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Fall 2000 Part I

Problem 8

Question

Estimate (a) the average speed (in m/s) and (b) the mean free path (in m) of a nitrogen molecule in this room.

Answer

(a) We relate the kinetic energy of an N_2 molecule with the thermal energy by the equipartition theorem. Since there are 3 translational degrees of freedom,

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$
$$v = \sqrt{\frac{3k_B T}{m}}$$

The mass of the molecule is twice that of a single nitrogen atom which is itself about 14 proton masses. Therefore

$$\boxed{v \approx \sqrt{\frac{3k_B T}{28m_p}} \approx 515\ \frac{\mathrm{m}}{\mathrm{s}}} \tag{F2000 I 8.1}$$

(b) Two particles collide if they come within $2r_0$ of each other where r_0 is the typical radius of the particle. For diatomic nitrogen, we assume $r_0 \approx 2a_0$ where a_0 is the Bohr radius. Then in the time τ that the particle is moving at velocity $\langle v \rangle$, the particle can collide with any other particle within the swept-out volume

$$\mathcal{V} = \pi \left(2r_0\right)^2 \cdot \langle v \rangle \tau$$

Since there are n particles per unit volume, there are $\mathcal{N}$ atoms to collide with:

$$\mathcal{N} = n\mathcal{V} = 4\pi n {r_0}^2 \langle v \rangle \tau$$

On average then, there are $\mathcal{N}$ collisions per length $\langle v \rangle \tau$ traversed, or in its reciprocal form, the mean free path λ is

$$\lambda = \frac{1}{4\pi n {r_0}^2}$$

To estimate the particle density, consider the ideal gas law $PV = Nk_B T$. We can assume atmospheric pressure at room temperature, so the density is

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Putting it all together,

$$\boxed{\lambda \approx \frac{k_B T}{4\pi {r_0}^2 P} \approx 2.90 \cdot 10^{-7}\ \mathrm{m}} \tag{F2000 I 8.2}$$

Spring 2000 Part I

Problem 1

Question

A satellite of mass $m = 500\,\mathrm{kg}$ is in a circular orbit at an altitude $h = 150\,\mathrm{km}$ above the Earth's surface. As a result of air friction, the satellite's orbit degrades. Protected by a heat shield, the satellite eventually impacts with a velocity of $2\,\mathrm{km/s}$. How much energy (in Joules) was released as heat in the process?

Answer

The soltuion method will be a simple energy balance equation. In its initial state, the satellite had gravitation potential energy that contributed to its energy equal to

$$V_0 = \frac{GM_E m}{R_E + h}$$

where M_E and R_E are the mass and radius of the Earth. The kinetic energy can be determined by making use of simple circle relations. The centriptal force must be provided by the gravitation force, so

$$\begin{aligned} \frac{{v_0}^2}{R_E + h} &= \frac{GM_E}{\left(R_E + h\right)^2} \\ v_0 &= \sqrt{\frac{GM_E}{R_E + h}} \end{aligned}$$

making the initial kinetic energy

$$T_0 = \frac{GM_E m}{2\left(R_E + h\right)}$$

In it's final state, the satellite consists of it's final given velocity's kinetic energy and more gravitational potential energy (with respect to the center of the Earth). They are given by

$$\begin{aligned} T_f &= \frac{1}{2} m {v_f}^2 \\ V_f &= \frac{GM_E m}{R_E} \end{aligned}$$

By conservation of energy, any energy different must result from the loss of energy between the intial and final states, so the lost energy E_{loss} is

$$\begin{aligned} E_{\text{loss}} &= T_0 + V_0 - T_f - V_f \\ &= \frac{GM_E m}{R_E\left(R_E + h\right)}\left(R_E - 2h\right) - \frac{1}{2} m {v_f}^2 \end{aligned}$$

Plugging in all given quantities and constants, we have that

$$\boxed{E_{\text{loss}} = 2.816 \cdot 10^{10}\,\mathrm{J} = 28.16\,\mathrm{GJ}} \tag{F2000 I 1.1}$$

Problem 2

Question

What is the velocity of recoil of an ^{57}Fe nucleus that emits a 100 keV photon, both in units of the speed of light in vacuum and in meters per second.

Answer

In the initial state observed from the iron atom's rest frame, the momentum is zero. After the emission, the photon has a momentum of $p_\gamma = E_\gamma/c$ and consequently, the nucleus must recoil with momentum $p_{Fe} = -p_\gamma$. Knowing that the energy is non-relativistic, then

$$m_{Fe} v_{Fe} = \frac{E_\gamma}{c}$$
$$v_{Fe} = \frac{E_\gamma}{57 m_p c}$$

where we've approximate the mass of the iron atom by a multiple of the proton mass. Plugging in the numbers

$$\boxed{v_{Fe} = 561.4\,\frac{\mathrm{m}}{\mathrm{s}}} \tag{F2000 I 2.1}$$

or in units of c

$$\boxed{v_{Fe} = 1.87 \cdot 10^{-6}\,\mathrm{c}} \tag{F2000 I 2.2}$$

Problem 3

Question

A resistance R and an inductance L are connected in series, and an alternating voltage $V_0 \cos \omega t$ is impressed across the combination. The resulting steady state voltage across the resistance can be written as $V_R \cos(\omega t + \beta)$. Find V_R and β, assuming both V_0 and V_R to be positive.

Answer

The problem is simplified by constucting the solution using complex voltages and currents and recovering the correct component at the end. Since the source voltage is a cosine, the real component will be kept at the end.

The complex impedance for a resistor is simply the resitance itself, so $Z_R = R$. For the inductor, it is $Z_L = i\omega L$. We then use the complex impedances together with Ohm's Law and the first Kirchoff rule to solve for the complex current I.

$$V_0 e^{i\omega t} = IR + i\omega L I$$
$$I = \frac{V_0}{R + i\omega L} e^{i\omega t}$$

Then by inserting this solution into the voltage law for inductors, we can solve for the complex voltage across the inductor.

$$V_L = L\frac{dI}{dt}$$
$$V_L = V_0 \frac{i\omega L}{R + i\omega L} e^{i\omega t}$$

The voltage drops across the resistor and inductor must equal the impressed voltage, so we can solve for the unknown voltage across the resistor.

$$V_0 e^{i\omega t} = V + V_L$$
$$V = V_0 e^{i\omega t} - V_0 \frac{i\omega L}{R + i\omega L} e^{i\omega t}$$
$$V = V_0 \left(\frac{R^2 - i\omega RL}{R^2 + \omega^2 L^2} \right) e^{i\omega t}$$

In order to make taking the real component simpler, we put the term in parentheses in complex exponential form according to the relation $z = |z| e^{i \arg z}$.

$$\left| \frac{R^2 - i\omega RL}{R^2 + \omega^2 L^2} \right| = \frac{R\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2}$$
$$\arg \left(\frac{R^2 - i\omega RL}{R^2 + \omega^2 L^2} \right) = \arctan \left(-\frac{\omega L}{R} \right)$$

Therefore the voltage across the resistor has the form

$$V = V_0 \frac{R\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2} e^{i\omega t + \arctan(-\omega L/R)}$$
$$V = V_R e^{i\omega t + i\beta}$$
$$\Re\{V\} = V_R \cos(\omega t + \beta)$$

where

$$\boxed{V_R = V_0 \frac{R\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2}} \qquad \boxed{\beta = \arctan \left(-\frac{\omega L}{R} \right)} \tag{F2000 I 3.1}$$

Problem 4

Question

Find the magnetic flux through a square loop of side a due to current I in a long straight wire. The geometry is as follows: the wire is coplanar with the loop and runs parallel to the loop's closest side, at a distance b away. Write your result as a formula in SI units.

Answer

By Ampère's law in integral form, the magnetic field at a radial distance r away from the wire is given by

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

The flux is then the total magnetic field through the loop. Integrating by lines of length a,

$$\Phi = a\int_b^{a+b} \frac{\mu_0 I}{2\pi r} dr$$

$$\boxed{\Phi = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{a+b}{a}\right)}$$

Problem 6

Question

By actually evaluating the integral, show that

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e}$$

Answer

Note that the integrand, so start by changing the limits of integration

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\,dx$$

Then expand the cosine into is complex exponential definition

$$\begin{aligned} &= \frac{1}{4}\left(\int_{-\infty}^\infty \frac{e^{ix}}{1+x^2}\,dx + \int_{-\infty}^\infty \frac{e^{-ix}}{1+x^2}\,dx\right) \\ &= \frac{1}{4}\left(\int_{-\infty}^\infty \frac{e^{ix}}{(x+i)(x-i)}\,dx + \int_{-\infty}^\infty \frac{e^{-ix}}{(x+i)(x-i)}\,dx\right) \end{aligned}$$

Complex contour integration lets us evaluate the integrals as limits of the coefficient of a pole as it approaches the pole, so

$$= \frac{1}{4}\left(2\pi i \cdot \lim_{z\to i}\left(\frac{e^{iz}}{z+i}\right) - 2\pi i \cdot \lim_{z\to -i}\left(\frac{e^{-iz}}{z-i}\right)\right)$$

After simplifying,

$$\boxed{\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e}} \tag{F2000 I 6.1}$$

Problem 7

Question

The drag force on a very high speed object of area A, passing through a gas of density ρ at a velocity v is expected to be of the form

$$\text{Force} \sim A^r \rho^s v^t$$

Determine the value of the exponents r, s, and t.

Answer

Force needs to have a unit of inverse time squared, therefore v as the only variable with a time unit sets $t = 2$. Similarly, ρ is the only one with a mass term, so we also immediately know that $s = 1$. That leaves

$$\left[\frac{\text{mass} \cdot \text{distance}}{\text{time}^2}\right] = \left[\frac{\text{mass}}{\text{distance} \cdot \text{time}^2}\right] \left[\text{distance}\right]^r$$

Therefore to have the two side have compatible units, $r = 2$.

$$\boxed{r = 2, \quad s = 1, \quad t = 2}$$

Problem 9

Question

For waves in shallow water, the relation between frequence ν and wavelength λ is

$$\nu = \left(\frac{2\pi T}{\rho \lambda^3} \right)^{1/2}$$

where ρ and T are the density and surface tension of water. What is the group velocity of these waves?

Answer

Transforming relation given to use the angular frequency $\omega = 2\pi\nu$ and wave number $k = 2\pi/\lambda$,

$$\omega = \sqrt{\frac{k^3 T}{\rho}}$$

Then the group velocity is simply the partial derivative with respect to k:

$$v_g = \frac{\partial \omega}{\partial k} = \frac{3}{2}\sqrt{\frac{kT}{\rho}}$$

Putting this back into the form which involves only ν and λ, we arrive at the answer

$$\boxed{v_g = \frac{3}{2}\sqrt{\frac{2\pi T}{\rho \lambda}}} \tag{F2000 I 9.1}$$

Problem 10

Question

Find the eigenvalues and corresponding eigenvectors (which need *not* be normalized) of the following matrix:

$$M = \begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & -1 \end{bmatrix}$$

Answer

The eigenvalue equation is found from the standard procedure of adding a parameter λ into the matrix and taking the determinant equal to zero:

$$\begin{aligned} \det\left(\begin{bmatrix} 1-\lambda & 0 & -i \\ 0 & 2-\lambda & 0 \\ i & 0 & -1-\lambda \end{bmatrix}\right) &= 0 = (1-\lambda)(2-\lambda)(-1-\lambda) - i(-i)(2-\lambda) \\ 0 &= -(1-\lambda)(1+\lambda)(2-\lambda) - (2-\lambda) \\ 0 &= -(2-\lambda)\left[(1-\lambda)(1+\lambda)+1\right] \end{aligned}$$

Solving the two equations $0 = (2-\lambda)$ and $(1-\lambda)(1+\lambda) = -1$ gives the three eigenvalues

$$\boxed{\lambda = \left\{-\sqrt{2}, \sqrt{2}, 2\right\}} \tag{F2000 I 10.1}$$

Starting with the eigenvalue $\lambda = 2$, we solve for its eigenvector using the usual Gaussian elimination approach:

$$\det\left(\begin{bmatrix} -1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -3 \end{bmatrix}\right)$$

The second variable is completely unconstrained, so we can set that component of the vector to 1. The first and last rows are incompatible, so that means that both the first and third variables must be zero. This gives us the eigenvector

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In a similar manner for the eignevalues $\lambda = \pm\sqrt{2}$, we perform Gaussian elimination. When all free parameters have been set arbitrarily to 1 and other constraints considered, we end up with the three eigen vectors:

$$\boxed{\lambda = -\sqrt{2} \quad \to \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ -i\left(1+\sqrt{2}\right) \end{bmatrix}} \tag{F2000 I 10.2}$$

$$\boxed{\lambda = \sqrt{2} \quad \to \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -i\left(1-\sqrt{2}\right) \end{bmatrix}} \tag{F2000 I 10.3}$$

$$\boxed{\lambda = 2 \quad \to \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \tag{F2000 I 10.4}$$

Problem 12

Question

Suppose the electron were to have spin $\frac{3}{2}$ instead of spin $\frac{1}{2}$. What would then be the atomic numbers Z of the *three* lowest-mass nobel gases, i.e. the equivalents of helium, neon, and argon?

Answer

In a spin $\frac{3}{2}$ particle, there are 4 possible spin configurations corresponding to the spin projections $s_z = \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$. Because of thise, each projection of the orbital angular momentum can hold 4 electrons instead of just two. This means we can make use of spectroscopic notation to easily count up to the atoms of interest.

$$\begin{aligned}
&\ell = 0 \rightarrow 1\ \ell_z \text{ state} && 1s^4 \\
&\ell = 1 \rightarrow 3\ \ell_z \text{ states} && 1s^4 2p^{12} 2s^4 \\
&\ell = 2 \rightarrow 5\ \ell_z \text{ states} && 1s^4 2p^{12} 2s^4 3d^{20} 3p^{12} 3s^4
\end{aligned}$$

Since all shells are filled at each level, we just have to sum the number of electrons in each line above to get the Z number of the new noble gases.

$$\boxed{Z = \{4, 20, 56\}} \tag{F2000 I 12.1}$$

Spring 2002 Part I

Problem 1

Question

A 1000 kg automobile has ground clearance of 18 cm but when loaded with an extra 500 kg from its 4 passengers it only clears the ground by 12 cm. The car's shock absorbers are ineffective. At what speed (in miles per hour) will the car bounce in resonance when it travels along a smooth road containing transverse tar patches every 15 m? Assume that the front and rear suspensions have the same bouncing frequency.

Answer

To find the natural resonant frequency of the vehicle, we use the two data points about it's clearance to obtain the spring constant k. We know that a constant force on a spring system does not change the dynamics, so it's only the change in mass and distance which are relevant. Therefore by simple equilibrium requirements,

$$\begin{aligned} k\Delta L &= mg \\ k &= \frac{mg}{\Delta L} \end{aligned}$$

where m and M are the masses of the passengers and empty car respectively and ΔL is the difference in the clearance between the unloaded and loaded car.

Next, we know that the resonant frequency of a simple harmonic oscillator is given by $\omega = \sqrt{k/m_{\mathrm{tot}}}$, so

$$\begin{aligned} \omega &= \sqrt{\frac{mg}{(M+m)\,\Delta L}} \\ f &= \frac{1}{2\pi}\sqrt{\frac{mg}{(M+m)\,\Delta L}} \end{aligned}$$

where the second line has converted from angular to linear frequency. We then simply relate that to how often the car hits a tar patch and solve for the velocity. Letting v be the velocity of the car and d be the separation distance between tar patches,

$$\begin{aligned} \frac{v}{d} &= f \\ v &= \frac{d}{2\pi}\sqrt{\frac{mg}{(M+m)\,\Delta L}} \end{aligned}$$

Plugging in the numbers,

$$\boxed{v = 17.62\,\frac{\mathrm{m}}{\mathrm{s}} = 39.42\,\mathrm{mph}} \tag{F2002 I 1.1}$$

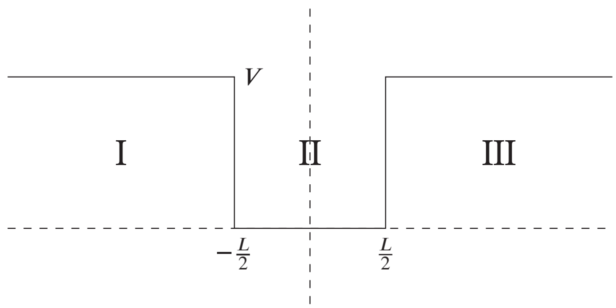
Problem 2

Question

Suppose a particle of mass m moves in a 1-dimensional square potential well of width L and depth V. What is the minimum depth of the well such that the particle will have two bound states?

Answer

Divide the problem into three regions according to the diagram below:



From the Schrödinger equation, we know that there will be two different types of solutions. Within region II, no potential exists, so the solution has the form

$$\psi = A\cos(kx) + B\sin(kx)$$
$$k^2 = \frac{2mE}{\hbar^2}$$

For regions I and III, we assume that the energy $E < V$ since we are only interested in the bound solutions and not ones where the particle is free. This gives the following exponential solution form:

$$\psi = Ae^{\kappa x} + Be^{-\kappa x}$$
$$\kappa^2 = \frac{2m(V - E)}{\hbar^2}$$

We immediately know that for regions I and III, the wavefunction must go to zero at $\pm\infty$, so that immediately removes two terms in the solution. This leaves us with the complete set of solutions

$$\psi_I = Ae^{\kappa x}$$
$$\psi_{II} = B\cos(kx) + C\sin(kx)$$
$$\psi_{III} = De^{-\kappa x}$$

We make a further simplification by noting that in this situation where there is a symmetric potential, the solution can also be divided into symmetric and antisymmetric solutions, corresponding to keeping either the sin or the cos solutions in region II. To solve the problem, we want to know the threshold potential for V that will maintain a second bound state. The first bound state is the symmetric case (which heuristically is true by its virtue of having only a single "hump" within region II), so the first excited state/second state is the antisymmetric case (heuristically expected since the sine solution will have two "humps" within region II). This lets us simplify the problem and immediately set $B = 0$ to isolate just the antisymmetric solution.

