velocity. Letting v be the velocity of the car and d be the separation distance between tar patches,

$$\begin{aligned} \frac{v}{d} &= f \\ v &= \frac{d}{2\pi}\sqrt{\frac{mg}{(M+m)\,\Delta L}} \end{aligned}$$

Plugging in the numbers,

$$\boxed{v = 17.62\,\frac{\mathrm{m}}{\mathrm{s}} = 39.42\,\mathrm{mph}} \tag{S2002 I 1.1}$$

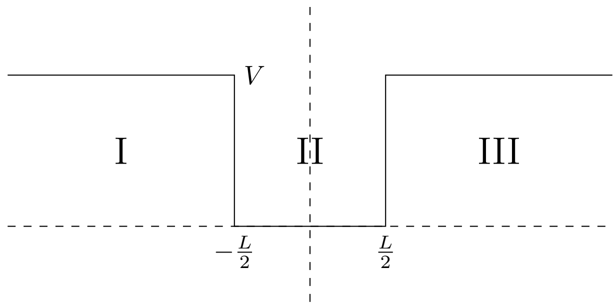
Problem 2

Question

Suppose a particle of mass m moves in a 1-dimensional square potential well of width L and depth V. What is the minimum depth of the well such that the particle will have two bound states?

Answer

Divide the problem into three regions according to the diagram below:



From the Schrödinger equation, we know that there will be two different types of solutions. Within region II, no potential exists, so the solution has the form

$$\psi = A\cos(kx) + B\sin(kx)$$
$$k^2 = \frac{2mE}{\hbar^2}$$

For regions I and III, we assume that the energy $E < V$ since we are only interested in the bound solutions and not ones where the particle is free. This gives the following exponential solution form:

$$\psi = Ae^{\kappa x} + Be^{-\kappa x}$$
$$\kappa^2 = \frac{2m\,(V-E)}{\hbar^2}$$

We immediately know that for regions I and III, the wavefunction must go to zero at $\pm\infty$, so that immediately removes two terms in the solution. This leaves us with the complete set of solutions

$$\psi_I = Ae^{\kappa x}$$
$$\psi_{II} = B\cos(kx) + C\sin(kx)$$
$$\psi_{III} = De^{-\kappa x}$$

We make a further simplification by noting that in this situation where there is a symmetric potential, the solution can also be divided into symmetric and antisymmetric solutions, corresponding to keeping either the sin or the cos solutions in region II. To solve the problem, we want to know the threshold potential for V that will maintain a second bound state. The first bound state is the symmetric case (which heuristically is true by its virtue of having only a single "hump" within region II), so the first excited state/second state is the antisymmetric case (heuristically expected since the sine solution will have two "humps" within region II). This lets us simplify the problem and immediately set $B = 0$ to isolate just the antisymmetric solution.

To continue, we make use of the fact that the wavefunction must be continuous and differentiably continuous at the boundaries between regions I-II and II-III. This means we find that:

$$Ae^{-\kappa L/2} = B\sin\left(-\frac{kL}{2}\right) \qquad B\sin\left(\frac{kL}{2}\right) = De^{-\kappa L/2}$$
$$A\kappa e^{-\kappa L/2} = Bk\cos\left(-\frac{kL}{2}\right) \qquad Bk\cos\left(\frac{kL}{2}\right) = -D\kappa e^{-\kappa L/2}$$

Dividing the lower equation by the upper equation in both cases leads to the condition

$$\kappa = -k \cot\left(\frac{kL}{2}\right)$$

If we perform a variable substitution where $v = kL/2$ and $u = \kappa L/2$, the equation above takes the slightly simpler form

$$u = -v \cot v \tag{S2002 I 2.1}$$

This transformation comes in more useful when we consider the energy equations that defined k and κ. Substituting into the equation for k and solving for E we have that

$$E = \frac{2v^2\hbar^2}{mL^2}$$

And doing the same for κ,

$$V - E = \frac{2u^2\hbar^2}{mL^2}$$

which combined gives the equation of a circle:

$$u^2 + v^2 = \frac{mL^2}{2\hbar^2}V$$

Therefore, the only valid solutions occur when both (S2002 I 2.1) and (S2002 I 2.2) are satisfied. The minimum value for a given cotangent line occurs at the roots which occur when $v = (2n+1)\,\pi/2$ for any integer n. The first excited state for $n = 1$ then occurs at $v = \pi/2$ and consequently $u = 0$. Substiting this into the equation above and solving for V, we find that the threshold energy for a second bound state corresponds to a potential depth of

$$\boxed{V = \frac{\pi^2\hbar^2}{2mL^2}} \tag{S2002 I 2.2}$$

Problem 3

Question

The cross-section for collisions between helium atoms is about $10^{-16}\,\mathrm{cm}^2$. Estimate the mean free path of helium atoms in helium gas at atmospheric pressure and temperature.

Answer

Consider the path traced out by a helium atom as it travels a path length L, colliding with other helium atoms along the way. Given that the cross section of helium is σ, than we can estimate the volume that contains probable interactions with our atom of interest as $\mathcal{V} = \sigma L$. To get the number of interactions, we make use of the fact that we're treating the gas as an ideal gas. From the ideal gas law,

$$PV = NkT$$

so that solving for the number density

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Combining the density with the volume, we get the number of other [point particle] helium atoms that are contained within the given helium atom's interaction volume. If we then assume that the atom interacts with all other atoms within the volume, and that the collisions are spaced out equally in time, we just have to normalize the value by the trajectory's path length to get an estimate of the mean free path of helium in a helium gas:

$$\lambda^{-1} = \frac{n\mathcal{V}}{L} = \frac{\sigma P}{k_B T}$$

Plugging in $\sigma = 10^{-16}\,\mathrm{cm}^2$, $P = 1.013 \cdot 10^5\,\mathrm{Pa}$, $k_B = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$, and $T = 298\,\mathrm{K}$, we get

$$\boxed{\lambda^{-1} = 4.06\,\mu\mathrm{m}} \tag{S2002 I 3.1}$$

Problem 4

Question

Consider an infinitely long cylindrical coaxial capacitor. The outer conductor has radius R and the applied voltage is V. For what radius of the inner conductor will the field strength at its surface be a minimum?

Answer

Problem 5

Question

A rigid container is filled with a classical gas of molecular mass m. The temperature inside the container is T and the pressure is P. Outside there is a vacuum. Suppose that a small hole of area A is punched in the outer wall. At what rate (molecules per unit time per unit area) will gas leave the container? Write the result as a function of $m, P,$ and T. Some possibly useful information

$$\int_0^\infty x^n e^{-\lambda x^2} \mathrm{d}x = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\lambda^{(n+1)/2}};\ \Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$$

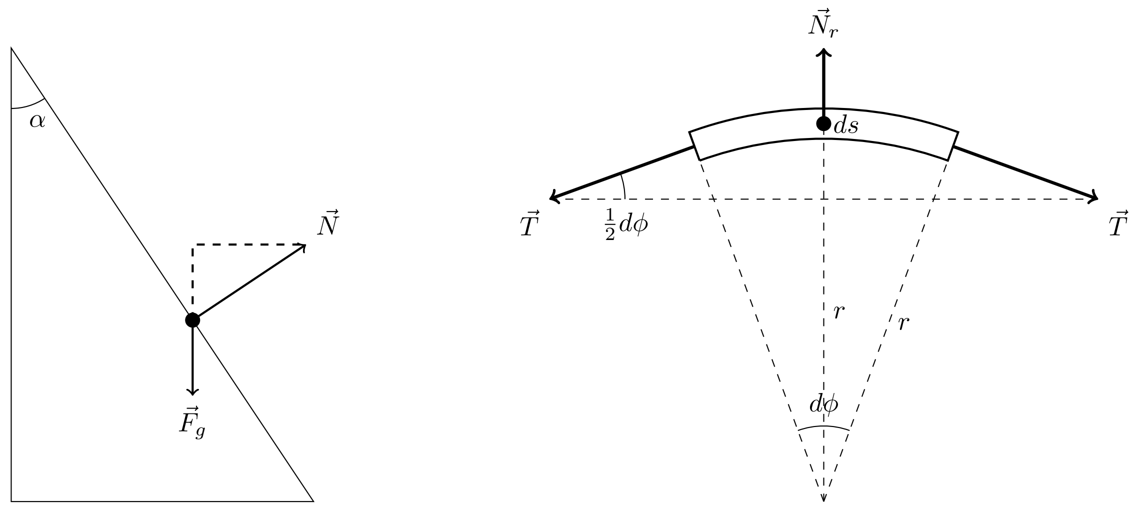
Answer

Problem 6

Question

A single closed loop of chain with mass m and length L rests horizontally on a smooth frictionless cone with half-angle α. What is the tension in the chain?

Answer



(a) Side View **(b)** Top View

Consider just a small element of the chain of arc length ds. It will have a corresponding mass $dm = \lambda\, ds$ where $\lambda = m/L$. Knowing that it's a statics problem, we can easily determine the normal force by balancing the vertical component with that of gravity.

$$F_g = N \sin\alpha$$
$$N = \frac{\lambda g\, ds}{\sin\alpha}$$

This leaves the horizontal component of the normal force to be balanced with the tension within the chain.

Now switching to the top view, we consider the short chain segment ds, shown above with an exagerated curvature. We note that the radial part of the normal force must be opposed by the sum of the radial components of the two tensions T acting on the end of the chain segment. By geometry, we know that the angle with respect to the midpoint's tangent is one half the differential angle change $d\phi = ds/r$. This means we balance the forces as

$$2T \sin\left(\frac{1}{2}d\phi\right) = N\cos\alpha$$
$$2T \sin\left(\frac{1}{2}d\phi\right) = \lambda g\, ds \cot\alpha$$

By the small angle approximation, $\sin\left(\frac{1}{2}d\phi\right) \approx \frac{1}{2}d\phi$, so after substituting for the fact that $dr = L/2\pi$ and $\lambda = M/L$,

$$\boxed{T = \frac{Mg}{2\pi}\cot\alpha} \qquad \text{(S2002 I 6.1)}$$

Problem 7

Question

A laser beam (photon energy 1 eV) collides head-on with a 50 GeV ultra-relativistic electron beam. What is the energy of the photons reflected backwards in the collision?

Answer

We'll be solving the problem using conservation of 4-momentum, so we define the following momenta with the assumption that the electron beam is moving to the right, and the photons are initially moving to the left. Let the unprimed and primed q^μ be the photon's 4-momentum before and after the collision respectively, and define the electron's momenta k^μ similarly. Then in terms of the energies E and 3-momenta p (actually taken to be 1D without loss of generality) for each of the photon γ and electron e:

$$q^\mu = \begin{pmatrix} E_\gamma/c \\ -E_\gamma/c \end{pmatrix} \qquad q'^\mu = \begin{pmatrix} E'_\gamma/c \\ E'_\gamma/c \end{pmatrix}$$

$$k^\mu = \begin{pmatrix} E_e/c \\ p_e \end{pmatrix} \qquad k'^\mu = \begin{pmatrix} E'_e/c \\ p'_e \end{pmatrix}$$

By conservation of momentum,

$$\begin{aligned} q'^\mu + k'^\mu &= q^\mu + k^\mu \\ k'^\mu &= q^\mu - q'^\mu + k^\mu \end{aligned}$$

Solving for the unknown electron momentum after the collision lets us eliminate it from the equation; when we square the equation, the squared quantities are Lorentz invariant, and therefore the product can be evaluated in any frame. A convenient choice is the rest frame where the electron evaluates to its mass energy and photons vanish.

$$\underbrace{k'^\mu k'_\mu}_{m_e c^2} = \underbrace{q^\mu q_\mu}_{0} - \underbrace{q'^\mu q'_\mu}_{0} + \underbrace{k^\mu k_\mu}_{m_e c^2} - q'^\mu q_\mu + q'^\mu k_\mu - q^\mu k_\mu$$

This greatly simplifies the rest of the problem to

$$q'^\mu q_\mu - q'^\mu k_\mu = -q^\mu k_\mu$$

Inserting the energy and 3-momentum components and performing the inner products,

$$2\frac{E_\gamma E'_\gamma}{c^2} - \frac{E'_\gamma E_e}{c^2} + \frac{E'_\gamma p_e}{c} = -\frac{E_\gamma E_e}{c^2} - \frac{E_\gamma p_e}{c}$$

Isolating E'_γ on the left and E_γ on the right,

$$-E'_\gamma \left(E_e - p_e c - 2E_\gamma\right) = -E_\gamma \left(E_e - p_e c\right)$$

which solving for the unknown photon energy gives

$$\boxed{E'_\gamma = E_\gamma \left(1 - \frac{2E_\gamma}{E_e - p_e c}\right)^{-1}} \qquad \text{(S2002 I 7.1)}$$

The solution is formally complete, but actually calculating a numerical answer can prove difficult because $E_e \approx p_e c$. Therefore, we will expand the solution. Beginning with the definition of the momentum from Einstein's energy relation,

$$p_e c = \sqrt{{E_e}^2 - {m_e}^2 c^4}$$

we can subtract it from E_e, leading to the useful form

$$E_e - p_e c = E_e \left(1 - \sqrt{1 - \frac{{m_e}^2 c^4}{{E_e}^2}}\right)$$

Expanding the root to first order in its argument,

$$\begin{aligned} E_e - p_e c &= E_e \cdot \frac{1}{2} \frac{{m_e}^2 c^4}{{E_e}^2} \\ &= \frac{1}{2} \frac{{m_e}^2 c^4}{E_e} \end{aligned}$$

Plugging this into the solution (S2002 I 7.1), the photon energy is then approximately given by

$$\boxed{E'_\gamma \approx E_\gamma \left(1 - 4\frac{E_\gamma E_e}{{m_e}^2 c^4}\right)^{-1}} \tag{S2002 I 7.2}$$

The numerics are much easier to calculate in this case, and we find that the final energy of the reflected photon is

$$\boxed{E'_\gamma = 4.273\,\mathrm{eV}} \tag{S2002 I 7.3}$$

Problem 8

Question

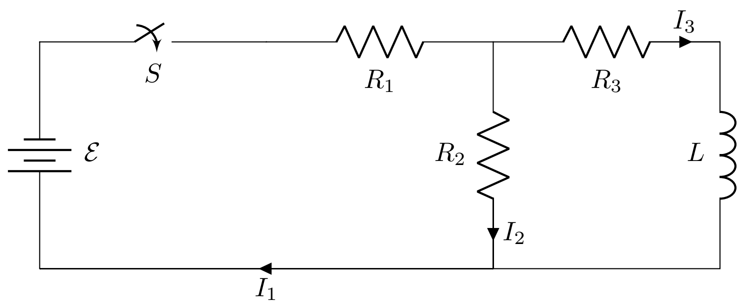
In the figure below, $\mathcal{E} = 100\,\text{V}$, $R_1 = 5\,\Omega$, $R_2 = 10\,\Omega$, $R_3 = 15\,\Omega$, and $L = 1.0\,\text{H}$. Find the values of the currents I_1 and I_2

a) immediately after the switch S is closed,

b) a long time later,

c) immediately after switch S is opened again,

d) and then how long must you wait, after the switch is opened, before I_2 falls by a factor of e?



Answer

Start by applying Kirchoff's rules to the circuit: current is conserved and the voltage changes must sum to zero around each loop, so

$$I_1 = I_2 + I_3 \tag{S2002 I 8.1}$$
$$0 = \mathcal{E} - I_1 R_1 - I_2 R_2 \tag{S2002 I 8.2}$$
$$0 = -I_3 R_3 - L\frac{dI_3}{dt} + I_2 R_2 \tag{S2002 I 8.3}$$

Since only (S2002 I 8.3) has a term involving a time derivative, we choose to first solve for the current I_3. By solving for $I_2 R_2$ in (S2002 I 8.2) and substituting, we eliminate I_2 and have

$$0 = -I_3 R_3 - L\frac{dI_3}{dt} + \mathcal{E} - I_1 R_1 \tag{S2002 I 8.4}$$

Furthermore, by also substituting the value of I_2 into (S2002 I 8.1):

$$I_1 = \frac{\mathcal{E}}{R_1 + R_2} + \frac{R_2}{R_1 + R_2} I_3 \tag{S2002 I 8.5}$$

Then by combining (S2002 I 8.4) and (S2002 I 8.5), we can produce a differential equation for I_3:

$$-I_3 R_3 - L\frac{dI_3}{dt} + \mathcal{E} - \frac{R_1}{R_1 + R_2}\mathcal{E} - \frac{R_1 R_2}{R_1 + R_2} I_3 = 0$$
$$-\underbrace{\frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 + R_2}}_{R'} I_3 = L\frac{dI_3}{dt} - \frac{R_2}{R_1 + R_2}\mathcal{E}$$

$$\frac{dI_3}{dt} = -\frac{R'}{L} I_3 + \frac{1}{L}\frac{R_2}{R_1 + R_2}\mathcal{E} \tag{S2002 I 8.6}$$

Considering just the homogeneous part, we easily solve it to find the standard exponential solution

$$I_{3h}(t) = I_{30} e^{-R' t/L}$$

And using the ansatz $I_{3p}(t) = At + B$ for the inhomogeneous part,

$$\begin{aligned} A &= -\frac{R'}{L} At - \frac{R'}{L} B + \frac{1}{L} \frac{R_2}{R_1 + R_2} \mathcal{E} \\ A &= 0 \\ B &= \frac{1}{R'} \frac{R_2}{R_1 + R_2} \mathcal{E} \end{aligned}$$

At $t = 0$, the inductor has no current passing through it, so when the switch is closed, the current must remain continuous. This gives us the initial condition necessary to solve for the unknown I_{30}, and after doing so and simplifying, the total solution is

$$I_3(t) = \frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2} \left(1 - e^{-R' t/L}\right) \tag{S2002 I 8.7}$$

Then by substituting this solution back into (S2002 I 8.5) we get the solution for I_1:

$$I_1(t) = \frac{\mathcal{E}}{R_1 + R_2} \left[1 + \frac{1}{R'} \frac{{R_2}^2}{R_1 + R_2} \left(1 - e^{-R' t/L}\right)\right] \tag{S2002 I 8.8}$$

Finally, combining both inserting both solutions for I_1 and I_3 into (S2002 I 8.1), the solution for I_2 is

$$I_2(t) = \frac{\mathcal{E}}{R_1 + R_2} \left[1 - \frac{1}{R'} \frac{R_1 R_2}{R_1 + R_2} \left(1 - e^{-R' t/L}\right)\right] \tag{S2002 I 8.9}$$

Plugging in all of the given values, we find that the currents at the instant the switch is closed are

$$\boxed{I_1(0) = 6.66\,\mathrm{A} \qquad S \text{ is closed}} \tag{S2002 I 8.10}$$

$$\boxed{I_2(0) = 6.66\,\mathrm{A} \qquad S \text{ is closed}} \tag{S2002 I 8.11}$$

For a long time later, we can let $t \to \infty$ and find that

$$\boxed{I_1(\infty) = 9.09\,\mathrm{A} \qquad S \text{ is closed}} \tag{S2002 I 8.12}$$

$$\boxed{I_2(\infty) = 5.54\,\mathrm{A} \qquad S \text{ is closed}} \tag{S2002 I 8.13}$$

Right after the switch is opened, the left loop is taken out of the circuit, so we immediately know that the value of I_1 is zero.

$$\boxed{I_1(0) = 0 \qquad S \text{ is open}} \tag{S2002 I 8.14}$$

For the right loop, we start by noting that the steady state current through the inductor will be needed. Taking the limit of (S2002 I 8.7), we have that the new initial condition is

$$I_3(0) = \frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2}$$

I_2 now is equal to $-I_3$ since there is no other path for the current to traverse. This loop's differential equation is then

$$-(R_2 + R_3) I_3 - L \frac{dI_3}{dt} = 0$$

Solving for the exponential and using the initial condition above, the time solution is

$$I_3(t) = \frac{\mathcal{E}}{R'}\frac{R_2}{R_1+R_2}e^{-(R_2+R_3)t/L}$$
$$I_2(t) = -\frac{\mathcal{E}}{R'}\frac{R_2}{R_1+R_2}e^{-(R_2+R_3)t/L}$$

Therefore the current in I_2 just after the switch is opened reverses direction

$$\boxed{I_2(0) = -3.64\,\mathrm{A}} \tag{S2002 I 8.15}$$

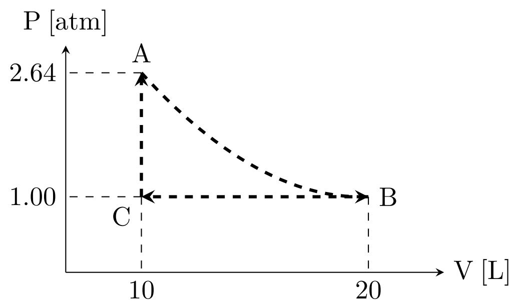
Then by simple exponential relations, we know that the time time to decay by a factor of e is given by the reciprocal of the coefficient of t, so inserting the appropriate numbers

$$\boxed{t_{\text{decay}} = 0.04\,\mathrm{s}} \tag{S2002 I 8.16}$$

Problem 9

Question

An engine using 1 mol of an ideal diatomic gas performs the cycle $A \to B \to C \to A$ as shown in the diagram below. $A \to B$ is an adiabatic expansion, $B \to C$ occurs at constant pressure, and $C \to A$ takes place at constant volume. What is the efficiency of the cycle?



Answer

Since we want to find the efficiency of the cycle, we only care about the heat exchanged during each stage of the cycle. Because the path $A \to B$ is adiabatic, we immediately know that $Q = 0$. Then proceeding to look at the stage $C \to A$, we know that the work done during this cycle is identically zero since there is no area under the curve. That means we are left simply with the equation

$$dU = dQ$$

Because this is an ideal [diatomic] classical gas, we combine the equations

$$U = \frac{5}{2}nRT$$

and

$$PV = nRT$$

to get that the difference in energy across the path is

$$\begin{aligned} Q_{CA} = U &= \frac{5}{2}nR\left(T_A - T_C\right) \\ &= \frac{5}{2}V_1\left(P_2 - P_1\right) \end{aligned}$$

For the remaining stage $B \to C$, we use the full thermodynamic identity:

$$dU = dQ - P\,dV$$

The pressure P_1 is constant, so both integration of dU and dV are simply the differences in each quantity. Again substituting for the temperature in U with the ideal gas law,

$$\begin{aligned} \frac{5}{2}nR\left(T_C - T_B\right) &= Q_{BC} - P_1\left(V_1 - V_2\right) \\ \frac{5}{2}P_1\left(V_1 - 2V_1\right) &= Q_{BC} + P\left(V_1 - 2V_1\right) \\ Q_{BC} &= -\frac{7}{2}P_1V_1 \end{aligned}$$

We've accounted for all the heat flow in the system. Q_{BC} is negative, so this is the heat flow out of the system, while Q_{CA} is positive and is the heat flow into the system. By definition then, the efficiency η of the system is

$$\begin{aligned}\eta &= 1 - \frac{Q_{out}}{Q_{in}} \\ &= 1 - \frac{\frac{7}{2}P_1V_1}{\frac{5}{2}V_1\left(P_2 - P_1\right)} \\ &= 1 - \frac{5}{7}\frac{P_1}{P_2 - P_1}\end{aligned}$$

Plugging in the given values, we find the efficiency to be

$$\boxed{\eta = 0.146 = 14.6\,\%} \tag{S2002 I 9.1}$$

Problem 10

Question

A thin circular hoop rolls down an inclined plane under the influence of gravity. What minimum coefficient of friction is required to ensure that it rolls rather than slides?

Answer

Begin first by finding the motion that describes the rolling without slipping state. We do this by solving the system's Lagrangian:

$$\mathcal{L} = \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2\right) - (mgx\sin\alpha)$$

where x is the length along the ramp with $x = 0$ at the bottom, I is the moment of inertia of the hoop, m is its mass, θ is the angle of rotation of the hoop about its center, and α is the angle of the incline plane. By noting that rolling without slipping requires that $r\dot{\theta} = \dot{x}$, we can reduce the problem to the single variable x. The result is the following differential equation, where $I = mR^2$ has been substituted in:

$$\begin{aligned} 2m\ddot{x} &= -mg\sin\alpha \\ \ddot{x} &= -\frac{1}{2}g\sin\alpha \end{aligned}$$

Therefore we know the linear acceleration will be $a = -\frac{1}{2}g\sin\alpha$ in the non-slipping case.

To find what coefficient of friction produces this motion, we consider the forces acting on the hoop with the coordinate system still oriented along and perpendicular to the plane. In the perpendicular direction, the normal force N is canceled by the perpendicular component of gravity, so

$$N = mg\cos\alpha$$

In the parallel direction, the frictional force and the parallel component of gravity must sum to give the requisite force, namely ma.

$$\begin{aligned} \mu N - mg\sin\alpha = ma &= -\frac{1}{2}mg\sin\alpha \\ \mu mg\cos\alpha &= \frac{1}{2}mg\sin\alpha \\ \mu &= \frac{1}{2}\tan\alpha \end{aligned}$$

Therefore we find that the coefficient of friction must be equal to half of the tangent of the inclined plane's angle.

Problem 11

Question

A particle is confined within a cubical box with sides of length L and is initially in the ground state. If the length of one side of the box (along the x-direction) is abruptly increased to a length $2L$, what is the probability that the particle remains in the ground state?

Answer

We start by recalling the solution for a particle in a box. In a 1D box with is left edge at the origin, the properly normalized wavefunction is given by

$$\psi\left(x\right)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

where L is the size of the box. The Cartesian extension into 3D is simple and is respectively for the $L\times L\times L$ and $2L\times L\times L$ boxes:

$$\begin{aligned}\psi\left(x,y,z\right)&=\left(\frac{2}{L}\right)^{3/2}\sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\\ \psi'\left(x,y,z\right)&=\frac{1}{\sqrt{2}}\left(\frac{2}{L}\right)^{3/2}\sin\left(\frac{n_x\pi x}{2L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\end{aligned}$$

To find the probability of remaining in the ground state, we simply must take the inner product of both wavefunctions in the ground state over an appropriate domain; this means that the initial, unexanded box's wavefunction is 0 within the new region.

$$\begin{aligned}\mathscr{P}&=\langle\psi_{111}|\psi'_{111}\rangle\\ &=\frac{1}{\sqrt{2}}\left(\frac{2}{L}\right)^{3}\left(\int_0^L\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi x}{2L}\right)dx\right)\left(\int_0^L\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi y}{L}\right)dy\right)\left(\int_0^L\sin\left(\frac{\pi z}{L}\right)\sin\left(\frac{\pi z}{L}\right)\right)\end{aligned}$$

The integrals over y and z are simple and simply evaluate to $L/2$ as we'd expect from the normalization factor. To evaluate the integral over x, use the trigonometric identity $\sin 2\theta=2\sin\theta\cos\theta$ and a change of variables with $u=\sin(\pi x/2L)$ to arrive at the integral

$$\mathscr{P}=\frac{2\sqrt{2}}{L}\int_0^1 u^2\cdot\frac{2L}{\pi}\,du$$

Evaluating this, we find the probability of remaining the ground state after the box is expanded suddenly to be

$$\boxed{\mathscr{P}=\frac{4\sqrt{2}}{3\pi}\approx 0.60}\tag{S2002 I 11.1}$$

Problem 12

Question

The frequency f of a deep water gravity wave (i.e. an ordinary ocean wave) is given by

$$f = \sqrt{\frac{1}{2\pi}\rho^a g^b \lambda^c}$$

where ρ, g, and λ are the water density, gravitational acceleration, and wavelength of the wave, respectively. What are the values of the exponents a, b, and c, and what is the ratio of the wave group velocity to phase velocity?

Answer

We proceed by dimensional analysis. Immediately we know that $a = 0$ since a frequency does not have a mass component, and neither g nor λ have a mass term to cancel the one in ρ. Furthermore, g is the only one with a time term, so it's exponent must then by $b = \frac{1}{2}$ in order to give f its $[\mathrm{s}^{-1}]$ unit. That leaves $c = \frac{1}{2}$ in order to cancel the $\sqrt{\mathrm{m}}$ dimension left over from g.

$$\boxed{f = \sqrt{\frac{g\lambda}{2\pi}} \qquad \text{with} \qquad a = 0,\, b = \frac{1}{2},\, c = \frac{1}{2}}$$

The phase velocity can be derived from the frequency given by noting that $v_p = \omega/k$ together with $k^{-1} = 2\pi\lambda$ and $\omega = 2\pi f$. Put together, this gives

$$v_p = \frac{1}{k}\sqrt{\frac{g}{k}}$$

The group velocity is given by $v_g = d\omega/dk$, so

$$v_g = -\frac{1}{2k}\sqrt{\frac{g}{k}}$$

Taking only the absolute values and finding the ratio

$$\boxed{\frac{v_g}{v_p} = \frac{1}{2}} \qquad \text{(S2002 I 12.1)}$$

Spring 2002 Part II

Problem 2

Question

A zipper has N links; each link has a closed state with zero energy and an open state with energy ε. We require, however, that the zipper can only unzip from the left end, and that the link number s can only open if all links to the left (i.e. $1, 2, \ldots s-1$) are already open.

a. Find an explicit expression for the partition function by doing the appropriate summation.

b. In the limit $\varepsilon \gg k_B T$ find the average number of open links. This model is a very simplified model of the unwinding of two-stranded DNA molecules.

Answer

a. Create the partition function by induction; start by assuming there is only a single link. Then the partition function is a simple two-state system:

$$Z_1 = e^0 + e^{-\varepsilon/k_BT} = 1 + e^{-\varepsilon/k_BT}$$

Adding a second link,

$$\begin{aligned} Z_2 &= \underbrace{e^{0+0}}_{\text{both closed}} + \underbrace{e^{(-\varepsilon+0)/k_BT}}_{\text{1 open, 1 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon)/k_BT}}_{\text{both open}} \\ &= 1 + e^{-\varepsilon/k_BT} + e^{-2\varepsilon/k_BT} \end{aligned}$$

Following, for three links:

$$\begin{aligned} Z_3 &= \underbrace{e^{0+0+0}}_{\text{all closed}} + \underbrace{e^{(-\varepsilon+0+0)/k_BT}}_{\text{1 open, 2 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon+0)/k_BT}}_{\text{2 open, 1 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon-\varepsilon)/k_BT}}_{\text{all open}} \\ &= 1 + e^{-\varepsilon/k_BT} + e^{-2\varepsilon/k_BT} + e^{-3\varepsilon/k_BT} \end{aligned}$$

By induction, we see that the maximum coefficient in the series of exponential factors is just the number of links, so by induction we conclude that

$$Z = \sum_{s=0}^{N} e^{-s\varepsilon/k_BT}$$

Applying the results of a finite geometric series, the closed-form solution for the partition function of the links is

$$\boxed{Z = \frac{1 - e^{-(N+1)\varepsilon/k_BT}}{1 - e^{-\varepsilon/k_BT}}} \tag{S2002 II 2.1}$$

b. To get the average number of open links, we use the standard procedure for finding expvalation values.

$$\langle s \rangle = \frac{1}{Z} \sum_{s=0}^{N} s e^{-s\varepsilon/k_BT}$$

By making use of differentiation under the summation trick, we can find the closed-form solution:

$$\begin{aligned} \langle s \rangle &= \frac{1}{Z} \sum_{s=0}^{N} \frac{d}{d\left(\frac{\varepsilon}{k_BT}\right)} \Big[-e^{-s\varepsilon/k_BT} \Big] \\ &= -\frac{1}{Z} \frac{\partial}{\partial\left(\frac{\varepsilon}{k_BT}\right)} \sum_{s=0}^{N} e^{-s\varepsilon/k_BT} \end{aligned}$$

Noting that the summation is the same as above,

$$\langle s \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)}$$

First considering just the derivative part:

$$\frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)} = \frac{(N+1)\, e^{-(N+1)\varepsilon/k_B T}}{1 - e^{-\varepsilon/k_B T}} - \frac{1 - e^{-(N+1)\varepsilon/k_B T}}{\left(1 - e^{-\varepsilon/k_B T}\right)^2} e^{-\varepsilon/k_B T}$$

which when combined with the factor $-1/Z$ simplifies to

$$\begin{aligned} -\frac{1}{Z} \frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)} &= -(N+1) \frac{e^{-(N+1)\varepsilon/k_B T}}{1 - e^{-(N+1)\varepsilon/k_B T}} + \frac{e^{-\varepsilon/k_B T}}{1 - e^{-\varepsilon/k_B T}} \\ &= \frac{1}{e^{\varepsilon/k_B T} - 1} - \frac{N+1}{e^{(N+1)\varepsilon/k_B T} - 1} \end{aligned}$$

Therefore the analytic solution is

$$\boxed{\langle s \rangle = \frac{1}{e^{\varepsilon/k_B T} - 1} - \frac{N+1}{e^{(N+1)\varepsilon/k_B T} - 1}} \qquad \text{(S2002 II 2.2)}$$

In the limit that $\varepsilon \gg k_B T$, though, the exponentials in the denominator are very large in comparison to 1, so we ignore the unity factors and make the approximation that

$$\langle s \rangle = e^{-\varepsilon/k_B T} - (N+1)\, e^{-(N+1)\varepsilon/k_B T}$$

Collecting like terms,

$$= \left[1 - (N+1)\, e^{-N\varepsilon/k_B T} \right] e^{-\varepsilon/k_B T}$$

The second term in the brackets approximate zero, so

$$= e^{-\varepsilon/k_B T}$$

Therefore in the low temperature limit where the thermal energy is much less than the energy of the open state,

$$\boxed{\langle s \rangle = e^{-\varepsilon/k_B T}} \qquad \text{(S2002 II 2.3)}$$

Fall 2002 Part I

Problem 1

Question

A cylindrical bucket is placed on the ground and filled with water to a height of 150 cm. How high from the ground should one punch a hole in the side of the bucket to make a stream of water that strikes the ground at the greatest distance from the bucket? What is that distance?

