Root locus technique:

open loop TF = $DGH = L(s)$.

D: controller TF

G: Plant TF.

H: F/b TF.

CE : $1+DGH=0$ or $1+L(s)=0$.

closed loop TF $\dfrac{Y(s)}{R(s)} = \dfrac{DG}{1+DGH}$.

$\boxed{\text{Roots of CE are also poles of closed loop TF}}$

Root locus is the locus of roots of CE (closed loop poles) when the system gain k is varied from 0 to ∞.

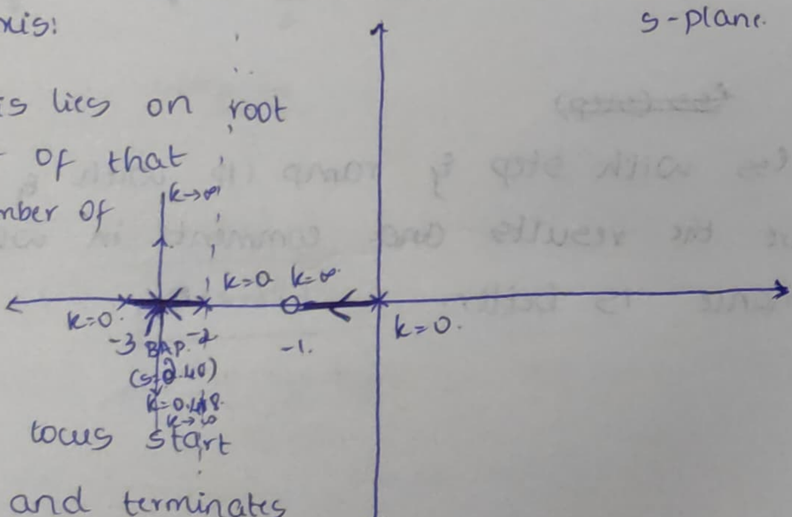
Set of rules for the construction of root locus:

Ex: The open loop TF $L(s) = DGH = \dfrac{K(s+1)}{s(s+2)(s+3)}$. construct the root locus when the system gain K is varied from 0 to ∞.

1. Open loop poles: $s=0,\ s=-2,\ s=-3$.
   Open loop zeros: $s=-1$.

2. Root locus on real axis:

A point on the real axis lies on root locus, if to the right of that point the sum of number of poles and zeros is an odd number.



3. Every branch of root locus start from open loop pole and terminates either on open loop zero or at infinity along the asymptotes.

4. No of asymptotes and asymptotic angles

No of asymptotes = No of open loop poles (P) − No of open loop zeros (Z).

$= 3 - 1$

$= 2.$

Two branches of root locus terminate at ∞ and guided by asymptotes.

Angles of asymptotes = ~~Σ real part of poles − Σ real part of zer~~

$$= \frac{(2q+1)180}{P-Z} \qquad q = 0, 1 \text{ (2 asymptotes)}.$$

$$= \frac{(2q+1)180}{2}$$

$q = 0 \Rightarrow \theta_1 = 90°$

$q = 1 \Rightarrow \theta_2 = 270°$

5. centroid: It is the point on real axis where the asymptotes meet.

$$\sigma = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{P - Z}$$

$$= \frac{(0-2-3) - (-1)}{2} = -2.$$

Asymptotes: They are the straight line paths which guides the root locus to terminate at ∞.

6. Break away point: It is the point where the root locus branches meet and breaks away.

$$\frac{dK}{ds} = 0.$$

CE : $1 + L(s) = 0$.

$$1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0.$$

$$K = -\frac{s(s+2)(s+3)}{(s+1)}.$$

$$= -\frac{(s^2+2s)(s+3)}{s+1}$$

$$= -\frac{(s^3+5s^2+6s)}{s+1}.$$

$$\frac{dK}{ds} = 0 \Rightarrow -\left\{\frac{(s+1)(3s^2+10s+6)-(s^3+5s^2+6s)}{(s+1)^2}\right\} = 0.$$

$$\Rightarrow 3s^3+13s^2+16s+6-s^3-5s^2-6s=0.$$

$$\Rightarrow 2s^3+8s^2+10s+6=0.$$

$s = -2.46$, $s = -0.767+0.792j$,

valid BAP $s = -0.767-0.792j$

Value of K at BAP: CE; $1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0$

Every point on root locus (RL) must satisfy CE.