To continue, we make use of the fact that the wavefunction must be continuous and differentiably continuous at the boundaries between regions I-II and II-III. This means we find that:

$$Ae^{-\kappa L/2} = B\sin\left(-\frac{kL}{2}\right) \qquad B\sin\left(\frac{kL}{2}\right) = De^{-\kappa L/2}$$
$$A\kappa e^{-\kappa L/2} = Bk\cos\left(-\frac{kL}{2}\right) \qquad Bk\cos\left(\frac{kL}{2}\right) = -D\kappa e^{-\kappa L/2}$$

Dividing the lower equation by the upper equation in both cases leads to the condition

$$\kappa = -k \cot\left(\frac{kL}{2}\right)$$

If we perform a variable substitution where $v = kL/2$ and $u = \kappa L/2$, the equation above takes the slightly simpler form

$$u = -v \cot v \tag{F2002 I 2.1}$$

This transformation comes in more useful when we consider the energy equations that defined k and κ. Substituting into the equation for k and solving for E we have that

$$E = \frac{2v^2\hbar^2}{mL^2}$$

And doing the same for κ,

$$V - E = \frac{2u^2\hbar^2}{mL^2}$$

which combined gives the equation of a circle:

$$u^2 + v^2 = \frac{mL^2}{2\hbar^2}V$$

Therefore, the only valid solutions occur when both (F2002 I 2.1) and (F2002 I 2.2) are satisfied. The minimum value for a given cotangent line occurs at the roots which occur when $v = (2n+1)\,\pi/2$ for any integer n. The first excited state for $n = 1$ then occurs at $v = \pi/2$ and consequently $u = 0$. Substiting this into the equation above and solving for V, we find that the threshold energy for a second bound state corresponds to a potential depth of

$$\boxed{V = \frac{\pi^2\hbar^2}{2mL^2}} \tag{F2002 I 2.2}$$

Problem 3

Question

The cross-section for collisions between helium atoms is about $10^{-16}\,\mathrm{cm}^2$. Estimate the mean free path of helium atoms in helium gas at atmospheric pressure and temperature.

Answer

Consider the path traced out by a helium atom as it travels a path length L, colliding with other helium atoms along the way. Given that the cross section of helium is σ, than we can estimate the volume that contains probable interactions with our atom of interest as $\mathcal{V} = \sigma L$. To get the number of interactions, we make use of the fact that we're treating the gas as an ideal gas. From the ideal gas law,

$$PV = NkT$$

so that solving for the number density

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Combining the density with the volume, we get the number of other [point particle] helium atoms that are contained within the given helium atom's interaction volume. If we then assume that the atom interacts with all other atoms within the volume, and that the collisions are spaced out equally in time, we just have to normalize the value by the trajectory's path length to get an estimate of the mean free path of helium in a helium gas:

$$\lambda = \frac{n\mathcal{V}}{L} = \frac{\sigma P}{k_B T}$$

Plugging in $\sigma = 10^{-16}\,\mathrm{cm}^2$, $P = 1.013 \cdot 10^5\,\mathrm{Pa}$, $k_B = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$, and $T = 298\,\mathrm{K}$, we get

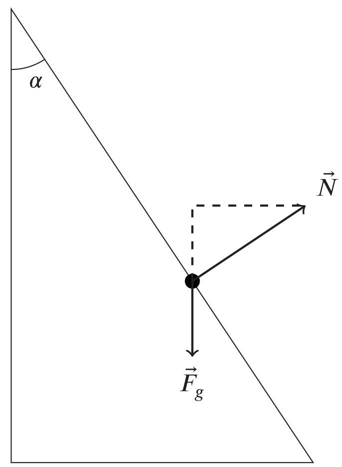
$$\boxed{\lambda = 4.06\,\mu\mathrm{m}} \tag{F2002 I 3.1}$$

Problem 6

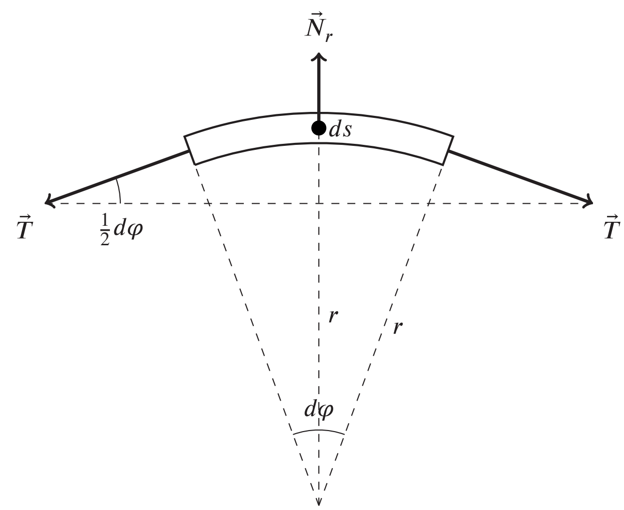
Question

A single closed loop of chain with mass m and length L rests horizontally on a smooth frictionless con with half-angle α. What is the tension in the chain?

Answer



(a) Side View



(b) Top View

Consider just a small element of the chain of arc length ds. It will have a corresponding mass $dm = \lambda\, ds$ where $\lambda = m/L$. Knowing that it's a statics problem, we can easily determine the normal force by balancing the vertical component with that of gravity.

$$F_g = N \sin\alpha$$
$$N = \frac{\lambda g\, ds}{\sin\alpha}$$

This leaves the horizontal component of the normal force to be balanced with the tension within the chain.

Now switching to the top view, we consider the short chain segment ds, shown above with an exagerated curvature. We note that the radial part of the normal force must be opposed by the sum of the radial components of the two tensions T acting on the end of the chain segment. By geometry, we know that the angle with respect to the midpoint's tangent is one half the differential angle change $d\varphi = ds/r$. This means we balance the forces as

$$2T \sin\left(\frac{1}{2}d\varphi\right) = N \cos\alpha$$
$$2T \sin\left(\frac{1}{2}d\varphi\right) = \lambda g\, ds \cot\alpha$$

By the small angle approximation, $\sin\left(\frac{1}{2}d\varphi\right) \approx \frac{1}{2}d\varphi$, so after substituting for the fact that $dr = L/2\pi$ and $\lambda = M/L$,

$$\boxed{T = \frac{Mg}{2\pi}\cot\alpha} \qquad \text{(F2002 I 6.1)}$$

Problem 7

Question

A laser beam (photon energy 1 eV) collides head-on with a 50 GeV ultra-relativistic electron beam. What is the energy of the photons reflected backwards in the collision?

Answer

We'll be solving the problem using conservation of 4-momentum, so we define the following momenta with the assumption that the electron beam is moving to the right, and the photons are initially moving to the left. Let the unprimed and primed q^μ be the photon's 4-momentum before and after the collision respectively, and define the electron's momenta k^μ similarly. Then in terms of the energies E and 3-momenta p (actually taken to be 1D without loss of generality) for each of the photon γ and electron e:

$$q^\mu = \begin{pmatrix} E_\gamma/c \\ -E_\gamma/c \end{pmatrix} \qquad q'^\mu = \begin{pmatrix} E'_\gamma/c \\ E'_\gamma/c \end{pmatrix}$$

$$k^\mu = \begin{pmatrix} E_e/c \\ p_e \end{pmatrix} \qquad k'^\mu = \begin{pmatrix} E'_e/c \\ p'_e \end{pmatrix}$$

By conservation of momentum,

$$\begin{aligned} q'^\mu + k'^\mu &= q^\mu + k^\mu \\ k'^\mu &= q^\mu - q'^\mu + k^\mu \end{aligned}$$

Solving for the unknown electron momentum after the collision lets us eliminate it from the equation; when we square the equation, the squared quantities are Lorentz invariant, and therefore the product can be evaluated in any frame. A convenient choice is the rest frame where the electron evaluates to its mass energy and photons vanish.

$$\underbrace{k'^\mu k'_\mu}_{m_e c^2} = \underbrace{q^\mu q_\mu}_{0} - \underbrace{q'^\mu q'_\mu}_{0} + \underbrace{k^\mu k_\mu}_{m_e c^2} - q'^\mu q_\mu + q'^\mu k_\mu - q^\mu k_\mu$$

This greatly simplifies the rest of the problem to

$$q'^\mu q_\mu - q'^\mu k_\mu = -q^\mu k_\mu$$

Inserting the energy and 3-momentum components and performing the inner products,

$$2\frac{E_\gamma E'_\gamma}{c^2} - \frac{E'_\gamma E_e}{c^2} + \frac{E'_\gamma p_e}{c} = -\frac{E_\gamma E_e}{c^2} - \frac{E_\gamma p_e}{c}$$

Isolating E'_γ on the left and E_γ on the right,

$$-E'_\gamma \left(E_e - p_e c - 2E_\gamma\right) = -E_\gamma \left(E_e - p_e c\right)$$

which solving for the unknown photon energy gives

$$\boxed{E'_\gamma = E_\gamma \left(1 - \frac{2E_\gamma}{E_e - p_e c}\right)^{-1}} \qquad \text{(F2002 I 7.1)}$$

The solution is formally complete, but actually calculating a numerical answer can prove difficult because $E_e \approx p_e c$. Therefore, we will expand the solution. Beginning with the definition of the momentum from Einstein's energy relation,

$$p_e c = \sqrt{{E_e}^2 - {m_e}^2 c^4}$$

we can subtract it from E_e, leading to the useful form

$$E_e - p_e c = E_e \left(1 - \sqrt{1 - \frac{{m_e}^2 c^4}{{E_e}^2}} \right)$$

Expanding the root to first order in its argument,

$$\begin{aligned} E_e - p_e c &= E_e \cdot \frac{1}{2} \frac{{m_e}^2 c^4}{{E_e}^2} \\ &= \frac{1}{2} \frac{{m_e}^2 c^4}{E_e} \end{aligned}$$

Plugging this into the solution (F2002 I 7.1), the photon energy is then approximately given by

$$\boxed{E'_\gamma \approx E_\gamma \left(1 - 4 \frac{E_\gamma E_e}{{m_e}^2 c^4} \right)^{-1}} \tag{F2002 I 7.2}$$

The numerics are much easier to calculate in this case, and we find that the final energy of the reflected photon is

$$\boxed{E'_\gamma = 4.273\,\text{eV}} \tag{F2002 I 7.3}$$

Problem 8

Question

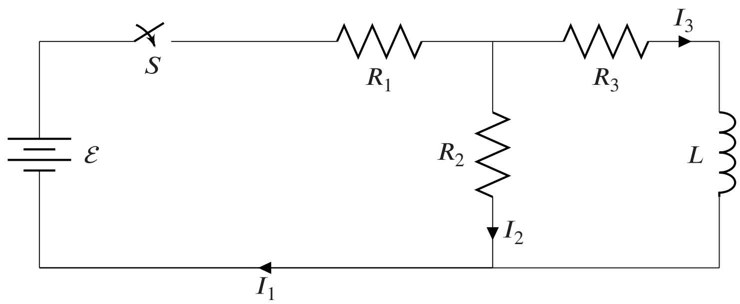
In the figure below, $\mathcal{E} = 100\,\mathrm{V}$, $R_1 = 5\,\Omega$, $R_2 = 10\,\Omega$, $R_3 = 15\,\Omega$, and $L = 1.0\,\mathrm{H}$. Find the values of the currents I_1 and I_2

a) immediately after the switch S is closed,

b) a long time later,

c) immediately after switch S is opened again,

d) and then how long must you wait, after the switch is opened, before I_2 falls by a factor of e?



Answer

Start by applying Kirchoff's rules to the circuit: current is conserved and the voltage changes must sum to zero around each loop, so

$$I_1 = I_2 + I_3 \tag{F2002 I 8.1}$$

$$0 = \mathcal{E} - I_1 R_1 - I_2 R_2 \tag{F2002 I 8.2}$$

$$0 = -I_3 R_3 - L\frac{dI_3}{dt} + I_2 R_2 \tag{F2002 I 8.3}$$

Since only (F2002 I 8.3) has a term involving a time derivative, we choose to first solve for the current I_3. By solving for $I_2 R_2$ in (F2002 I 8.2) and substituting, we eliminate I_2 and have

$$0 = -I_3 R_3 - L\frac{dI_3}{dt} + \mathcal{E} - I_1 R_1 \tag{F2002 I 8.4}$$

Furthermore, by also substituting the value of I_2 into (F2002 I 8.1):

$$I_1 = \frac{\mathcal{E}}{R_1 + R_2} + \frac{R_2}{R_1 + R_2} I_3 \tag{F2002 I 8.5}$$

Then by combining (F2002 I 8.4) and (F2002 I 8.5), we can produce a differential equation for I_3:

$$-I_3 R_3 - L\frac{dI_3}{dt} + \mathcal{E} - \frac{R_1}{R_1 + R_2}\mathcal{E} - \frac{R_1 R_2}{R_1 + R_2} I_3 = 0$$

$$-\underbrace{\frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 + R_2}}_{R'} I_3 = L\frac{dI_3}{dt} - \frac{R_2}{R_1 + R_2}\mathcal{E}$$

$$\frac{dI_3}{dt} = -\frac{R'}{L} I_3 + \frac{1}{L}\frac{R_2}{R_1 + R_2}\mathcal{E} \tag{F2002 I 8.6}$$

Considering just the homogenous part, we easily solve it to find the standard exponential solution

$$I_{3h}(t) = I_{30} e^{-R' t/L}$$

And using the ansatz $I_{3p}(t) = At + B$ for the inhomogenous part,

$$\begin{aligned} A &= -\frac{R'}{L}At - \frac{R'}{L}B + \frac{1}{L}\frac{R_2}{R_1+R_2}\mathcal{E} \\ A &= 0 \\ B &= \frac{1}{R'}\frac{R_2}{R_1+R_2}\mathcal{E} \end{aligned}$$

At $t = 0$, the inductor has no current passing through it, so when the switch is closed, the current must remain continuous. This gives us the initial condition necessary to solve for the unknown I_{30}, and after doing so and simplifying, the total solution is

$$I_3(t) = \frac{\mathcal{E}}{R'}\frac{R_2}{R_1+R_2}\left(1 - e^{-R' t/L}\right) \tag{F2002 I 8.7}$$

Then by substiting this solution back into (F2002 I 8.5) we get the solution for I_1:

$$I_1(t) = \frac{\mathcal{E}}{R_1+R_2}\left[1 + \frac{1}{R'}\frac{{R_2}^2}{R_1+R_2}\left(1 - e^{-R' t/L}\right)\right] \tag{F2002 I 8.8}$$

Finally, combining both inserting both solutions for I_1 and I_3 into (F2002 I 8.1), the solution for I_2 is

$$I_2(t) = \frac{\mathcal{E}}{R_1+R_2}\left[1 - \frac{1}{R'}\frac{R_1 R_2}{R_1+R_2}\left(1 - e^{-R' t/L}\right)\right] \tag{F2002 I 8.9}$$

Plugging in all of the given values, we find that the currents at the instant the switch is closed are

$$\boxed{I_1(0) = 6.66\,\mathrm{A} \qquad S \text{ is closed}} \tag{F2002 I 8.10}$$

$$\boxed{I_2(0) = 6.66\,\mathrm{A} \qquad S \text{ is closed}} \tag{F2002 I 8.11}$$

For a long time later, we can let $t \to \infty$ and find that

$$\boxed{I_1(\infty) = 9.09\,\mathrm{A} \qquad S \text{ is closed}} \tag{F2002 I 8.12}$$

$$\boxed{I_2(\infty) = 5.54\,\mathrm{A} \qquad S \text{ is closed}} \tag{F2002 I 8.13}$$

Right after the switch is opened, the left loop is taken out of the circuit, so we immediately know that the value of I_1 is zero.

$$\boxed{I_1(0) = 0 \qquad S \text{ is open}} \tag{F2002 I 8.14}$$

For the right loop, we start by noting that the steady state current through the inductor will be needed. Taking the limit of (F2002 I 8.7), we have that the new initial condition is

$$I_3(0) = \frac{\mathcal{E}}{R'}\frac{R_2}{R_1+R_2}$$

I_2 now is equal to $-I_3$ since there is no other path for the current to traverse. This loop's differential equation is then

$$-\left(R_2 + R_3\right) I_3 - L\frac{dI_3}{dt} = 0$$

Solving for the exponential and using the initial condition above, the time solution is

$$I_3(t) = \frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2} e^{-(R_2+R_3)t/L}$$
$$I_2(t) = -\frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2} e^{-(R_2+R_3)t/L}$$

Therefore the current in I_2 just after the switch is opened reverses direction

$$\boxed{I_2(0) = -3.64\,\mathrm{A}} \tag{F2002 I 8.15}$$

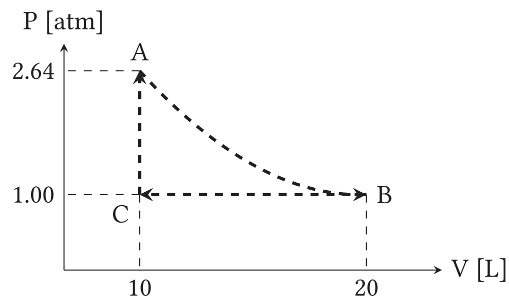
Then by simple exponential relations, we know that the time time to decay by a factor of e is given by the reciprocal of the coefficient of t, so inserting the appropriate numbers

$$\boxed{t_{\mathrm{decay}} = 0.04\,\mathrm{s}} \tag{F2002 I 8.16}$$

Problem 9

Question

An engine using 1 mol of an ideal diatomic gas performs the cycle $A \to B \to C \to A$ as shown in the diagram below. $A \to B$ is an adiabatic expansion, $B \to C$ occurs at constant pressure, and $C \to A$ takes place at constant volume. What is the efficiency of the cycle?



Answer

Since we want to find the efficiency of the cycle, we only care about the heat exchanged during each stage of the cycle. Because the path $A \to B$ is adiabatic, we immediately know that $Q = 0$. Then proceeding to look at the stage $C \to A$, we know that the work done during this cycle is identically zero since there is no area under the curve. That means we are left simply with the equation

$$dU = dQ$$

Because this is an ideal [diatomic] classical gas, we combine the equations

$$U = \frac{5}{2} nRT$$

and

$$PV = nRT$$

to get that the difference in energy across the path is

$$\begin{aligned} Q_{CA} = U &= \frac{5}{2} nR \left(T_A - T_C\right) \\ &= \frac{5}{2} V_1 \left(P_2 - P_1\right) \end{aligned}$$

For the remaining stage $B \to C$, we use the full thermodynamic identity:

$$dU = dQ - P\,dV$$

The pressure P_1 is constant, so both integration of dU and dV are simply the differences in each quantity. Again substituting for the temperature in U with the ideal gas law,

$$\begin{aligned} \frac{5}{2} nR \left(T_C - T_B\right) &= Q_{BC} - P_1 \left(V_1 - V_2\right) \\ \frac{5}{2} P_1 \left(V_1 - 2V_1\right) &= Q_{BC} + P \left(V_1 - 2V_1\right) \\ Q_{BC} &= -\frac{7}{2} P_1 V_1 \end{aligned}$$

We've accounted for all the heat flow in the system. Q_{BC} is negative, so this is the heat flow out of the system, while Q_{CA} is positive and is the heat flow into the system. By definition then, the efficiency η of the system is

$$\begin{aligned}\eta &= 1 - \frac{Q_{out}}{Q_{in}} \\ &= 1 - \frac{\frac{7}{2} P_1 V_1}{\frac{5}{2} V_1 \left(P_2 - P_1\right)} \\ &= 1 - \frac{5}{7} \frac{P_1}{P_2 - P_1}\end{aligned}$$

Plugging in the given values, we find the efficency to be

$$\boxed{\eta = 0.146 = 14.6\,\%} \tag{F2002 I 9.1}$$

Problem 10

Question

A thin circular hoop rolls down an inclined plane under the influence of gravity. What minimum coefficient of friction is required to ensure that it rolls rather than slides?

