Contents

Fall 2000 Part I 2
Problem 8 . . . 2

Spring 2002 Part I 2
Problem 3 . . . 3
Problem 9 . . . 3

Spring 2002 Part II 5
Problem 2 . . . 6

Fall 2002 Part I 7
Problem 5 . . . 8
Problem 10 . . . 9
Problem 11 . . . 11

Fall 2011 Part I 12
Problem 1 . . . 13
Problem 2 . . . 13
Problem 3 . . . 15
Problem 4 . . . 16
Problem 5 . . . 17
Problem 8 . . . 18

Fall 2011 Part II 19
Problem 1 . . . 20

Spring 2012 Part I 21
Problem 1 . . . 22
Problem 2 . . . 22
Problem 3 . . . 23
Problem 4 . . . 24
Problem 5 . . . 25
Problem 6 . . . 26
Problem 7 . . . 27
Problem 8 . . . 28
Problem 10 . . . 30

Spring 2012 Part II 31
Problem 1 . . . 32
Problem 5 . . . 35

Fall 2012 Part I 38
Problem 2 . . . 39
Problem 4 . . . 40

Index

circuits
Parallel capacitors with switches, 25

dimensional analysis
Freezing ice, 10
Vacuum (Casimir) force, 24

electrostatics
Charges in Conductor Cavities, 16
Dielectric Breakdown of Air, 17
Hall effect, 31

Gauss' Law
Charges in Conductor Cavities, 16

Lagrangian
Central Forces, 14, 27
Elastic collision on spring-connected blocks, 8
Pendulum from free horizontal support, 20

mechanics
Angular momentum of a planet, 26
Central Forces, 14, 27
Elastic collision on spring-connected blocks, 8
Pendulum from free horizontal support, 20
Pendulum in Elevator, 13

orbits
Angular momentum of a planet, 26
Central Forces, 14, 27

particle
Compton scattering, 41
Proton Collision, 18

pendulum
Pendulum in Elevator, 13

quantum
Expectation values, 32
Helium ionization, 23
Infinite square-well periodicity, 39
Significance in limits, 22

relativity
Compton scattering, 41
Proton Collision, 18

solid state
Hall effect, 31

statistical mechanics
Carbon-14 dating, 12
Fermi gas properties, 29
Magnetic moments, 36
Mean free path of helium, 3
Particle velocity and mean free path, 2
Zipper partition function, 6

thermodynamics
Arbitrary engine efficiency, 4
Atmospheric Scale Height (Pressure), 19
Fermi gas properties, 29
Freezing ice, 10
Isentropic compression, 28
Magnetic moments, 36
Mean free path of helium, 3
Particle velocity and mean free path, 2
Zipper partition function, 6

Fall 2000 Part I

Problem 8

Question

Estimate (a) the average speed (in m/s) and (b) the mean free path (in m) of a nitrogen molecule in this room.

Answer

(a) We relate the kinetic energy of an N_2 molecule with the thermal energy by the equipartition theorem. Since there are 3 translational degrees of freedom,

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$
$$v = \sqrt{\frac{3k_B T}{m}}$$

The mass of the molecule is twice that of a single nitrogen atom which is itself about 14 proton masses. Therefore

$$\boxed{v \approx \sqrt{\frac{3k_B T}{28m_p}} \approx 515\ \frac{\mathrm{m}}{\mathrm{s}}} \tag{F2000 I 8.1}$$

(b) Two particles collide if they come within $2r_0$ of each other where r_0 is the typical radius of the particle. For diatomic nitrogen, we assume $r_0 \approx 2a_0$ where a_0 is the Bohr radius. Then in the time τ that the particle is moving at velocity $\langle v\rangle$, the particle can collide with any other particle within the swept-out volume

$$\mathcal{V} = \pi\left(2r_0\right)^2 \cdot \langle v\rangle\tau$$

Since there are n particles per unit volume, there are $\mathcal{N}$ atoms to collide with:

$$\mathcal{N} = n\mathcal{V} = 4\pi n {r_0}^2 \langle v\rangle\tau$$

On average then, there are $\mathcal{N}$ collisions per length $\langle v\rangle\tau$ traversed, or in its reciprocal form, the mean free path λ is

$$\lambda = \frac{1}{4\pi n {r_0}^2}$$

To estimate the particle density, consider the ideal gas law $PV = Nk_B T$. We can assume atmospheric pressure at room temperature, so the density is

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Putting it all together,

$$\boxed{\lambda \approx \frac{k_B T}{4\pi {r_0}^2 P} \approx 2.90 \cdot 10^{-7}\ \mathrm{m}} \tag{F2000 I 8.2}$$

Spring 2002 Part I

Problem 3

Question

The cross-section for collisions between helium atoms is about $10^{-16}\,\mathrm{cm}^2$. Estimate the mean free path of helium atoms in helium gas at atmospheric pressure and temperature.

Answer

Consider the path traced out by a helium atom as it travels a path length L, colliding with other helium atoms along the way. Given that the cross section of helium is σ, than we can estimate the volume that contains probable interactions with our atom of interest as $\mathcal{V} = \sigma L$. To get the number of interactions, we make use of the fact that we're treating the gas as an ideal gas. From the ideal gas law,

$$PV = NkT$$

so that solving for the number density

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Combining the density with the volume, we get the number of other [point particle] helium atoms that are contained within the given helium atom's interaction volume. If we then assume that the atom interacts with all other atoms within the volume, and that the collisions are spaced out equally in time, we just have to normalize the value by the trajectory's path length to get an estimate of the mean free path of helium in a helium gas:

$$\lambda = \frac{n\mathcal{V}}{L} = \frac{\sigma P}{k_B T}$$

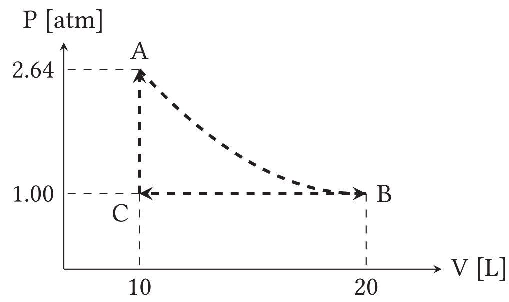
Plugging in $\sigma = 10^{-16}\,\mathrm{cm}^2$, $P = 1.013 \cdot 10^5\,\mathrm{Pa}$, $k_B = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$, and $T = 298\,\mathrm{K}$, we get

$$\boxed{\lambda = 4.06\,\mu\mathrm{m}} \tag{F2002 I 3.1}$$

Problem 9

Question

An engine using 1 mol of an ideal diatomic gas performs the cycle $A \to B \to C \to A$ as shown in the diagram below. $A \to B$ is an adiabatic expansion, $B \to C$ occurs at constant pressure, and $C \to A$ takes place at constant volume. What is the efficiency of the cycle?



Answer

Since we want to find the efficiency of the cycle, we only care about the heat exchanged during each stage of the cycle. Because the path $A \to B$ is adiabatic, we immediately know that $Q = 0$. Then proceeding to look at the stage $C \to A$, we know that the work done during this cycle is identically zero since there is no area under the curve. That means we are left simply with the equation

$$dU = dQ$$

Because this is an ideal [diatomic] classical gas, we combine the equations

$$U = \frac{5}{2} nRT$$

and

$$PV = nRT$$

to get that the difference in energy across the path is

$$\begin{aligned} Q_{CA} = U &= \frac{5}{2} nR \left(T_A - T_C \right) \\ &= \frac{5}{2} V_1 \left(P_2 - P_1 \right) \end{aligned}$$

For the remaining stage $B \to C$, we use the full thermodynamic identity:

$$dU = dQ - P\,dV$$

The pressure P_1 is constant, so both integration of dU and dV are simply the differences in each quantity. Again substituting for the temperature in U with the ideal gas law,

$$\begin{aligned} \frac{5}{2} nR \left(T_C - T_B \right) &= Q_{BC} - P_1 \left(V_1 - V_2 \right) \\ \frac{5}{2} P_1 \left(V_1 - 2V_1 \right) &= Q_{BC} + P \left(V_1 - 2V_1 \right) \\ Q_{BC} &= -\frac{7}{2} P_1 V_1 \end{aligned}$$

We've accounted for all the heat flow in the system. Q_{BC} is negative, so this is the heat flow out of the system, while Q_{CA} is positive and is the heat flow into the system. By definition then, the efficiency η of the system is

$$\begin{aligned}
\eta &= 1 - \frac{Q_{out}}{Q_{in}} \\
&= 1 - \frac{\frac{7}{2} P_1 V_1}{\frac{5}{2} V_1 \left(P_2 - P_1\right)} \\
&= 1 - \frac{5}{7} \frac{P_1}{P_2 - P_1}
\end{aligned}$$

Plugging in the given values, we find the efficency to be

$$\boxed{\eta = 0.146 = 14.6\,\%} \tag{F2002 I 9.1}$$

Spring 2002 Part II

Problem 2

Question

A zipper has N links; each link has a closed state with zero energy and an open state with energy ε. We require, however, that the zipper can only unzip from the left end, and that the link number s can only open if all links to the left (i.e. $1, 2, \dots s-1$) are already open.

a. Find an explicit expression for the partition function by doing the appropriate summation.

b. In the limit $\varepsilon \gg k_B T$ find the average number of open links. This model is a very simplified model of the unwinding of two-stranded DNA molecules.