Answer

To solve and optimize the kinematic equation to determine the maximum range of the water jet, we first need to determine what the velocity of the water exiting the hole is as a function of the hole's depth below the surface of the water. To do this, we make use of Bernoulli's principle.

$$\frac{v^2}{2} + gz + \frac{p_0}{\rho} = \text{constant}$$

We begin by determining the value of the constant by evaluating the equation at the surface of the water where we know all the properties. The water is at the ambient air pressure, the height is given in the problem statement, and the velocity can be assumed to be zero in the limit that the hole leaks at a rate too slowly to change the height of the surface. Doing so,

$$\text{constant} = gz_0 + \frac{p_0}{\rho}$$

Then at some height z' below the surface of the water, we puncture a hole. Again the water is moving into a body at atmospheric pressure, so we again use p_0. That leaves the velocity we're searching for remaining, so equating with the surface value,

$$gz_0 + \frac{p_0}{\rho} = \frac{1}{2}v^2 + gz' + \frac{p_0}{\rho}$$
$$v^2 = 2g\left(z_0 - z'\right)$$

Now we're ready to solve the kinematic equation. Begin as usual by finding the time of flight from the vertical components.

$$0 = z' - \frac{1}{2}gt^2 \qquad \rightarrow \qquad t = \sqrt{\frac{2z'}{g}}$$

Then solving the horizontal equation:

$$\ell = vt = 2\sqrt{z'\left(z_0 - z'\right)}$$

Then finding the extrema:

$$\frac{d\ell}{dz'} = 0 = \frac{z_0 - 2z'}{\sqrt{z'\left(z_0 - z'\right)}}$$
$$z' = \frac{1}{2}z_0$$

That height that maximizes distance, and that distance, is

$$\boxed{z' = 75\,\text{cm}} \qquad \boxed{\ell = 150\,\text{cm}} \tag{F2002 I 1.1}$$

Problem 5

Question

Blocks of mass m and $2m$ are free to slide without friction on a horizontal wire. They are connected by a massless spring of equilibrium length L and force constant k. A projectile of mass m is fired with velocity v into the block with mass m and sticks to it. If the blocks are initially at rest, what is the maximum displacement between them in the subsequent motion?

Answer

Take time $t = 0$ to be the moment the projectile collides with the mass m, and let the subsequent transfer of momentum be instantaneous. In this case, the initial conditions of the problem are then:

$$\begin{aligned} x_1(0) &= 0 & \qquad \dot{x}_1(0) &= u \\ x_2(0) &= L & \qquad \dot{x}_2(0) &= 0 \end{aligned}$$

where u is the initial velocity of the combined project-mass system. We get u from conservation of momentum:

$$\begin{aligned} 2mu &= mv + 0 \\ u &= \frac{1}{2}v \end{aligned}$$

Now solve the mechanics problem using the Lagrangian approach. Both masses have kinetic energy, and the spring stores potential energy, so

$$\begin{aligned} T &= m\dot{x}_1^2 + m\dot{x}_2^2 \\ V &= \frac{1}{2}m\left(x_2 - x_1\right)^2 \\ L &= m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k\left({x_1}^2 + {x_2}^2 + 2x_1x_2\right) \end{aligned}$$

Setting up the differential equation, we get

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= -kx_1 + kx_2 & \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= 2m\ddot{x}_1 \\ \frac{\partial L}{\partial x_2} &= kx_1 - kx_2 & \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= 2m\ddot{x}_2 \end{aligned}$$

Leading to the system of equations where $\omega^2 = k/2m$,

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solving the eigensystem, we find the eigenfrequencies to be $\lambda = \{0, -2\omega^2\}$. Letting ${\omega'}^2 = 2\omega^2$, the eigenfunction equations are then

$$\begin{aligned} \ddot{\psi}_1 &= 0 & \qquad &\rightarrow & \qquad \psi_1 &= A_1 t + B_1 \\ \ddot{\psi}_2 &= -2\omega^2\psi_2 & \qquad &\rightarrow & \qquad \psi_2 &= A_2\cos(\omega' t) + B_2\sin(\omega' t) \end{aligned}$$

From the eigenvectors, we express the solutions of x_1 and x_2 in terms of ψ_1 and ψ_2:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$x_1 = A_1 t + B_1 + A_2 \cos(\omega' t) + B_2 \sin(\omega' t)$$
$$x_2 = A_1 t + B_1 - A_2 \cos(\omega' t) - B_2 \sin(\omega' t)$$

Applying the boundary conditions, we find that

$$x_1(t) = \frac{1}{4} vt + \frac{1}{2} L - \frac{1}{2} L \cos(\omega' t) + \frac{v}{4\omega'} \sin(\omega' t)$$
$$x_2(t) = \frac{1}{4} vt + \frac{1}{2} L + \frac{1}{2} L \cos(\omega' t) - \frac{v}{4\omega'} \sin(\omega' t)$$

The distance $\ell(t) = x_2(t) - x_1(t)$ between the two masses maximizes when

$$\frac{d\ell}{dt} = 0 = \frac{d}{dt}\left[L \cos(\omega' t) - \frac{v}{2\omega'} \sin(\omega' t)\right]$$
$$t = -\frac{1}{\omega'} \arctan\left(\frac{v}{2L\omega'}\right)$$

Plugging back into the function $\ell(t)$,

$$\ell = L \cos\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right] - \frac{v}{2\omega'} \sin\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right]$$
$$\ell = L \frac{2L\omega'}{\sqrt{v^2 + 4L^2\omega'^2}} + \frac{v}{2\omega'} \frac{v}{\sqrt{v^2 + 4L^2\omega'^2}}$$
$$\ell = \frac{\sqrt{v^2 + 4L^2\omega'^2}}{2\omega'}$$

Finally, substituting back in $\omega' = \sqrt{2k/m}$ and simplifying, we get the final solution that maximum distance between the two masses is

$$\boxed{\ell = \sqrt{L^2 + \frac{\frac{1}{2}mv^2}{8k}}} \tag{F2002 I 5.1}$$

which agrees qualitatively with the fact that a larger spring constant should stiffen the system and decrease the maximum displacement, while launching the projectile with a greater velocity would increase it.

Problem 7

Question

In the state $\psi_{\ell,m}$ with angular momentum ℓ and its projection m, determine the average values ${l_x}^2$ and ${l_y}^2$.

Answer

Begin by noting that the angular momentum operators are related as,

$$L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$$

If we assume that the average x and y components will be equal based on symmetry where only the z direction is identifiable, then we can let $L_x = L_y$ and rearrange the equation to get

$${L_x}^2 = \frac{1}{2}\left(L^2 - {L_z}^2\right)$$

The expectation value is then found by the standard method:

$$\begin{aligned}\langle\psi_{\ell,m}|L_x^2|\psi_{\ell,m}\rangle &= \langle\psi_{\ell,m}|\frac{1}{2}\left(L^2 - {L_z}^2\right)|\psi_{\ell,m}\rangle \\ &= \frac{1}{2}\left(\langle\psi_{\ell,m}|L^2|\psi_{\ell,m}\rangle - \langle\psi_{\ell,m}|{L_z}^2|\psi_{\ell,m}\rangle\right)\end{aligned}$$

Then because we know these quantum numbers

$$\langle\psi_{\ell,m}|L_x^2|\psi_{\ell,m}\rangle = \frac{1}{2}\hbar^2\left(\ell\left(\ell+1\right) - m^2\right)$$

Therefore with both ${\ell_x}^2$ and ${\ell_y}^2$ being assumed equation, we conclude that

$$\left\langle{\ell_x}^2\right\rangle = \left\langle{\ell_y}^2\right\rangle = \frac{1}{2}\hbar^2\left(\ell\left(\ell+1\right) - m^2\right) \tag{F2002 I 7.1}$$

Problem 10

Question

Ice on a pond is 10 cm thick and the water temperature just below the ice is 0 °C. If the air temperature is −20 °C, by how much will the ice thickness increase in 1 hour? Assuming that the air temperature stays the same over a long period, how will the ice thickness increase with time? Comment on any approximation that you make in your calculation. Density of ice $= 0.9\,\mathrm{g/cm^3}$. Thermal conductivity of ice $= 0.0005\,\mathrm{cal/(cm\,s\,{}^\circ C)}$. Latent heat of fusion of water $= 80\,\mathrm{cal/g}$.

Answer

Since the thermal heat flow is a one dimensional problem, immediately consider everything with respect to a small area element with its normal perpendicular to the ice-water interface dA. Then we want to know how much ice is generated on the surface of the ice. This small ice element's mass is simply

$$dm = \rho\, dA\, dz$$

where dz is the thickness of the new ice layer. To generate this ice, the latent heat of fusion must be conducted away, so the energy released is,

$$\begin{aligned} dE_f &= L_f\, dm \\ &= L_f \rho\, dA\, dz \end{aligned}$$

The energy flow is through the ice, and we expval this to increase with the temperature differential across the ice sheet, suggesting that the thermal conductivity κ be multiplied by the temperature difference ΔT. Furthermore, the ice will decrease the rate of heat flow as it becomes thicker, so the quantity should also be divided by the thickness z. This gives

$$\frac{\kappa \Delta T}{z} = \left[\frac{\mathrm{cal}}{\mathrm{cm^2\,s}}\right]$$

This energy is flowing through a surface element dA, giving the power flow due to heat as

$$\frac{\kappa \Delta T\, dA}{z} = \left[\frac{\mathrm{cal}}{\mathrm{s}}\right]$$

This power can be matched in units with the energy released from the ice calculated above by taking the time derivative of dE_f, so equating the two we have

$$\begin{aligned} L_f \rho\, dA \frac{dz}{dt} &= \frac{\kappa \Delta T\, dA}{z} \\ \int_{z_0}^{z_0+\Delta z} z\, dz &= \int_0^t \frac{\kappa \Delta T}{L_f \rho}\, dt \\ 2 z_0 \Delta z + (\Delta z)^2 &= \frac{\kappa \Delta T}{L_f \rho} t \end{aligned}$$

Solving for the length the ice grows Δz,

$$\begin{aligned} \Delta z &= \frac{-2z_0 \pm \sqrt{4{z_0}^2 - 4\left(\frac{\kappa \Delta T}{L_f \rho}\right) t}}{2} \\ \Delta z &= z_0 \left(1 \pm \sqrt{1 - \frac{\kappa \Delta T}{L_f \rho {z_0}^2} t}\right) \end{aligned}$$

The two roots give solutions $\Delta z = \{0.0501\,\mathrm{cm}, 19.950\,\mathrm{cm}\}$. Since the second root is unrealistic, we know that the solution must then be

$$\boxed{\Delta z = 0.0501\,\mathrm{cm} \quad \text{in an hour}} \tag{F2002 I 10.1}$$

Problem 11

Question

Carbon-14 is produced by cosmic rays interacting with the nitrogen in the Earth's atmosphere. It is eventually incorporated into all living things, and since it has a half-life of $(5730 \pm 40)\,\mathrm{yr}$, it is useful for dating archaeological specimens up to several tens of thousands of years old. The radioactivity of a particular specimen of wood containing $3\,\mathrm{g}$ of carbon was measured with a counter whose efficiency was $18\,\%$; a count rate of $(12.8 \pm 0.1)\,\mathrm{min}^{-1}$ was measured. It is known that in $1\,\mathrm{g}$ of living wood, there are $16.1\,\mathrm{min}^{-1}$ radioactive carbon-14 decays. What is the age of this specimen, and its uncertainty? (Where errors are not quoted, they can be assumed to be negligible).

Answer

The rate N after a given time is given by the exponential decay formula

$$N(t) = N_0 e^{-t/\tau}$$

Since we have the half-life $t_{1/2}$ instead of the decay constant τ, we use the relation $t_{1/2} = \tau \ln 2$ to simplify the expression instead to

$$N(t) = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The counter use has an efficiency of $\varepsilon = 0.18$, so the measured counting rate N_m must be corrected for that. Furthermore, the sample has a mass of $3\,\mathrm{g}$ whereas we know the rate for a one gram sample, so we also normalize the count rate by the mass of the sample. Plugging this all into the exponential decay function above gives

$$\frac{N_m}{3\varepsilon} = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The only unknown left in the equation is the time, so solving for it,

$$\begin{aligned} t &= t_{1/2} \log_{1/2}\left(\frac{N_m}{3\varepsilon N_0}\right) \\ t &= t_{1/2} \frac{\ln\left(\frac{N_m}{3\varepsilon N_0}\right)}{\ln 2} \\ t &= \frac{t_{1/2}}{\ln 2} \ln\left(\frac{N_m}{3\varepsilon N_0}\right) \end{aligned}$$

To find the uncertainty, we note that only the quantities N_m and $t_{1/2}$ have non-negligible uncertainties, so we propagate the errors only over these two terms:

$$\begin{aligned} {\sigma_t}^2 &= \left(-\frac{t_{1/2}}{N_m \ln 2}\right)^2 {\sigma_{N_m}}^2 + \left(\frac{1}{\ln 2} \ln\left(\frac{3\varepsilon N_0}{N_m}\right)\right)^2 {\sigma_{t_{1/2}}}^2 \\ \sigma_t &= \frac{t_{1/2}}{\ln 2} \sqrt{\left(\frac{\sigma_{N_m}}{N_m}\right)^2 + \left(\frac{\sigma_{t_{1/2}}}{t_{1/2}}\right)^2 \left[\ln\left(\frac{3\varepsilon N_0}{N_m}\right)\right]^2} \end{aligned}$$

Plugging in all the numbers, we get $t = 4248.8435\,\mathrm{yr}$ and $\sigma_t = 161.717\,\mathrm{yr}$. The given uncertainties have a single significant digit, so adding an extra significant figure to the uncertainty and matching decimal places in the answer, we conclude that the sample has an age of

$$\boxed{t = (4250 \pm 160)\,\mathrm{yr}} \tag{F2002 I 11.1}$$

Fall 2007 Part I

Problem 9

Question

An electric bulb is rated at 100 W when used with a DC voltage of 110 V. What total power is dissipated if this voltage is applied to two such bulbs connected in series? It can be assumed that each bulb dissipates heat by radiation from its filament similar to a black body and that the resistance of the filament is proportional to its absolute temperature.

Answer

Beginning from the known properties of a single bulb, we know that the dissipated power in a single bulb P_0 is like a black body, so the power must follow the Stefan-Boltzmann law:

$$P_0 = \sigma_B {T_0}^4$$

where T_0 is operating equilibrium temperature. In addition, we are told that the bulb is like a resistor with a resistance proportional to its temperature:

$$P_0 = \frac{V^2}{R} = \frac{V_0^2}{CT_0}$$

Combining the two equations gives the proportionality constant for a single bulb in the circuit.

$$C = \frac{{V_0}^2}{\sigma_B {T_0}^5}$$

When a second bulb is added to the circuit, the voltage across each bulb is dropped and a corresponding change in the equilibrium temperature is created. Since the bulbs are in series, the voltage $V = \frac{1}{2}V_0$ across each resistor sums in series. Repeating the same procedure as in the first case,

$$\begin{aligned}
\sigma_B T^4 &= 2\frac{V^2}{R} = 2\frac{\left(\frac{1}{2}V_0\right)^2}{CT} \\
\sigma_B T^5 &= \frac{1}{2}\frac{{V_0}^2}{C}
\end{aligned}$$

and inserting the constant C,

$$\begin{aligned}
T^5 &= \frac{1}{8}{T_0}^5 \\
T &= \frac{1}{2^{1/5}}T_0
\end{aligned}$$

Therefore from the Stefan-Boltmann law, the total power dissipated is

$$P = \sigma_B T^4 = \frac{1}{2^{4/5}}\sigma_B {T_0}^4$$

$$\boxed{P = \frac{100\,\mathrm{W}}{2^{4/5}} = 57.435\,\mathrm{W}} \tag{F2007 I 9.1}$$

Fall 2008 Part I

Problem 2

Question

If an impulse is delivered to the end of a uniform rod of length ℓ, lying on a frictionless plane, how far will it travel while making one revolution? The impulse is in the plane of the table and perpendicular to the rod.

Answer

For a given impulse $\vec{J}$, the change in the motion is $\vec{J} = \Delta\vec{p}$. If the rod start at rest, then the final momentum must be $\vec{p} = \vec{J}$. This means the rod is moving laterally with a velocity

$$V = \frac{1}{m}\vec{J}$$

which when integrated over a time t gives the distance it has moved $\vec{x}$.

$$\vec{x} = \frac{1}{m}\vec{J}t$$

The impulse also imparts a rotation on the rod because the force was not applied at the rod's center of mass. The torque $\vec{\tau}$ relates the force to the angular momentum $\vec{L}$ by

$$\vec{r} \times \vec{F} = \vec{\tau} = \dot{\vec{L}}$$

Integrating both sides of the equation, we can write the equation in terms of the given impulse:

$$\vec{r} \times \int \vec{F}\,dt = \int \dot{\vec{L}}\,dt$$
$$\vec{r} \times \vec{J} = \Delta\vec{L}$$

Again, since the rod starts at rest, we know that the final angular momentum must be

$$\vec{L} = \vec{r} \times \vec{J}$$

The rotation about the rod's center of mass occurs at a rate $\vec{\omega}$ dependent on the moment of inertia $I = \frac{1}{12}m\ell^2$, so

$$\vec{\omega} = \frac{12}{m\ell^2}\vec{r} \times \vec{J}$$

We know that the impulse is applied perpendicular to the rod, so we can easily integrate the expression in time and solve for the time it takes to revolve 2π radians:

$$\theta = 2\pi = \frac{12}{m\ell^2} rJt$$
$$t = \frac{\pi m\ell^2}{6rJ}$$

Plugging this back into the linear motion equation, the rod travels

$$\vec{x} = \frac{1}{m}\vec{J} \cdot \frac{\pi m\ell^2}{6rJ}$$

where we can set $r = \frac{1}{2}\ell$ and therefore simplifies to

$$\boxed{\vec{x} = \frac{\pi\ell}{3}\hat{J}} \tag{F2008 I 2.1}$$

where $\hat{J}$ is the direction of the applied impulse.

Problem 3

Question

A time-independent magnetic field is given by $\vec{B} = 2bxy\,\hat{\imath} + ay^2\,\hat{\jmath}$.

a) What is the relationship between the constants a and b?

b) Determine the steady current density J that gives rise to this field.

Answer

For part (a), we realize that all magnetic fields must be divergence-less. Therefore we can find the requirements on the constants a and b by constraining the divergence to be zero.

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} = 0 &= \frac{\partial}{\partial x}(2bxy) + \frac{\partial}{\partial y}\left(ay^2\right) \\ 0 &= 2by + 2ay \\ b &= -a\end{aligned}$$

Therefore the relation between the constants is that

$$\boxed{b = -a} \tag{F2008 I 3.1}$$

For the second part, we make use of Maxwell's equations. Assuming that none of the field is due to a time-varying electric field, we make use of

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

to calculate the current that generates the field. Doing so, we find that the solution is

$$\boxed{\vec{J} = \frac{2a}{\mu_0} x\,\hat{k}} \tag{F2008 I 3.2}$$

Problem 4

Question

A set of four point charges q_1, q_2, q_3, and q_4 are arranged collinearly along the z-axis at $z_1 = 0$, $z_2 = a$, $z_3 = 2a$, $z_4 = 4a$, respectively and the resulting electric field at a distant point $\vec{r}$ ($r \gg a$) decays *faster* than $1/r^3$. Determine the values of q_1 and q_4 which $q_2 = +2$ and $q_3 = +4$. Units for all charges are Coulombs.

Answer

Given that the electric field must fall off faster than $1/r^3$, this corresponds to a potential which drops off faster than $1/R^2$. We know that the monopole moment drops off like $1/r$ and the dipole like $1/R^2$, so we conclude that the first configuration which could satisfy the given requirement is that of a quadrupole moment.

Making use of the fact that he monopole and dipole moments are vanishing, we can use them to generate constraint equations for what the charges must be: we have two unknown charges and the two equations will allow us to solve them.

For the monopole, the sum of all charges must simply equal zero. Therefore we immediately know that

$$\begin{aligned} 0 &= q_1 + q_4 + 6 \\ -6 &= q_1 + q_4 \end{aligned}$$

The dipole moment (where we take the dipole considered at the origin) is given by

$$\vec{p} = \sum_i \vec{r_i} q_i$$

This gives us the equation

$$\begin{aligned} 0 &= 10a + 4aq_4 \\ q_4 &= -\frac{5}{2} \end{aligned}$$

The charge q_1 does not show up in the equation since it is located at the origin. This lets us very simply then solve for q_1 as

$$-6 = q_1 - \frac{5}{2}$$

Therefore, the solution is that the charges have values of

$$\boxed{q_1 = -\frac{7}{2}} \tag{F2008 I 4.1}$$

$$\boxed{q_4 = -\frac{5}{2}} \tag{F2008 I 4.2}$$

Problem 5

Question

The Lyman-α transition in atomic hydrogen has a wavelength $\lambda = 121.5\,\text{nm}$, and a transition rate of $0.6 \cdot 10^9\,\text{s}^{-1}$. Estimate the minimum value of $\Delta\lambda/\lambda$.

Answer

We can make an estimate of the spread $\Delta\lambda$ by making use of the Heisenberg uncertainty relation for energy-time. Starting with the variation in wavelength,

$$\begin{aligned}\Delta\lambda &= \lambda - \lambda' \\ &= \frac{hc}{E} - \frac{hc}{E'} \\ &= \frac{hc\,(E' - E)}{EE'}\end{aligned}$$

Making use of the approximation that $E \approx E'$,

$$= \frac{hc\Delta E}{E^2}$$

Dividing by the frequency and substituting in the uncertainty relation $\Delta E \Delta t = \frac{\hbar}{2}$,

$$\begin{aligned}\frac{\Delta\lambda}{\lambda} &= \frac{hc}{\lambda} \cdot \frac{1}{E^2} \frac{\hbar}{2\Delta t} \\ &= \frac{\lambda}{4\pi c \Delta t}\end{aligned}$$

For the time, we estimate the transition rate is occurring as fast as it can within the limits of the uncertainty relation, so we can let $\Delta t \approx 0.6 \cdot 10^9\,\text{s}^{-1}$. Plugging in the other values, we find the fractional line width to be estimated as

$$\boxed{\frac{\Delta\lambda}{\lambda} \approx 1.935 \cdot 10^{-8} \approx 1 \text{ part in 50 million}} \tag{F2008 I 5.1}$$

Problem 11

Question

A rock is found to contain 4.20 mg of ^{238}U and 2.00 mg of ^{206}Pb. Assume that the rock contained no lead at the time of its formation, so that all the lead now present is due to the decay of the Uranium originally present in the rock. Find the age of the rock given that the half-life of ^{238}U is $4.47 \cdot 10^9$ yr. The decay times of all intermediate elements are negligibly short and ignore any differences in the binding energies.

Answer

From decay processes, we know that the uranium atom count will decrease as an exponential according to

$$N_U = N_{U0} e^{-t/\tau}$$

where $\tau = t_{1/2}/\ln 2$. Likewise, the number of lead atoms will increase according to

$$N_{Pb} = N_{U0}\left(1 - e^{-t/\tau}\right)$$

Solving for N_{U0} in the first equation and substituting it into the second, we can solve for the time required to generate a specific number of uranium and lead atoms in a sample.

$$\begin{aligned} N_{Pb} &= N_U e^{t/\tau}\left(1 - e^{-t/\tau}\right) \\ t &= \tau \ln\left(\frac{N_{Pb}}{N_U} + 1\right) \\ t &= \frac{t_{1/2}}{\ln 2} \ln\left(\frac{N_{Pb}}{N_U} + 1\right) \end{aligned}$$

We were only given the masses, though, so we approximate the mass of each atom by the number of nucleons in the nucleus; each uranium atom has a mass of $m_U = 238 m_N$ making the N_U atoms have a mass of $M_U = 238 N_U m_N$, and similar for the lead. This gives us the final equation

$$t = \frac{t_{1/2}}{\ln 2} \ln\left(\frac{238}{206}\frac{M_{Pb}}{M_U} + 1\right)$$

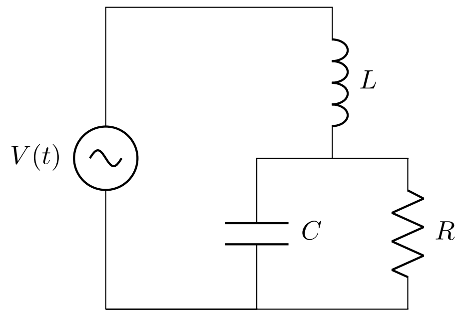
Plugging in all the numbers,

$$\boxed{t = 2.83 \cdot 10^9 \text{ yr}} \tag{F2008 I 11.1}$$

Problem 12

Question

The applied AC voltage in the circuit is given by $V(t) = V_0 \sin \omega t$, with a frequency fixed at $\omega = 1/(LC)^{1/2}$. Determine the steady state amplitude and phase of the current through the resistor R. Express your answer in terms of the amplitude V_0 of the applied voltage and the other circuit parameters.

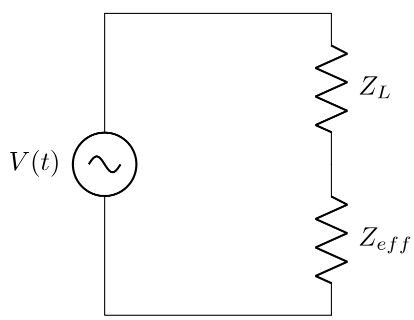


Answer

AC problems are simplified by using complex impedances, so we first convert the given voltage into a complex one:

$$\tilde{V}(t) = V_0 e^{i\omega t}$$

where the physical solution can be recovered by keeping the imaginary component of the complex solution. Then to solve the problem, we realize that there is another complimentary circuit diagram which is helpful: the one with the resistor and capacitor replaced by an effective resistor (impedance). The circuit looks like



The inductor has been been replaced by an effective resistor with impedance $Z_L = i\omega L$. The effective resistor that replaced the capacitor and resistor is a complex impedance that is calculated the same as for traditional resistors in parallel:

$$\begin{aligned} Z_{eff} &= \left(\frac{1}{Z_C} + \frac{1}{Z_R}\right)^{-1} \\ &= \left(i\omega C + \frac{1}{R}\right)^{-1} \\ &= \frac{R}{i\omega CR + 1} \end{aligned}$$

Now making use of Kirchoff's loop rule on this simplified circuit where the total current passing through the voltage source is labeled $\tilde{I}_0$,

$$\begin{aligned}
0 &= \tilde{V} - \tilde{I}_0 \left(Z_L + Z_{eff}\right) \\
\tilde{V} &= \left(i\omega L + \frac{R}{i\omega CR + 1}\right) \tilde{I}_0 \\
\tilde{V} &= \frac{R\left(1 - \omega^2 LC\right) + i\omega L}{i\omega RC + 1} \tilde{I}_0
\end{aligned}$$

The first term in the numerator goes to zero since $\omega^2 = 1/LC$, leaving

$$\tilde{I}_0 = \frac{i\omega RC + 1}{i\omega L} V_0 e^{i\omega t}$$

To isolate the current passing through the resistor, we return to the original unsimplified circuit diagram and apply Kirchoff's loop rule to only the inner loop. If we define the current through capacitor to be I_1 and through the resistor to be I_2, we get

$$\begin{aligned}
0 &= -\tilde{I}_2 Z_R + \tilde{I}_1 Z_C \\
\tilde{I}_1 &= \frac{Z_R}{Z_C} I_2 \\
\tilde{I}_1 &= i\omega RC I_2
\end{aligned}$$

Remembering the the current passing into a junction must be conserved, we know that $I_0 = I_1 + I_2$ and therefore,

$$\begin{aligned}
\tilde{I}_0 &= i\omega RC \tilde{I}_2 + \tilde{I}_2 \\
\tilde{I}_2 &= \frac{1}{i\omega RC + 1} \tilde{I}_0
\end{aligned}$$

Inserting the solution for I_0 from the previous part leaves

$$\tilde{I}_2 = \frac{V_0}{i\omega L} e^{i\omega t}$$

To prepare for finding the physical solution, we transform the coefficient complex polar form.

$$\begin{aligned}
\tilde{I}_2 &= \left|-\frac{iV_0}{\omega L}\right| e^{i \arg(-iV_0/\omega L)} e^{i\omega t} \\
&= \frac{V_0}{\omega L} e^{-i\pi/2} e^{i\omega t}
\end{aligned}$$

Therefore taking the imaginary part of the solution,

$$\boxed{I_R\left(t\right) = V_0 \sqrt{\frac{C}{L}} \sin\left(\omega t - \frac{\pi}{2}\right)} \tag{F2008 I 12.1}$$

The current's amplitude is $V_0\sqrt{C/L}$ and has a phase of $-\pi/2$ with respect to the voltage.

Fall 2008 Part II

Problem 1

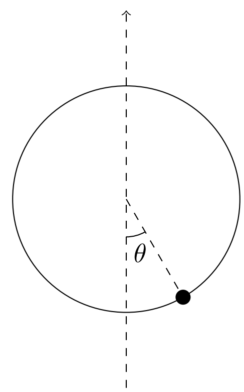
Question

A particle of mass m is constrained to move without friction on a circular wire of radius R rotating with constant angular frequency ω about a vertical diameter. Gravity can not be neglected.

a) Write down the Lagrangian for the system and the equations of motion.

b) Find the equilibrium position(s) of the particle and determine whether this position is stable.

c) Calculate the frequency of small oscillations about any stable points.



