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## Fall 2000 Part I

## Problem 8

## Question

Estimate (a) the average speed (in m/s) and (b) the mean free path (in m) of a nitrogen molecule in this room.

## Answer

- (a) We relate the kinetic energy of an  $N_2$  molecule with the thermal energy by the equipartition theorem. Since there are 3 translational degrees of freedom,

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$

$$v = \sqrt{\frac{3k_B T}{m}}$$

The mass of the molecule is twice that of a single nitrogen atom which is itself about 14 proton masses. Therefore

$$v \approx \sqrt{\frac{3k_B T}{28m_p}} \approx 515 \frac{\text{m}}{\text{s}} \quad (\text{F2000 I 8.1})$$

- (b) Two particles collide if they come within  $2r_0$  of each other where  $r_0$  is the typical radius of the particle. For diatomic nitrogen, we assume  $r_0 \approx 2a_0$  where  $a_0$  is the Bohr radius. Then in the time  $\tau$  that the particle is moving at velocity  $\langle v \rangle$ , the particle can collide with any other particle within the swept-out volume

$$\mathcal{V} = \pi (2r_0)^2 \cdot \langle v \rangle \tau$$

Since there are  $n$  particles per unit volume, there are  $\mathcal{N}$  atoms to collide with:

$$\mathcal{N} = n\mathcal{V} = 4\pi n r_0^2 \langle v \rangle \tau$$

On average then, there are  $\mathcal{N}$  collisions per length  $\langle v \rangle \tau$  traversed, or in its reciprocal form, the mean free path  $\lambda$  is

$$\lambda = \frac{1}{4\pi n r_0^2}$$

To estimate the particle density, consider the ideal gas law  $PV = Nk_B T$ . We can assume atmospheric pressure at room temperature, so the density is

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Putting it all together,

$$\lambda \approx \frac{k_B T}{4\pi r_0^2 P} \approx 2.90 \cdot 10^{-7} \text{ m} \quad (\text{F2000 I 8.2})$$

## Spring 2000 Part I

### Problem 1

#### Question

A satellite of mass  $m = 500 \text{ kg}$  is in a circular orbit at an altitude  $h = 150 \text{ km}$  above the Earth's surface. As a result of air friction, the satellite's orbit degrades. Protected by a heat shield, the satellite eventually impacts with a velocity of  $2 \text{ km/s}$ . How much energy (in Joules) was released as heat in the process?

#### Answer

The solution method will be a simple energy balance equation. In its initial state, the satellite had gravitational potential energy that contributed to its energy equal to

$$V_0 = \frac{GM_E m}{R_E + h}$$

where  $M_E$  and  $R_E$  are the mass and radius of the Earth. The kinetic energy can be determined by making use of simple circle relations. The centripetal force must be provided by the gravitation force, so

$$\frac{v_0^2}{R_E + h} = \frac{GM_E}{(R_E + h)^2}$$

$$v_0 = \sqrt{\frac{GM_E}{R_E + h}}$$

making the initial kinetic energy

$$T_0 = \frac{GM_E m}{2(R_E + h)}$$

In its final state, the satellite consists of its final given velocity's kinetic energy and more gravitational potential energy (with respect to the center of the Earth). They are given by

$$T_f = \frac{1}{2}mv_f^2$$

$$V_f = \frac{GM_E m}{R_E}$$

By conservation of energy, any energy difference must result from the loss of energy between the initial and final states, so the lost energy  $E_{\text{loss}}$  is

$$E_{\text{loss}} = T_0 + V_0 - T_f - V_f$$

$$= \frac{GM_E m}{R_E(R_E + h)}(R_E - 2h) - \frac{1}{2}mv_f^2$$

Plugging in all given quantities and constants, we have that

$$E_{\text{loss}} = 2.816 \cdot 10^{10} \text{ J} = 28.16 \text{ GJ}$$

(F2000 I 1.1)

## Problem 2

### Question

What is the velocity of recoil of an  $^{57}\text{Fe}$  nucleus that emits a 100 keV photon, both in units of the speed of light in vacuum and in meters per second.

### Answer

In the initial state observed from the iron atom's rest frame, the momentum is zero. After the emission, the photon has a momentum of  $p_\gamma = E_\gamma/c$  and consequently, the nucleus must recoil with momentum  $p_{Fe} = -p_\gamma$ . Knowing that the energy is non-relativistic, then

$$m_{Fe}v_{Fe} = \frac{E_\gamma}{c}$$

$$v_{Fe} = \frac{E_\gamma}{75m_p c}$$

where we've approximate the mass of the iron atom by a multiple of the proton mass. Plugging in the numbers

$$v_{Fe} = 561.4 \frac{\text{m}}{\text{s}} \quad (\text{F2000 I 2.1})$$

or in units of  $c$

$$v_{Fe} = 1.87 \cdot 10^{-6} c \quad (\text{F2000 I 2.2})$$

**Problem 4****Question**

Find the magnetic flux through a square loop of side  $a$  due to current  $I$  in a long straight wire. The geometry is as follows: the wire is coplanar with the loop and runs parallel to the loop's closest side, at a distance  $b$  away. Write your result as a formula in SI units.

**Answer**

By Ampère's law in integral form, the magnetic field at a radial distance  $r$  away from the wire is given by

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

The flux is then the total magnetic field through the loop. Integrating by lines of length  $a$ ,

$$\Phi = a \int_b^{a+b} \frac{\mu_0 I}{2\pi r} dr$$

$$\Phi = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{a+b}{a}\right)$$

## Problem 6

### Question

By actually evaluating the integral, show that

$$\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$$

### Answer

Note that the integrand, so start by changing the limits of integration

$$\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$$

Then expand the cosine into its complex exponential definition

$$\begin{aligned} &= \frac{1}{4} \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx + \int_{-\infty}^{\infty} \frac{e^{-ix}}{1+x^2} dx \right) \\ &= \frac{1}{4} \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{(x+i)(x-i)} dx + \int_{-\infty}^{\infty} \frac{e^{-ix}}{(x+i)(x-i)} dx \right) \end{aligned}$$

Complex contour integration lets us evaluate the integrals as limits of the coefficient of a pole as it approaches the pole, so

$$= \frac{1}{4} \left( 2\pi i \cdot \lim_{z \rightarrow i} \left( \frac{e^{iz}}{z+i} \right) - 2\pi i \cdot \lim_{z \rightarrow -i} \left( \frac{e^{-iz}}{z-i} \right) \right)$$

After simplifying,

$$\boxed{\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}}$$

(F2000 I 6.1)

**Problem 7****Question**

The drag force on a very high speed object of area  $A$ , passing through a gas of density  $\rho$  at a velocity  $v$  is expected to be of the form

$$\text{Force} \sim A^r \rho^s v^t$$

Determine the value of the exponents  $r$ ,  $s$ , and  $t$ .

**Answer**

Force needs to have a unit of inverse time squared, therefore  $v$  as the only variable with a time unit sets  $t = 2$ . Similarly,  $\rho$  is the only one with a mass term, so we also immediately know that  $s = 1$ . That leaves

$$\left[ \frac{\text{mass} \cdot \text{distance}}{\text{time}^2} \right] = \left[ \frac{\text{mass}}{\text{distance} \cdot \text{time}^2} \right] [\text{distance}]^r$$

Therefore to have the two side have compatible units,  $r = 2$ .

$$\boxed{r = 2, \quad s = 1, \quad t = 2}$$



**Problem 9****Question**

For waves in shallow water, the relation between frequency  $\nu$  and wavelength  $\lambda$  is

$$\nu = \left( \frac{2\pi T}{\rho \lambda^3} \right)^{1/2}$$

where  $\rho$  and  $T$  are the density and surface tension of water. What is the group velocity of these waves?

**Answer**

Transforming relation given to use the angular frequency  $\omega = 2\pi\nu$  and wave number  $k = 2\pi/\lambda$ ,

$$\omega = \sqrt{\frac{k^3 T}{\rho}}$$

Then the group velocity is simply the partial derivative with respect to  $k$ :

$$v_g = \frac{\partial \omega}{\partial k} = \frac{3}{2} \sqrt{\frac{kT}{\rho}}$$

Putting this back into the form which involves only  $\nu$  and  $\lambda$ , we arrive at the answer

$$v_g = \sqrt{\frac{2\pi T}{\rho \lambda}}$$

(F2000 I 9.1)

## Problem 10

### Question

Find the eigenvalues and corresponding eigenvectors (which need *not* be normalized) of the following matrix:

$$M = \begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & -1 \end{bmatrix}$$

### Answer

The eigenvalue equation is found from the standard procedure of adding a parameter  $\lambda$  into the matrix and taking the determinant equal to zero:

$$\det \begin{bmatrix} 1-\lambda & 0 & -i \\ 0 & 2-\lambda & 0 \\ i & 0 & -1-\lambda \end{bmatrix} = 0 = (1-\lambda)(2-\lambda)(-1-\lambda) - i(-i)(2-\lambda)$$

$$0 = -(1-\lambda)(1+\lambda)(2-\lambda) - (2-\lambda)$$

$$0 = -(2-\lambda)[(1-\lambda)(1+\lambda) + 1]$$

Solving the two equations  $0 = (2-\lambda)$  and  $(1-\lambda)(1+\lambda) = -1$  gives the three eigenvalues

$$\lambda = \{-\sqrt{2}, \sqrt{2}, 2\} \quad (\text{F2000 I 10.1})$$

Starting with the eigenvalue  $\lambda = 2$ , we solve for its eigenvector using the usual Gaussian elimination approach:

$$\det \begin{bmatrix} -1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -3 \end{bmatrix}$$

The second variable is completely unconstrained, so we can set that component of the vector to 1. The first and last rows are incompatible, so that means that both the first and third variables must be zero. This gives us the eigenvector

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In a similar manner for the eigenvalues  $\lambda = \pm\sqrt{2}$ , we perform Gaussian elimination. When all free parameters have been set arbitrarily to 1 and other constraints considered, we end up with the three eigen vectors:

$$\lambda = -\sqrt{2} \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ -i(1+\sqrt{2}) \end{bmatrix} \quad (\text{F2000 I 10.2})$$

$$\lambda = \sqrt{2} \rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ -i(1-\sqrt{2}) \end{bmatrix} \quad (\text{F2000 I 10.3})$$

$$\lambda = 2 \rightarrow v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{F2000 I 10.4})$$

## Spring 2002 Part I

### Problem 1

#### Question

A 1000 kg automobile has ground clearance of 18 cm but when loaded with an extra 500 kg from its 4 passengers it only clears the ground by 12 cm. The car's shock absorbers are ineffective. At what speed (in miles per hour) will the car bounce in resonance when it travels along a smooth road containing transverse tar patches every 15 m? Assume that the front and rear suspensions have the same bouncing frequency.

#### Answer

To find the natural resonant frequency of the vehicle, we use the two data points about its clearance to obtain the spring constant  $k$ . We know that a constant force on a spring system does not change the dynamics, so it's only the change in mass and distance which are relevant. Therefore by simple equilibrium requirements,

$$k\Delta L = mg$$

$$k = \frac{mg}{\Delta L}$$

where  $m$  and  $M$  are the masses of the passengers and empty car respectively and  $\Delta L$  is the difference in the clearance between the unloaded and loaded car.

Next, we know that the resonant frequency of a simple harmonic oscillator is given by  $\omega = \sqrt{k/m_{\text{tot}}}$ , so

$$\omega = \sqrt{\frac{mg}{(M+m)\Delta L}}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{mg}{(M+m)\Delta L}}$$

where the second line has converted from angular to linear frequency. We then simply relate that to how often the car hits a tar patch and solve for the velocity. Letting  $v$  be the velocity of the car and  $d$  be the separation distance between tar patches,

$$\frac{v}{d} = f$$

$$v = \frac{d}{2\pi} \sqrt{\frac{mg}{(M+m)\Delta L}}$$

Plugging in the numbers,

$v = 17.62 \frac{\text{m}}{\text{s}} = 39.42 \text{ mph}$
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(F2002 I 1.1)

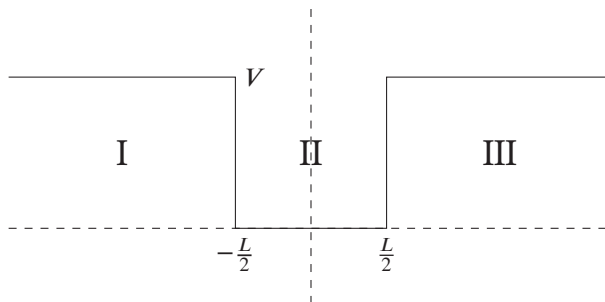
## Problem 2

### Question

Suppose a particle of mass  $m$  moves in a 1-dimensional square potential well of width  $L$  and depth  $V$ . What is the minimum depth of the well such that the particle will have two bound states?

### Answer

Divide the problem into three regions according to the diagram below:



From the Schrödinger equation, we know that there will be two different types of solutions. Within region II, no potential exists, so the solution has the form

$$\psi = A \cos(kx) + B \sin(kx)$$

$$k^2 = \frac{2mE}{\hbar^2}$$

For regions I and III, we assume that the energy  $E < V$  since we are only interested in the bound solutions and not ones where the particle is free. This gives the following exponential solution form:

$$\psi = Ae^{\kappa x} + Be^{-\kappa x}$$

$$\kappa^2 = \frac{2m(V - E)}{\hbar^2}$$

We immediately know that for regions I and III, the wavefunction must go to zero at  $\pm\infty$ , so that immediately removes two terms in the solution. This leaves us with the complete set of solutions

$$\psi_I = Ae^{\kappa x}$$

$$\psi_{II} = B \cos(kx) + C \sin(kx)$$

$$\psi_{III} = De^{-\kappa x}$$

We make a further simplification by noting that in this situation where there is a symmetric potential, the solution can also be divided into symmetric and antisymmetric solutions, corresponding to keeping either the sin or the cos solutions in region II. To solve the problem, we want to know the threshold potential for  $V$  that will maintain a second bound state. The first bound state is the symmetric case (which heuristically is true by its virtue of having only a single “hump” within region II), so the first excited state/second state is the antisymmetric case (heuristically expected since the sine solution will have two “humps” within region II). This lets us simplify the problem and immediately set  $B = 0$  to isolate just the antisymmetric solution.