Answer

a. Create the partion function by induction; start by assuming there is only a single link. Then the partition function is a simple two-state system:

$$Z_1 = e^0 + e^{-\varepsilon/k_B T} = 1 + e^{-\varepsilon/k_B T}$$

Adding a second link,

$$\begin{aligned} Z_2 &= \underbrace{e^{0+0}}_{\text{both closed}} + \underbrace{e^{(-\varepsilon+0)/k_B T}}_{\text{1 open, 1 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon)/k_B T}}_{\text{both open}} \\ &= 1 + e^{-\varepsilon/k_B T} + e^{-2\varepsilon/k_B T} \end{aligned}$$

Following, for three links:

$$\begin{aligned} Z_3 &= \underbrace{e^{0+0+0}}_{\text{all closed}} + \underbrace{e^{(-\varepsilon+0+0)/k_B T}}_{\text{1 open, 2 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon+0)/k_B T}}_{\text{2 open, 1 closed}} + \underbrace{e^{(-\varepsilon-\varepsilon-\varepsilon)/k_B T}}_{\text{all open}} \\ &= 1 + e^{-\varepsilon/k_B T} + e^{-2\varepsilon/k_B T} + e^{-3\varepsilon/k_B T} \end{aligned}$$

By induction, we see that the maximum coefficient in the series of exponential factors is just the number of links, so by induction we conclude that

$$Z = \sum_{s=0}^{N} e^{-s\varepsilon/k_B T}$$

Applying the results of a finite geometric series, the closed-form solution for the partition function of the links is

$$\boxed{Z = \frac{1 - e^{-(N+1)\varepsilon/k_B T}}{1 - e^{-\varepsilon/k_B T}}} \tag{F2002 II 2.1}$$

b. To get the average number of open links, we use the standard procedure for finding expectation values.

$$\langle s \rangle = \frac{1}{Z} \sum_{s=0}^{N} s e^{-s\varepsilon/k_B T}$$

By making use of differentiation under the summation trick, we can find the closed-form solution:

$$\begin{aligned}\langle s\rangle &= \frac{1}{Z}\sum_{s=0}^{N}\frac{d}{d\left(\frac{\varepsilon}{k_BT}\right)}\left[-e^{-s\varepsilon/k_BT}\right]\\ &= -\frac{1}{Z}\frac{\partial}{\partial\left(\frac{\varepsilon}{k_BT}\right)}\sum_{s=0}^{N}e^{-s\varepsilon/k_BT}\end{aligned}$$

Noting that the summation is the same as above,

$$\langle s\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial\left(\frac{\varepsilon}{k_BT}\right)}$$

First considering just the derivative part:

$$\frac{\partial Z}{\partial\left(\frac{\varepsilon}{k_BT}\right)} = \frac{(N+1)\,e^{-(N+1)\varepsilon/k_BT}}{1-e^{-\varepsilon/k_BT}} - \frac{1-e^{-(N+1)\varepsilon/k_BT}}{\left(1-e^{-\varepsilon/k_BT}\right)^2}e^{-\varepsilon/k_BT}$$

which when combined with the factor $-1/Z$ simplifies to

$$\begin{aligned}-\frac{1}{Z}\frac{\partial Z}{\partial\left(\frac{\varepsilon}{k_BT}\right)} &= -(N+1)\frac{e^{-(N+1)\varepsilon/k_BT}}{1-e^{-(N+1)\varepsilon/k_BT}} + \frac{e^{-\varepsilon/k_BT}}{1-e^{-\varepsilon/k_BT}}\\ -\frac{1}{Z}\frac{\partial Z}{\partial\left(\frac{\varepsilon}{k_BT}\right)} &= \frac{1}{e^{\varepsilon/k_BT}-1} - \frac{N+1}{e^{(N+1)\varepsilon/k_BT}-1}\end{aligned}$$

Therefore the analytic solution is

$$\boxed{\langle s\rangle = \frac{1}{e^{\varepsilon/k_BT}-1} - \frac{N+1}{e^{(N+1)\varepsilon/k_BT}-1}} \tag{F2002 II 2.2}$$

In the limit that $\varepsilon \gg k_BT$, though, the exponentials in the denominator are very large in comparison to 1, so we ignore the unity factors and make the approximation that

$$\langle s\rangle = e^{-\varepsilon/k_BT} - (N+1)\,e^{-(N+1)\varepsilon/k_BT}$$

Collecting like terms,

$$= \left[1-(N+1)\,e^{-N\varepsilon/k_BT}\right]e^{-\varepsilon/k_BT}$$

The second term in the brackets approximate zero, so

$$= e^{-\varepsilon/k_BT}$$

Therefore in the low temperature limit where the thermal energy is much less than the energy of the open state,

$$\boxed{\langle s\rangle = e^{-\varepsilon/k_BT}} \tag{F2002 II 2.3}$$

Fall 2002 Part I

Problem 5

Question

Blocks of mass m and $2m$ are free to slide without friction on a horizontal wire. They are connected by a massless spring of equilibrium length L and force constant k. A projectile of mass m is fired with velocity v into the block with mass m and sticks to it. If the blocks are initially at rest, what is the maximum displacement between them in the subsequent motion?

Answer

Take time $t = 0$ to be the moment the projectile collides with the mass m, and let the subsequent transfer of momentum be instantaneous. In this case, the initial conditions of the problem are then:

$$
\begin{aligned}
x_1(0) &= 0 & \qquad \dot{x}_1(0) &= u \\
x_2(0) &= L & \qquad \dot{x}_2(0) &= 0
\end{aligned}
$$

where u is the initial velocity of the combined project-mass system. We get u from conservation of mometum:

$$
\begin{aligned}
2mu &= mv + 0 \\
u &= \frac{1}{2}v
\end{aligned}
$$

Now solve the mechanics problem using the Lagrangian approach. Both masses have kinetic energy, and the spring stores potential energy, so

$$
\begin{aligned}
T &= m\dot{x}_1^2 + m\dot{x}_2^2 \\
V &= \frac{1}{2}m\left(x_2 - x_1\right)^2 \\
L &= m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k\left({x_1}^2 + {x_2}^2 + 2x_1x_2\right)
\end{aligned}
$$

Setting up the differential equation, we get

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= -kx_1 + kx_2 & \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= 2m\ddot{x}_1 \\
\frac{\partial L}{\partial x_2} &= kx_1 - kx_2 & \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] &= 2m\ddot{x}_2
\end{aligned}
$$

Leading to the system of equations where $\omega^2 = k/2m$,

$$
\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Solving the eigensystem, we find the eigenfrequencies to be $\lambda = \{0, -2\omega^2\}$. Letting $\omega'^2 = 2\omega^2$, the eigenfunction equations are then

$$
\begin{aligned}
\ddot{\psi}_1 &= 0 & \qquad &\rightarrow & \qquad \psi_1 &= A_1t + B_1 \\
\ddot{\psi}_2 &= -2\omega^2\psi_2 & \qquad &\rightarrow & \qquad \psi_2 &= A_2\cos\left(\omega' t\right) + B_2\sin\left(\omega' t\right)
\end{aligned}
$$

From the eigenvectors, we express the solutions of x_1 and x_2 in terms of ψ_1 and ψ_2:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= A_1 t + B_1 + A_2 \cos\left(\omega' t\right) + B_2 \sin\left(\omega' t\right) \\ x_2 &= A_1 t + B_1 - A_2 \cos\left(\omega' t\right) - B_2 \sin\left(\omega' t\right) \end{aligned}$$

Applying the boundary conditions, we find that

$$\begin{aligned} x_1(t) &= \frac{1}{4} v t + \frac{1}{2} L - \frac{1}{2} L \cos\left(\omega' t\right) + \frac{v}{4\omega'} \sin\left(\omega' t\right) \\ x_2(t) &= \frac{1}{4} v t + \frac{1}{2} L + \frac{1}{2} L \cos\left(\omega' t\right) - \frac{v}{4\omega'} \sin\left(\omega' t\right) \end{aligned}$$

The distance $\ell(t) = x_2(t) - x_1(t)$ between the two masses maximizes when

$$\begin{aligned} \frac{d\ell}{dt} = 0 &= \frac{d}{dt}\left[L \cos\left(\omega' t\right) - \frac{v}{2\omega'} \sin\left(\omega' t\right)\right] \\ t &= -\frac{1}{\omega'} \arctan\left(\frac{v}{2L\omega'}\right) \end{aligned}$$

Plugging back into the function $\ell(t)$,

$$\begin{aligned} \ell &= L \cos\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right] - \frac{v}{2\omega'} \sin\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right] \\ \ell &= L \frac{2L\omega'}{\sqrt{v^2 + 4L^2\omega'^2}} + \frac{v}{2\omega'} \frac{v}{\sqrt{v^2 + 4L^2\omega'^2}} \\ \ell &= \frac{\sqrt{v^2 + 4L^2\omega'^2}}{2\omega'} \end{aligned}$$

Finally, substituting back in $\omega' = \sqrt{2k/m}$ and simplifying, we get the final solution that maximum distance between the two masses is

$$\boxed{\ell = \sqrt{L^2 + \frac{\frac{1}{2} m v^2}{8k}}} \tag{F2002 I 5.1}$$

which agrees qualitatively with the fact that a larger spring constant should stiffen the system and decrease the maximum displacement, while launching the projectile with a greater velocity would increase it.