Answer

To start constructing the Lagrangian and equations of motion, we first specify the kinetic and potential energies. For the kinetic energy, there is an energy associated with the rotation about the axis and one along the bead. These combined to give

$$\begin{aligned} T &= \frac{1}{2}m\left(R\omega\sin\theta\right)^2 + \frac{1}{2}m\left(R\dot{\theta}\right)^2 \\ &= \frac{1}{2}mR^2\omega^2\sin^2\theta + \frac{1}{2}mR^2\dot{\theta}^2 \end{aligned}$$

The potential energy is all gravitational, so

$$V = -mgR\cos\theta$$

where the zero point was taken to be at the center of the hoop to avoid adding extra constant terms to the Lagrangian. Combining the two, we get

$$\boxed{\mathcal{L} = \frac{1}{2}mR^2\omega^2\sin^2\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta} \qquad \text{(F2008 II 1.1)}$$

Taking the appropriate derivatives in θ, the equation of motion is

$$\boxed{\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta} \qquad \text{(F2008 II 1.2)}$$

In order to determine any possible stable points, we note that a stable point is a place where the angle does not change in time. Since this also equates to $\ddot{\theta} = 0$, we set the equation above equal to zero and solve for the angles which satisfy this condition. They end up being the trivial $\theta = \{0, \pi\}$ where the sine function is zero as well as

$$\cos\theta_0 = \frac{g}{R\omega^2}$$

The three stable points are then

$$\boxed{\theta_0 = \left\{0, \arccos\left(\frac{g}{R\omega^2}\right), \pi\right\}} \tag{F2008 II 1.3}$$

To determine the stability of each, we must determine whether we get oscillatory or exponential solutions to the differential equation of motion. To do this, we suppose the angle θ is composed of the equilibrium angle θ_0 and a small perturbation Δ. Expanding the equation in terms of this,

$$\ddot{\Delta} = \omega^2 \sin(\theta_0 + \Delta)\cos(\theta_0 + \Delta) - \frac{g}{R}\sin(\theta_0 + \Delta)$$

Using several trigonometric expansions, the equation can be expanded into the form

$$\ddot{\Delta} = \omega^2 \left[\cos\theta_0 \sin\theta_0 \left(\cos^2\Delta - \sin^2\Delta\right) + \cos\Delta \sin\Delta \left(\cos^2\theta_0 - \sin^2\theta_0\right)\right] - \frac{g}{R}\left[\sin\theta_0 \cos\Delta + \cos\theta_0 \sin\Delta\right]$$

For the case where $\theta_0 = 0$,

$$\ddot{\Delta} = \omega^2 \cos\Delta \sin\Delta - \frac{g}{R}\sin\Delta$$

Expanding to first order in $\Delta \approx 0$,

$$\ddot{\Delta} = -\left(\frac{g}{R} - \omega^2\right)\Delta$$

Therefore, the equilibrium point $\theta_0 = 0$ is only stable if $\omega < \sqrt{\frac{g}{R}}$.

Likewise for for $\theta_0 = \pi$,

$$\ddot{\Delta} = \left(\frac{g}{R} + \omega^2\right)\Delta$$

The coefficient on Δ will never be negative, so the angle $\theta_0 = \pi$ will be unstable under all conditions.

For the final angle where $\theta_0 = \arccos\left(\frac{g}{R\omega^2}\right)$, we must do several substitutions and expansions. $\cos\theta_0$ is trivial. $\sin\theta_0$ ends up being $\sqrt{R\omega^2 - g^2}/\left(R\omega^2\right)$ by triangle relations. If we substitute these in plus do an expansion to first order for small Δ, we get the equation

$$\ddot{\Delta} = \omega^2 \left[\frac{g\sqrt{R\omega^2 - g^2}}{R^2\omega^2} + \Delta\frac{2g^2 - R\omega^2}{R^2\omega^2}\right] - \frac{g}{R}\left[\frac{\sqrt{R\omega^2 - g^2}}{R^2\omega^2} + \Delta\frac{g}{R\omega^2}\right]$$

If we consider only the homogeneous terms dependent on Δ,

$$\ddot{\Delta} = \frac{g^2 - R\omega^2}{R^2\omega^2}\Delta$$

This equation is stable if and only if the coefficient on Δ is negative, so it must be that $\omega > \frac{g}{\sqrt{R}}$.

In summary, the equilibrium points have the following conditions:

$$\boxed{\theta_0 = 0 \qquad \text{Stable iff } \omega < \sqrt{\frac{g}{R}}} \tag{F2008 II 1.4}$$

$$\boxed{\theta_0 = \arccos\left(\frac{g}{R\omega^2}\right) \qquad \text{Stable iff } \omega > \frac{g}{\sqrt{R}}} \tag{F2008 II 1.5}$$

$$\boxed{\theta_0 = \pi \qquad \text{Never stable}} \tag{F2008 II 1.6}$$

About the two stable points, we simply use the coefficient that has already been isolated to determine the frequency of the oscillations about that point.

$$\boxed{\omega_1 = \sqrt{\frac{g}{R} - \omega^2} \qquad \text{for } \theta_0 = 0} \tag{F2008 II 1.7}$$

$$\boxed{\omega_2 = \sqrt{\frac{R\omega^2 - g^2}{R^2\omega^2}} \qquad \text{for } \theta_0 = \arccos\left(\frac{g}{R\omega^2}\right)} \tag{F2008 II 1.8}$$

Problem 2

Question

The general solution of the Laplace's equation for an electrostatic problem having azimuthal symmetry can be written as

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Now consider the following problem. A solid spherical conductor of radius R having charge Q is placed in an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$.

(a) Qualitatively describe the electric field inside and outside of the sphere.

(b) Solve the problem and find the electric potential in the region outside the sphere.

Answer

To provide a qualitative description, we make use of several properties of conductors. The electric field inside the conductor is guaranteed to be zero in the limit of a perfect conductor which can move its electrons anywhere they're needed to cancel any applied fields. For the region outside the sphere, it is easiest to describe the region just outside the surface and the region at infinitely large distances. Far away, the effects of the the sphere are negligible and the electric field is the external uniform field. Near the surface, though, all field lines are perpendicular to the surface; therefore, the external field's lines are curved so that any intersections occur perpendicular to the surface.

In summary

1. The field is uniform at large distances
2. The field is perpendicular to the surface of the conductor at the conductor's surface
3. There is no field within the interior of the conductor

In order to analyze the problem analytically, we make use of the superposition principle to simplify the problem. Because the sphere carries its own charge, we treat this case as a superposition of the two simpler cases of a charged sphere in vacuum and that of a perfectly conducting, grounded sphere in a uniform electric field.

Since we are only concerned with the potential outside the sphere, we can use Gauss' Law to get the potential due to the charge Q. It is

$$V_Q(r, \theta) = \frac{Q}{4\pi\varepsilon_0 r} \qquad\qquad r > R$$

The uniform field is the considered by satisfying the appropriate boundary conditions to solve for the coefficients A_ℓ and B_ℓ in the general solution given above. We start by converting the given electric field to a potential. In Cartesian coordinates,

$$\vec{E} = E_0 \hat{z} = -\vec{\nabla} V_\infty \qquad \Rightarrow \qquad V_\infty = -E_0 z$$

which when converted to spherical coordinates gives the potential as $r \to \infty$

$$V_\infty = -E_0 r \cos\theta$$

In the infinite distance limit, $B_\ell / r^{\ell+1} \to 0$ so the boundary condition equation becomes

$$-E_0 r \cos\theta = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

By the orthogonality of the Legendre polynomials, only A_1 is non-zero:

$$-E_0 r\cos\theta = A_1 r\cos\theta$$
$$A_1 = -E_0$$

The general solution has thus been simplified to

$$V_0\left(r,\theta\right) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell\left(\cos\theta\right)$$

By our choice of making use of the superposition principle, we have set the potential to be zero at the surface, so at $r = R$, the boundary conditions lets us solve for the values of the B_ℓ:

$$0 = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{R^{\ell+1}}P_\ell\left(\cos\theta\right)$$

The Legendre polynomial orthogonality again eliminates all coefficients except B_1.

$$E_0 R\cos\theta = \frac{B_1}{R^2}\cos\theta$$
$$B_1 = E_0 R^3$$

This gives us the solution to the grounded sphere as

$$V_0\left(r,\theta\right) = -E_0 r\cos\theta\left[1-\left(\frac{R}{r}\right)^3\right]$$

Therefore by superposition of both solutions, the potential in this situation at all points outside the sphere is

$$\boxed{V\left(r,\theta\right) = \frac{Q}{4\pi\varepsilon_0 r} - E_0 r\cos\theta\left[1-\left(\frac{R}{r}\right)^3\right]} \qquad \text{(F2008 II 2.1)}$$

Problem 3

Question

Consider the transmission of a beam of particles of mass m and momentum $p = \hbar k$, in one dimension, incident on a rectangular potential barrier of height V_0 and extending from $x = 0$ to $x = L$, in the special case that the energy E of the incident particle *is exactly equal* to the barrier height V_0.

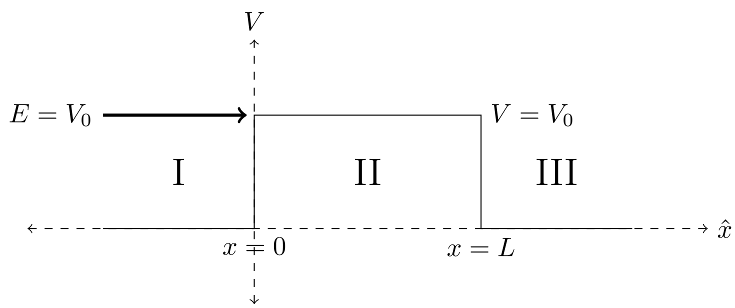
(a) Calculate the transmission and reflection coefficients T and R.

(b) Check some properties of your answers in (a): is probability conserved? Do T and R have the expected limiting values for L very large or very small?

(c) For what values of the de Broglie wavelength of the particles is the transmitted fraction equal to 1/2?

Answer

Consider a beam of particles incident from the left as shown in the figure below:



Ignoring normalization for a minute, we know that in regions I and III that the wavefunction is that of a free particle:

$$\psi(x) = Ae^{ikx} + B^{-ikx} \qquad\qquad k^2 = \frac{2mE}{\hbar^2}$$

In region II, the energy E cancels with the potential V in the Schrödinger equation, so the solution takes the form of a first order polynomial

$$\psi(x) = Ax + B$$

We will only be concerned with the reflection and transmission coefficients, and knowing that they are defined in terms of a ratio of the wavefunction amplitude for the reflected and transmitted components with respect to the incident amplitude, we simplify our solution by directly setting the incident particle amplitude to unity. Furthermore, we know that there is no leftward traveling component in region III. Assigning each component a unique and appropriate unknown coefficient, the three wavefunctions are

$$\psi_{\mathrm{I}} = e^{ikx} + re^{-ikx}$$
$$\psi_{\mathrm{II}} = ax + b$$
$$\psi_{\mathrm{III}} = te^{ikx}$$

We find the values for r and t by applying continuity boundary conditions at the interfaces between each solution. Starting at $x = 0$,

$$\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0) \qquad\qquad \psi'_{\mathrm{I}}(0) = \psi'_{\mathrm{II}}(0)$$
$$1 + r = b \qquad\qquad ik(1 - r) = a$$

Then putting the values into ψ_{I} and solving the boundary conditions at $x = L$,

$$\begin{aligned} \psi_{\mathrm{II}}(L) &= \psi_{\mathrm{III}}(L) & \psi'_{\mathrm{II}}(L) &= \psi'_{\mathrm{III}}(L) \\ ik(1-r)L + 1 + r &= te^{ikL} & ik(1-r) &= ikte^{ikL} \end{aligned}$$

From the equation on the right, we solve for t as a function of r and insert it into the condition on the left:

$$\begin{aligned} t &= (1-r)\, e^{-ikL} \\ ik(1-r)L + 1 + r &= \left((1-r)\, e^{-ikL}\right) e^{ikL} \\ r &= \frac{-ikL}{2-ikL} \end{aligned}$$

Plugged back into t gives

$$t = \frac{2}{2-ikL} e^{-ikL}$$

We then just take the complex square of both amplitudes to get the reflection and transmission coefficients:

$$\boxed{R = |r|^2 = \frac{k^2L^2}{4+k^2L^2}} \tag{F2008 II 3.1}$$

$$\boxed{T = |t|^2 = \frac{4}{4+k^2L^2}} \tag{F2008 II 3.2}$$

These satisfy the requisite properties: the probabilities sum to unity so all particles are accounted for, in the limit that the barrier vanishes no particles are reflected and all are transmitted, and in the limit that the barrier grows to infinite depth, all particles are reflected.

$$\boxed{R + T = 1} \tag{F2008 II 3.3}$$

$$\boxed{T \underset{L\to 0}{\longrightarrow} 1 \qquad T \underset{L\to\infty}{\longrightarrow} 0} \tag{F2008 II 3.4}$$

$$\boxed{R \underset{L\to 0}{\longrightarrow} 0 \qquad R \underset{L\to\infty}{\longrightarrow} 1} \tag{F2008 II 3.5}$$

This system can be tuned such that half of the particles are transmitted through the barrier by changing the energy of the particles. To do so, we set the transmission probability to $\frac{1}{2}$ and solve for the particles' corresponding de Broglie wavelength.

$$\begin{aligned} \frac{1}{2} &= \frac{4}{4+k^2L^2} \\ 4 &= k^2L^2 \\ k^2 &= \frac{4}{L^2} \end{aligned}$$

Making use of the definition of k^2 in terms of the energy,

$$\frac{2mE}{\hbar^2} = \frac{4}{L^2}$$

Then writing the energy in terms of the de Broglie wavelength:

$$\frac{2m}{\hbar^2}\frac{4\pi^2\hbar^2}{2m\lambda^2} = \frac{4}{L^2}$$

Therefore, the particles' incident momentum can be tuned and half the particles will be transmitted when

$$\boxed{\lambda = \pi L} \tag{F2008 II 3.6}$$

Problem 4

Question

Consider a one-dimensional infinite array of points labeled by an index n and separated by a fixed unit distance. At each point there is an identical very deep and narrow potential well. Let $|n\rangle$ denote an eigenstate of a *single* well, with energy E.

(a) Argue that if the wells are so narrow that the different sites can be considered uncoupled, then $|n\rangle$ is an eigenstate of the total Hamiltonian H with eigenvalue E. What is its degeneracy? Then show that the state $|k\rangle$ defined as

$$|k\rangle = \sum_{n=-\infty}^{\infty} e^{ink} |n\rangle$$

with $-\pi < k < \pi$ is an eigenstate of both H and the translation operator T defined as $T|n\rangle = |n+1\rangle$. Find the respective eigenvalues.

(b) Assume now that neighboring sites are weakly coupled so that the total Hamiltonian can now be written as

$$H = \sum_{n=-\infty}^{\infty} \left(|n\rangle E \langle n| - |n+1\rangle D \langle n| - |n\rangle D \langle n+1|\right)$$

where the coupling parameter D is real and we assume that $\langle n|n'\rangle = \Delta_{n,n'}$. Show that $|n\rangle$ is no longer an eigenstate of H but that $|k\rangle$ still is. Find the eigenvalue.

Answer

The only reasonable choice for the form of the total Hamiltonian H is a superposition of the Hamiltonian of individual sites.

$$H = \sum_i H_i$$

If we then operate on a state $|n\rangle$ with the total Hamiltonian,

$$\begin{aligned} H|n\rangle &= \left(\sum_i H_i\right)|n\rangle \\ &= \sum_i H_i |n\rangle \\ H|n\rangle &= H_i \Delta_{i,n} |n\rangle \end{aligned}$$

Only the n-th Hamiltonian will operate on $|n\rangle$, so the state is in fact an eigenstate of the total Hamiltonian with an eigenvalue of E.

$$\boxed{H|n\rangle = E|n\rangle} \tag{F2008 II 4.1}$$

$$\boxed{N\text{-fold degeneracy}} \tag{F2008 II 4.2}$$

Because each state n has the same eigenvalue of E, the degeneracy is equal to the number of sites. If there are N sites in the array, then that is also the degeneracy of the total system.

Similarly for the state $|k\rangle$ as defined will can be operated on by the total Hamiltonian:

$$\begin{aligned} H|k\rangle &= H\left(\sum_n e^{ink}|n\rangle\right) \\ &= \sum_n e^{ink} H|n\rangle \end{aligned}$$

Then because we've already shown that $|n\rangle$ is an eigenstate of H with eigenvalue E

$$= \sum_n e^{ink} E |n\rangle$$
$$H|k\rangle = E\left(\sum_n e^{ink}|n\rangle\right)$$

We find that $|k\rangle$ is also an eigenstate of the total Hamiltonian with ane eigenvalue of E as well.

$$\boxed{H|k\rangle = E|k\rangle} \tag{F2008 II 4.3}$$

Finally, we define a translation operator T for $|n\rangle$ and determine its effect on the state $|k\rangle$.

$$\begin{aligned} T|k\rangle &= T\left(\sum_n e^{ink}|n\rangle\right) \\ &= \sum_n e^{ink} T|n\rangle \\ &= \sum_n e^{ink}|n+1\rangle \end{aligned}$$

We can insert a factor of unity to extract a more useful form

$$\begin{aligned} &= \sum_n e^{i(n+1)k} e^{-ik}|n+1\rangle \\ &= e^{-ik}\sum_n e^{i(n+1)k}|n+1\rangle \end{aligned}$$

and since $n \in (-\infty, \infty)$, the distinction between n and $n+1$ is inconsequential to the definition of $|k\rangle$. Therefore

$$\boxed{T|k\rangle = e^{-ik}|k\rangle} \tag{F2008 II 4.4}$$

The translation operator has a phase eigenvalue of e^{-ik} when operating on the Bloch wave function $|k\rangle$.

If the total Hamiltonian is then modified include nearest neighbor interactions, the individual site wavefunctions $|n\rangle$ are no longer eigenstates of the total Hamiltonian as shown by explicit calculation:

$$\begin{aligned} H|n\rangle &= \left[\sum_{n'} |n'\rangle E \langle n'| - |n'+1\rangle D \langle n'| - |n'\rangle D \langle n'+1|\right]|n\rangle \\ &= \sum_{n'} |n'\rangle E \langle n'|n\rangle - |n'+1\rangle D \langle n'|n\rangle - |n'\rangle D \langle n'+1|n\rangle \\ &= \sum_{n'} \Delta_{n,n'}\left(E|n'\rangle - D|n'+1\rangle\right) - D\Delta_{n,n'+1}|n'\rangle \\ &= E|n\rangle - D|n+1\rangle - D|n-1\rangle \end{aligned}$$

There are now three wavefunctions left with two different coefficients, so the problem is not an eigenvalue problem.

$$\boxed{E|n\rangle - D|n+1\rangle - D|n-1\rangle = H|n\rangle \neq \lambda|n\rangle} \tag{F2008 II 4.5}$$

Operating on the Bloch wavefunction, though

$$\begin{aligned}
H\left|k\right\rangle &= \left[\sum_{n'}\left|n'\right\rangle E\left\langle n'\right| - \left|n'+1\right\rangle D\left\langle n'\right| - \left|n'\right\rangle D\left\langle n'+1\right|\right]\left(\sum_{n} e^{ink}\left|n\right\rangle\right) \\
&= \sum_{n,n'}\left|n'\right\rangle Ee^{ink}\left\langle n'|n\right\rangle - \left|n'+1\right\rangle De^{ink}\left\langle n'|n\right\rangle - \left|n'\right\rangle De^{ink}\left\langle n'+1|n\right\rangle \\
&= \sum_{n,n'} Ee^{ink}\Delta_{n,n'}\left|n'\right\rangle - De^{ink}\Delta_{n,n'}\left|n'+1\right\rangle - De^{ink}\Delta_{n,n'+1}\left|n'\right\rangle
\end{aligned}$$

Consuming the summation over n' to select a non-zero term in the Kronecker delta leaves

$$= \sum_{n} Ee^{ink}\left|n\right\rangle - De^{ink}\left|n+1\right\rangle - De^{ink}\left|n-1\right\rangle$$

We can then separate each term into a summation:

$$= E\sum_{n}\left(e^{ink}\left|n\right\rangle\right) - D\sum_{n}\left(e^{ink}\left|n+1\right\rangle\right) - D\sum_{n}\left(e^{ink}\left|n-1\right\rangle\right)$$

Then using the same unity-factor trick as in demonstrating that $\left|k\right\rangle$ is an eigenstate of T,

$$= E\sum_{n}\left(e^{ink}\left|n\right\rangle\right) - De^{-ik}\sum_{n}\left(e^{i(n+1)k}\left|n+1\right\rangle\right) - De^{ik}\sum_{n}\left(e^{i(n-1)k}\left|n-1\right\rangle\right)$$

Each of these terms is the definition of $\left|k\right\rangle$, so we find that $\left|k\right\rangle$ is still an eigenstate of this new Hamiltonian with an eigenvalue of $E - 2D\cos k$.

$$\boxed{H\left|k\right\rangle = \left(E - 2D\cos k\right)\left|k\right\rangle} \tag{F2008 II 4.6}$$

Problem 5

Question

Two mono-atomic ideal gases, each occupying a volume $V = 1\,\mathrm{m}^3$, are separated by a removable insulating partition. They have different temperatures $T_1 = 350\,\mathrm{K}$ and $T_2 = 450\,\mathrm{K}$, and different pressures $p_1 = 10^3\,\mathrm{N/m^2}$ and $p_2 = 5 \cdot 10^3\,\mathrm{N/m^2}$. The partition is removed, and the gases are allowed to mix while remaining thermally isolated from the outside.

(a) What are the final temperature T_f (in K) and pressures p_f (in $\mathrm{N/m^2}$)?

(b) What is the net change in entropy due to mixing (in J/K)?

Answer

Starting with conservation of energy, the total internal energy after the partition is removed must be the same as the sum of the internal energies of both starting gases.

$$
\begin{aligned}
U &= U_1 + U_2 \\
\frac{3}{2} N k_B T_f &= \frac{3}{2} N_1 k_B T_1 + \frac{3}{2} N_2 k_B T_2 \\
T_f &= \frac{N_1 T_1 + N_2 T_2}{N_1 + N_2}
\end{aligned}
$$

where we made use of the fact that particle number must also be a conserved quantity so that $N = N_1 + N_2$. The original particle numbers N_1 and N_2 can be determined from the ideal gas law in the initial state:

$$
\begin{aligned}
p_1 V &= N_1 k_B T_1 \\
N_1 &= \frac{p_1 V}{k_B T_1}
\end{aligned}
\qquad\qquad
\begin{aligned}
p_2 V &= N_2 k_B T_2 \\
N_2 &= \frac{p_2 V}{k_B T_2}
\end{aligned}
$$

Plugging these into the final temperature T_f above and simplifying gives the value in terms of known quantities as

$$
T_f = \frac{p_1 + p_2}{p_1 T_2 + p_2 T_1} T_1 T_2
$$

which when the numbers are plugged in gives a final temperature of

$$
\boxed{T_f = 429.55\,\mathrm{K}} \tag{F2008 II 5.1}
$$

We can then plug this temperature into a formulation of the ideal gas law for the system after the partition has been removed to get the final pressure.

$$
\begin{aligned}
p_f\,(2V) &= (N_1 + N_2)\, k_B T_f \\
p_f &= \frac{(N_1 + N_2)\, k_B}{2V} \frac{N_1 T_1 + N_2 T_2}{N_1 + N_2} \\
p_f &= \frac{k_B}{2V} (N_1 T_1 + N_2 T_2)
\end{aligned}
$$

Again substitution for N_1 and N_2 in terms of the original pressures and temperatures gives

$$
p_f = \frac{p_1 + p_2}{2}
$$

which when the values are plugged in

$$
\boxed{p_f = 3 \cdot 10^3\,\mathrm{N/m^2}} \tag{F2008 II 5.2}
$$

From the Sackur-Tetrode equation, we can calculate the change in the entropy from the beginning state to the final one. We start by simplifying the equation to isolate constant factors:

$$\begin{aligned} S &= Nk_B \left\{ \frac{5}{2} + \ln\left[\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right] \right\} \\ &= \frac{5}{2} Nk_B + Nk_B \ln\left[\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right] \\ &= Nk_B \left\{ \frac{5}{2} + \frac{3}{2} \ln\left(\frac{4\pi m}{3h^2} \right) + \ln\left[\frac{V}{N} \left(\frac{U}{N} \right)^{3/2} \right] \right\} \end{aligned}$$

Then with the change in entropy defined as

$$\Delta S = S - S_1 - S_2$$

we start calculating the sum term-by-term. For the first two constant terms, the fact that $N = N_1 + N_2$ causes these terms to cancel with those in S_1 and S_2. That leaves us just with the last term.

$$\Delta S = (N_1 + N_2)\, k_B \ln\left[\frac{2V}{N} \left(\frac{\frac{3}{2} N k_B T_f}{N} \right)^{3/2} \right] - N_1 k_B \ln\left[\frac{V}{N_1} \left(\frac{\frac{3}{2} N_1 k_B T_1}{N_1} \right)^{3/2} \right] - N_2 k_B \ln\left[\frac{V}{N_2} \left(\frac{\frac{3}{2} N_2 k_B T_2}{N_2} \right)^{3/2} \right]$$

By the ideal gas law, $V/N = k_B T/p$ which simplifies the expression to

$$\begin{aligned} &= N_1 k_B \ln\left[\frac{2k_B T_f}{p_f} \left(\frac{3}{2} k_B T_f \right)^{3/2} \cdot \frac{p_1}{k_B T_1} \left(\frac{1}{\frac{3}{2} k_B T_1} \right)^{3/2} \right] + N_2 k_B \ln\left[\frac{2k_B T_f}{p_f} \left(\frac{3}{2} k_B T_f \right)^{3/2} \cdot \frac{p_2}{k_B T_2} \left(\frac{1}{\frac{3}{2} k_B T_2} \right)^{3/2} \right] \\ &= N_1 k_B \ln\left[\frac{p_1}{p_f} \left(\frac{T_f}{T_1} \right)^{5/2} \right] + N_2 k_B \ln\left[\frac{p_2}{p_f} \left(\frac{T_f}{T_2} \right)^{5/2} \right] + (N_1 + N_2)\, k_B \ln 2 \end{aligned}$$

Using the ideal gas law again to manipulate the coefficients $Nk_B = pV/T$,

$$\Delta S = V \left\{ \frac{p_1}{T_1} \ln\left[\frac{p_1}{p_f} \left(\frac{T_f}{T_1} \right)^{5/2} \right] + \frac{p_2}{T_2} \ln\left[\frac{p_2}{p_f} \left(\frac{T_f}{T_2} \right)^{5/2} \right] + \frac{p_f}{T_f} \ln 2 \right\}$$

Plugging in the values for all these numbers as given or we determined, the change in entropy is

$$\boxed{\Delta S = 7.55\,\mathrm{J/K}} \tag{F2008 II 5.3}$$

Problem 6

Question

A rocket passes Earth at a speed $v = 0.6c$. When a clock on the rocket says that one hour has elapsed since passing, the rocket sends a light signal back to Earth.

(a) Suppose that the Earth and rocket clocks were synchronized at zero at the time passing. According to the *Earth* clocks, when was the signal sent?

(b) According to the *Earth* clocks, when did the signa arrive back on Earth?

(c) According to the *rocket* clocks, how long after the rocket passed did the signal arrive back on Earth?

Answer

Let $\beta = v/c$.

(a) Let $t'_{sent} = 1\,\mathrm{h}$ be the time at which the rocket sent the signal according to its own clock. Because from the Earth's reference frame the rocket's clocks are running slow, the clocks on Earth must show a time greater than the rocket's by a factor of γ.

$$\begin{aligned} t_{sent} &= \gamma t'_{sent} \\ &= \frac{1\,\mathrm{h}}{\sqrt{1-\beta^2}} \end{aligned}$$

$$\boxed{t_{sent} = \frac{5}{4}\,\mathrm{h} = 1.25\,\mathrm{h}} \tag{F2008 II 6.1}$$

(b) The returning light will traverse the intermediate distance at c, so $t_{ret} = x/c$. The distance from the Earth is simply the velocity times the time (in Earth's reference frame), so together,

$$\begin{aligned} t_{ret} &= \frac{vt_{sent}}{c} \\ &= \beta t_{sent} \\ t_{ret} &= \frac{3}{4}\,\mathrm{h} = 0.75\,\mathrm{h} \end{aligned}$$

The total round trip time is then

$$\boxed{t_{tot} = t_{sent} + t_{ret} = 2\,\mathrm{h}} \tag{F2008 II 6.2}$$

(c) Because the rocket is flying away from the Earth, the distance behind it towards the earth appears to have been length expanded by a factor of γ compared to the distance that Earth would report. Therefore the time for the signal to be sent from the rocket to Earth appears to the rocket to be

$$\begin{aligned} t'_{ret} &= \frac{\gamma x}{c} \\ &= \gamma t_{ret} \\ t'_{ret} &= \frac{15}{16}\,\mathrm{h} = 0.9375\,\mathrm{h} \end{aligned}$$

Added to the 1 hour that the rocket observes as the time before it sent the signal, the signal would arrive at earth at time

$$\boxed{t'_{tot} = \frac{31}{16}\,\mathrm{h} = 1.9375\,\mathrm{h}}$$

Spring 2010 Part I

Problem 1

Question

Consider a thin uniform and rigid rod of mass m and length L. A small ball of mass M is attached to one end of the rod. The other end of the rod is suspended from the ceiling and the system is free to oscillate about the suspension point without friction. Compute the period of the small oscillations (in a plane) of this system. Verify that you obtain the expected result when $M \gg m$.

Answer

Problem 2

Question

Consider a semi-infinite one dimensional potential well. The potential is infinite at $x = 0$, it is zero for $0 < x < a$, and it has the finite value $V_0 > 0$ for all $x > a$. Compute the minimum value of a for which such a potential can confine a particle of mass m.

Answer

Problem 3

Question

The electron has mass $m_e = 0.511$ MeV$/c^2$. The top quark has mass $m_t = 173$ GeV$/c^2$. A machine produces a beam of electrons, of energy E_1 each. A second machine produces a beam of positrons, of energy E_2 each. The two beams are made to collide head on. A total amount of energy can be given to the electrons and positrons beams, in whatever ratio; namely $E_1 = xE$, $E_2 = (1-x)\,E$. (i) For any value of x, compute the threshold energy E that allows the production of pairs of real top and anti-top quarks when one electron collides with a positron (you may disregard m_e when compared to m_t). (ii) Which choice of x gives the smallest value of E?

Answer

Problem 4

Question

A very long linear solenoid is made of n circular loops per unit length. The area of each loop is A. The current in this solenoid is increased linearly with time, $I = \alpha t$, where α is a constant. (i) What is the magnetic field inside this solenoid?

The solenoid is placed perpendicularly to a planar circuit, as shown in Figure 1 (the solenoid extends both inside and outside the page, for a distance much greater than the dimensions of the circuit; the arrows on the figure show the direction of the current in the loops of the solenoid). The circuit shown in the figure consists of two resistors, of resistances R_1 and R_2 , and two voltmeters. The internal resistances of the two voltmeters are much greater than R_1 and R_2 . (ii) What are the magnitudes of the potential differences V_1 and V_2 measured by the two voltmeters?

Answer

Problem 5

Question

Consider two identical and coaxial superconducting loops. Each loop has self-inductance L. Initially, the two loops are very far apart from each other, and a current I flows in each of them; the currents in the two loops have the same direction. Starting from this initial configuration, the two loops are then "translated" one on the top of the other, and superimposed (you can assume that they do not touch, although their distance becomes negligible). What is the final current in each of them? What are the initial and final energies of the system?

Answer

Problem 6

Question

Two parallel perfectly black planes are in a vacuum, and are kept at constant and different temperatures T_1 and T_2. Denote by Φ the heat flux between these two planes. If a third perfectly black plane is inserted between these two planes, the system reaches a new steady state, for which the flux between the two external plates is $\Phi' = 6\Phi$. Compute the ratio Φ'/Φ.

Answer

Problem 7

Question

A nonrelativistic particle of mass m and electric charge q is in the ground state of a one-dimensional simple harmonic oscillator potential $V(x) = m\omega^2 x^2/2$. Recall that the normalized wavefunction for this state is

$$\psi = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \tag{S2010 I 7.1}$$

At some moment, a uniform electric field in the x direction is switched on very quickly (i.e., on a timescale which can be regarded as instantaneous for this problem), and is then kept constant. (i) Show that the new (i.e., with the electric field switched on) potential to which the particle is subject is of the simple harmonic oscillator type. (ii) Compute the probability that the particle is found in the ground state of this new potential.

Answer

Problem 8

Question

The latent heat of melting for ordinary ice is 334 J/g. Use this and your own experience on how the volumes of ice and water differ to determine the sign and estimate the slope of the melting curve for water in the $p - T$ (pressure and temperature) plane.

Answer

Problem 9

Question

An engine with 1 mol of an ideal gas starts at $V_1 = 26.9$ liters and performs a cycle consisting of four steps:

1. Heating at constant pressure to twice its initial volume, $V_2 = 2V_1$.
2. Isothermal expansion at T_2 to $V_3 = 3V_1$.
3. Cooling at constant volume to $T_1 = 250\text{K}$.
4. Isothermal compression to its original volume V_1.