To continue, we make use of the fact that the wavefunction must be continuous and differentiable continuous at the boundaries between regions I-II and II-III. This means we find that:

$$Ae^{-\kappa L/2} = B \sin\left(-\frac{kL}{2}\right) \qquad B \sin\left(\frac{kL}{2}\right) = De^{-\kappa L/2}$$

$$A\kappa e^{-\kappa L/2} = Bk \cos\left(-\frac{kL}{2}\right) \qquad Bk \cos\left(\frac{kL}{2}\right) = -D\kappa e^{-\kappa L/2}$$

Dividing the lower equation by the upper equation in both cases leads to the condition

$$\kappa = -k \cot\left(\frac{kL}{2}\right)$$

If we perform a variable substitution where  $v = kL/2$  and  $u = \kappa L/2$ , the equation above takes the slightly simpler form

$$u = -v \cot v \quad (\text{F2002 I 2.1})$$

This transformation comes in more useful when we consider the energy equations that defined  $k$  and  $\kappa$ . Substituting into the equation for  $k$  and solving for  $E$  we have that

$$E = \frac{2v^2 \hbar^2}{mL^2}$$

And doing the same for  $\kappa$ ,

$$V - E = \frac{2u^2 \hbar^2}{mL^2}$$

which combined gives the equation of a circle:

$$u^2 + v^2 = \frac{mL^2}{2\hbar^2} V$$

Therefore, the only valid solutions occur when both (F2002 I 2.1) and (F2002 I 2.2) are satisfied. The minimum value for a given cotangent line occurs at the roots which occur when  $v = (2n + 1) \pi/2$  for any integer  $n$ . The first excited state for  $n = 1$  then occurs at  $v = \pi/2$  and consequently  $u = 0$ . Substituting this into the equation above and solving for  $V$ , we find that the threshold energy for a second bound state corresponds to a potential depth of

$$\boxed{V = \frac{\pi^2 \hbar^2}{2mL^2}} \quad (\text{F2002 I 2.2})$$

### Problem 3

#### Question

The cross-section for collisions between helium atoms is about  $10^{-16} \text{ cm}^2$ . Estimate the mean free path of helium atoms in helium gas at atmospheric pressure and temperature.

#### Answer

Consider the path traced out by a helium atom as it travels a path length  $L$ , colliding with other helium atoms along the way. Given that the cross section of helium is  $\sigma$ , then we can estimate the volume that contains probable interactions with our atom of interest as  $\mathcal{V} = \sigma L$ . To get the number of interactions, we make use of the fact that we're treating the gas as an ideal gas. From the ideal gas law,

$$PV = NkT$$

so that solving for the number density

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

Combining the density with the volume, we get the number of other [point particle] helium atoms that are contained within the given helium atom's interaction volume. If we then assume that the atom interacts with all other atoms within the volume, and that the collisions are spaced out equally in time, we just have to normalize the value by the trajectory's path length to get an estimate of the mean free path of helium in a helium gas:

$$\lambda = \frac{n\mathcal{V}}{L} = \frac{\sigma P}{k_B T}$$

Plugging in  $\sigma = 10^{-16} \text{ cm}^2$ ,  $P = 1.013 \cdot 10^5 \text{ Pa}$ ,  $k_B = 1.38 \cdot 10^{-23} \text{ J/K}$ , and  $T = 298 \text{ K}$ , we get

$$\lambda = 4.06 \mu\text{m}$$

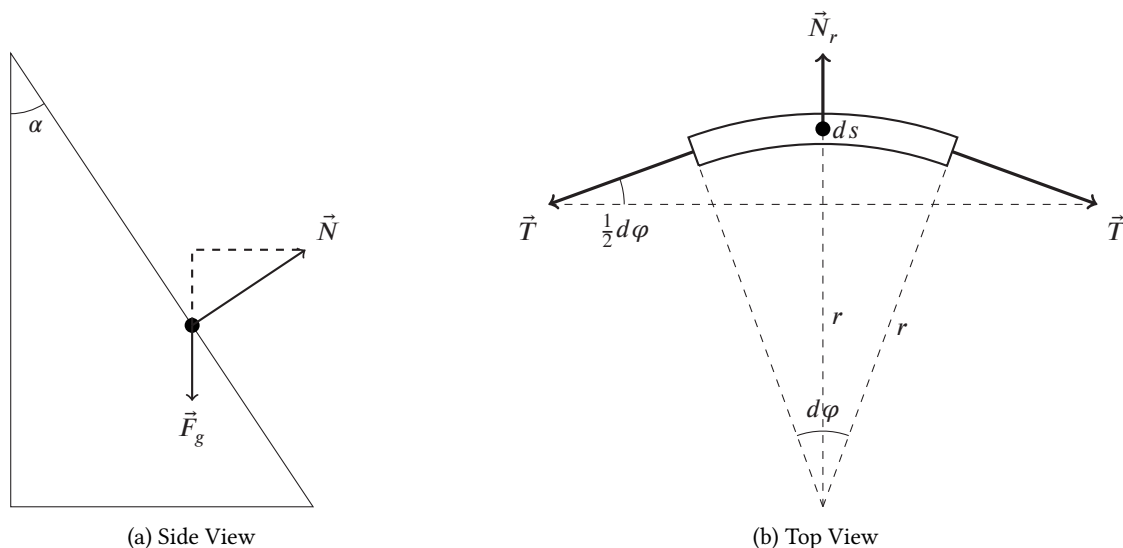
(F2002 I 3.1)

# Problem 6

## Question

A single closed loop of chain with mass  $m$  and length  $L$  rests horizontally on a smooth frictionless con with half-angle  $\alpha$ . What is the tension in the chain?

## Answer



Consider just a small element of the chain of arc length  $ds$ . It will have a corresponding mass  $dm = \lambda ds$  where  $\lambda = m/L$ . Knowing that it's a statics problem, we can easily determine the normal force by balancing the vertical component with that of gravity.

$$F_g = N \sin \alpha$$

$$N = \frac{\lambda g ds}{\sin \alpha}$$

This leaves the horizontal component of the normal force to be balanced with the tension within the chain.

Now switching to the top view, we consider the short chain segment  $ds$ , shown above with an exaggerated curvature. We note that the radial part of the normal force must be opposed by the sum of the radial components of the two tensions  $T$  acting on the end of the chain segment. By geometry, we know that the angle with respect to the midpoint's tangent is one half the differential angle change  $d\phi = ds/r$ . This means we balance the forces as

$$2T \sin\left(\frac{1}{2}d\phi\right) = N \cos \alpha$$

$$2T \sin\left(\frac{1}{2}d\phi\right) = \lambda g ds \cot \alpha$$

By the small angle approximation,  $\sin\left(\frac{1}{2}d\phi\right) \approx \frac{1}{2}d\phi$ , so after substituting for the fact that  $dr = L/2\pi$  and  $\lambda = M/L$ ,

$$T = \frac{Mg}{2\pi} \cot \alpha \quad (\text{F2002 I 6.1})$$

**Problem 7****Question**

A laser beam (photon energy 1 eV) collides head-on with a 50 GeV ultra-relativistic electron beam. What is the energy of the photons reflected backwards in the collision?

**Answer**

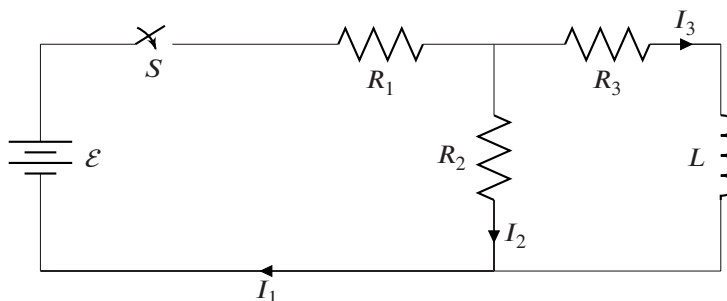


# Problem 8

## Question

In the figure below,  $\mathcal{E} = 100\text{ V}$ ,  $R_1 = 5\ \Omega$ ,  $R_2 = 10\ \Omega$ ,  $R_3 = 15\ \Omega$ , and  $L = 1.0\text{ H}$ . Find the values of the currents  $I_1$  and  $I_2$

- immediately after the switch  $S$  is closed,
- a long time later,
- immediately after switch  $S$  is opened again,
- and then how long must you wait, after the switch is opened, before  $I_2$  falls by a factor of  $e$ ?



## Answer

Start by applying Kirchoff's rules to the circuit: current is conserved and the voltage changes must sum to zero around each loop, so

$$I_1 = I_2 + I_3 \quad (\text{F2002 I 8.1})$$

$$0 = \mathcal{E} - I_1 R_1 - I_2 R_2 \quad (\text{F2002 I 8.2})$$

$$0 = -I_3 R_3 - L \frac{dI_3}{dt} + I_2 R_2 \quad (\text{F2002 I 8.3})$$

Since only (F2002 I 8.3) has a term involving a time derivative, we choose to first solve for the current  $I_3$ . By solving for  $I_2 R_2$  in (F2002 I 8.2) and substituting, we eliminate  $I_2$  and have

$$0 = -I_3 R_3 - L \frac{dI_3}{dt} + \mathcal{E} - I_1 R_1 \quad (\text{F2002 I 8.4})$$

Furthermore, by also substituting the value of  $I_2$  into (F2002 I 8.1):

$$I_1 = \frac{\mathcal{E}}{R_1 + R_2} + \frac{R_2}{R_1 + R_2} I_3 \quad (\text{F2002 I 8.5})$$

Then by combining (F2002 I 8.4) and (F2002 I 8.5), we can produce a differential equation for  $I_3$ :

$$\begin{aligned} -I_3 R_3 - L \frac{dI_3}{dt} + \mathcal{E} - \frac{R_1}{R_1 + R_2} \mathcal{E} - \frac{R_1 R_2}{R_1 + R_2} I_3 &= 0 \\ -\underbrace{\frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 + R_2}}_{R'} I_3 &= L \frac{dI_3}{dt} - \frac{R_2}{R_1 + R_2} \mathcal{E} \\ \frac{dI_3}{dt} &= -\frac{R'}{L} I_3 + \frac{1}{L} \frac{R_2}{R_1 + R_2} \mathcal{E} \end{aligned} \quad (\text{F2002 I 8.6})$$

Considering just the homogenous part, we easily solve it to find the standard exponential solution

$$I_{3h}(t) = I_{30}e^{-R't/L}$$

And using the ansatz  $I_{3p}(t) = At + B$  for the inhomogenous part,

$$A = -\frac{R'}{L}At - \frac{R'}{L}B + \frac{1}{L} \frac{R_2}{R_1 + R_2} \mathcal{E}$$

$$A = 0$$

$$B = \frac{1}{R'} \frac{R_2}{R_1 + R_2} \mathcal{E}$$

At  $t = 0$ , the inductor has no current passing through it, so when the switch is closed, the current must remain continuous. This gives us the initial condition necessary to solve for the unknown  $I_{30}$ , and after doing so and simplifying, the total solution is

$$I_3(t) = \frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2} \left(1 - e^{-R't/L}\right) \quad (\text{F2002 I 8.7})$$

Then by substituting this solution back into (F2002 I 8.5) we get the solution for  $I_1$ :

$$I_1(t) = \frac{\mathcal{E}}{R_1 + R_2} \left[1 + \frac{1}{R'} \frac{R_2^2}{R_1 + R_2} \left(1 - e^{-R't/L}\right)\right] \quad (\text{F2002 I 8.8})$$

Finally, combining both inserting both solutions for  $I_1$  and  $I_3$  into (F2002 I 8.1), the solution for  $I_2$  is

$$I_2(t) = \frac{\mathcal{E}}{R_1 + R_2} \left[1 - \frac{1}{R'} \frac{R_1 R_2}{R_1 + R_2} \left(1 - e^{-R't/L}\right)\right] \quad (\text{F2002 I 8.9})$$

Plugging in all of the given values, we find that the currents at the instant the switch is closed are

$$\boxed{I_1(0) = 6.66 \text{ A} \quad S \text{ is closed}} \quad (\text{F2002 I 8.10})$$

$$\boxed{I_2(0) = 6.66 \text{ A} \quad S \text{ is closed}} \quad (\text{F2002 I 8.11})$$

For a long time later, we can let  $t \rightarrow \infty$  and find that

$$\boxed{I_1(\infty) = 9.09 \text{ A} \quad S \text{ is closed}} \quad (\text{F2002 I 8.12})$$

$$\boxed{I_2(\infty) = 5.54 \text{ A} \quad S \text{ is closed}} \quad (\text{F2002 I 8.13})$$

Right after the switch is opened, the left loop is taken out of the circuit, so we immediately know that the value of  $I_1$  is zero.

$$\boxed{I_1(0) = 0 \quad S \text{ is open}} \quad (\text{F2002 I 8.14})$$

For the right loop, we start by noting that the steady state current through the inductor will be needed. Taking the limit of (F2002 I 8.7), we have that the new initial condition is

$$I_3(0) = \frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2}$$

$I_2$  now is equal to  $-I_3$  since there is no other path for the current to traverse. This loop's differential equation is then

$$-(R_2 + R_3) I_3 - L \frac{dI_3}{dt} = 0$$

Solving for the exponential and using the initial condition above, the time solution is

$$I_3(t) = \frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2} e^{-(R_2 + R_3)t/L}$$

$$I_2(t) = -\frac{\mathcal{E}}{R'} \frac{R_2}{R_1 + R_2} e^{-(R_2 + R_3)t/L}$$

Therefore the current in  $I_2$  just after the switch is opened reverses direction

$$I_2(0) = -3.64 \text{ A} \quad (\text{F2002 I 8.15})$$

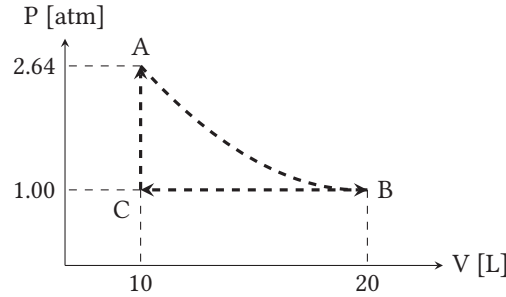
Then by simple exponential relations, we know that the time to decay by a factor of  $e$  is given by the reciprocal of the coefficient of  $t$ , so inserting the appropriate numbers

$$t_{\text{decay}} = 0.04 \text{ s} \quad (\text{F2002 I 8.16})$$

## Problem 9

### Question

An engine using 1 mol of an ideal diatomic gas performs the cycle  $A \rightarrow B \rightarrow C \rightarrow A$  as shown in the diagram below.  $A \rightarrow B$  is an adiabatic expansion,  $B \rightarrow C$  occurs at constant pressure, and  $C \rightarrow A$  takes place at constant volume. What is the efficiency of the cycle?