$$\left|\frac{K(s+1)}{s(s+2)(s+3)}\right| = |-1|.$$

$$\left|\frac{K(s+1)}{s(s+2)(s+3)}\right| = 1.$$

$$K = \left|\frac{s(s+2)(s+3)}{s+1}\right|$$

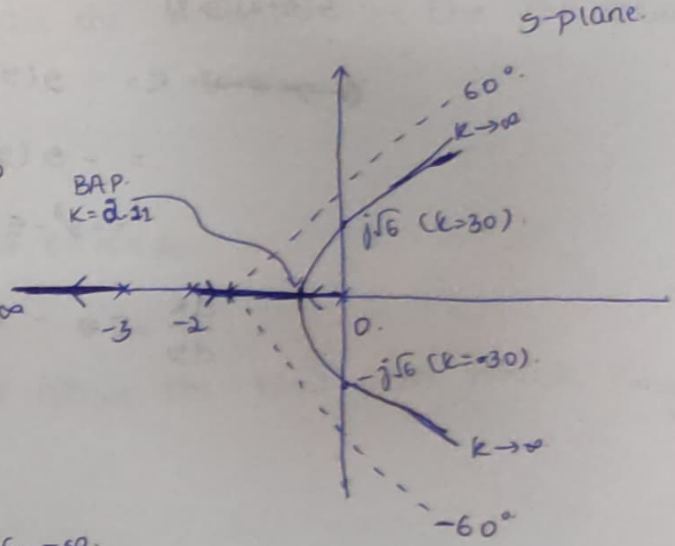
$= 0.418$

RL is symmetrical wrt real axis

For all values of K from 0 to ∞, the RL lies to the left half of the s-plane.

∴ The system is absolutely stable.

1. $L(s) = \dfrac{K}{s(s+2)(s+3)}$



Open loop poles, $s = 0, -2, -3$

$P = 3$

Open loop zeros: nil

$Z = 0$

<u>Root locus on real axis:</u>

Lies b/w $s=0$ & -2 & $s=-3$ & $-\infty$.

<u>No of asymptotes:</u>

$P - Z = 3$

Angle of asymptotes:

$$\theta_A = \frac{(2q+1)180}{P-Z} \qquad q = 0, 1, 2.$$

$q = 0 \Rightarrow \theta_A = 60°$

$q = 1 \Rightarrow \theta_A = 180°$

$q = 2 \Rightarrow \theta_A = 300° = -60°$

<u>Centroid:</u>

$$\sigma = \frac{\Sigma \text{RP of poles} - \Sigma \text{RP of zeros}}{P-Z}$$

$$= \frac{(-3-2) - 0}{3}$$

$$= -\frac{5}{3} = -1.66$$

Break away points: $\frac{dK}{ds} = 0$.

$CE = 1 + L(s) = 0$.

$$1 + \frac{K}{s(s+2)(s+3)} = 0.$$

$$\cancel{s\ s(s+2)}\ K = -s(s+2)(s+3).$$

$$= -s(s^2+5s+6)$$

$$= -s^3 - 5s^2 - 6s.$$

$$\frac{dK}{ds} = 0 \Rightarrow -3s^2 - 10s - 6 = 0.$$

$$3s^2 + 10 + 6 = 0.$$

$$s = \frac{-10 \pm \sqrt{100 - 72}}{6}.$$

$$s_1 = -0.784 \checkmark \quad , s_2 = -2.5485. \times.$$

Value of at BAP: ~~K = 2.11~~

$$\left| \frac{K}{s(s+2)(s+3)} \right| = 1 - 11$$

$$\left| \frac{K}{s(s+2)(s+3)} \right| = 1.$$

$$K = |s(s+2)(s+3)|.$$

$$K = 2.11.$$

Intersection of root locus with imaginary axis. This can be obtained using RH criteria.

$$CE: 1 + \frac{K}{s(s+2)(s+3)} = 0$$

$$s^3 + 5s^2 + 6s + K. = 0.$$

s^3	1	6
s^2	5	K
s	$\frac{30-K}{5}$	0
s^0	K	

For the system to be stable, all the elements in the 1st column must be +ve

$\frac{30-K}{5} > 0$ & $K > 0$

$K < 30$ $K > 0 \Rightarrow 0 < K < 30$.

$K_{marginal} = 30$

consider s^2 row which is just above the row from which $K_{marginal}$ is obtained.