Answer

Begin first by finding the motion that describes the rolling without slipping state. We do this by solving the system's Lagrangian:

$$\mathcal{L} = \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2\right) - (mgx\sin\alpha)$$

where x is the length along the ramp with $x = 0$ at the bottom, I is the moment of inertia of the hoop, m is its mass, θ is the angle of rotation of the hoop about its center, and α is the angle of the incline plane. By noting that rolling without slipping requires that $r\dot{\theta} = \dot{x}$, we can reduce the problem to the single variable x. The result is the following differential equation, where $I = mr^2$ has been substituted in:

$$\begin{aligned} 2m\ddot{x} &= -mg\sin\alpha \\ \ddot{x} &= -\frac{1}{2}g\sin\alpha \end{aligned}$$

Therefore we know the linear acceleration will be $a = -\frac{1}{2}g\sin\alpha$ in the non-slipping case.

To find what coefficient of friction produces this motion, we consider the forces acting on the hoop with the coordinate system still oriented along and perpendicular to the plane. In the perpendicular direction, the normal force N is canceled by the perpendicular component of gravity, so

$$N = mg\cos\alpha$$

In the parallel direction, the frictioanl force and the parallel component of gravity must sum to give the requisitve force, namely ma.

$$\begin{aligned} \mu N - mg\sin\alpha = a &= -\frac{1}{2}mg\sin\alpha \\ \mu mg\cos\alpha &= \frac{1}{2}mg\sin\alpha \\ \mu &= \frac{1}{2}\tan\alpha \end{aligned}$$

Therefore we find that the coefficient of friction must be equal to half of the tangent of the inclined plane's angle.

Problem 11

Question

A particle is confined within a cubical box with sides of length L and is initially in the ground state. If the length of one side of the box (along the x-direction) is abruptly increased to a length $2L$, what is the probability that the particle remains in the ground state?

Answer

We start by recalling the solution for a particle in a box. In a 1D box with is left edge at the origin, the properly normalized wavefunction is given by

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

where L is the size of the box. The Cartesian extension into 3D is simple and is respectively for the $L \times L \times L$ and $2L \times L \times L$ boxes:

$$\begin{aligned}
\psi(x, y, z) &= \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \\
\psi'(x, y, z) &= \frac{1}{\sqrt{2}} \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{2L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right)
\end{aligned}$$

To find the probability of remaining in the ground state, we simply must take the inner product of both wavefunctions in the ground state over an appropriate domain; this means that the initial, unexanded box's wavefunction is 0 within the new region.

$$\begin{aligned}
\mathscr{P} &= \left\langle \psi_{111} \middle| \psi'_{111} \right\rangle \\
&= \frac{1}{\sqrt{2}} \left(\frac{2}{L}\right)^3 \left(\int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{2L}\right) \, dx \right) \left(\int_0^L \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{\pi y}{L}\right) \, dy \right) \left(\int_0^L \sin\left(\frac{\pi z}{L}\right) \sin\left(\frac{\pi z}{L}\right) \right)
\end{aligned}$$

The integrals over y and z are simple and simply evaluate to $L/2$ as we'd expect from the normalization factor. To evaluate the integral over x, use the trigonometric identity $\sin 2\theta = 2 \sin\theta \cos\theta$ and a change of variables with $u = \sin(\pi x/2L)$ to arrive at the integral

$$\mathscr{P} = \frac{2\sqrt{2}}{L} \int_0^1 u^2 \cdot \frac{2L}{\pi} \, du$$

Evaluating this, we find the probability of remaining the ground state after the box is expanded suddenly to be

$$\boxed{\mathscr{P} = \frac{4\sqrt{2}}{3\pi} \approx 0.60} \tag{F2002 I 11.1}$$

Problem 12

Question

The frequency f of a deep water gravity wave (i.e. an ordinary ocean wave) is given by

$$f = \sqrt{\frac{1}{2\pi}\rho^a g^b \lambda^c}$$

where ρ, g, and λ are the water density, gravitational acceleration, and wavelength of the wave, respectively. What are the values of the exponents a, b, and c, and what is the ratio of the wave group velocity to phase velocity?

Answer

We proceed by dimesional analysis. Immediately we know that $a = 0$ since a frequency does not have a mass component, and neither g nor λ have a mass term to cancel the one in ρ. Furthermore, g is the only one with a time term, so it's exponent must then by $b = \frac{1}{2}$ in order to give f its $[\mathrm{s}^{-1}]$ unit. That leaves $c = \frac{1}{2}$ in order to cancel the $\sqrt{\mathrm{m}}$ dimensin left over from g.

$$\boxed{f = \sqrt{\frac{g\lambda}{2\pi}} \qquad \text{with} \qquad a = 0,\ b = \frac{1}{2},\ c = \frac{1}{2}}$$

The phase velocity can be derived from the frequency given by noting that $v_g = \omega/k$ together with $k^{-1} = 2\pi\lambda$ and $\omega = 2\pi f$. Put together, this gives

$$v_p = \frac{1}{k}\sqrt{\frac{g}{k}}$$

The group velocity is given by $v_g = d\omega/dk$, so

$$v_g = -\frac{1}{2k}\sqrt{\frac{g}{k}}$$

Taking only the absolute values and finding the ratio

$$\boxed{\frac{v_g}{v_p} = \frac{1}{2}} \tag{F2002 I 12.1}$$

Spring 2002 Part II

Problem 2

Question

A zipper has N links; each link has a closed state with zero energy and an open state with energy ε. We require, however, that the zipper can only unzip from the left end, and that the link number s can only open if all links to the left (i.e. $1, 2, \ldots s-1$) are already open.

a. Find an explicit expression for the partition function by doing the appropriate summation.

b. In the limit $\varepsilon \gg k_B T$ find the average number of open links. This model is a very simplified model of the unwinding of two-stranded DNA molecules.

Answer

a. Create the partion function by induction; start by assuming there is only a single link. Then the partition function is a simple two-state system:

$$Z_1 = e^0 + e^{-\varepsilon/k_B T} = 1 + e^{-\varepsilon/k_B T}$$

Adding a second link,

$$\begin{aligned} Z_2 &= \underbrace{e^{0+0}}_{\text{both closed}} + \underbrace{e^{(-\varepsilon+0)/k_B T}}_{\text{1 open, 1 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon)/k_B T}}_{\text{both open}} \\ &= 1 + e^{-\varepsilon/k_B T} + e^{-2\varepsilon/k_B T} \end{aligned}$$

Following, for three links:

$$\begin{aligned} Z_3 &= \underbrace{e^{0+0+0}}_{\text{all closed}} + \underbrace{e^{(-\varepsilon+0+0)/k_B T}}_{\text{1 open, 2 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon+0)/k_B T}}_{\text{2 open, 1 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon-\varepsilon)/k_B T}}_{\text{all open}} \\ &= 1 + e^{-\varepsilon/k_B T} + e^{-2\varepsilon/k_B T} + e^{-3\varepsilon/k_B T} \end{aligned}$$

By induction, we see that the maximum coefficient in the series of exponential factors is just the number of links, so by induction we conclude that

$$Z = \sum_{s=0}^{N} e^{-s\varepsilon/k_B T}$$

Applying the results of a finite geometric series, the closed-form solution for the partition function of the links is

$$\boxed{Z = \frac{1 - e^{-(N+1)\varepsilon/k_B T}}{1 - e^{-\varepsilon/k_B T}}} \tag{F2002 II 2.1}$$

b. To get the average number of open links, we use the standard procedure for finding expectation values.

$$\langle s \rangle = \frac{1}{Z} \sum_{s=0}^{N} s e^{-s\varepsilon/k_B T}$$

By making use of differentiation under the summation trick, we can find the closed-form solution:

$$\begin{aligned} \langle s \rangle &= \frac{1}{Z} \sum_{s=0}^{N} \frac{d}{d\left(\frac{\varepsilon}{k_B T}\right)} \left[- e^{-s\varepsilon/k_B T} \right] \\ &= -\frac{1}{Z} \frac{\partial}{\partial\left(\frac{\varepsilon}{k_B T}\right)} \sum_{s=0}^{N} e^{-s\varepsilon/k_B T} \end{aligned}$$

Noting that the summation is the same as above,

$$\langle s \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)}$$

First considering just the derivative part:

$$\frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)} = \frac{(N+1)\, e^{-(N+1)\varepsilon/k_B T}}{1 - e^{-\varepsilon/k_B T}} - \frac{1 - e^{-(N+1)\varepsilon/k_B T}}{\left(1 - e^{-\varepsilon/k_B T}\right)^2} e^{-\varepsilon/k_B T}$$

which when combined with the factor $-1/Z$ simplifies to

$$-\frac{1}{Z} \frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)} = -(N+1) \frac{e^{-(N+1)\varepsilon/k_B T}}{1 - e^{-(N+1)\varepsilon/k_B T}} + \frac{e^{-\varepsilon/k_B T}}{1 - e^{-\varepsilon/k_B T}}$$

$$-\frac{1}{Z} \frac{\partial Z}{\partial \left(\frac{\varepsilon}{k_B T} \right)} = \frac{1}{e^{\varepsilon/k_B T} - 1} - \frac{N+1}{e^{(N+1)\varepsilon/k_B T} - 1}$$

Therefore the analytic solution is

$$\boxed{\langle s \rangle = \frac{1}{e^{\varepsilon/k_B T} - 1} - \frac{N+1}{e^{(N+1)\varepsilon/k_B T} - 1}} \tag{F2002 II 2.2}$$

In the limit that $\varepsilon \gg k_B T$, though, the exponentials in the denominator are very large in comparison to 1, so we ignore the unity factors and make the approximation that

$$\langle s \rangle = e^{-\varepsilon/k_B T} - (N+1)\, e^{-(N+1)\varepsilon/k_B T}$$

Collecting like terms,

$$= \left[1 - (N+1)\, e^{-N\varepsilon/k_B T}\right] e^{-\varepsilon/k_B T}$$

The second term in the brackets approximate zero, so

$$= e^{-\varepsilon/k_B T}$$

Therefore in the low temperature limit where the thermal energy is much less than the energy of the open state,

$$\boxed{\langle s \rangle = e^{-\varepsilon/k_B T}} \tag{F2002 II 2.3}$$

Fall 2002 Part I

Problem 1

Question

A cylindrical bucket is placed on the ground and filled with water to a height of 150 cm. HOw high from the ground should one punch a hole in the side of the bucket to make a stream of water that strikes the ground at the greatest distance from the bucket? What is that distance?

Answer

To solve and optimize the kinematic equation to determine the maximum range of the water jet, we first need to determine what the velocity of the water exiting the hole is as a function of the hole's depth below the surface of the water. To do this, we make use of Bernoulli's principle.

$$\frac{v^2}{2} + gz + \frac{p_0}{\rho} = \text{constant}$$

We begin by determining the value of the constant by evaluating the equation at the surface of the water where we know all the properties. The water is at the ambient air pressure, the height is given in the problem statement, and the velocity can be assumed to be zero in the limit that the hole leaks at a rate too slowly to change the height of the surface. Doing so,

$$\text{constant} = gz_0 + \frac{p_0}{\rho}$$

Then at some height z' below the surface of the water, we puncture a hole. Again the water is moving into a body at atmospheric pressure, so we again use p_0. That leaves the velocity we're searching for remaining, so equating with the surface value,

$$\begin{aligned} gz_0 + \frac{p_0}{\rho} &= \frac{1}{2}v^2 + gz' + \frac{p_0}{\rho} \\ v^2 &= 2g\left(z_0 - z'\right) \end{aligned}$$

Now we're ready to solve the kinematic equation. Begin as usual by finding the time of flight from the vertical components.

$$0 = z' - \frac{1}{2}gt^2 \qquad \rightarrow \qquad t = \sqrt{\frac{2z'}{g}}$$

Then solving the horizontal equation:

$$\ell = vt = 2\sqrt{z'\left(z_0 - z'\right)}$$

Then finding the extrema:

$$\begin{aligned} \frac{d\ell}{dz'} &= 0 = \frac{z_0 - 2z'}{\sqrt{z'\left(z_0 - z'\right)}} \\ z' &= \frac{1}{2}z_0 \end{aligned}$$

That height that maximizes distance, and that distance, is

$$\boxed{z' = 75\,\text{cm}} \qquad \boxed{\ell = 150\,\text{cm}} \tag{F2002 I 1.1}$$

Problem 5

Question

Blocks of mass m and $2m$ are free to slide without friction on a horizontal wire. They are connected by a massless spring of equilibrium length L and force constant k. A projectile of mass m is fired with velocity v into the block with mass m and sticks to it. If the blocks are initially at rest, what is the maximum displacement between them in the subsequent motion?

Answer

Take time $t = 0$ to be the moment the projectile collides with the mass m, and let the subsequent transfer of momentum be instantaneous. In this case, the initial conditions of the problem are then:

$$
\begin{aligned}
x_1(0) &= 0 & \dot{x}_1(0) &= u \\
x_2(0) &= L & \dot{x}_2(0) &= 0
\end{aligned}
$$

where u is the initial velocity of the combined project-mass system. We get u from conservation of mometum:

$$
\begin{aligned}
2mu &= mv + 0 \\
u &= \frac{1}{2}v
\end{aligned}
$$

Now solve the mechanics problem using the Lagrangian approach. Both masses have kinetic energy, and the spring stores potential energy, so

$$
\begin{aligned}
T &= m\dot{x}_1^2 + m\dot{x}_2^2 \\
V &= \frac{1}{2}m\left(x_2 - x_1\right)^2 \\
L &= m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k\left({x_1}^2 + {x_2}^2 + 2x_1x_2\right)
\end{aligned}
$$

Setting up the differential equation, we get

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= -kx_1 + kx_2 & \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= 2m\ddot{x}_1 \\
\frac{\partial L}{\partial x_2} &= kx_1 - kx_2 & \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= 2m\ddot{x}_2
\end{aligned}
$$

Leading to the system of equations where $\omega^2 = k/2m$,

$$
\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Solving the eigensystem, we find the eigenfrequencies to be $\lambda = \{0, -2\omega^2\}$. Letting $\omega'^2 = 2\omega^2$, the eigenfunction equations are then

$$
\begin{aligned}
\ddot{\psi}_1 &= 0 & &\rightarrow & \psi_1 &= A_1 t + B_1 \\
\ddot{\psi}_2 &= -2\omega^2\psi_2 & &\rightarrow & \psi_2 &= A_2\cos\left(\omega' t\right) + B_2\sin\left(\omega' t\right)
\end{aligned}
$$

From the eigenvectors, we express the solutions of x_1 and x_2 in terms of ψ_1 and ψ_2:

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}
$$

$$x_1 = A_1 t + B_1 + A_2 \cos\left(\omega' t\right) + B_2 \sin\left(\omega' t\right)$$
$$x_2 = A_1 t + B_1 - A_2 \cos\left(\omega' t\right) - B_2 \sin\left(\omega' t\right)$$

Applying the boundary conditions, we find that

$$x_1(t) = \frac{1}{4} v t + \frac{1}{2} L - \frac{1}{2} L \cos\left(\omega' t\right) + \frac{v}{4\omega'} \sin\left(\omega' t\right)$$
$$x_2(t) = \frac{1}{4} v t + \frac{1}{2} L + \frac{1}{2} L \cos\left(\omega' t\right) - \frac{v}{4\omega'} \sin\left(\omega' t\right)$$

The distance $\ell(t) = x_2(t) - x_1(t)$ between the two masses maximizes when

$$\frac{d\ell}{dt} = 0 = \frac{d}{dt}\left[L \cos\left(\omega' t\right) - \frac{v}{2\omega'} \sin\left(\omega' t\right)\right]$$
$$t = -\frac{1}{\omega'} \arctan\left(\frac{v}{2L\omega'}\right)$$

Plugging back into the function $\ell(t)$,

$$\ell = L \cos\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right] - \frac{v}{2\omega'} \sin\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right]$$
$$\ell = L \frac{2L\omega'}{\sqrt{v^2 + 4L^2\omega'^2}} + \frac{v}{2\omega'} \frac{v}{\sqrt{v^2 + 4L^2\omega'^2}}$$
$$\ell = \frac{\sqrt{v^2 + 4L^2\omega'^2}}{2\omega'}$$

Finally, substituting back in $\omega' = \sqrt{2k/m}$ and simplifying, we get the final solution that maximum distance between the two masses is

$$\boxed{\ell = \sqrt{L^2 + \frac{\frac{1}{2} m v^2}{8k}}} \qquad \text{(F2002 I 5.1)}$$

which agrees qualitatively with the fact that a larger spring constant should stiffen the system and decrease the maximum displacement, while launching the projectile with a greater velocity would increase it.

Problem 7

Question

In the state $\psi_{\ell,m}$ with angular momentum ℓ and its projection m, determine the average values ${l_x}^2$ and ${l_y}^2$.

Answer

Begin by noting that the angular momentum operators are related as,

$$L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$$

If we assume that the average x and y components will be equal based on symmetry where only the z direction is identifiable, then we can let $L_x = L_y$ and rearrage the equation to get

$$ {L_x}^2 = \frac{1}{2}\left(L^2 - {L_z}^2\right)$$

The expectation value is then found by the standard method:

$$\begin{aligned}\left\langle \psi_{\ell,m} \middle| L_x^2 \middle| \psi_{\ell,m} \right\rangle &= \left\langle \psi_{\ell,m} \middle| \frac{1}{2}\left(L^2 - {L_z}^2\right) \middle| \psi_{\ell,m} \right\rangle \\ &= \frac{1}{2}\left(\left\langle \psi_{\ell,m} \middle| L^2 \middle| \psi_{\ell,m} \right\rangle - \left\langle \psi_{\ell,m} \middle| {L_z}^2 \middle| \psi_{\ell,m} \right\rangle\right)\end{aligned}$$

Then because we know these quantum numbers

$$\left\langle \psi_{\ell,m} \middle| L_x^2 \middle| \psi_{\ell,m} \right\rangle = \frac{1}{2}\left(\ell - m^2\right)$$

Therefore with both q ${\ell_x}^2$ and ${\ell_y}^2$ being assumed equation, we conclude that

$$\left\langle {\ell_x}^2 \right\rangle = \left\langle {\ell_y}^2 \right\rangle = \frac{1}{2}\left(\ell - m^2\right) \tag{F2002 I 7.1}$$

Problem 10

Question

Ice on a pond is 10 cm thick and the water temperature just below the ice is 0 ℃. If the air temperature is −20 ℃, by how much will the ice thickness increase in 1 hour? Assuming that the air temperature stays the same over a long period, how will the ice thickness increase with time? Comment on any approximation that you make in your calculation.