### Answer

Since we want to find the efficiency of the cycle, we only care about the heat exchanged during each stage of the cycle. Because the path  $A \rightarrow B$  is adiabatic, we immediately know that  $Q = 0$ . Then proceeding to look at the stage  $C \rightarrow A$ , we know that the work done during this cycle is identically zero since there is no area under the curve. That means we are left simply with the equation

$$dU = dQ$$

Because this is an ideal [diatomic] classical gas, we combine the equations

$$U = \frac{5}{2}nRT$$

and

$$PV = nRT$$

to get that the difference in energy across the path is

$$\begin{aligned} Q_{CA} = U &= \frac{5}{2}nR(T_A - T_C) \\ &= \frac{5}{2}V_1(P_2 - P_1) \end{aligned}$$

For the remaining stage  $B \rightarrow C$ , we use the full thermodynamic identity:

$$dU = dQ - P dV$$

The pressure  $P_1$  is constant, so both integration of  $dU$  and  $dV$  are simply the differences in each quantity. Again substituting for the temperature in  $U$  with the ideal gas law,

$$\begin{aligned} \frac{5}{2}nR(T_C - T_B) &= Q_{BC} - P_1(V_1 - V_2) \\ \frac{5}{2}P_1(V_1 - 2V_1) &= Q_{BC} + P_1(V_1 - 2V_1) \\ Q_{BC} &= -\frac{7}{2}P_1V_1 \end{aligned}$$

We've accounted for all the heat flow in the system.  $Q_{BC}$  is negative, so this is the heat flow out of the system, while  $Q_{CA}$  is positive and is the heat flow into the system. By definition then, the efficiency  $\eta$  of the system is

$$\begin{aligned}\eta &= 1 - \frac{Q_{out}}{Q_{in}} \\ &= 1 - \frac{\frac{7}{2}P_1V_1}{\frac{5}{2}V_1(P_2 - P_1)} \\ &= 1 - \frac{5}{7} \frac{P_1}{P_2 - P_1}\end{aligned}$$

Plugging in the given values, we find the efficiency to be

$$\boxed{\eta = 0.146 = 14.6 \%}$$

(F2002 I 9.1)

## Problem 10

### Question

A thin circular hoop rolls down an inclined plane under the influence of gravity. What minimum coefficient of friction is required to ensure that it rolls rather than slides?

### Answer

Begin first by finding the motion that describes the rolling without slipping state. We do this by solving the system's Lagrangian:

$$\mathcal{L} = \left( \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 \right) - (mgx \sin \alpha)$$

where  $x$  is the length along the ramp with  $x = 0$  at the bottom,  $I$  is the moment of inertia of the hoop,  $m$  is its mass,  $\theta$  is the angle of rotation of the hoop about its center, and  $\alpha$  is the angle of the incline plane. By noting that rolling without slipping requires that  $r\dot{\theta} = \dot{x}$ , we can reduce the problem to the single variable  $x$ . The result is the following differential equation, where  $I = mr^2$  has been substituted in:

$$\begin{aligned} 2m\ddot{x} &= -mg \sin \alpha \\ \ddot{x} &= -\frac{1}{2}g \sin \alpha \end{aligned}$$

Therefore we know the linear acceleration will be  $a = -\frac{1}{2}g \sin \alpha$  in the non-slipping case.

To find what coefficient of friction produces this motion, we consider the forces acting on the hoop with the coordinate system still oriented along and perpendicular to the plane. In the perpendicular direction, the normal force  $N$  is canceled by the perpendicular component of gravity, so

$$N = mg \cos \alpha$$

In the parallel direction, the frictional force and the parallel component of gravity must sum to give the requisite force, namely  $ma$ .

$$\begin{aligned} \mu N - mg \sin \alpha &= a = -\frac{1}{2}g \sin \alpha \\ \mu mg \cos \alpha &= \frac{1}{2}g \sin \alpha \\ \mu &= \frac{1}{2} \tan \alpha \end{aligned}$$

Therefore we find that the coefficient of friction must be equal to half of the tangent of the inclined plane's angle.

## Problem 11

### Question

A particle is confined within a cubical box with sides of length  $L$  and is initially in the ground state. If the length of one side of the box (along the  $x$ -direction) is abruptly increased to a length  $2L$ , what is the probability that the particle remains in the ground state?

### Answer

We start by recalling the solution for a particle in a box. In a 1D box with its left edge at the origin, the properly normalized wavefunction is given by

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

where  $L$  is the size of the box. The Cartesian extension into 3D is simple and is respectively for the  $L \times L \times L$  and  $2L \times L \times L$  boxes:

$$\begin{aligned}\psi(x, y, z) &= \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \\ \psi'(x, y, z) &= \frac{1}{\sqrt{2}} \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{2L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)\end{aligned}$$

To find the probability of remaining in the ground state, we simply must take the inner product of both wavefunctions in the ground state over an appropriate domain; this means that the initial, unexpanded box's wavefunction is 0 within the new region.

$$\begin{aligned}\mathcal{P} &= \langle \psi_{111} | \psi'_{111} \rangle \\ &= \frac{1}{\sqrt{2}} \left(\frac{2}{L}\right)^3 \left( \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{2L}\right) dx \right) \left( \int_0^L \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{\pi y}{L}\right) dy \right) \left( \int_0^L \sin\left(\frac{\pi z}{L}\right) \sin\left(\frac{\pi z}{L}\right) dz \right)\end{aligned}$$

The integrals over  $y$  and  $z$  are simple and simply evaluate to  $L/2$  as we'd expect from the normalization factor. To evaluate the integral over  $x$ , use the trigonometric identity  $\sin 2\theta = 2 \sin \theta \cos \theta$  and a change of variables with  $u = \sin(\pi x/2L)$  to arrive at the integral

$$\mathcal{P} = \frac{2\sqrt{2}}{L} \int_0^1 u^2 \cdot \frac{2L}{\pi} du$$

Evaluating this, we find the probability of remaining the ground state after the box is expanded suddenly to be

$$\mathcal{P} = \frac{4\sqrt{2}}{3\pi} \approx 0.15$$

(F2002 I 11.1)

## Problem 12

### Question

The frequency  $f$  of a deep water gravity wave (i.e. an ordinary ocean wave) is given by

$$f = \sqrt{\frac{1}{2\pi}} \rho^a g^b \lambda^c$$

where  $\rho$ ,  $g$ , and  $\lambda$  are the water density, gravitational acceleration, and wavelength of the wave, respectively. What are the values of the exponents  $a$ ,  $b$ , and  $c$ , and what is the ratio of the wave group velocity to phase velocity?

### Answer

We proceed by dimensional analysis. Immediately we know that  $a = 0$  since a frequency does not have a mass component, and neither  $g$  nor  $\lambda$  have a mass term to cancel the one in  $\rho$ . Furthermore,  $g$  is the only one with a time term, so its exponent must then be  $b = \frac{1}{2}$  in order to give  $f$  its  $[s^{-1}]$  unit. That leaves  $c = \frac{1}{2}$  in order to cancel the  $\sqrt{m}$  dimension left over from  $g$ .

$$f = \sqrt{\frac{g\lambda}{2\pi}} \quad \text{with} \quad a = 0, b = \frac{1}{2}, c = \frac{1}{2}$$

The phase velocity can be derived from the frequency given by noting that  $v_g = \omega/k$  together with  $k^{-1} = 2\pi\lambda$  and  $\omega = 2\pi f$ . Put together, this gives

$$v_p = \frac{1}{k} \sqrt{\frac{g}{k}}$$

The group velocity is given by  $v_g = d\omega/dk$ , so

$$v_g = -\frac{1}{2k} \sqrt{\frac{g}{k}}$$

Taking only the absolute values and finding the ratio

$$\frac{v_g}{v_p} = \frac{1}{2}$$

(F2002 I 12.1)



## Spring 2002 Part II

### Problem 2

#### Question

A zipper has  $N$  links; each link has a closed state with zero energy and an open state with energy  $\epsilon$ . We require, however, that the zipper can only unzip from the left end, and that the link number  $s$  can only open if all links to the left (i.e.  $1, 2, \dots, s-1$ ) are already open.

- Find an explicit expression for the partition function by doing the appropriate summation.
- In the limit  $\epsilon \gg k_B T$  find the average number of open links. This model is a very simplified model of the unwinding of two-stranded DNA molecules.

#### Answer

- Create the partition function by induction; start by assuming there is only a single link. Then the partition function is a simple two-state system:

$$Z_1 = e^0 + e^{-\epsilon/k_B T} = 1 + e^{-\epsilon/k_B T}$$

Adding a second link,

$$\begin{aligned} Z_2 &= \underbrace{e^{0+0}}_{\text{both closed}} + \underbrace{e^{(-\epsilon+0)/k_B T}}_{1 \text{ open, 1 closed}} + \underbrace{e^{(-\epsilon-\epsilon)/k_B T}}_{\text{both open}} \\ &= 1 + e^{-\epsilon/k_B T} + e^{-2\epsilon/k_B T} \end{aligned}$$

Following, for three links:

$$\begin{aligned} Z_3 &= \underbrace{e^{0+0+0}}_{\text{all closed}} + \underbrace{e^{(-\epsilon+0+0)/k_B T}}_{1 \text{ open, 2 closed}} + \underbrace{e^{(-\epsilon-\epsilon+0)/k_B T}}_{2 \text{ open, 1 closed}} + \underbrace{e^{(-\epsilon-\epsilon-\epsilon)/k_B T}}_{\text{all open}} \\ &= 1 + e^{-\epsilon/k_B T} + e^{-2\epsilon/k_B T} + e^{-3\epsilon/k_B T} \end{aligned}$$

By induction, we see that the maximum coefficient in the series of exponential factors is just the number of links, so by induction we conclude that

$$Z = \sum_{s=0}^N e^{-s\epsilon/k_B T}$$

Applying the results of a finite geometric series, the closed-form solution for the partition function of the links is

$$Z = \frac{1 - e^{-(N+1)\epsilon/k_B T}}{1 - e^{-\epsilon/k_B T}} \quad (\text{F2002 II 2.1})$$

- To get the average number of open links, we use the standard procedure for finding expectation values.

$$\langle s \rangle = \frac{1}{Z} \sum_{s=0}^N s e^{-s\epsilon/k_B T}$$

By making use of differentiation under the summation trick, we can find the closed-form solution:

$$\begin{aligned} \langle s \rangle &= \frac{1}{Z} \sum_{s=0}^N \frac{d}{d\left(\frac{\epsilon}{k_B T}\right)} \left[ -e^{-s\epsilon/k_B T} \right] \\ &= -\frac{1}{Z} \frac{\partial}{\partial\left(\frac{\epsilon}{k_B T}\right)} \sum_{s=0}^N e^{-s\epsilon/k_B T} \end{aligned}$$

Noting that the summation is the same as above,

$$\langle s \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \left( \frac{\epsilon}{k_B T} \right)}$$

First considering just the derivative part:

$$\frac{\partial Z}{\partial \left( \frac{\epsilon}{k_B T} \right)} = \frac{(N+1)e^{-(N+1)\epsilon/k_B T}}{1 - e^{-\epsilon/k_B T}} - \frac{1 - e^{-(N+1)\epsilon/k_B T}}{(1 - e^{-\epsilon/k_B T})^2} e^{-\epsilon/k_B T}$$

which when combined with the factor  $-1/Z$  simplifies to

$$\begin{aligned} -\frac{1}{Z} \frac{\partial Z}{\partial \left( \frac{\epsilon}{k_B T} \right)} &= -(N+1) \frac{e^{-(N+1)\epsilon/k_B T}}{1 - e^{-(N+1)\epsilon/k_B T}} + \frac{e^{-\epsilon/k_B T}}{1 - e^{-\epsilon/k_B T}} \\ -\frac{1}{Z} \frac{\partial Z}{\partial \left( \frac{\epsilon}{k_B T} \right)} &= \frac{1}{e^{\epsilon/k_B T} - 1} - \frac{N+1}{e^{(N+1)\epsilon/k_B T} - 1} \end{aligned}$$

Therefore the analytic solution is

$$\boxed{\langle s \rangle = \frac{1}{e^{\epsilon/k_B T} - 1} - \frac{N+1}{e^{(N+1)\epsilon/k_B T} - 1}} \quad (\text{F2002 II 2.2})$$

In the limit that  $\epsilon \gg k_B T$ , though, the exponentials in the denominator are very large in comparison to 1, so we ignore the unity factors and make the approximation that

$$\langle s \rangle = e^{-\epsilon/k_B T} - (N+1)e^{-(N+1)\epsilon/k_B T}$$

Collecting like terms,

$$= [1 - (N+1)e^{-N\epsilon/k_B T}] e^{-\epsilon/k_B T}$$

The second term in the brackets approximate zero, so

$$= e^{-\epsilon/k_B T}$$

Therefore in the low temperature limit where the thermal energy is much less than the energy of the open state,

$$\boxed{\langle s \rangle = e^{-\epsilon/k_B T}} \quad (\text{F2002 II 2.3})$$

## Fall 2002 Part I

### Problem 1

#### Question

A cylindrical bucket is placed on the ground and filled with water to a height of 150 cm. How high from the ground should one punch a hole in the side of the bucket to make a stream of water that strikes the ground at the greatest distance from the bucket? What is that distance?