$A(s) = 5s^2 + K = 0$

$5s^2 + 30 = 0$

$s^2 = -6$

$s = \pm j\sqrt{6} = \pm j\omega$

$\omega = \sqrt{6}$ r/s

when $K = 30$, branches of RL cuts $j\omega$ axis at $\pm j\omega = \pm j\sqrt{6}$ and the system becomes marginally stable.

For $K > 30$, the RL enters into RH of s-plane and the system becomes unstable

$L(s) = \frac{K(s+1)}{s(s+2)(s+3)}$ $\xleftarrow{\text{Added 1 zero}}$ $L(s) = \frac{K}{s(s+2)(s+30)}$

Range of K for stability	$0 < K < \infty$	$\Rightarrow$ stability improves.	$0 < K < 30$

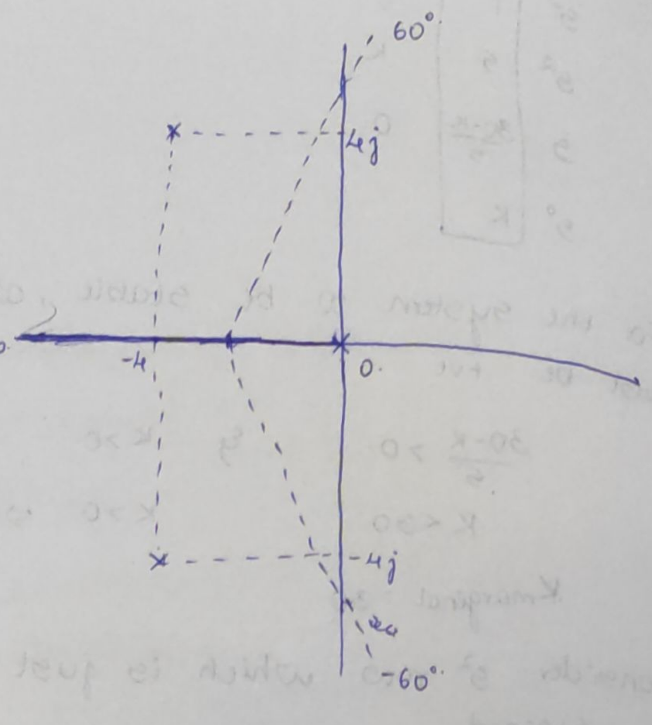
Addition of zero improves stability of system.

Complex poles:

$$L(s) = \frac{K}{s((s+4)^2+16)}$$

$$= \frac{K}{s(s^2+8s+32)}$$

$$= \frac{K}{s(s-4-4j)(s-4+4j)}$$



Open loop poles:

$s=0$,

$s=-4+4j$

$s=-4-4j$

RL on real axis lies b/w $s=0$ & $-\infty$.

NO of asymptotes: $P-Z=3$.

$\theta_A = 60°, 180°, 300°$.

$$\sigma = \frac{(0-4-4)-(0)}{3}$$

$$= \frac{-8}{3}$$

$$= -2.66.$$

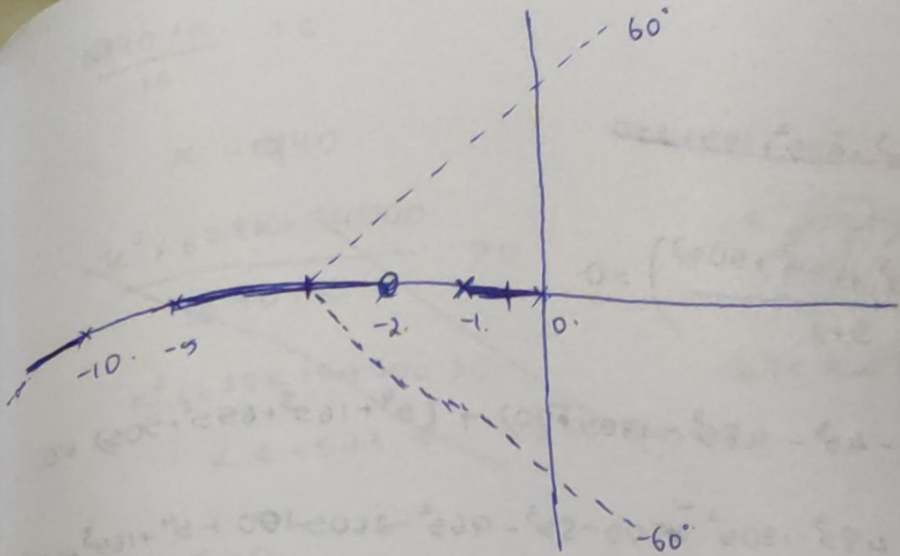
Angle of departure: For complex poles, it is necessary to find the angle of departure. It is given by $\boxed{\theta_d = 180 - \theta_p + \theta_z}$

