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# Fall 2011 Part I

## Problem 1

#### Question

An elevator operator in a skyscraper, being a very meticulous person, put a pendulum clock on the wall of the elevator to make sure that he spends exactly 8 hours a day at his work place. Over the course of his work day, he records that the time during which the elevator has acceleration $a$ is exactly equal to the time during which it has acceleration $-a$. Does the elevator operator work, in actual time, (1) more than 8 hours, (2) exactly 8 hours, or (3) less than 8 hours? Why?

#### Answer

The nominal period of a pendulum is

$$T_{nom} = 2\pi\sqrt{\frac{\ell}{g}}$$

but within the elevator, the acceleration $g$ is not going to be constant and will rather depend on the acceleration of the elevator. Therefore,

$$T_{\uparrow} = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad\qquad T_{\downarrow} = 2\pi\sqrt{\frac{\ell}{g-a}}$$

for the upward and downward cases, respectively.

Since the elevator operator observed that equal time was spent going up as was spent going down, so he must have observed $N$ oscillations in both cases. In order to compare to the actual time, we simply compare the elevator's total time measurement with that of a stationary clock.

$$NT_{\uparrow} + NT_{\downarrow} \stackrel{?}{=} 2NT_{nom}$$

$$\begin{aligned}
2\pi N\sqrt{\frac{\ell}{g+a}} + 2\pi N\sqrt{\frac{\ell}{g-a}} &\stackrel{?}{=} 4\pi N\sqrt{\frac{\ell}{g}} \\
\sqrt{\frac{1}{g+a}} + \sqrt{\frac{1}{g-a}} &\stackrel{?}{=} 2\sqrt{\frac{1}{g}} \\
\sqrt{\frac{g}{g+a}} + \sqrt{\frac{g}{g-a}} &\stackrel{?}{=} 2
\end{aligned}$$

Use the test value $a = 5$ for comparison (with $g = 10$)

$$\boxed{2.23 > 2} \qquad \text{(F2011 I 1.1)}$$

Therefore the elevator operator actually spends more than 8 hours in the elevator during his shift.

# Problem 2

### Question

A classical particle is subject to an attractive central force proportional to $r^\alpha$, where $r$ is the radius and $\alpha$ is a constant. Show by perturbation analysis what is required of $\alpha$ in order for the particle to have a stable circular orbit.

### Answer

Construct the Lagrangian for the system in order to determine the equations of motion for the given central force (noting that we were given the *force* so we need to make an appropriate potential).

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad\qquad V = \frac{k}{\alpha+1}r^{\alpha+1}$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{\alpha+1}r^{\alpha+1}$$

Conservation of angular momentum is a consequence of the $\theta$ and $\dot{\theta}$ coordinates:

$$0 = \frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right]$$
$$0 = \frac{d}{dt}\left[mr^2\dot{\theta}\right]$$

Nothing that

$$\ell = \left|\frac{\vec{r}\times\vec{p}}{m}\right| = r^2\dot{\theta}$$

we can say that

$$\dot{\theta} = \frac{\ell}{r^2}$$

Then returning to the $r$ and $\dot{r}$ coordinates in the Lagrangian,

$$\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - kr^\alpha \qquad\qquad \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right] = m\ddot{r}$$

Putting the differential equation together and substituting for the angular momentum per unit mass gives

$$m\ddot{r} = \frac{m\ell^2}{r^3} - kr^\alpha \tag{F2011 I 2.1}$$

In the case that the orbit is circular, $r$ must be a constant, so let $r = a$ and note that $\ddot{r} = 0$ necessarily.

$$\frac{m\ell^2}{a^3} = ka^\alpha$$

Returning to the differential equation, let the actual distance $r$ be a perturbation from a circular orbit, and Taylor expand in $x$ where $x = r - a$.

$$
\begin{aligned}
m\ddot{x} &= \frac{m\ell^2}{a^3}\left(1 + \frac{x}{a}\right)^{-3} - ka^{\alpha}\left(1 + \frac{x}{a}\right)^{\alpha} \\
m\ddot{x} &\approx \frac{m\ell^2}{a^3}\left(1 - 3\frac{x}{a} + \dots\right) - ka^{\alpha}\left(1 + \alpha\frac{x}{a} + \dots\right) \\
m\ddot{x} &\approx ka^{\alpha}\left(1 - 3\frac{x}{a}\right) - ka^{\alpha}\left(1 + \alpha\frac{x}{a}\right) \\
m\ddot{x} &\approx -3ka^{\alpha}\frac{x}{a} - \alpha ka^{\alpha}\frac{x}{a} \\
m\ddot{x} &\approx -ka^{\alpha-1}\left(3 + \alpha\right)x
\end{aligned}
$$

To form a stable orbit, the coefficient on $x$ must be negative, giving a simple harmonic solution. Therefore $3 + \alpha > 0$ to keep the coefficient negative and

$$\boxed{a > -3} \tag{F2011 I 2.2}$$

# Problem 3

### Question

A neutral conductor A with a spherical outer surface of radius $R$ contains three cavities B, C, and D, but is solid otherwise. B and C are spherical, and D is hemispherical. Without touching A, positive charges $q_B$ and $q_C$ are introduced at the centers of B and C, respectively.

1. Give the amount and the distribution of the induced charges on the surfaces of A, B, C, and D.
2. Now another positive charge $q_E$ is introduced at a distance $r > R$ from the center of A. Describe qualitatively the distribution of induced charges on the surfaces of A, B, C, and D.
3. Give the amount of the induced charges on the surfaces of A, B, C, and D for the situation in (2).

### Answer (1)

An ideal conductor will not support an electric field inside the solid, so each of cavities B and C will have a surface charge to cancel the electric fields emminating from $q_B$ and $q_C$ respectively.

- Cavity B will have a uniform surface charge density of $-q_B/4\pi r_B^2$, where $r_B$ is the radius of cavity B, with total induced charge $-q_B$ (because of symmetry and use of a Gaussian surface).
- Cavity C will have a uniform surface charge density of $-q_C/4\pi r_C^2$, where $r_C$ is the radius of cavity B, with total induced charge $-q_C$ (because of symmetry and use of a Gaussian surface).

Cavity D will not have a surface charge since a Guassian surface coincident with its boundary contains no charge.

The surface A will have total charge $q_B + q_C$ with uniform surface charge density of $\left(q_B + q_C\right)/4\pi R^2$ in accordance with the symmetry of a Gaussian surface containing the sphere as well as properties of an ideal conductor.