Assume that the molar heat capacity at constant volume for this gas is $C_V = 21$ J/K. (i) Calculate the P, V, T (pressure, volume, temperature) points, and draw the engine cycle on a $P - V$ diagram. (ii) Determine the efficiency of this engine.

Answer

Problem 10

Question

A particle of mass M is initially moving along the $x-$axis, with constant speed v (as measured in the laboratory frame), which can vary from 0 to near the speed of light. This particle decays into two identical particles of mass m, with isotropic probability in its rest frame. In the following, unprimed quantities are in the laboratory frame, while primed quantities are in the rest frame of the initial particle. Choose the x and x' axis of these two frames to coincide. Denote by $\mathbf{p}$ the momentum of one of the decay products. Choose the axes such that $p_z = p'_z = 0$. Denote by θ the angle in the laboratory frame between the x-axis and the velocity of this decay product ($\theta = 0$ if the decay product moves in the direction of the initial particle). The corresponding angle θ' in the rest frame of the initial particle is shown in Figure 2. (i) For a given θ', compute p'_x , p'_y, p_x, p_y, and determine the relation between θ and θ'. (ii) Consider a beam of many such initial particles, all moving along the same straight line with velocity v. Consider the value of v for which, in the laboratory frame, half of the decay products are emitted inside a cone forming an angle $\theta \leq \theta_0$ with the direction of the initial beam. Find the relation between θ_0 and v, as v varies from 0 to the speed of light.

Answer

Problem 11

Question

A cook has a spherically shaped soup spoon. On looking into the concave side he sees his inverted image 4 cm from the bottom of the spoon (see Figure 3). Without changing his distance to the spoon, he turns it over, and sees an erect image of himself 3 cm from the bottom of the spoon. What is the radius of curvature of the spoon?

Answer

Problem 12

Question

In one day in 1987, the IMB detector observed 8 neutrino interactions. The normal background interaction rate in the detector was two a day. (i) What is the probability of eight background events being detected in one day? (ii) In fact, all those neutrinos occurred in a 10 second period. What is the probability that all those events were due to a background fluctuation?

Answer

Spring 2010 Part II

Problem 1

Question

Two identical objects A and B, of mass m each, are connected by a spring, of spring constant k. At $t = 0$ the two objects are at rest, and the spring is in its equilibrium position. For $t > 0$, the object A is subject to an external force $F_{ext} = F\cos(\omega t)$, with F and ω constant, as shown in Figure 1. Compute the motion of the object B for any $t \geq 0$. Neglect all friction.

Answer

Problem 2

Question

The neutron has the magnetic dipole moment

$$\boldsymbol{\mu} = \gamma \mathbf{S}$$

where γ is a constant, and $\mathbf{S}$ is the spin of the neutron.

A nonrelativistic neutron with momentum k is moving in a uniform and constant magnetic field. The interaction between the neutron magnetic dipole moment and the magnetic field gives a term in the Hamiltonian H of this system. (i) Write down the complete Hamiltonian H.

(ii) Assume that the magnetic field is $\mathbf{B} = (0, 0, B_z)$. What are the possible energies for the neutron? What are the corresponding normalized wave functions? (To get the normalization, require that there is a probability one that the neutron is at some place in a large volume L^3).

(iii) Answer the same questions as in part (ii), in the case of a magnetic field $\mathbf{B} = (B_x, 0, B_z) \equiv |\mathbf{B}| (\sin(\theta), 0, \cos(\theta))$.

(iv) Assume now that B_x is very small, and can be treated as a perturbation on the problem solved at point (ii), where B_x was taken to vanish. Starting from the unperturbed solutions obtained in (ii), compute the possible energies of the neutron to first order in B_x. Compare with the exact energies obtained in (iii).

Answer

Problem 3

Question

Let $Z_1(m)$ be the partition function of a single (quantum) particle of mass m in a volume L^3, at the temperature T.

(i) Consider a system of two such particles, assuming that they do not interact. Denote by $Z_{2,\text{dist}}(m)$ the partition function of the system assuming that the two particles are distinguishable. Express this quantity in terms of $Z_1(m)$.

(ii) Assume now that the two particles are indistinguishable spin zero bosons. Denote by $Z_{2,\text{bose}}(m)$ the partition function for this system. Express this quantity in terms of $Z_1(m)$ and $Z_1(m/2)$.

(iii) Comparing the cases (i) and (ii), calculate (to lowest order in the quantum effects) the correction to the expectation value of the energy of the two particle system due to Bose statistics. In which regime is the correction negligible?

Answer

Problem 4

Question

Consider a system of two particles, with identical masses, orbiting in a circle around their center of mass. (i) Show that the gravitational potential energy of the system is -2 times the total kinetic energy.

(ii) This relation is true, on average, for any system of particles held together by their mutual gravitational attraction: $\bar{U}_{\text{potential}} = -2\bar{U}_{\text{kinetic}}$, where $\bar{U}$'s are the total amount of potential and kinetic energies, averaged over some sufficiently long time. Suppose that you add a small amount of energy to such system, and then you wait until it equilibrates. Will the particles in the system, on average, move faster, or more slowly? Explain.

(iii) Compute the potential energy for a uniform spherical distribution of particles of radius R and total mass M.

(iv) Assume that a star can be modeled by an ideal gas of particles obeying classical statistics, at the same temperature T, which interact among themselves only gravitationally. Estimate the temperature of a star of mass $M = 2 \times 10^{30}$Kg and radius $R = 7 \times 10^{8}$m. Assume for simplicity that the star contains only protons and electrons.

Answer

Problem 5

Question

Consider a uniform infinitely long cylindrical wire, of cross section area A, with a current I flowing through it. Consider a charged object, of charge $q > 0$, moving parallel to the wire, with speed v. The object is outside the wire, at the distance d from it $d \gg \sqrt{A}$. The wire is neutral, and the object moves in the direction opposite to the flow of the current in the wire.

(i) Compute the magnitude and direction of the magnetic force $\mathbf{F}$ acting on the charged object.

(ii) Assume the following idealized situation for the wire: the wire is made of only protons and electrons, uniformly distributed within it. The proton and electrons have the same number density n (n has dimension of inverse volume). The protons are at rest, while all the electrons move with the same velocity $\mathbf{v}$. Assume that this velocity is equal (both in magnitude and direction) to that of the outside object. Express the current I in terms of v (and of any other relevant parameter), and insert this expression in the formula for the magnetic force computed in (i).

All the above statements are made by an observer $\mathcal{O}$ at rest with respect to the wire. Consider now the same situation in the rest-frame of the outside charged object.

(iii) Does the object experience a magnetic force in this frame?

(iv) Compute the number densities of protons $\left(n'_+\right)$ and electrons $\left(n'_-\right)$ inside the wire in this frame (hint 1: the electric charge of any individual particle is the same in both frames; hint 2: notice that, due to the symmetry of the problem, there is a simple relation between the ratio n'_+/n and the ratio n'_-/n).

(v) Compute the linear charge density of the wire in this frame (charge per unit length along the wire). Compute the force $\mathbf{F}'$ acting on the outside object in this frame.

(vi) Show that the resulting ratio $\mathbf{F}'/\mathbf{F}$ is only function of the γ factor between the two frames, and of no other parameters. Show that this result is the one you would have expected, given that $\mathbf{F} = \Delta\mathbf{p}/\Delta t$, $\mathbf{F}' = \Delta\mathbf{p}'/\Delta t'$, and how $\Delta\mathbf{p}$ and Δt are related to $\Delta\mathbf{p}'$ and $\Delta t'$.

Answer

Problem 6

Question

A particle of mass m is confined to slide on the surface of an "upside-down" cone with semi-angle α, as shown in Figure 2, and is subject to the constant gravitational field of the Earth surface. The axis of the cone is on the z-axis. Neglect any form of friction for points (i) and (ii).

(i) Write down the Lagrangian for this particle, using the coordinates r and θ, defined by $x = r\cos(\theta)$ and $y = r\sin(\theta)$ (notice that r and θ completely specify the position of the particle on the surface of the cone). Write down the Euler-Lagrange equations, obtained from this Lagrangian, that describe the motion of the particle.

(ii) For appropriate speed $|\mathbf{v}|$, the particle can move on a horizontal, and therefore circular, trajectory with $z = \bar{z}$ =constant. Write down the relation between $\bar{z}$ and the speed. Write down the total energy for the particle in this motion.

(iii) For this part only, assume that the cone is filled by some viscous medium, so that the particle is subject to a dragging force $\mathbf{F}_{\text{drag}} = -b\mathbf{v}$, where b is constant and $\mathbf{v}$ is the velocity of the particle. Assuming that the particle is initially (at $t = 0$) on a circular horizontal orbit, with height $\bar{z}_0$, and that the effect of the drag is small, so that the orbits of the particle can be approximated as circular at all times (with a very slowly decreasing radius, due to the drag), compute the time evolution of the height of the particle $\bar{z}\,(t)$.

Answer

Fall 2010 Part I

Problem 1

Question

A positively charge particle Σ^+ decays into a neutron n and a pion π^+. Both the neutron and pion are observed to move in the same direction as the Σ^+ was originally moving, with momentum $|\mathbf{p}_n| = 4702\text{MeV/c}$, and $|\mathbf{p}_{\pi^+}| = 171\text{MeV/c}$. What is the mass of Σ^+? The rest energy of the neutron and the charged pion are, respectively, $E_{0n} = 940\text{MeV}$, and $E_{0\pi^+} = 140\text{MeV}$.

Answer

We need to use the invariance of the relativistic four momentum for this problem. Let $p^\mu_{\Sigma^+} = p^\mu_n + p^\mu_{\pi^+}$, then the relativistic invariant is

$$p^2_{\Sigma^+} = m^2_{\Sigma^+} = m^2_n + m^2_{\pi^+} + 2p_n \cdot p_{\pi^+},$$

where $p_n \cdot p_{\pi^+} = E_n E_{\pi^+} - |\mathbf{p}_n||\mathbf{p}_{\pi^+}|$. Plugging in for the above values, and taking the positive square root, one finds

$$m_{\Sigma^+} = 1189.3\text{MeV/c}.$$

Problem 2

Question

The spin-orbit interaction for an electron in a hydrogen atom is governed by the Hamiltonian

$$H_{SO} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{2m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

where ϵ_0 is the vacuum permitivity, e and m are the electron's electric charge and mass, respectively, c is the speed of light, r is the radial distance of the electron from the proton, and $\mathbf{L}$ and $\mathbf{S}$ are the orbital and spin angular momentum of the electron.

1. Compute the energy correction due to the spin-orbit term. You may need the following identity:

   $$\left\langle\frac{1}{r^3}\right\rangle = \frac{2}{a^3n^3l\left(l+1\right)\left(2l+1\right)}$$

   where a is the Bohr radius and n and l are the principle and orbital quantum numbers, respectively (notice the identity holds for $l \neq 0$).

2. Describe the energy level splitting for $n = 2$ (use spectroscopic notation).

Answer

Problem 3

Question

Consider the positronium, namely a bound state of a positron and an electron. What is the corresponding Bohr radius? What is the energy corresponding to the $(n = 2) \to (n = 1)$ transition?

Answer

In the hydrogen atom, the reduced mass may be neglected in the calculations since m_{e^-}/m_p. In this case, the Bohr radius is instead

$$a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

where $\mu = m_{e^-}m_{e^+}/\left(m_{e^-} + m_{e^+}\right) = \frac{1}{2}m_{e^-}$ is the reduced mass. The energy corresponding to the $(n = 2) \to (n = 1)$ transition is given by

$$\Delta E = E_2 - E_1 = E_1\left[\frac{1}{2^2} - \frac{1}{1^2}\right] = -\frac{3}{4}E_1$$

where the ground state binding energy is $E_1 = -\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = -\frac{m_e}{4\hbar}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = -6.8$ eV. Thus

$$\boxed{\Delta E = -\frac{3E_1}{4} = 5.1\text{eV}} \tag{F2010 I 3.1}$$

Problem 4

Question

Two conductors of arbitrary shape are placed (without touching each other) in a liquid with a uniform conductivity σ. At $t = 0$ a total charge of $+Q_0$ is placed on one of the conductors, and $-Q_0$ on the other. Derive the time dependence of the charge on the conductors as a function of time.

Answer

Problem 5

Question

Wind-driven currents in a body of water on the Earth spiral down due to a combination of viscous and Coriolis forces. The pitch of this spiral (called Ekman spiral in oceanography) is a length λ which depends on the water density ρ, the viscosity η, and the angular speed of the Earth rotation ω. Assuming that $\lambda \propto \rho^a \eta^b \omega^c$, use dimensional analysis to find the exponents a, b, and c.

Answer

Problem 6

Question

A container of volume V is divided into two parts by a sliding partition. On one side there is one mole of an ideal gas made of spin 1/2 particles, and on the other side one mole of an ideal gas consisting of spin zero particles. The two gases have the same temperature T. Find the ratio of the volumes occupied by the two gases (i) at $T = 0$ and (ii) at very high T.

Answer

Problem 7

Question

A cylindrical container of total volume $4V$, thermally insulated, is separated into two compartments of volumes V and $3V$ by a non-insulated partition. The partition is initially fixed: the smaller compartment holds one mole of an ideal gas, and the larger one six moles of a different ideal gas. The system has the temperature T_0. The partition is then allowed to slide inside the container (see Figure 2) until the system reaches equilibrium.

1. What is the final temperature?
2. What are the final volumes?
3. What is the change in entropy in this process?

Answer

Problem 8

Question

Consider a monovalent simple cubic metal in which the interactions between the electrons and the lattice are so weak that the electrons can be treated as free. (i) Calculate the Fermi wavevector k_F in terms of the lattice spacing a. (ii) Show that the minimum energy that a photon must have to be absorbed by this metal is (approximately) 0.063 times the Fermi energy. Hint: You may use free electron bands in the reduced zone scheme.

(Partial) Answer

The definition of the Fermi wavevector is given by $k_F = \left(3\rho\pi^2\right)^{1/3}$ where $\rho = Nq/V$ is the free electron density; N is the number of atoms, q is the number of electrons (typically 1 or 2), and V is the lattice volume. Since the simple cubic metal is monovalent, $q = 1$ and $V = a^3$, thus

$$k_F = \left(\frac{3N\pi^2}{a^3}\right)^{1/3}. \tag{F2010 I 8.1}$$

Problem 9

Question

An astronaut is on the surface of a spherical asteroid, with a uniform density equal to the average density of Earth. Estimate the condition on the radius of the asteroid, for which the astronaut can escape from the asteroid with a jump (give a numerical answer).

Answer

Problem 10

Question

A yo-yo consists of two disks of radius R_2 connected by an axle of radius $R_1 < R_2$ (see Figure 3). The yo-yo descends under the influence of gravity by the unwinding of a string wrapped around its axle (the top end of the string is kept fixed). The total mass of the yo-yo is m, and the axle and the string have negligible mass. Compute the downward acceleration of the yo-yo's center of mass, and the tension in the string.

Answer

Fall 2010 Part II

Problem 1

Question

Consider the system shown in Figure 1: Two objects, of mass m_1 and m_2 , can be treated as point-like. Each of them is suspended from the ceiling by a wire of negligible mass, and of length L. The two objects are connected to each other by a spring, of spring constant k. The spring is relaxed when the two wires are vertical, as shown in the Figure. Denote by θ_1 and θ_2 the angles that the two wires form with the vertical, for an arbitrary position of the two masses ($\theta_1 = \theta_2 = 0$ in the figure). Consider small oscillations of the two masses in the plane of the Figure, about the equilibrium position shown in the Figure.

(i) Find the angular frequencies ω of the two normal modes of the system (in other terms, the eigenfrequencies of the system). (ii) Provide the physical explanation of why the eigenfrequencies are what they are. (iii) Find the time evolution $\theta_1\left(t\right)$ and $\theta_2\left(t\right)$ for small oscillations of the two objects, starting at rest from the initial values $\theta_1 = 0$, and $\theta_2 = \epsilon$.

Answer

Problem 2

Question

The Hamiltonian for a rigid body is

$$H = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$$

where I_i are the principal moments of inertia and $(L_1, L_2, L_3) = (L_x, L_y, L_z)$ are the angular momentum operators. This Hamiltonian describes the rotational spectrum of molecules.

1. First consider the case $I_1 = I_2 = I_3 = I$ - that is, a spherical top, such as methane. Write down a formula for the energy levels in terms of an appropriate quantum number.
2. Next consider the case $I_1 = I_2 = I_0 \neq I_3$ - that is, a symmetric top, such as ammonia. Show that $[L_3, H] = 0$ and $[L_2, H] = 0$, and write the Hamiltonian in terms of L_3 and L_2. Then write down a formula for the energy levels in terms of appropriate quantum numbers. Indicate the allowed ranges of the quantum numbers and indicate if there are any energy degeneracies.
3. Now consider a slightly asymetric top, such as water molecule, for which we can write

$$\begin{aligned} I_1 &= I_0 - \alpha \\ I_2 &= I_0 + \alpha \end{aligned}$$

where α is small. Write down the Hamiltonian for this system as a sum of the symmetric top Hamiltonian (from part (ii)) and a perturbation term. Find the shifts in energy levels relative to those of the symmetric top, using perturbation theory to first order in α. Be complete and list the energy shifts for all values of the two quantum numbers in part (ii).

Answer

Problem 3

Question

A nonrelativistic particle of mass m moves in one dimension in a square well potential with walls of infinite height at a distance L apart (to be concrete, you may take $V = 0$ for $0 < x < L$ and $V = +\infty$ elsewhere). At time $t = 0$ the particle is in a state with equal admixtures (with zero relative phase) of the two lowest energy eigenstates of the system.

1. Write down the wavefunction for this particle at $t = 0$ and at an arbitrary later time t.
2. Calculate and plot as a function of time the probability that the particle will be found in the right-hand side of the well, i.e. at some $x > L/2$.

Answer

Problem 4

Question

Consider a gas of (nonrelativistic) spin 1/2 fermions, of mass m, in the volume V. Denote by $n_\pm$ the densities of the fermions with spin up and spin down, respectively.

1. Assuming that the ground state for this system can be described as two Fermi spheres, one for spin up and the other for spin down, find the ground state kinetic energy in terms of n_+ and n_-.
2. Assuming now that the average densities deviate slightly from average, $n_\pm \approx (n/2)(1 \pm \delta)$, expand the kinetic energy to the lowest nontrivial order in δ.
3. Assume that the interactions between these fermions can be described in terms of a potential energy term U of the form:
$$\frac{U}{V} = \alpha n_+ n_- \tag{F2010 II 4.1}$$
Add the potential energy to the kinetic energy obtained above, and find the total energy to lowest nontrivial order in δ. Show that for small α one has $\delta = 0$ in the ground state, but that for $\alpha > \alpha_c$ where α_c is a critical value that you should find as a function of n, the ground state is ferromagnetic (that is, $\delta \neq 0$).

Answer

Problem 5

Question

A solenoid, of radius a and length $L \gg a$, has n turns per unit length of wire carrying the current I. An insulating cylindrical shell with negligible thickness, radius $b > a$, and length $L \gg b$, is placed with the axis of the shell coinciding with the axis of the solenoid. The shell has a total mass M and a total charge Q, distributed uniformly on it. The current through the solenoid is reduced from $I = I_0$ to $I = 0$ over some time interval.

1. Discuss (very briefly) what happens to the shell during this time interval. Compute the velocity acquired by the shell after the current of the solenoid has been dropped to zero. Ignore any effects of the magnetic field produced by the shell (assume that the shell can move without any mechanical friction).
2. Answer the same questions in part (i), including the magnetic field induced by the shell. Discuss under which condition the magnetic field produced by the shell can be ignored.
3. Derive a relation between the initial vector potential of the magnetic field of the solenoid at the radius of the shell and the final velocity of the shell (the approximated one obtained in part (i)). Notice that the vector potential is not uniquely defined from the magnetic field; however, choose the vector potential for which the relation to be derived is as simple as possible.

Answer

Spring 2011 Part I

Problem 1

Question

A monoenergetic muon beam is produced at point A and directed to point B, which is 20 km away. Suppose that only 2/3 of the muons reach point B and the remainder decay in flight. Find the energy of the muons. In the rest frame of a muon, its mass and lifetime are $105.7\ \mathrm{MeV}/c^2$ and 2.2×10^{-6} s, respectively.

Answer

Problem 2

Question

A wheel consists of a thin cylindrical steel shell of radius R, mass M, and density ρ mounted on light but very strong spokes. The wheel rotates around a vertical axle, which coincides with its axis of symmetry, and is used to store energy.

1. Find the stress in the steel shell when the angular speed of rotation is ω.
2. If steel breaks apart when the stress reaches 3×10^8 N/m^2 , what is the maximum energy that can be stored per unit volume?

Answer

Problem 3

Question

Suppose that Chicago and Minneapolis are connected by an underground train that is confined to run on a straight track of 600 km between the two cities. The train is released from rest at either city and powered solely by gravity (ignore all dissipative forces). Ignore the rotation of the earth and assume that it is a uniform sphere of $6 \times 10^2 4$ kg in mass and 6400 km in radius. How long does a one-way trip last and what is the maximum speed of the train?

Answer

Problem 4

Question

A long cylindrical capacitor has an inner cylinder of adjustable radius and a thin outer cylindrical shell of fixed radius b. The inner cylinder is at a fixed potential V and the outer shell is at a fixed zero potential. You need to adjust the radius of the inner cylinder so as to minimize the electric field at the surface of the inner cylinder. Find this radius R_{in} of the inner cylinder.

Answer

Problem 5

Question

A parallel-plate capacitor has two plates each of area A and is connected to a battery of voltage V . The initial separation between the two plates is d.

1. If one plate is fixed and the other is moved by an external force to double the separation, what is the work done by the external force?
2. Show that the work in (1) equals the net change of energy in the battery-capacitor system.

Answer

Problem 6

Question

A non-relativistic particle of mass m is in the potential $V(x) = -\alpha\delta(x)$ with $\alpha > 0$. How many bound states are there? Find the energy eigenvalue and the expectation value of x^2 for each bound state.

Answer

Problem 7

Question

The sun radiates photons at a total power of $4 \times 10^2 6$ Watts, which is supplied by a series of reactions effectively converting four protons into a ^{4}He nucleus.

1. Write down the effective net reaction of converting four protons into a 4 He nucleus by considering conservation laws.
2. By using the values of the particle masses provided on the cover sheet, estimate the total number of neutrinos emitted by the sun per second.

Answer

Problem 8

Question

The speed of sound in a gas is $v = \sqrt{(\partial P/\partial \rho)_S}$, where P, ρ, and S are the pressure, mass density, and entropy of the gas, respectively. A standing sound wave forms in a tube filled with iodine vapor (an ideal gas) at 400 K. The frequency of the wave is 1000 Hz and the distance between the adjacent nodes is 6.77 cm. Determine whether iodine vapor is a monoatomic or diatomic gas. The atomic weight of iodine is 127.

Answer

Problem 9

Question

Derive the relation between the pressure and energy density of black-body radiation. You may do this by considering the collisions of photons with the wall of a cavity enclosing such radiation, or by any other way that you may choose.

Answer

Problem 10

Question

A windmill has blades of 15 meters in diameter. If the wind is blowing into the windmill at a speed of 12 m/s and the downstream air moves at a speed of 10 m/s, estimate the power generated by the windmill. The density of air blown into the windmill is 1.3 kg/m^3 .

Answer

Spring 2011 Part II

Problem 1

Question

A pendulum, which consists of a light rod of length l and a bob of mass m, is attached to the ceiling. A second identical pendulum is attached to the bob of the first. Denote the angle each rod makes with respect to the vertical as θ_1 and θ_2, respectively (see Figure 1). Assume that both angles are small for all time t. Find the relative amplitudes and phases of $\theta_1(t)$ and $\theta_2(t)$ when the system is oscillating in the normal modes, as well as the corresponding frequencies.

Answer

Problem 2

Question

In a region with a uniform vertically upward magnetic field of magnitude B, two long straight conducting rails with negligible resistance are set up in a horizontal plane with distance l between them (see Figure 2). Their left ends are connected to a capacitor of capacitance C. A metal rod of mass m and resistance R can move on the rails without friction. At time $t = 0$, there is no charge on the capacitor and the rod is released with a velocity v_0 to the right.

1. Find the charge on the capacitor as a function of t.
2. Find the terminal velocity of the rod.
3. Show that energy is conserved by considering changes between the initial and terminal states.

Answer

Problem 3

Question

1. A non-relativistic particle of mass m is in the ground state of the potential

$$V\left(x\right)=\begin{cases}0, & 0<x<a\\ \infty, & \text{elsewhere.}\end{cases}$$

At some point of time, the potential suddenly changes to

$$V\left(x\right)=\begin{cases}0, & 0<x<2a\\ \infty, & \text{elsewhere.}\end{cases}$$

Some time later, a measurement of the energy of the particle is made. Find the probability that the particle will be measured to have the ground-state energy for the new potential.

2. Do the same for the three-dimensional case where the initial potential is

$$V\left(x\right)=\begin{cases}0, & 0<r<a\\ \infty, & \text{elsewhere.}\end{cases} \tag{S2011 II 3.1}$$

and the new potential is

$$V\left(x\right)=\begin{cases}0, & 0<r<a\\ \infty, & \text{elsewhere.}\end{cases} \tag{S2011 II 3.2}$$

It is suggested that you (a) derive or write down the radial part of the Schrödinger equation that needs to be satisfied, and (b) note that the equation only needs to be solved for the ground state.

Answer

Problem 4

Question

Consider a single electron in the Coulomb potential of a nucleus with proton number Z. Assume that the electric charge of the nucleus is distributed uniformly within a sphere of radius R.

1. Calculate the Coulomb potential energy $V(r)$ for the electron as a function of radius r (with the origin at the center of the nucleus).
2. The ground-state wave function for the electron in a hydrogen-like atom with a pointlike nucleus is $\psi(\mathbf{r}) \propto \exp(-r/a)$, where a is a constant. Use this information to make a leading-order estimate of the shift in the ground-state energy due to the Coulomb interaction between the electron and a nucleus of finite R (relative to the case of a point-like nucleus).
3. Evaluate the energy shift in (2) for $Z = 81$ and $R = 7$ fm.

Answer

Problem 5

Question

An ideal gas of N spinless atoms has volume V, temperature T, and partition function Z_0 .

1. By assuming that the atoms obey Maxwell-Boltzmann statistics, find an expression for Z_0 .
2. Now consider that each atom has two internal energy levels with energy ϵ and $\epsilon + \Delta$, respectively. Find the new partition function.
3. Calculate the specific heat at constant volume for the case in (2).

Answer

Fall 2011 Part I

Problem 1

Question

An elevator operator in a skyscraper, being a very meticulous person, put a pendulum clock on the wall of the elevator to make sure that he spends exactly 8 hours a day at his work place. Over the course of his work day, he records that the time during which the elevator has acceleration a is exactly equal to the time during which it has acceleration $-a$. Does the elevator operator work, in actual time, (1) more than 8 hours, (2) exactly 8 hours, or (3) less than 8 hours? Why?

Answer

The nominal period of a pendulum is

$$T_{nom} = 2\pi\sqrt{\frac{\ell}{g}}$$

but within the elevator, the acceleration g is not going to be constant and will rather depend on the acceleration of the elevator. Therefore,

$$T_{\uparrow} = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad T_{\downarrow} = 2\pi\sqrt{\frac{\ell}{g-a}}$$

for the upward and downward cases, respectively.

Since the elevator operator observed that equal time was spent going up as was spent going down, so he must have observed N oscillations in both cases. In order to compare to the actual time, we simply compare the elevator's total time measurement with that of a stationary clock.

$$NT_{\uparrow} + NT_{\downarrow} \stackrel{?}{=} 2NT_{nom}$$

$$\begin{aligned}
2\pi N\sqrt{\frac{\ell}{g+a}} + 2\pi N\sqrt{\frac{\ell}{g-a}} &\stackrel{?}{=} 4\pi N\sqrt{\frac{\ell}{g}} \\
\sqrt{\frac{1}{g+a}} + \sqrt{\frac{1}{g-a}} &\stackrel{?}{=} 2\sqrt{\frac{1}{g}} \\
\sqrt{\frac{g}{g+a}} + \sqrt{\frac{g}{g-a}} &\stackrel{?}{=} 2
\end{aligned}$$

Use the test value $a = 5$ for comparison (with $g = 10$)

$$\boxed{2.23 > 2} \qquad \text{(F2011 I 1.1)}$$

Therefore the elevator operator actually spends more than 8 hours in the elevator during his shift.

Problem 2

Question

A classical particle is subject to an attractive central force proportional to r^α, where r is the radius and α is a constant. Show by perturbation analysis what is required of α in order for the particle to have a stable circular orbit.

Answer

Construct the Lagrangian for the system in order to determine the equations of motion for the given central force (noting that we were given the *force* so we need to make an appropriate potential).

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad\qquad V = \frac{k}{\alpha+1}r^{\alpha+1}$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{\alpha+1}r^{\alpha+1}$$

Conservation of angular momentum is a consequence of the θ and $\dot{\theta}$ coordinates:

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] \\ 0 &= \frac{d}{dt}\left[mr^2\dot{\theta}\right] \end{aligned}$$

Noting that

$$L = |\vec{r}\times\vec{p}| = mr^2\dot{\theta}$$

we can say that

$$\dot{\theta} = \frac{L}{mr^2}$$

Then returning to the r and $\dot{r}$ coordinates in the Lagrangian,

$$\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - kr^\alpha \qquad\qquad \begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{r}} &= m\dot{r} \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right] &= m\ddot{r} \end{aligned}$$

Putting the differential equation together and substituting for the angular momentum gives

$$m\ddot{r} = \frac{L^2}{mr^3} - kr^\alpha \tag{F2011 I 2.1}$$

In the case that the orbit is circular, r must be a constant, so let $r = a$ and note that $\ddot{r} = 0$ necessarily.

$$\frac{L^2}{ma^3} = ka^\alpha$$

For simplicity, we introduce $\ell = L/m$ for the following calculation.

Returning to the differential equation, let the actual distance r be a perturbation from a circular orbit, and

Taylor expand in x where $x = r - a$.

$$
\begin{aligned}
m\ddot{x} &= \frac{m\ell^2}{a^3}\left(1 + \frac{x}{a}\right)^{-3} - ka^{\alpha}\left(1 + \frac{x}{a}\right)^{\alpha} \\
m\ddot{x} &\approx \frac{m\ell^2}{a^3}\left(1 - 3\frac{x}{a} + \dots\right) - ka^{\alpha}\left(1 + \alpha\frac{x}{a} + \dots\right) \\
m\ddot{x} &\approx ka^{\alpha}\left(1 - 3\frac{x}{a}\right) - ka^{\alpha}\left(1 + \alpha\frac{x}{a}\right) \\
m\ddot{x} &\approx -3ka^{\alpha}\frac{x}{a} - \alpha ka^{\alpha}\frac{x}{a} \\
m\ddot{x} &\approx -ka^{\alpha-1}\left(3 + \alpha\right)x
\end{aligned}
$$

To form a stable orbit, the coefficient on x must be negative, giving a simple harmonic solution. Therefore $3 + \alpha > 0$ to keep the coefficient negative and

$$\boxed{a > -3} \tag{F2011 I 2.2}$$

Problem 3

Question

A neutral conductor A with a spherical outer surface of radius R contains three cavities B, C, and D, but is solid otherwise. B and C are spherical, and D is hemispherical. Without touching A, positive charges q_B and q_C are introduced at the centers of B and C, respectively.

1. Give the amount and the distribution of the induced charges on the surfaces of A, B, C, and D.
2. Now another positive charge q_E is introduced at a distance $r > R$ from the center of A. Describe qualitatively the distribution of induced charges on the surfaces of A, B, C, and D.
3. Give the amount of the induced charges on the surfaces of A, B, C, and D for the situation in (2).

Answer (1)

An ideal conductor will not support an electric field inside the solid, so each of cavities B and C will have a surface charge to cancel the electric fields emminating from q_B and q_C respectively.