1. Sketch the root locus WRT K for the equation $1+KL(s)=0$ and the following $L(s)$. Be sure to give asymptotes & arrival and departure angles at any complex zeros or poles and the frequency of any imaginary axis crossing

i) $L(s) = \frac{s+2}{s(s+1)(s+5)(s+10)}$.

poles: $0, -1, -5, -10$.

zeros: -2.



Number of asymptotes: $p-k = 4-1 = 3$

angle of asymptotes : $\frac{(2q+1)180}{p-z}$

$q=0 \Rightarrow \theta_A = 60°$

$q=1 \Rightarrow \theta_B = 180°$

$q=2 \Rightarrow \theta_C = -60°$

centroid:

$$\sigma = \frac{(-1-5-10)-(-2)}{3}$$

$$= \frac{-14}{3}$$

$$= -4.66.$$

BAP:

$CE = 1 + KL(s) = 0.$

$$1 + \frac{(s+2)K}{s(s+1)(s+5)(s+10)} = 0.$$

$$K = -\frac{(s^2+s)(s^2+15s+50)}{s+2}.$$

$$= -\cancel{\frac{(s^4+5s^2+15s^3+75s+50s^2+250)}{s+2}}.$$

$$= -\frac{(s^4+s^3+15s^3+15s^2+50s^2+50s)}{s+2}.$$

$$\frac{dK}{ds} = 0.$$

$$\cancel{\frac{d}{ds}\left(-s^4+15s^3+55s^2+75s+250\right.}$$

$$\frac{d}{ds}\left(-\frac{(s^4+16s^3+65s^2+50s)}{s+2}\right) = 0.$$

$$(s+2)(-4s^3-48s^2-130s-50) + (s^4+16s^3+65s^2+50s) = 0.$$

$$-4s^4-48s^3-130s^2-50s-8s^3-96s^2-260s-100+s^4+16s^3+65s^2+50s = 0$$

$$-3s^4-40s^3-161s^2-260s-100 = 0$$

$$3s^4+40s^3+161s^2+260s+100=0.$$

$$s=-0.54$$

$$1+\frac{K(s+2)}{s(s+1)(s+5)(s+10)} = 0.$$

$$1+\frac{K(s+2)}{s(s+1)(s+5)(s+10)} = 0.$$

$$\cancel{K = \frac{s^4+16s^3+65s^2+50s}{s+2}}$$

$$s^4+16s^3+65s^2+50s+Ks+2K=0.$$

s^4	1	65	2K.
s^3	16.	50+K	
s^2	$\frac{990-K}{16}$	2K.	
s	$\frac{-K^2+428K+49500}{990-K}$	0	
s^0	2K.		