#### Answer

To solve and optimize the kinematic equation to determine the maximum range of the water jet, we first need to determine what the velocity of the water exiting the hole is as a function of the hole's depth below the surface of the water. To do this, we make use of Bernoulli's principle.

$$\frac{v^2}{2} + gz + \frac{p_0}{\rho} = \text{constant}$$

We begin by determining the value of the constant by evaluating the equation at the surface of the water where we know all the properties. The water is at the ambient air pressure, the height is given in the problem statement, and the velocity can be assumed to be zero in the limit that the hole leaks at a rate too slowly to change the height of the surface. Doing so,

$$\text{constant} = gz_0 + \frac{p_0}{\rho}$$

Then at some height  $z'$  below the surface of the water, we puncture a hole. Again the water is moving into a body at atmospheric pressure, so we again use  $p_0$ . That leaves the velocity we're searching for remaining, so equating with the surface value,

$$gz_0 + \frac{p_0}{\rho} = \frac{1}{2}v^2 + gz' + \frac{p_0}{\rho}$$

$$v^2 = 2g(z_0 - z')$$

Now we're ready to solve the kinematic equation. Begin as usual by finding the time of flight from the vertical components.

$$0 = z' - \frac{1}{2}gt^2 \quad \rightarrow \quad t = \sqrt{\frac{2z'}{g}}$$

Then solving the horizontal equation:

$$\ell = vt = 2\sqrt{z'(z_0 - z')}$$

Then finding the extrema:

$$\frac{d\ell}{dz'} = 0 = \frac{z_0 - 2z'}{\sqrt{z'(z_0 - z')}}}$$

$$z' = \frac{1}{2}z_0$$

That height that maximizes distance, and that distance, is

$$z' = 75 \text{ cm}$$

$$\ell = 150 \text{ cm}$$

(F2002 I 1.1)

## Problem 5

### Question

Blocks of mass  $m$  and  $2m$  are free to slide without friction on a horizontal wire. They are connected by a massless spring of equilibrium length  $L$  and force constant  $k$ . A projectile of mass  $m$  is fired with velocity  $v$  into the block with mass  $m$  and sticks to it. If the blocks are initially at rest, what is the maximum displacement between them in the subsequent motion?

### Answer

Take time  $t = 0$  to be the moment the projectile collides with the mass  $m$ , and let the subsequent transfer of momentum be instantaneous. In this case, the initial conditions of the problem are then:

$$\begin{aligned} x_1(0) &= 0 & \dot{x}_1(0) &= u \\ x_2(0) &= L & \dot{x}_2(0) &= 0 \end{aligned}$$

where  $u$  is the initial velocity of the combined project-mass system. We get  $u$  from conservation of momentum:

$$\begin{aligned} 2mu &= mv + 0 \\ u &= \frac{1}{2}v \end{aligned}$$

Now solve the mechanics problem using the Lagrangian approach. Both masses have kinetic energy, and the spring stores potential energy, so

$$\begin{aligned} T &= m\dot{x}_1^2 + m\dot{x}_2^2 \\ V &= \frac{1}{2}m(x_2 - x_1)^2 \\ L &= m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 + x_2^2 + 2x_1x_2) \end{aligned}$$

Setting up the differential equation, we get

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= -kx_1 + kx_2 & \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}_1} \right] &= 2m\ddot{x}_1 \\ \frac{\partial L}{\partial x_2} &= kx_1 - kx_2 & \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}_2} \right] &= 2m\ddot{x}_2 \end{aligned}$$

Leading to the system of equations where  $\omega^2 = k/2m$ ,

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solving the eigensystem, we find the eigenfrequencies to be  $\lambda = \{0, -2\omega^2\}$ . Letting  $\omega'^2 = 2\omega^2$ , the eigenfunction equations are then

$$\begin{aligned} \ddot{\psi}_1 &= 0 & \rightarrow & \psi_1 = A_1 t + B_1 \\ \ddot{\psi}_2 &= -2\omega^2 \psi_2 & \rightarrow & \psi_2 = A_2 \cos(\omega' t) + B_2 \sin(\omega' t) \end{aligned}$$

From the eigenvectors, we express the solutions of  $x_1$  and  $x_2$  in terms of  $\psi_1$  and  $\psi_2$ :

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\begin{aligned}x_1 &= A_1 t + B_1 + A_2 \cos(\omega' t) + B_2 \sin(\omega' t) \\x_2 &= A_1 t + B_1 - A_2 \cos(\omega' t) - B_2 \sin(\omega' t)\end{aligned}$$

Applying the boundary conditions, we find that

$$\begin{aligned}x_1(t) &= \frac{1}{4}vt + \frac{1}{2}L - \frac{1}{2}L \cos(\omega' t) + \frac{v}{4\omega'} \sin(\omega' t) \\x_2(t) &= \frac{1}{4}vt + \frac{1}{2}L + \frac{1}{2}L \cos(\omega' t) - \frac{v}{4\omega'} \sin(\omega' t)\end{aligned}$$

The distance  $\ell(t) = x_2(t) - x_1(t)$  between the two masses maximizes when

$$\begin{aligned}\frac{d\ell}{dt} = 0 &= \frac{d}{dt} \left[ L \cos(\omega' t) - \frac{v}{2\omega'} \sin(\omega' t) \right] \\t &= -\frac{1}{\omega'} \arctan\left(\frac{v}{2L\omega'}\right)\end{aligned}$$

Plugging back into the function  $\ell(t)$ ,

$$\begin{aligned}\ell &= L \cos\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right] - \frac{v}{2\omega'} \sin\left[-\arctan\left(\frac{v}{2L\omega'}\right)\right] \\ \ell &= L \frac{2L\omega'}{\sqrt{v^2 + 4L^2\omega'^2}} + \frac{v}{2\omega'} \frac{v}{\sqrt{v^2 + 4L^2\omega'^2}} \\ \ell &= \frac{\sqrt{v^2 + 4L^2\omega'^2}}{2\omega'}\end{aligned}$$

Finally, substituting back in  $\omega' = \sqrt{2k/m}$  and simplifying, we get the final solution that maximum distance between the two masses is

$$\boxed{\ell = \sqrt{L^2 + \frac{\frac{1}{2}mv^2}{8k}}} \quad (\text{F2002 I 5.1})$$

which agrees qualitatively with the fact that a larger spring constant should stiffen the system and decrease the maximum displacement, while launching the projectile with a greater velocity would increase it.

## Problem 7

### Question

In the state  $\psi_{\ell,m}$  with angular momentum  $\ell$  and its projection  $m$ , determine the average values  $\ell_x^2$  and  $\ell_y^2$ .

### Answer

Begin by noting that the angular momentum operators are related as,

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

If we assume that the average  $x$  and  $y$  components will be equal based on symmetry where only the  $z$  direction is identifiable, then we can let  $L_x = L_y$  and rearrange the equation to get

$$L_x^2 = \frac{1}{2} (L^2 - L_z^2)$$

The expectation value is then found by the standard method:

$$\begin{aligned} \langle \psi_{\ell,m} | L_x^2 | \psi_{\ell,m} \rangle &= \langle \psi_{\ell,m} | \frac{1}{2} (L^2 - L_z^2) | \psi_{\ell,m} \rangle \\ &= \frac{1}{2} (\langle \psi_{\ell,m} | L^2 | \psi_{\ell,m} \rangle - \langle \psi_{\ell,m} | L_z^2 | \psi_{\ell,m} \rangle) \end{aligned}$$

Then because we know these quantum numbers

$$\langle \psi_{\ell,m} | L_x^2 | \psi_{\ell,m} \rangle = \frac{1}{2} (\ell(\ell+1) - m^2)$$

Therefore with both  $\ell_x^2$  and  $\ell_y^2$  being assumed equation, we conclude that

$$\langle \ell_x^2 \rangle = \langle \ell_y^2 \rangle = \frac{1}{2} (\ell(\ell+1) - m^2) \quad (\text{F2002 I 7.1})$$

## Problem 10

### Question

Ice on a pond is 10 cm thick and the water temperature just below the ice is  $0^\circ\text{C}$ . If the air temperature is  $-20^\circ\text{C}$ , by how much will the ice thickness increase in 1 hour? Assuming that the air temperature stays the same over a long period, how will the ice thickness increase with time? Comment on any approximation that you make in your calculation.

Density of ice  $= 0.9 \text{ g/cm}^3$

Thermal conductivity of ice  $= 0.0005 \text{ cal/(cm}\cdot\text{s}\cdot^\circ\text{C)}$

Latent heat of fusion of water  $= 80 \text{ cal/g}$

### Answer

Since the thermal heat flow is a one dimensional problem, immediately consider everything with respect to a small area element with its normal perpendicular to the ice-water interface  $dA$ . Then we want to know how much ice is generated on the surface of the ice. This small ice element's mass is simply

$$dm = \rho dA dz$$

where  $dz$  is the thickness of the new ice layer. To generate this ice, the latent heat of fusion must be conducted away, so the energy released is,

$$\begin{aligned} dE_f &= L_f dm \\ &= L_f \rho dA dz \end{aligned}$$

The energy flow is through the ice, and we expect this to increase with the temperature differential across the ice sheet, suggesting that the thermal conductivity  $\kappa$  be multiplied by the temperature difference  $\Delta T$ . Furthermore, the ice will decrease the rate of heat flow as it becomes thicker, so the quantity should also be divided by the thickness  $z$ . This gives

$$\frac{\kappa \Delta T}{z} = \left[ \frac{\text{cal}}{\text{cm}^2 \cdot \text{s}} \right]$$

This energy is flowing through a surface element  $dA$ , giving the power flow due to heat as

$$\frac{\kappa \Delta T dA}{z} = \left[ \frac{\text{cal}}{\text{s}} \right]$$

This power can be matched in units with the energy released from the ice calculated above by taking the time derivative of  $dE_f$ , so equating the two we have

$$\begin{aligned} L_f \rho dA \frac{dz}{dt} &= \frac{\kappa \Delta T dA}{z} \\ \int_{z_0}^{z_0+\delta z} z dz &= \int_0^t \frac{\kappa \Delta T}{L_f \rho} dt \\ 2z_0 \delta z + (\delta z)^2 &= \frac{\kappa \Delta T}{L_f \rho} t \end{aligned}$$

Solving for the length the ice grows  $\delta z$ ,

$$\begin{aligned} \delta z &= \frac{-2z_0 \pm \sqrt{4z_0^2 - 4\left(\frac{\kappa \Delta T}{L_f \rho}\right)t}}{2} \\ \delta z &= z_0 \left( 1 \pm \sqrt{1 - \frac{\kappa \Delta T}{L_f \rho z_0^2} t} \right) \end{aligned}$$

The two roots give solutions  $\delta z = \{0.0501 \text{ cm}, 19.950 \text{ cm}\}$ . Since the second root is unrealistic, we know that the solution must then be

$$\delta z = 0.0501 \text{ cm} \quad \text{in an hour}$$

(F2002 I 10.1)



## Problem 11

### Question

Carbon-14 is produced by cosmic rays interacting with the nitrogen in the Earth's atmosphere. It is eventually incorporated into all living things, and since it has a half-life of  $(5730 \pm 40)$  yr, it is useful for dating archaeological specimens up to several tens of thousands of years old. The radioactivity of a particular specimen of wood containing 3 g of carbon was measured with a counter whose efficiency was 18 %; a count rate of  $(12.8 \pm 0.1) \text{ min}^{-1}$  was measured. It is known that in 1 g of living wood, there are  $16.1 \text{ min}^{-1}$  radioactive carbon-14 decays. What is the age of this specimen, and its uncertainty? (Where errors are not quoted, they can be assumed to be negligible).