Problem 10

Question

Ice on a pond is 10 cm thick and the water temperature just below the ice is 0 ℃. If the air temperature is −20 ℃, by how much will the ice thickness increase in 1 hour? Assuming that the air temperature stays the same over a long period, how will the ice thickness increase with time? Comment on any approximation that you make in your calculation.

Density of ice $= 0.9\,\mathrm{g/cm^3}$

Thermal conductivity of ice $= 0.0005\,\mathrm{cal/(cm{\cdot}s{\cdot}℃)}$

Latent heat of fusion of water $= 80\,\mathrm{cal/g}$

Answer

Since the thermal heat flow is a one dimensional problem, immediately consider everything with respect to a small area element with its normal perpendicular to the ice-water interface dA. Then we want to know how much ice is generated on the surface of the ice. This small ice element's mass is simply

$$dm = \rho\, dA\, dz$$

where dz is the thickness of the new ice layer. To generate this ice, the latent heat of fusion must be conducted away, so the energy released is,

$$\begin{aligned} dE_f &= L_f\, dm \\ &= L_f \rho\, dA\, dz \end{aligned}$$

The energy flow is through the ice, and we expect this to increase with the temperature differential across the ice sheet, suggesting that the thermal conductivity κ be multiplied by the temperature difference ΔT. Furthermore, the ice will decrease the rate of heat flow as it becomes thicker, so the quantity should also be divided by the thickness z. This gives

$$\frac{\kappa \Delta T}{z} = \left[\frac{\mathrm{cal}}{\mathrm{cm^2{\cdot}s}}\right]$$

This energy is flowing through a surface element dA, giving the power flow due to heat as

$$\frac{\kappa \Delta T\, dA}{z} = \left[\frac{\mathrm{cal}}{\mathrm{s}}\right]$$

This power can be matched in units with the energy released from the ice calculated above by taking the time derivative of dE_f, so equating the two we have

$$\begin{aligned} L_f \rho\, dA \frac{dz}{dt} &= \frac{\kappa \Delta T\, dA}{z} \\ \int_{z_0}^{z_0+\delta z} z\, dz &= \int_0^t \frac{\kappa \Delta T}{L_f \rho}\, dt \\ 2z_0 \delta z + (\delta z)^2 &= \frac{\kappa \Delta T}{L_f \rho} t \end{aligned}$$

Solving for the length the ice grows δz,

$$\begin{aligned} \delta z &= \frac{-2z_0 \pm \sqrt{4{z_0}^2 - 4\left(\frac{\kappa \Delta T}{L_f \rho}\right) t}}{2} \\ \delta z &= z_0 \left(1 \pm \sqrt{1 - \frac{\kappa \Delta T}{L_f \rho {z_0}^2} t}\right) \end{aligned}$$

The two roots give solutions $\delta z = \{0.0501\,\text{cm}, 19.950\,\text{cm}\}$. Since the second root is unrealistic, we know that the solution must then be

$$\boxed{\delta z = 0.0501\,\text{cm} \quad \text{in an hour}} \tag{F2002 I 10.1}$$

Problem 11

Question

Carbon-14 is produced by cosmic rays interacting with the nitrogen in the Earth's atmosphere. It is eventaully incorporated into all living things, and since it has a half-life of (5730 ± 40) yr, it is useful for dating archaeological specimens up to several tens of thousands of years old. The radioactivity of a particular specimen of wood containing 3 g of carbon was measured with a counter whose efficiency was 18 %; a count rate of $(12.8 \pm 0.1)\,\mathrm{min}^{-1}$ was measured. It is known that in 1 g of living wood, there are $16.1\,\mathrm{min}^{-1}$ radioactive carbon-14 decays. What is the age of this specimen, and its uncertainity? (Where errors are not quoted, they can be assumed to be negligible).

Answer

The rate N after a given time is given by the exponential decay formula

$$N\left(t\right) + = N_0 e^{-t/\tau}$$

Since we have the half-life $t_{1/2}$ instead of the decay constant τ, we use the relation $t_{1/2} = \tau \ln 2$ to simplify the expression instead to

$$N\left(t\right) = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The counter use has an efficiency of $\varepsilon = 0.18$, so the measured counting rate N_m must be corrected for that. Furthermore, the sample has a mass of 3 g whereas we know the rate for a one gram sample, so we also normalize the count rate by the mass of the sample. Plugging this all into the exponential decay function above gives

$$\frac{N_m}{3\varepsilon} = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The only unknown left in the equation is the time, so solving for it,

$$t = t_{1/2} \log_{1/2}\left(\frac{N_m}{3\varepsilon N_0}\right)$$
$$t = t_{1/2} \frac{\ln\left(\frac{N_m}{3\varepsilon N_0}\right)}{\ln 2}$$
$$t = \frac{t_{1/2}}{\ln 2} \ln\left(\frac{N_m}{3\varepsilon N_0}\right)$$

To find the uncertainty, we note that only the quantities N_m and $t_{1/2}$ have non-negligible uncertainties, so we propagate the errors only over these two terms:

$$\sigma_t{}^2 = \left(-\frac{t_{1/2}}{N_m \ln 2}\right)^2 \sigma_{N_m}{}^2 + \left(\frac{1}{\ln 2} \ln\left(\frac{3\varepsilon N_0}{N_m}\right)\right)^2 \sigma_{t_{1/2}}{}^2$$
$$\sigma_t = \frac{t_{1/2}}{\ln 2} \sqrt{\left(\frac{\sigma_{N_m}}{N_m}\right)^2 + \left(\frac{\sigma_{t_{1/2}}}{t_{1/2}}\right)^2 \left[\ln\left(\frac{3\varepsilon N_0}{N_m}\right)\right]^2}$$

Plugging in all the numbers, we get $t = 4248.8435$ yr and $\sigma_t = 161.717$ yr. The given uncertainties have a single significant digit, so adding an extra significant figure to the uncertainity and matching decimal places in the answer, we conclude that the sample has an age of

$$\boxed{t = (4250 \pm 160)\,\mathrm{yr}} \qquad \text{(F2002 I 11.1)}$$

Fall 2011 Part I

Problem 1

Question

An elevator operator in a skyscraper, being a very meticulous person, put a pendulum clock on the wall of the elevator to make sure that he spends exactly 8 hours a day at his work place. Over the course of his work day, he records that the time during which the elevator has acceleration a is exactly equal to the time during which it has acceleration $-a$. Does the elevator operator work, in actual time, (1) more than 8 hours, (2) exactly 8 hours, or (3) less than 8 hours? Why?

Answer

The nominal period of a pendulum is

$$T_{nom} = 2\pi\sqrt{\frac{\ell}{g}}$$

but within the elevator, the acceleration g is not going to be constant and will rather depend on the acceleration of the elevator. Therefore,

$$T_{\uparrow} = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad\qquad T_{\downarrow} = 2\pi\sqrt{\frac{\ell}{g-a}}$$

for the upward and downward cases, respectively.

Since the elevator operator observed that equal time was spent going up as was spent going down, so he must have observed N oscillations in both cases. In order to compare to the actual time, we simply compare the elevator's total time measurement with that of a stationary clock.

$$NT_{\uparrow} + NT_{\downarrow} \stackrel{?}{=} 2NT_{nom}$$

$$\begin{aligned}
2\pi N\sqrt{\frac{\ell}{g+a}} + 2\pi N\sqrt{\frac{\ell}{g-a}} &\stackrel{?}{=} 4\pi N\sqrt{\frac{\ell}{g}} \\
\sqrt{\frac{1}{g+a}} + \sqrt{\frac{1}{g-a}} &\stackrel{?}{=} 2\sqrt{\frac{1}{g}} \\
\sqrt{\frac{g}{g+a}} + \sqrt{\frac{g}{g-a}} &\stackrel{?}{=} 2
\end{aligned}$$

Use the test value $a = 5$ for comparison (with $g = 10$)

$$\boxed{2.23 > 2} \qquad \text{(F2011 I 1.1)}$$

Therefore the elevator operator actually spends more than 8 hours in the elevator during his shift.

Problem 2

Question

A classical particle is subject to an attractive central force proportional to r^α, where r is the radius and α is a constant. Show by perturbation analysis what is required of α in order for the particle to have a stable circular orbit.

Answer

Construct the Lagrangian for the system in order to determine the equations of motion for the given central force (noting that we were given the *force* so we need to make an appropriate potential).