$$\frac{\frac{54500+1140K+K^2}{16}-32K}{\frac{1090+K}{16}}$$

$$\cancel{\frac{54500+1140K+K^2}{1090+K}} - 512K.$$

$$\frac{\frac{49500+940K-K^2}{16}-32K}{990-K/16}$$

$$\frac{-K^2+428K+49500}{990-K}$$

$$\frac{1990+K}{16} > 0$$

$K < 1990$

$$\frac{K^2+628K+54500}{1990+K} > 0$$

$K^2+628K+54500 > 0$

$K < -523.9 \qquad K > -104.00$

$-K^2 + 428K + 49900 > 0$

$K^2 - 428K - 49500 < 0$

$-94.7 < K < 522.7$

$K > 0$

$A(s) = 16s^3 + 1040s = 0$

$s(16s^2+1040) = 0$

$s = 0, \sqrt{\frac{-1040}{16}}$

$s = 8.06j$

$s = \sigma + j\omega$

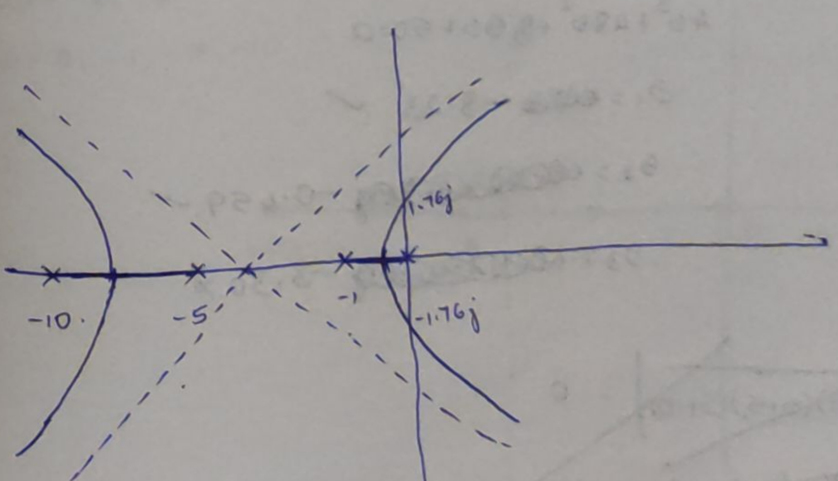
$s = j\omega$

$\omega = 8.06$ r/s

2. $L(s) = \dfrac{1}{s(s+1)(s+5)(s+10)}$

zeros: nil

poles: 0, −1, −5, −10.



no of Asymptotes = 4.

angle of asymptotes = $\frac{(2q+1)180}{4}$

θ_A : $q=0 \Rightarrow 45$.

θ_B : $q=1 \Rightarrow 135$

θ_C : $q=2 \Rightarrow 225 = -135°$

θ_D : $q=2 \Rightarrow 315 = -45°$

Centroid : $\frac{-1-5-10}{4}$

$= -4$.

BAP: ~~[illegible]~~

$$1+\frac{K}{s(s+1)(s+5)(s+10)}=0$$

$$K = -s(s+1)(s+5)(s+10)$$

$$= -(s^2+s)(s^2+15s+50)$$

$$= -s^4-s^3-15s^3-15s^2-50s^2-50s$$

$$K = -s^4-16s^3-65s^2-50s$$

$$\frac{dK}{ds}=0$$

$$\frac{d}{ds}(-s^4-16s^3-65s^2-50s)=0$$

$$4s^3+48s^2+130s+50=0$$

$s_1 =$ -8.23 ✓

$s_2 =$ -0.459 ✓

$s_3 =$ -3.30 ✗

~~$1+\frac{K}{s(s+1)(s+5)(s+10)} = 0$~~

~~$K = -(s(s+1)(s+5)(s+10))$~~

=

$$1+\frac{K}{s(s+1)(s+5)(s+10)}=0.$$

$$s^4+16s^3+65s^2+50s+K=0.$$

s^4	1	65	K.
s^3	16	50	
s^2	61.875	K.	
s	$\frac{3093.75-16K}{61.875}$		
s^0	K.		