### Answer

The rate  $N$  after a given time is given by the exponential decay formula

$$N(t) = N_0 e^{-t/\tau}$$

Since we have the half-life  $t_{1/2}$  instead of the decay constant  $\tau$ , we use the relation  $t_{1/2} = \tau \ln 2$  to simplify the expression instead to

$$N(t) = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The counter use has an efficiency of  $\epsilon = 0.18$ , so the measured counting rate  $N_m$  must be corrected for that. Furthermore, the sample has a mass of 3 g whereas we know the rate for a one gram sample, so we also normalize the count rate by the mass of the sample. Plugging this all into the exponential decay function above gives

$$\frac{N_m}{3\epsilon} = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

The only unknown left in the equation is the time, so solving for it,

$$\begin{aligned} t &= t_{1/2} \log_{1/2} \left( \frac{N_m}{3\epsilon N_0} \right) \\ t &= t_{1/2} \frac{\ln \left( \frac{N_m}{3\epsilon N_0} \right)}{\ln 2} \\ t &= \frac{t_{1/2}}{\ln 2} \ln \left( \frac{N_m}{3\epsilon N_0} \right) \end{aligned}$$

To find the uncertainty, we note that only the quantities  $N_m$  and  $t_{1/2}$  have non-negligible uncertainties, so we propagate the errors only over these two terms:

$$\begin{aligned} \sigma_t^2 &= \left( -\frac{t_{1/2}}{N_m \ln 2} \right)^2 \sigma_{N_m}^2 + \left( \frac{1}{\ln 2} \ln \left( \frac{3\epsilon N_0}{N_m} \right) \right)^2 \sigma_{t_{1/2}}^2 \\ \sigma_t &= \frac{t_{1/2}}{\ln 2} \sqrt{\left( \frac{\sigma_{N_m}}{N_m} \right)^2 + \left( \frac{\sigma_{t_{1/2}}}{t_{1/2}} \right)^2 \left[ \ln \left( \frac{3\epsilon N_0}{N_m} \right) \right]^2} \end{aligned}$$

Plugging in all the numbers, we get  $t = 4248.8435$  yr and  $\sigma_t = 161.717$  yr. The given uncertainties have a single significant digit, so adding an extra significant figure to the uncertainty and matching decimal places in the answer, we conclude that the sample has an age of

$$t = (4250 \pm 160) \text{ yr}$$

(F2002 I 11.1)

## Fall 2011 Part I

### Problem 1

#### Question

An elevator operator in a skyscraper, being a very meticulous person, put a pendulum clock on the wall of the elevator to make sure that he spends exactly 8 hours a day at his work place. Over the course of his work day, he records that the time during which the elevator has acceleration  $a$  is exactly equal to the time during which it has acceleration  $-a$ . Does the elevator operator work, in actual time, (1) more than 8 hours, (2) exactly 8 hours, or (3) less than 8 hours? Why?

#### Answer

The nominal period of a pendulum is

$$T_{nom} = 2\pi\sqrt{\frac{\ell}{g}}$$

but within the elevator, the acceleration  $g$  is not going to be constant and will rather depend on the acceleration of the elevator. Therefore,

$$T_{\uparrow} = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad T_{\downarrow} = 2\pi\sqrt{\frac{\ell}{g-a}}$$

for the upward and downward cases, respectively.

Since the elevator operator observed that equal time was spent going up as was spent going down, so he must have observed  $N$  oscillations in both cases. In order to compare to the actual time, we simply compare the elevator's total time measurement with that of a stationary clock.

$$NT_{\uparrow} + NT_{\downarrow} \stackrel{?}{=} 2NT_{nom}$$

$$\begin{aligned} 2\pi N\sqrt{\frac{\ell}{g+a}} + 2\pi N\sqrt{\frac{\ell}{g-a}} &\stackrel{?}{=} 4\pi N\sqrt{\frac{\ell}{g}} \\ \sqrt{\frac{1}{g+a}} + \sqrt{\frac{1}{g-a}} &\stackrel{?}{=} 2\sqrt{\frac{1}{g}} \\ \sqrt{\frac{g}{g+a}} + \sqrt{\frac{g}{g-a}} &\stackrel{?}{=} 2 \end{aligned}$$

Use the test value  $a = 5$  for comparison (with  $g = 10$ )

$$\boxed{2.23 > 2}$$

(F2011 I 1.1)

Therefore the elevator operator actually spends more than 8 hours in the elevator during his shift.

## Problem 2

### Question

A classical particle is subject to an attractive central force proportional to  $r^\alpha$ , where  $r$  is the radius and  $\alpha$  is a constant. Show by perturbation analysis what is required of  $\alpha$  in order for the particle to have a stable circular orbit.

### Answer

Construct the Lagrangian for the system in order to determine the equations of motion for the given central force (noting that we were given the *force* so we need to make an appropriate potential).

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad V = \frac{k}{\alpha+1}r^{\alpha+1}$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{\alpha+1}r^{\alpha+1}$$

Conservation of angular momentum is a consequence of the  $\theta$  and  $\dot{\theta}$  coordinates:

$$0 = \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right]$$

$$0 = \frac{d}{dt} [mr^2\dot{\theta}]$$

Nothing that

$$\ell = \left| \frac{\vec{r} \times \vec{p}}{m} \right| = r^2\dot{\theta}$$

we can say that

$$\dot{\theta} = \frac{\ell}{r^2}$$

Then returning to the  $r$  and  $\dot{r}$  coordinates in the Lagrangian,

$$\frac{\partial \mathcal{L}}{\partial r} = m\dot{\theta}^2 - kr^\alpha \qquad \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

$$\frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right] = m\ddot{r}$$

Putting the differential equation together and substituting for the angular momentum per unit mass gives

$$m\ddot{r} = \frac{m\ell^2}{r^3} - kr^\alpha \qquad (\text{F2011 I 2.1})$$

In the case that the orbit is circular,  $r$  must be a constant, so let  $r = a$  and note that  $\ddot{r} = 0$  necessarily.

$$\frac{m\ell^2}{a^3} = ka^\alpha$$

Returning to the differential equation, let the actual distance  $r$  be a perturbation from a circular orbit, and Taylor expand in  $x$  where  $x = r - a$ .

$$\begin{aligned}
 m\ddot{x} &= \frac{m\ell^2}{a^3} \left(1 + \frac{x}{a}\right)^{-3} - ka^\alpha \left(1 + \frac{x}{a}\right)^\alpha \\
 m\ddot{x} &\approx \frac{m\ell^2}{a^3} \left(1 - 3\frac{x}{a} + \dots\right) - ka^\alpha \left(1 + \alpha\frac{x}{a} + \dots\right) \\
 m\ddot{x} &\approx ka^\alpha \left(1 - 3\frac{x}{a}\right) - ka^\alpha \left(1 + \alpha\frac{x}{a}\right) \\
 m\ddot{x} &\approx -3ka^\alpha \frac{x}{a} - \alpha ka^\alpha \frac{x}{a} \\
 m\ddot{x} &\approx -ka^{\alpha-1} (3 + \alpha) x
 \end{aligned}$$

To form a stable orbit, the coefficient on  $x$  must be negative, giving a simple harmonic solution. Therefore  $3 + \alpha > 0$  to keep the coefficient negative and

$$\boxed{a > -3}$$

(F2011 I 2.2)

### Problem 3

#### Question

A neutral conductor A with a spherical outer surface of radius  $R$  contains three cavities B, C, and D, but is solid otherwise. B and C are spherical, and D is hemispherical. Without touching A, positive charges  $q_B$  and  $q_C$  are introduced at the centers of B and C, respectively.

1. Give the amount and the distribution of the induced charges on the surfaces of A, B, C, and D.
2. Now another positive charge  $q_E$  is introduced at a distance  $r > R$  from the center of A. Describe qualitatively the distribution of induced charges on the surfaces of A, B, C, and D.
3. Give the amount of the induced charges on the surfaces of A, B, C, and D for the situation in (2).

#### Answer (1)

An ideal conductor will not support an electric field inside the solid, so each of cavities B and C will have a surface charge to cancel the electric fields emanating from  $q_B$  and  $q_C$  respectively.

- Cavity B will have a uniform surface charge density of  $-q_B/4\pi r_B^2$ , where  $r_B$  is the radius of cavity B, with total induced charge  $-q_B$  (because of symmetry and use of a Gaussian surface).
- Cavity C will have a uniform surface charge density of  $-q_C/4\pi r_C^2$ , where  $r_C$  is the radius of cavity C, with total induced charge  $-q_C$  (because of symmetry and use of a Gaussian surface).

Cavity D will not have a surface charge since a Gaussian surface coincident with its boundary contains no charge.

The surface A will have total charge  $q_B + q_C$  with uniform surface charge density of  $(q_B + q_C)/4\pi R^2$  in accordance with the symmetry of a Gaussian surface containing the sphere as well as properties of an ideal conductor.

#### Answer (2)

The surfaces B, C, and D will remain unaffected since the surrounding conductor shields the cavities from electric fields produced by charge  $q_E$ . The distribution on surface A will shift so that the negative charge concentration is greatest on the side nearest to  $q_E$  with an increasingly positive distribution towards the opposite side.

#### Answer (3)

The surface of A will still contain the same total charge  $q_B + q_C$  since only a redistribution of induced charges occurred along the surface. Similarly, because surface B, C, and D are shielded from the electric field of  $q_E$  by conductor A, the total charges along their surfaces remains unchanged as well.

## Problem 4

### Question

The dielectric strength of air at standard temperature and pressure is  $3 \cdot 10^6$  V/m. What is the maximum intensity in units of  $\text{W/m}^2$  for a monochromatic laser that can be used in the laboratory?

### Answer

Failure of a dielectric occurs when the energy density in the dielectric is great enough to overcome the ionization energy of the constituent atoms. This suggests that an electric field of greater than  $3 \cdot 10^6$  V/m would cause this ionization to occur.

Starting here, We can calculate the energy density of the electric field at any point in space by

$$U_{em} = \frac{\epsilon_0}{2} E^2$$

(where we've used the vacuum energy density since air differs very little from the vacuum permittivity).

Then the power transmitted by the laser is  $P = cU_{em}$ , so plugging in the numbers,

$$P = \frac{1}{2} \left( 8.854 \cdot 10^{-12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2} \right) \left( 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \right) \left( 3 \cdot 10^6 \frac{\text{V}}{\text{m}} \right)^2$$

$$P = 1.19 \cdot 10^{10} \frac{\text{W}}{\text{m}^2}$$

The maximum power of a laser usable in the lab is  $1.19 \cdot 10^{10} \text{ W/m}^2$ .

## Problem 5

### Question

What is the minimum energy of the projectile proton required to induce the reaction  $p + p \rightarrow p + p + p + \bar{p}$  if the target proton is at rest?

### Answer

Energy and momentum must be conserved. At the minimum allowed energy, the resultant 4 proton/anti-protons will be colinear with no relative momentum with respect to one another, so the momentum equation in the lab frame is simply

$$p_i = 4p_f \quad (\text{F2011 I 5.1})$$

Similarly, the resultant (anti-)protons are indistinguishable, so they will all have equivalent energy  $E_f$ . The initial protons have different energies since one is at rest in the lab frame while the other is moving, leading to the energy equation

$$\sqrt{p_i^2 c^2 + m_p^2 c^4} + m_p c^2 = 4\sqrt{p_f^2 c^2 + m_p^2 c^4}$$

Substituting the momentum relation into the equation, squaring, and simplifying,

$$\begin{aligned} \sqrt{16p_f^2 c^2 + m_p^2 c^4} + m_p c^2 &= 4\sqrt{p_f^2 c^2 + m_p^2 c^4} \\ 16p_f^2 c^2 + m_p^2 c^4 + m_p^2 c^4 + 2\sqrt{m_p^2 c^4 (16p_f^2 c^2 + m_p^2 c^4)} &= 16p_f^2 c^2 + 16m_p^2 c^4 \\ 2\sqrt{m_p^2 c^4 (16p_f^2 c^2 + m_p^2 c^4)} &= 14m_p^2 c^4 \\ 16p_f^2 c^2 + m_p^2 c^4 &= 49m_p^2 c^4 \\ p_f^2 &= 3m_p^2 c^4 \end{aligned}$$

Therefore,

$$p_i^2 = 48m_p^2 c^4$$

and

$$E_1 = \sqrt{49m_p^2 c^4}$$

$$E_1 \approx 6.567 \text{ GeV}/c^2$$

(F2011 I 5.2)

## Problem 8

### Question

Assume that the atmosphere near the earth's surface is in approximate hydrostatic equilibrium, where any movement of air parcels is gentle and adiabatic. Find an expression for the pressure  $P$  of the atmosphere as a function of the height  $z$ .

### Answer

Note that the pressure at a given point is due to the mass of air above the given point. Then by moving an infinitesimal distance vertically, the total mass is changed by the density of the air (which is affected by the gravitational force). This leads to the differential equation

$$\frac{dP}{dz} = \rho g$$

Then using the ideal gas equation

$$PV = Nk_B T$$

multiply and divide by the average molecular mass  $m$  of the air (in kg) which combined with the number of molecules  $N$  gives the total mass

$$PV = (Nm) \frac{1}{m} k_B T$$

and then divide by the volume to get the ideal gas equation in terms of the mass density

$$\begin{aligned} P &= \frac{Nm}{V} \frac{1}{m} k_B T \\ P &= \rho \frac{k_B T}{m} \\ \rho &= \frac{Pm}{k_B T} \end{aligned}$$

Finally, substitute this into the differential equation above and solve to get the atmospheric scale height equation.

$$\begin{aligned} \frac{dP}{dz} &= \frac{Pm}{k_B T} g \\ \frac{dP}{P} &= \frac{mg}{k_B T} dz \end{aligned}$$

$P(z) = P_0 e^{z/\xi} \quad \text{where } \xi = \frac{k_B T}{mg}$	(F2011 I 8.1)
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## Fall 2011 Part II

### Problem 1

#### Question

Mass  $m_1$  moves freely along a fixed, long, horizontal rod. The position of  $m_1$  on the rod is  $x$ . A massless string of length  $\ell$  is attached to  $m_1$  at the end and to mass  $m_2$  at the other. Mass  $m_2$  executes pendulum motion in the vertical plane containing the rod.

1. Find the Lagrangian of the system.
2. Derive the equations of motion and the corresponding conservation laws.
3. Assume that  $x(0) = x_0$ ,  $\dot{x}(0) = 0$ ,  $\varphi(0) = \varphi_0$  ( $|\varphi_0| \ll 1$ ), and  $\dot{\varphi}(0) = 0$ . Find  $x(t)$  and  $\varphi(t)$  for  $t > 0$ .