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad\qquad V = \frac{k}{\alpha+1}r^{\alpha+1}$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{\alpha+1}r^{\alpha+1}$$

Conservation of angular momentum is a consequence of the θ and $\dot{\theta}$ coordinates:

$$0 = \frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right]$$
$$0 = \frac{d}{dt}\left[mr^2\dot{\theta}\right]$$

Nothing that

$$\ell = \left|\frac{\vec{r}\times\vec{p}}{m}\right| = r^2\dot{\theta}$$

we can say that

$$\dot{\theta} = \frac{\ell}{r^2}$$

Then returning to the r and $\dot{r}$ coordinates in the Lagrangian,

$$\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - kr^\alpha \qquad\qquad \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right] = m\ddot{r}$$

Putting the differential equation together and substituting for the angular momentum per unit mass gives

$$m\ddot{r} = \frac{m\ell^2}{r^3} - kr^\alpha \tag{F2011 I 2.1}$$

In the case that the orbit is circular, r must be a constant, so let $r = a$ and note that $\ddot{r} = 0$ necessarily.

$$\frac{m\ell^2}{a^3} = ka^\alpha$$

Returning to the differential equation, let the actual distance r be a perturbation from a circular orbit, and Taylor expand in x where $x = r - a$.

$$\begin{aligned}
m\ddot{x} &= \frac{m\ell^2}{a^3}\left(1 + \frac{x}{a}\right)^{-3} - ka^\alpha\left(1 + \frac{x}{a}\right)^\alpha \\
m\ddot{x} &\approx \frac{m\ell^2}{a^3}\left(1 - 3\frac{x}{a} + \ldots\right) - ka^\alpha\left(1 + \alpha\frac{x}{a} + \ldots\right) \\
m\ddot{x} &\approx ka^\alpha\left(1 - 3\frac{x}{a}\right) - ka^\alpha\left(1 + \alpha\frac{x}{a}\right) \\
m\ddot{x} &\approx -3ka^\alpha\frac{x}{a} - \alpha ka^\alpha\frac{x}{a} \\
m\ddot{x} &\approx -ka^{\alpha-1}(3 + \alpha)\,x
\end{aligned}$$

To form a stable orbit, the coefficient on x must be negative, giving a simple harmonic solution. Therefore $3 + \alpha > 0$ to keep the coefficient negative and

$$\boxed{a > -3} \qquad \text{(F2011 I 2.2)}$$

Problem 3

Question

A neutral conductor A with a spherical outer surface of radius R contains three cavities B, C, and D, but is solid otherwise. B and C are spherical, and D is hemispherical. Without touching A, positive charges q_B and q_C are introduced at the centers of B and C, respectively.

1. Give the amount and the distribution of the induced charges on the surfaces of A, B, C, and D.
2. Now another positive charge q_E is introduced at a distance $r > R$ from the center of A. Describe qualitatively the distribution of induced charges on the surfaces of A, B, C, and D.
3. Give the amount of the induced charges on the surfaces of A, B, C, and D for the situation in (2).

Answer (1)

An ideal conductor will not support an electric field inside the solid, so each of cavities B and C will have a surface charge to cancel the electric fields emminating from q_B and q_C respectively.

- Cavity B will have a uniform surface charge density of $-q_B/4\pi r_B^2$, where r_B is the radius of cavity B, with total induced charge $-q_B$ (because of symmetry and use of a Gaussian surface).
- Cavity C will have a uniform surface charge density of $-q_C/4\pi r_C^2$, where r_C is the radius of cavity B, with total induced charge $-q_C$ (because of symmetry and use of a Gaussian surface).

Cavity D will not have a surface charge since a Guassian surface coincident with its boundary contains no charge.

The surface A will have total charge $q_B + q_C$ with uniform surface charge density of $\left(q_B + q_C\right)/4\pi R^2$ in accordance with the symmetry of a Gaussian surface containing the sphere as well as properties of an ideal conductor.

Answer (2)

The surfaces B, C, and D will remain unaffected since the surrounding conductor shields the cavities from electric fields produced by charge q_E. The distribution on surface A will shift so that the negative charge concentration is greatest on the side nearest to q_E with an increasingly positive distribution towards the opposite side.

Answer (3)

The surface of A will still contain the same total charge $q_B + q_C$ since only a redistribution of induced charges occurred along the surface. Similarly, because surface B, C, and D are shielded from the electric field of q_E by conductor A, the total charges along their surfaces remains unchanged as well.

Problem 4

Question

The dielectric strength of air at standard temperature and pressure is $3 \cdot 10^6$ V/m. What is the maximum intensity in units of W/m^2 for a monocromatic laser that can be used in the laboratory?

Answer

Failure of a dielectric occurs when the energy density in the dielectric is great enough to overcome the ionization energy of the constituent atoms. This suggests that an electric field of greater than $3 \cdot 10^6$ V/m would cause this ionization to occur.

Starting here, We can calculate the energy density of the electric field at any point in space by

$$U_{em} = \frac{\varepsilon_0}{2} E^2$$

(where we've used the vacuum energy density since air differs very little from the vacuum permittivity).

Then the power transmitted by the laser is $P = cU_{em}$, so plugging in the numbers,

$$P = \frac{1}{2} \left(8.854 \cdot 10^{-12} \, \frac{\mathrm{C}^2}{\mathrm{N{\cdot}m}^2} \right) \left(3 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}} \right) \left(3 \cdot 10^6 \, \frac{\mathrm{V}}{\mathrm{m}} \right)^2$$
$$P = 1.19 \cdot 10^{10} \, \frac{\mathrm{W}}{\mathrm{m}^2}$$

$$\boxed{\text{The maximum power of a laser usable in the lab is } 1.19 \cdot 10^{10}\ \mathrm{W/m^2}.}$$

Problem 5

Question

What is the minimum energy of the projectile proton required to induce the reaction $p + p \to p + p + p + \bar{p}$ if the target proton is at rest?

Answer

Energy and momentum must be conserved. At the minimum allowed energy, the resultant 4 proton/anti-protons will be colinear with no relative momentum with respect to one another, so the momenum equation in the lab frame is simply

$$p_i = 4p_f \tag{F2011 I 5.1}$$

Similarly, the resultant (anti-)protons are indistinguishable, so they will all have equivalent energy E_f. The initial protons have different energies since one is at rest in the lab frame while the other is moving, leading to the energy equation

$$\sqrt{p_i^2c^2 + m_p^2c^4} + m_pc^2 = 4\sqrt{p_f^2c^2 + m_p^2c^4}$$

Substituting the momentum relation into the equation, squaring, and simplifying,

$$\begin{aligned}
\sqrt{16p_f^2c^2 + m_p^2c^4} + m_pc^2 &= 4\sqrt{p_f^2c^2 + m_p^2c^4} \\
16p_f^2c^2 + m_p^2c^4 + m_p^2c^4 + 2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 16p_f^2c^2 + 16m_p^2c^4 \\
2\sqrt{m_p^2c^4\left(16p_f^2c^2 + m_p^2c^4\right)} &= 14m_p^2c^4 \\
16p_f^2c^2 + m_p^2c^4 &= 49m_p^2c^4 \\
p_f^2 &= 3m_p^2c^4
\end{aligned}$$

Therefore,

$$p_i^2 = 48m_p^2c^4$$

and

$$E_1 = \sqrt{49m_p^2c^4}$$

$$\boxed{E_1 \approx 6.567\,\mathrm{GeV/c^2}} \tag{F2011 I 5.2}$$

Problem 8

Question

Assume that the atmosphere near the earth's surface is in approximate hydrostatic equilibrium, where any movement of air parcels is gentle and adiabatic. Find an expression for the pressure P of the atmosphere as a function of the height z.

Answer

Note that the pressure at a given point is due to the mass of air above the given point. Then by moving an infinitesimal distance vertically, the total mass is changed by the density of the air (which is affected by the gravitational force). This leads to the differential equation

$$\frac{dP}{dz} = \rho g$$

Then using the ideal gas equation

$$PV = Nk_B T$$

multiply and divide by the average molecular mass m of the air (in kg) which combined with the number of molecules N gives the total mass

$$PV = (Nm)\,\frac{1}{m}k_B T$$

and then divide by the volume to get the ideal gas equation in terms of the mass density

$$P = \frac{Nm}{V}\frac{1}{m}k_B T$$
$$P = \rho\frac{k_B T}{m}$$
$$\rho = \frac{Pm}{k_B T}$$

Finally, substitute this into the differential equation above and solve to get the atmospheric scale height equation.

$$\frac{dP}{dz} = \frac{Pm}{k_B T}g$$
$$\frac{dP}{P} = \frac{mg}{k_B T}dz$$

$$\boxed{P(z) = P_0 e^{z/\xi} \qquad \text{where } \xi = \frac{k_B T}{mg}} \qquad \text{(F2011 I 8.1)}$$

Fall 2011 Part II

Problem 1

Question

Mass m_1 moves freely along a fixed, long, horizontal rod. The position of m_1 on the rod is x. A massless string of length ℓ is attached to m_1 at the end and to mass m_2 at the other. Mass m_2 executes pendulum motion in the vertical plane containing the rod.

1. Find the Lagrangian of the system.
2. Derive the equations of motion and the corresponding conservation laws.
3. Assume that $x(0) = x_0$, $\dot{x}(0) = 0$, $\varphi(0) = \varphi_0$ $\left(|\varphi_0| \ll 1\right)$, and $\dot{\varphi}(0) = 0$. Find $x(t)$ and $\varphi(t)$ for $t > 0$.