$3093.75-16K>0.$

$K<193.35.$

$A(s)=61.875s^2+193.35=0.$

$$s^2=\sqrt{\frac{-193.35}{61.875}}$$

$s=\pm 1.76j$

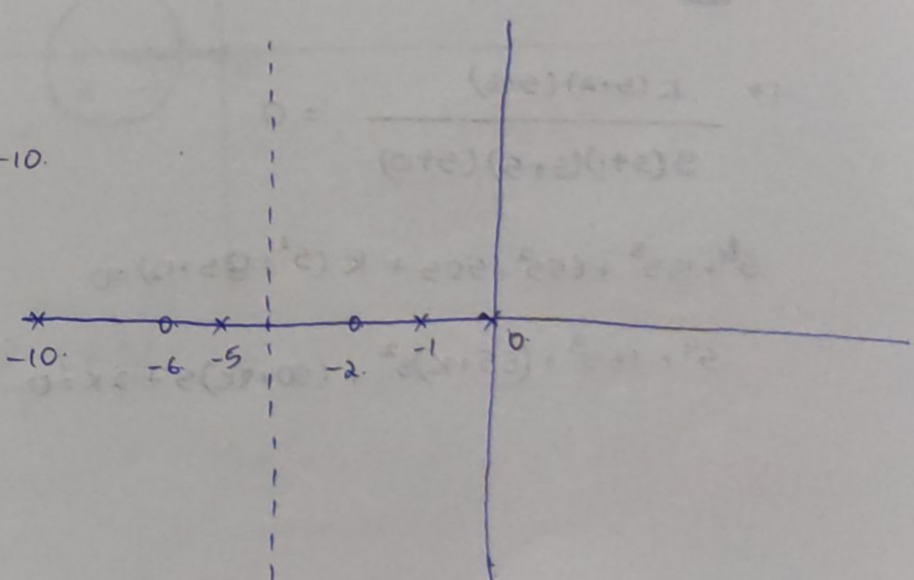
$\omega=1.76.$

3. $L(s)=\frac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

~~$1+L(s)=0$~~.

zeros $=-2,-6$

poles $=0,-1,-5,-10.$



no of asymptotes $= 4-2 = 2$.

$\theta_A = 90^\circ$

$\theta_B = -90^\circ$

centroid: $\dfrac{(-1-5-10)-(-2-6)}{2}$

$= -4$.

BAP: $1 + \dfrac{K(s+2)(s+6)}{s(s+1)(s+5)(s+10)} = 0$.

$K = \dfrac{-[s(s+1)(s+5)(s+10)]}{(s+2)(s+6)} = -\dfrac{[s^4+16s^3+65s^2+50s]}{s^2+8s+12}$

$\dfrac{dK}{ds} = -\left[\dfrac{(s^2+8s+12)(4s^3+48s^2+130s+50)-(s^4+16s^3+65s^2+50s)(2s+8)}{(s+2)^2(s+6)^2}\right]$

$4s^5+32s^4+48s^4+48s^3+384s^3+576s^2+130s^3+1040s^2+1560s+50s^2+400s+600$
$-2s^5-8s^4-32s^4-128s^3-130s^3-520s^2-100s-400=0$

$2s^5+40s^4+304s^3+1146s^2+1860s+200=0$

$s=-0.568$

$1 + \dfrac{K(s+2)(s+6)}{s(s+1)(s+5)(s+10)} = 0$.

$s^4+16s^3+65s^2+50s+K(s^2+8s+12)=0$.

$s^4+16s^3+(65+K)s^2+(50+8K)s+12K=0$.

s^4	1	$65+K$	$12K$
s^3	16	$50+8K$	
s^2	$\frac{990+8K}{16}$	$12K$	
s	$\frac{64K^2+8128K+49500}{990+8K}$		
s^0	$12K$		