Density of ice $= 0.9\,\mathrm{g/cm^3}$

Thermal conductivity of ice $= 0.0005\,\mathrm{cal/(cm{\cdot}s{\cdot}{}^\circ C)}$

Latent heat of fusion of water $= 80\,\mathrm{cal/g}$

Answer

Since the thermal heat flow is a one dimensional problem, immediately consider everything with respect to a small area element with its normal perpendicular to the ice-water interface dA. Then we want to know how much ice is generated on the surface of the ice. This small ice element's mass is simply

$$dm = \rho\, dA\, dz$$

where dz is the thickness of the new ice layer. To generate this ice, the latent heat of fusion must be conducted away, so the energy released is,

$$\begin{aligned} dE_f &= L_f\, dm \\ &= L_f \rho\, dA\, dz \end{aligned}$$

The energy flow is through the ice, and we expect this to increase with the temperature differential across the ice sheet, suggesting that the thermal conductivity κ be multiplied by the temperature difference ΔT. Furthermore, the ice will decrease the rate of heat flow as it becomes thicker, so the quantity should also be divided by the thickness z. This gives

$$\frac{\kappa \Delta T}{z} = \left[\frac{\mathrm{cal}}{\mathrm{cm^2{\cdot}s}}\right]$$

This energy is flowing through a surface element dA, giving the power flow due to heat as

$$\frac{\kappa \Delta T\, dA}{z} = \left[\frac{\mathrm{cal}}{\mathrm{s}}\right]$$

This power can be matched in units with the energy released from the ice calculated above by taking the time derivative of dE_f, so equating the two we have

$$\begin{aligned} L_f \rho\, dA \frac{dz}{dt} &= \frac{\kappa \Delta T\, dA}{z} \\ \int_{z_0}^{z_0+\delta z} z\, dz &= \int_0^t \frac{\kappa \Delta T}{L_f \rho}\, dt \\ 2z_0 \delta z + (\delta z)^2 &= \frac{\kappa \Delta T}{L_f \rho} t \end{aligned}$$

Solving for the length the ice grows δz,

$$\begin{aligned} \delta z &= \frac{-2z_0 \pm \sqrt{4{z_0}^2 - 4\left(\frac{\kappa \Delta T}{L_f \rho}\right) t}}{2} \\ \delta z &= z_0 \left(1 \pm \sqrt{1 - \frac{\kappa \Delta T}{L_f \rho {z_0}^2} t}\right) \end{aligned}$$

The two roots give solutions $\delta z = \{0.0501\,\mathrm{cm}, 19.950\,\mathrm{cm}\}$. Since the second root is unrealistic, we know that the solution must then be

$$\boxed{\delta z = 0.0501\,\mathrm{cm} \quad \text{in an hour}} \tag{F2002 I 10.1}$$

Problem 11

Question

Carbon-14 is produced by cosmic rays interacting with the nitrogen in the Earth's atmosphere. It is eventaully incorporated into all living things, and since it has a half-life of (5730 ± 40) yr, it is useful for dating archaeological specimens up to several tens of thousands of years old. The radioactivity of a particular specimen of wood containing 3 g of carbon was measured with a counter whose efficiency was 18 %; a count rate of $(12.8 \pm 0.1)\,\mathrm{min}^{-1}$ was measured. It is known that in 1 g of living wood, there are $16.1\,\mathrm{min}^{-1}$ radioactive carbon-14 decays. What is the age of this specimen, and its uncertainity? (Where errors are not quoted, they can be assumed to be negligible).

Answer

The rate N after a given time is given by the exponential decay formula

$$N(t) + = N_0 e^{-t/\tau}$$

Since we have the half-life $t_{1/2}$ instead of the decay constant τ, we use the relation $t_{1/2} = \tau \ln 2$ to simplify the expression instead to

$$N(t) = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The counter use has an efficiency of $\varepsilon = 0.18$, so the measured counting rate N_m must be corrected for that. Furthermore, the sample has a mass of 3 g whereas we know the rate for a one gram sample, so we also normalize the count rate by the mass of the sample. Plugging this all into the exponential decay function above gives

$$\frac{N_m}{3\varepsilon} = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The only unknown left in the equation is the time, so solving for it,

$$t = t_{1/2} \log_{1/2}\left(\frac{N_m}{3\varepsilon N_0}\right)$$
$$t = t_{1/2} \frac{\ln\left(\frac{N_m}{3\varepsilon N_0}\right)}{\ln 2}$$
$$t = \frac{t_{1/2}}{\ln 2} \ln\left(\frac{N_m}{3\varepsilon N_0}\right)$$

To find the uncertainty, we note that only the quantities N_m and $t_{1/2}$ have non-negligible uncertainties, so we propagate the errors only over these two terms:

$$\sigma_t{}^2 = \left(-\frac{t_{1/2}}{N_m \ln 2}\right)^2 \sigma_{N_m}{}^2 + \left(\frac{1}{\ln 2} \ln\left(\frac{3\varepsilon N_0}{N_m}\right)\right)^2 \sigma_{t_{1/2}}{}^2$$
$$\sigma_t = \frac{t_{1/2}}{\ln 2} \sqrt{\left(\frac{\sigma_{N_m}}{N_m}\right)^2 + \left(\frac{\sigma_{t_{1/2}}}{t_{1/2}}\right)^2 \left[\ln\left(\frac{3\varepsilon N_0}{N_m}\right)\right]^2}$$

Plugging in all the numbers, we get $t = 4248.8435$ yr and $\sigma_t = 161.717$ yr. The given uncertainties have a single significant digit, so adding an extra significant figure to the uncertainity and matching decimal places in the answer, we conclude that the sample has an age of

$$\boxed{t = (4250 \pm 160)\,\mathrm{yr}} \qquad \text{(F2002 I 11.1)}$$

Fall 2008 Part I

Problem 2

Question

If an impulse is delivered to the end of a uniform rod of length ℓ, lying on a frictionless plane, how far will it travel while making one revolution? The impulse is in the plane of the table and perpendicular to the rod.

Answer

For a given impulse $\vec{J}$, the change in the motion is $\vec{J} = \Delta\vec{p}$. If the rod start at rest, then the final momentum must be $\vec{p} = \vec{J}$. This means the rod is moving laterally with a velocity

$$V = \frac{1}{m}\vec{J}$$

which when integrated over a time t gives the distance it has moved $\vec{x}$.

$$\vec{x} = \frac{1}{m}\vec{J}t$$

The impulse also imparts a rotation on the rod because the force was not applied at the rod's center of mass. The torque $\vec{\tau}$ relates the force to the angular momentum $\vec{L}$ by

$$\vec{r} \times \vec{F} = \vec{\tau} = \dot{\vec{L}}$$

Integrating both sides of the equation, we can write the equation in terms of the given impulse:

$$\vec{r} \times \int \vec{F}\,dt = \int \dot{\vec{L}}\,dt$$
$$\vec{r} \times \vec{J} = \Delta\vec{L}$$

Again, since the rod starts at rest, we know that the final angular momentum must be

$$\vec{L} = \vec{r} \times \vec{J}$$

The rotation about the rod's center of mass occurs at a rate $\vec{\omega}$ dependent on the moment of inertia $I = \frac{1}{12}m\ell^2$, so

$$\vec{\omega} = \frac{12}{m\ell^2}\vec{r} \times \vec{J}$$

We know that the impulse is applied perpendicular to the rod, so we can easily integrate the expression in time and solve for the time it takes to revolve 2π radians:

$$\theta = 2\pi = \frac{12}{m\ell^2} rJt$$
$$t = \frac{\pi m\ell^2}{6rJ}$$

Plugging this back into the linear motion equation, the rod travels

$$\vec{x} = \frac{1}{m}\vec{J} \cdot \frac{\pi m\ell^2}{6rJ}$$

where we can set $r = \frac{1}{2}\ell$ and therefore simplifies to

$$\boxed{\vec{x} = \frac{\pi\ell}{3}\hat{J}} \tag{F2008 I 2.1}$$

where $\hat{J}$ is the direction of the applied impulse.

Problem 3

Question

A time-indpendent magnetic field is given by $\vec{B} = 2bxy\,\hat{\imath} + ay^2\,\hat{\jmath}$.

a) What is the relationship between the constants a and b?

b) Determine the steady current density J that gives rise to this field.

Answer

For part (a), we realize that all magnetic fields must be divergenceless. Therefore we can find the requirements on the constants a and b by constraining the divergence to be zero.

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} = 0 &= \frac{\partial}{\partial x}(2bxy) + \frac{\partial}{\partial y}\left(ay^2\right) \\ 0 &= 2by + 2ay \\ b &= -a\end{aligned}$$

Therefore the relation between the constants is that

$$\boxed{b = -a} \tag{F2008 I 3.1}$$

For the second part, we make use of Maxwell's equations. Assuming that none of the field is due to a time-varying electric field, we make use of

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

to calculate the current that generates the field. Doing so, we find that the solution is

$$\boxed{\vec{J} = \frac{2a}{\mu_0} x\,\hat{k}} \tag{F2008 I 3.2}$$

Problem 4

Question

A set of four point charges q_1, q_2, q_3, and q_4 are arranged collinearly along the z-axis at $z_1 = 0$, $z_2 = a$, $z_3 = 2a$, $z_4 = 4a$, respectively and the resulting electric field at a distant point $\vec{r}$ ($r \gg a$) decays *faster* than $1/r^3$. Determine the values of q_1 and q_4 which $q_2 = +2$ and $q_3 = +4$. Units for all charges are Coulombs.

Answer

Given that the electric field must fall off faster than $1/r^3$, this corresponds to a potential which drops off faster than $1/r^2$. We know that the monopole moment drops off like $1/r$ and the dipole like $1/r^2$, so we conclude that the first configuration which could satisfy the given requirement is that of a quadrupole moment.

Making use of the fact that he monopole and dipole moments are vanishing, we can use them to generate constraint equations for what the charges must be: we have two unknown charges and the two equations will allow us to solve them.

For the monopole, the sum of all charges must simply equal zero. Therefore we immediately know that

$$\begin{aligned} 0 &= q_1 + q_4 + 6 \\ -6 &= q_1 + q_4 \end{aligned}$$

The dipole moment (where we take the dipole considered at the origin) is given by

$$\vec{p} = \sum_i \vec{r}_i q_i$$

This gives us the equation

$$\begin{aligned} 0 &= 10a + 4aq_4 \\ q_4 &= -\frac{5}{2} \end{aligned}$$

The charge q_1 does not show up in the equation since it is located at the origin. This lets us very simply then solve for q_1 as

$$-6 = q_1 - \frac{5}{2}$$

Therefore, the solution is that the charges have values of

$$\boxed{q_1 = -\frac{7}{2}} \tag{F2008 I 4.1}$$

$$\boxed{q_4 = -\frac{5}{2}} \tag{F2008 I 4.2}$$

Problem 5

Question

The Lyman-α transition in atomic hydrogen has a wavelength $\lambda = 121.5\,\mathrm{nm}$, and a transition rate of $0.6 \cdot 10^9\,\mathrm{s}^{-1}$. Estimate the minimum value of $\Delta\lambda/\lambda$.

Answer

We can make an estimate of the spread $\Delta\lambda$ by making use of the Heisenberg uncertainty relation for energy-time. Starting with the variation in wavelength,

$$\begin{aligned}\Delta\lambda &= \lambda - \lambda' \\ &= \frac{hc}{E} - \frac{hc}{E'} \\ &= \frac{hc\left(E' - E\right)}{EE'}\end{aligned}$$

Making use of the approximation that $E \approx E'$,

$$= \frac{hc\Delta E}{E^2}$$

Dividing by the frequency and substituting in the uncertainty relation $\Delta E \Delta t = \frac{\hbar}{2}$,

$$\begin{aligned}\frac{\Delta\lambda}{\lambda} &= \frac{hc}{\lambda} \cdot \frac{1}{E^2}\frac{\hbar}{2\Delta t} \\ &= \frac{\lambda}{4\pi c \Delta t}\end{aligned}$$

For the time, we estimate the transition rate is occuring as fast as it can within the limits of the uncertainty relation, so we can let $\Delta t \approx 0.6 \cdot 10^9\,\mathrm{s}^{-1}$. Plugging in the other values, we find the fractional line width to be estimated as

$$\boxed{\frac{\Delta\lambda}{\lambda} \approx 1.935 \cdot 10^{-8} \approx 1 \text{ part in 50 million}} \tag{F2008 I 5.1}$$

Problem 11

Question

A rock is found to contain 4.20 mg of ^{238}U and 2.00 mg of ^{206}Pb. Assume tha the rock contained no lead at the time of its formation, so that all the lead now present is due to th decay of the uranium orignally present in the rock. Find the age of the rock given that the half-life of ^{238}U is $4.47 \cdot 10^9$ yr. The decay times of all intermediate elements are negligibly short and ignore any differences in the binding energies.

Answer

From decay processes, we know that the uranium atom count will decrease as an exponential according to

$$N_U = N_{U0} e^{-t/\tau}$$

where $\tau = t_{1/2}/\ln 2$. Likewise, the number of lead atoms will increase according to

$$N_{Pb} = N_{U0}\left(1 - e^{-t/\tau}\right)$$

Solveing for N_{U0} in the first equation and substituting it into the second, we can solve for the time required to generate a specific number of uranium and lead atoms in a sample.

$$\begin{aligned} N_{Pb} &= N_U e^{t/\tau}\left(1 - e^{-t/\tau}\right) \\ t &= \tau \ln\left(\frac{N_{Pb}}{N_U} + 1\right) \\ t &= \frac{t_{1/2}}{\ln 2} \ln\left(\frac{N_{Pb}}{N_U} + 1\right) \end{aligned}$$

We were only given the masses, though, so we approximate the mass of each atom by the number of nucleons in the nucleus; each uranium atom has a mass of $m_U = 238 m_N$ making the N_U atoms have a mass of $M_U = 238 N_U m_N$, and similar for the lead. This gives us the final equation

$$t = \frac{t_{1/2}}{\ln 2} \ln\left(\frac{238}{206}\frac{M_{Pb}}{M_U} + 1\right)$$

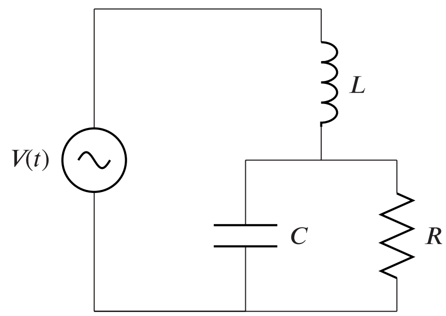
Plugging in all the numbers,

$$\boxed{t = 2.83 \cdot 10^9 \text{ yr}} \tag{F2008 I 11.1}$$

Problem 12

Question

The applied AC voltage in the circuit is given by $V(t) = V_0 \sin \omega t$, with a frequency fixed at $\omega = 1/(LC)^{1/2}$. Determine the steady state amplitude and phase of the current through the resistor R. Express your answer in terms of the amplitude V_0 of the applied voltage and the other circuit parameters.

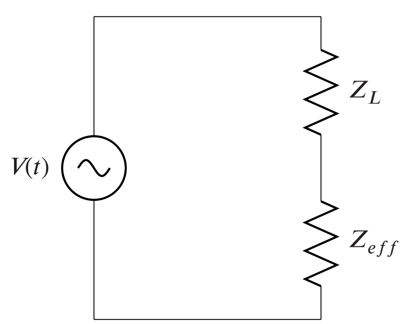


Answer

AC problems are simplified by using complex impedances, so we first convert the given voltage into a complex one:

$$\tilde{V}(t) = V_0 e^{i\omega t}$$

where the physical solution can be recovered by keeping the imaginary component of the complex solution. Then to solve the problem, we realize that there is another complimentary circuit diagram which is helpful: the one with the resistor and capacitor replaced by an effective resistor (impedance). The circuit looks like



The inductor has been been replaced by an effective resistor with impedance $Z_L = i\omega L$. The effective resistor that replaced the capacitor and resistor is a complex impedance that is calculated the same as for traditional resistors in parallel:

$$\begin{aligned} Z_{eff} &= \left(\frac{1}{Z_C} + \frac{1}{Z_R} \right)^{-1} \\ &= \left(i\omega C + \frac{1}{R} \right)^{-1} \\ &= \frac{R}{i\omega CR + 1} \end{aligned}$$

Now making use of Kirchoff's loop rule on this simplified circuit where the total current passing through the voltage source is labeled $\tilde{I}_0$,

$$\begin{aligned} 0 &= \tilde{V} - \tilde{I}_0\left(Z_L + Z_{eff}\right) \\ \tilde{V} &= \left(i\omega L + \frac{R}{i\omega CR + 1}\right)\tilde{I}_0 \\ \tilde{V} &= \frac{R\left(1 - \omega^2 LC\right) + i\omega L}{i\omega RC + 1}\tilde{I}_0 \end{aligned}$$

The first term in the numerator goes to zero since $\omega^2 = 1/LC$, leaving

$$\tilde{I}_0 = \frac{i\omega RC + 1}{i\omega L} V_0 e^{i\omega t}$$

To isolate the current passing through the resistor, we return to the original unsimplified circuit diagram and apply Kirchoff's loop rule to only the inner loop. If we define the current through capacitor to be I_1 and through the resistor to be I_2, we get

$$\begin{aligned} 0 &= -\tilde{I}_2 Z_R + \tilde{I}_1 Z_C \\ \tilde{I}_1 &= \frac{Z_R}{Z_C} I_2 \\ \tilde{I}_1 &= i\omega RC I_2 \end{aligned}$$

Remembering the the current passing into a junction must be conserved, we know that $I_0 = I_1 + I_2$ and therefore,

$$\begin{aligned} \tilde{I}_0 &= i\omega RC\tilde{I}_2 + \tilde{I}_2 \\ \tilde{I}_2 &= \frac{1}{i\omega RC + 1}\tilde{I}_0 \end{aligned}$$

Inserting the solution for I_0 from the previous part leaves

$$\tilde{I}_2 = \frac{V_0}{i\omega L} e^{i\omega t}$$

To prepare for finding the physical solution, we transform the coefficient complex polar form.

$$\begin{aligned} \tilde{I}_2 &= \left|-\frac{iV_0}{\omega L}\right| e^{i\arg\left(-iV_0/\omega L\right)} e^{i\omega t} \\ &= \frac{V_0}{\omega L} e^{-i\pi/2} e^{i\omega t} \end{aligned}$$

Therefore taking the imaginary part of the solution,

$$\boxed{I_R(t) = V_0\sqrt{\frac{C}{L}}\sin\left(\omega t - \frac{\pi}{2}\right)} \qquad \text{(F2008 I 12.1)}$$

The current's amplitude is $V_0\sqrt{C/L}$ and has a phase of $-\pi/2$ with respect to the voltage.

Fall 2008 Part II

Problem 1

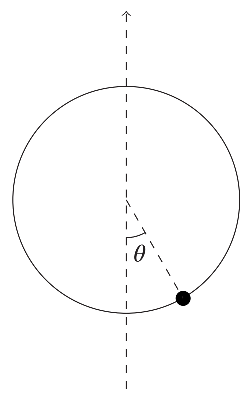
Question

A particle of mass m is constrained to move without friction on a circular wire of radius R rotating with constant angular frequency ω about a vertical diameter. Gravity can not be neglected.

a) Write down the Lagrangian for the system and the equations of motion.

b) Find the equilibrium position(s) of the particle and determine whether this position is stable.

c) Calculate the frequency of small oscillations about any stable points.