### Answer (2)

The surfaces B, C, and D will remain unaffected since the surrounding conductor shields the cavities from electric fields produced by charge $q_E$. The distribution on surface A will shift so that the negative charge concentration is greatest on the side nearest to $q_E$ with an increasingly positive distribution towards the opposite side.

### Answer (3)

The surface of A will still contain the same total charge $q_B + q_C$ since only a redistribution of induced charges occurred along the surface. Similarly, because surface B, C, and D are shielded from the electric field of $q_E$ by conductor A, the total charges along their surfaces remains unchanged as well.

# Problem 4

### Question

The dielectric strength of air at standard temperature and pressure is $3 \cdot 10^6$ V/m. What is the maximum intensity in units of W/m$^2$ for a monocromatic laser that can be used in the laboratory?

### Answer

Failure of a dielectric occurs when the energy density in the dielectric is great enough to overcome the ionization energy of the constituent atoms. This suggests that an electric field of greater than $3 \cdot 10^6$ V/m would cause this ionization to occur.

Starting here, We can calculate the energy density of the electric field at any point in space by

$$U_{em} = \frac{\varepsilon_0}{2} E^2$$

(where we've used the vacuum energy density since air differs very little from the vacuum permittivity).

Then the power transmitted by the laser is $P = cU_{em}$, so plugging in the numbers,

$$P = \frac{1}{2}\left(8.854 \cdot 10^{-12} \, \frac{\mathrm{C}^2}{\mathrm{N{\cdot}m}^2}\right)\left(3 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}}\right)\left(3 \cdot 10^6 \, \frac{\mathrm{V}}{\mathrm{m}}\right)^2$$
$$P = 1.19 \cdot 10^{10} \, \frac{\mathrm{W}}{\mathrm{m}^2}$$

$\boxed{\text{The maximum power of a laser usable in the lab is } 1.19 \cdot 10^{10}\ \mathrm{W/m^2}.}$

# Problem 5

### Question

What is the minimum energy of the projectile proton required to induce the reaction $p + p \to p + p + p + \bar{p}$ if the target proton is at rest?

### Answer

Energy and momentum must be conserved. At the minimum allowed energy, the resultant 4 proton/anti-protons will be colinear with no relative momentum with respect to one another, so the momenum equation in the lab frame is simply

$$p_i = 4p_f \tag{F2011 I 5.1}$$

Similarly, the resultant (anti-)protons are indistinguishable, so they will all have equivalent energy $E_f$. The initial protons have different energies since one is at rest in the lab frame while the other is moving, leading to the energy equation

$$\sqrt{p_i^2c^2 + m_p^2c^4} + m_pc^2 = 4\sqrt{p_f^2c^2 + m_p^2c^4}$$

Substituting the momentum relation into the equation, squaring, and simplifying,

$$\begin{aligned}
\sqrt{16p_f^2c^2 + m_p^2c^4} + m_pc^2 &= 4\sqrt{p_f^2c^2 + m_p^2c^4} \\
16p_f^2c^2 + m_p^2c^4 + m_p^2c^4 + 2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 16p_f^2c^2 + 16m_p^2c^4 \\
2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 14m_p^2c^4 \\
16p_f^2c^2 + m_p^2c^4 &= 49m_p^2c^4 \\
p_f^2 &= 3m_p^2c^4
\end{aligned}$$

Therefore,

$$p_i^2 = 48m_p^2c^4$$

and

$$E_1 = \sqrt{49m_p^2c^4}$$

$$\boxed{E_1 \approx 6.567\,\mathrm{GeV/c^2}} \tag{F2011 I 5.2}$$

# Problem 6

### Question

A subatomic particle has spin 1 and negative parity. It decays at rest into an $e^+e^-$ pair, which is produced in the $s$ and $d$ waves. From these data determine (1) the total spin of the $e^+e^-$ pair and (2) the intrinsic parity of $e^+$ relative to $e^-$.

### Answer

Let $p$ denote the original particle which decayed.

The $s$ state indicates an angular momentum $\ell = 0$ while $d$ indicates $\ell = 2$, so the relative angular momentum between the $e^+$ and $e^-$ is $\Delta\ell = 2$. To conserve total angular momentum (including spin) which was originally 1, both electron and positron must be in the spin down state.

$$\boxed{(1)_p = (2)_{\Delta\ell} + \left(-\tfrac{1}{2}\right)_{e^+} + \left(-\tfrac{1}{2}\right)_{e^-}} \tag{F2011 I 6.1}$$

Parity is a multiplicative quantum number, so because the original particle has negative parity, the electron and positron must have opposite relative parities (the electron is odd with respect to the positron and vice versa).

$$\boxed{\begin{aligned} (-1)_p &= (-1)_{e^+} \cdot (+1)_{e^-} \\ &\text{or} \\ (-1)_p &= (+1)_{e^+} \cdot (-1)_{e^-} \end{aligned}} \tag{F2011 I 6.2}$$

# Problem 8

### Question

Assume that the atmosphere near the earth's surface is in approximate hydrostatic equilibrium, where any movement of air parcels is gentle and adiabatic. Find an expression for the pressure $P$ of the atmosphere as a function of the height $z$.

### Answer

Note that the pressure at a given point is due to the mass of air above the given point. Then by moving an infinitesimal distance vertically, the total mass is changed by the density of the air (which is affected by the gravitational force). This leads to the differential equation

$$\frac{dP}{dz} = \rho g$$

Then using the ideal gas equation

$$PV = Nk_B T$$

multiply and divide by the average molecular mass $m$ of the air (in kg) which combined with the number of molecules $N$ gives the total mass

$$PV = (Nm)\frac{1}{m}k_B T$$

and then divide by the volume to get the ideal gas equation in terms of the mass density

$$\begin{aligned} P &= \frac{Nm}{V}\frac{1}{m}k_B T \\ P &= \rho\frac{k_B T}{m} \\ \rho &= \frac{Pm}{k_B T} \end{aligned}$$

Finally, substitute this into the differential equation above and solve to get the atmospheric scale height equation.

$$\begin{aligned} \frac{dP}{dz} &= \frac{Pm}{k_B T}g \\ \frac{dP}{P} &= \frac{mg}{k_B T}dz \end{aligned}$$

$$\boxed{\begin{aligned} &P(z) = P_0 e^{z/\xi} \\ &\qquad \text{where } \xi = \frac{k_B T}{mg} \end{aligned}} \qquad \text{(F2011 I 8.1)}$$

# Fall 2011 Part II

## Problem 1

### Question

Mass $m_1$ moves freely along a fixed, long, horizontal rod. The position of $m_1$ on the rod is $x$. A massless string of length $\ell$ is attached to $m_1$ at the end and to mass $m_2$ at the other. Mass $m_2$ executes pendulum motion in the vertical plane containing the rod.