Answer (1)

For the sliding support mass m_1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{x}^2 \\ V_1 &= 0 \end{aligned}$$

For the pendulum mass m_2:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} m_2 \left(\dot{x} + \dot{x}_2\right)^2 \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Then using $x_2 = \ell \sin\varphi$ and $y_2 = -\ell \cos\varphi$,

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \left(\ell^2 \dot{\varphi}^2 + \dot{x}^2 + 2\ell\dot{\varphi}\dot{x}\cos\varphi\right) \\ V_2 &= -m_2 g y_2 \end{aligned}$$

Putting the Lagrangian together equals the first line. Applying the small angle approximation gives the second line where the kinetic energy term involving $\cos\varphi$ can be simply expanded as $\cos\varphi \approx 1$, but the potential energy term must be expanded to second order so that $\cos\varphi \approx 1 - \frac{1}{2}\varphi^2$.

$$\boxed{\mathcal{L} = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + 2\ell\dot{\varphi}\dot{x}\cos\varphi\right) + m_2 g \ell \cos\varphi} \tag{F2011 II 1.1}$$

$$\boxed{\mathcal{L} \approx \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2} m_2 \left(\ell^2\dot{\varphi}^2 + 2\ell\dot{\varphi}\dot{x}\right) + m_2 g \ell - \frac{1}{2} m_2 g \ell \varphi^2} \tag{F2011 II 1.2}$$

Answer (2)

Constructing the Euler-Lagrange equations for x and $\dot{x}$:

$$\frac{\partial\mathcal{L}}{\partial x} = 0 \qquad \begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{x}} &= \left(m_1 + m_2\right)\dot{x} + m_2\ell\dot{\varphi} \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] &= \left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\varphi} \end{aligned}$$

$$\left(m_1 + m_2\right)\ddot{x} + m_2\ell\ddot{\varphi} = 0 \tag{F2011 II 1.3}$$

and for φ and $\dot{\varphi}$:

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -m_2 g\ell\varphi \qquad \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m_2\ell^2\dot{\varphi} + m_2\ell\dot{x}$$

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right] = m_2\ell^2\ddot{\varphi} + m_2\ell\ddot{x}$$

$$-m_2 g\ell\varphi - m_2\ell^2\ddot{\varphi} + m_2\ell\ddot{x} = 0 \tag{F2011 II 1.4}$$

The equations of motion are:

$$\boxed{\ddot{x} + \frac{m_2}{m_1 + m_2}\ell\ddot{\varphi} = 0} \tag{F2011 II 1.5}$$

$$\boxed{\ddot{\varphi} + \frac{1}{\ell}\ddot{x} + \frac{g}{\ell}\varphi = 0} \tag{F2011 II 1.6}$$

Answer (3)

Solve for $\ddot{x}$ and substitute into the other differential equation

$$\ddot{\varphi} - \frac{1}{\ell}\frac{m_2}{m_1 + m_2}\ell\ddot{\varphi} + \frac{g}{\ell}\varphi = 0$$

$$\frac{m_1}{m_1 + m_2}\ddot{\varphi} + \frac{g}{\ell}\varphi = 0$$

$$\ddot{\varphi} + \frac{g}{\ell}\frac{m_1 + m_2}{m_1}\varphi = 0 \tag{F2011 II 1.7}$$

This is just the differential equation for a simple harmonic oscillator, so considering the given boundary conditions,

$$\boxed{\varphi(t) = \varphi_0\cos(\omega t) \qquad \text{where } \omega^2 = \frac{g}{\ell}\frac{m_1 + m_2}{m_1}} \tag{F2011 II 1.8}$$

Then differentiating $\varphi(t)$ twice and substituting into the first equation,

$$\ddot{x} = \ell\varphi_0\omega^2\frac{m_2}{m_1 + m_2}\cos(\omega t)$$

Then integrating twice and applying the boundary conditions,

$$\boxed{x(t) = x_0 - \frac{g}{\omega^2}\frac{m_2}{m_1}\cos(\omega t)} \tag{F2011 II 1.9}$$

Spring 2012 Part I

Problem 1

Question

For a many particle system of weekly interacting particles, will quantum effects be more important for (a) high densities or low densities and (b) high temperatures or low temperatures for a system. Explain your answers in terms of the de Broglie wavelength λ defined as $\lambda^2 \equiv h^2 / \left(3mk_B T\right)$ where m is the mass of the particles and k_b Boltzmann's constant.

Answer

(a) High density — The de Broglie wavelength gives a "size" of the particle, and in the high density limit, the wavefunctions overlap significantly so quantum effects and interactions are critical to the behavior of the system.

(b) Low temperature — Since $\lambda^2 \propto T^{-1}$, as $T \to 0$, λ increases so that again the wavefunctions overlap and quantum effects are significant.

Problem 2

Question

The ground state energy of Helium is −79 eV. What is its ionization energy, which is the energy required to remove just one electron?

Answer

Using the Hydrogen solution with modifications for single-electron atoms of higher Z, we know that the ground state energy of singly ionized Helium is

$$E_{He}^{1} = 2^2 \left(-13.6\,\mathrm{eV}\right) = -54.4\,\mathrm{eV}$$

Therefore, the difference between the singly-ionized and neutral ground state energies gives the first ionization energy of the Helium atom.

$$\boxed{E_i = -24.6\,\mathrm{eV}} \tag{F2012 I 2.1}$$

Problem 3

Question

It is known that the force per unit area (F/A) between two neutral conducting plates due to polarization fluctuations of the vacuum (namely, the Casimir force) is a function of h (Planck's constant), c (speed of light), and z (distance between the plates) only. Using only dimensional analysis, obtain F/A as a function of h, c, and z.

Answer

The units of F/A are

$$\frac{F}{A} = \left[\frac{\text{kg}}{\text{m}\cdot\text{s}^2}\right]$$

The kg suggests a factor proportional to h, making the equation

$$\frac{F}{A} \sim \left[\frac{1}{\text{m}^3\cdot\text{s}}\right] h$$

Accounting for the factor of seconds requires a c:

$$\frac{F}{A} \sim \left[\frac{1}{\text{m}^4}\right] hc$$

Finally, account for all the factors of distance:

$$\frac{F}{A} \sim \frac{hc}{z^4}$$

Therefore,

$$\boxed{\frac{F}{A} \sim \frac{hc}{z^4}} \tag{F2012 I 3.1}$$

Problem 4

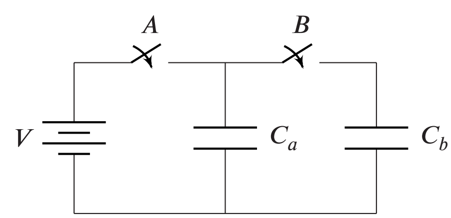
Question

In the circuit diagram opposite, initially the two identical capacitors with capacitance C are uncharged. The connections between the components are all made with short copper wires. The battery is an ideal EMF and supplies a voltage V.

(a) At first Switch A is closed and Switch B is kept open. What is the final sotred energy on capacitor C_a?

(b) Switch A is opened and afterwards Switch B is closed. What is the final energy stored in both capacitors?

(c) Provide a physical explanation for any difference between the results of parts (a) and (b), if there is one.



Answer

(a) Initially, the right side of the circuit with C_b can be ignored, so the total energy is simply the energy stored within C_a.

$$\boxed{E = \frac{1}{2}CV^2} \tag{F2012 I 4.1}$$

(b) The system is now effectively just the two capacitors on the right. Because the voltage difference is supported across both capacitors, the system can be modeled as an effective capacitor in parallel

$$C_{eff} = 2C$$

The total charge stored by the capacitors must remain the same when switching from Switch A being closed to Switch B. Initially,

$$Q = CV_i$$

and afterwards it is

$$Q = C_{eff}V = 2CV_f$$

so the final voltage across the capacitors is

$$V_f = \frac{1}{2}V_i$$

This means the total energy is

$$E = \frac{1}{2}C_{eff}{V_f}^2$$

$$\boxed{E = \frac{1}{4}CV^2} \tag{F2012 I 4.2}$$

(c) The energy is dissipated (heat, fields, etc).

Problem 5

Question

A planet of mass m moves around the sun, mass M, in an elliptical orbit with minimum and maximum distances of r_1 and r_2, respectively. Find the angular momentum of the planet relative to the center of the sun in terms of these quantities and the gravitational constant G.

Answer

We solve the problem using conservation of energy since we know that stable elliptical orbits have constant energy. The generic equation is

$$E = \frac{L^2}{2I} - \frac{GMm}{r}$$

where L is the angular momentum and I the moment of inertia. Substituting for the values at both r_1 and r_2 and equating,

$$\frac{L^2}{2m{r_1}^2} - \frac{GMm}{r_1} = \frac{L^2}{2m{r_2}^2} - \frac{GMm}{r_2}$$
$$\frac{L^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right) = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

which leads to the solution

$$\boxed{L = \sqrt{\frac{2GMm^2 r_1 r_2}{r_1 + r_2}}} \qquad \text{(F2012 I 5.1)}$$

Problem 6

Question

A particle moves in a circular orbit under the influence of a central force that varies as the n-th power of the distance. Show that this motion is unstable if $n < -3$. (Hint: Consider the centrifugal potential.)

Answer

See solution for Fall 2011 Part I, Problem 2 with the condition inverted so that *in*stability is $n < -3$ rather than stability requiring $n > -3$.

Problem 7

Question

A classical, ideal, monatomic gas of N particles is reversibly compressed *isentropically*, i.e. with the entropy kept constant, from an initial temperature T_0 and pressure P to a pressure $2P$. Find (a) the work done on the system, and (b) the net change in entropy of the system and its surroundings.

(a) An isentropic process is the same as an adiabatic process since no heat can be exchanged ($T\,\mathrm{d}S = Q = 0$), so we begin with the relation that PV^γ is a constant. Combining this with the ideal gas law, we can determine that

$$P^{1-\gamma}T^\gamma = \text{const}$$

where $\gamma = C_p/C_v$ is the ratio of heat capacities with $C_p = \frac{5}{2}Nk_B$ and $C_v = \frac{3}{2}Nk_B$ for a monatomic ideal gas. Using this, we solve for the final temperature of the system after compressions as

$$T_f = 2^{2/5}T_0 \approx 1.32T_0$$

Combining both of

$$\begin{aligned} \Delta U &= C_v \Delta T \\ \Delta U &= Q + W \end{aligned}$$

where $Q = 0$, we get that

$$\boxed{W = \frac{3}{2}Nk_BT_0\left(2^{2/5} - 1\right)} \tag{F2012 I 7.1}$$

(b) Because the compression is done reversibly, by definition, $\Delta S = 0$.