Answer

To start constructing the Lagrangian and equations of motion, we first specify the kinetic and potential energies. For the kinetic energy, there is an energy associated with the rotation about the axis and one along the bead. These combined to give

$$\begin{aligned} T &= \frac{1}{2} m \left(R\omega \sin\theta\right)^2 + \frac{1}{2} m \left(R\dot{\theta}\right)^2 \\ &= \frac{1}{2} m R^2 \omega^2 \sin^2\theta + \frac{1}{2} m R^2 \dot{\theta}^2 \end{aligned}$$

The potential energy is all gravitational, so

$$V = -mgR\cos\theta$$

where the zero point was taken to be at the center of the hoop to avoid adding extra constant terms to the Lagrangian. Combining the two, we get

$$\boxed{\mathcal{L} = \frac{1}{2} m R^2 \omega^2 \sin^2\theta + \frac{1}{2} m R^2 \dot{\theta}^2 + mgR\cos\theta} \tag{F2008 II 1.1}$$

Taking the appropriate derivatives in θ, the equation of motion is

$$\boxed{\ddot{\theta} = \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta} \tag{F2008 II 1.2}$$

In order to determine any possible stable points, we note that a stable point is a place where the angle does not change in time. Since this also equates to $\ddot{\theta} = 0$, we set the equation above equal to zero and solve for the angles which satisfy this condition. They end up being the trivial $\theta = \{0, \pi\}$ where the sine function is zero as well as

$$\cos\theta_0 = \frac{g}{R\omega^2}$$

The three stable points are then

$$\boxed{\theta_0 = \left\{0, \arccos\left(\frac{g}{R\omega^2}\right), \pi\right\}} \qquad \text{(F2008 II 1.3)}$$

To determine the stability of each, we must determine whether we get oscillatory or exponential solutions to the differential equation of motion. To do this, we suppose the angle θ is composed of the equilibrium angle θ_0 and a small perturbation δ. Expanding the equation in terms of this,

$$\ddot{\delta} = \omega^2 \sin\left(\theta_0 + \delta\right)\cos\left(\theta_0 + \delta\right) - \frac{g}{R}\sin\left(\theta_0 + \delta\right)$$

Using several trigonometric expansions, the equation can be expanded into the form

$$\ddot{\delta} = \omega^2\left[\cos\theta_0\sin\theta_0\left(\cos^2\delta - \sin^2\delta\right) + \cos\delta\sin\delta\left(\cos^2\theta_0 - \sin^2\theta_0\right)\right] - \frac{g}{R}\left[\sin\theta_0\cos\delta + \cos\theta_0\sin\delta\right]$$

For the case where $\theta_0 = 0$,

$$\ddot{\delta} = \omega^2\cos\delta\sin\delta - \frac{g}{R}\sin\delta$$

Expanding to first order in $\delta \approx 0$,

$$\ddot{\delta} = -\left(\frac{g}{R} - \omega^2\right)\delta$$

Therefore, the equilibrium point $\theta_0 = 0$ is only stable if $\omega < \sqrt{\frac{g}{R}}$.

Likewise for for $\theta_0 = \pi$,

$$\ddot{\delta} = \left(\frac{g}{R} + \omega^2\right)\delta$$

The coefficient on δ will never be negative, so the angle $\theta_0 = \pi$ will be unstable under all conditions.

For the final angle where $\theta_0 = \arccos\left(\frac{g}{R\omega^2}\right)$, we must do several substitutions and expansions. $\cos\theta_0$ is trivial. $\sin\theta_0$ ends up being $\sqrt{R\omega^2 - g^2}/\left(R\omega^2\right)$ by triangle relations. If we substitute these in plus do an expansion to first order for small δ, we get the equation

$$\ddot{\delta} = \omega^2\left[\frac{g\sqrt{R\omega^2 - g^2}}{R^2\omega^2} + \delta\frac{2g^2 - R\omega^2}{R^2\omega^2}\right] - \frac{g}{R}\left[\frac{\sqrt{R\omega^2 - g^2}}{R^2\omega^2} + \delta\frac{g}{R\omega^2}\right]$$

If we consider only the homogeneous terms dependent on δ,

$$\ddot{\delta} = \frac{g^2 - R\omega^2}{R^2\omega^2}\delta$$

This equation is stable if and only if the coefficient on δ is negative, so it must be that $\omega > \frac{g}{\sqrt{R}}$.

In summary, the equilibrium points have the following conditions:

$$\boxed{\theta_0 = 0 \qquad \text{Stable iff } \omega < \sqrt{\frac{g}{R}}} \qquad \text{(F2008 II 1.4)}$$

$$\boxed{\theta_0 = \arccos\left(\frac{g}{R\omega^2}\right) \qquad \text{Stable iff } \omega > \frac{g}{\sqrt{R}}} \tag{F2008 II 1.5}$$

$$\boxed{\theta_0 = \pi \qquad \text{Never stable}} \tag{F2008 II 1.6}$$

About the two stable points, we simply use the coefficient that has already been isolated to determine the frequency of the oscillations about that point.

$$\boxed{\omega_1 = \sqrt{\frac{g}{R} - \omega^2} \qquad \text{for } \theta_0 = 0} \tag{F2008 II 1.7}$$

$$\boxed{\omega_2 = \sqrt{\frac{R\omega^2 - g^2}{R^2\omega^2}} \qquad \text{for } \theta_0 = \arccos\left(\frac{g}{R\omega^2}\right)} \tag{F2008 II 1.8}$$

Problem 2

Question

The general solution of the Laplace's equation for an electrostatic problem having azimuthal symmetry can be written as

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Now consider the follwoing problem. A solid spherical conductor of radius R having charge Q is placed in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$.

(a) Qualtitatively describe the electric field inside and outside of the sphere.

(b) Solve the problem and find the electric potential in th eregion outside the sphere.

Answer

To provide a qualitative description, we make use of several properties of conductors. The electric field inside the conductor is guaranteed to be zero in the limit of a perfect conductor which can move its electrons anywhere they're needed to cancel any applied fields. For the region outside the sphere, it is easiest to describe the region just outside the surface and the region at infinitely large distances. Far away, the effects of the the sphere are negligible and the electric field is the external uniform field. Near the surface, though, all field lines are perpendicular to the surface; therefore, the external field's lines are curved so that any intersections occur perpendicular to the surface.

In summary

1. The field is uniform at large distances
2. The field is perpendicular to the surface of the conductor at the conductor's surface
3. There is no field within the interior of the conductor

In order to analyze the problem analytically, we make use of the superposition principle to simplify the problem. Because the sphere carries its own charge, we treat this case as a superposition of the two simpler cases of a charged sphere in vacuum and that of a perfectly conducting, grounded sphere in a uniform electric field.

Since we are only concerned with the potential outside the sphere, we can use Gauss' Law to get the potential due to the charge Q. It is

$$V_Q(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r} \qquad\qquad r > R$$

The uniform field is the considered by satisfying the appropriate boundary conditions to solve for the coefficients A_ℓ and B_ℓ in the general solution given above. We start by converting the given electric field to a potential. In Cartesian coordinates,

$$\vec{E} = E_0\hat{z} = -\vec{\nabla}V_\infty \qquad \Rightarrow \qquad V_\infty = -E_0 z$$

which when converted to spherical coordinates gives the potential as $r \to \infty$

$$V_\infty = -E_0 r\cos\theta$$

In the infinite distance limit, $B_\ell/r^{\ell+1} \to 0$ so the boundary condition equation becomes

$$-E_0 r\cos\theta = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

By the orthogonality of the Legendre polynomials, only A_1 is non-zero:

$$-E_0 r \cos\theta = A_1 r \cos\theta$$
$$A_1 = -E_0$$

The general solution has thus been simplified to

$$V_0(r,\theta) = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

By our choice of making use of the superposition principle, we have set the potential to be zero at the surface, so at $r = R$, the boundary conditions lets us solve for the values of the B_ℓ:

$$0 = -E_0 R \cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

The Legendre polynomial orthogonality again eliminates all coefficients except B_1.

$$E_0 R \cos\theta = \frac{B_1}{R^2}\cos\theta$$
$$B_1 = E_0 R^3$$

This gives us the solution to the grounded sphere as

$$V_0(r,\theta) = -E_0 r \cos\theta \left[1 - \left(\frac{R}{r}\right)^3\right]$$

Therefore by superposition of both solutions, the potential in this situation at all points outside the sphere is

$$\boxed{V(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r} - E_0 r \cos\theta \left[1 - \left(\frac{R}{r}\right)^3\right]} \tag{F2008 II 2.1}$$

Problem 3

Question

Consider the transmission of a beam of particles of mass m and momentum $p = \hbar k$, in one dimension, incident on a rectangular potential barrier of height V_0 and extending from $x = 0$ to $x = L$, in the special case that the energy E of the incident particle *is exactly equal* to the barrier height V_0.

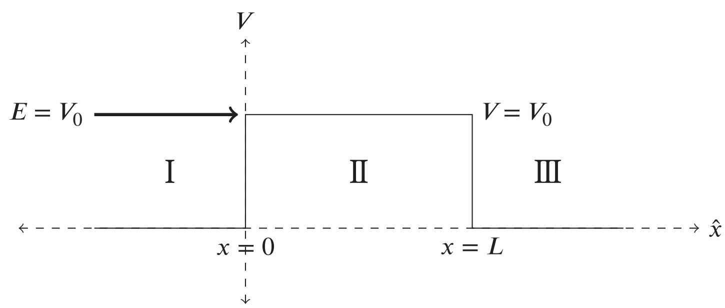
(a) Calculate the transmission and reflection coefficients T and R.

(b) Check some properties of your answers in (a): is probability conserved? Do T and R have the expected limiting values for L very large or very small?

(c) For what values of the de Broglie wavelength of th eparticles is th etransmitted fraction equal to 1/2?

Answer

Consider a beam of particles incident from the left as shown in the figure below:



Ignoring normalization for a minute, we know that in regions I and III that the wavefunction is that of a free particle:

$$\psi(x) = Ae^{ikx} + B^{-ikx} \qquad k^2 = \frac{2mE}{\hbar^2}$$

In region II, the energy E cancels with the potential V in the Schrödinger equation, so the solution takes the form of a first order polynomial

$$\psi(x) = Ax + B$$

We will only be concerned with the reflection and transmission coefficients, and knowing that they are defined in terms of a ratio of the wavefunction amplitude for the reflected and transmitted components with respect to the incident amplitude, we simplify our solution by directly setting the incident particle amplitude to unity. Furthermore, we know that there is no leftward traveling component in region III. Assigning each component a unique and appropriate unknown coefficient, the three wavefunctions are

$$\psi_{\mathrm{I}} = e^{ikx} + re^{-ikx}$$
$$\psi_{\mathrm{II}} = ax + b$$
$$\psi_{\mathrm{III}} = te^{ikx}$$

We find the values for r and t by applying continuity boundary conditions at the interfaces between each solution. Starting at $x = 0$,

$$\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0) \qquad \psi_{\mathrm{I}}'(0) = \psi_{\mathrm{II}}'(0)$$
$$1 + r = b \qquad ik(1 - r) = a$$

Then putting the values into ψ_{I} and solving the boundary conditions at $x = L$,

$$\psi_{\mathrm{II}}(L) = \psi_{\mathrm{III}}(L) \qquad \psi'_{\mathrm{II}}(L) = \psi'_{\mathrm{III}}(L)$$
$$ik(1-r)L + 1 + r = te^{ikL} \qquad ik(1-r) = ikte^{ikL}$$

From the equation on the right, we solve for t as a function of r and insert it into the condition on the left:

$$t = (1-r)e^{-ikL}$$
$$ik(1-r)L + 1 + r = \left((1-r)e^{-ikL}\right)e^{ikL}$$
$$r = \frac{-ikL}{2-ikL}$$

Plugged back into t gives

$$t = \frac{2}{2-ikL}e^{-ikL}$$

We then just take the complex square of both amplitudes to get the reflection and transmission coefficients:

$$\boxed{R = |r|^2 = \frac{k^2L^2}{4+k^2L^2}} \tag{F2008 II 3.1}$$

$$\boxed{T = |t|^2 = \frac{4}{4+k^2L^2}} \tag{F2008 II 3.2}$$

These satisfy the requisite properties: the probabilities sum to unity so all particles are accounted for, in the limit that the barrier vanishes no particles are reflected and all are transmitted, and in the limit that the barrier grows to infinite depth, all particles are reflected.

$$\boxed{R + T = 1} \tag{F2008 II 3.3}$$

$$\boxed{T \xrightarrow[L\to 0]{} 1 \qquad T \xrightarrow[L\to\infty]{} 0} \tag{F2008 II 3.4}$$

$$\boxed{R \xrightarrow[L\to 0]{} 0 \qquad R \xrightarrow[L\to\infty]{} 1} \tag{F2008 II 3.5}$$

This system can be tuned such that half of the particles are transmitted through the barrier by changing the energy of the particles. To do so, we set the transmission probability to $\frac{1}{2}$ and solve for the particles' corresponding de Broglie wavelength.

$$\frac{1}{2} = \frac{4}{4+k^2L^2}$$
$$4 = k^2L^2$$
$$k^2 = \frac{4}{L^2}$$

Making use of the definition of k^2 in terms of the energy,

$$\frac{2mE}{\hbar^2} = \frac{4}{L^2}$$

Then writing the energy in terms of the de Broglie wavelength:

$$\frac{2m}{\hbar^2}\frac{4\pi^2\hbar^2}{2m\lambda^2} = \frac{4}{L^2}$$

Therefore, the particles' incident momentum can be tuned and half the particles will be transmitted when

$$\boxed{\lambda = \pi L} \tag{F2008 II 3.6}$$

Problem 4

Question

Consider a one-dimensional infinite array of points labeled by an index n and separated by a fixed unit distance. At each point there is an identical very deep and narrow potential well. Let $|n\rangle$ denote an eigenstate of a *single* well, with energy E.

(a) Argue that if the wells are so narrow that the different sites can be considered uncoupled, then $|n\rangle$ is an eigenstate of the total Hamiltonian H with eigenvalue E. What is its degeneracy? Then show that the state $|k\rangle$ defined as

$$|k\rangle = \sum_{n=-\infty}^{\infty} e^{ink} |n\rangle$$

with $-\pi < k < \pi$ is an eigenstate of both H and the translation operator T defined as $T|n\rangle = |n+1\rangle$. Find the respective eigenvalues.

(b) Assume now that neighboring sites are weakly coupled so that the total Hamiltonian can now be written as

$$H = \sum_{n=-\infty}^{\infty} \left(|n\rangle E \langle n| - |n+1\rangle D \langle n| - |n\rangle D \langle n+1| \right)$$

where the coupling parameter D is real and we assume that $\langle n|n'\rangle = \delta_{n,n'}$. Show that $|n\rangle$ is no longer an eigenstate of H but that $|k\rangle$ still is. Find the eigenvalue.

Answer

The only reasonable choice for the form of the total Hamiltonian H is a superposition of the Hamiltonian of individual sites.

$$H = \sum_i H_i$$

If we then operate on a state $|n\rangle$ with the total Hamiltonian,

$$\begin{aligned} H|n\rangle &= \left(\sum_i H_i \right) |n\rangle \\ &= \sum_i H_i |n\rangle \\ H|n\rangle &= H_i \delta_{i,n} |n\rangle \end{aligned}$$

Only the n-th Hamiltonian will operate on $|n\rangle$, so the state is in fact an eigenstate of the total Hamiltonian with an eigenvalue of E.

$$\boxed{H|n\rangle = E|n\rangle} \tag{F2008 II 4.1}$$

$$\boxed{N\text{-fold degeneracy}} \tag{F2008 II 4.2}$$

Because each state n has the same eigenvalue of E, the degeneracy is equal to the number of sites. If there are N sites in the array, then that is also the degeneracy of the total system.

Similarly for the state $|k\rangle$ as defined will can be operated on by the total Hamiltonian:

$$\begin{aligned} H|k\rangle &= H\left(\sum_n e^{ink} |n\rangle \right) \\ &= \sum_n e^{ink} H |n\rangle \end{aligned}$$

Then because we've already shown that $|n\rangle$ is an eigenstate of H with eigenvalue E

$$= \sum_n e^{ink} E\,|n\rangle$$
$$H\,|k\rangle = E\left(\sum_n e^{ink}\,|n\rangle\right)$$

We find that $|k\rangle$ is also an eigenstate of the total Hamiltonian with ane eigenvalue of E as well.

$$\boxed{H\,|k\rangle = E\,|k\rangle} \tag{F2008 II 4.3}$$

Finally, we define a translation operator T for $|n\rangle$ and determine its effect on the state $|k\rangle$.

$$\begin{aligned} T\,|k\rangle &= T\left(\sum_n e^{ink}\,|n\rangle\right) \\ &= \sum_n e^{ink} T\,|n\rangle \\ &= \sum_n e^{ink}\,|n+1\rangle \end{aligned}$$

We can insert a factor of unity to extract a more useful form

$$\begin{aligned} &= \sum_n e^{i(n+1)k} e^{-ik}\,|n+1\rangle \\ &= e^{-ik} \sum_n e^{i(n+1)k}\,|n+1\rangle \end{aligned}$$

and since $n \in (-\infty, \infty)$, the distinction between n and $n+1$ is inconsequential to the definition of $|k\rangle$. Therefore

$$\boxed{T\,|k\rangle = e^{-ik}\,|k\rangle} \tag{F2008 II 4.4}$$

The translation operator has a phase eigenvalue of e^{-ik} when operating on the Bloch wave function $|k\rangle$.