- Cavity B will have a uniform surface charge density of $-q_B/4\pi r_B^2$, where r_B is the radius of cavity B, with total induced charge $-q_B$ (because of symmetry and use of a Gaussian surface).
- Cavity C will have a uniform surface charge density of $-q_C/4\pi r_C^2$, where r_C is the radius of cavity B, with total induced charge $-q_C$ (because of symmetry and use of a Gaussian surface).

Cavity D will not have a surface charge since a Gaussian surface coincident with its boundary contains no charge.

The surface A will have total charge $q_B + q_C$ with uniform surface charge density of $(q_B + q_C)/4\pi r^2$ in accordance with the symmetry of a Gaussian surface containing the sphere as well as properties of an ideal conductor.

Answer (2)

The surfaces B, C, and D will remain unaffected since the surrounding conductor shields the cavities from electric fields produced by charge q_E. The distribution on surface A will shift so that the negative charge concentration is greatest on the side nearest to q_E with an increasingly positive distribution towards the opposite side.

Answer (3)

The surface of A will still contain the same total charge $q_B + q_C$ since only a redistribution of induced charges occurred along the surface. Similarly, because surface B, C, and D are shielded from the electric field of q_E by conductor A, the total charges along their surfaces remains unchanged as well.

Problem 4

Question

The dielectric strength of air at standard temperature and pressure is $3 \cdot 10^6$ V/m. What is the maximum intensity in units of $\mathrm{W/m^2}$ for a monochromatic laser that can be used in the laboratory?

Answer

Failure of a dielectric occurs when the energy density in the dielectric is great enough to overcome the ionization energy of the constituent atoms. This suggests that an electric field of greater than $3 \cdot 10^6$ V/m would cause this ionization to occur.

Starting here, We can calculate the energy density of the electric field at any point in space by

$$U_{em} = \frac{\varepsilon_0}{2} E^2$$

(where we've used the vacuum energy density since air differs very little from the vacuum permitivity).

Then the power transmitted by the laser is $P = cU_{em}$, so plugging in the numbers,

$$P = \frac{1}{2}\left(8.854 \cdot 10^{-12}\,\frac{\mathrm{C^2}}{\mathrm{N\,m^2}}\right)\left(3 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}\right)\left(3 \cdot 10^6\,\frac{\mathrm{V}}{\mathrm{m}}\right)^2$$
$$P = 1.19 \cdot 10^{10}\,\frac{\mathrm{W}}{\mathrm{m^2}}$$

$\boxed{\text{The maximum power of a laser usable in the lab is } 1.19 \cdot 10^{10}\ \mathrm{W/m^2}.}$

Problem 5

Question

What is the minimum energy of the projectile proton required to induce the reaction $p + p \to p + p + p + \bar{p}$ if the target proton is at rest?

Answer

Energy and momentum must be conserved. At the minimum allowed energy, the resultant 4 proton/anti-protons will be collinear with no relative momentum with respect to one another, so the momentum equation in the lab frame is simply

$$p_i = 4p_f \tag{F2011 I 5.1}$$

Similarly, the resultant (anti-)protons are indistinguishable, so they will all have equivalent energy E_f. The initial protons have different energies since one is at rest in the lab frame while the other is moving, leading to the energy equation

$$\sqrt{p_i^2c^2 + m_p^2c^4} + m_pc^2 = 4\sqrt{p_f^2c^2 + m_p^2c^4}$$

Substituting the momentum relation into the equation, squaring, and simplifying,

$$\begin{aligned}
\sqrt{16p_f^2c^2 + m_p^2c^4} + m_pc^2 &= 4\sqrt{p_f^2c^2 + m_p^2c^4} \\
16p_f^2c^2 + m_p^2c^4 + m_p^2c^4 + 2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 16p_f^2c^2 + 16m_p^2c^4 \\
2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 14m_p^2c^4 \\
16p_f^2c^2 + m_p^2c^4 &= 49m_p^2c^4 \\
p_f^2 &= 3m_p^2c^4
\end{aligned}$$

Therefore,

$$p_i^2 = 48m_p^2c^4$$

and

$$E_1 = \sqrt{49m_p^2c^4}$$

$$\boxed{E_1 \approx 6.567\,\mathrm{GeV/c^2}} \tag{F2011 I 5.2}$$

Problem 6

Question

A subatomic particle has spin 1 and negative parity. It decays at rest into an e^+e^- pair, which is produced in the s and d waves. From these data determine (1) the total spin of the e^+e^- pair and (2) the intrinsic parity of e^+ relative to e^-.

Answer

Problem 7

Question

There is a uniform, vertical gravitational field with a downward acceleration of gravity g above a horizontal, perfectly elastic surface. A particle of mass m can only move above the surface. Give a rough estimate of the energy eigenvalue for the ground state of the particle.

Answer

Problem 8

Question

Assume that the atmosphere near the earth's surface is in approximate hydrostatic equilibrium, where any movement of air parcels is gentle and adiabatic. Find an expression for the pressure P of the atmosphere as a function of the height z.

Answer

Note that the pressure at a given point is due to the mass of air above the given point. Then by moving an infinitesimal distance vertically, the total mass is changed by the density of the air (which is affected by the gravitational force). This leads to the differential equation

$$\frac{dP}{dz} = \rho g$$

Then using the ideal gas equation

$$PV = Nk_BT$$

multiply and divide by the average molecular mass m of the air (in kg) which combined with the number of molecules N gives the total mass

$$PV = (Nm)\,\frac{1}{m}k_BT$$

and then divide by the volume to get the ideal gas equation in terms of the mass density

$$\begin{aligned} P &= \frac{Nm}{V}\frac{1}{m}k_BT \\ P &= \rho\frac{k_BT}{m} \\ \rho &= \frac{Pm}{k_BT} \end{aligned}$$

Finally, substitute this into the differential equation above and solve to get the atmospheric scale height equation.

$$\begin{aligned} \frac{dP}{dz} &= \frac{Pm}{k_BT}g \\ \frac{dP}{P} &= \frac{mg}{k_BT}dz \end{aligned}$$

$$\boxed{P\,(z) = P_0e^{z/\xi} \qquad \text{where } \xi = \frac{k_BT}{mg}} \tag{F2011 I 8.1}$$

Problem 9

Question

Consider a gas of atoms in a magnetic field of 10 Tesla. The nucleus of the atom has spin 1/2, magnetic moment $\mu \approx 10^{-26}$J/Tesla, and mass $m \approx 5 \times 10^{-27}$kg. The electrons in the atom have zero total angular momentum. What is the maximum number density of the gas for which the nuclei are completely polarized by the magnetic field at zero temperature?

Answer

Problem 10

Question

A new long-lived particle X is observed to decay via $X \to K^+ + K^-$. The mass of X is about 1.2 GeV$/c^2$. You wish to determine this mass to within 1% using the momenta of the K^+ and K^- from decay of X at rest. What should be the maximum relative error on your momentum measurements if you use only a single decay event? You know that the mass of K^+ and K^- is $493.677 \pm 0.013\text{MeV}/c^2$.

Answer

Fall 2011 Part II

Problem 1

Question

Mass m_1 moves freely along a fixed, long, horizontal rod. The position of m_1 on the rod is x. A massless string of length ℓ is attached to m_1 at the end and to mass m_2 at the other. Mass m_2 executes pendulum motion in the vertical plane containing the rod.

1. Find the Lagrangian of the system.
2. Derive the equations of motion and the corresponding conservation laws.
3. Assume that $x(0) = x_0$, $\dot{x}(0) = 0$, $\phi(0) = \phi_0$ ($|\phi_0| \ll 1$), and $\dot{\phi}(0) = 0$. Find $x(t)$ and $\phi(t)$ for $t > 0$.

Answer (1)

For the sliding support mass m_1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{x}^2 \\ V_1 &= 0 \end{aligned}$$

For the pendulum mass m_2:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} m_2 \left(\dot{x} + \dot{x}_2\right)^2 \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Then using $x_2 = \ell \sin\phi$ and $y_2 = -\ell\cos\phi$,

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \left(\ell^2 \dot{\phi}^2 + \dot{x}^2 + 2\ell\dot{\phi}\dot{x}\cos\phi\right) \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Putting the Lagrangian together equals the first line. Applying the small angle approximation gives the second line where the kinetic energy term involving $\cos\phi$ can be simply expanded as $\cos\phi \approx 1$, but the potential energy term must be expanded to second order so that $\cos\phi \approx 1 - \frac{1}{2}\phi^2$.

$$\boxed{\mathcal{L} = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\phi}^2 + 2\ell\dot{\phi}\dot{x}\cos\phi\right) + m_2 g\ell\cos\phi} \tag{F2011 II 1.1}$$

$$\boxed{\mathcal{L} \approx \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\phi}^2 + 2\ell\dot{\phi}\dot{x}\right) + m_2 g\ell - \frac{1}{2} m_2 g\ell\phi^2} \tag{F2011 II 1.2}$$

Answer (2)

Constructing the Euler-Lagrange equations for x and $\dot{x}$:

$$\frac{\partial\mathcal{L}}{\partial x} = 0 \qquad \begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{x}} &= \left(m_1 + m_2\right)\dot{x} + m_2\ell\dot{\phi} \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] &= \left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\phi} \end{aligned}$$

$$\left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\phi} = 0 \tag{F2011 II 1.3}$$

and for ϕ and $\dot{\phi}$:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m_2 g \ell \phi \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m_2 \ell^2 \dot{\phi} + m_2 \ell \dot{x}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] = m_2 \ell^2 \ddot{\phi} + m_2 \ell \ddot{x}$$

$$-m_2 g \ell \phi - m_2 \ell^2 \ddot{\phi} + m_2 \ell \ddot{x} = 0 \tag{F2011 II 1.4}$$

The equations of motion are:

$$\boxed{\ddot{x} + \frac{m_2}{m_1 + m_2} \ell \ddot{\phi} = 0} \tag{F2011 II 1.5}$$

$$\boxed{\ddot{\phi} + \frac{1}{\ell}\ddot{x} + \frac{g}{\ell}\phi = 0} \tag{F2011 II 1.6}$$

Answer (3)

Solve for $\ddot{x}$ and substitute into the other differential equation

$$\begin{aligned} \ddot{\phi} - \frac{1}{\ell}\frac{m_2}{m_1 + m_2}\ell\ddot{\phi} + \frac{g}{\ell}\phi &= 0 \\ \frac{m_1}{m_1 + m_2}\ddot{\phi} + \frac{g}{\ell}\phi &= 0 \\ \ddot{\phi} + \frac{g}{\ell}\frac{m_1 + m_2}{m_1}\phi &= 0 \end{aligned} \tag{F2011 II 1.7}$$

This is just the differential equation for a simple harmonic oscillator, so considering the given boundary conditions,

$$\boxed{\phi(t) = \phi_0 \cos(\omega t) \qquad \text{where } \omega^2 = \frac{g}{\ell}\frac{m_1 + m_2}{m_1}} \tag{F2011 II 1.8}$$

Then differentiating $\phi(t)$ twice and substituting into the first equation,

$$\ddot{x} = \ell \phi_0 \omega^2 \frac{m_2}{m_1 + m_2} \cos(\omega t)$$

Then integrating twice and applying the boundary conditions,

$$\boxed{x(t) = x_0 - \frac{g}{\omega^2}\frac{m_2}{m_1}\cos(\omega t)} \tag{F2011 II 1.9}$$

Problem 2

Question

A light bulb has a tungsten filament formed into a coil of 60 turns. The coil is a straight column of 3 mm in diameter and 20 mm in length. The bulb is rated 75 W for an AC source of 110 V with a frequency of 60 Hz. Find the current in the bulb as a function of time t after it is connected to the AC source at $t = 0$. Assume that the voltage of the source is of the form $V(t) = V_0 \cos(\omega t)$ and the resistance of the filament stays constant.

Answer

Problem 3

Question

A particle of mass m is in the potential

$$V(x) = \begin{cases} -\Omega_0 \delta(x) & -a/2 < x < a/2 \\ \infty, & \text{otherwise} \end{cases} \tag{F2011 II 3.1}$$

where $\delta(x)$ is the Dirac delta function and Ω_0 and a are positive parameters.

1. For the energy eigenstates that have wave functions with odd parity, find these wave functions and the corresponding eigenvalues.
2. For the rest of the energy eigenstates, find the approximate eigenvalues by treating $-\Omega_0\delta(x)$ as a perturbation. What is required of Ω_0 for the approximation to be valid?
3. For a special value of Ω_0, the energy of the ground state is exactly zero. Find this special value of Ω_0

Answer

Problem 4

Question

For the simple harmonic oscillator with the Hamiltonian

$$H_{\text{sho}} = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

where P and X are the momentum and position operators, the wave functions for the energy eigenstates are $\phi_n\left(x/a\right)$, where n is a non-negative integer, a is a constant with the dimension R of length, and $\int_{-\infty}^{\infty}\left|\psi_n\left(x/a\right)\right|^2 \mathrm{d}x = 1$. Now consider two distinguishable particles of mass m with the total Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2}m\omega^2\left[X_1^2 + X_2^2 + \left(X_1 - X_2\right)^2\right].$$

1. Find the normalized wave functions for the eigenstates of H in terms of $\psi_{n'}\left(\eta\right)$ and $\psi_{n''}$ by choosing two appropriate dimensionless position variables η and ξ.
2. Find the eigenalues of H.
3. How would the answers to the above two problems change if the two particles are identical and have spin 0?

Answer

Problem 5

Question

Consider only the rotational motion of a diatomic molecule with moment of inertia I.

1. What is the specific heat for a classical system of N such molecules?
2. In the quantum mechanical case, the energy levels for an individual molecule are

$$E_{lm} = \frac{l\,(l+1)\,\hbar^2}{2I}$$

   where $l = 0, 1, 2, \cdots$ and for each $l, m = -l, -l+1, \cdots, l$. For a system of N such molecules, express the partition function Z and the energy E as sums of well-defined quantities.
3. Calculate the specific heat in the low-temperature and high-temperature limits for the quantum mechanical case.
4. For what range of temperature is the classical result in (1) valid?

Answer

Spring 2012 Part I

Problem 1

Question

For a many particle system of weakly interacting particles, will quantum effects be more important for (a) high densities or low densities and (b) high temperatures or low temperatures for a system. Explain your answers in terms of the de Broglie wavelength λ defined as $\lambda^2 \equiv h^2/(3mk_bT)$ where m is the mass of the particles and k_b Boltzmann's constant.

Answer

(a) High density — The de Broglie wavelength gives a "size" of the particle, and in the high density limit, the wavefunctions overlap significantly so quantum effects and interactions are critical to the behavior of the system.

(b) Low temperature — Since $\lambda^2 \propto T^{-1}$, as $T \to 0$, λ increases so that again the wavefunctions overlap and quantum effects are significant.

Problem 2

Question

The ground state energy of Helium is $-79\,\text{eV}$. What is its ionization energy, which is the energy required to remove just one electron?

Answer

Using the Hydrogen solution with modifications for single-electron atoms of higher Z, we know that the ground state energy of singly ionized Helium is

$$E_{He}^{1} = 2^2\,(-13.6\,\text{eV}) = -54.4\,\text{eV}$$

Therefore, the difference between the singly-ionized and neutral ground state energies gives the first ionization energy of the Helium atom.

$$\boxed{E_i = -24.6\,\text{eV}} \tag{S2012 I 2.1}$$

Problem 3

Question

It is known that the force per unit area (F/A) between two neutral conducting plates due to polarization fluctuations of the vacuum (namely, the Casimir force) is a function of h (Planck's constant), c (speed of light), and z (distance between the plates) only. Using only dimensional analysis, obtain F/A as a function of h, c, and z.

Answer

The units of F/A are

$$\frac{F}{A} = \left[\frac{\mathrm{kg}}{\mathrm{m\,s^2}}\right]$$

The kg suggests a factor proportional to h, making the equation

$$\frac{F}{A} \sim \left[\frac{1}{\mathrm{m^3\,s}}\right] h$$

Accounting for the factor of seconds requires a c:

$$\frac{F}{A} \sim \left[\frac{1}{\mathrm{m^4}}\right] hc$$

Finally, account for all the factors of distance:

$$\frac{F}{A} \sim \frac{hc}{z^4}$$

Therefore,

$$\boxed{\frac{F}{A} \sim \frac{hc}{z^4}} \tag{S2012 I 3.1}$$

Problem 4

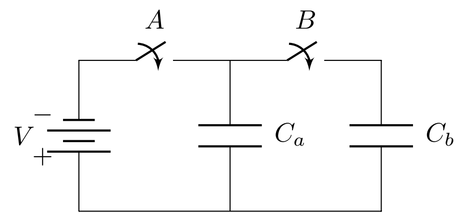
Question

In the circuit diagram opposite, initially the two identical capacitors with capacitance C are uncharged. The connections between the components are all made with short copper wires. The battery is an ideal EMF and supplies a voltage V.

(a) At first Switch A is closed and Switch B is kept open. What is the final stored g energy on capacitor C_a?

(b) Switch A is opened and afterwards Switch B is closed. What is the final energy stored in both capacitors?

(c) Provide a physical explanation for any difference between the results of parts (a) and (b), if there is one.



Answer

(a) Initially, the right side of the circuit with C_b can be ignored, so the total energy is simply the energy stored within C_a.

$$\boxed{E = \frac{1}{2}CV^2} \tag{S2012 I 4.1}$$

(b) The system is now effectively just the two capacitors on the right. Because the voltage difference is supported across both capacitors, the system can be modeled as an effective capacitor in parallel

$$C_{eff} = 2C$$

The total charge stored by the capacitors must remain the same when switching from Switch A being closed to Switch B. Initially,

$$Q = CV_i$$

and afterwards it is

$$Q = C_{eff}V = 2CV_f$$

so the final voltage across the capacitors is

$$V_f = \frac{1}{2}V_i$$

This means the total energy is

$$E = \frac{1}{2}C_{eff}{V_f}^2$$

$$\boxed{E = \frac{1}{4}CV^2} \tag{S2012 I 4.2}$$

(c) The energy is dissipated (heat, fields, etc).

Problem 5

Question

A planet of mass m moves around the sun, mass M, in an elliptical orbit with minimum and maximum distances of r_1 and r_2, respectively. Find the angular momentum of the planet relative to the center of the sun in terms of these quantities and the gravitational constant G.

Answer

We solve the problem using conservation of energy since we know that stable elliptical orbits have constant energy. The generic equation is

$$E = \frac{L^2}{2I} - \frac{GMm}{r}$$

where L is the angular momentum and I the moment of inertia. Substituting for the values at both r_1 and r_2 and equating,

$$\frac{L^2}{2m{r_1}^2} - \frac{GMm}{r_1} = \frac{L^2}{2m{r_2}^2} - \frac{GMm}{r_2}$$
$$\frac{L^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right) = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

which leads to the solution

$$\boxed{L = \sqrt{\frac{2GMm^2r_1r_2}{r_1+r_2}}} \tag{S2012 I 5.1}$$

Problem 6

Question

A particle moves in a circular orbit under the influence of a central force that varies as the n-th power of the distance. Show that this motion is unstable if $n < -3$. (Hint: Consider the centrifugal potential.)

Answer

See solution for Fall 2011 Part I, Problem 2 with the condition inverted so that *in*stability is $n < -3$ rather than stability requiring $n > -3$.

Problem 7

Question

A classical, ideal, monatomic gas of N particles is reversibly compressed *isentropically*, i.e. with the entropy kept constant, from an initial temperature T_0 and pressure P to a pressure $2P$. Find (a) the work done on the system, and (b) the net change in entropy of the system and its surroundings.

Answer

(a) An isentropic process is the same as an adiabatic process since no heat can be exchanged ($T\mathrm{d}S = Q = 0$), so we begin with the relation that PV^γ is a constant. Combining this with the ideal gas law, we can determine that

$$P^{1-\gamma}T^\gamma = \text{const}$$

where $\gamma = C_p/C_v$ is the ratio of heat capacities with $C_p = \frac{5}{2}Nk_B$ and $C_v = \frac{3}{2}Nk_B$ for a monatomic ideal gas. Using this, we solve for the final temperature of the system after compressions as

$$T_f = 2^{2/5}T_0 \approx 1.32T_0$$

Combining both of

$$\Delta U = C_v \Delta T$$
$$\Delta U = Q + W$$

where $Q = 0$, we get that

$$\boxed{W = \frac{3}{2}Nk_BT_0\left(2^{2/5} - 1\right)} \tag{S2012 I 7.1}$$

(b) Because the compression is done reversibly, by definition, $\Delta S = 0$.

Problem 8

Question

For an ideal Fermi gas of N neutral spin-$\frac{1}{2}$ particles in a volume V at $T = 0$, calculate the following:

(a) The chemical potential

(b) The average energy per particle

(c) The pressure

Answer

(a) At $T = 0$, the particles are all in the lowest state allowed by Fermi-Dirac statistics, so the chemical potential, defined by the energy required to add another particle to the system, is equal to the Fermi energy. For a particle contained within a box V, the energy per particle is

$$\varepsilon_n = \frac{\pi^2\hbar^2}{2mV^{2/3}}n^2$$

Given a Fermi energy ε_F, the maximum occupied state is

$$n_F = \sqrt{\frac{2mV^{2/3}}{\pi^2\hbar^2}}\sqrt{\varepsilon_F}$$

Equally we know that all N particles must exist within the eighth-sphere of n space, where the extra factor of 2 is because there are two spin states per n:

$$N = 2\cdot\frac{1}{8}\cdot\frac{4}{3}\pi {n_F}^3$$
$$N = \frac{1}{3}\pi\left(\frac{2m}{\pi^2\hbar^2}\right)^{3/2} V{\varepsilon_F}^{3/2}$$
$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

Therefore $\mu = \varepsilon_F$,

$$\boxed{\mu = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}} \tag{S2012 I 8.1}$$

(b) To get the total energy, we can imagine filling all N particles one at a time, so that at each step, there are N' total particles:

$$U = \int_0^N \varepsilon_F \mathrm{d}N'$$
$$U = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3}\int_0^N N^{2/3}\mathrm{d}N'$$
$$U = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3}\cdot\frac{3}{5}N^{5/3}\mathrm{d}N'$$

Therefore, the average energy per particle is U/N or

$$\boxed{\langle\varepsilon\rangle = \frac{3}{5}\varepsilon_F} \tag{S2012 I 8.2}$$

(c) From the thermodynamic relation

$$dU = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$$

we can read off the derivative that defines the pressure P as

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

Doing so, we get that

$$\frac{\partial U}{\partial V} = \frac{3}{5}N \cdot \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \left(-\frac{2}{3V}\right)$$

making the pressure

$$\boxed{P = \frac{2}{5}\frac{N}{V}\varepsilon_F} \tag{S2012 I 8.3}$$

Problem 9

Question

A transparent spherical lens, with a uniform index of refraction n, has the image of a distance object on the opposite surface of the lens when it is in vacuum. What is the index of refraction n?

Answer

Problem 10

Question

A piece of p-doped silicon has a carrier density $n = 10^{15}\,\text{cm}^{-3}$ and dimensions of $\Delta x = 10\,\text{mm}$, $\Delta y = 2\,\text{mm}$, and $\Delta z = 1\,\text{mm}$. A magnetic field of $B_z = 1\,\text{T}$ is applied in the z-direction and a current $I_x = 1\,\text{A}$ flows in the x-direction, and the voltage V_y is measured.

(a) Express the current density j_x in terms of the carrier density n and the carrier velocity v_x.

(b) Write down the equilibrium force condition that determines V_y.

(c) Find V_y in volts.

Answer

(a) The current passing through each thin cross-sectional slice of the conductor is dependent on the charge of a carrier, carrier density, and velocity of the flow.

$$I_x = en\Delta y\Delta z v_x$$

The current density is just the current passing through each point, so

$$j_x = \frac{I_x}{\Delta y\Delta z}$$

$$\boxed{j_x = nev_x} \tag{S2012 I 10.1}$$

(b) The positive carriers drift to the edge of the conductor due to the magnetic field and the holes accumulate on the opposite edge. An electric field is created between the charge separation, so an equilibrium is set up between the electric field trying to bring the opposite charges together and the magnetic drift separating them.

$$0 = e\vec{E} + \vec{v}\times\vec{B}$$

By the right-hand rule, the positive charges accumulate along $y = 0$, so $\vec{E} = E\hat{y}$. Similarly, $\vec{v}\times\vec{B} = -v_x B_z\hat{y}$:

$$0 = eE\hat{y} - ev_x B_z\hat{y}$$

Written in terms of the potential $V_y = E\Delta y$, the equilibrium condition becomes

$$\boxed{V_y = v_x B_z \Delta y} \tag{S2012 I 10.2}$$

(c) Substituting in for given quantities

$$V_y = \frac{I_x B_z}{ne\Delta z}$$
$$V_y = \frac{(1\,\text{A})\,(1\,\text{T})}{\left(10^{15}\,\text{cm}^{-3}\right)\left(1.612\cdot 10^{-19}\,\text{C}\right)(1\,\text{mm})}$$

$$\boxed{V_y = 6.24\,\text{V}} \tag{S2012 I 10.3}$$

Spring 2012 Part II

Problem 1

Question

An electron in a hydrogen atom occupies a state:

$$|\psi\rangle = \sqrt{\frac{1}{3}}\,|3,1,0,+\rangle + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle$$

where the properly normalized states are specified by the quantum numbers $|n,\ell,m,\pm\rangle$ and the $\pm$ specifies whether the spin is up or down.

(a) What is the expectation value of the energy in terms of the ground state energy?

(b) If you measured the expectation values of the orbital momentum squared $\langle L^2\rangle$, the square of the spin $\langle S^2\rangle$, and their z-components $\langle L_z\rangle$ and $\langle S_z\rangle$, what would be the result?

(c) Show that if you measure the position of the electron, the probability density for finding it an an angle specified by θ and ϕ integrated over all values of r is independent of θ and ϕ. Note, for this part you will need $Y_1^0 = \sqrt{3/4\pi}\cos\theta$ and $Y_1^1 = -\sqrt{3/8\pi}\sin\theta\exp(i\phi)$. You do *not*, however, need to know the radial functions, only that they are properly normalized and orthogonal to each other.

(d) List all additional possible states that are degenerate with the first state in the linear combination above. Note: this part can be done even if you have not answered the previous parts.

Assume now that the state $|\psi\rangle$, given above, is the initial state of an electron in a hydrogen atom.

(e) Write down the electron's state as a function of time for all $t > 0$.

(f) go through the results you obtained in parts (a) through (c) and determine which of them are time independent.

Answer

(a) Calculate the energy by sandwiching the Hamiltonian between the wavefunction:

$$\begin{aligned}
\langle E\rangle &= \langle\psi|H|\psi\rangle \\
&= \left(\sqrt{\frac{1}{3}}\,\langle 3,1,0,+| + \sqrt{\frac{2}{3}}\,\langle 2,1,1,-|\right) H \left(\sqrt{\frac{1}{3}}\,|3,1,0,+\rangle + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle\right) \\
&= \frac{1}{3}\,\langle 3,1,0,+|H|3,1,0,+\rangle + \frac{\sqrt{2}}{3}\,\langle 2,1,1,-|H|3,1,0,+\rangle \\
&\quad + \frac{\sqrt{2}}{3}\,\langle 3,1,0,+|H|2,1,1,-\rangle + \frac{2}{3}\,\langle 2,1,1,-|H|2,1,1,-\rangle
\end{aligned}$$

For every term, the wavefunctions are eigenstates of the Hamiltonian, so we extract the appropriate energy term from every bra-ket sandwhich. Then the middle two terms integrate to zero since states with different n are orthogonal while the first and last terms integrate to unity since they are properly normalized.

$$\langle E\rangle = \frac{1}{3}E_3 + 0 + 0 + \frac{2}{3}E_2$$

Each energy is related to the ground state energy by $E_n = E_0/n^2$, so

$$= \frac{1}{3}\frac{E_0}{9} + \frac{2}{3}\frac{E_0}{4}$$

$$\boxed{\langle E\rangle = \frac{11}{54}E_0 \approx -2.77\,\mathrm{eV}} \qquad \text{(S2012 II 1.1)}$$

(b) For each of the other expectation values, the process is very similar with an appropriate change for eigenvalues; specifically,

$$
\begin{aligned}
L^2 \left|n, \ell, m, \pm\right\rangle &= \ell\left(\ell+1\right)\hbar^2 \left|n, \ell, m, \pm\right\rangle \\
S^2 \left|n, \ell, m, \pm\right\rangle &= \frac{1}{2}\left(\frac{1}{2}\pm 1\right)\hbar^2 \left|n, \ell, m, \pm\right\rangle \\
L_z \left|n, \ell, m, \pm\right\rangle &= m\hbar \left|n, \ell, m, \pm\right\rangle \\
S_z \left|n, \ell, m, \pm\right\rangle &= \pm\frac{1}{2}\hbar \left|n, \ell, m, \pm\right\rangle
\end{aligned}
$$

The same restrictions that the middle terms integrate to zero because of orthogonality and the first and last terms integrate to unity still applies, so we can almost immediately conclude that

$$\boxed{\left\langle L^2\right\rangle = 2\hbar^2} \tag{S2012 II 1.2}$$

$$\boxed{\left\langle S^2\right\rangle = \frac{\hbar^2}{12}} \tag{S2012 II 1.3}$$

$$\boxed{\left\langle L_z\right\rangle = \frac{2\hbar}{3}} \tag{S2012 II 1.4}$$

$$\boxed{\left\langle S_z\right\rangle = -\frac{\hbar}{6}} \tag{S2012 II 1.5}$$

(c) In the $\left|r, \theta, \phi\right\rangle$ basis,

$$
\begin{aligned}
\left|3, 1, 0\right\rangle &= R_{3,1}\left(r\right) Y_1^0\left(\theta, \phi\right) = R_{31}\left(r\right)\sqrt{\frac{3}{4\pi}}\cos\theta \\
\left|2, 1, 1\right\rangle &= R_{2,1}\left(r\right) Y_1^1\left(\theta, \phi\right) = -R_{21}\left(r\right)\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta
\end{aligned}
$$

This means that the probability density is

$$
\begin{aligned}
\left\langle\psi|\psi\right\rangle &= \frac{1}{3}\left\langle 3,1,0|3,1,0\right\rangle + \frac{\sqrt{2}}{3}\left\langle 2,1,1|3,1,0\right\rangle + \frac{\sqrt{2}}{3}\left\langle 3,1,0|2,1,1\right\rangle + \frac{2}{3}\left\langle 2,1,1|2,1,1\right\rangle \\
&= \frac{1}{4\pi}\cos^2\theta R_{31}^2\left(r\right) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}\left(r\right)R_{31}\left(r\right) + \frac{1}{4\pi}\sin^2\theta R_{21}^2\left(r\right)
\end{aligned}
$$

Integrating over r,

$$
\begin{aligned}
\int_0^\infty \left\langle\psi|\psi\right\rangle \mathrm{d}r = \int_0^\infty &\frac{1}{4\pi}\cos^2\theta R_{31}^2\left(r\right) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}\left(r\right)R_{31}\left(r\right) \\
&+ \frac{1}{4\pi}\sin^2\theta R_{21}^2\left(r\right)\mathrm{d}r
\end{aligned}
$$

Integrating over all r, we know that $R_{n\ell}R_{n'\ell'}$ are orthonormal, so again the first and last terms' R integrates to unity and the middle term integrates to zero.

$$= \frac{1}{4\pi}\left(\cos^2\theta + \sin^2\theta\right)$$

Therefore we find that the probability density is constant in θ and ϕ when integrated over all r.

$$\boxed{\int_0^\infty \left\langle\psi|\psi\right\rangle \mathrm{d}r = \frac{1}{4\pi}} \tag{S2012 II 1.6}$$

(d) The states degenerate with the first term in ψ are all combinations of allowed ℓ, m, and $\pm$: n must remain at $n = 3$ since it is the n quantum number which determines the energy of the state. The angular momentum number ℓ has to be in the range $[0, n-1]$, so there are at least 3 cases.

$$|3, 0, m, \pm\rangle$$
$$|3, 1, m, \pm\rangle$$
$$|3, 2, m, \pm\rangle$$

Then for each ℓ, the projection m can take a range of values $m \in [-\ell, \ell]$ so using $\{..., -1, 0, -1, ...\}$ to denote a set of options,

$$|3, 0, m, \pm\rangle \to |3, 0, \{0\}, \pm\rangle \qquad \text{2 states}$$
$$|3, 1, m, \pm\rangle \to |3, 1, \{-1, 0, 1\}, \pm\rangle \qquad \text{6 states}$$
$$|3, 2, m, \pm\rangle \to |3, 2, \{-2, -1, 0, 1, 2\}, \pm\rangle \qquad \text{10 states}$$

$$\boxed{\text{In total, there are 18 degenerate states}}$$

(e) To get the time evolution, we simply use the fact that for each basis eigenstate, we can add the time evolution component

$$\exp\left(-\frac{iE_n t}{\hbar}\right)$$

to get (in terms of the ground state energy E_0)

$$\boxed{|\psi(t)\rangle = \sqrt{\frac{1}{3}}\,|3, 1, 0, +\rangle\, e^{-iE_0 t/9\hbar} + \sqrt{\frac{2}{3}}\,|2, 1, 1, -\rangle\, e^{-iE_0 t/4\hbar}} \qquad \text{(S2012 II 1.7)}$$

(f) From Ehrenfest's Theorem, we can quickly find the answers to most of the question without worrying about the wavefunction. Ehrenfest's Theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\omega\rangle = -\frac{i}{\hbar}\langle[\omega, H]\rangle + \left\langle\frac{\partial\omega}{\partial t}\right\rangle$$

None of the operators L^2, S^2, L_z, and S_z are explicit in time, so the second term on the right can be dropped. Then because each of these operators commute with the Hamiltonian, the first term on the right is also dropped. Therefore, the expectation values are constant in time, so

$$\boxed{\langle L^2\rangle \quad \text{Time independent}} \qquad \text{(S2012 II 1.8)}$$
$$\boxed{\langle S^2\rangle \quad \text{Time independent}} \qquad \text{(S2012 II 1.9)}$$
$$\boxed{\langle L_z\rangle \quad \text{Time independent}} \qquad \text{(S2012 II 1.10)}$$
$$\boxed{\langle S_z\rangle \quad \text{Time independent}} \qquad \text{(S2012 II 1.11)}$$

For the probability density, we return to the integral in part (c) and insert the appropriate exponential terms. The first and last terms' exponentials cancel each other out, leaving

$$\begin{aligned}\int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \int_0^\infty & \frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi} + e^{-i\phi}\right) R_{21}(r)\, R_{31}(r) \\ & \cdot\left[\exp\left(\frac{i(E_2 - E_3)t}{\hbar}\right) + \exp\left(-\frac{i(E_2 - E_3)t}{\hbar}\right)\right] \\ & + \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)\,\mathrm{d}r\end{aligned}$$

The integral is unaffected by the new time factors, though, so integrating over r, the middle term still goes to zero and we're left with the same result previously of $1/4\pi$, therefore

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle \,\mathrm{d}r \quad \text{Time independent}} \tag{S2012 II 1.12}$$

Problem 2

Question

The heat equation $\partial_t T = \frac{\lambda}{c\rho}\partial_x^2 T$ describes the flow of heat in one dimension, where λ is the thermal conductivity, ρ is the density and c is the specific heat. Derive the dispersion relation for harmonic solutions that obey this equation. For harmonic seasonal variations in surface temperature described by $T = T_0 + T_1 \cos(\omega t)$, where $\omega = 2\pi$ radians per year, at what depth will the temperature phase lag be 6 months if $\lambda = 1$ W/m·K, $c = 10^3$ J/kg·K and $\rho = 2 \times 10^3$ kg/m^3? It may be useful to remember that $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$

Answer

Problem 3

Question

A charge $+Q$ is located at the center of a concentric hollow cavity (radius a) inside spherical conductor with radius b. The conductor is surrounded by a linear dielectric material of permitivity ϵ out to radius c.