Problem 8

Question

For an idea Fermi gas of N neutral spin-$\frac{1}{2}$ particles in a volume V at $T = 0$, calculate the following:

(a) The chemical potential

(b) The average energy per particle

(c) The pressure

Answer

(a) At $T = 0$, the particles are all in the lowest state allowed by Fermi-Dirac statistics, so the chemical potential, defined by the energy required to add another particle to the system, is equal to the Fermi energy. For a particle contained within a box V, the energy per particle is

$$\varepsilon_n = \frac{\pi^2\hbar^2}{2mV^{2/3}}n^2$$

Given a Fermi energy ε_F, the maximum occupied state is

$$n_F = \sqrt{\frac{2mV^{2/3}}{\pi^2\hbar^2}}\sqrt{\varepsilon_F}$$

Equally we know that all N particles must exist within the eighth-sphere of n space, where the extra factor of 2 is because there are two spin states per n:

$$N = 2 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi {n_F}^3$$
$$N = \frac{1}{3}\pi \left(\frac{2m}{\pi^2\hbar^2}\right)^{3/2} V{\varepsilon_F}^{3/2}$$
$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

Therefore $\mu = \varepsilon_F$,

$$\boxed{\mu = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}} \tag{F2012 I 8.1}$$

(b) To get the total energy, we can imagine filling all N particles one at a time, so that at each step, there are N' total particles:

$$U = \int_0^N \varepsilon_F \, \mathrm{d}N'$$
$$U = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \int_0^N N^{2/3}\, \mathrm{d}N'$$
$$U = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \frac{3}{5}N^{5/3}\, \mathrm{d}N'$$

Therefore, the average energy per particle is U/N or

$$\boxed{\langle\varepsilon\rangle = \frac{3}{5}\varepsilon_F} \tag{F2012 I 8.2}$$

(c) From the thermodynamic relation

$$dU = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N$$

we can read off the derivative that defines the pressure P as

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

Doing so, we get that

$$\frac{\partial U}{\partial V} = \frac{3}{5}N \cdot \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} \cdot \left(-\frac{2}{3V}\right)$$

making the pressure

$$\boxed{P = \frac{2}{5}\frac{N}{V}\varepsilon_F} \qquad \text{(F2012 I 8.3)}$$

Problem 10

Question

A piece of p-doped silicon has a carrier density $n = 10^{15}\,\mathrm{cm}^{-3}$ and dimensions of $\Delta x = 10\,\mathrm{mm}$, $\Delta y = 2\,\mathrm{mm}$, and $\Delta z = 1\,\mathrm{mm}$. A magnetic field of $B_z = 1\,\mathrm{T}$ is applied in the z-direction and a current $I_x = 1\,\mathrm{A}$ flows in the x-direction, and the voltage V_y is measured.

(a) Express the current density j_x in terms of the carrier density n and the carrier velocity v_x.

(b) Write down the equilibrium force condition that determins V_y.

(c) Find V_y in volts.

Answer

(a) The current passing through each thin cross-sectional slice of the conductor is dependent on the charge of a carrier, carrier density, and velocity of the flow.

$$I_x = en\Delta y\Delta z v_x$$

The current density is just the current passing through each point, so

$$j_x = \frac{I_x}{\Delta y\Delta z}$$

$$\boxed{j_x = nev_x} \qquad \text{(F2012 I 10.1)}$$

(b) The positive carriers drift to the edge of the conductor due to the magnetic field and the holes accumulate on the opposite edge. An electric field is created between the charge separation, so an equilibrium is set up between the electric field trying to bring the opposite charges together and the magnetic drift separating them.

$$0 = e\vec{E} + \vec{v} \times \vec{B}$$

By the right-hand rule, the positive charges accumulate along $y = 0$, so $\vec{E} = E\hat{y}$. Similarly, $\vec{v} \times \vec{B} = -v_x B_z \hat{y}$:

$$0 = eE\hat{y} - ev_x B_z \hat{y}$$

Written in terms of the potential $V_y = E\Delta y$, the equilibrium condition becomes

$$\boxed{V_y = v_x B_z \Delta y} \qquad \text{(F2012 I 10.2)}$$

(c) Substituting in for given quantities

$$V_y = \frac{I_x B_z}{ne\Delta z}$$
$$V_y = \frac{(1\,\mathrm{A})\,(1\,\mathrm{T})}{\left(10^{15}\,\mathrm{cm}^{-3}\right)\left(1.612 \cdot 10^{-19}\,\mathrm{C}\right)(1\,\mathrm{mm})}$$

$$\boxed{V_y = 6.24\,\mathrm{V}} \qquad \text{(F2012 I 10.3)}$$

Spring 2012 Part II

Problem 1

Question

An electron in a hydrogen atom occupies a state:

$$|\psi\rangle = \sqrt{\frac{1}{3}}\,|3,1,0,+\rangle + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle$$

where the properly normalized states are specified by the quantum numbers $|n,\ell,m,\pm\rangle$ and the $\pm$ specifies whether the spin is up or down.

(a) What is the expectation value of the energy in terms of the ground state energy?

(b) If you meausred the expectation values of the orbital momentum squared $\left\langle L^2 \right\rangle$, the square of the spin $\left\langle S^2 \right\rangle$, and their z-components $\langle L_z\rangle$ and $\langle S_z\rangle$, what would be the result?

(c) Show that if you measure the position of the electron, the probability density for finding it an an angle specified by θ and ϕ integrated over all values of r is independent of θ and ϕ. Note, for this part you will need $Y_1^0 = \sqrt{3/4\pi}\cos\theta$ and $Y_1^1 = -\sqrt{3/8\pi}\sin\theta\exp(i\phi)$. You do *not*, however, need to know the radial functions, only that they are properly normalized and orthogonal to each other.

(d) List all additional possible states that are degenerate with the first state in the linear combination above. Note: this part can be done even if you have not answered the previous parts.

Assume now that the state $|\psi\rangle$, given above, is the initial state of an electron in a hydrogen atom.

(e) Write down the elctron's state as a function of time for all $t > 0$.

(f) go through the results you obtained in parts (a) through (c) and determine which of them are time independent.

Answer

(a) Calculate the energy by sandwhiching the Hamiltonian between the wavefunction:

$$\begin{aligned}
\langle E\rangle &= \langle\psi|H|\psi\rangle \\
&= \left(\sqrt{\frac{1}{3}}\,\langle 3,1,0,+| + \sqrt{\frac{2}{3}}\,\langle 2,1,1,-|\right) H \left(\sqrt{\frac{1}{3}}\,|3,1,0,+\rangle + \sqrt{\frac{2}{3}}\,|2,1,1,-\rangle\right) \\
&= \frac{1}{3}\,\langle 3,1,0,+|H|3,1,0,+\rangle + \frac{\sqrt{2}}{3}\,\langle 2,1,1,-|H|3,1,0,+\rangle \\
&\quad + \frac{\sqrt{2}}{3}\,\langle 3,1,0,+|H|2,1,1,-\rangle + \frac{2}{3}\,\langle 2,1,1,-|H|2,1,1,-\rangle
\end{aligned}$$

For every term, the wavefunctions are eigenstates of the Hamiltonian, so we extract the appropriate energy term from every bra-ket sandwhich. Then the middle two terms integrate to zero since states with different n are orthogonal while the first and last terms integrate to unity since they are properly normalized.

$$\langle E\rangle = \frac{1}{3}E_3 + 0 + 0 + \frac{2}{3}E_2$$

Each energy is related to the ground state energy by $E_n = E_0/n^2$, so

$$= \frac{1}{3}\frac{E_0}{9} + \frac{2}{3}\frac{E_0}{4}$$

$$\boxed{\langle E\rangle = \frac{11}{54}E_0 \approx -2.77\,\mathrm{eV}} \tag{F2012 II 1.1}$$

(b) For each of the other expectation values, the process is very similar with an appropriate change for eigenvalues; specifically,

$$\begin{aligned}
L^2\left|n,\ell,m,\pm\right\rangle &= \ell\left(\ell+1\right)\hbar^2\left|n,\ell,m,\pm\right\rangle \\
S^2\left|n,\ell,m,\pm\right\rangle &= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\left|n,\ell,m,\pm\right\rangle \\
L_z\left|n,\ell,m,\pm\right\rangle &= \ell\hbar\left|n,\ell,m,\pm\right\rangle \\
S_z\left|n,\ell,m,\pm\right\rangle &= \pm\frac{1}{2}\hbar\left|n,\ell,m,\pm\right\rangle
\end{aligned}$$

The same restrictions that the middle terms integrate to zero because of orthogonality and the first and last terms integrate to unity still applies, so we can almost immediately conclude that

$$\boxed{\left\langle L^2\right\rangle = 2\hbar^2} \tag{F2012 II 1.2}$$

$$\boxed{\left\langle S^2\right\rangle = \frac{3\hbar^2}{4}} \tag{F2012 II 1.3}$$

$$\boxed{\left\langle L_z\right\rangle = \frac{2\hbar}{3}} \tag{F2012 II 1.4}$$

$$\boxed{\left\langle S_z\right\rangle = -\frac{\hbar}{6}} \tag{F2012 II 1.5}$$

(c) In the $\left|r,\theta,\phi\right\rangle$ basis,

$$\begin{aligned}
\left|3,1,0\right\rangle &= R_{3,1}\left(r\right)Y_1^0\left(\theta,\phi\right) = R_{31}\left(r\right)\sqrt{\frac{3}{4\pi}}\cos\theta \\
\left|2,1,1\right\rangle &= R_{2,1}\left(r\right)Y_1^1\left(\theta,\phi\right) = -R_{21}\left(r\right)\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta
\end{aligned}$$