If the total Hamiltonian is then modified include nearest neighbor interactions, the individual site wavefunctions $|n\rangle$ are no longer eigenstates of the total Hamiltonian as shown by explicit calculation:

$$\begin{aligned} H\,|n\rangle &= \left[\sum_{n'} \left|n'\right\rangle E \left\langle n'\right| - \left|n'+1\right\rangle D \left\langle n'\right| - \left|n'\right\rangle D \left\langle n'+1\right|\right] |n\rangle \\ &= \sum_{n'} \left|n'\right\rangle E \left\langle n' \middle| n\right\rangle - \left|n'+1\right\rangle D \left\langle n' \middle| n\right\rangle - \left|n'\right\rangle D \left\langle n'+1 \middle| n\right\rangle \\ &= \sum_{n'} \delta_{n,n'} \left(E\left|n'\right\rangle - D\left|n'+1\right\rangle\right) - D\delta_{n,n'+1}\left|n'\right\rangle \\ &= E\,|n\rangle - D\,|n+1\rangle - D\,|n-1\rangle \end{aligned}$$

There are now three wavefunctions left with two different coeffients, so the problem is not an eigenvalue problem.

$$\boxed{E\,|n\rangle - D\,|n+1\rangle - D\,|n-1\rangle = H\,|n\rangle \neq \lambda\,|n\rangle} \tag{F2008 II 4.5}$$

Operating on the Bloch wavefunction, though

$$\begin{aligned} H\,|k\rangle &= \left[\sum_{n'} \left|n'\right\rangle E \left\langle n'\right| - \left|n'+1\right\rangle D \left\langle n'\right| - \left|n'\right\rangle D \left\langle n'+1\right|\right] \left(\sum_n e^{ink}\,|n\rangle\right) \\ &= \sum_{n,n'} \left|n'\right\rangle E e^{ink} \left\langle n' \middle| n\right\rangle - \left|n'+1\right\rangle D e^{ink} \left\langle n' \middle| n\right\rangle - \left|n'\right\rangle D e^{ink} \left\langle n'+1 \middle| n\right\rangle \\ &= \sum_{n,n'} E e^{ink} \delta_{n,n'} \left|n'\right\rangle - D e^{ink} \delta_{n,n'} \left|n'+1\right\rangle - D e^{ink} \delta_{n,n'+1} \left|n'\right\rangle \end{aligned}$$

Consuming the summation over n' to select a non-zero term in the Kronecker delta leaves

$$= \sum_n Ee^{ink} |n\rangle - De^{ink} |n+1\rangle - De^{ink} |n-1\rangle$$

We can then separate each term into a summation:

$$= E \sum_n \left(e^{ink} |n\rangle\right) - D \sum_n \left(e^{ink} |n+1\rangle\right) - D \sum_n \left(e^{ink} |n-1\rangle\right)$$

Then using the same unity-factor trick as in demonstrating that $|k\rangle$ is an eigenstate of T,

$$= E \sum_n \left(e^{ink} |n\rangle\right) - De^{-ik} \sum_n \left(e^{i(n+1)k} |n+1\rangle\right) - De^{ik} \sum_n \left(e^{i(n-1)k} |n-1\rangle\right)$$

Each of these terms is the definition of $|k\rangle$, so we find that $|k\rangle$ is still an eigenstate of this new Hamiltonian with an eigenvalue of $E - 2D \cos k$.

$$\boxed{H |k\rangle = (E - 2D \cos k) |k\rangle} \qquad \text{(F2008 II 4.6)}$$

Problem 5

Question

Two monatomic ideal gases, each occupying a volume $V = 1\,\mathrm{m}^3$, are separated by a removeable insulating partition. They have different temperatures $T_1 = 350\,\mathrm{K}$ and $T_2 = 450\,\mathrm{K}$, and different pressures $p_1 = 10^3\,\mathrm{N/m^2}$ and $p_2 = 5 \cdot 10^3\,\mathrm{N/m^2}$. The partition is removed, and the gases are allowed to mix while remaining thermally isolated from the outside.

(a) What are the final temperature T_f (in K) and pressures p_f (in N/m²)?

(b) What is the net change in entropy due to mixing (in J/K)?

Answer

Starting with conservation of energy, the total internal energy after the partition is removed must be the same as the sum of the internal energies of both starting gases.

$$\begin{aligned} U &= U_1 + U_2 \\ \frac{3}{2} N k_B T_f &= \frac{3}{2} N_1 k_B T_1 + \frac{3}{2} N_2 k_B T_2 \\ T_f &= \frac{N_1 T_1 + N_2 T_2}{N_1 + N_2} \end{aligned}$$

where we made use of the fact that particle number must also be a conserved quantity so that $N = N_1 + N_2$. The original particle numbers N_1 and N_2 can be determined from the ideal gas law in the initial state:

$$\begin{aligned} p_1 V &= N_1 k_B T_1 \\ N_1 &= \frac{p_1 V}{k_B T_1} \end{aligned} \qquad\qquad \begin{aligned} p_2 V &= N_2 k_B T_2 \\ N_2 &= \frac{p_2 V}{k_B T_2} \end{aligned}$$

Plugging these into the final temperature T_f above and simplifying gives the value in terms of known quantities as

$$T_f = \frac{p_1 + p_2}{p_1 T_2 + p_2 T_1} T_1 T_2$$

which when the numbers are plugged in gives a final temperature of

$$\boxed{T_f = 429.55\,\mathrm{K}} \tag{F2008 II 5.1}$$

We can then plug this temperature into a formulation of the ideal gas law for the system after the partition has been removed to get the final pressure.

$$\begin{aligned} p_f\,(2V) &= \left(N_1 + N_2\right) k_B T_f \\ p_f &= \frac{\left(N_1 + N_2\right) k_B}{2V} \frac{N_1 T_1 + N_2 T_2}{N_1 + N_2} \\ p_f &= \frac{k_B}{2V} \left(N_1 T_1 + N_2 T_2\right) \end{aligned}$$

Again substitution for N_1 and N_2 in terms of the original pressures and temperatures gives

$$p_f = \frac{p_1 + p_2}{2}$$

which when the values are plugged in

$$\boxed{p_f = 3 \cdot 10^3\,\mathrm{N/m^2}} \tag{F2008 II 5.2}$$

From the Sackur-Tetrode euation, we can calculate the change in the entropy from the beginning state to the final one. We start by simplifying the equation to isolate constant factors:

$$
\begin{aligned}
S &= Nk_B \left\{ \frac{5}{2} + \ln\left[\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right] \right\} \\
&= \frac{5}{2} Nk_B + Nk_B \ln\left[\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right] \\
&= Nk_B \left\{ \frac{5}{2} + \frac{3}{2} \ln\left(\frac{4\pi m}{3h^2} \right) + \ln\left[\frac{V}{N} \left(\frac{U}{N} \right)^{3/2} \right] \right\}
\end{aligned}
$$

Then with the change in entropy defined as

$$
\Delta S = S - S_1 - S_2
$$

we start calculating the sum term-by-term. For the first two constant terms, the fact that $N = N_1 + N_2$ causes these terms to cancel with those in S_1 and S_2. That leaves us just with the last term.

$$
\Delta S = \left(N_1 + N_2\right) k_B \ln\left[\frac{2V}{N} \left(\frac{\frac{3}{2} N k_B T_f}{N} \right)^{3/2} \right] - N_1 k_B \ln\left[\frac{V}{N_1} \left(\frac{\frac{3}{2} N_1 k_B T_1}{N_1} \right)^{3/2} \right] - N_2 k_B \ln\left[\frac{V}{N_2} \left(\frac{\frac{3}{2} N_2 k_B T_2}{N_2} \right)^{3/2} \right]
$$

By the ideal gas law, $V/N = k_B T/p$ which simplifies the expression to

$$
\begin{aligned}
&= N_1 k_B \ln\left[\frac{2k_B T_f}{p_f} \left(\frac{3}{2} k_B T_f \right)^{3/2} \cdot \frac{p_1}{k_B T_1} \left(\frac{1}{\frac{3}{2} k_B T_1} \right)^{3/2} \right] + N_2 k_B \ln\left[\frac{2k_B T_f}{p_f} \left(\frac{3}{2} k_B T_f \right)^{3/2} \cdot \frac{p_2}{k_B T_2} \left(\frac{1}{\frac{3}{2} k_B T_2} \right)^{3/2} \right] \\
&= N_1 k_B \ln\left[\frac{p_1}{p_f} \left(\frac{T_f}{T_1} \right)^{5/2} \right] + N_2 k_B \ln\left[\frac{p_2}{p_f} \left(\frac{T_f}{T_2} \right)^{5/2} \right] + \left(N_1 + N_2\right) k_B \ln 2
\end{aligned}
$$

Using the ideal gas law again to manipulate the coefficients $Nk_B = pV/T$,

$$
\Delta S = V \left\{ \frac{p_1}{T_1} \ln\left[\frac{p_1}{p_f} \left(\frac{T_f}{T_1} \right)^{5/2} \right] + \frac{p_2}{T_2} \ln\left[\frac{p_2}{p_f} \left(\frac{T_f}{T_2} \right)^{5/2} \right] + \frac{p_f}{T_f} \ln 2 \right\}
$$

Plugging in the values for all these numbers as given or we determined, the change in entropy is

$$
\boxed{\Delta S = 7.55\,\mathrm{J/K}} \tag{F2008 II 5.3}
$$

Problem 6

Question

A rocket passes Earth at a speed $v = 0.6c$. When a clock on the rocket says that one hour has elapsed since passing, the rocket sends a light signal back to Earth.

(a) Suppose that the Earth and rocket clocks were synchronized at zero at the time passing. According to the *Earth* clocks, when was the signal sent?

(b) According to the *Earth* clocks, when did the signa arrive back on Earth?

(c) According to the *rocket* clocks, how long after the rocket passed did the signal arrive back on Earth?

Answer

Let $\beta = v/c$.

(a) Let $t'_{sent} = 1\,\text{h}$ be the time at which the rocket sent the signal according to its own clock. Because from the Earth's reference frame the rocket's clocks are running slow, the clocks on Earth must show a time greater than the rocket's by a factor of γ.

$$\begin{aligned} t_{sent} &= \gamma t'_{sent} \\ &= \frac{1\,\text{h}}{\sqrt{1-\beta^2}} \end{aligned}$$

$$\boxed{t_{sent} = \frac{5}{4}\,\text{h} = 1.25\,\text{h}} \tag{F2008 II 6.1}$$

(b) The returning light will traverse the intermediate distance at c, so $t_{ret} = x/c$. The distance from the Earth is simply the velocity times the time (in Earth's reference frame), so together,

$$\begin{aligned} t_{ret} &= \frac{v t_{sent}}{c} \\ &= \beta t_{sent} \\ t_{ret} &= \frac{3}{4}\,\text{h} = 0.75\,\text{h} \end{aligned}$$

The total round trip time is then

$$\boxed{t_{tot} = t_{sent} + t_{ret} = 2\,\text{h}} \tag{F2008 II 6.2}$$

(c) Because the rocket is flying away from the Earth, the distance behind it towards the earth appears to have been length expanded by a factor of γ compared to the distance that Earth would report. Therefore the time for the signal to be sent from the rocket to Earth appears to the rocket to be

$$\begin{aligned} t'_{ret} &= \frac{\gamma x}{c} \\ &= \gamma t_{ret} \\ t'_{ret} &= \frac{15}{16}\,\text{h} = 0.9375\,\text{h} \end{aligned}$$

Added to the 1 hour that the rocket observes as the time before it sent the signal, the signal would arrive at earth at time

$$\boxed{t'_{tot} = \frac{31}{16}\,\text{h} = 1.9375\,\text{h}}$$

Fall 2011 Part I

Problem 1

Question

An elevator operator in a skyscraper, being a very meticulous person, put a pendulum clock on the wall of the elevator to make sure that he spends exactly 8 hours a day at his work place. Over the course of his work day, he records that the time during which the elevator has acceleration a is exactly equal to the time during which it has acceleration $-a$. Does the elevator operator work, in actual time, (1) more than 8 hours, (2) exactly 8 hours, or (3) less than 8 hours? Why?

Answer

The nominal period of a pendulum is

$$T_{nom} = 2\pi\sqrt{\frac{\ell}{g}}$$

but within the elevator, the acceleration g is not going to be constant and will rather depend on the acceleration of the elevator. Therefore,

$$T_{\uparrow} = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad\qquad T_{\downarrow} = 2\pi\sqrt{\frac{\ell}{g-a}}$$

for the upward and downward cases, respectively.

Since the elevator operator observed that equal time was spent going up as was spent going down, so he must have observed N oscillations in both cases. In order to compare to the actual time, we simply compare the elevator's total time measurement with that of a stationary clock.

$$NT_{\uparrow} + NT_{\downarrow} \stackrel{?}{=} 2NT_{nom}$$

$$\begin{aligned}
2\pi N\sqrt{\frac{\ell}{g+a}} + 2\pi N\sqrt{\frac{\ell}{g-a}} &\stackrel{?}{=} 4\pi N\sqrt{\frac{\ell}{g}} \\
\sqrt{\frac{1}{g+a}} + \sqrt{\frac{1}{g-a}} &\stackrel{?}{=} 2\sqrt{\frac{1}{g}} \\
\sqrt{\frac{g}{g+a}} + \sqrt{\frac{g}{g-a}} &\stackrel{?}{=} 2
\end{aligned}$$

Use the test value $a = 5$ for comparison (with $g = 10$)

$$\boxed{2.23 > 2} \tag{F2011 I 1.1}$$

Therefore the elevator operator actually spends more than 8 hours in the elevator during his shift.

Problem 2

Question

A classical particle is subject to an attractive central force proportional to r^α, where r is the radius and α is a constant. Show by perturbation analysis what is required of α in order for the particle to have a stable circular orbit.

Answer

Construct the Lagrangian for the system in order to determine the equations of motion for the given central force (noting that we were given the *force* so we need to make an appropriate potential).

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad\qquad V = \frac{k}{\alpha+1}r^{\alpha+1}$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{\alpha+1}r^{\alpha+1}$$

Conservation of angular momentum is a consequence of the θ and $\dot{\theta}$ coordinates:

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] \\ 0 &= \frac{d}{dt}\left[mr^2\dot{\theta}\right] \end{aligned}$$

Nothing that

$$\ell = \left|\frac{\vec{r}\times\vec{p}}{m}\right| = r^2\dot{\theta}$$

we can say that

$$\dot{\theta} = \frac{\ell}{r^2}$$

Then returning to the r and $\dot{r}$ coordinates in the Lagrangian,

$$\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - kr^\alpha \qquad\qquad \begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{r}} &= m\dot{r} \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right] &= m\ddot{r} \end{aligned}$$

Putting the differential equation together and substituting for the angular momentum per unit mass gives

$$m\ddot{r} = \frac{m\ell^2}{r^3} - kr^\alpha \tag{F2011 I 2.1}$$

In the case that the orbit is circular, r must be a constant, so let $r = a$ and note that $\ddot{r} = 0$ necessarily.

$$\frac{m\ell^2}{a^3} = ka^\alpha$$

Returning to the differential equation, let the actual distance r be a perturbation from a circular orbit, and Taylor expand in x where $x = r - a$.

$$\begin{aligned}
m\ddot{x} &= \frac{m\ell^2}{a^3}\left(1 + \frac{x}{a}\right)^{-3} - ka^\alpha\left(1 + \frac{x}{a}\right)^\alpha \\
m\ddot{x} &\approx \frac{m\ell^2}{a^3}\left(1 - 3\frac{x}{a} + \ldots\right) - ka^\alpha\left(1 + \alpha\frac{x}{a} + \ldots\right) \\
m\ddot{x} &\approx ka^\alpha\left(1 - 3\frac{x}{a}\right) - ka^\alpha\left(1 + \alpha\frac{x}{a}\right) \\
m\ddot{x} &\approx -3ka^\alpha\frac{x}{a} - \alpha ka^\alpha\frac{x}{a} \\
m\ddot{x} &\approx -ka^{\alpha-1}(3 + \alpha)\,x
\end{aligned}$$

To form a stable orbit, the coefficient on x must be negative, giving a simple harmonic solution. Therefore $3 + \alpha > 0$ to keep the coefficient negative and

$$\boxed{a > -3} \tag{F2011 I 2.2}$$

Problem 3

Question

A neutral conductor A with a spherical outer surface of radius R contains three cavities B, C, and D, but is solid otherwise. B and C are spherical, and D is hemispherical. Without touching A, positive charges q_B and q_C are introduced at the centers of B and C, respectively.

1. Give the amount and the distribution of the induced charges on the surfaces of A, B, C, and D.
2. Now another positive charge q_E is introduced at a distance $r > R$ from the center of A. Describe qualitatively the distribution of induced charges on the surfaces of A, B, C, and D.
3. Give the amount of the induced charges on the surfaces of A, B, C, and D for the situation in (2).

Answer (1)

An ideal conductor will not support an electric field inside the solid, so each of cavities B and C will have a surface charge to cancel the electric fields emminating from q_B and q_C respectively.

- Cavity B will have a uniform surface charge density of $-q_B/4\pi r_B^2$, where r_B is the radius of cavity B, with total induced charge $-q_B$ (because of symmetry and use of a Gaussian surface).
- Cavity C will have a uniform surface charge density of $-q_C/4\pi r_C^2$, where r_C is the radius of cavity B, with total induced charge $-q_C$ (because of symmetry and use of a Gaussian surface).

Cavity D will not have a surface charge since a Guassian surface coincident with its boundary contains no charge.

The surface A will have total charge $q_B + q_C$ with uniform surface charge density of $\left(q_B + q_C\right)/4\pi R^2$ in accordance with the symmetry of a Gaussian surface containing the sphere as well as properties of an ideal conductor.

Answer (2)

The surfaces B, C, and D will remain unaffected since the surrounding conductor shields the cavities from electric fields produced by charge q_E. The distribution on surface A will shift so that the negative charge concentration is greatest on the side nearest to q_E with an increasingly positive distribution towards the opposite side.

Answer (3)

The surface of A will still contain the same total charge $q_B + q_C$ since only a redistribution of induced charges occurred along the surface. Similarly, because surface B, C, and D are shielded from the electric field of q_E by conductor A, the total charges along their surfaces remains unchanged as well.

Problem 4

Question

The dielectric strength of air at standard temperature and pressure is $3 \cdot 10^6$ V/m. What is the maximum intensity in units of W/m^2 for a monocromatic laser that can be used in the laboratory?

Answer

Failure of a dielectric occurs when the energy density in the dielectric is great enough to overcome the ionization energy of the constituent atoms. This suggests that an electric field of greater than $3 \cdot 10^6$ V/m would cause this ionization to occur.

Starting here, We can calculate the energy density of the electric field at any point in space by

$$U_{em} = \frac{\varepsilon_0}{2} E^2$$

(where we've used the vacuum energy density since air differs very little from the vacuum permittivity).

Then the power transmitted by the laser is $P = cU_{em}$, so plugging in the numbers,

$$\begin{aligned} P &= \frac{1}{2} \left(8.854 \cdot 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N{\cdot}m}^2} \right) \left(3 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}} \right) \left(3 \cdot 10^6 \frac{\mathrm{V}}{\mathrm{m}} \right)^2 \\ P &= 1.19 \cdot 10^{10} \frac{\mathrm{W}}{\mathrm{m}^2} \end{aligned}$$

$$\boxed{\text{The maximum power of a laser usable in the lab is } 1.19 \cdot 10^{10}\ \mathrm{W/m^2}.}$$

Problem 5

Question

What is the minimum energy of the projectile proton required to induce the reaction $p + p \to p + p + p + \bar{p}$ if the target proton is at rest?

Answer

Energy and momentum must be conserved. At the minimum allowed energy, the resultant 4 proton/anti-protons will be colinear with no relative momentum with respect to one another, so the momenum equation in the lab frame is simply

$$p_i = 4p_f \tag{F2011 I 5.1}$$

Similarly, the resultant (anti-)protons are indistinguishable, so they will all have equivalent energy E_f. The initial protons have different energies since one is at rest in the lab frame while the other is moving, leading to the energy equation

$$\sqrt{p_i^2c^2 + m_p^2c^4} + m_pc^2 = 4\sqrt{p_f^2c^2 + m_p^2c^4}$$

Substituting the momentum relation into the equation, squaring, and simplifying,

$$\begin{aligned}
\sqrt{16p_f^2c^2 + m_p^2c^4} + m_pc^2 &= 4\sqrt{p_f^2c^2 + m_p^2c^4} \\
16p_f^2c^2 + m_p^2c^4 + m_p^2c^4 + 2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 16p_f^2c^2 + 16m_p^2c^4 \\
2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 14m_p^2c^4 \\
16p_f^2c^2 + m_p^2c^4 &= 49m_p^2c^4 \\
p_f^2 &= 3m_p^2c^4
\end{aligned}$$

Therefore,

$$p_i^2 = 48m_p^2c^4$$

and

$$E_1 = \sqrt{49m_p^2c^4}$$

$$\boxed{E_1 \approx 6.567\,\mathrm{GeV/c^2}} \tag{F2011 I 5.2}$$

Problem 8

Question

Assume that the atmosphere near the earth's surface is in approximate hydrostatic equilibrium, where any movement of air parcels is gentle and adiabatic. Find an expression for the pressure P of the atmosphere as a function of the height z.