1. Find the Lagrangian of the system.
2. Derive the equations of motion and the corresponding conservation laws.
3. Assume that $x(0) = x_0$, $\dot{x}(0) = 0$, $\varphi(0) = \varphi_0$ $\left(|\varphi_0| \ll 1\right)$, and $\dot{\varphi}(0) = 0$. Find $x(t)$ and $\varphi(t)$ for $t > 0$.

### Answer (1)

For the sliding support mass $m_1$:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{x}^2 \\ V_1 &= 0 \end{aligned}$$

For the pendulum mass $m_2$:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} m_2 \left(\dot{x} + \dot{x}_2\right)^2 \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Then using $x_2 = \ell \sin\varphi$ and $y_2 = -\ell\cos\varphi$,

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \left(\ell^2 \dot{\varphi}^2 + \dot{x}^2 + 2\ell\dot{\varphi}\dot{x}\cos\varphi\right) \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Putting the Lagrangian together equals the first line. Applying the small angle approximation gives the second line where the kinetic energy term involving $\cos\varphi$ can be simply expanded as $\cos\varphi \approx 1$, but the potential energy term must be expanded to second order so that $\cos\varphi \approx 1 - \frac{1}{2}\varphi^2$.

$$\boxed{\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + 2\ell\dot{\varphi}\dot{x}\cos\varphi\right) + m_2 g\ell\cos\varphi \\ \mathcal{L} &\approx \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + 2\ell\dot{\varphi}\dot{x}\right) + m_2 g\ell - \frac{1}{2} m_2 g\ell\varphi^2 \end{aligned}}$$

(F2011 II 1.1)
(F2011 II 1.2)

### Answer (2)

Constructing the Euler-Lagrange equations for $x$ and $\dot{x}$:

$$\frac{\partial\mathcal{L}}{\partial x} = 0 \qquad \begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{x}} &= \left(m_1 + m_2\right)\dot{x} + m_2\ell\dot{\varphi} \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] &= \left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\varphi} \end{aligned}$$

$$\left(m_1 + m_2\right)\ddot{x} + m_2 \ell \ddot{\varphi} = 0 \tag{F2011 II 1.3}$$

and for $\varphi$ and $\dot{\varphi}$:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m_2 g \ell \varphi \qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m_2 \ell^2 \dot{\varphi} + m_2 \ell \dot{x}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right] = m_2 \ell^2 \ddot{\varphi} + m_2 \ell \ddot{x}$$

$$-m_2 g \ell \varphi - m_2 \ell^2 \ddot{\varphi} + m_2 \ell \ddot{x} = 0 \tag{F2011 II 1.4}$$

The equations of motion are:

$$\boxed{\begin{aligned} \ddot{x} + \frac{m_2}{m_1 + m_2} \ell \ddot{\varphi} = 0 \\ \ddot{\varphi} + \frac{1}{\ell}\ddot{x} + \frac{g}{\ell}\varphi = 0 \end{aligned}} \qquad \begin{aligned} &(\text{F2011 II 1.5}) \\ &(\text{F2011 II 1.6}) \end{aligned}$$

**Answer (3)**

Solve for $\ddot{x}$ and substitute into the other differential equation

$$\begin{aligned} \ddot{\varphi} - \frac{1}{\ell}\frac{m_2}{m_1 + m_2}\ell\ddot{\varphi} + \frac{g}{\ell}\varphi &= 0 \\ \frac{m_1}{m_1 + m_2}\ddot{\varphi} + \frac{g}{\ell}\varphi &= 0 \\ \ddot{\varphi} + \frac{g}{\ell}\frac{m_1 + m_2}{m_1}\varphi &= 0 \qquad (\text{F2011 II 1.7}) \end{aligned}$$

This is just the differential equation for a simple harmonic oscillator, so considering the given boundary conditions,

$$\boxed{\begin{aligned} &\varphi(t) = \varphi_0 \cos(\omega t) \\ &\qquad \text{where } \omega^2 = \frac{g}{\ell}\frac{m_1 + m_2}{m_1} \end{aligned}} \tag{F2011 II 1.8}$$

Then differentiating $\varphi(t)$ twice and substituting into the first equation,

$$\ddot{x} = \ell \varphi_0 \omega^2 \frac{m_2}{m_1 + m_2}\cos(\omega t)$$

Then integrating twice and applying the boundary conditions,

$$\boxed{x(t) = x_0 - \frac{g}{\omega^2}\frac{m_2}{m_1}\cos(\omega t)} \tag{F2011 II 1.9}$$

# Spring 2012 Part I

## Problem 1

### Question

For a many particle system of weekly interacting particles, will quantum effects be more important for (a) high densities or low densities and (b) high temperatures or low temperatures for a system. Explain your answers in terms of the de Broglie wavelength $\lambda$ defined as $\lambda^2 \equiv h^2/\left(3mk_B T\right)$ where $m$ is the mass of the particles and $k_b$ Boltzmann's constant.

### Answer

(a) High density — The de Broglie wavelength gives a "size" of the particle, and in the high density limit, the wavefunctions overlap significantly so quantum effects and interactions are critical to the behavior of the system.

(b) Low temperature — Since $\lambda^2 \propto T^{-1}$, as $T \to 0$, $\lambda$ increases so that again the wavefunctions overlap and quantum effects are significant.

## Problem 2

### Question

The ground state energy of Helium is $-79\,\text{eV}$. What is its ionization energy, which is the energy required to remove just one electron?

### Answer

Using the Hydrogen solution with modifications for single-electron atoms of higher $Z$, we know that the ground state energy of singly ionized Helium is

$$E^1_{He} = 2^2\,(-13.6\,\text{eV}) = -54.4\,\text{eV}$$

Therefore, the difference between the singly-ionized and neutral ground state energies gives the first ionization energy of the Helium atom.

$$\boxed{E_i = -24.6\,\text{eV}} \tag{F2012 I 2.1}$$

# Problem 3

### Question

It is known that the force per unit area ($F/A$) between two neutral conducting plates due to polarization fluctuations of the vacuum (namely, the Casimir force) is a function of $h$ (Planck's constant), $c$ (speed of light), and $z$ (distance between the plates) only. Using only dimensional analysis, obtain $F/A$ as a function of $h$, $c$, and $z$.