(a) Derive expressions for electric fields in the four regions, $0 < r < a$, $a < r < b$, $b < r < c$, $r > c$.

(b) Find the potential at $r = a$ (relative to the potential at infinity).

(c) Show that the polarization density in the dielectric can be expressed as $\mathbf{P} = [(\epsilon - \epsilon_0) Q / (4\pi\epsilon r^2)]\hat{\mathbf{r}}$.

(d) What is the bound charge in the dielectric?

(e) What is the bound surface charge (at $r = b$ and $r = c$) in the dielectric?

Answer

Problem 4

Question

A bead of mass m slides on a straight wire. The wire rotates with an angular frequency ω around an axis perpendicular to the wire. The bead is connected to the axis by a spring of natural length L and spring constant k.

(a) Give the Lagrangian of the system.

(b) Derive the equation of motion of the bead.

(c) Under what conditions will the bead have a stable equilibrium position along the wire?

Answer

Problem 5

Question

Consider N non-interacting, stationary particles, each with magnetic moment μ at temperature T in a uniform external magnetic field $\mathbf{B}$. Their energy is $-\mu \cdot \mathbf{B}$. Calculate the partition function Z, the internal energy, and magnetization for two distinct cases (a and b below):

(a) The magnetic moment of each particle can be oriented only parallel or anti-parallel to the magnetic field.

(b) The magnetic moment of each particle can rotate freely.

(c) Show that, in both cases, the total magnetization $\mathbf{M}$ can be written as a derivative of the partition function.

(d) In each case, calculate the fluctuations of magnetization $\left\langle (\Delta\mathbf{u})^2 \right\rangle$.

Question

(a) Begin by constructing the partition function for a single particle. Since there are only two energy states, the sum is simply over the two Boltzmann factors:

$$Z_1 = e^{\mu B/kT} + e^{-\mu B/kT}$$

This can be simplified using trigonometric identities to

$$Z_1 = 2\cosh\left(\frac{\mu B}{kT}\right)$$

For fixed site particles, the partition function for N particles is simply $Z = Z^N$, so

$$\boxed{Z = 2^N \cosh^N\left(\frac{\mu B}{kT}\right)} \tag{S2012 II 5.1}$$

The total energy can be calculated either by finding the expectation energy per particle $\langle \varepsilon \rangle$ and multiplying by N using the Boltzmann factors directly, or by using the thermodynamic identity

$$U = kT^2 \frac{\partial \ln Z}{\partial T}$$

Doing so,

$$U = kT^2 \frac{N}{2\cosh\left(\frac{\mu B}{kT}\right)} \cdot 2\sinh\left(\frac{\mu B}{kT}\right) \cdot \left(-\frac{\mu B}{kT^2}\right)$$

$$\boxed{U = -N\mu B \tanh\left(\frac{\mu B}{kT}\right)} \tag{S2012 II 5.2}$$

To find the Magnetization, we use the Boltzmann factors directly since we don't know a thermodynamic relation. Let

$$\begin{aligned}
\langle m \rangle &= \sum_{\mu} \mu \frac{e^{-\varepsilon_\mu/kT}}{Z_1} \\
&= \frac{1}{Z_1}\left(-\mu e^{\mu B/kT} + \mu e^{-\mu B/kT}\right) \\
&= -\mu \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\cosh\left(\frac{\mu B}{kT}\right)}
\end{aligned}$$

So knowing that $\langle M \rangle = N \langle m \rangle$,

$$\boxed{\langle M \rangle = -N\mu \tanh\left(\frac{\mu B}{kT}\right)} \tag{S2012 II 5.3}$$

(b) In the continuous case, the sum needs to be changed into an integral, remembering to keep Z unitless. This requires dividing by the volume of the energy state, which in this case is μB. (Justification: think of the energy vector $\vec{\mu} \cdot \vec{B} = \mu B \cos\theta$ on the unit circle of length μB. From geometry, the unitless $\mathrm{d}\theta$ is related to $\mathrm{d}\varepsilon$ by the factor μB.)

$$Z_1 = \int_{-\mu B}^{\mu B} e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B}$$

Letting $u = -\frac{\varepsilon}{kT}$,

$$\begin{aligned} Z_1 &= -\frac{kT}{\mu B} \int_{\mu B/kT}^{-\mu B/kT} e^u \mathrm{d}u \\ &= 2\frac{kT}{\mu B} \sinh\left(\frac{\mu B}{kT}\right) \end{aligned}$$

Therefore the partition function for all N particles is

$$\boxed{Z = \left(\frac{2kT}{\mu B}\right)^N \sinh^N\left(\frac{\mu B}{kT}\right)} \tag{S2012 II 5.4}$$

The total energy is found in the same way as the previous case, giving

$$\boxed{U = NkT - N\mu B \coth\left(\frac{\mu B}{kT}\right)} \tag{S2012 II 5.5}$$

For the magnetization, we also calculate the expectation value from integrating the probability distribution, again making sure to keep the correct units. This time we work with the relevant projection of the magnetic moment $m = \mu \cos\theta$ so that when combined with the energy $\varepsilon = -\mu B \cos\theta$, the magnetization per particle in each state is $m = -\varepsilon/B$.

$$\begin{aligned} \langle m \rangle &= \int_{-\mu B}^{\mu B} -\frac{\varepsilon}{B} \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B} \\ &= \frac{1}{\mu Z_1} \left(\frac{kT}{B}\right)^2 \int_{-\mu B/kT}^{\mu B/kT} e^u \mathrm{d}u \\ &= \frac{kT}{B} \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\sinh\left(\frac{\mu B}{kT}\right)} \\ &= \frac{kT}{B} \end{aligned}$$

The total magnetization $\langle M \rangle = N \langle m \rangle$ is

$$\boxed{\langle M \rangle = \frac{NkT}{B}} \tag{S2012 II 5.6}$$

Note that this is to be expected for the continuous case limit which corresponds to the classical limit. We'd expect the total energy to be related to the magnetization by $U = MB$. Rearranging the terms,

$$\begin{aligned} \langle M \rangle B &= NkT \\ U &= NkT \end{aligned}$$

which is the expected result from the equipartition theorem for a stationary particle with two rotational degrees of freedom.

(c) Proving the discrete case only differs from the continuous case proof by the obvious substitutions, so only the continuous case will be presented here. Begin by writing the first starting integral from the previous problem

$$\langle m\rangle = \int_{-\mu B}^{\mu B} m \frac{e^{-\varepsilon/kT}}{Z_1}\frac{\mathrm{d}\varepsilon}{\mu B}$$

The Z_1 can be pulled outside the integral since it is a constant. Then note that per our definition $\varepsilon = -mB$, it follows that

$$\frac{\partial \varepsilon}{\partial B} = -m$$

We identify the integral above to be a result of using the chain rule, so we undo that and get

$$\langle m\rangle = \frac{1}{Z_1}\int_{-\mu B}^{\mu B} \frac{\partial}{\partial B}\left(-e^{-\varepsilon/kT}\right)\frac{\mathrm{d}\varepsilon}{\mu B}$$

Changing the order of integration and differentiation,

$$= \frac{1}{Z_1}\frac{\partial}{\partial B}\left(\int_{-\mu B}^{\mu B} -e^{-\varepsilon/kT}\frac{\mathrm{d}\varepsilon}{\mu B}\right)$$

The term within the brackets is simply the definition of the partition function, so

$$\langle m\rangle = \frac{1}{Z_1}\frac{\partial Z_1}{\partial B} = \frac{\partial \ln Z_1}{\partial B}$$

To then get the total magnetization $\langle M\rangle$, we use several properties of differentiation and logarithms:

$$\begin{aligned}\langle M\rangle &= N\langle m\rangle \\ &= N\frac{\partial \ln Z_1}{\partial B} \\ &= \frac{\partial\left(N\ln Z_1\right)}{\partial B} \\ &= \frac{\partial \ln(Z_1)^N}{\partial B}\end{aligned}$$

Giving us the final expression

$$\boxed{\langle M\rangle = \frac{\partial \ln Z}{\partial B}} \tag{S2012 II 5.7}$$

(d) Using the definition

$$\left\langle (\Delta\mu)^2\right\rangle = \left\langle \mu^2\right\rangle - \langle\mu\rangle^2$$

we already know $\langle\mu\rangle^2$ for both cases from the previous problems, so we must only calculate $\left\langle\mu^2\right\rangle$.

Fall 2012 Part I

Problem 1

Question

Assume you have three identical particles and three single particle states $|a\rangle$, $|b\rangle$, and $|c\rangle$ available for them. Count how many different three-particle states there can be if the particles are (a) fermions, and (b) bosons.

Answer

Fermionic statistics exclude two fermions living in the same energy state, thus there are 3! different particle states for fermions. Bosonic statistics allow an arbitrary number of bosons to live in the same energy, thus there are 3^3 different particle states for bosons.

Problem 2

Question

Show that a particle in a one-dimensional infinite square well initially in a state $\psi(x, 0)$ will always return to that state after a time $T = 4ma^2/\pi\hbar$ where a is the width of the well.

Answer

Use the standard time independent Schödinger equation

$$\psi(x,t) = \psi(x)\, e^{iEt/\hbar}$$

with associated differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi$$

For an infinite square well, the potential has the form

$$V(x) = \begin{cases} 0 & |x| < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$$

so that the only region to consider is $-\frac{a}{2} < x < \frac{a}{2}$. In this region, the differential equation takes the form of a harmonic oscillator

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

leading to solutions

$$\psi(x) = A\cos kx + B\sin kx$$
$$\text{where } k^2 = \frac{2mE}{\hbar^2}$$

The boundary conditions $\psi\left(-\frac{a}{2}\right) = 0$ and $\psi\left(\frac{a}{2}\right) = 0$ impose

$$\psi\left(-\frac{a}{2}\right) = 0 = A\cos\frac{ka}{2} - B\sin\frac{ka}{2}$$
$$\psi\left(\frac{a}{2}\right) = 0 = A\cos\frac{ka}{2} + B\sin\frac{ka}{2}$$

so that $B = 0$ and

$$0 = 2A\cos\frac{ka}{2}$$
$$\frac{(2n+1)\pi}{2} = \frac{ka}{2}$$
$$k = \frac{(2n+1)\pi}{a}$$

We already had a relation for k defined, so substitute and solve for the energies E_n.

$$\frac{(2n+1)^2\pi^2}{a^2} = \frac{2mE}{\hbar^2}$$
$$E_n = \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2}$$

Then considering $\psi(x,t)$, the complex exponential is periodic in time with period

$$T_n = \frac{2\pi\hbar}{E}$$

where $n = 0$ will be the case with the longest periodicity, so

$$\begin{aligned} T &= \frac{2\pi\hbar \cdot 2ma^2}{\pi^2\hbar^2} \\ &= \frac{4ma^2}{\pi\hbar} \end{aligned}$$

Therefore, the function is periodic in time with a periodicity

$$\boxed{T = \frac{4ma^2}{\pi\hbar}} \tag{F2012 I 2.1}$$

Problem 3

Question

Consider a system of N independent classical molecules, each at a fixed position, with magnetic moment μ in an external magnetic field B. Determine the partition function, and hence find the free energy and the magnetization at temperature T, when the molecules can only be oriented parallel or antiparallel to the external magnetic field.

Answer

Problem 4

Question

A photon collides with a stationary electron. If the photon scatters at an angle θ, show that the resulting wavelength λ' is given in terms of the original wavelength λ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron.

Answer

Start by considering conservation of momentum for the system. The initial values are

$$\begin{aligned} p_{\gamma x} &= \frac{h}{\lambda} & p'_{\gamma x} &= \frac{h}{\lambda'}\cos\theta \\ p_{\gamma y} &= 0 & p'_{\gamma y} &= \frac{h}{\lambda'}\sin\theta \\ p_{ex} &= 0 & p'_{ex} &= ? \\ p_{ey} &= 0 & p'_{ey} &= ? \end{aligned}$$

and considering each component in turn:

$$\begin{aligned} \frac{h}{\lambda} + 0 &= \frac{h}{\lambda'}\cos\theta + p'_{ex} & 0 &= \frac{h}{\lambda'}\sin\theta + p'_{ey} \\ p'_{ex} &= \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta & p'_{ey} &= -\frac{h}{\lambda'}\sin\theta \end{aligned}$$

The total momentum of the electron is then

$$\begin{aligned} p_e^2 &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta\right)^2 + \left(-\frac{h}{\lambda'}\sin\theta\right)^2 \\ &= \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta + \frac{h^2}{\lambda'^2}\cos^2\theta + \frac{h^2}{\lambda'^2}\sin^2\theta \\ p_e^2 &= h^2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{\lambda\lambda'}\cos\theta \end{aligned} \tag{F2012 I 4.1}$$

Then consider energy conservation, with initial values

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} & E'_\gamma &= \frac{hc}{\lambda'} \\ E_e &= mc^2 & E'_e &= \frac{{p'_e}^2}{2m} + mc^2 \end{aligned}$$

leading to the equation

$$
\begin{aligned}
\frac{hc}{\lambda} + mc^2 &= \frac{hc}{\lambda'} + \frac{{p'_e}^2}{2m} + mc^2 \\
\frac{hc}{\lambda} &= \frac{hc}{\lambda'} + \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{hc}{\lambda} - \frac{hc}{\lambda'} &= \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{\lambda' - \lambda}{\lambda\lambda'} &= \frac{h}{2mc}\frac{\lambda'^2 + \lambda^2}{\lambda^2{\lambda'}^2} - \frac{h}{mc\lambda\lambda'}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{\left(\lambda' - \lambda\right)^2 + 2\lambda\lambda'}{\lambda\lambda'}\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} + 2\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} + \frac{h}{mc}\left(1 - \cos\theta\right)
\end{aligned}
$$

The difference in the wavelengths is small, so

$$\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} \approx 0$$

leading to the final Compton scattering equation

$$\boxed{\lambda' = \lambda + \frac{h}{mc}\left(1 - \cos\theta\right)} \qquad \text{(F2012 I 4.2)}$$

Problem 5

Question

The composition of a glass block varies as a function of the distance x from the top surface. Thus the index of refraction $n(x)$ increases as a function of x according to the relationship $n(x) = 1.50 - (0.20)/(x+1)^2$, where x is expressed in centimeters. A beam of light strikes the surface with an angle of incidence θ_i , measured from the vertical, as shown in the figure opposite. What will be the direction of the beam deep inside the block?

Answer

Problem 6

Question

You want to measure a current when using a voltmeter and a precision resistor. You measure a voltage of 0.85 V across a $10,000\Omega$ resistor and -0.05 V when the input to the voltmeter is short-circuited. The precision of the voltmeter is 0.001 V and the resistor is rated at 0.1%. What is the value of the current and the precision of the measurement?

Answer

Problem 7

Question

Find the capacitance per unit length of two coaxial conducting cylindrical tubes, of radii a and b.

Answer

Problem 8

Question

Suppose that the radius of the Earth were to gravitationally collapse uniformly by one percent, with its mass remaining the same. What would happen to the Earth's kinetic energy of rotation? If it changes, how does it change and by how much? Assume that the Earth is a uniform sphere.

Answer

Problem 9

Question

A neutron star has a radius of 10 km, a mass of 3.0×10^{30} kg. Find the nearest distance to the surface that a person 2 m tall could approach the pulsar without being pulled apart. Assume a uniform mass distribution, feet toward the pulsar and that a person starts to come apart when the force that each half of the body exerts on the other exceeds ten times the body weight on Earth. What is the period of revolution in a circular orbit about the pulsar at this distance?

Answer

Problem 10

Question

Given that the latent heat of fusion for water is 334 kJ/kg and the density of water and ice are 1000 kg/m^3 and 917.0 kg/m^3, respectively. At what pressure will ice melt at -0.1° C? It may be useful to remember that the Gibbs free energy is the same across a phase boundary.

Answer

Fall 2012 Part II

Problem 1

Question

In the middle of an infinite square well extending from $x = 0$ to $x = a$ we put a delta function potential $H = \kappa\delta\left(x - a/2\right)$ where κ is constant. (a) Find the first order perturbation correction to the energies. (b) For the ground state, find also the second order correction to the energy.

Answer

Problem 2

Question

(a) Beginning with the Lorentz transformations show that when an object is moving with velocity v_2 in a reference frame that is moving with velocity v_1 with respect to an observer, the velocity v of the object as seen by the observer is:

$$v = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2} \tag{F2012 II 2.1}$$

(b) A spaceship is initially at rest with respect to frame S. At a given instant, it starts to accelerate with a constant acceleration, a, in the instantaneous rest frame of the spaceship. What is the relative speed of the spaceship in frame S when the spaceship's clock reads time t? (c) The command center, which is stationary in frame S, communicates with the spaceship by using a laser with a wavelength of λ_s. What would be the wavelength of the laser beam observed on board the spaceship at time t?

Answer

Problem 3

Question

Consider a classical model of a CO_2 molecule where the masses are connected by springs of spring constant k. Assume all motion to be linear along the axis of the molecule. (a) Find the relative frequencies for the two vibrational modes. (b) Find the eigenvectors for the modes of the molecule, including any zero frequency mode.

Answer

Problem 4

Question

Consider a free Fermi gas of N electrons in two dimensions confined to a square of area A. (a) Find the Fermi energy (ϵ_F) in terms of N and A. (b) Derive the formula for the density of states and show that it is independent of energy. (c) Use this to find the chemical potential μ as a function of N and the temperature. (d) What is the behavior of the system at low temperature?

(Partial) Answer

To find the Fermi energy, we only need to solve the Schroedinger equation for an infinite potential well in two dimensions. The energy levels for this system are well known;

$$E_{n_1 n_2} = \frac{\hbar^2}{2m}\left[\frac{n_x^2}{\ell_x^2} + \frac{n_y^2}{\ell_y^2}\right] = \frac{\hbar^2}{2m}k^2 \tag{F2012 II 4.1}$$

where k^2 is the magnitude of the wave vector in 2D. In order to obtain the Fermi energy, first split the 2D surface into squares of width L. Then center a circle on the corner of a square. Since electrons are fermions, they will fill up one a quadrant of the circle in k space whose radius k_F is determined by the fact that each pair of electrons requires an area $A \sim L^2$. Let $N_q = qN$ denote the number of free electrons, then the Fermi relation says (area of the circle contained in the square) = (number of electrons per unit lattice area)

$$\frac{1}{4}\left(\pi k_F^2\right) = \frac{N_e}{2}\left(\frac{\pi^2}{A}\right) \implies \boxed{k_F^2 = \frac{2\pi}{A}N_e = 2\pi n_e.} \tag{F2012 II 4.2}$$

Solving for the energy leads us to $\epsilon_F = \frac{\hbar^2}{2m}k_F^2 = \frac{\hbar^2}{2m}2\pi n_e$.

Problem 5

Question

If magnetic charges were found, (a) write down the proper set of four Maxwell equations in vacuum to include the electric and magnetic charges as well as electric and magnetic currents. (b) Suppose a magnetic monopole of strength, q_m , passes through a zero resistance loop of wire and moves far away. If the self-inductance of the loop is a, what would be the electric current induced in the loop? (c) Set up the equation of motion of an electrically charged particle (charge q_e , mass m) about a fixed magnetic monopole of strength, q_m.

Answer

Spring 2013 Part I

Problem 1

Question

Three perfectly cylindrical frictionless oil pipes are packed inside an inclined railway car as shown on the figure. Find the range of angles θ where such a packing is stable.

Answer

Problem 2

Question

An "air molecule" at room temperature and atmospheric pressure is moving with the average speed of 450 m/sec. It travels about 7×10^{-6} cm between collisions. Estimate how much time it takes for a molecule to travel 1cm?

Answer

Problem 3

Question

A particle of mass m, moving with velocity v, crosses a boundary between the region where its potential energy is equal to U_1 to a region where its potential energy is equal to U_2. Derive a "Snell's law" of refraction for this boundary.

Answer

Problem 4

Question

A coaxial cable made of ideal conductor (no resistance) has an inner cylindrical conductor of radius a, and air gap between $r = a$ and $r = b$, and another conductor between $r = b$ and $r = c$, as shown. The inner conductor is connected to a voltage source at one of its ends such that its voltage with respect to the outer conductor is V_0 . The inner conductor carries a current I_0 in the $\breve{z}$ direction (into the page) which is returned along the outer conductor via a resister which connects the inner and outer conductors at the other end. Calculate the magnitude and direction of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in the region in the air gap where $a < r < b$.

Answer

Problem 5

Question

An ideal voltage generator with output voltage $V = V_0 \sin(\omega t)$ where $V_0 = 100$ Volts is connected to the circuit shown below. Calculate the time-averaged power dissipated in the 100 Ω resistor as a function of the generator frequency Ω.

Answer

Problem 6

Question

A beam of monoenergetic π^+ of kinetic energy $T = 140$ MeV and rest mass $m_0 = 140$ MeV$/c^2$ is to traverse a total flight path of length $D = 20$ m. Calculate the fraction of Pions that survive the 20 m trip, provided that the mean lifetime of charged Pions is $\tau = 2.6 \times 10^{-8}$ sec (in the rest frame of the pion).

Answer

Problem 7

Question

Some organic molecules have a spin triplet ($S = 1$) excited state at an energy $k_B\Delta$ above a singlet ($S = 0$) ground state. Find an expression for the magnetic moment, $\langle\mu\rangle$, in a field $\mathbf{B}$. What is the susceptibility at high temperature?

Answer

Problem 8

Question

Two identical perfect gases with the same pressure P and the same number of particles N, but with different temperatures, T_1 and T_2 are confined in two vessels, of volume V_1 and V_1 that are then connected. Find the change in entropy after the system has reached equilibrium.

Answer

Problem 9

Question

A charged particle is residing in an infinite one-dimensional square well potential.

(a) Write down an expression for the matrix element of the electric dipole moment for the particle when making a transition from one quantized level to another.

(b) From consideration of your answer in part (a), what would be the corresponding selection rules governing the allowed transitions for this particle? Provide a physical justification for your answer.

Answer

Problem 10

Question

Consider a diatomic molecule of two dissimilar nuclei that have the following properties: a reduced mass $\mu = 20m_p$ (where m_p is the mass of a proton), interatomic spacing, $r_0 = 3.0 \times 10^{-10}$ m, and the force constant, $C = 8 \times 10^{18}$ eV$/cm^2$.

(a) What is the energy difference, in eV, between the $r = 0$ and $r = 1$ rotational levels for this molecule in the vibrational ground state?

(b) What is the energy difference, in eV, between the $\nu = 0$ and $\nu = 1$ vibrational levels for this molecule? [Assume that the rotational state does not change.]

Answer

Spring 2013 Part II

Problem 1

Question

A uniform cylindrical drum of mass M and radius R is free to rotate about its axis, which is horizontal. An elastic cable with spring constant κ and negligible mass is wound on the drum. (In real life, κ will decrease as the cable unwinds, but ignore this effect and assume that it is a constant.) On its free end, it carries a mass m, which is allowed to fall down unwinding the elastic cable.

(a) Write down a Lagrangian function in terms of the drum rotation angle θ and the vertical displacement x of the mass m. Derive the corresponding equations of motions.

(b) Find proper normal modes and determine oscillation frequency (or frequencies) of the system.

Answer

Problem 2

Question

An electron is oscillating in a simple harmonic oscillator potential with an angular frequency $\omega = 10^{15}$ rad/sec and amplitude $x_0 = 10^{\breve{}10}$ m.

(a) Calculate the amount of energy radiated per cycle. If you don't remember the radiation equation, you may want to think about what quantities should be included, and dimensional analysis may be useful.

(b) What is the ratio of the radiated energy per cycle to the average mechanical energy?

(c) How long will it take the system to radiate away half of its energy?

Answer

Problem 3

Question

An engine, that uses a photon gas as the working substance, operates in accordance with the Carnot cycle. The energy of a photon gas is given by Stefan-Boltzmann law $U = \alpha V \tau^4$, and the entropy, σ, is given by $4U/3\tau$, where U is the energy, α is the Stefan-Boltzmann constant, V is the volume of the gas, and τ is the temperature.

Given τ_h, τ_l, the high and low temperatures of the cycle, and starting with isothermal compression at (V_l, τ_l) calculate the work done by the gas for each stage of the cycle and compute the total work. Use this information to calculate the efficiency of the engine.

Answer

Problem 4

Question

Consider a one-particle system capable of three states $(-\epsilon, 0, \epsilon)$ in thermal contact with a reservoir at temperature T. Find:

(a) Partition function

(b) Average Energy

(c) Heat capacity at constant volume

(d) Entropy

(e) Free energy

In addition, find the leading temperature dependence of (b), (c), and (d) when $T \gg \epsilon$ and $T \ll \epsilon$.

Answer

Problem 5

Question

The nucleus of a hydrogen atom isotope of mass 3 is radioactive, and changes suddenly into a helium nucleus of mass 3 with the emission of an electron that escapes the nucleus. If the initial hydrogen atom was in its ground state, what is the probability that the single-electron helium ion formed by this radioactive decay is in the 1s state? Which other state(s) will it be found in, other than the 1s state? Useful information:

$$\psi_{nlm} = R_{nl}\left(r\right) Y_l^m\left(\theta, \varphi\right);\ R_{10}\left(r\right) = 2\left(\frac{z}{a_0}\right)^{3/2} e^{-Zr/a_0};\ \int_0^\infty e^{-x} x^n \mathrm{d}x = n!$$

Answer

Fall 2013 Part I

Problem 1

Question

Assuming the electron to be a classical particle, a sphere of radius 10^{-15} m and of a uniform mass density with an intrinsic angular momentum of order $\hbar$, compute the speed of rotation at the electron's equator. How does your result compare with the speed of light?

Answer

Since we're assuming the electron is a classical sphere, we can equate the angular momentum of a spinning sphere to the intrinsic angular momentum $\hbar$. Knowing that the moment of inertia of a sphere is $I = \frac{2}{5}mR^2$ for a sphere of uniform density, that means the momentum equation is,

$$\hbar = \frac{2}{5}mR^2\omega$$

Then solving for the frequency and relating it to the velocity of a point on the equator,

$$v = \frac{5}{2}\frac{\hbar}{mR}$$

Plugging in the values,

$$\boxed{v = 2.89 \cdot 10^{11}\,\frac{\mathrm{m}}{\mathrm{s}} = 965\,\mathrm{c}} \tag{F2013 I 1.1}$$

Problem 2

Question

Two electrons can be considered distinguishable if they are well separated in space from each other, that is, their single particle wavefunctions are non-overlapping. In that case, for every possible x_1 value, either $\psi_\alpha(x_1)$ or $\psi_\beta(x_1)$ is zero. Show that for non-overlapping wavefunctions as defined above, the probability density for the total antisymmetric wavefunction $\psi_A^*\psi_A$ is equal to the probability density of the total symmetric wavefunction $\psi_S^*\psi_S$.

Answer

We start by constructing the antisymmetric and symmetric wavefunctions:

$$\begin{aligned}\psi_A &= \frac{1}{\sqrt{2}}\left(\psi_\alpha(x_1)\,\psi_\beta(x_2) - \psi_\alpha(x_2)\,\psi_\beta(x_1)\right)\\ \psi_S &= \frac{1}{\sqrt{2}}\left(\psi_\alpha(x_1)\,\psi_\beta(x_2) + \psi_\alpha(x_2)\,\psi_\beta(x_1)\right)\end{aligned}$$

Then calculating the complex square of both (where the A/S indicates the choice of $\pm$),

$$\begin{aligned}\psi_{A/S}^*\psi_{A/S} = \frac{1}{2}\Big(&\psi_\alpha^*(x_1)\,\psi_\beta^*(x_2)\,\psi_\alpha(x_1)\,\psi_\beta(x_2)\\ &\mp \psi_\alpha^*(x_1)\,\psi_\beta^*(x_2)\,\psi_\alpha(x_2)\,\psi_\beta(x_1)\\ &\mp \psi_\alpha^*(x_2)\,\psi_\beta^*(x_1)\,\psi_\alpha(x_1)\,\psi_\beta(x_2)\\ &+ \psi_\alpha^*(x_2)\,\psi_\beta^*(x_1)\,\psi_\alpha(x_2)\,\psi_\beta(x_1)\Big)\end{aligned}$$

The first and fourth lines are simply the complex square of each wavefunction. The second and third, though, contain the cross-terms, and since we are given that the wavefunctions are non-overlapping, that means necessarily the product of different wavefunctions at the same point in space must be zero. Therefore both the second and third lines result in a value of 0, thus simplifying the expression to,

$$\boxed{\psi^*\psi = \frac{1}{2}\Big(|\psi_\alpha(x_1)|^2|\psi_\beta(x_2)|^2 + |\psi_\alpha(x_2)|^2|\psi_\beta(x_1)|^2\Big)} \tag{F2013 I 2.1}$$

Problem 3

Question

A particle of mass m moves in a circular orbit of radius r in a hypothetical atom where the force on the particle is in the form of a generalized Hooke's law: $F = -Cr$ directed towards the center of the atom, where C is the 'spring constant'. Assuming that Bohr's postulates for the atom apply in this case, in particular, that the orbital angular momentum is quantized with a quantum number n, derive:

(a) The radii of the allowed orbits

(b) The energies of these orbits in terms of the quantum number n. (You may take the potential energy of this "spring atom" to be $V(r) = \frac{1}{2}Cr^2$.)