Answer

Note that the pressure at a given point is due to the mass of air above the given point. Then by moving an infinitesimal distance vertically, the total mass is changed by the density of the air (which is affected by the gravitational force). This leads to the differential equation

$$\frac{dP}{dz} = \rho g$$

Then using the ideal gas equation

$$PV = Nk_B T$$

multiply and divide by the average molecular mass m of the air (in kg) which combined with the number of molecules N gives the total mass

$$PV = (Nm)\frac{1}{m}k_B T$$

and then divide by the volume to get the ideal gas equation in terms of the mass density

$$P = \frac{Nm}{V}\frac{1}{m}k_B T$$
$$P = \rho\frac{k_B T}{m}$$
$$\rho = \frac{Pm}{k_B T}$$

Finally, substitute this into the differential equation above and solve to get the atmospheric scale height equation.

$$\frac{dP}{dz} = \frac{Pm}{k_B T}g$$
$$\frac{dP}{P} = \frac{mg}{k_B T}dz$$

$$\boxed{P(z) = P_0 e^{z/\xi} \qquad \text{where } \xi = \frac{k_B T}{mg}} \qquad \text{(F2011 I 8.1)}$$

Fall 2011 Part II

Problem 1

Question

Mass m_1 moves freely along a fixed, long, horizontal rod. The position of m_1 on the rod is x. A massless string of length ℓ is attached to m_1 at the end and to mass m_2 at the other. Mass m_2 executes pendulum motion in the vertical plane containing the rod.

1. Find the Lagrangian of the system.
2. Derive the equations of motion and the corresponding conservation laws.
3. Assume that $x(0) = x_0$, $\dot{x}(0) = 0$, $\varphi(0) = \varphi_0$ $\left(|\varphi_0| \ll 1\right)$, and $\dot{\varphi}(0) = 0$. Find $x(t)$ and $\varphi(t)$ for $t > 0$.

Answer (1)

For the sliding support mass m_1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{x}^2 \\ V_1 &= 0 \end{aligned}$$

For the pendulum mass m_2:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} m_2 \left(\dot{x} + \dot{x}_2\right)^2 \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Then using $x_2 = \ell \sin\varphi$ and $y_2 = -\ell\cos\varphi$,

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + \dot{x}^2 + 2\ell\dot{\varphi}\dot{x}\cos\varphi\right) \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Putting the Lagrangian together equals the first line. Applying the small angle approximation gives the second line where the kinetic energy term involving $\cos\varphi$ can be simply expanded as $\cos\varphi \approx 1$, but the potential energy term must be expanded to second order so that $\cos\varphi \approx 1 - \frac{1}{2}\varphi^2$.

$$\boxed{\mathcal{L} = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + 2\ell\dot{\varphi}\dot{x}\cos\varphi\right) + m_2 g\ell\cos\varphi} \tag{F2011 II 1.1}$$

$$\boxed{\mathcal{L} \approx \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + 2\ell\dot{\varphi}\dot{x}\right) + m_2 g\ell - \frac{1}{2} m_2 g\ell\varphi^2} \tag{F2011 II 1.2}$$

Answer (2)

Constructing the Euler-Lagrange equations for x and $\dot{x}$:

$$\frac{\partial\mathcal{L}}{\partial x} = 0 \qquad \begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{x}} &= \left(m_1 + m_2\right)\dot{x} + m_2\ell\dot{\varphi} \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] &= \left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\varphi} \end{aligned}$$

$$\left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\varphi} = 0 \tag{F2011 II 1.3}$$

and for φ and $\dot{\varphi}$:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m_2 g \ell \varphi \qquad\qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m_2 \ell^2 \dot{\varphi} + m_2 \ell \dot{x}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right] = m_2 \ell^2 \ddot{\varphi} + m_2 \ell \ddot{x}$$

$$-m_2 g \ell \varphi - m_2 \ell^2 \ddot{\varphi} + m_2 \ell \ddot{x} = 0 \tag{F2011 II 1.4}$$

The equations of motion are:

$$\boxed{\ddot{x} + \frac{m_2}{m_1 + m_2} \ell \ddot{\varphi} = 0} \tag{F2011 II 1.5}$$

$$\boxed{\ddot{\varphi} + \frac{1}{\ell}\ddot{x} + \frac{g}{\ell}\varphi = 0} \tag{F2011 II 1.6}$$

Answer (3)

Solve for $\ddot{x}$ and substitute into the other differential equation

$$\begin{aligned} \ddot{\varphi} - \frac{1}{\ell}\frac{m_2}{m_1 + m_2}\ell\ddot{\varphi} + \frac{g}{\ell}\varphi &= 0 \\ \frac{m_1}{m_1 + m_2}\ddot{\varphi} + \frac{g}{\ell}\varphi &= 0 \\ \ddot{\varphi} + \frac{g}{\ell}\frac{m_1 + m_2}{m_1}\varphi &= 0 \end{aligned} \tag{F2011 II 1.7}$$

This is just the differential equation for a simple harmonic oscillator, so considering the given boundary conditions,

$$\boxed{\varphi(t) = \varphi_0 \cos(\omega t) \qquad \text{where } \omega^2 = \frac{g}{\ell}\frac{m_1 + m_2}{m_1}} \tag{F2011 II 1.8}$$

Then differentiating $\varphi(t)$ twice and substituting into the first equation,

$$\ddot{x} = \ell \varphi_0 \omega^2 \frac{m_2}{m_1 + m_2} \cos(\omega t)$$

Then integrating twice and applying the boundary conditions,

$$\boxed{x(t) = x_0 - \frac{g}{\omega^2}\frac{m_2}{m_1}\cos(\omega t)} \tag{F2011 II 1.9}$$

Spring 2012 Part I

Problem 1

Question

For a many particle system of weekly interacting particles, will quantum effects be more important for (a) high densities or low densities and (b) high temperatures or low temperatures for a system. Explain your answers in terms of the de Broglie wavelength λ defined as $\lambda^2 \equiv h^2/\left(3mk_B T\right)$ where m is the mass of the particles and k_b Boltzmann's constant.

Answer

(a) High density — The de Broglie wavelength gives a "size" of the particle, and in the high density limit, the wavefunctions overlap significantly so quantum effects and interactions are critical to the behavior of the system.

(b) Low temperature — Since $\lambda^2 \propto T^{-1}$, as $T \to 0$, λ increases so that again the wavefunctions overlap and quantum effects are significant.

Problem 2

Question

The ground state energy of Helium is −79 eV. What is its ionization energy, which is the energy required to remove just one electron?

Answer

Using the Hydrogen solution with modifications for single-electron atoms of higher Z, we know that the ground state energy of singly ionized Helium is

$$E^1_{He} = 2^2\,(-13.6\,\text{eV}) = -54.4\,\text{eV}$$

Therefore, the difference between the singly-ionized and neutral ground state energies gives the first ionization energy of the Helium atom.

$$\boxed{E_i = -24.6\,\text{eV}} \tag{F2012 I 2.1}$$

Problem 3

Question

It is known that the force per unit area (F/A) between two neutral conducting plates due to polarization fluctuations of the vacuum (namely, the Casimir force) is a function of h (Planck's constant), c (speed of light), and z (distance between the plates) only. Using only dimensional analysis, obtain F/A as a function of h, c, and z.

Answer

The units of F/A are

$$\frac{F}{A} = \left[\frac{\text{kg}}{\text{m}\cdot\text{s}^2}\right]$$

The kg suggests a factor proportional to h, making the equation

$$\frac{F}{A} \sim \left[\frac{1}{\text{m}^3\cdot\text{s}}\right] h$$

Accounting for the factor of seconds requires a c:

$$\frac{F}{A} \sim \left[\frac{1}{\text{m}^4}\right] hc$$

Finally, account for all the factors of distance:

$$\frac{F}{A} \sim \frac{hc}{z^4}$$

Therefore,

$$\boxed{\frac{F}{A} \sim \frac{hc}{z^4}} \tag{F2012 I 3.1}$$

Problem 4

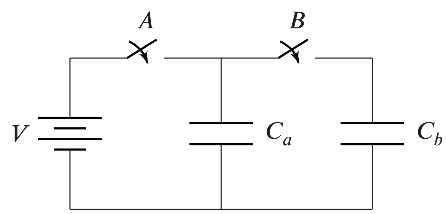
Question

In the circuit diagram opposite, initially the two identical capacitors with capacitance C are uncharged. The connections between the components are all made with short copper wires. The battery is an ideal EMF and supplies a voltage V.

(a) At first Switch A is closed and Switch B is kept open. What is the final sotred energy on capacitor C_a?

(b) Switch A is opened and afterwards Switch B is closed. What is the final energy stored in both capacitors?

(c) Provide a physical explanation for any difference between the results of parts (a) and (b), if there is one.



Answer

(a) Initially, the right side of the circuit with C_b can be ignored, so the total energy is simply the energy stored within C_a.

$$\boxed{E = \frac{1}{2}CV^2} \tag{F2012 I 4.1}$$

(b) The system is now effectively just the two capacitors on the right. Because the voltage difference is supported across both capacitors, the system can be modeled as an effective capacitor in parallel

$$C_{eff} = 2C$$

The total charge stored by the capacitors must remain the same when switching from Switch A being closed to Switch B. Initially,

$$Q = CV_i$$

and afterwards it is

$$Q = C_{eff}V = 2CV_f$$

so the final voltage across the capacitors is

$$V_f = \frac{1}{2}V_i$$

This means the total energy is

$$E = \frac{1}{2}C_{eff}{V_f}^2$$

$$\boxed{E = \frac{1}{4}CV^2} \tag{F2012 I 4.2}$$

(c) The energy is dissipated (heat, fields, etc).

Problem 5

Question

A planet of mass m moves around the sun, mass M, in an elliptical orbit with minimum and maximum distances of r_1 and r_2, respectively. Find the angular momentum of the planet relative to the center of the sun in terms of these quantities and the gravitational constant G.

Answer

We solve the problem using conservation of energy since we know that stable elliptical orbits have constant energy. The generic equation is

$$E = \frac{L^2}{2I} - \frac{GMm}{r}$$

where L is the angular momentum and I the moment of inertia. Substituting for the values at both r_1 and r_2 and equating,

$$\frac{L^2}{2m{r_1}^2} - \frac{GMm}{r_1} = \frac{L^2}{2m{r_2}^2} - \frac{GMm}{r_2}$$
$$\frac{L^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right) = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

which leads to the solution

$$\boxed{L = \sqrt{\frac{2GMm^2 r_1 r_2}{r_1 + r_2}}} \tag{F2012 I 5.1}$$

Problem 6

Question

A particle moves in a circular orbit under the influence of a central force that varies as the n-th power of the distance. Show that this motion is unstable if $n < -3$. (Hint: Consider the centrifugal potential.)

Answer

See solution for Fall 2011 Part I, Problem 2 with the condition inverted so that *in*stability is $n < -3$ rather than stability requiring $n > -3$.

Problem 7

Question

A classical, ideal, monatomic gas of N particles is reversibly compressed *isentropically*, i.e. with the entropy kept constant, from an initial temperature T_0 and pressure P to a pressure $2P$. Find (a) the work done on the system, and (b) the net change in entropy of the system and its surroundings.

(a) An isentropic process is the same as an adiabatic process since no heat can be exchanged ($T\,\mathrm{d}S = Q = 0$), so we begin with the relation that PV^γ is a constant. Combining this with the ideal gas law, we can determine that

$$P^{1-\gamma}T^\gamma = \text{const}$$

where $\gamma = C_p/C_v$ is the ratio of heat capacities with $C_p = \frac{5}{2}Nk_B$ and $C_v = \frac{3}{2}Nk_B$ for a monatomic ideal gas. Using this, we solve for the final temperature of the system after compressions as

$$T_f = 2^{2/5}T_0 \approx 1.32T_0$$

Combining both of

$$\begin{aligned} \Delta U &= C_v \Delta T \\ \Delta U &= Q + W \end{aligned}$$

where $Q = 0$, we get that

$$\boxed{W = \frac{3}{2}Nk_B T_0 \left(2^{2/5} - 1\right)} \tag{F2012 I 7.1}$$

(b) Because the compression is done reversibly, by definition, $\Delta S = 0$.

Problem 8

Question

For an idea Fermi gas of N neutral spin-$\frac{1}{2}$ particles in a volume V at $T = 0$, calculate the following:

(a) The chemical potential

(b) The average energy per particle

(c) The pressure

Answer

(a) At $T = 0$, the particles are all in the lowest state allowed by Fermi-Dirac statistics, so the chemical potential, defined by the energy required to add another particle to the system, is equal to the Fermi energy. For a particle contained within a box V, the energy per particle is

$$\varepsilon_n = \frac{\pi^2\hbar^2}{2mV^{2/3}}n^2$$

Given a Fermi energy ε_F, the maximum occupied state is

$$n_F = \sqrt{\frac{2mV^{2/3}}{\pi^2\hbar^2}}\sqrt{\varepsilon_F}$$

Equally we know that all N particles must exist within the eighth-sphere of n space, where the extra factor of 2 is because there are two spin states per n:

$$\begin{aligned} N &= 2 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi {n_F}^3 \\ N &= \frac{1}{3}\pi\left(\frac{2m}{\pi^2\hbar^2}\right)^{3/2} V{\varepsilon_F}^{3/2} \\ \varepsilon_F &= \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \end{aligned}$$

Therefore $\mu = \varepsilon_F$,

$$\boxed{\mu = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}} \tag{F2012 I 8.1}$$

(b) To get the total energy, we can imagine filling all N particles one at a time, so that at each step, there are N' total particles:

$$\begin{aligned} U &= \int_0^N \varepsilon_F \, \mathrm{d}N' \\ U &= \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \int_0^N N^{2/3} \, \mathrm{d}N' \\ U &= \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \frac{3}{5} N^{5/3} \, \mathrm{d}N' \end{aligned}$$

Therefore, the average energy per particle is U/N or

$$\boxed{\langle\varepsilon\rangle = \frac{3}{5}\varepsilon_F} \tag{F2012 I 8.2}$$

(c) From the thermodynamic relation

$$dU = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N$$

we can read off the derivative that defines the pressure P as

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

Doing so, we get that

$$\frac{\partial U}{\partial V} = \frac{3}{5}N \cdot \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \left(-\frac{2}{3V}\right)$$

making the pressure

$$\boxed{P = \frac{2}{5}\frac{N}{V}\varepsilon_F} \tag{F2012 I 8.3}$$

Problem 10

Question

A piece of p-doped silicon has a carrier density $n = 10^{15}\,\mathrm{cm}^{-3}$ and dimensions of $\Delta x = 10\,\mathrm{mm}$, $\Delta y = 2\,\mathrm{mm}$, and $\Delta z = 1\,\mathrm{mm}$. A magnetic field of $B_z = 1\,\mathrm{T}$ is applied in the z-direction and a current $I_x = 1\,\mathrm{A}$ flows in the x-direction, and the voltage V_y is measured.

(a) Express the current density j_x in terms of the carrier density n and the carrier velocity v_x.

(b) Write down the equilibrium force condition that determins V_y.

(c) Find V_y in volts.

Answer

(a) The current passing through each thin cross-sectional slice of the conductor is dependent on the charge of a carrier, carrier density, and velocity of the flow.

$$I_x = en\Delta y\Delta z v_x$$

The current density is just the current passing through each point, so

$$j_x = \frac{I_x}{\Delta y\Delta z}$$

$$\boxed{j_x = nev_x} \tag{F2012 I 10.1}$$

(b) The positive carriers drift to the edge of the conductor due to the magnetic field and the holes accumulate on the opposite edge. An electric field is created between the charge separation, so an equilibrium is set up between the electric field trying to bring the opposite charges together and the magnetic drift separating them.

$$0 = e\vec{E} + \vec{v} \times \vec{B}$$

By the right-hand rule, the positive charges accumulate along $y = 0$, so $\vec{E} = E\hat{y}$. Similarly, $\vec{v} \times \vec{B} = -v_x B_z \hat{y}$:

$$0 = eE\hat{y} - ev_x B_z \hat{y}$$

Written in terms of the potential $V_y = E\Delta y$, the equilibrium condition becomes

$$\boxed{V_y = v_x B_z \Delta y} \tag{F2012 I 10.2}$$

(c) Substituting in for given quantities

$$V_y = \frac{I_x B_z}{ne\Delta z}$$
$$V_y = \frac{(1\,\mathrm{A})\,(1\,\mathrm{T})}{\left(10^{15}\,\mathrm{cm}^{-3}\right)\left(1.612 \cdot 10^{-19}\,\mathrm{C}\right)(1\,\mathrm{mm})}$$

$$\boxed{V_y = 6.24\,\mathrm{V}} \tag{F2012 I 10.3}$$

Spring 2012 Part II

Problem 1

Question

An electron in a hydrogen atom occupies a state:

$$|\psi\rangle = \sqrt{\frac{1}{3}}\,|3,1,0,+\rangle + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle$$

where the properly normalized states are specified by the quantum numbers $|n,\ell,m,\pm\rangle$ and the $\pm$ specifies whether the spin is up or down.

(a) What is the expectation value of the energy in terms of the ground state energy?

(b) If you meausred the expectation values of the orbital momentum squared $\left\langle L^2\right\rangle$, the square of the spin $\left\langle S^2\right\rangle$, and their z-components $\langle L_z\rangle$ and $\langle S_z\rangle$, what would be the result?

(c) Show that if you measure the position of the electron, the probability density for finding it an an angle specified by θ and ϕ integrated over all values of r is independent of θ and ϕ. Note, for this part you will need $Y_1^0 = \sqrt{3/4\pi}\cos\theta$ and $Y_1^1 = -\sqrt{3/8\pi}\sin\theta\exp(i\phi)$. You do *not*, however, need to know the radial functions, only that they are properly normalized and orthogonal to each other.

(d) List all additional possible states that are degenerate with the first state in the linear combination above. Note: this part can be done even if you have not answered the previous parts.

Assume now that the state $|\psi\rangle$, given above, is the initial state of an electron in a hydrogen atom.

(e) Write down the elctron's state as a function of time for all $t > 0$.

(f) go through the results you obtained in parts (a) through (c) and determine which of them are time independent.