### Answer

The units of $F/A$ are

$$\frac{F}{A} = \left[\frac{\text{kg}}{\text{m}\cdot\text{s}^2}\right]$$

The kg suggests a factor proportional to $h$, making the equation

$$\frac{F}{A} \sim \left[\frac{1}{\text{m}^3\cdot\text{s}}\right] h$$

Accounting for the factor of seconds requires a $c$:

$$\frac{F}{A} \sim \left[\frac{1}{\text{m}^4}\right] hc$$

Finally, account for all the factors of distance:

$$\frac{F}{A} \sim \frac{hc}{z^4}$$

Therefore,

$$\boxed{\frac{F}{A} \sim \frac{hc}{z^4}} \tag{F2012 I 3.1}$$

# Problem 4

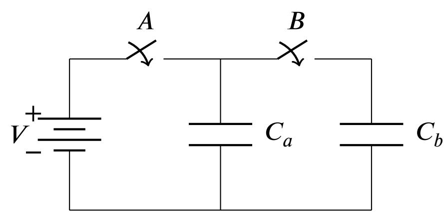
### Question

In the circuit diagram opposite, initially the two identical capacitors with capacitance $C$ are uncharged. The connections between the components are all made with short copper wires. The battery is an ideal EMF and supplies a voltage $V$.

(a) At first Switch A is closed and Switch B is kept open. What is the final sotred energy on capacitor $C_a$?

(b) Switch A is opened and afterwards Switch B is closed. What is the final energy stored in both capacitors?

(c) Provide a physical explanation for any difference between the results of parts (a) and (b), if there is one.




### Answer

(a) Initially, the right side of the circuit with $C_b$ can be ignored, so the total energy is simply the energy stored within $C_a$.

$$\boxed{E = \frac{1}{2}CV^2} \tag{F2012 I 4.1}$$

(b) The system is now effectively just the two capacitors on the right. Because the voltage difference is supported across both capacitors, the system can be modeled as an effective capacitor in parallel

$$C_{eff} = 2C$$

The total charge stored by the capacitors must remain the same when switching from Switch A being closed to Switch B. Initially,

$$Q = CV_i$$

and afterwards it is

$$Q = C_{eff}V = 2CV_f$$

so the final voltage across the capacitors is

$$V_f = \frac{1}{2}V_i$$

This means the total energy is

$$E = \frac{1}{2}C_{eff}{V_f}^2$$

$$\boxed{E = \frac{1}{4}CV^2} \tag{F2012 I 4.2}$$

(c) The energy is dissipated (heat, fields, etc).

# Problem 5

### Question

A planet of mass $m$ moves around the sun, mass $M$, in an elliptical orbit with minimum and maximum distances of $r_1$ and $r_2$, respectively. Find the angular momentum of the planet relative to the center of the sun in terms of these quantities and the gravitational constant $G$.

### Answer

We solve the problem using conservation of energy since we know that stable elliptical orbits have constant energy. The generic equation is

$$E = \frac{L^2}{2I} - \frac{GMm}{r}$$

where $L$ is the angular momentum and $I$ the moment of inertia. Substituting for the values at both $r_1$ and $r_2$ and equating,

$$\frac{L^2}{2m{r_1}^2} - \frac{GMm}{r_1} = \frac{L^2}{2m{r_2}^2} - \frac{GMm}{r_2}$$
$$\frac{L^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right) = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

which leads to the solution

$$\boxed{L = \sqrt{\frac{2GMm^2 r_1 r_2}{r_1 + r_2}}} \tag{F2012 I 5.1}$$

# Problem 6

### Question

A particle moves in a circular orbit under the influence of a central force that varies as the $n$-th power of the distance. Show that this motion is unstable if $n < -3$. (Hint: Consider the centrifugal potential.)

### Answer

See solution for Fall 2011 Part I, Problem 2 with the condition inverted so that *in*stability is $n < -3$ rather than stability requiring $n > -3$.

# Problem 7

### Question

A classical, ideal, monatomic gas of $N$ particles is reversibly compressed *isentropically*, i.e. with the entropy kept constant, from an initial temperature $T_0$ and pressure $P$ to a pressure $2P$. Find (a) the work done on the system, and (b) the net change in entropy of the system and its surroundings.

(a) An isentropic process is the same as an adiabatic process since no heat can be exchanged ($T\,\mathrm{d}S = Q = 0$), so we begin with the relation that $PV^\gamma$ is a constant. Combining this with the ideal gas law, we can determine that

$$P^{1-\gamma}T^\gamma = \text{const}$$

where $\gamma = C_p/C_v$ is the ratio of heat capacities with $C_p = \frac{5}{2}Nk_B$ and $C_v = \frac{3}{2}Nk_B$ for a monatomic ideal gas. Using this, we solve for the final temperature of the system after compressions as

$$T_f = 2^{2/5}T_0 \approx 1.32T_0$$

Combining both of

$$\Delta U = C_v \Delta T$$
$$\Delta U = Q + W$$

where $Q = 0$, we get that

$$\boxed{W = \frac{3}{2}Nk_B T_0 \left(2^{2/5} - 1\right)} \tag{F2012 I 7.1}$$

(b) Because the compression is done reversibly, by definition, $\Delta S = 0$.

# Problem 8

### Question

For an idea Fermi gas of $N$ neutral spin-$\frac{1}{2}$ particles in a volume $V$ at $T = 0$, calculate the following:

(a) The chemical potential

(b) The average energy per particle

(c) The pressure

### Answer

(a) At $T = 0$, the particles are all in the lowest state allowed by Fermi-Dirac statistics, so the chemical potential, defined by the energy required to add another particle to the system, is equal to the Fermi energy. For a particle contained within a box $V$, the energy per particle is

$$\varepsilon_n = \frac{\pi^2\hbar^2}{2mV^{2/3}}n^2$$

Given a Fermi energy $\varepsilon_F$, the maximum occupied state is

$$n_F = \sqrt{\frac{2mV^{2/3}}{\pi^2\hbar^2}}\sqrt{\varepsilon_F}$$

Equally we know that all $N$ particles must exist within the eighth-sphere of $n$ space, where the extra factor of 2 is because there are two spin states per $n$:

$$\begin{aligned}
N &= 2 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi {n_F}^3 \\
N &= \frac{1}{3}\pi \left(\frac{2m}{\pi^2\hbar^2}\right)^{3/2} V {\varepsilon_F}^{3/2} \\
\varepsilon_F &= \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}
\end{aligned}$$

Therefore $\mu = \varepsilon_F$,

$$\boxed{\mu = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}} \qquad \text{(F2012 I 8.1)}$$