Answer

With the assumption that the orbits are circular, we know that the Hooke force must provide the centripetal acceleration. Setting them equal and solving for the angular frequency by $v = \omega r$,

$$\begin{aligned} m\frac{v^2}{r} &= Cr \\ \omega &= \sqrt{\frac{C}{m}} \end{aligned}$$

The point-particle electron then has an associated moment of inertia and angular momentum as it orbits the nucleus.

$$\begin{aligned} L &= mr^2 \cdot \sqrt{\frac{C}{m}} \\ n\hbar &= r^2\sqrt{Cm} \end{aligned}$$

The quantization condition on the orbital radii is

$$\boxed{r = \left(\frac{\hbar^2}{Cm}\right)^{1/4}\sqrt{n}} \tag{F2013 I 3.1}$$

To find the energies, we simply sum the kinetic energy from the orbital motion with the potential energy contained in the "spring".

$$\begin{aligned} E_n &= \frac{1}{2}I\omega^2 + \frac{1}{2}CR^2 \\ &= \frac{1}{2}mR^2\frac{C}{m} + \frac{1}{2}CR^2 \\ &= CR^2 = C \cdot \sqrt{\frac{\hbar^2}{Cm}}n \end{aligned}$$

Therefore the associated energy states are

$$\boxed{E_n = \hbar n\sqrt{\frac{C}{m}}} \tag{F2013 I 3.2}$$

Problem 4

Question

A vertical cylinder contains 1.0 mole of an ideal gas at temperature T under a light piston. The top of the piston is at atmospheric pressure. Find the work needed to increase the ideal gas volume by a factor of β at $T =$ const.

Answer

Problem 5

Question

The index of refraction of glass can be increased by diffusing in impurities. It is then possible to make a lens of constant thickness. Given a disk of radius a and thickness d, find the radial variation of the index of refraction, $n(r)$, that will bring rays emitted from A in the diagram below to a focus at B. Assume that the lens is thin ($d \ll a$ or b). Note that there are two approaches to image focusing problems. You may be more familiar with one, where you would trace light rays using Snell's law, and two rays converge to a point. This approach will lead to a very complex solution.

Answer

Problem 6

Question

Consider one-particle system capable of four states ($\epsilon m = m\Delta$ where $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$, and $\frac{3}{2}$) in thermal contact with a reservoir at temperature T. For specific cases of $T = 0$ and $T \to \infty$, find the average energy, entropy and heat capacity.

Answer

Problem 7

Question

A lambda baryon traveling through a laboratory decays into a proton and a π meson. The π meson is left at rest. Find the initial laboratory kinetic energy of the lambda baryon. $m_p = 940\text{MeV}/c^2$, $m_\pi = 140\text{MeV}/c^2$, $m_\Lambda = 1120\ \text{MeV}/c^2$.

Answer

For the decay process $A \to B + C$ to occur, the energy of the outgoing particle (say B) must be

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2.$$

where A was at rest and B and C are moving in opposite directions. Since A is moving and C is stationary, we perform a cyclic rotation $C \to B \to A \to C$ to obtain

$$E_A = \frac{m_C^2 + m_A^2 - m_B^2}{2m_C} c^2.$$

To find the kinetic energy of A, we need to subtract the rest energy from E_A thus

$$K = E_\Lambda - m_\Lambda c^2 = \left(\frac{m_\Lambda^2 + m_\pi^2 - m_p^2}{2m_\pi} \right) - m_\Lambda c^2 \approx 274 \frac{\text{MeV}}{c^2} \tag{F2013 I 7.1}$$

Problem 8

Question

A metal can be thought of as a box filled with atoms. For each atom one electron is bound with a spring-like force (i.e. like Hookes' Law) such that when the electron is displaced from its equilibrium location it experiences a restoring force that is proportional to the displacement. This gives rise to a natural frequency of oscillation ω_0. Now consider these atoms under the influence of an external electric field of amplitude E_0 and frequency ω. Calculate the electric polarization of the metal as a function of frequency and E_0 if the density of these electrons is n cm^{-3}.

Answer

Problem 9

Question

In the circuit depicted on the right, the voltage source generates an AC voltage $V(t) = V_0 \cos(\omega t)$, where V_0 is fixed to a specific value while ω can be adjusted to any value. What is the power (averaged over a long time, $T \gg 1/\omega$) that is consumed by the circuit as a function of ω? When $\omega \to 0$ or $\omega \to \infty$, what will happen? Is there any other case(s) when an interesting thing will happen to the power consumption?

Answer

Problem 10

Question

A planet of mass m orbiting a star of mass M revolves around the center-of-mass of the system. This introduces a "reflex oscillation" of the star of amplitude R (see the diagram below where the motion of the star and the planet is illustrated as seen by the observer at the center of mass of the system. R and r are the radii of the orbits of m and M) and T is the period of the stellar motion, viewed from outside the system.

An observer searching for extrasolar planets with an astrometric telescope (measuring positions of astronomical objects with high precision) detects a stellar reflex motion of period $T = 1$ year and an amplitude R that subtend an angle of 7.0×10^{-9} radians in a star 10 light years from the earth. Assume circular orbit.

If the reflex motion is caused by an orbiting planet with Jupiter's mass (10^{-3} of the solar mass), what is the planet-star distance in AU (one AU = earth-sun distance = 8.3 light minutes)?

Answer

Fall 2013 Part II

Problem 1

Question

A free particle of mass m and spin s is initially (at $t = 0$) in a state corresponding to the wave function

$$\psi(r) = \left(\frac{\gamma}{\pi}\right)^{3/4} e^{-\gamma r^2/2}$$

(a) Calculate the probability density of finding the particle with momentum $\hbar\kappa$ at any time t. Is it isotropic?

(b) What is the probability of finding the particle with energy E?

(c) Examine whether the particle is in an eigenstate of the square of the angular momentum $\mathbf{L}$, and of its z-component $\mathbf{L}_z$, for any time t.

Answer

Problem 2

Question

Consider an elastic film (2D square lattice of N atoms) stretched over a rigid square with side L. Assume that only transverse modes, i.e. corresponding to displacements orthogonal to the film, can be excited and that sound velocity u is frequency independent. Find:

(a) The dispersion relationship between ω and k (wave number of the sound)?

(b) The maximum energy of an atom (often referred to as the Debye energy $\theta_D = k_B T_D$, where T_D is the Debye temperature).

(c) The energy U for extreme temperatures, $T \gg T_D$ and $T \ll T_D$.

(d) The heat capacity C_V , for $T \gg T_D$ and $T \ll T_D$.

Answer

Problem 3

Question

Two identical pendulae ($\omega_0 = g/l$) are connected by a light coupling spring. With the coupling spring connected, one pendulum is clamped and the period of the other is found to be T seconds. With neither of the connected pendulum clamped, that is, free to oscillate, what are the periods of the two normal modes?

Answer

Problem 4

Question

Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (with dielectric constant ϵ and mass density ρ). The inner cylinder is maintained at potential V, while the outer one is grounded. To what height h does the oil rise in the space between the tubes? For simplicity, you can assume that the top surface of the oil between the cylinders is flat and horizontal.

Answer

Problem 5

Question

At Brookhaven National Laboratory, a beam of heavy nuclei smash into a target composed of the same, identical nuclei. The kinetic energy of the beam particles is 14.5 GeV per nucleon. What is the Lorentz factor γ of each nucleus as seen by an observer in the center of mass frame?

Assume that these nuclei stop each other and form one composite nuclear system, what nucleon density results? Compare this density with the average density of a neutron star with mass 1.4 solar masses and radius $R = 10$ km. Atomic nuclei have a normal density of $n_0 = 0.15$ nucleons/fm^3 , $m_N = 1.7 \times 10^{-27}$ kg and the solar mass, $M_O = 2 \times 10^{30}$ kg.

Answer

Spring 2014 Part I

Problem 1

Question

Consider the one-dimensional motion at positive displacement x of a particle subject to a force $R = -\frac{b_1}{x^3}$ where b_1 is a positive constant.

(a) Find the escape velocity from position x_0.

(b) Suppose the force was $F = -b_2/x$, (b_2 is also positive). Show that the escape velocity is infinite.

Answer

Problem 2

Question

A classical particle of mass m moves in a closed orbit in the gravitational field of a mass $M \gg m$; M is at the origin and the potential energy is $V = -k/r$. Besides energy and angular momentum there is another conserved quantity, the Laplace-Runge-Lenz vector $\mathbf{A}$

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk}{r}\mathbf{r}$$

where $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{L}$ denote the linear momentum, position, and angular momentum of the particle, respectively.

(a) In one or two lines argue why $\mathbf{A}$ is in the plane of the orbit.

(b) The magnitude of the vector $\mathbf{A}$ is related to the eccentricity ϵ of the orbit. Recalling that when the origin coincides with one of the foci the equation of an ellipse can be written as

$$\frac{1}{r} = C\left(1 + \epsilon\cos(\theta)\right)$$

, r where θ is the azimuthal angle measured relative to $\mathbf{A}$ ($\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta$) and C is a constant, relate the magnitude A to the eccentricity ϵ, the mass m, and k.

Answer

Problem 3

Question

Find the magnetic flux through a square loop due to a current I in a long straight wire. The loop is coplanar with the wire and has two sides parallel to it. The loop has side length a and its side nearest to the wire is a distance b from the wire.

Answer

Problem 4

Question

A system is described by the wave function $\Psi = A\cos^2(\phi)$, where A is a normalization constant and ϕ is the azimuthal angle. You measure the angular momentum of this system along the z axis. Compute which results you can obtain from such a measurement, and their probabilities.

Answer

Problem 5

Question

Derive the average momentum $\langle \hat{p} \rangle$ for a packet with a normalizable wave function of the form

$$\Psi(x) = C\phi(x)\, e^{ikx}$$

where C is a normalization constant and $\phi(x)$ is a real function.

Answer

Problem 6

Question

Galaxy A moves away from our galaxy at a speed of $0.6c$. Galaxy B moves away at $0.7c$ with a trajectory that is 45 degrees to A. Write down the transformation of space and time differential coordinates $(\Delta t, \Delta x, \Delta y)$ between two objects moving at constant velocity v_0 relative to each other (orient the axes as you see fit), derive the velocity transformation, and express the velocity of B as observed by the civilization in A.

Answer

Problem 7

Question

A circular storage ring with an orbit diameter of 10 meters stores protons with 1 GeV kinetic energy.

(a) How long do they take to complete one orbit?

(b) What magnetic field strength is needed to constrain them in orbit?

Answer

Problem 8

Question

Consider a planar square lattice with N classical spins at each site i represented by the unit vector $\mathbf{S}_i$. Each spin is restricted to point along only four directions: $\pm\hat{\mathbf{x}}$ and $\pm\hat{\mathbf{y}}$. Each spin interacts only with its nearest neighbors according to the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

where $J > 0$ is the exchange interaction and $\langle ij \rangle$ denotes a pair of nearest neighbor sites. Since we are ultimately interested in the thermodynamic limit $N \to \infty$ the effect of the boundary is negligible.

By computing the internal energy of the ferromagnetic ordered state (in which all spins are perfectly parallel to each other) and the entropy of the paramagnetic state (in which the spins are completely disordered), and by considering the free energy of the system give an estimate for the temperature at which the system undergoes a phase transition from the ferromagnetic to the paramagnetic state in the limit $N \to \infty$.

Answer

Problem 9

Question

Two protons are at large distance from each other. Initially one proton is at rest and the other is moving head-on toward the first with kinetic energy $\mathcal{E}$. The motion is nonrelativistic, $v \ll c$. Find the minimal distance r_{min} between the protons. At what energy will the minimum distance between the protons be $r_{min} = 10^{-13}$ cm?

Answer

Problem 10

Question

Mirrors 1 and 4 in Figure 1 are 'half-silvered' so half the intensity of light incident upon them is transmitted and half is reflected. Ignore multiple reflections at all the mirrors and assume that other reflections and attenuations are negligible.

(a) Find the dependence of the intensity of transmission of the device on L, the indices of refraction n_1 and n_2, and the wavelength λ of the incident monochromatic, phase coherent light.

(b) You are provided with a crude light detector that can only measure the maximum and minimum of intensity with any confidence (but not intermediate light levels). You are starting with both chambers evacuated and you have a pressure gauge. Show how to determine the index of refraction of a gas as a function of pressure by monitoring the transmission intensity as gas is added to one of the tubes.

Answer

Spring 2014 Part II

Problem 1

Question

A bead of mass m is constrained to move on a frictionless hoop (see Figure 1). The hoop of radius R is forced to rotate about the vertical diameter with angular speed ω. The position of the bead is specified by coordinate θ. Write the Lagrangian for the system and derive the differential equation of motion for θ. Include the gravitational force in your analysis. Determine the equilibrium solutions for this problem, that is, the values of constant θ that solve the differential equation of motion. For each solution, discuss the conditions under which this solution exists and indicate whether it is a stable or an unstable equilibrium solution.

Answer

Problem 2

Question

A charge q is uniformly distributed throughout a spherical volume of radius a. Surrounding and concentric with this charge distribution is an uncharged, conducting spherical shell of inner radius b and outer radius $c\,(c > b > a)$. The space between the charge distribution and the conducting shell is filled with a linear isotropic, homogeneous dielectric, having dielectric constant κ. Find the polarization vector $\mathbf{P}$ in the dielectric, and find the electrostatic field $\mathbf{E}$ and potential V everywhere. Take V to approach zero at large distances from this configuration. Make two sketches highlighting the main features of the functional dependence of $\mathbf{E}$ and V on the radial coordinate.

Answer

Problem 3

Question

An ideal gas engine is working in a reversible Joule cycle shown in the T vs. S diagram (figure 2). The gas is monatomic and has n moles.

1. Draw the corresponding P-V diagram for the cycle.
2. Express the work done in terms of temperatures $T_1, T_2, T_3,$ and T_4.
3. Find the efficiency of the engine in terms of these temperatures.

Answer

Problem 4

Question

A particle is in an infinitely deep one-dimensional potential well of the width a located at $0 \le x \le a$

$$\begin{aligned} V(x) &= 0, && 0 \le x \le a \\ V(x) &= \infty, && x \le 0,\ x \ge a \end{aligned}$$

(a) Find the normalized wave functions that describe its energy states.

(b) Find the first order corrections ΔE_n to the energy levels for a perturbation of the form

$$\begin{aligned} \Delta V(x) &= V_0 \frac{2x}{a}, && 0 \le x \le \frac{a}{2}, \\ \Delta V(x) &= V_0 \left(2 - \frac{2x}{a}\right), && \frac{a}{2} \le x \le a \end{aligned}$$

(Hint: to solve integrals of the form $\int x\cos(kx)\mathrm{d}x$ you may use the technique of differentiating with respect to a parameter, as in $\int x\cos(kx)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}k}\int \sin(kx)\mathrm{d}x$.)

Answer

Problem 5

Question

It has been proposed to create a terrestrial source of high energy photons by having low energy laser photons collide with accelerator-generated high energy electrons. Consider a laser photon with energy $\hbar\omega = 1$ eV that collides head-to-head with an electron and reflects backward. The energy of the electron is $E = 50$ GeV $\left(= 50{\times}10^9 \text{ eV}\right)$. Find the energy of the photon after the collision.

[Hints: 1. note that you cannot neglect the electron mass because it is much larger than the photon energy; 2. to simplify calculations you may want to transform to (and back from) the center of mass frame. A different approach (with the same goal) is to make use of $(E + pc)\,(E - pc) = E^2 - p^2c^2 = m^2c^4$].

Answer

Fall 2014 Part I

Problem 1

Question

A photon source is at the focus of a parabolic mirror and both are attached to a rocket, see Figure 1. Upon reflection the photons form a parallel beam. Find the final velocity of a rocket if it starts from rest with mass m_1 at and its final rest mass is m_2. (Be sure to use relativistic expressions throughout.)

Answer

Problem 2

Question

In a laboratory on Earth mass m_1 is on a table and connected by a string, which runs over a frictionless pulley, to mass m_2 that is hanging from the side of the table. The table moves horizontally with acceleration a such that m_2 is tilted at a constant angle away from the table and m_1 is sliding. Find the tension in the string if the coefficient of kinetic friction between m_1 and the table is μ.

Answer

Problem 3

Question

Consider a two-dimensional anisotropic oscillator where the potential energy is given by

$$V = \frac{1}{2}\left(k_1 x^2 + k_2 y^2\right)$$

(a) Find the position and velocity as functions of time if the initial position and velocity are given by $r = (x_0, y_0)$ and $v = (0, 0)$.

(b) What is the condition on k_1 and k_2 for the particle's trajectory to be a closed Lissajous figure?

Answer

Problem 4

Question

The refractive index of glass can be represented approximately by the empirical relation

$$n = A + B\lambda^{-2} \tag{F2014 I 4.1}$$

where λ is the wavelength of light in vacuum. What are the corresponding phase and group velocities of light in glass? Do your formulae reduce to what you expect if there is no dispersion?

Answer

Problem 5

Question

Two identical bodies, each characterized by a heat capacity at constant pressure C which is independent of temperature, are used as heat reservoirs for a heat engine. The bodies remain at constant pressure and undergo no change of phase. Initially, their temperature are T_1 and T_2, $T_1 > T_2$. At the final state, as a result of the operation of the heat engine, the bodies will attain a common final temperature T_f.

(a) What is the total amount of work W done by the engine? Express the answer in terms of C, T_1, T_2, and T_f.

(b) Use arguments based on entropy considerations to derive an inequality relating T_f to the initial temperatures.

(c) For given initial temperatures, what is the maximum amount of work obtainable from the engine?

Answer

Problem 6

Question

Consider a model for the hydrogen atom in which the proton is assumed to be a point charge located at the origin and the electron is described by a continuous charge distribution with spherical symmetry:

$$\rho\left(r\right) = -\frac{e}{\pi a^3}\exp\left(-\frac{2r}{a}\right) \tag{F2014 I 6.1}$$

where a is the Bohr radius and r is the distance from the origin. Here e is the absolute value of the charge of the electron. In the presence of an electric field $\mathbf{E}$, the proton is displaced from the origin to a new equilibrium position a distance $d \ll a$ from the origin. Calculate d and the induced dipole moment p to find an expression for the polarizability of the hydrogen atom. Hint: use the approximation $d \ll a$ before evaluating the integral.

Answer

Problem 7

Question

At low enough temperatures, the thermodynamic properties of a two-dimensional d-wave superconductor can be described in terms of a gas of non-interacting fermions that follow the p_2 dispersion relation $E(\mathbf{k}) = \sqrt{a^2k_x^2 + b^2k_y^2}$, with a, b denoting positive constants. The total number of these fermions is not conserved. Determine how the specific heat of this system depends on the temperature T in this low-temperature regime. Hint: you do not need to evaluate the pre-factors and you can ignore the spin degeneracy.

Answer

Problem 8

Question

The wavefunction of a particle in the ground state of a one-dimensional oscillator with potential energy $U(x) = m^2\omega^2 x^2/2$ is

$$\psi(x) = \frac{e^{-x^2/\left(2\ell^2\right)}}{\left(\pi\ell^2\right)^{1/4}}$$

where $\ell = \sqrt{\hbar/(m\omega)}$. The oscillator potential is abruptly shifted by distance a so that the potential energy becomes $U(x) = m^2\omega^2 (x-a)^2/2$. What is the probability that the particle will stay in the ground state?

Answer

After the shift, the wave function should read

$$\psi(x) = \frac{e^{-(x-a)^2/\left(2\ell^2\right)}}{\left(\pi\ell^2\right)^{1/4}}.$$

To find the probability of finding the new wave function in the old ground state, we need to calculate

$$P = |\langle\psi_{g,o}|\psi_{g,n}\rangle| \tag{F2014 I 8.1}$$

Problem 9

Question

Answer

Problem 10

Question

The magnetic field inside a large piece of magnetic material is $\mathbf{B}_0$, so that $\mathbf{H}_0 = (1/\mu_0)\,\mathbf{B}_0 - \mathbf{M}$, where $\mathbf{M}$ is a magnetization that is frozen in the material.

(a) A long narrow cylinder with the long axis parallel to $\mathbf{M}$ is hollowed out of the material. Find the field $\mathbf{B}$ at the center of the cavity in terms of $\mathbf{B}_0$, and $\mathbf{M}$. Also find $\mathbf{H}$ at the center of the cavity in terms of $\mathbf{H}_0$, and $\mathbf{M}$.

(b) Do the same assuming that the cavity is a thin disc with the symmetry axis parallel to $\mathbf{M}$.

(c) Do the same assuming that a small spherical cavity is hollowed out of the material.

Note: the magnetic field inside a magnetized sphere with frozen-in magnetization $\mathbf{M}$ is $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$.

Answer

Fall 2014 Part II

Problem 1

Question

A particle of mass m and electric charge q is constrained to move along a smooth vertical hoop of radius R. The hoop is in a laboratory on Earth. At the lowest point of the hoop there is another fixed charge q. Find the equilibrium positions of m and the frequency of small oscillations about the equilibrium.

Answer

Problem 2

Question

Consider a long cylindrical co-axial capacitor with an inner conductor of radius a, outer conductor of radius b, and a linear dielectric in between that has permittivity $\epsilon(\rho) = \epsilon_r(\rho)\,\epsilon_0$, where ρ is the radial coordinate of a cylindrical coordinate system. The capacitor has a positive surface charge density σ on the inner conductor and is charged to a voltage V. You can tune the function $\epsilon(\rho)$.

(a) What is the function $\epsilon(\rho)$ such that the energy density in the capacitor is independent of ρ?

(b) Assuming this energy density, calculate the electric field, the electric displacement, and the polarization.

(c) Calculate the bound volume and surface charges.

Note: you may need to solve part B before you can completely determine $\epsilon(\rho)$ in terms of the parameters provided. Once $\epsilon(\rho)$ is determined, use its explicit form in all other expressions (such as fields, polarization, etc).

Answer

Problem 3

Question

Consider a system of N non-interacting atoms in contact with a thermal reservoir at a temperature T. Each one of these atoms can be only in one of two states: the ground state, with zero energy, and the excited state, with energy $\epsilon > 0$.

(a) Find the general expression for the free energy F of the system, and evaluate it in the limiting cases $T \to 0$ and $kT \gg \epsilon$.

(b) Compute the specific heat of the system and determine how it depends on the temperature in the cases $T \to 0$ and $kT \gg \epsilon$.

(c) The energy ϵ of the excited state of an atom depends on its average distance from the other atoms such that $\epsilon = b/v^\gamma$, where b is a constant, $v = V/N$ is the volume of the system per atom, and $\gamma > 1$ is the so-called Gruneisen parameter. Find the equation of state relating the pressure P, the volume V, and the total energy E of the system.

Answer

Problem 4

Question

The probability of nuclear decay when an α-particle (bound state of two protons and two neutrons) is emitted has a very strong dependence on the energy of the α-particle E. Empirically it was formulated in 1911 as the Geiger-Nuttal law:

$$\ln t_{1/2} = a_1 \frac{Z}{\sqrt{E}} - a_2$$

where $t_{1/2}$ is the half-life (in seconds), Z is the electric charge of the final nucleus, and $a_{1,2}$ are constants. The law was explained in 1928 by George Gamow who used the WKB approximation to calculate the transmission coefficient for the potential

$$U(r) = \begin{cases} -U_0 \text{ at } r < r_0 \\ Z\alpha/r \text{ at } r > r_0, \end{cases}$$

which represents a potential well at $r < r_0$ and a Coulomb repulsion at $r > r_0$.

Use this potential to find the coefficient a_1 for $E \ll Z\alpha/r_0$, where you can neglect the size of the nucleus, i.e. take the limit $r_0 \to 0$. Determine the numerical value of a_1 when the energy E is measured in MeV.

Reminder: the WKB approximation consists of assuming that the phase of the wave function in a weakly inhomogeneous potential $U(x)$ can be written as $\int k(x)\,dx$ where $k(x)$ is the wave number corresponding to the value of the potential at point x. Hint: an integral of the form $\int_0^1 \sqrt{\frac{1}{x} - 1}dx$ can be solved by substituting $x = \sin^2(\phi)$.

Answer

Problem 5

Question

A slow neutron hits a hydrogen atom in its ground state and forms a final state containing a deuterium atom and a photon, $n + H \to D + \gamma$.

The nucleus of the deuterium atom, a deuteron, is the bound state of a neutron and proton with a binding energy of 2.23 MeV.

What is the velocity of the deuterium in the final state in the limit of zero kinetic energy of the initial neutron? In this limit, what is the total probability to find the electron in any excited state of the deuterium atom?

Hints: (1) You can treat the deuterium motion as non-relativistic, $v \ll c$; (2) transition to a coordinate system moving with velocity $\mathbf{v}$ leads to multiplication of the wave function $\phi(\mathbf{r})$ by a factor $\exp(-im\mathbf{v}\mathbf{r}/\hbar)$; (3) the hydrogen wave function for the ground state is $\psi_0 = \left(1/\sqrt{\pi a^3}\right)\exp(-r/a)$, where a is the Bohr radius of the atom: $a = \hbar/(mc\alpha)$ with $\alpha = 1/137$.

Answer

Fall 2015 Part I

Problem 1

Question

A positive point charge q is fixed at a distance $a = 10$ cm above a grounded and conducting plane. An equal negative charge $-q$ is at a distance b above the plane, on the segment perpendicular to the plane that goes from the plane to the positive charge. Compute the value of b for which no force is acting on the negative charge, neglecting gravity.

Answer

The trick with this problem is that the charge at b is placed in between the conducting plane and the charge at a. First, for convenience, let us name each of these point charges. Take $P_1 = (q, a)$, and $P_2 = (-q, b)$. To solve this problem, use the method of images to remove the conducting plane and place charges $P_3 = (q, -b)$ and $P_4 = (-q, -a)$. Since the force on $(-q, b)$ should be zero, apply the superposition principle along with Coulomb's law to find

$$0 = \sum_i \mathbf{F}\left(P_i, P_1\right) = (-q) \sum_i \mathbf{E}\left(P_i, P_1\right).$$

Thus the sum of the electric fields should vanish. Since these point charges are colinear, this significantly reduces the problem. Then

$$\begin{aligned}
\mathbf{E}_{12} + \mathbf{E}_{32} + \mathbf{E}_{42} = \frac{q}{4\pi\epsilon_0}\left[\frac{(-\hat{\mathbf{z}})}{(a-b)^2} - \frac{(\hat{\mathbf{z}})}{(b-(-a))^2} + \frac{\hat{\mathbf{z}}}{(b-(-b))^2}\right] = 0 \\
\implies -\frac{1}{(b-a)^2} - \frac{1}{(b+a)^2} + \frac{1}{(2b)^2} = 0
\end{aligned}$$

This is an algebraic problem now. After the dust settles, we find the equation $-7b^4 - 10a^2b^2 + a^4 = 0$. The solutions are

$$b^2 = \frac{5a^2 \pm a^2\sqrt{32}}{-7} \implies b \approx 65.7\text{cm}$$

where we have taken the negative solution and used $a = 10$ cm.

Problem 2

Question

Consider a free Fermi gas consisting of N spin $\frac{1}{2}$ particles of mass m in 2 dimensions confined to a square with area $A = L^2$.

(a) Find the Fermi energy ϵ_F (in terms of N, A, and m).

(b) Derive a formula for the density of states. (Hint: You should find that it is a constant, independent of ϵ).

(c) Find the average energy per particle in terms of ϵ_F.

Answer

Problem 3

Question

An electron in a hydrogen atom does not fall to the proton because of quantum motion (which may be accounted for by the Heisenberg uncertainty relation for an electron localized in the volume with size r). This is true because the absolute value of the Coulomb potential energy goes to minus infinity with decreasing distance to the center r relatively slowly, like $-1/r$. Is such an "atom" stable for any potential behaving as $-1/r^s$? If not, find the range of values of s at which the "atom" is stable, so that "the electron" does not fall to the center.

Answer

Problem 4

Question

Estimate the average velocity (in m/s) and the mean free path (in m) of nitrogen molecules in this room. Hint: Recall that atmospheric pressure is about 10^5 N/m^2.

Answer

See solution for Fall 2000 Part I, Problem 8. These problems are identical.

Problem 5

Question

A coaxial cable of length $l = 1$ m is made of two thin coaxial copper cylinders with diameters $a = 1$ cm and $b = 2$ cm separated by air. At one end of the cable the internal and external cylinders are connected by a short wire. At the other end internal and external cylinders are connected to opposite poles of a battery. The current runs on the external cylinder to the opposite end of the cable and then returns back to the battery via the internal cylinder. Calculate the cable inductance L in H (Henry).

Answer

Problem 6

Question

A particle of mass m can move in 1 spatial dimension x. Its wave function is

$$\psi(x,t) = Ne^{-a|x|-ibt}$$

where t is time, and N, a, and b are positive constants. Find the potential $V(x)$ which governs the motion of this particle. (Hint: Take into account the discontinuity in the slope of the wave function at $x = 0$.)

Answer

Problem 7

Question

Consider the arrangement shown at right with a rectangular cart with mass M that can move horizontally along a rail on massless pulleys. A sphere with mass m is attached too the cart by a massless string of length L. If the string is always taut and there are no dissipative forces, what is the period of small oscillations for the sphere?

Answer

Problem 8

Question

Consider a particle which has only two energy states, $E_1 = 0$, $E_2 = \epsilon$.

(a) Compute the aver energy $\langle E \rangle$ of such particle in a reservoir with temperature T.

(b) Calculate the heat capacity C_V of a system of N such non-interacting particles.

Answer

Problem 9

Question

A solid cylinder with a uniform density has a mass M and radius R is released with zero initial speed from the top of an inclined surface, as shown in the figure.

(a) Calculate explicitly that moment of inertia off the cylinder.

(b) How long does it take for it to reach the bottom of the surface, assuming that it rolls without sliding?

(c) Suppose a hollow cylindrical shell with the same mass M and radius R is released at the same time as the cylinder. Does it reach the bottom of the inclined surface sooner or later than the cylinder? Explain your reasoning.

Answer

Problem 10

Question

A particular Cerenkov particle detector consists of a tank full of water. A tau passes through the detector and collides with a nucleus of a hydrogen atom in the detector. The tau-antineutrino and proton annihilate in the collision to produce an anti-tau and a neutron.

(a) What is that minimum energy the tau-antineutrino needs to have in the rest frame of the detector in order for this interaction to happen?

(b) Using the minimum energy from part (a), what is the magnitude and direction (with respect to the incoming anti-neutrino) of the momentum of the produced anti-tau?

That anti-tau has a mass of 1.8 GeV/c^2. The proton and neutron each have a mass of 0.90 GeV/c^2. Assume the antineutrino is massless.

Answer

Fall 2015 Part II

Problem 1

Question

A long linear solenoid has length ℓ, radius r, and n turns per unit length. The solenoid is part of a circuit having resistor of resistance R and a generator of emf $\mathcal{E}$. The emf of the generator is very slowly increased in such a way that the current I in the windings of the solenoid grows as $I = at$, where a is a small positive constant and t is time.

(a) Compute the total magnetic energy inside the solenoid.

(b) Compute the induced electric field inside and outside the solenoid.

(c) Compute the flux of the Poynting vector through the solenoid (evaluate $\mathbf{B}$ just inside the solenoid), and show that is it equal to the rate of increase with time of the magnetic energy inside the solenoid.