Answer

(a) Calculate the energy by sandwhiching the Hamiltonian between the wavefunction:

$$\begin{aligned}
\langle E\rangle &= \langle\psi|H|\psi\rangle \\
&= \left(\sqrt{\frac{1}{3}}\,\langle 3,1,0,+| + \sqrt{\frac{2}{3}}\,\langle 2,1,1,-|\right) H \left(\sqrt{\frac{1}{3}}\,|3,1,0,+\rangle + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle\right) \\
&= \frac{1}{3}\,\langle 3,1,0,+|H|3,1,0,+\rangle + \frac{\sqrt{2}}{3}\,\langle 2,1,1,-|H|3,1,0,+\rangle \\
&\quad + \frac{\sqrt{2}}{3}\,\langle 3,1,0,+|H|2,1,1,-\rangle + \frac{2}{3}\,\langle 2,1,1,-|H|2,1,1,-\rangle
\end{aligned}$$

For every term, the wavefunctions are eigenstates of the Hamiltonian, so we extract the appropriate energy term from every bra-ket sandwhich. Then the middle two terms integrate to zero since states with different n are orthogonal while the first and last terms integrate to unity since they are properly normalized.

$$\langle E\rangle = \frac{1}{3}E_3 + 0 + 0 + \frac{2}{3}E_2$$

Each energy is related to the ground state energy by $E_n = E_0/n^2$, so

$$= \frac{1}{3}\frac{E_0}{9} + \frac{2}{3}\frac{E_0}{4}$$

$$\boxed{\langle E\rangle = \frac{11}{54}E_0 \approx -2.77\,\mathrm{eV}} \qquad \text{(F2012 II 1.1)}$$

(b) For each of the other expectation values, the process is very similar with an appropriate change for eigenvalues; specifically,

$$L^2 |n,\ell,m,\pm\rangle = \ell(\ell+1)\hbar^2 |n,\ell,m,\pm\rangle$$
$$S^2 |n,\ell,m,\pm\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 |n,\ell,m,\pm\rangle$$
$$L_z |n,\ell,m,\pm\rangle = \ell\hbar |n,\ell,m,\pm\rangle$$
$$S_z |n,\ell,m,\pm\rangle = \pm\frac{1}{2}\hbar |n,\ell,m,\pm\rangle$$

The same restrictions that the middle terms integrate to zero because of orthogonality and the first and last terms integrate to unity still applies, so we can almost immediately conclude that

$$\boxed{\left\langle L^2\right\rangle = 2\hbar^2} \tag{F2012 II 1.2}$$

$$\boxed{\left\langle S^2\right\rangle = \frac{3\hbar^2}{4}} \tag{F2012 II 1.3}$$

$$\boxed{\langle L_z\rangle = \frac{2\hbar}{3}} \tag{F2012 II 1.4}$$

$$\boxed{\langle S_z\rangle = -\frac{\hbar}{6}} \tag{F2012 II 1.5}$$

(c) In the $|r,\theta,\phi\rangle$ basis,

$$|3,1,0\rangle = R_{3,1}(r)\,Y_1^0(\theta,\phi) = R_{31}(r)\sqrt{\frac{3}{4\pi}}\cos\theta$$
$$|2,1,1\rangle = R_{2,1}(r)\,Y_1^1(\theta,\phi) = -R_{21}(r)\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta$$

This means that the probability density is

$$\begin{aligned}\langle\psi|\psi\rangle &= \frac{1}{3}\langle 3,1,0|3,1,0\rangle + \frac{\sqrt{2}}{3}\langle 2,1,1|3,1,0\rangle + \frac{\sqrt{2}}{3}\langle 3,1,0|2,1,1\rangle + \frac{2}{3}\langle 2,1,1|2,1,1\rangle \\ &= \frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}(r)\,R_{31}(r) + \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)\end{aligned}$$

Integrating over r,

$$\begin{aligned}\int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \int_0^\infty &\frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}(r)\,R_{31}(r) \\ &+ \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)\,\mathrm{d}r\end{aligned}$$

Integrating over all r, we know that $R_{n\ell}R_{n'\ell'}$ are orthonormal, so again the first and last terms' R integrates to unity and the middle term integrates to zero.

$$= \frac{1}{4\pi}\left(\cos^2\theta + \sin^2\theta\right)$$

Therefore we find that the probability density is constant in θ and ϕ when integrated over all r.

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \frac{1}{4\pi}} \tag{F2012 II 1.6}$$

(d) The states degenerate with the first term in ψ are all combinations of allowed ℓ, m, and $\pm$: n must remain at $n = 3$ since it is the n quantum number which determines the energy of the state. The angular momentum number ℓ has to be in the range $[0, n-1]$, so there are at least 3 cases.

$$|3,0,m,\pm\rangle$$
$$|3,1,m,\pm\rangle$$
$$|3,2,m,\pm\rangle$$

Then for each ℓ, the projection m can take a range of values $m \in [-\ell, \ell]$ so using $\{\ldots,-1,0,-1,\ldots\}$ to denote a set of options,

$$\begin{aligned} |3,0,m,\pm\rangle &\to |3,0,\{0\},\pm\rangle && \text{2 states} \\ |3,1,m,\pm\rangle &\to |3,1,\{-1,0,1\},\pm\rangle && \text{6 states} \\ |3,2,m,\pm\rangle &\to |3,2,\{-2,-1,0,1,2\},\pm\rangle && \text{10 states} \end{aligned}$$

$$\boxed{\text{In total, there are 18 degenerate states}}$$

(e) To get the time evolution, we simply use the fact that for each basis eigenstate, we can add the time evolution component

$$\exp\left(-\frac{iE_n t}{\hbar}\right)$$

to get (in terms of the ground state energy E_0)

$$\boxed{|\psi(t)\rangle = \sqrt{\frac{1}{3}}\,|3,1,0,+\rangle\, e^{-iE_0t/9\hbar} + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle\, e^{-iE_0t/4\hbar}} \quad \text{(F2012 II 1.7)}$$

(f) From Ehrenfest's Theorem, we can quickly find the answers to most of the question without worrying about the wavefunction. Ehrenfest's Theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle E\rangle = -\frac{i}{\hbar}\langle[\Omega, H]\rangle + \left\langle\frac{\partial\Omega}{\partial t}\right\rangle$$

None of the operators L^2, S^2, L_z, and S_z are explicit in time, so the second term on the right can be dropped. Then because each of these operators commute with the Hamiltonian, the first term on the right is also dropped. Therefore, the expectation values are constant in time, so

$$\boxed{\left\langle L^2\right\rangle \quad \text{Time independent}} \quad \text{(F2012 II 1.8)}$$

$$\boxed{\left\langle S^2\right\rangle \quad \text{Time independent}} \quad \text{(F2012 II 1.9)}$$

$$\boxed{\langle L_z\rangle \quad \text{Time independent}} \quad \text{(F2012 II 1.10)}$$

$$\boxed{\langle S_z\rangle \quad \text{Time independent}} \quad \text{(F2012 II 1.11)}$$

For the probabilty density, we return to the integral in part (c) and insert the appropriate exponential terms. The first and last terms' exponentials cancel each other out, leaving

$$\begin{aligned} \int_0^\infty \langle\psi|\psi\rangle\,\mathrm{d}r = \int_0^\infty & \frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}(r)R_{31}(r) \\ & \cdot\left[\exp\left(\frac{i\left(E_2-E_3\right)t}{\hbar}\right) + \exp\left(-\frac{i\left(E_2-E_3\right)t}{\hbar}\right)\right] \\ & + \frac{1}{4\pi}\sin^2\theta R_{21}^2(r)\,\mathrm{d}r \end{aligned}$$

The integral is unaffected by the new time factors, though, so integrating over r, the middle term still goes to zero and we're left with the same result previously of $1/4\pi$, therefore

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle \, \mathrm{d}r \quad \text{Time independent}} \tag{F2012 II 1.12}$$

Problem 5

Question

Coinsider N non-interacting, stationary particles, each with magnetic moment $\vec{\mu}$ at temperature T in a uniform external magnetic field $\vec{B}$. Their energy is $-\vec{\mu} \cdot \vec{B}$. Calculate the partition function Z, the internal energy, and magnetization for two distinct cases (a and b below):

(a) The mgnetic moment of each particle can be oriented only parallel or anti-parallel to the magnetic field.

(b) The magnetic moment of each particle can rotate freely.

(c) Show that, in both cases, the total magnetization $\vec{M}$ can be written as a derivative of the parition function.

(d) In each case, calculate the fluctuations of magnetization $\left\langle (\Delta\vec{u})^2 \right\rangle$.

Question

(a) Begin by constructing the partition function for a single particle. Since there are only two energy states, the sum is simply over the two Boltzmann factors:

$$Z_1 = e^{\mu B/kT} + e^{-\mu B/kT}$$

This can be simplified using trigonometric identities to

$$Z_1 = 2\cosh\left(\frac{\mu B}{kT}\right)$$

For fixed site particles, the partition function for N particles is simply $Z = Z^N$, so

$$\boxed{Z = 2^N \cosh^N\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.1}$$

The total energy can be calculated either by finding the expectation energy per particle $\langle \varepsilon \rangle$ and multiplying by N using the Boltzmann factors directly, or by using the thermodynamic identity

$$U = kT^2 \frac{\partial \ln Z}{\partial T}$$

Doing so,

$$U = kT^2 \frac{N}{2\cosh\left(\frac{\mu B}{kT}\right)} \cdot 2\sinh\left(\frac{\mu B}{kT}\right) \cdot \left(-\frac{\mu B}{kT^2}\right)$$

$$\boxed{U = -N\mu B \tanh\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.2}$$

To find the Magnetization, we use the Boltzmann factors directly since we don't know a thermodynamic relation. Let

$$\begin{aligned}
\langle m \rangle &= \sum_\mu \mu \frac{e^{-\varepsilon_\mu/kT}}{Z_1} \\
&= \frac{1}{Z_1}\left(-\mu e^{\mu B/kT} + \mu e^{-\mu B/kT}\right) \\
&= -\mu \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\cosh\left(\frac{\mu B}{kT}\right)}
\end{aligned}$$

So knowing that $\langle M \rangle = N \langle m \rangle$,

$$\boxed{\langle M \rangle = -N\mu \tanh\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.3}$$

(b) In the continuous case, the sum needs to be changed into an integral, remembering to keep Z unitless. This requires dividing by the volume of the energy state, which in this case is μB. (Justification: think of the energy vector $\vec{\mu} \cdot \vec{B} = \mu B \cos\theta$ on the unit circle of length μB. From geometry, the unitless $\mathrm{d}\theta$ is related to $\mathrm{d}\varepsilon$ by the factor μB.)

$$Z_1 = \int_{-\mu B}^{\mu B} e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B}$$

Letting $u = -\frac{\varepsilon}{kT}$,

$$\begin{aligned} Z_1 &= -\frac{kT}{\mu B} \int_{\mu B/kT}^{-\mu B/kT} e^u \,\mathrm{d}u \\ &= 2\frac{kT}{\mu B} \sinh\left(\frac{\mu B}{kT}\right) \end{aligned}$$

Therefore the partition function for all N particles is

$$\boxed{Z = \left(\frac{2kT}{\mu B}\right)^N \sinh^N\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.4}$$

The total energy is found in the same way as the previous case, giving

$$\boxed{U = NkT - N\mu B \coth\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.5}$$

For the magnetization, we also calculate the expectation value from integrating the probability distribution, again making sure to keep the correct units. This time we work with the relevant projection of the magnetic moment $m = \mu \cos\theta$ so that when combined with the energy $\varepsilon = -\mu B \cos\theta$, the magnetization per particle in each state is $m = -\varepsilon/B$.

$$\begin{aligned} \langle m \rangle &= \int_{-\mu B}^{\mu B} -\frac{\varepsilon}{B} \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B} \\ &= \frac{1}{\mu Z_1} \left(\frac{kT}{B}\right)^2 \int_{-\mu B/kT}^{\mu B/kT} e^u \,\mathrm{d}u \\ &= \frac{kT}{B} \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\sinh\left(\frac{\mu B}{kT}\right)} \\ &= \frac{kT}{B} \end{aligned}$$

The total magnetization $\langle M \rangle = N \langle m \rangle$ is

$$\boxed{\langle M \rangle = \frac{NkT}{B}} \tag{F2012 II 5.6}$$

Note that this is to be expected for the continuous case limit which corresponds to the classical limit. We'd expect the total energy to be related to the magnetization by $U = MB$. Rearranging the terms,

$$\begin{aligned} \langle M \rangle B &= NkT \\ U &= NkT \end{aligned}$$

which is the expected result from the equipartition theorem for a stationary particle with two rotational degrees of freedom.

(c) Proving the discrete case only differs from the continuous case proof by the obvious substitutions, so only the continuous case will be presented here. Begin by writing the first starting integral from the previous problem

$$\langle m \rangle = \int_{-\mu B}^{\mu B} m \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B}$$

The Z_1 can be pulled outside the integral since it is a constant. Then note that per our definition $\varepsilon = -mB$, it follows that

$$\frac{\partial \varepsilon}{\partial B} = -m$$

We identify the integral above to be a result of using the chain rule, so we undo that and get

$$\langle m \rangle = \frac{1}{Z_1} \int_{-\mu B}^{\mu B} \frac{\partial}{\partial B} \left(-e^{-\varepsilon/kT} \right) \frac{\mathrm{d}\varepsilon}{\mu B}$$

Changing the order of integration and differentiation,

$$= \frac{1}{Z_1} \frac{\partial}{\partial B} \left(\int_{-\mu B}^{\mu B} -e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B} \right)$$

The term within the brackets is simply the definition of the partition function, so

$$\langle m \rangle = \frac{1}{Z_1} \frac{\partial Z_1}{\partial B} = \frac{\partial \ln Z_1}{\partial B}$$

To then get the total magnetization $\langle M \rangle$, we use several properties of differentiation and logarithms:

$$\begin{aligned} \langle M \rangle &= N \langle m \rangle \\ &= N \frac{\partial \ln Z_1}{\partial B} \\ &= \frac{\partial \left(N \ln Z_1 \right)}{\partial B} \\ &= \frac{\partial \ln \left(Z_1 \right)^N}{\partial B} \end{aligned}$$

Giving us the final expression

$$\boxed{\langle M \rangle = \frac{\partial \ln Z}{\partial B}} \tag{F2012 II 5.7}$$

(d) Using the definition

$$\left\langle (\Delta\mu)^2 \right\rangle = \left\langle \mu^2 \right\rangle - \langle \mu \rangle^2$$

we already know $\langle \mu \rangle^2$ for both cases from the previous problems, so we must only calculate $\left\langle \mu^2 \right\rangle$.

Fall 2012 Part I

Problem 2

Question

Show that a particle in a one-dimensional infinite square well initially in a state $\Psi(x,0)$ will always return to that state after a time $T = 4ma^2/\pi\hbar$ where a is the width of the well.

Answer

Use the standard time independent Schödinger equation

$$\Psi(x,t) = \psi(x)\, e^{iEt/\hbar}$$

with associated differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi$$

For an infinite square well, the potential has the form

$$V(x) = \begin{cases} 0 & |x| < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$$

so that the only region to consider is $-\frac{a}{2} < x < \frac{a}{2}$. In this region, the differential equation takes the form of a harmonic oscillator

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

leading to solutions

$$\psi(x) = A\cos kx + B\sin kx$$
$$\text{where } k^2 = \frac{2mE}{\hbar^2}$$

The boundary conditions $\psi\left(-\frac{a}{2}\right) = 0$ and $\psi\left(\frac{a}{2}\right) = 0$ impose

$$\psi\left(-\frac{a}{2}\right) = 0 = A\cos\frac{ka}{2} - B\sin\frac{ka}{2}$$
$$\psi\left(\frac{a}{2}\right) = 0 = A\cos\frac{ka}{2} + B\sin\frac{ka}{2}$$

so that $B = 0$ and

$$0 = 2A\cos\frac{ka}{2}$$
$$\frac{(2n+1)\pi}{2} = \frac{ka}{2}$$
$$k = \frac{(2n+1)\pi}{a}$$

We already had a relation for k defined, so substitute and solve for the energies E_n.

$$\frac{(2n+1)^2\pi^2}{a^2} = \frac{2mE}{\hbar^2}$$
$$E_n = \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2}$$

Then considering $\Psi(x,t)$, the complex exponential is periodic in time with period

$$T_n = \frac{2\pi\hbar}{E}$$

where $n = 0$ will be the case with the longest periodicity, so

$$\begin{aligned} T &= \frac{2\pi\hbar \cdot 2ma^2}{\pi^2\hbar^2} \\ &= \frac{4ma^2}{\pi\hbar} \end{aligned}$$

Therefore, the function is periodic in time with a periodicity

$$\boxed{T = \frac{4ma^2}{\pi\hbar}} \tag{F2012 I 2.1}$$

Problem 4

Question

A photon collides with a stationary electron. If the photon scatters at an angle θ, show that the resulting wavelength λ' is given in terms of the original wavelength λ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron.

Answer

Start by considering conservation of momentum for the system. The initial values are

$$\begin{aligned} p_{\gamma x} &= \frac{h}{\lambda} & p'_{\gamma x} &= \frac{h}{\lambda'}\cos\theta \\ p_{\gamma y} &= 0 & p'_{\gamma y} &= \frac{h}{\lambda'}\sin\theta \\ p_{ex} &= 0 & p'_{ex} &= ? \\ p_{ey} &= 0 & p'_{ey} &= ? \end{aligned}$$

and considering each component in turn:

$$\begin{aligned} \frac{h}{\lambda} + 0 &= \frac{h}{\lambda'}\cos\theta + p'_{ex} & 0 &= \frac{h}{\lambda'}\sin\theta + p'_{ey} \\ p'_{ex} &= \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta & p'_{ey} &= -\frac{h}{\lambda'}\sin\theta \end{aligned}$$

The total momentum of the electron is then

$$\begin{aligned} p_e^2 &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta\right)^2 + \left(-\frac{h}{\lambda'}\sin\theta\right)^2 \\ &= \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta + \frac{h^2}{\lambda'^2}\cos^2\theta + \frac{h^2}{\lambda'^2}\sin^2\theta \\ p_e^2 &= h^2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{\lambda\lambda'}\cos\theta \end{aligned} \tag{F2012 I 4.1}$$

Then consider energy conservation, with initial values

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} & E'_\gamma &= \frac{hc}{\lambda'} \\ E_e &= mc^2 & E'_e &= \frac{p_e'^2}{2m} + mc^2 \end{aligned}$$

leading to the equation

$$
\begin{aligned}
\frac{hc}{\lambda} + mc^2 &= \frac{hc}{\lambda'} + \frac{p_e'^2}{2m} + mc^2 \\
\frac{hc}{\lambda} &= \frac{hc}{\lambda'} + \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{hc}{\lambda} - \frac{hc}{\lambda'} &= \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{\lambda' - \lambda}{\lambda\lambda'} &= \frac{h}{2mc}\frac{\lambda'^2 + \lambda^2}{\lambda^2\lambda'^2} - \frac{h}{mc\lambda\lambda'}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{\left(\lambda' - \lambda\right)^2 + 2\lambda\lambda'}{\lambda\lambda'}\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} + 2\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} + \frac{h}{mc}\left(1 - \cos\theta\right)
\end{aligned}
$$

The difference in the wavelengths is small, so

$$\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} \approx 0$$

leading to the final Compton scattering equation

$$\boxed{\lambda' = \lambda + \frac{h}{mc}\left(1 - \cos\theta\right)} \tag{F2012 I 4.2}$$