(b) To get the total energy, we can imagine filling all $N$ particles one at a time, so that at each step, there are $N'$ total particles:

$$\begin{aligned}
U &= \int_0^N \varepsilon_F \,\mathrm{d}N' \\
U &= \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \int_0^N N^{2/3}\,\mathrm{d}N' \\
U &= \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \frac{3}{5}N^{5/3}\,\mathrm{d}N'
\end{aligned}$$

Therefore, the average energy per particle is $U/N$ or

$$\boxed{\langle\varepsilon\rangle = \frac{3}{5}\varepsilon_F} \qquad \text{(F2012 I 8.2)}$$

(c) From the thermodynamic relation

$$dU = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N$$

we can read off the derivative that defines the pressure $P$ as

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

Doing so, we get that

$$\frac{\partial U}{\partial V} = \frac{3}{5}N \cdot \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \left(-\frac{2}{3V}\right)$$

making the pressure

$$\boxed{P = \frac{2}{5}\frac{N}{V}\varepsilon_F} \tag{F2012 I 8.3}$$

# Problem 9

### Question

A transparent spherical lens, with a uniform index of refraction $n$ has the image of a distant object on the opposite surface of the lense when it is in a vacuum. What is the index of refraction $n$?

### Answer

# Problem 10

### Question

A piece of $p$-doped silicon has a carrier density $n = 10^{15}\,\mathrm{cm}^{-3}$ and dimensions of $\Delta x = 10\,\mathrm{mm}$, $\Delta y = 2\,\mathrm{mm}$, and $\Delta z = 1\,\mathrm{mm}$. A magnetic field of $B_z = 1\,\mathrm{T}$ is applied in the $z$-direction and a current $I_x = 1\,\mathrm{A}$ flows in the $x$-direction, and the voltage $V_y$ is measured.

(a) Express the current density $j_x$ in terms of the carrier density $n$ and the carrier velocity $v_x$.

(b) Write down the equilibrium force condition that determins $V_y$.

(c) Find $V_y$ in volts.

### Answer

(a) The current passing through each thin cross-sectional slice of the conductor is dependent on the charge of a carrier, carrier density, and velocity of the flow.

$$I_x = en\Delta y\Delta z v_x$$

The current density is just the current passing through each point, so

$$j_x = \frac{I_x}{\Delta y\Delta z}$$

$$\boxed{j_x = nev_x} \tag{F2012 I 10.1}$$

(b) The positive carriers drift to the edge of the conductor due to the magnetic field and the holes accumulate on the opposite edge. An electric field is created between the charge separation, so an equilibrium is set up between the electric field trying to bring the opposite charges together and the magnetic drift separating them.

$$0 = e\vec{E} + \vec{v} \times \vec{B}$$

By the right-hand rule, the positive charges accumulate along $y = 0$, so $\vec{E} = E\hat{y}$. Similarly, $\vec{v} \times \vec{B} = -v_x B_z \hat{y}$:

$$0 = eE\hat{y} - ev_x B_z \hat{y}$$

Written in terms of the potential $V_y = E\Delta y$, the equilibrium condition becomes

$$\boxed{V_y = v_x B_z \Delta y} \tag{F2012 I 10.2}$$

(c) Substituting in for given quantities

$$V_y = \frac{I_x B_z}{ne\Delta z}$$

$$V_y = \frac{(1\,\mathrm{A})\,(1\,\mathrm{T})}{\left(10^{15}\,\mathrm{cm}^{-3}\right)\left(1.612 \cdot 10^{-19}\,\mathrm{C}\right)(1\,\mathrm{mm})}$$

$$\boxed{V_y = 6.24\,\mathrm{V}} \tag{F2012 I 10.3}$$

# Spring 2012 Part II

## Problem 1

### Question

An electron in a hydrogen atom occupies a state:

$$|\psi\rangle = \sqrt{\frac{1}{3}}\,|3, 1, 0, +\rangle + \sqrt{\frac{2}{3}}\,|2, 1, 1, -\rangle$$

where the properly normalized states are specified by the quantum numbers $|n, \ell, m, \pm\rangle$ and the $\pm$ specifies whether the spin is up or down.

(a) What is the expectation value of the energy in terms of the ground state energy?

(b) If you meausred the expectation values of the orbital momentum squared $\langle L^2\rangle$, the square of the spin $\langle S^2\rangle$, and their $z$-components $\langle L_z\rangle$ and $\langle S_z\rangle$, what would be the result?

(c) Show that if you measure the position of the electron, the probability density for finding it an an angle specified by $\theta$ and $\phi$ integrated over all values of $r$ is independent of $\theta$ and $\phi$. Note, for this part you will need $Y_1^0 = \sqrt{3/4\pi}\cos\theta$ and $Y_1^1 = -\sqrt{3/8\pi}\sin\theta\exp(i\phi)$. You do *not*, however, need to know the radial functions, only that they are properly normalized and orthogonal to each other.

(d) List all additional possible states that are degenerate with the first state in the linear combination above. Note: this part can be done even if you have not answered the previous parts.

Assume now that the state $|\psi\rangle$, given above, is the initial state of an electron in a hydrogen atom.

(e) Write down the elctron's state as a function of time for all $t > 0$.

(f) go through the results you obtained in parts (a) through (c) and determine which of them are time independent.

### Answer

(a) Calculate the energy by sandwhiching the Hamiltonian between the wavefunction:

$$\begin{aligned}
\langle E\rangle &= \langle\psi|H|\psi\rangle \\
&= \left(\sqrt{\frac{1}{3}}\,\langle 3, 1, 0, +| + \sqrt{\frac{2}{3}}\,\langle 2, 1, 1, -|\right) H \left(\sqrt{\frac{1}{3}}\,|3, 1, 0, +\rangle + \sqrt{\frac{2}{3}}\,|2, 1, 1, -\rangle\right) \\
&= \frac{1}{3}\,\langle 3, 1, 0, +|H|3, 1, 0, +\rangle + \frac{\sqrt{2}}{3}\,\langle 2, 1, 1, -|H|3, 1, 0, +\rangle \\
&\quad + \frac{\sqrt{2}}{3}\,\langle 3, 1, 0, +|H|2, 1, 1, -\rangle + \frac{2}{3}\,\langle 2, 1, 1, -|H|2, 1, 1, -\rangle
\end{aligned}$$

For every term, the wavefunctions are eigenstates of the Hamiltonian, so we extract the appropriate energy term from every bra-ket sandwhich. Then the middle two terms integrate to zero since states with different $n$ are orthogonal while the first and last terms integrate to unity since they are properly normalized.