#### Answer (1)

For the sliding support mass  $m_1$ :

$$T_1 = \frac{1}{2} m_1 \dot{x}^2$$

$$V_1 = 0$$

For the pendulum mass  $m_2$ :

$$T_2 = \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} m_2 (\dot{x} + \dot{x}_2)^2$$

$$V_2 = -m_2 g y_2$$

Then using  $x_2 = \ell \sin \varphi$  and  $y_2 = -\ell \cos \varphi$ ,

$$T_2 = \frac{1}{2} m_2 (\ell^2 \dot{\varphi}^2 + \dot{x}^2 + 2\ell \dot{\varphi} \dot{x} \cos \varphi)$$

$$V_2 = -m_2 g y_2$$

Putting the Lagrangian together equals the first line. Applying the small angle approximation gives the second line where the kinetic energy term involving  $\cos \varphi$  can be simply expanded as  $\cos \varphi \approx 1$ , but the potential energy term must be expanded to second order so that  $\cos \varphi \approx 1 - \frac{1}{2} \varphi^2$ .

$$\mathcal{L} = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 (\ell^2 \dot{\varphi}^2 + 2\ell \dot{\varphi} \dot{x} \cos \varphi) + m_2 g \ell \cos \varphi \quad (\text{F2011 II 1.1})$$

$$\mathcal{L} \approx \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 (\ell^2 \dot{\varphi}^2 + 2\ell \dot{\varphi} \dot{x}) + m_2 g \ell - \frac{1}{2} m_2 g \ell \varphi^2 \quad (\text{F2011 II 1.2})$$

#### Answer (2)

Constructing the Euler-Lagrange equations for  $x$  and  $\dot{x}$ :

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1 + m_2) \dot{x} + m_2 \ell \dot{\varphi}$$

$$\frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}} \right] = (m_1 + m_2) \ddot{x} + m_2 \ell \ddot{\varphi}$$

$$(m_1 + m_2) \ddot{x} + m_2 \ell \ddot{\varphi} = 0 \quad (\text{F2011 II 1.3})$$

and for  $\varphi$  and  $\dot{\varphi}$ :

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \varphi} &= -m_2 g \ell \varphi & \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} &= m_2 \ell^2 \dot{\varphi} + m_2 \ell \dot{x} \\ \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right] &= m_2 \ell^2 \ddot{\varphi} + m_2 \ell \ddot{x}\end{aligned}$$

$$-m_2 g \ell \varphi - m_2 \ell^2 \ddot{\varphi} + m_2 \ell \ddot{x} = 0 \quad (\text{F2011 II 1.4})$$

The equations of motion are:

$$\ddot{x} + \frac{m_2}{m_1 + m_2} \ell \ddot{\varphi} = 0 \quad (\text{F2011 II 1.5})$$

$$\ddot{\varphi} + \frac{1}{\ell} \ddot{x} + \frac{g}{\ell} \varphi = 0 \quad (\text{F2011 II 1.6})$$

### Answer (3)

Solve for  $\ddot{x}$  and substitute into the other differential equation

$$\begin{aligned}\ddot{\varphi} - \frac{1}{\ell} \frac{m_2}{m_1 + m_2} \ell \ddot{\varphi} + \frac{g}{\ell} \varphi &= 0 \\ \frac{m_1}{m_1 + m_2} \ddot{\varphi} + \frac{g}{\ell} \varphi &= 0 \\ \ddot{\varphi} + \frac{g}{\ell} \frac{m_1 + m_2}{m_1} \varphi &= 0\end{aligned} \quad (\text{F2011 II 1.7})$$

This is just the differential equation for a simple harmonic oscillator, so considering the given boundary conditions,

$$\varphi(t) = \varphi_0 \cos(\omega t) \quad \text{where } \omega^2 = \frac{g}{\ell} \frac{m_1 + m_2}{m_1} \quad (\text{F2011 II 1.8})$$

Then differentiating  $\varphi(t)$  twice and substituting into the first equation,

$$\ddot{x} = \ell \varphi_0 \omega^2 \frac{m_2}{m_1 + m_2} \cos(\omega t)$$

Then integrating twice and applying the boundary conditions,

$$x(t) = x_0 - \frac{g}{\omega^2} \frac{m_2}{m_1} \cos(\omega t) \quad (\text{F2011 II 1.9})$$

## Spring 2012 Part I

### Problem 1

#### Question

For a many particle system of weakly interacting particles, will quantum effects be more important for (a) high densities or low densities and (b) high temperatures or low temperatures for a system. Explain your answers in terms of the de Broglie wavelength  $\lambda$  defined as  $\lambda^2 \equiv h^2 / (3mk_B T)$  where  $m$  is the mass of the particles and  $k_b$  Boltzmann's constant.

#### Answer

- (a) High density — The de Broglie wavelength gives a “size” of the particle, and in the high density limit, the wavefunctions overlap significantly so quantum effects and interactions are critical to the behavior of the system.
- (b) Low temperature — Since  $\lambda^2 \propto T^{-1}$ , as  $T \rightarrow 0$ ,  $\lambda$  increases so that again the wavefunctions overlap and quantum effects are significant.

## Problem 2

### Question

The ground state energy of Helium is  $-79$  eV. What is its ionization energy, which is the energy required to remove just one electron?

### Answer

Using the Hydrogen solution with modifications for single-electron atoms of higher  $Z$ , we know that the ground state energy of singly ionized Helium is

$$E_{He}^1 = 2^2 (-13.6 \text{ eV}) = -54.4 \text{ eV}$$

Therefore, the difference between the singly-ionized and neutral ground state energies gives the first ionization energy of the Helium atom.

$$E_i = -24.6 \text{ eV}$$

(F2012 I 2.1)

### Problem 3

#### Question

It is known that the force per unit area ( $F/A$ ) between two neutral conducting plates due to polarization fluctuations of the vacuum (namely, the Casimir force) is a function of  $h$  (Planck's constant),  $c$  (speed of light), and  $z$  (distance between the plates) only. Using only dimensional analysis, obtain  $F/A$  as a function of  $h$ ,  $c$ , and  $z$ .

#### Answer

The units of  $F/A$  are

$$\frac{F}{A} = \left[ \frac{\text{kg}}{\text{m} \cdot \text{s}^2} \right]$$

The kg suggests a factor proportional to  $h$ , making the equation

$$\frac{F}{A} \sim \left[ \frac{1}{\text{m}^3 \cdot \text{s}} \right] h$$

Accounting for the factor of seconds requires a  $c$ :

$$\frac{F}{A} \sim \left[ \frac{1}{\text{m}^4} \right] hc$$

Finally, account for all the factors of distance:

$$\frac{F}{A} \sim \frac{hc}{z^4}$$

Therefore,

$$\boxed{\frac{F}{A} \sim \frac{hc}{z^4}}$$

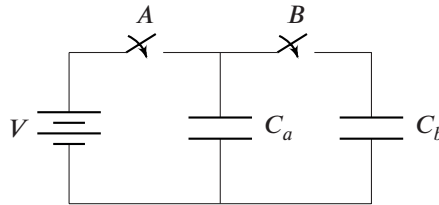
(F2012 I 3.1)

### Problem 4

#### Question

In the circuit diagram opposite, initially the two identical capacitors with capacitance  $C$  are uncharged. The connections between the components are all made with short copper wires. The battery is an ideal EMF and supplies a voltage  $V$ .

- At first Switch A is closed and Switch B is kept open. What is the final stored energy on capacitor  $C_a$ ?
- Switch A is opened and afterwards Switch B is closed. What is the final energy stored in both capacitors?
- Provide a physical explanation for any difference between the results of parts (a) and (b), if there is one.



#### Answer

- Initially, the right side of the circuit with  $C_b$  can be ignored, so the total energy is simply the energy stored within  $C_a$ .

$$E = \frac{1}{2}CV^2 \quad (\text{F2012 I 4.1})$$

- The system is now effectively just the two capacitors on the right. Because the voltage difference is supported across both capacitors, the system can be modeled as an effective capacitor in parallel

$$C_{eff} = 2C$$

The total charge stored by the capacitors must remain the same when switching from Switch A being closed to Switch B. Initially,

$$Q = CV_i$$

and afterwards it is

$$Q = C_{eff}V_f = 2CV_f$$

so the final voltage across the capacitors is

$$V_f = \frac{1}{2}V_i$$

This means the total energy is

$$E = \frac{1}{2}C_{eff}V_f^2$$

$$E = \frac{1}{4}CV^2 \quad (\text{F2012 I 4.2})$$

- The energy is dissipated (heat, fields, etc).

## Problem 5

### Question

A planet of mass  $m$  moves around the sun, mass  $M$ , in an elliptical orbit with minimum and maximum distances of  $r_1$  and  $r_2$ , respectively. Find the angular momentum of the planet relative to the center of the sun in terms of these quantities and the gravitational constant  $G$ .

### Answer

We solve the problem using conservation of energy since we know that stable elliptical orbits have constant energy. The generic equation is

$$E = \frac{L^2}{2I} - \frac{GMm}{r}$$

where  $L$  is the angular momentum and  $I$  the moment of inertia. Substituting for the values at both  $r_1$  and  $r_2$  and equating,

$$\begin{aligned} \frac{L^2}{2mr_1^2} - \frac{GMm}{r_1} &= \frac{L^2}{2mr_2^2} - \frac{GMm}{r_2} \\ \frac{L^2}{2m} \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) &= GMm \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \end{aligned}$$

which leads to the solution

$$L = \sqrt{\frac{2GMm^2r_1r_2}{r_1 + r_2}}$$

(F2012 I 5.1)

**Problem 6****Question**

A particle moves in a circular orbit under the influence of a central force that varies as the  $n$ -th power of the distance. Show that this motion is unstable if  $n < -3$ . (Hint: Consider the centrifugal potential.)

**Answer**

See solution for Fall 2011 Part I, Problem 2 with the condition inverted so that *instability* is  $n < -3$  rather than stability requiring  $n > -3$ .



## Problem 7

### Question

A classical, ideal, monatomic gas of  $N$  particles is reversibly compressed *isentropically*, i.e. with the entropy kept constant, from an initial temperature  $T_0$  and pressure  $P$  to a pressure  $2P$ . Find (a) the work done on the system, and (b) the net change in entropy of the system and its surroundings.

- (a) An isentropic process is the same as an adiabatic process since no heat can be exchanged ( $T dS = Q = 0$ ), so we begin with the relation that  $PV^\gamma$  is a constant. Combining this with the ideal gas law, we can determine that

$$P^{1-\gamma} T^\gamma = \text{const}$$

where  $\gamma = C_p/C_v$  is the ratio of heat capacities with  $C_p = \frac{5}{2} N k_B$  and  $C_v = \frac{3}{2} N k_B$  for a monatomic ideal gas. Using this, we solve for the final temperature of the system after compressions as

$$T_f = 2^{2/5} T_0 \approx 1.32 T_0$$

Combining both of

$$\Delta U = C_v \Delta T$$

$$\Delta U = Q + W$$

where  $Q = 0$ , we get that

$$W = \frac{3}{2} N k_B T_0 (2^{2/5} - 1) \quad (\text{F2012 I 7.1})$$

- (b) Because the compression is done reversibly, by definition,  $\Delta S = 0$ .

## Problem 8

### Question

For an idea Fermi gas of  $N$  neutral spin- $\frac{1}{2}$  particles in a volume  $V$  at  $T = 0$ , calculate the following:

- (a) The chemical potential
- (b) The average energy per particle
- (c) The pressure

### Answer

- (a) At  $T = 0$ , the particles are all in the lowest state allowed by Fermi-Dirac statistics, so the chemical potential, defined by the energy required to add another particle to the system, is equal to the Fermi energy. For a particle contained within a box  $V$ , the energy per particle is

$$\epsilon_n = \frac{\pi^2 \hbar^2}{2mV^{2/3}} n^2$$

Given a Fermi energy  $\epsilon_F$ , the maximum occupied state is

$$n_F = \sqrt{\frac{2mV^{2/3}}{\pi^2 \hbar^2}} \sqrt{\epsilon_F}$$

Equally we know that all  $N$  particles must exist within the eighth-sphere of  $n$  space, where the extra factor of 2 is because there are two spin states per  $n$ :

$$\begin{aligned} N &= 2 \cdot \frac{1}{8} \cdot \frac{4}{3} \pi n_F^3 \\ N &= \frac{1}{3} \pi \left( \frac{2m}{\pi^2 \hbar^2} \right)^{3/2} V \epsilon_F^{3/2} \\ \epsilon_F &= \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \end{aligned}$$

Therefore  $\mu = \epsilon_F$ ,

$$\boxed{\mu = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}} \quad (\text{F2012 I 8.1})$$

- (b) To get the total energy, we can imagine filling all  $N$  particles one at a time, so that at each step, there are  $N'$  total particles:

$$\begin{aligned} U &= \int_0^N \epsilon_F dN' \\ U &= \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{V} \right)^{2/3} \int_0^N N'^{2/3} dN' \\ U &= \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{V} \right)^{2/3} \cdot \frac{3}{5} N^{5/3} \end{aligned}$$

Therefore, the average energy per particle is  $U/N$  or

$$\boxed{\langle \epsilon \rangle = \frac{3}{5} \epsilon_F} \quad (\text{F2012 I 8.2})$$

(c) From the thermodynamic relation

$$dU = T dS - P dV + \mu dN$$

we can read off the derivative that defines the pressure  $P$  as

$$P = - \left( \frac{\partial U}{\partial V} \right)_{S,N}$$

Doing so, we get that

$$\frac{\partial U}{\partial V} = \frac{3}{5} N \cdot \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{V} \right)^{2/3} \cdot \left( -\frac{2}{3V} \right)$$

making the pressure

$$P = \frac{2}{5} \frac{N}{V} \epsilon_F$$

(F2012 I 8.3)

## Problem 10

### Question

A piece of  $p$ -doped silicon has a carrier density  $n = 10^{15} \text{ cm}^{-3}$  and dimensions of  $\Delta x = 10 \text{ mm}$ ,  $\Delta y = 2 \text{ mm}$ , and  $\Delta z = 1 \text{ mm}$ . A magnetic field of  $B_z = 1 \text{ T}$  is applied in the  $z$ -direction and a current  $I_x = 1 \text{ A}$  flows in the  $x$ -direction, and the voltage  $V_y$  is measured.