$$\frac{\frac{990+8K}{16} \times 50+8K - 192K}{\frac{990+8K}{16}}$$

$$\frac{49500+8320K+64K^2-192K}{990+8K}$$

$$\frac{64K^2+8128K+49500}{990+8K}$$

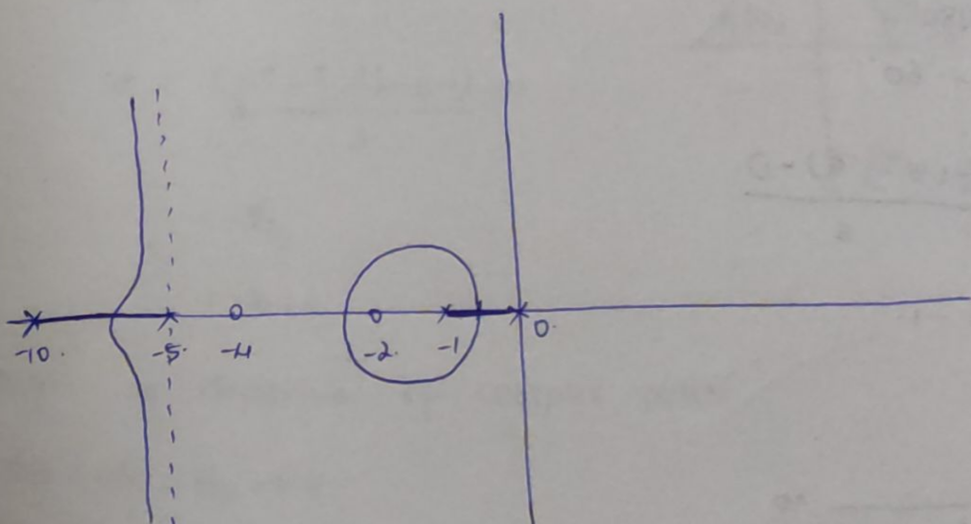
$$\frac{990+8K}{16} > 0$$

$$K \gtrless \frac{-990}{8} = -123.75$$

4. $L(s) = \frac{(s+2)(s+4)}{s(s+1)(s+5)(s+10)}$

zeros: $-2, -4$

poles: $0, -1, -5, -10$

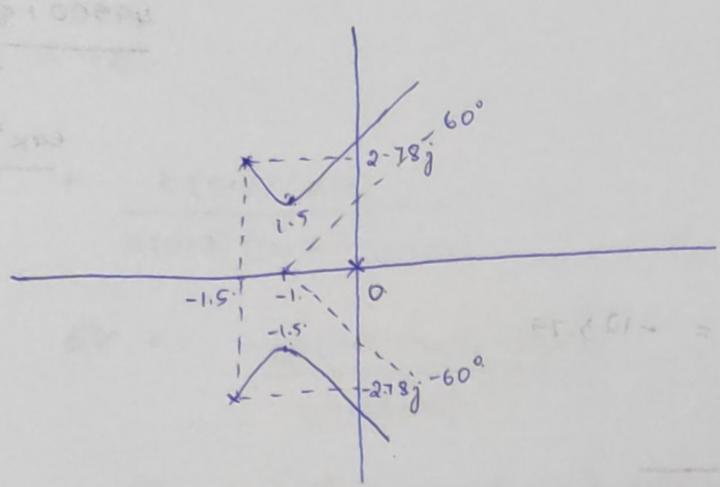


centroid: $\frac{(-1-5-10)-(-2-4)}{2} = 5$

5. $L(s) = \dfrac{1}{s(s^2+3s+10)}$

$\dfrac{1}{s(s+1.5-2.78j)(s+1.5+2.78j)}$

poles $= 0, \; -1.5-2.78j, \; -1.5+2.78j$



Asymptotes $= 3$.

$\theta_A = 60°$

$\theta_B = 180°$

$\theta_C = -60°$

centroid $= \dfrac{(-1.5-1.5)-0}{3}$

$= -1$.

$1 + KL(s) = 0$.

$1 + \dfrac{K}{s(s^2+3s+10)} = 0$

$K = -s^3 - 3s^2 - 10s$

$\dfrac{dK}{ds} = 0 \Rightarrow -3s^2 - 6s - 10 = 0$

$3s^2 + 6s + 10 = 0$.

$s = -1 + 1.52j \;, \; s = -1 - 1.52j$

$s^3+3s^2+10s+K=0$

s^3	1	10
s^2	3	K
s	$\frac{30-K}{3}$	0
s^0	K	

$30-K>0$

$K<30$

$3s^2+30=0$

$s^2=-10$

$s=\pm 3.16j$

6. $L(s)=\dfrac{s+1}{s(s+5)(s^2+4s+8)}$

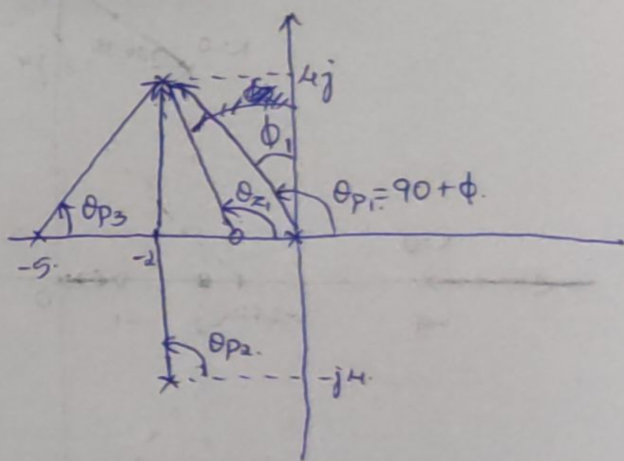
$p-z=3$

$\theta_A=60^\circ, 180^\circ, 300^\circ$

$\sigma=\dfrac{(-5-2-2)-(-1)}{3}$

$=-8/3$

$=-2.66$



Angle of departure of complex poles

$\theta_d=180-\theta_p+\theta_z$

$\theta_p=\Sigma$ angle contribution from all other poles at the complex pole.

$\theta_z=\Sigma$ angle contribution from all other zeros at the complex pole.