$$\langle E\rangle = \frac{1}{3}E_3 + 0 + 0 + \frac{2}{3}E_2$$

Each energy is related to the ground state energy by $E_n = E_0/n^2$, so

$$= \frac{1}{3}\frac{E_0}{9} + \frac{2}{3}\frac{E_0}{4}$$

$$\boxed{\langle E\rangle = \frac{11}{54}E_0 \approx -2.77\,\mathrm{eV}} \tag{F2012 II 1.1}$$

(b) For each of the other expectation values, the process is very similar with an appropriate change for eigenvalues; specifically,

$$\begin{aligned}
L^2\,|n,\ell,m,\pm\rangle &= \ell(\ell+1)\,\hbar^2\,|n,\ell,m,\pm\rangle \\
S^2\,|n,\ell,m,\pm\rangle &= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\,|n,\ell,m,\pm\rangle \\
L_z\,|n,\ell,m,\pm\rangle &= \ell\hbar\,|n,\ell,m,\pm\rangle \\
S_z\,|n,\ell,m,\pm\rangle &= \pm\frac{1}{2}\hbar\,|n,\ell,m,\pm\rangle
\end{aligned}$$

The same restrictions that the middle terms integrate to zero because of orthogonality and the first and last terms integrate to unity still applies, so we can almost immediately conclude that

$$\boxed{\begin{aligned}
\left\langle L^2\right\rangle &= 2\hbar^2 && \text{(F2012 II 1.2)}\\
\left\langle S^2\right\rangle &= \frac{3\hbar^2}{4} && \text{(F2012 II 1.3)}\\
\langle L_z\rangle &= \frac{2\hbar}{3} && \text{(F2012 II 1.4)}\\
\langle S_z\rangle &= -\frac{\hbar}{6} && \text{(F2012 II 1.5)}
\end{aligned}}$$

(c) In the $|r,\theta,\phi\rangle$ basis,

$$\begin{aligned}
|3,1,0\rangle &= R_{3,1}(r)\,Y_1^0(\theta,\phi) = R_{31}(r)\sqrt{\frac{3}{4\pi}}\cos\theta \\
|2,1,1\rangle &= R_{2,1}(r)\,Y_1^1(\theta,\phi) = -R_{21}(r)\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta
\end{aligned}$$

This means that the probability density is

$$\begin{aligned}
\langle\psi|\psi\rangle &= \frac{1}{3}\langle 3,1,0|3,1,0\rangle + \frac{\sqrt{2}}{3}\langle 2,1,1|3,1,0\rangle + \frac{\sqrt{2}}{3}\langle 3,1,0|2,1,1\rangle + \frac{2}{3}\langle 2,1,1|2,1,1\rangle \\
&= \frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}(r)\,R_{31}(r) + \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)
\end{aligned}$$

Integrating over $r$,

$$\begin{aligned}
\int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \int_0^\infty &\frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}(r)\,R_{31}(r) \\
&+ \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)\,\mathrm{d}r
\end{aligned}$$

Integrating over all $r$, we know that $R_{n\ell}R_{n'\ell'}$ are orthonormal, so again the first and last terms' $R$ integrates to unity and the middle term integrates to zero.

$$= \frac{1}{4\pi}\left(\cos^2\theta + \sin^2\theta\right)$$

Therefore we find that the probability density is constant in $\theta$ and $\phi$ when integrated over all $r$.

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \frac{1}{4\pi}} \tag{F2012 II 1.6}$$

(d) The states degenerate with the first term in $\psi$ are all combinations of allowed $\ell$, $m$, and $\pm$: $n$ must remain at $n = 3$ since it is the $n$ quantum number which determines the energy of the state. The angular momentum number $\ell$ has to be in the range $[0, n-1]$, so there are at least 3 cases.

$$|3, 0, m, \pm\rangle$$
$$|3, 1, m, \pm\rangle$$
$$|3, 2, m, \pm\rangle$$

Then for each $\ell$, the projection $m$ can take a range of values $m \in [-\ell, \ell]$ so using $\{\ldots, -1, 0, -1, \ldots\}$ to denote a set of options,

$$\begin{aligned} |3, 0, m, \pm\rangle &\to |3, 0, \{0\}, \pm\rangle && \text{2 states} \\ |3, 1, m, \pm\rangle &\to |3, 1, \{-1, 0, 1\}, \pm\rangle && \text{6 states} \\ |3, 2, m, \pm\rangle &\to |3, 2, \{-2, -1, 0, 1, 2\}, \pm\rangle && \text{10 states} \end{aligned}$$

$$\boxed{\text{In total, there are 18 degenerate states}}$$

(e) To get the time evolution, we simply use the fact that for each basis eigenstate, we can add the time evolution component

$$\exp\left(-\frac{iE_n t}{\hbar}\right)$$

to get (in terms of the ground state energy $E_0$)

$$\boxed{|\psi(t)\rangle = \sqrt{\frac{1}{3}}\,|3, 1, 0, +\rangle\, e^{-iE_0 t/9\hbar} + \sqrt{\frac{2}{3}}\,|2, 1, 1, -\rangle\, e^{-iE_0 t/4\hbar}} \tag{F2012 II 1.7}$$

(f) From Ehrenfest's Theorem, we can quickly find the answers to most of the question without worrying about the wavefunction. Ehrenfest's Theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle E\rangle = -\frac{i}{\hbar}\langle[\Omega, H]\rangle + \left\langle\frac{\partial\Omega}{\partial t}\right\rangle$$

None of the operators $L^2$, $S^2$, $L_z$, and $S_z$ are explicit in time, so the second term on the right can be dropped. Then because each of these operators commute with the Hamiltonian, the first term on the right is also dropped. Therefore, the expectation values are constant in time, so

$$\boxed{\begin{aligned} \left\langle L^2\right\rangle &\quad \text{Time independent} && \text{(F2012 II 1.8)} \\ \left\langle S^2\right\rangle &\quad \text{Time independent} && \text{(F2012 II 1.9)} \\ \langle L_z\rangle &\quad \text{Time independent} && \text{(F2012 II 1.10)} \\ \langle S_z\rangle &\quad \text{Time independent} && \text{(F2012 II 1.11)} \end{aligned}}$$