- Express the current density  $j_x$  in terms of the carrier density  $n$  and the carrier velocity  $v_x$ .
- Write down the equilibrium force condition that determines  $V_y$ .
- Find  $V_y$  in volts.

### Answer

- The current passing through each thin cross-sectional slice of the conductor is dependent on the charge of a carrier, carrier density, and velocity of the flow.

$$I_x = en\Delta y\Delta z v_x$$

The current density is just the current passing through each point, so

$$j_x = \frac{I_x}{\Delta y\Delta z}$$

$$\boxed{j_x = nev_x} \quad (\text{F2012 I 10.1})$$

- The positive carriers drift to the edge of the conductor due to the magnetic field and the holes accumulate on the opposite edge. An electric field is created between the charge separation, so an equilibrium is set up between the electric field trying to bring the opposite charges together and the magnetic drift separating them.

$$0 = e\vec{E} + \vec{v} \times \vec{B}$$

By the right-hand rule, the positive charges accumulate along  $y = 0$ , so  $\vec{E} = E\hat{y}$ . Similarly,  $\vec{v} \times \vec{B} = -v_x B_z \hat{y}$ :

$$0 = eE\hat{y} - ev_x B_z \hat{y}$$

Written in terms of the potential  $V_y = E\Delta y$ , the equilibrium condition becomes

$$\boxed{V_y = v_x B_z \Delta y} \quad (\text{F2012 I 10.2})$$

- Substituting in for given quantities

$$V_y = \frac{I_x B_z}{ne\Delta z}$$

$$V_y = \frac{(1 \text{ A})(1 \text{ T})}{(10^{15} \text{ cm}^{-3})(1.612 \cdot 10^{-19} \text{ C})(1 \text{ mm})}$$

$$\boxed{V_y = 6.24 \text{ V}} \quad (\text{F2012 I 10.3})$$

## Spring 2012 Part II

### Problem 1

#### Question

An electron in a hydrogen atom occupies a state:

$$|\psi\rangle = \sqrt{\frac{1}{3}} |3, 1, 0, +\rangle + \sqrt{\frac{2}{3}} |2, 1, 1, -\rangle$$

where the properly normalized states are specified by the quantum numbers  $|n, \ell, m, \pm\rangle$  and the  $\pm$  specifies whether the spin is up or down.

- What is the expectation value of the energy in terms of the ground state energy?
- If you measured the expectation values of the orbital momentum squared  $\langle L^2 \rangle$ , the square of the spin  $\langle S^2 \rangle$ , and their  $z$ -components  $\langle L_z \rangle$  and  $\langle S_z \rangle$ , what would be the result?
- Show that if you measure the position of the electron, the probability density for finding it at an angle specified by  $\theta$  and  $\phi$  integrated over all values of  $r$  is independent of  $\theta$  and  $\phi$ . Note, for this part you will need  $Y_1^0 = \sqrt{3/4\pi} \cos \theta$  and  $Y_1^1 = -\sqrt{3/8\pi} \sin \theta \exp(i\phi)$ . You do *not*, however, need to know the radial functions, only that they are properly normalized and orthogonal to each other.
- List all additional possible states that are degenerate with the first state in the linear combination above. Note: this part can be done even if you have not answered the previous parts.

Assume now that the state  $|\psi\rangle$ , given above, is the initial state of an electron in a hydrogen atom.

- Write down the electron's state as a function of time for all  $t > 0$ .
- go through the results you obtained in parts (a) through (c) and determine which of them are time independent.

#### Answer

- Calculate the energy by sandwiching the Hamiltonian between the wavefunction:

$$\begin{aligned} \langle E \rangle &= \langle \psi | H | \psi \rangle \\ &= \left( \sqrt{\frac{1}{3}} \langle 3, 1, 0, + | + \sqrt{\frac{2}{3}} \langle 2, 1, 1, - | \right) H \left( \sqrt{\frac{1}{3}} | 3, 1, 0, + \rangle + \sqrt{\frac{2}{3}} | 2, 1, 1, - \rangle \right) \\ &= \frac{1}{3} \langle 3, 1, 0, + | H | 3, 1, 0, + \rangle + \frac{\sqrt{2}}{3} \langle 2, 1, 1, - | H | 3, 1, 0, + \rangle \\ &\quad + \frac{\sqrt{2}}{3} \langle 3, 1, 0, + | H | 2, 1, 1, - \rangle + \frac{2}{3} \langle 2, 1, 1, - | H | 2, 1, 1, - \rangle \end{aligned}$$

For every term, the wavefunctions are eigenstates of the Hamiltonian, so we extract the appropriate energy term from every bra-ket sandwich. Then the middle two terms integrate to zero since states with different  $n$  are orthogonal while the first and last terms integrate to unity since they are properly normalized.

$$\langle E \rangle = \frac{1}{3} E_3 + 0 + 0 + \frac{2}{3} E_2$$

Each energy is related to the ground state energy by  $E_n = E_0/n^2$ , so

$$= \frac{1}{3} \frac{E_0}{9} + \frac{2}{3} \frac{E_0}{4}$$

$$\langle E \rangle = \frac{11}{54} E_0 \approx -2.77 \text{ eV} \quad (\text{F2012 II 1.1})$$

- (b) For each of the other expectation values, the process is very similar with an appropriate change for eigenvalues; specifically,

$$\begin{aligned} L^2 |n, \ell, m, \pm\rangle &= \ell(\ell+1) \hbar^2 |n, \ell, m, \pm\rangle \\ S^2 |n, \ell, m, \pm\rangle &= \frac{1}{2} \left( \frac{1}{2} + 1 \right) \hbar^2 |n, \ell, m, \pm\rangle \\ L_z |n, \ell, m, \pm\rangle &= \ell \hbar |n, \ell, m, \pm\rangle \\ S_z |n, \ell, m, \pm\rangle &= \pm \frac{1}{2} \hbar |n, \ell, m, \pm\rangle \end{aligned}$$

The same restrictions that the middle terms integrate to zero because of orthogonality and the first and last terms integrate to unity still applies, so we can almost immediately conclude that

$$\langle L^2 \rangle = 2\hbar^2 \quad (\text{F2012 II 1.2})$$

$$\langle S^2 \rangle = \frac{3\hbar^2}{4} \quad (\text{F2012 II 1.3})$$

$$\langle L_z \rangle = \frac{2\hbar}{3} \quad (\text{F2012 II 1.4})$$

$$\langle S_z \rangle = -\frac{\hbar}{6} \quad (\text{F2012 II 1.5})$$

- (c) In the  $|r, \theta, \phi\rangle$  basis,

$$\begin{aligned} |3, 1, 0\rangle &= R_{3,1}(r) Y_1^0(\theta, \phi) = R_{31}(r) \sqrt{\frac{3}{4\pi}} \cos \theta \\ |2, 1, 1\rangle &= R_{2,1}(r) Y_1^1(\theta, \phi) = -R_{21}(r) \sqrt{\frac{3}{8\pi}} e^{i\phi} \sin \theta \end{aligned}$$

This means that the probability density is

$$\begin{aligned} \langle \psi | \psi \rangle &= \frac{1}{3} \langle 3, 1, 0 | 3, 1, 0 \rangle + \frac{\sqrt{2}}{3} \langle 2, 1, 1 | 3, 1, 0 \rangle + \frac{\sqrt{2}}{3} \langle 3, 1, 0 | 2, 1, 1 \rangle + \frac{2}{3} \langle 2, 1, 1 | 2, 1, 1 \rangle \\ &= \frac{1}{4\pi} \cos^2 \theta R_{31}^2(r) - \frac{1}{\pi} \sin \theta \cos \theta (e^{i\phi} + e^{-i\phi}) R_{21}(r) R_{31}(r) + \frac{1}{4\pi} \sin^2 \theta R_{21}^2(r) \end{aligned}$$

Integrating over  $r$ ,

$$\begin{aligned} \int_0^\infty \langle \psi | \psi \rangle dr &= \int_0^\infty \frac{1}{4\pi} \cos^2 \theta R_{31}^2(r) - \frac{1}{\pi} \sin \theta \cos \theta (e^{i\phi} + e^{-i\phi}) R_{21}(r) R_{31}(r) \\ &\quad + \frac{1}{4\pi} \sin^2 \theta R_{21}^2(r) dr \end{aligned}$$

Integrating over all  $r$ , we know that  $R_{n\ell} R_{n'\ell'}$  are orthonormal, so again the first and last terms'  $R$  integrates to unity and the middle term integrates to zero.

$$= \frac{1}{4\pi} (\cos^2 \theta + \sin^2 \theta)$$

Therefore we find that the probability density is constant in  $\theta$  and  $\phi$  when integrated over all  $r$ .

$$\int_0^\infty \langle \psi | \psi \rangle dr = \frac{1}{4\pi} \quad (\text{F2012 II 1.6})$$

- (d) The states degenerate with the first term in  $\psi$  are all combinations of allowed  $\ell$ ,  $m$ , and  $\pm$ :  $n$  must remain at  $n = 3$  since it is the  $n$  quantum number which determines the energy of the state. The angular momentum number  $\ell$  has to be in the range  $[0, n - 1]$ , so there are at least 3 cases.

$$|3, 0, m, \pm\rangle$$

$$|3, 1, m, \pm\rangle$$

$$|3, 2, m, \pm\rangle$$

Then for each  $\ell$ , the projection  $m$  can take a range of values  $m \in [-\ell, \ell]$  so using  $\{\dots, -1, 0, 1, \dots\}$  to denote a set of options,

$$|3, 0, m, \pm\rangle \rightarrow |3, 0, \{0\}, \pm\rangle \quad 2 \text{ states}$$

$$|3, 1, m, \pm\rangle \rightarrow |3, 1, \{-1, 0, 1\}, \pm\rangle \quad 6 \text{ states}$$

$$|3, 2, m, \pm\rangle \rightarrow |3, 2, \{-2, -1, 0, 1, 2\}, \pm\rangle \quad 10 \text{ states}$$

In total, there are 18 degenerate states

- (e) To get the time evolution, we simply use the fact that for each basis eigenstate, we can add the time evolution component

$$\exp\left(-\frac{iE_n t}{\hbar}\right)$$

to get (in terms of the ground state energy  $E_0$ )

$$|\psi(t)\rangle = \sqrt{\frac{1}{3}} |3, 1, 0, +\rangle e^{-iE_0 t/9\hbar} + \sqrt{\frac{2}{3}} |2, 1, 1, -\rangle e^{-iE_0 t/4\hbar} \quad (\text{F2012 II 1.7})$$

- (f) From Ehrenfest's Theorem, we can quickly find the answers to most of the question without worrying about the wavefunction. Ehrenfest's Theorem is

$$\frac{d}{dt} \langle E \rangle = -\frac{i}{\hbar} \langle [\Omega, H] \rangle + \left\langle \frac{\partial \Omega}{\partial t} \right\rangle$$

None of the operators  $L^2$ ,  $S^2$ ,  $L_z$ , and  $S_z$  are explicit in time, so the second term on the right can be dropped. Then because each of these operators commute with the Hamiltonian, the first term on the right is also dropped. Therefore, the expectation values are constant in time, so

$$\langle L^2 \rangle \quad \text{Time independent} \quad (\text{F2012 II 1.8})$$

$$\langle S^2 \rangle \quad \text{Time independent} \quad (\text{F2012 II 1.9})$$

$$\langle L_z \rangle \quad \text{Time independent} \quad (\text{F2012 II 1.10})$$

$$\langle S_z \rangle \quad \text{Time independent} \quad (\text{F2012 II 1.11})$$

For the probability density, we return to the integral in part (c) and insert the appropriate exponential terms. The first and last terms' exponentials cancel each other out, leaving

$$\begin{aligned} \int_0^\infty \langle \psi | \psi \rangle dr &= \int_0^\infty \frac{1}{4\pi} \cos^2 \theta R_{31}^2(r) - \frac{1}{\pi} \sin \theta \cos \theta (e^{i\phi} + e^{-i\phi}) R_{21}(r) R_{31}(r) \\ &\quad \cdot \left[ \exp\left(\frac{i(E_2 - E_3)t}{\hbar}\right) + \exp\left(-\frac{i(E_2 - E_3)t}{\hbar}\right) \right] \\ &\quad + \frac{1}{4\pi} \sin^2 \theta R_{21}^2(r) dr \end{aligned}$$

The integral is unaffected by the new time factors, though, so integrating over  $r$ , the middle term still goes to zero and we're left with the same result previously of  $1/4\pi$ , therefore

$$\int_0^\infty \langle \psi | \psi \rangle dr \quad \text{Time independent} \quad (\text{F2012 II 1.12})$$



## Problem 5

### Question

Consider  $N$  non-interacting, stationary particles, each with magnetic moment  $\vec{\mu}$  at temperature  $T$  in a uniform external magnetic field  $\vec{B}$ . Their energy is  $-\vec{\mu} \cdot \vec{B}$ . Calculate the partition function  $Z$ , the internal energy, and magnetization for two distinct cases (a and b below):

- The magnetic moment of each particle can be oriented only parallel or anti-parallel to the magnetic field.
- The magnetic moment of each particle can rotate freely.
- Show that, in both cases, the total magnetization  $\vec{M}$  can be written as a derivative of the partition function.
- In each case, calculate the fluctuations of magnetization  $\langle (\Delta \vec{u})^2 \rangle$ .