(d) When the current reaches the value I_0, the generator is switched off. Compute how the current evolves from this time on, and show that energy is conserved.

Answer

Problem 2

Question

You decide to do a Rutherford-type scattering experiment to find the composition of an unknown material. You shoot a beam of oxygen nuclei ($m = 16m_p$, where m_p is the proton mass) at the material, and find that the oxygen nuclei that are scattered by $60°$ have 54% of their initial kinetic energy. What is the mass of the nucleus that scattered the oxygen in units of the proton mass? You can assume that the velocity of the oxygen beam is much less than the speed of light.

Answer

Problem 3

Question

Consider a particle of mass m in an infinite one dimensional potential well of width $2L$ (i.e., $V = 0$ for $-L < x < L$ and is infinite otherwise).

(a) Compute the (properly normalized) eigenfunctions and the corresponding eigenenergies.

(b) Assume that the potential is perturbed by a small potential of height V_0 and width $2a$, where $a \leq L$. Compute the corrections to the eigenenergies to leading order in V_0.

(c) Compute the perturbed energies in the limit of $a = L$, and compare them with exact solutions in this limit.

Answer

Problem 4

Question

A metallic ball with radius R is immersed and suspended in a weakly conducting medium with a uniform conductivity σ in the middle of a large metallic vessel (e.g., salty water in a metallic bathtub).

(a) One wire from the battery is attached t the ball and the second wire is attached to the vessel. Calculate the resistance of the media. You can assume that the current is spherically symmetric around the ball. (This is how a standard plasma probe tests the ionization degree of a plasma).

(b) After the ball is charge to a charge Q_0, the batter is disconnected at $t = 0$ and the ball discharges with time. Find how the ball charge Q depends on time and the characteristic time of this discharge by writing a simple differential equation for $Q\,(t)$ (this characteristic time is called the Maxwell time).

Answer

Problem 5

Question

A hemoglobin molecule can bind four O_2 molecules. Assume that ϵ is the energy of each bound O_2, relative to O_2 at rest at an infinite distance. Let $\lambda = \exp(\mu/kT)$ denote the absolute activity of the free O_2, where μ is the chemical potential.

(a) Find the appropriate partition function.

(b) What is the probability that one and only one O_2 is adsorbed on a hemoglobin molecule? Carefully sketch the result qualitatively as a function of λ.

(c) What is the probability that all four O_2 are adsorbed? Sketch this result also as a function of λ.

Answer

Fall 2016 Part I

Problem 1

Question

A ball of mass m move with no friction along half a circle of center O and radius R (figure below). AB is the diameter of the circle, O_x is the horizontal axis and O_y the vertical axis pointing down. We call θ the angle between O_x and OM. The ball is attached to a spring of elastic constant k, attached to B on the other side. The ball moves along the arc of circle.

1. Determine the potential energy $E_p^* = E_p/E_0$ (with $E_0 = mgR$) as a function of θ and p, where p is function of k, R, g and m.

2. Determine the equilibrium positions(s) θ_e of the system. Draw the plot $E_p^*(\theta)$ as a function of θ for $p = 2$. Determine θ_e for $p = 2$.

Answer

Problem 2

Question

A hollow cylinder of mass m and radius a rolls without slipping down a movable wedge of mass M. The angle of the wedge relative to the horizontal surface is α, and the wedge is free to slide on this smooth horizontal surface. The contact between the cylinder and the wedge is perfectly rough. Find the acceleration of the wedge.

Answer

Problem 3

Question

A cylinder with adiabatic walls (i.e., thermally insulated walls) which is closed at both ends is initially divided into two equal volumes by a frictionless piston that is also thermally insulating. Initially the volume, pressure and temperature of the ideal gas in each side of the cylinder are V_0, p_0, and T_0 respectively. A heater in the right-hand volume is used to slowly heat the gas on that side until the pressure there reaches $64p_0/27$. If the heat capacity C_V of the gas is independent of temperature and $C_P/C_V = \gamma = 1.5$, find the following in terms of V_0, p_0, and T_0:

(a) The entropy change of the gas on the left.

(b) The final left-hand volume.

(c) The final left-hand temperature.

Answer

Problem 4

Question

In many cases, graphene can be modeled as a two-dimensional gas of non-interacting electrons with energy $\epsilon(\boldsymbol{k}) = \hbar\nu|\boldsymbol{k}|$, where $\hbar|\boldsymbol{k}|$ is the momentum and ν is the effective velocity. Each state is fourfold degenerate due to the spin and the so-called valley degrees of freedom. Consider a positive chemical potential corresponding to an electronic density n. Calculate the ratio between the average energy per particle and the Fermi energy of the system at zero temperature.

Answer

Problem 5

Question

A very long wire of radius a is suspended a distance d above an infinite conducting plane. The wire is uniformly charged with uniform charge density λ. In the case that $d \gg a$, find approximate expression for:

(a) The capacitance per unit length of the wire, conducting plane system.

(b) The surface charge density on the conducting plane as a function of y, the distance along the plane lateral to the wire.

Answer

Problem 6

Question

A thin rod of length a, mass m, and resistance R moves vertically with no friction and closes an electrical circuit with an inductance L. In this problem, we consider that the total resistance of the circuit is R and the total inductance of the circuit is L. A magnetic field $\boldsymbol{B}$ is applied perpendicularly to the circuit (out of page, the figure below). The rod is dropped at $t = 0$ with no initial velocity.

1. Determine the differential equation of the current in the rod.

2. If $R = 0$, determine the current intensity $i(t)$ and the speed of the rod $v(t)$.

Answer

Problem 7

Question

Consider a thin divergent lens with a focal distance $f' = -30\text{cm}$.

(a) Determine the distance between the lens and the virtual image of a point A located 30cm in front of the lens.

(b) If a vertical object AB of height 1mm is placed at point A, what is the height of its virtual image?

Answer

Problem 8

Question

A particle is confined between two planes at $x = 0$ and $x = L$ in an infinite well. The wave function is:

$$\psi\left(x,t\right) = A\sin(kx)e^{-i\omega t}$$

(a) Determine the possible values for k as a function of L and a positive integer n.

(b) Find A as a function of L.

(c) Draw $\left|\psi\left(x,t\right)\right|^2$ as a function of x in the case of $n = 1$ and $n = 2$.

Answer

Problem 9

Question

A particle of mass m is in the ground state in one-dimensional box of length L with impenetrable walls. Find the distribution of the probability P of the momentum p. (This means finding the function $\rho(p)$ such that the probability $\mathrm{d}P$ of measuring the momentum in the differential interval $\mathrm{d}p$ is given by $\mathrm{d}P = \rho(p)\,\mathrm{d}p$).

Answer

Problem 10

Question

A τ lepton decays into muon (μ^-), muon antineutrino ($\tilde{\mu}_\mu$) and τ neutrino (ν_τ). What is the maximal possible momentum of the muon (in MeV/c^2) in the rest frame of the τ lepton. The mass of τ^- is M= 1777MeV/c^2, the mass of the muon $m = 106$ MeV/c^2 and both neutrinos are very light, so that their masses can be neglected.

Answer

Fall 2016 Part II

Problem 1

Question

A simple pendulum of length L and mass M is suspended from the middle of a horizontal, rigid rod in such a way that the pendulum can swing only in the plane perpendicular to the plane of the support frame, as shown in the figure. The pendulum is a massless, rigid rod with the mass M at the end. The horizontal rod is rotated counterclockwise (when viewed from above) at rotation rate $\mathbf{\Omega}$. Let θ be the angle between the pendulum and the vertical.

(a) Find the angles of equilibrium (whether stable or unstable). Under what conditions are the equilibrium angles stable points?

(b) Using the reference frame that rotates with the support (i.e., in which the support frame is stationary), find the oscillation frequency for small amplitude oscillations about $\theta = 0$. For what rotation rates $\mathbf{\Omega}$ is $\theta = 0$ a stable equilibrium point?

(c) If there are other stable equilibrium angles(s) θ, find the frequency of small amplitude oscillations about these equilibrium point(s).

Answer

Problem 2

Question

A Helmholtz coil consists of two parallel circular current loops of identical radius R and separated by a distance d from each other. Each loop carries a uniform current I along the same direction. Define the z axis to be the common axis that crosses the centers of the two coils, such that $z = 0$ is at the point midway between them.

(a) What is the value of d for which the amplitude of the magnetic field $\boldsymbol{B}$ produced by the Helmholtz coil is such that $z\frac{\partial B}{\partial z} = 0$ and $\frac{\partial^2 B}{\partial z^2} = 0$ at $z = 0$.

(b) For the value of d that you obtained above, calculate $\boldsymbol{B}\left(\boldsymbol{z} = \mathbf{0}\right)$

Answer

Problem 3

Question

A futuristic starship with the mass one million metric tons departs from a base in the outer space. The cruising speed of the starship corresponds to the time dilation (as observed from the base) 10 times. The starship accelerates in a straight line and reaches the cruising speed and then decelerates (also in straight line) reaching its destination near a star in a distant galaxy, which moves very slowly with respect to the ship's home base. On the way back the ship again accelerates to its cruising speed and then decelerates returning to the base. Assuming that all the starships in the future are propelled by converting their fuel into light with 100% efficiency and perfectly directing all the generated light in the direction opposite to the thrust, find the mass of the starship after it returns to the base. Ignore gravitational effects.

Answer

Problem 4

Question

A one-dimensional simple harmonic oscillator of angular frequency ω is acted upon by a spatially uniform but time-dependent perturbation force:

$$F(t) = \frac{F_0 \tau / \omega}{(\tau^2 + t^2)} \qquad -\infty < t < +\infty.$$

At $t = -\infty$, the oscillator is known to be in the ground state. Calculate, to leading order in the perturbation potential derived from the force above, the probability that the oscillator is in the first excited state at $t = +\infty$. You may find useful the following relationships between the position and momentum of the harmonic oscillator and the creation and annihilation operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right) \qquad p = i\sqrt{\frac{m\omega\hbar}{2}} \left(-a + a^\dagger\right).$$

Answer

Problem 5

Question

Consider a classical statistical mechanics system consisting of N subsystems labeled by $i = 1, \cdots, N$, each of which can exists in two states $s_i = \pm 1$. Call n_+ the number of $+$ values and n_- the number of $-$ values. Let the total energy of the system be given by

$$E = -J \sum_{i=1}^{N} s_i$$

(a) Let the energy of the system be fixed at $E_0 = N\epsilon$ where $-J < \epsilon < 0$ (microcanonical ensemble). Express the number of configuration in terms of N and $x = n_+/N$. Express x in terms of E_0, N and J.

(b) Focus on a particular subsystem, $i = 1$. Compute the probability r (as a function of x) that $s_1 = +1$ divided by the probability that $s_1 = -1$, directly in the microcanonical ensemble, again assuming that N is large.

(c) Now suppose that the entire system is used as a heat bath for the subsystem considered in part (b). What temperature does the entire system have as a function of J and x?

(d) Repeat part (b) above, treating the subsystems $s_2, \cdots, s_N$ as a heat bath for system s_1 and then working in the canonical ensemble. Are your answers consistent?

(e) Give a qualitative sketch of the temperature as a function of energy. Note a peculiarity in the system for $E_0 > 0$.

You may find Sterling's approximation useful: $log\,(N!) \approx N\log(N) - N$.

Answer

Fall 2017 Part I

Problem 1

Question

Assume that the atmosphere near the earth's surface is in approximate hydrostatic equilibrium, where any movement of air parcels is isothermal. Derive an expression for the pressure P of the atmosphere as a function of the height z using the ideal gas law.

Answer

Problem 2

Question

You are asked to design a yo-yo that accelerates, as it drops, with a value tenth that of the acceleration due to gravity g. Yo-yo's are formed by two disks of radius R and connected by a spindle, of a smaller radius r, around which a string is wound (the string goes down vertically from your hand to the spindle). The spindle and the string have negligible mass, and the combined mass of the two disks is m. What should be the ratio R/r needed to achieve this acceleration as the yo-yo drops from a stationary hand holding onto the yo-yo string?

Answer

Problem 3

Question

An electric potential is given by the expression, $V(\boldsymbol{r}) = Ae^{-\alpha r}/r$. Here A and α are constant. Find

(a) The corresponding electric field $\boldsymbol{E}(\boldsymbol{r})$.

(b) The charge density $\rho(\boldsymbol{r})$ that generates this potential.

Answer

Problem 4

Question

The flux of solar radiation arriving at the earth is 1.4 kW/m^2. Estimate the total number flux of solar neutrinos arriving at the earth.

Answer

Problem 5

Question

In the state with given angular momentum ℓ and its projection on the z-axis m_z (i.e., with the wavefunction ψ_{ℓ,m_z}) find the average values of ℓ_x^2 and ℓ_y^2.

Answer

Problem 6

Question

A plank of length L and height W is in equilibrium, balanced on a cylinder of diameter D. What condition must be satisfied by L, W, and D if this equilibrium is to be stable? Hint: Consider the change in height of the mass center of the plank as it rotates through a small angle θ.

Answer

Problem 7

Question

Find the energy eigenvalues and normalized wave functions of the bound state in the potential

$$V\left(x\right) = -\alpha\delta\left(x\right)$$

Here α is a positive constant and δ is the Dirac delta function. Find the average value of kinetic and potential energies in the bound state.

Answer

Problem 8

Question

Let us consider two identical linear harmonic oscillators 1 and 2 separated by R. Each oscillator bears charges $\pm e$ with separation x_1 and x_2, as shown in Fig. The particles oscillate along the x axis. Let p_1 and p_2 denote the momenta. The force constant is C. All particles have mass m. Then the Hamiltonian of the non-interacting system is

$$H_0 = \frac{1}{2m}p_1^2 + \frac{1}{2}Cx_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}Cx_2^2$$

(a) Write down the Coulomb interaction energy (H_1) between all the charged particles. Then take the approximation, $|x_{1,2}| \ll R$ and obtain the leading term.

(b) By applying the normal mode transformation to the total Hamiltonian ($H_0 + H_1$),show the characteristic frequencies of these coupled oscillators are

$$\omega = \left[\left(C \pm \frac{2e^2}{R^3}\right)/m\right]^{1/2}$$

Answer

Problem 9

Question

Consider a system of N non-interacting particles each with a spin $S = 1$ and fixed position in an external magnetic field H. The particles have magnetic moment μ_B. If no magnetic field is present all spin projections S_z of a single particle are degenerate with an energy $E = 0$.

a) Plot the energy for all spin projections as a function of H.

b) Calculate the partition function of a single spin and then of the system as a function of the temperature and magnetic field H.

c) Calculate the average energy $\langle E \rangle$ of the system and find its form for $T \to 0$.

Answer

Problem 10

Question

A particle of mass m_0 decays at rest in the lab frame, producing particle 1 of mass m_1 and particle 2 of mass m_2. Find the energy of the particle 1 in the lab frame.

Answer

Fall 2017 Part II

Problem 1

Question

The Coriolis force is a force that acts on objects that are in motion relative to a rotating reference frame. The magnitude of the Coriolis acceleration of the object is a $\boldsymbol{a}_C = -2m\boldsymbol{\Omega} \times \boldsymbol{v}$, where $\boldsymbol{a}_C$ is the acceleration of the particle in the rotating system, $\boldsymbol{v}$ is the velocity of the particle with respect to the rotating system, and $\boldsymbol{\Omega}$ is the angular velocity vector having magnitude equal to the rotation rate $\boldsymbol{\Omega}$. The effect of the Coriolis force on the pendulum produces a precession, or rotation with time of the plane of oscillation. Describe the motion of this system, know as a Foucault pendulum. Assume that the oscillations have small amplitude with the horizontal excursions small compared with the length of the pendulum.

Answer

Problem 2

Question

Consider a tank that is divided by a barrier into two equal volumes V.

(a) One side of the tank is filled with N molecules of an ideal gas. The other side is empty. What is the change in entropy of the entire system if the barrier is removed and the system is equilibrated? Assume that the tank is thermally insulated.

(b) The barrier is closed again and the gas on one side is instantaneously heated up to a higher temperature T_{i1} while the gas on the other side remains at the initial temperature T_{i2}. Both sides of the tank are in thermal contact and allowed to equilibrate. What is the final temperature T_f of the gas?

(c) Calculate the change in entropy of the entire system in b) as a function of the initial temperatures.

(d) Now suppose that the barrier is removed and that the entire equilibrated gas with the heat capacity C_P and C_V undergoes the following quasi-static cycle:

  (a) $a \to b$ adiabatic expansion from V_1 to V_2.

  (b) $b \to c$ cooling at constant volume from T_1 to T_2.

  (c) $c \to d$ adiabatic compression from V_2 to V_1.

  (d) $d \to a$ heating at constant volume V_1.

  Calculate the thermal efficiency from the ratio of the entire work performed by the gas and the heat taken up in 4).

Answer

Problem 3

Question

An infinite straight wire carries a constant current I. A square loop, of total resistance R, has sides of length L, two of which are parallel to the wire. The closest side of the loop is initially at distance D_0 from the wire, and the loop is moving away from the wire, with a constant velocity v, under the influence of an external force.

Compute the following

(a) The flux of the magnetic field through the loop at a generic time t;

(b) The induced current in the loop and its direction;

(c) The energy dissipated through the loop per unit time;

(d) The net magnetic force on the loop;

(e) The power input by the external agent.

Answer

Problem 4

Question

A particle of spin 1/2 has a magnetic moment $\mu = \gamma \vec{s}$, where γ is the gyromagnetic ratio and $\vec{s}$ is the spin operator. The particle is placed in a magnetic field $B(t) = B(t)\,\hat{\boldsymbol{k}}$ with its spin in the $+x$ direction at time $t = 0$. Here $\hat{\boldsymbol{k}}$ is the unit vector in the $+z$ direction.

(a) Find the energy eigenvalues of the particle as functions of t.

(b) Find the spin wave function of the particle at time $t > 0$.

Answer

Problem 5

Question

Spaceships A, B, and C have the same proper length ℓ, and according to an observer on Earth, are moving with same relativistic speed v in the $+x$, $-x$, and $-y$ direction, respectively. Find the length of each spaceship as measured by an observer in the Spaceship A.

Answer

Spring 2018 Part I

Problem 1

Question

The electron in a hydrogen atom is in a state described by the following superposition of normalized energy eignestates u, with real $A > 0$,

$$\psi(r, \theta, \phi) = \frac{1}{5}(3u_{100} + Au_{211} - 2u_{21-1} + 3u_{321})$$

where the subscripts represent the quantum numbers $\{n, l, m_l\}$.

(a) Calculate A such that this wavefunction is normalized.

(b) Find the expectation value of the energy in this state, in terms of the ground state energy of hydrogen E_1.

(c) Find the expectation values of L^2 and L_z in this state.

Answer

Problem 2

Question

An electron is confined to the interior of a hollow spherical cavity of radius R with impenetrable walls. Find an expression for the pressure exerted on the walls of the cavity by the electron in its ground state, recalling that the Laplacian in spherical polar coordinates is given by

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial^2}{\partial r^2} r^2 + (angular\ part)$$

Answer

Problem 3

Question

Consider a "ballistic pendulum" made up of a heavy uniform rod 1 meter long suspended vertically from one end. A bullet traveling horizontally hits this rod at a distance of 75cm from the pivot point and becomes embedded in the rod, causing the rod to deflect by an angle of 30° . Assuming that the mass of the bullet is 1% of the mass of the rod, find the speed of the bullet. (Hint: you can neglect the mass of the bullet compared to the rod when determining the moment of inertia).

Answer

Problem 4

Question

Find the value of c for which the force given below is conservative. Determine the potential energy function for this force.

Answer

Problem 5

Question

Consider a spherical shell of radius R that has an imposed potential given by

$$V(\theta) = V_0 \cos^2(\theta)$$

(a) Determine the potential inside and outside of the sphere in terms of a Legendre polynomial expansion.

(b) Calculate the radial electric field inside and outside, and find the surface charge density.

(c) What is the total charge on the spherical shell?

Answer

Problem 6

Question

Consider a co-axial cable made up of an inner conductor of radius a and an outer conductor of radius b. The inner conductor is solid and contains a constant current density j. The outer conductor is a thin shell and contains a surface current that exactly balances the current of the inner conductor.

(a) Determine the surface current density on the outer conductor.

(b) Determine the magnetic field everywhere in space and calculate the total magnetic energy per unit length.

(c) Find the inductance per unit length of this cable.

Answer

Problem 7

Question

A system with an equidistant energy spectrum ($\epsilon_n = n \cdot \Delta, n = 0, 1, 2, \cdots$) is populated with identical bosons of spin $s = 0$. If $N \gg 1$ is the number of bosons occupying the lowest two orbitals and the occupation of the ground state is twice the occupation of the lowest excited state, find the:

(a) Temperature.

(b) Chemical Potential.

(c) Occupancy o the second excited orbital ($n = 2$).

Answer

Problem 8

Question

Two identical classical monoatomic ideal gases with the same temperature τ and the same number of atoms N are contained in two vessels of volumes V_1 and V_2 which are then connected. The combined system is isolated from the environment. After the system has reached equilibrium, what is the:

(a) Total energy.

(b) Pressure.

(c) Change in entropy.

(d) Change in temperature.

Answer

Problem 9

Question

A particle of mass m is given a kinetic energy equal to n times its rest energy. What is its speed and momentum?

Answer

Problem 10

Question

A typical neutron star has approximately the same mass as the sun but is as dense as a proton. Estimate the radius of a neutron star and the gravitational binding energy released in its formation.

Answer

Spring 2018 Part II

Problem 1

Question

Consider the infinite square well in one dimension, extending from $-a/2$ to $+a/2$. A particle of mass m is sitting in its ground state. At time $t = 0$ the size of the well doubles, so that instead of going from $-a/2$ to $a/2$, it goes from $-a$ to a. If one then measures the energy, what is the most probable result, and what is the probability of getting it?

Answer

Problem 2

Question

Consider a system as shown in the figure with three equal masses m connected with identical springs of spring constant k, with the whole system attached to two immovable walls. The masses are constrained to move only in the horizontal direction.

(a) Write down the equations of motion for each mass and find the normal mode frequencies (Hint: this gives a cubic equation, but you should be able to find an obvious common factor).

(b) Determine the eigenvectors (i.e., the ratio between the amplitudes of the three mases in each mode). Assume that the first mass has an amplitude of 1 for each mode. Sketch the motion in each mode.

Answer

Problem 3

Question

Consider a "sliding bar" generator made up of a U-shaped wire with width w and having a resistance R with the current closed by a movable bar with initial velocity v_0 as in the figure. A magnetic field $\mathbf{B} = B_0\hat{\boldsymbol{z}}$ comes out of the page.

(a) Find the emf generated and the current that flows through the circuit. Draw a diagram showing the direction of the current flow.

(b) Determine the magnetic force on the bar, and solve for its motion, assuming the bar has mass M.

(c) Show that the energy dissipated by the resistor is equal to the kinetic energy lost by the bar.

Answer

Problem 4

Question

The energy of a photon gas contained to a volume V and in thermal equilibrium with a reservoir at temperature τ is given by a Stefan-Boltzmann law, $U = \alpha V \tau^4$.

(a) Find the heat capacity C_V and the entropy σ.

(b) Find the free energy.

(c) Derive the "photon gas law", i.e., find $p\,(V, \tau)$.

(d) Calculate the work performed by the photon gas in isothermal expansion from $V_1 = V$ to $V_2 = 2V$.

(e) Calculate the heat transferred to the photon gas when it expands at constant pressure from $V_1 = V$ to $V_2 = V$.

Answer

Problem 5

Question

A radioactive isotope of bismuth, ${}^{210}_{83}Bi$ undergoes beta-decay into polonium with a mean lifetime τ_1 of 7.2 days. In turn, the polonium alpha-decays into lead with mean lifetime τ_2 of 200 days. Denote the number of bismuth and polonium nuclei at time t respectively by $N_1(t)$ and $N_2(t)$.

(a) Write out the nuclear reactions corresponding to both decays, carefully accounting for the atomic and mass numbers Z and A of the nuclei involved.

(b) The number of parent bismuth nuclei evolves with time according to the differential equation

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1$$

where $\lambda_1 = 1/\tau_1$ is the decay constant of bismuth. Write down the corresponding equation for the number of polonium nuclei $N_2(t)$ produced by bismuth decay, with λ_2 as the polonium decay constant.

(c) Solve these equations for $N_2(t)$, given the initial conditions $N_1(0) = N$ and $N_2(0) = 0$. Hint: First solve for $N_1(t)$ and use this result together with a simple integrating factor in the equation for $\frac{\mathrm{d}N_2}{\mathrm{d}t}$.

(d) Since we start with no polonium at $t = 0$, and that after a long enough time all the polonium produced will have decayed, there will be a time t^* at which the number of polonium nuclei and correspondingly, the rate of $\alpha-$particle emission will reach a maximum. What is that time (in days)?

Answer

Fall 2018 Part I

Problem 1

Question

A function $\Psi(x,t)$ is a solution to the Schroedinger equation for a potential $V(x)$:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t)$$

Consider now the potential $V'(x) = V(x) + V_0$ where V_0 is a constant.

(a) Is $\Psi(x,t)$ a solution of (1) with the potential $V(x)$ replaced by $V'(x)$? If not, can it be modified so that it is a solution?

(b) Compare with the situation in classical mechanics.

(c) Discuss whether the above results would have experimental consequences.

Answer

Problem 2

Question

Consider the Earth-Sun system as a quantum gravitational analog of a hydrogen atom.

(a) What would be, in actual units, the Bohr radius of the system?

(b) By equating the actual energy of the system to the Bohr formula, estimate the quantum number n of the Earth.

Answer

Problem 3

Question

A particle moves in one dimension under the action of a quadratic velocity dependent retarding force. (Assume the initial velocity is v_0 , and the motion is horizontal.)

a) Derive the expression for the position as a function of time.

b) Does this particle ever stop (i.e., at $t = \infty$ is the distance traveled finite)?

Answer

Problem 4

Question

A bullet of mass m and velocity v_0 strikes a target object of mass M, which is resting on a frictionless support. If the bullet emerges with velocity $v_0/2$, find the fraction of the initial kinetic energy which is deposited into the target as frictional heat in terms of the ratio of bullet mass to target mass $\gamma = m/M$.

Answer

Problem 5

Question

Two conducting metal objects are embedded in a weakly conducting material of conductivity σ.

(a) Show that the resistance between them is related to the capacitance by $R = \epsilon_0/\sigma C$.

(b) Suppose that you connect the two objects with a battery so that the potential difference is V_0. Show that when you disconnect the battery, the potential decreases exponentially and determine the time constant.

Answer

Problem 6

Question

A satellite in geostationary orbit is used to transmit data via electromagnetic radiation. The satellite is at a height of $35,000$ km above the surface of the earth, and we assume it has an isotropic power output of 1 kW.

a) What is the amplitude E_0 of the electric field vector of the satellite broadcast as measured at the surface of the earth?

b) A receiving dish of (projected) radius R focuses the electromagnetic energy incident from the satellite onto a receiver which has a surface area of 5 cm^2. How large does the radius R need to be to achieve an electric field vector amplitude of 0.1 mV/m at the receiver?

Answer

Problem 7

Question

Consider a one-particle system capable of three states ($\epsilon_n = n \cdot \Delta, n = 0, 1, 2$) in thermal contact with a reservoir at temperature τ. In the limits $\tau/\Delta \to \infty$ and $\tau/\Delta \to 0$, find the

(a) Energy.

(b) Free Energy.

(c) Heat Capacity.

Answer

Problem 8

Question

Consider a non-interacting gas of $N \gg 1$ ^4He atoms at temperature τ in a volume V at pressure p. If half of the atoms are in the ground state ($\epsilon = 0$), find the chemical potential μ.

Answer

Problem 9

Question

A radioactive nucleus in an excited state decays (at rest) with a lifetime of $\tau = 10^{-7}$s to its ground state through the emission of a gamma ray of energy $E = 15$ keV.

a) What is the wavelength of the photon emitted in this decay (in nm)?

b) What is the natural line width of the excited level (in eV)?

c) What is the length of the photon wave train (in meters)?

Answer

Problem 10

Question

The laboratory differential cross section for proton scattering in a certain process is $d\sigma/d\Omega = a + b\cos^2(\theta)$ with a and b constants with the numerical values $a = 420\mu$b/sterad and $b = 240\mu$b/sterad. Note that 1 barn $(b) = 10^{-24}$ cm^2.

a) What is the total cross section (numerical value)?

b) A counter mounted at 10 cm from the target at an angle of 60° has an effective area of 0.1 cm^2. The target thickness is 10^{-4} cm and the number of atoms per cm^3 in the target is 10^{22}. What is the counting rate, when the beam current is 1μA?

Answer

Fall 2018 Part II

Problem 1

Question

Recall that for a charged particle in a magnetic field the quantum Hamiltonian is:

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{p} \to -i\hbar\boldsymbol{\nabla}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{r} \rangle = \langle (\boldsymbol{p} - r\boldsymbol{A})/m \rangle$$

Is this result consistent with classical mechanics?

Answer

Problem 2

Question

A simple pendulum (mass m) is suspended from a cart (mass M) which moves without friction along a horizontal track.

(a) Taking the position of the cart on the tract as X and the angle of the pendulum as θ, find the equations of motion.

(b) Show how one of these reduces to the expected special case solutions if we (i) fix the position of the cart or (ii) imagine the pendulum. mass is stationary with respect to the cart.

Answer

Problem 3

Question

Tritium ions are confined inside a thin wall spherical vessel with a radius of R. (Tritium ions are positively charged isotopes of Hydrogen with one proton and two neutrons, and you can assume that they have the mass of three protons). When the hot gas of tritium ions is stored inside the vessel, it forms layers with different concentrations such that the charge density is $\rho = a_0 r$ where a_0 is a positive constant and r is the radial position from the center of the vessel. The goal is to inject additional Tritium ions into the container so that they can reach the center of the vessel. To know the requirements for your injection system, you need to map out the potential created by the ionic gas.

(a) What is the potential as a function of radial distance r from the center of the vessel? Make sure you determine the function for all regions of space and assume $V = 0$ at infinity. Give your answer in terms of R and a_0.

(b) Sketch a graph of the potential from part a).

(c) Assume the vessel has a radius of 1.00m and $a_0 = 1.88\times10^{-6}$ C/m^4 . If you are a long distance away from the vessel (you can use the approximation that $r = \infty$), at what minimum speed would you need to shoot a Tritium ion towards the vessel if you want it to reach the center of the vessel?

Answer

Problem 4

Question

The energy spectrum of a system consists of two orbitals with one-particle energies $\epsilon_1 = -\epsilon$ and $\epsilon_2 = \epsilon$. The system is in thermal and diffusive contact with a reservoir at temperature τ and chemical potential μ. Assuming that each orbital can be occupied by no more than one particle and that $\mu = 0$, find:

(a) All possible states of a system, (ϵ, N), and the grand partition function $\mathcal{Z}$.

(b) The probability that the system is in a state with zero energy.

(c) The probability for the system to be occupied by one particle.

(d) The average number of particles in the system, $\langle N \rangle$, and average energy, $\langle U \rangle$.

(e) $\langle N \rangle$ and $\langle U \rangle$ in the limits $\tau \to 0$ and $\tau \to \infty$.

Answer

Problem 5

Question

A spaceship travels at a constant velocity $v = 0.8c$ with respect to the Earth. Denote the spaceship frame coordinates by a prime ($'$). At time $t = t' = 0$ by Earth and spaceship clocks, respectively, a light signal is sent from the tail (back end) of the spaceship towards the nose (front end) of the spaceship, just as the tail of the spaceship (at $x' = 0$ in the spaceship frame) passes the Earth (at $x = 0$ in the Earth frame). The length of the spaceship, measured in a frame in which it is at rest, is L.

(a) At what time, by spaceship clocks, does the light signal reach the nose of the spaceship?

(b) At what time, by Earth clocks, does the light signal reach the nose of the spaceship?

Now suppose there is a mirror at the nose of the spaceship which immediately reflects the light signal back to the tail of the spaceship.

(a) At what time, by spaceship clocks, does the light signal finally return to the tail of the spaceship?

(b) At what time, by earth clocks, does the light signal finally return to the tail of the spaceship? All answers should be expressed in terms of L and c.

Answer