For the probabilty density, we return to the integral in part (c) and insert the appropriate exponential terms. The first and last terms' exponentials cancel each other out, leaving

$$\begin{aligned} \int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \int_0^\infty &\frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi} + e^{-i\phi}\right) R_{21}(r)\, R_{31}(r) \\ &\cdot\left[\exp\left(\frac{i\left(E_2 - E_3\right)t}{\hbar}\right) + \exp\left(-\frac{i\left(E_2 - E_3\right)t}{\hbar}\right)\right] \\ &+ \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)\,\mathrm{d}r \end{aligned}$$

The integral is unaffected by the new time factors, though, so integrating over $r$, the middle term still goes to zero and we're left with the same result previously of $1/4\pi$, therefore

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r \quad \text{Time independent}} \tag{F2012 II 1.12}$$

# Problem 5

### Question

Coinsider $N$ non-interacting, stationary particles, each with magnetic moment $\vec{\mu}$ at temperature $T$ in a uniform external magnetic field $\vec{B}$. Their energy is $-\vec{\mu}\cdot\vec{B}$. Calculate the partition function $Z$, the internal energy, and magnetization for two distinct cases (a and b below):

(a) The mgnetic moment of each particle can be oriented only parallel or anti-parallel to the magnetic field.

(b) The magnetic moment of each particle can rotate freely.

(c) Show that, in both cases, the total magnetization $\vec{M}$ can be written as a derivative of the parition function.

(d) In each case, calculate the fluctuations of magnetization $\left\langle (\Delta\vec{u})^2 \right\rangle$.

### Question

(a) Begin by constructing the partition function for a single particle. Since there are only two energy states, the sum is simply over the two Boltzmann factors:

$$Z_1 = e^{\mu B/kT} + e^{-\mu B/kT}$$

This can be simplified using trigonometric identities to

$$Z_1 = 2\cosh\left(\frac{\mu B}{kT}\right)$$

For fixed site particles, the partition function for $N$ particles is simply $Z = Z^N$, so

$$\boxed{Z = 2^N \cosh^N\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.1}$$

The total energy can be calculated either by finding the expectation energy per particle $\langle\varepsilon\rangle$ and multiplying by $N$ using the Boltzmann factors directly, or by using the thermodynamic identity

$$U = kT^2 \frac{\partial \ln Z}{\partial T}$$

Doing so,

$$U = kT^2 \frac{N}{2\cosh\left(\frac{\mu B}{kT}\right)} \cdot 2\sinh\left(\frac{\mu B}{kT}\right)\cdot\left(-\frac{\mu B}{kT^2}\right)$$

$$\boxed{U = -N\mu B\tanh\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.2}$$

To find the Magnetization, we use the Boltzmann factors directly since we don't know a thermodynamic relation. Let

$$\begin{aligned}
\langle m\rangle &= \sum_\mu \mu \frac{e^{-\varepsilon_\mu/kT}}{Z_1} \\
&= \frac{1}{Z_1}\left(-\mu e^{\mu B/kT} + \mu e^{-\mu B/kT}\right) \\
&= -\mu \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\cosh\left(\frac{\mu B}{kT}\right)}
\end{aligned}$$

So knowing that $\langle M \rangle = N \langle m \rangle$,

$$\boxed{\langle M \rangle = -N\mu \tanh\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.3}$$

(b) In the continuous case, the sum needs to be changed into an integral, remembering to keep $Z$ unitless. This requires dividing by the volume of the energy state, which in this case is $\mu B$. (Justification: think of the energy vector $\vec{\mu} \cdot \vec{B} = \mu B \cos\theta$ on the unit circle of length $\mu B$. From geometry, the unitless $\mathrm{d}\theta$ is related to $\mathrm{d}\varepsilon$ by the factor $\mu B$.)

$$Z_1 = \int_{-\mu B}^{\mu B} e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B}$$

Letting $u = -\frac{\varepsilon}{kT}$,

$$\begin{aligned} Z_1 &= -\frac{kT}{\mu B} \int_{\mu B/kT}^{-\mu B/kT} e^u \,\mathrm{d}u \\ &= 2\frac{kT}{\mu B} \sinh\left(\frac{\mu B}{kT}\right) \end{aligned}$$

Therefore the partition function for all $N$ particles is

$$\boxed{Z = \left(\frac{2kT}{\mu B}\right)^N \sinh^N\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.4}$$

The total energy is found in the same way as the previous case, giving

$$\boxed{U = NkT - N\mu B \coth\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.5}$$

For the magnetization, we also calculate the expectation value from integrating the probability distribution, again making sure to keep the correct units. This time we work with the relevant projection of the magnetic moment $m = \mu\cos\theta$ so that when combined with the energy $\varepsilon = -\mu B \cos\theta$, the magnetization per particle in each state is $m = -\varepsilon/B$.

$$\begin{aligned} \langle m \rangle &= \int_{-\mu B}^{\mu B} -\frac{\varepsilon}{B} \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B} \\ &= \frac{1}{\mu Z_1} \left(\frac{kT}{B}\right)^2 \int_{-\mu B/kT}^{\mu B/kT} e^u \,\mathrm{d}u \\ &= \frac{kT}{B} \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\sinh\left(\frac{\mu B}{kT}\right)} \\ &= \frac{kT}{B} \end{aligned}$$

The total magnetization $\langle M \rangle = N \langle m \rangle$ is

$$\boxed{\langle M \rangle = \frac{NkT}{B}} \tag{F2012 II 5.6}$$

Note that this is to be expected for the continuous case limit which corresponds to the classical limit. We'd expect the total energy to be related to the magnetization by $U = MB$. Rearranging the terms,

$$\begin{aligned} \langle M \rangle B &= NkT \\ U &= NkT \end{aligned}$$

which is the expected result from the equipartition theorem for a stationary particle with two rotational degrees of freedom.