### Question

- Begin by constructing the partition function for a single particle. Since there are only two energy states, the sum is simply over the two Boltzmann factors:

$$Z_1 = e^{\mu B/kT} + e^{-\mu B/kT}$$

This can be simplified using trigonometric identities to

$$Z_1 = 2 \cosh \left( \frac{\mu B}{kT} \right)$$

For fixed site particles, the partition function for  $N$  particles is simply  $Z = Z_1^N$ , so

$$Z = 2^N \cosh^N \left( \frac{\mu B}{kT} \right) \quad (\text{F2012 II 5.1})$$

The total energy can be calculated either by finding the expectation energy per particle  $\langle \epsilon \rangle$  and multiplying by  $N$  using the Boltzmann factors directly, or by using the thermodynamic identity

$$U = kT^2 \frac{\partial \ln Z}{\partial T}$$

Doing so,

$$U = kT^2 \frac{N}{2 \cosh \left( \frac{\mu B}{kT} \right)} \cdot 2 \sinh \left( \frac{\mu B}{kT} \right) \cdot \left( -\frac{\mu B}{kT^2} \right)$$

$$U = -N \mu B \tanh \left( \frac{\mu B}{kT} \right) \quad (\text{F2012 II 5.2})$$

To find the Magnetization, we use the Boltzmann factors directly since we don't know a thermodynamic relation. Let

$$\begin{aligned} \langle m \rangle &= \sum_{\mu} \mu \frac{e^{-\epsilon_{\mu}/kT}}{Z_1} \\ &= \frac{1}{Z_1} (-\mu e^{\mu B/kT} + \mu e^{-\mu B/kT}) \\ &= -\mu \frac{2 \sinh \left( \frac{\mu B}{kT} \right)}{2 \cosh \left( \frac{\mu B}{kT} \right)} \end{aligned}$$

So knowing that  $\langle M \rangle = N \langle m \rangle$ ,

$$\langle M \rangle = -N\mu \tanh\left(\frac{\mu B}{kT}\right) \quad (\text{F2012 II 5.3})$$

- (b) In the continuous case, the sum needs to be changed into an integral, remembering to keep  $Z$  unitless. This requires dividing by the volume of the energy state, which in this case is  $\mu B$ . (Justification: think of the energy vector  $\vec{\mu} \cdot \vec{B} = \mu B \cos \theta$  on the unit circle of length  $\mu B$ . From geometry, the unitless  $d\theta$  is related to  $d\varepsilon$  by the factor  $\mu B$ .)

$$Z_1 = \int_{-\mu B}^{\mu B} e^{-\varepsilon/kT} \frac{d\varepsilon}{\mu B}$$

Letting  $u = -\frac{\varepsilon}{kT}$ ,

$$\begin{aligned} Z_1 &= -\frac{kT}{\mu B} \int_{\mu B/kT}^{-\mu B/kT} e^u du \\ &= 2 \frac{kT}{\mu B} \sinh\left(\frac{\mu B}{kT}\right) \end{aligned}$$

Therefore the partition function for all  $N$  particles is

$$Z = \left(\frac{2kT}{\mu B}\right)^N \sinh^N\left(\frac{\mu B}{kT}\right) \quad (\text{F2012 II 5.4})$$

The total energy is found in the same way as the previous case, giving

$$U = NkT - N\mu B \coth\left(\frac{\mu B}{kT}\right) \quad (\text{F2012 II 5.5})$$

For the magnetization, we also calculate the expectation value from integrating the probability distribution, again making sure to keep the correct units. This time we work with the relevant projection of the magnetic moment  $m = \mu \cos \theta$  so that when combined with the energy  $\varepsilon = -\mu B \cos \theta$ , the magnetization per particle in each state is  $m = -\varepsilon/B$ .

$$\begin{aligned} \langle m \rangle &= \int_{-\mu B}^{\mu B} -\frac{\varepsilon}{B} \frac{e^{-\varepsilon/kT}}{Z_1} \frac{d\varepsilon}{\mu B} \\ &= \frac{1}{\mu Z_1} \left(\frac{kT}{B}\right)^2 \int_{-\mu B/kT}^{\mu B/kT} e^u du \\ &= \frac{kT}{B} \frac{2 \sinh\left(\frac{\mu B}{kT}\right)}{2 \sinh\left(\frac{\mu B}{kT}\right)} \\ &= \frac{kT}{B} \end{aligned}$$

The total magnetization  $\langle M \rangle = N \langle m \rangle$  is

$$\langle M \rangle = \frac{NkT}{B} \quad (\text{F2012 II 5.6})$$

Note that this is to be expected for the continuous case limit which corresponds to the classical limit. We'd expect the total energy to be related to the magnetization by  $U = MB$ . Rearranging the terms,

$$\begin{aligned} \langle M \rangle B &= NkT \\ U &= NkT \end{aligned}$$

which is the expected result from the equipartition theorem for a stationary particle with two rotational degrees of freedom.

- (c) Proving the discrete case only differs from the continuous case proof by the obvious substitutions, so only the continuous case will be presented here. Begin by writing the first starting integral from the previous problem

$$\langle m \rangle = \int_{-\mu B}^{\mu B} m \frac{e^{-\epsilon/kT}}{Z_1} \frac{d\epsilon}{\mu B}$$

The  $Z_1$  can be pulled outside the integral since it is a constant. Then note that per our definition  $\epsilon = -mB$ , it follows that

$$\frac{\partial \epsilon}{\partial B} = -m$$

We identify the integral above to be a result of using the chain rule, so we undo that and get

$$\langle m \rangle = \frac{1}{Z_1} \int_{-\mu B}^{\mu B} \frac{\partial}{\partial B} (-e^{-\epsilon/kT}) \frac{d\epsilon}{\mu B}$$

Changing the order of integration and differentiation,

$$= \frac{1}{Z_1} \frac{\partial}{\partial B} \left( \int_{-\mu B}^{\mu B} -e^{-\epsilon/kT} \frac{d\epsilon}{\mu B} \right)$$

The term within the brackets is simply the definition of the partition function, so

$$\langle m \rangle = \frac{1}{Z_1} \frac{\partial Z_1}{\partial B} = \frac{\partial \ln Z_1}{\partial B}$$

To then get the total magnetization  $\langle M \rangle$ , we use several properties of differentiation and logarithms:

$$\begin{aligned} \langle M \rangle &= N \langle m \rangle \\ &= N \frac{\partial \ln Z_1}{\partial B} \\ &= \frac{\partial (N \ln Z_1)}{\partial B} \\ &= \frac{\partial \ln (Z_1)^N}{\partial B} \end{aligned}$$

Giving us the final expression

$$\boxed{\langle M \rangle = \frac{\partial \ln Z}{\partial B}} \quad (\text{F2012 II 5.7})$$

- (d) Using the definition

$$\langle (\Delta \mu)^2 \rangle = \langle \mu^2 \rangle - \langle \mu \rangle^2$$

we already know  $\langle \mu \rangle^2$  for both cases from the previous problems, so we must only calculate  $\langle \mu^2 \rangle$ .

## Fall 2012 Part I

### Problem 2

#### Question

Show that a particle in a one-dimensional infinite square well initially in a state  $\Psi(x, 0)$  will always return to that state after a time  $T = 4ma^2/\pi\hbar$  where  $a$  is the width of the well.

#### Answer

Use the standard time independent Schödinger equation

$$\Psi(x, t) = \psi(x) e^{iEt/\hbar}$$

with associated differential equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

For an infinite square well, the potential has the form

$$V(x) = \begin{cases} 0 & |x| < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$$

so that the only region to consider is  $-\frac{a}{2} < x < \frac{a}{2}$ . In this region, the differential equation takes the form of a harmonic oscillator

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

leading to solutions

$$\psi(x) = A \cos kx + B \sin kx$$

where  $k^2 = \frac{2mE}{\hbar^2}$

The boundary conditions  $\psi(-\frac{a}{2}) = 0$  and  $\psi(\frac{a}{2}) = 0$  impose

$$\begin{aligned} \psi\left(-\frac{a}{2}\right) &= 0 = A \cos \frac{ka}{2} - B \sin \frac{ka}{2} \\ \psi\left(\frac{a}{2}\right) &= 0 = A \cos \frac{ka}{2} + B \sin \frac{ka}{2} \end{aligned}$$

so that  $B = 0$  and

$$\begin{aligned} 0 &= 2A \cos \frac{ka}{2} \\ \frac{(2n+1)\pi}{2} &= \frac{ka}{2} \\ k &= \frac{(2n+1)\pi}{a} \end{aligned}$$

We already had a relation for  $k$  defined, so substitute and solve for the energies  $E_n$ .

$$\begin{aligned} \frac{(2n+1)^2 \pi^2}{a^2} &= \frac{2mE}{\hbar^2} \\ E_n &= \frac{(2n+1)^2 \pi^2 \hbar^2}{2ma^2} \end{aligned}$$

Then considering  $\Psi(x, t)$ , the complex exponential is periodic in time with period

$$T_n = \frac{2\pi\hbar}{E}$$

where  $n = 0$  will be the case with the longest periodicity, so

$$\begin{aligned} T &= \frac{2\pi\hbar \cdot 2ma^2}{\pi^2\hbar^2} \\ &= \frac{4ma^2}{\pi\hbar} \end{aligned}$$

Therefore, the function is periodic in time with a periodicity

$$T = \frac{4ma^2}{\pi\hbar}$$

(F2012 I 2.1)

## Problem 4

### Question

A photon collides with a stationary electron. If the photon scatters at an angle  $\theta$ , show that the resulting wavelength  $\lambda'$  is given in terms of the original wavelength  $\lambda$  by

$$\lambda' = \lambda + \frac{h}{mc} (1 - \cos \theta)$$

where  $m$  is the mass of the electron.

### Answer

Start by considering conservation of momentum for the system. The initial values are

$$\begin{aligned} p_{\gamma x} &= \frac{h}{\lambda} & p'_{\gamma x} &= \frac{h}{\lambda'} \cos \theta \\ p_{\gamma y} &= 0 & p'_{\gamma y} &= \frac{h}{\lambda'} \sin \theta \\ p_{ex} &= 0 & p'_{ex} &= ? \\ p_{ey} &= 0 & p'_{ey} &= ? \end{aligned}$$

and considering each component in turn:

$$\begin{aligned} \frac{h}{\lambda} + 0 &= \frac{h}{\lambda'} \cos \theta + p'_{ex} & 0 &= \frac{h}{\lambda'} \sin \theta + p'_{ey} \\ p'_{ex} &= \frac{h}{\lambda} - \frac{h}{\lambda'} \cos \theta & p'_{ey} &= -\frac{h}{\lambda'} \sin \theta \end{aligned}$$

The total momentum of the electron is then

$$\begin{aligned} p_e^2 &= \left( \frac{h}{\lambda} - \frac{h}{\lambda'} \cos \theta \right)^2 + \left( -\frac{h}{\lambda'} \sin \theta \right)^2 \\ &= \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'} \cos \theta + \frac{h^2}{\lambda'^2} \cos^2 \theta + \frac{h^2}{\lambda'^2} \sin^2 \theta \\ p_e^2 &= h^2 \left( \frac{1}{\lambda^2} + \frac{1}{\lambda'^2} \right) - \frac{2h^2}{\lambda\lambda'} \cos \theta \end{aligned} \tag{F2012 I 4.1}$$

Then consider energy conservation, with initial values

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} & E'_\gamma &= \frac{hc}{\lambda'} \\ E_e &= mc^2 & E'_e &= \frac{p_e'^2}{2m} + mc^2 \end{aligned}$$

leading to the equation

$$\begin{aligned}
 \frac{hc}{\lambda} + mc^2 &= \frac{hc}{\lambda'} + \frac{p_e'^2}{2m} + mc^2 \\
 \frac{hc}{\lambda} &= \frac{hc}{\lambda'} + \frac{h^2}{2m} \left( \frac{1}{\lambda^2} + \frac{1}{\lambda'^2} \right) - \frac{2h^2}{2m\lambda\lambda'} \cos \theta \\
 \frac{hc}{\lambda} - \frac{hc}{\lambda'} &= \frac{h^2}{2m} \left( \frac{1}{\lambda^2} + \frac{1}{\lambda'^2} \right) - \frac{2h^2}{2m\lambda\lambda'} \cos \theta \\
 \frac{\lambda' - \lambda}{\lambda\lambda'} &= \frac{h}{2mc} \frac{\lambda'^2 + \lambda^2}{\lambda^2\lambda'^2} - \frac{h}{mc\lambda\lambda'} \cos \theta \\
 \lambda' - \lambda &= \frac{h}{2mc} \left( \frac{(\lambda' - \lambda)^2 + 2\lambda\lambda'}{\lambda\lambda'} \right) - \frac{h}{mc} \cos \theta \\
 \lambda' - \lambda &= \frac{h}{2mc} \left( \frac{(\lambda' - \lambda)^2}{\lambda\lambda'} + 2 \right) - \frac{h}{mc} \cos \theta \\
 \lambda' - \lambda &= \frac{h}{2mc} \frac{(\lambda' - \lambda)^2}{\lambda\lambda'} + \frac{h}{mc} (1 - \cos \theta)
 \end{aligned}$$

The difference in the wavelengths is small, so

$$\frac{(\lambda' - \lambda)^2}{\lambda\lambda'} \approx 0$$

leading to the final Compton scattering equation

$$\lambda' = \lambda + \frac{h}{mc} (1 - \cos \theta)$$

(F2012 I 4.2)