This means that the probability density is

$$\begin{aligned}
\left\langle\psi|\psi\right\rangle &= \frac{1}{3}\left\langle 3,1,0|3,1,0\right\rangle + \frac{\sqrt{2}}{3}\left\langle 2,1,1|3,1,0\right\rangle + \frac{\sqrt{2}}{3}\left\langle 3,1,0|2,1,1\right\rangle + \frac{2}{3}\left\langle 2,1,1|2,1,1\right\rangle \\
&= \frac{1}{4\pi}\cos^2\theta R_{31}^2\left(r\right) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}\left(r\right)R_{31}\left(r\right) + \frac{1}{4\pi}\sin^2\theta R_{21}^2\left(r\right)
\end{aligned}$$

Integrating over r,

$$\begin{aligned}
\int_0^\infty \left\langle\psi|\psi\right\rangle\,\mathrm{d}r = \int_0^\infty &\frac{1}{4\pi}\cos^2\theta R_{31}^2\left(r\right) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right)R_{21}\left(r\right)R_{31}\left(r\right) \\
&+ \frac{1}{4\pi}\sin^2\theta R_{21}^2\left(r\right)\,\mathrm{d}r
\end{aligned}$$

Integrating over all r, we know that $R_{n\ell} R_{n'\ell'}$ are orthonormal, so again the first and last terms' R integrates to unity and the middle term integrates to zero.

$$= \frac{1}{4\pi} \left(\cos^2\theta + \sin^2\theta\right)$$

Therefore we find that the probability density is constant in θ and ϕ when integrated over all r.

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle \, \mathrm{d}r = \frac{1}{4\pi}} \tag{F2012 II 1.6}$$

(d) The states degenerate with the first term in ψ are all combinations of allowed ℓ, m, and $\pm$: n must remain at $n = 3$ since it is the n quantum number which determines the energy of the state. The angular momentum number ℓ has to be in the range $[0, n-1]$, so there are at least 3 cases.

$$|3, 0, m, \pm\rangle$$
$$|3, 1, m, \pm\rangle$$
$$|3, 2, m, \pm\rangle$$

Then for each ℓ, the projection m can take a range of values $m \in [-\ell, \ell]$ so using $\{..., -1, 0, -1, ...\}$ to denote a set of options,

$$|3, 0, m, \pm\rangle \to |3, 0, \{0\}, \pm\rangle \qquad \text{2 states}$$
$$|3, 1, m, \pm\rangle \to |3, 1, \{-1, 0, 1\}, \pm\rangle \qquad \text{6 states}$$
$$|3, 2, m, \pm\rangle \to |3, 2, \{-2, -1, 0, 1, 2\}, \pm\rangle \qquad \text{10 states}$$

$$\boxed{\text{In total, there are 18 degenerate states}}$$

(e) To get the time evolution, we simply use the fact that for each basis eigenstate, we can add the time evolution component

$$\exp\left(-\frac{iE_n t}{\hbar}\right)$$

to get (in terms of the ground state energy E_0)

$$\boxed{|\psi(t)\rangle = \sqrt{\frac{1}{3}}\, |3, 1, 0, +\rangle\, e^{-iE_0 t/9\hbar} + \sqrt{\frac{2}{3}}\, |2, 1, 1, -\rangle\, e^{-iE_0 t/4\hbar}} \tag{F2012 II 1.7}$$

(f) From Ehrenfest's Theorem, we can quickly find the answers to most of the question without worrying about the wavefunction. Ehrenfest's Theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle E\rangle = -\frac{i}{\hbar} \langle[\Omega, H]\rangle + \left\langle \frac{\partial\Omega}{\partial t} \right\rangle$$

None of the operators L^2, S^2, L_z, and S_z are explicit in time, so the second term on the right can be dropped. Then because each of these operators commute with the Hamiltonian, the first term on the right is also dropped. Therefore, the expectation values are constant in time, so

$$\boxed{\left\langle L^2 \right\rangle \quad \text{Time independent}} \tag{F2012 II 1.8}$$

$$\boxed{\left\langle S^2 \right\rangle \quad \text{Time independent}} \tag{F2012 II 1.9}$$

$$\boxed{\langle L_z \rangle \quad \text{Time independent}} \tag{F2012 II 1.10}$$

$$\boxed{\langle S_z \rangle \quad \text{Time independent}} \tag{F2012 II 1.11}$$

For the probabilty density, we return to the integral in part (c) and insert the appropriate exponential terms. The first and last terms' exponentials cancel each other out, leaving

$$\begin{aligned}\int_0^\infty \langle\psi|\psi\rangle \, \mathrm{d}r = \int_0^\infty & \frac{1}{4\pi}\cos^2\theta R_{31}^2(r) - \frac{1}{\pi}\sin\theta\cos\theta\left(e^{i\phi}+e^{-i\phi}\right) R_{21}(r) R_{31}(r) \\ & \cdot\left[\exp\left(\frac{i\left(E_2-E_3\right)t}{\hbar}\right)+\exp\left(-\frac{i\left(E_2-E_3\right)t}{\hbar}\right)\right] \\ & + \frac{1}{4\pi}\sin^2\theta R_{21}^2(r) \, \mathrm{d}r\end{aligned}$$

The integral is unaffected by the new time factors, though, so integrating over r, the middle term still goes to zero and we're left with the same result previously of $1/4\pi$, therefore

$$\boxed{\int_0^\infty \langle\psi|\psi\rangle \, \mathrm{d}r \quad \text{Time independent}} \tag{F2012 II 1.12}$$

Problem 5

Question

Coinsider N non-interacting, stationary particles, each with magnetic moment $\vec{\mu}$ at temperature T in a uniform external magnetic field $\vec{B}$. Their energy is $-\vec{\mu} \cdot \vec{B}$. Calculate the partition function Z, the internal energy, and magnetization for two distinct cases (a and b below):

(a) The mgnetic moment of each particle can be oriented only parallel or anti-parallel to the magnetic field.

(b) The magnetic moment of each particle can rotate freely.

(c) Show that, in both cases, the total magnetization $\vec{M}$ can be written as a derivative of the parition function.

(d) In each case, calculate the fluctuations of magnetization $\left\langle \left(\Delta \vec{u}\right)^2 \right\rangle$.

Question

(a) Begin by constructing the partition function for a single particle. Since there are only two energy states, the sum is simply over the two Boltzmann factors:

$$Z_1 = e^{\mu B/kT} + e^{-\mu B/kT}$$

This can be simplified using trigonometric identities to

$$Z_1 = 2\cosh\left(\frac{\mu B}{kT}\right)$$

For fixed site particles, the partition function for N particles is simply $Z = Z^N$, so

$$\boxed{Z = 2^N \cosh^N\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.1}$$

The total energy can be calculated either by finding the expectation energy per particle $\langle \varepsilon \rangle$ and multiplying by N using the Boltzmann factors directly, or by using the thermodynamic identity

$$U = kT^2 \frac{\partial \ln Z}{\partial T}$$

Doing so,

$$U = kT^2 \frac{N}{2\cosh\left(\frac{\mu B}{kT}\right)} \cdot 2\sinh\left(\frac{\mu B}{kT}\right) \cdot \left(-\frac{\mu B}{kT^2}\right)$$

$$\boxed{U = -N\mu B \tanh\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.2}$$

To find the Magnetization, we use the Boltzmann factors directly since we don't know a thermodynamic relation. Let

$$\begin{aligned}
\langle m \rangle &= \sum_\mu \mu \frac{e^{-\varepsilon_\mu/kT}}{Z_1} \\
&= \frac{1}{Z_1}\left(-\mu e^{\mu B/kT} + \mu e^{-\mu B/kT}\right) \\
&= -\mu \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\cosh\left(\frac{\mu B}{kT}\right)}
\end{aligned}$$

So knowing that $\langle M \rangle = N \langle m \rangle$,

$$\boxed{\langle M \rangle = -N\mu \tanh\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.3}$$

(b) In the continuous case, the sum needs to be changed into an integral, remembering to keep Z unitless. This requires dividing by the volume of the energy state, which in this case is μB. (Justification: think of the energy vector $\vec{\mu} \cdot \vec{B} = \mu B \cos\theta$ on the unit circle of length μB. From geometry, the unitless $\mathrm{d}\theta$ is related to $\mathrm{d}\varepsilon$ by the factor μB.)

$$Z_1 = \int_{-\mu B}^{\mu B} e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B}$$

Letting $u = -\frac{\varepsilon}{kT}$,

$$\begin{aligned} Z_1 &= -\frac{kT}{\mu B} \int_{\mu B/kT}^{-\mu B/kT} e^u \,\mathrm{d}u \\ &= 2\frac{kT}{\mu B} \sinh\left(\frac{\mu B}{kT}\right) \end{aligned}$$

Therefore the partition function for all N particles is

$$\boxed{Z = \left(\frac{2kT}{\mu B}\right)^N \sinh^N\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.4}$$