(c) Proving the discrete case only differs from the continuous case proof by the obvious substitutions, so only the continuous case will be presented here. Begin by writing the first starting integral from the previous problem

$$\langle m \rangle = \int_{-\mu B}^{\mu B} m \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B}$$

The $Z_1$ can be pulled outside the integral since it is a constant. Then note that per our definition $\varepsilon = -mB$, it follows that

$$\frac{\partial \varepsilon}{\partial B} = -m$$

We identify the integral above to be a result of using the chain rule, so we undo that and get

$$\langle m \rangle = \frac{1}{Z_1} \int_{-\mu B}^{\mu B} \frac{\partial}{\partial B} \left( -e^{-\varepsilon/kT} \right) \frac{\mathrm{d}\varepsilon}{\mu B}$$

Changing the order of integration and differentiation,

$$= \frac{1}{Z_1} \frac{\partial}{\partial B} \left( \int_{-\mu B}^{\mu B} -e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B} \right)$$

The term within the brackets is simply the definition of the partition function, so

$$\langle m \rangle = \frac{1}{Z_1} \frac{\partial Z_1}{\partial B} = \frac{\partial \ln Z_1}{\partial B}$$

To then get the total magnetization $\langle M \rangle$, we use several properties of differentiation and logarithms:

$$\begin{aligned} \langle M \rangle &= N \langle m \rangle \\ &= N \frac{\partial \ln Z_1}{\partial B} \\ &= \frac{\partial \left( N \ln Z_1 \right)}{\partial B} \\ &= \frac{\partial \ln \left( Z_1 \right)^N}{\partial B} \end{aligned}$$

Giving us the final expression

$$\boxed{\langle M \rangle = \frac{\partial \ln Z}{\partial B}} \tag{F2012 II 5.7}$$

(d) Using the definition

$$\left\langle (\Delta\mu)^2 \right\rangle = \left\langle \mu^2 \right\rangle - \langle \mu \rangle^2$$

we already know $\langle \mu \rangle^2$ for both cases from the previous problems, so we must only calculate $\left\langle \mu^2 \right\rangle$.

# Fall 2012 Part I

## Problem 2

### Question

Show that a particle in a one-dimensional infinite square well initially in a state $\Psi(x,0)$ will always return to that state after a time $T = 4ma^2/\pi\hbar$ where $a$ is the width of the well.

### Answer

Use the standard time independent Schödinger equation

$$\Psi(x,t) = \psi(x)\, e^{iEt/\hbar}$$

with associated differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi$$

For an infinite square well, the potential has the form

$$V(x) = \begin{cases} 0 & |x| < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$$

so that the only region to consider is $-\frac{a}{2} < x < \frac{a}{2}$. In this region, the differential equation takes the form of a harmonic oscillator

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

leading to solutions

$$\psi(x) = A\cos kx + B\sin kx$$
$$\text{where } k^2 = \frac{2mE}{\hbar^2}$$

The boundary conditions $\psi\left(-\frac{a}{2}\right) = 0$ and $\psi\left(\frac{a}{2}\right) = 0$ impose

$$\psi\left(-\frac{a}{2}\right) = 0 = A\cos\frac{ka}{2} - B\sin\frac{ka}{2}$$
$$\psi\left(\frac{a}{2}\right) = 0 = A\cos\frac{ka}{2} + B\sin\frac{ka}{2}$$

so that $B = 0$ and

$$0 = 2A\cos\frac{ka}{2}$$
$$\frac{(2n+1)\pi}{2} = \frac{ka}{2}$$
$$k = \frac{(2n+1)\pi}{a}$$

We already had a relation for $k$ defined, so substitute and solve for the energies $E_n$.

$$\begin{aligned} \frac{(2n+1)^2 \pi^2}{a^2} &= \frac{2mE}{\hbar^2} \\ E_n &= \frac{(2n+1)^2 \pi^2 \hbar^2}{2ma^2} \end{aligned}$$

Then considering $\Psi(x,t)$, the complex exponential is periodic in time with period

$$T_n = \frac{2\pi\hbar}{E}$$

where $n = 0$ will be the case with the longest periodicity, so

$$\begin{aligned} T &= \frac{2\pi\hbar \cdot 2ma^2}{\pi^2\hbar^2} \\ &= \frac{4ma^2}{\pi\hbar} \end{aligned}$$

Therefore, the function is periodic in time with a periodicity

$$\boxed{T = \frac{4ma^2}{\pi\hbar}} \tag{F2012 I 2.1}$$

# Problem 4

### Question

A photon collides with a stationary electron. If the photon scatters at an angle $\theta$, show that the resulting wavelength $\lambda'$ is given in terms of the original wavelength $\lambda$ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where $m$ is the mass of the electron.

### Answer

Start by considering conservation of momentum for the system. The initial values are

$$\begin{aligned} p_{\gamma x} &= \frac{h}{\lambda} & p'_{\gamma x} &= \frac{h}{\lambda'}\cos\theta \\ p_{\gamma y} &= 0 & p'_{\gamma y} &= \frac{h}{\lambda'}\sin\theta \\ p_{ex} &= 0 & p'_{ex} &= ? \\ p_{ey} &= 0 & p'_{ey} &= ? \end{aligned}$$

and considering each component in turn:

$$\begin{aligned} \frac{h}{\lambda} + 0 &= \frac{h}{\lambda'}\cos\theta + p'_{ex} & 0 &= \frac{h}{\lambda'}\sin\theta + p'_{ey} \\ p'_{ex} &= \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta & p'_{ey} &= -\frac{h}{\lambda'}\sin\theta \end{aligned}$$

The total momentum of the electron is then

$$\begin{aligned} p_e^2 &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta\right)^2 + \left(-\frac{h}{\lambda'}\sin\theta\right)^2 \\ &= \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta + \frac{h^2}{\lambda'^2}\cos^2\theta + \frac{h^2}{\lambda'^2}\sin^2\theta \\ p_e^2 &= h^2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{\lambda\lambda'}\cos\theta \end{aligned} \qquad \text{(F2012 I 4.1)}$$

Then consider energy conservation, with initial values

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} & E'_\gamma &= \frac{hc}{\lambda'} \\ E_e &= mc^2 & E'_e &= \frac{p_e'^2}{2m} + mc^2 \end{aligned}$$

leading to the equation

$$\begin{aligned}
\frac{hc}{\lambda} + mc^2 &= \frac{hc}{\lambda'} + \frac{p_e'^2}{2m} + mc^2 \\
\frac{hc}{\lambda} &= \frac{hc}{\lambda'} + \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{hc}{\lambda} - \frac{hc}{\lambda'} &= \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{\lambda' - \lambda}{\lambda\lambda'} &= \frac{h}{2mc}\frac{\lambda'^2 + \lambda^2}{\lambda^2\lambda'^2} - \frac{h}{mc\lambda\lambda'}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{(\lambda' - \lambda)^2 + 2\lambda\lambda'}{\lambda\lambda'}\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{(\lambda' - \lambda)^2}{\lambda\lambda'} + 2\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\frac{(\lambda' - \lambda)^2}{\lambda\lambda'} + \frac{h}{mc}(1 - \cos\theta)
\end{aligned}$$

The difference in the wavelengths is small, so

$$\frac{(\lambda' - \lambda)^2}{\lambda\lambda'} \approx 0$$

leading to the final Compton scattering equation

$$\boxed{\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)} \tag{F2012 I 4.2}$$