The total energy is found in the same way as the previous case, giving

$$\boxed{U = NkT - N\mu B \coth\left(\frac{\mu B}{kT}\right)} \tag{F2012 II 5.5}$$

For the magnetization, we also calculate the expectation value from integrating the probability distribution, again making sure to keep the correct units. This time we work with the relevant projection of the magnetic moment $m = \mu \cos\theta$ so that when combined with the energy $\varepsilon = -\mu B \cos\theta$, the magnetization per particle in each state is $m = -\varepsilon/B$.

$$\begin{aligned} \langle m \rangle &= \int_{-\mu B}^{\mu B} -\frac{\varepsilon}{B} \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B} \\ &= \frac{1}{\mu Z_1} \left(\frac{kT}{B}\right)^2 \int_{-\mu B/kT}^{\mu B/kT} e^u \,\mathrm{d}u \\ &= \frac{kT}{B} \frac{2\sinh\left(\frac{\mu B}{kT}\right)}{2\sinh\left(\frac{\mu B}{kT}\right)} \\ &= \frac{kT}{B} \end{aligned}$$

The total magnetization $\langle M \rangle = N \langle m \rangle$ is

$$\boxed{\langle M \rangle = \frac{NkT}{B}} \tag{F2012 II 5.6}$$

Note that this is to be expected for the continuous case limit which corresponds to the classical limit. We'd expect the total energy to be related to the magnetization by $U = MB$. Rearranging the terms,

$$\begin{aligned} \langle M \rangle B &= NkT \\ U &= NkT \end{aligned}$$

which is the expected result from the equipartition theorem for a stationary particle with two rotational degrees of freedom.

(c) Proving the discrete case only differs from the continuous case proof by the obvious substitutions, so only the continuous case will be presented here. Begin by writing the first starting integral from the previous problem

$$\langle m \rangle = \int_{-\mu B}^{\mu B} m \frac{e^{-\varepsilon/kT}}{Z_1} \frac{\mathrm{d}\varepsilon}{\mu B}$$

The Z_1 can be pulled outside the integral since it is a constant. Then note that per our definition $\varepsilon = -mB$, it follows that

$$\frac{\partial \varepsilon}{\partial B} = -m$$

We identify the integral above to be a result of using the chain rule, so we undo that and get

$$\langle m \rangle = \frac{1}{Z_1} \int_{-\mu B}^{\mu B} \frac{\partial}{\partial B} \left(-e^{-\varepsilon/kT} \right) \frac{\mathrm{d}\varepsilon}{\mu B}$$

Changing the order of integration and differentiation,

$$= \frac{1}{Z_1} \frac{\partial}{\partial B} \left(\int_{-\mu B}^{\mu B} -e^{-\varepsilon/kT} \frac{\mathrm{d}\varepsilon}{\mu B} \right)$$

The term within the brackets is simply the definition of the partition function, so

$$\langle m \rangle = \frac{1}{Z_1} \frac{\partial Z_1}{\partial B} = \frac{\partial \ln Z_1}{\partial B}$$

To then get the total magnetization $\langle M \rangle$, we use several properties of differentiation and logarithms:

$$\begin{aligned} \langle M \rangle &= N \langle m \rangle \\ &= N \frac{\partial \ln Z_1}{\partial B} \\ &= \frac{\partial \left(N \ln Z_1 \right)}{\partial B} \\ &= \frac{\partial \ln \left(Z_1 \right)^N}{\partial B} \end{aligned}$$

Giving us the final expression

$$\boxed{\langle M \rangle = \frac{\partial \ln Z}{\partial B}} \qquad \text{(F2012 II 5.7)}$$

(d) Using the definition

$$\left\langle (\Delta\mu)^2 \right\rangle = \left\langle \mu^2 \right\rangle - \langle \mu \rangle^2$$

we already know $\langle \mu \rangle^2$ for both cases from the previous problems, so we must only calculate $\left\langle \mu^2 \right\rangle$.

Fall 2012 Part I

Problem 2

Question

Show that a particle in a one-dimensional infinite square well initially in a state $\Psi(x,0)$ will always return to that state after a time $T = 4ma^2/\pi\hbar$ where a is the width of the well.

Answer

Use the standard time independent Schödinger equation

$$\Psi(x,t) = \psi(x)\, e^{iEt/\hbar}$$

with associated differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi$$

For an infinite square well, the potential has the form

$$V(x) = \begin{cases} 0 & |x| < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$$

so that the only region to consider is $-\frac{a}{2} < x < \frac{a}{2}$. In this region, the differential equation takes the form of a harmonic oscillator

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

leading to solutions

$$\begin{aligned} \psi(x) &= A\cos kx + B\sin kx \\ \text{where } k^2 &= \frac{2mE}{\hbar^2} \end{aligned}$$

The boundary conditions $\psi\left(-\frac{a}{2}\right) = 0$ and $\psi\left(\frac{a}{2}\right) = 0$ impose

$$\begin{aligned} \psi\left(-\frac{a}{2}\right) &= 0 = A\cos\frac{ka}{2} - B\sin\frac{ka}{2} \\ \psi\left(\frac{a}{2}\right) &= 0 = A\cos\frac{ka}{2} + B\sin\frac{ka}{2} \end{aligned}$$

so that $B = 0$ and

$$\begin{aligned} 0 &= 2A\cos\frac{ka}{2} \\ \frac{(2n+1)\pi}{2} &= \frac{ka}{2} \\ k &= \frac{(2n+1)\pi}{a} \end{aligned}$$

We already had a relation for k defined, so substitute and solve for the energies E_n.

$$\begin{aligned}\frac{(2n+1)^2\,\pi^2}{a^2} &= \frac{2mE}{\hbar^2} \\ E_n &= \frac{(2n+1)^2\,\pi^2\hbar^2}{2ma^2}\end{aligned}$$

Then considering $\Psi\,(x,t)$, the complex exponential is periodic in time with period

$$T_n = \frac{2\pi\hbar}{E}$$

where $n = 0$ will be the case with the longest periodicity, so

$$\begin{aligned}T &= \frac{2\pi\hbar \cdot 2ma^2}{\pi^2\hbar^2} \\ &= \frac{4ma^2}{\pi\hbar}\end{aligned}$$

Therefore, the function is periodic in time with a periodicity

$$\boxed{T = \frac{4ma^2}{\pi\hbar}} \tag{F2012 I 2.1}$$

Problem 4

Question

A photon collides with a stationary electron. If the photon scatters at an angle θ, show that the resulting wavelength λ' is given in terms of the original wavelength λ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron.

Answer

Start by considering conservation of momentum for the system. The initial values are

$$\begin{aligned} p_{\gamma x} &= \frac{h}{\lambda} & p'_{\gamma x} &= \frac{h}{\lambda'}\cos\theta \\ p_{\gamma y} &= 0 & p'_{\gamma y} &= \frac{h}{\lambda'}\sin\theta \\ p_{ex} &= 0 & p'_{ex} &= ? \\ p_{ey} &= 0 & p'_{ey} &= ? \end{aligned}$$

and considering each component in turn:

$$\begin{aligned} \frac{h}{\lambda} + 0 &= \frac{h}{\lambda'}\cos\theta + p'_{ex} & 0 &= \frac{h}{\lambda'}\sin\theta + p'_{ey} \\ p'_{ex} &= \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta & p'_{ey} &= -\frac{h}{\lambda'}\sin\theta \end{aligned}$$

The total momentum of the electron is then

$$\begin{aligned} p_e^2 &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta\right)^2 + \left(-\frac{h}{\lambda'}\sin\theta\right)^2 \\ &= \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta + \frac{h^2}{\lambda'^2}\cos^2\theta + \frac{h^2}{\lambda'^2}\sin^2\theta \\ p_e^2 &= h^2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{\lambda\lambda'}\cos\theta \end{aligned} \tag{F2012 I 4.1}$$

Then consider energy conservation, with initial values

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} & E'_\gamma &= \frac{hc}{\lambda'} \\ E_e &= mc^2 & E'_e &= \frac{p_e'^2}{2m} + mc^2 \end{aligned}$$

leading to the equation

$$
\begin{aligned}
\frac{hc}{\lambda} + mc^2 &= \frac{hc}{\lambda'} + \frac{p_e'^2}{2m} + mc^2 \\
\frac{hc}{\lambda} &= \frac{hc}{\lambda'} + \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{hc}{\lambda} - \frac{hc}{\lambda'} &= \frac{h^2}{2m}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2}{2m\lambda\lambda'}\cos\theta \\
\frac{\lambda' - \lambda}{\lambda\lambda'} &= \frac{h}{2mc}\frac{\lambda'^2 + \lambda^2}{\lambda^2\lambda'^2} - \frac{h}{mc\lambda\lambda'}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{\left(\lambda' - \lambda\right)^2 + 2\lambda\lambda'}{\lambda\lambda'}\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\left(\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} + 2\right) - \frac{h}{mc}\cos\theta \\
\lambda' - \lambda &= \frac{h}{2mc}\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} + \frac{h}{mc}(1 - \cos\theta)
\end{aligned}
$$

The difference in the wavelengths is small, so

$$
\frac{\left(\lambda' - \lambda\right)^2}{\lambda\lambda'} \approx 0
$$

leading to the final Compton scattering equation

$$
\boxed{\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)} \tag{F2012 I 4.2}
$$