$\theta_p=\theta_{p_1}+\theta_{p_2}+\theta_{p_3}$

$\theta_z=\theta_{z_1}$

$\phi = \tan^{-1}\left(\frac{2}{4}\right) = 26.56.$

$\theta_{p1} = 90 + 26.56 = 116.56.$

$\theta_{p2} = 90^\circ.$

$\theta_{p3} = \tan^{-1}\left(\frac{4}{3}\right) = 53.13.$

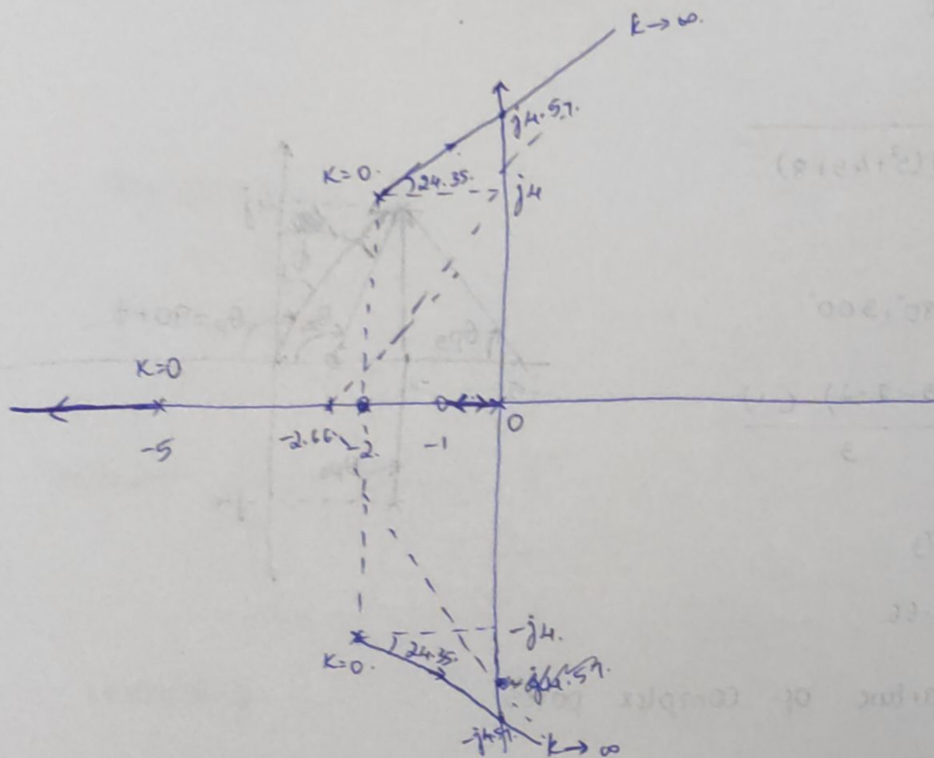
$\theta_p = 116.56 + 90 + 53.13 = 259.69.$

$\theta_{z1} =$ ~~180~~ $180 - \tan^{-1}\left(\frac{4}{1}\right) = 104.03.$

$\theta_d = 180 - \theta_p + \theta_z.$

$= 180 - 259.69 + 104.03$

$\theta_d = 24.34.$



Intersection of root locus with imaginary axis.

$CE = 1 + L(s) = 0.$

$s(s+5)(s^2+4s+8) + K(s+1) = 0.$

$(s^2+5s)(s^2+4s+8) + K(s+1) = 0.$

$s^4 + 9s^3 + 28s^2 + 40s + Ks + K = 0$

$s^4 + 9s^3 + 28s^2 + (40+K)s + K = 0$

s^4 1 28 K.

s^3 9 40+K.

s^2 $\frac{212-K}{9}$ K.

s $\frac{-K^2+91K+8480}{212-K}$

s^0 K.

$$\frac{\frac{8480+172K-K^2}{9}-9K}{\frac{212-K}{9}}$$

$$\frac{8480+172K-K^2-81K}{212-K}$$

$$\frac{-K^2+91K+8480}{212-K}$$

$212-K>0$

$K<212.$

$-K^2+91K+8480>0.$

$K^2-91K-8480<0.$

$-57.21<K<148.21$

~~for the system to be stable all elements~~

$K_{marginal}=148.21.$

$$A(s)=\frac{212-148.2}{9}s^2+148.2=0.$$

$$7.08s^2=-148.2.$$

$$s^2=\sqrt{-20.90}$$

$$s=\pm j4.57.$$

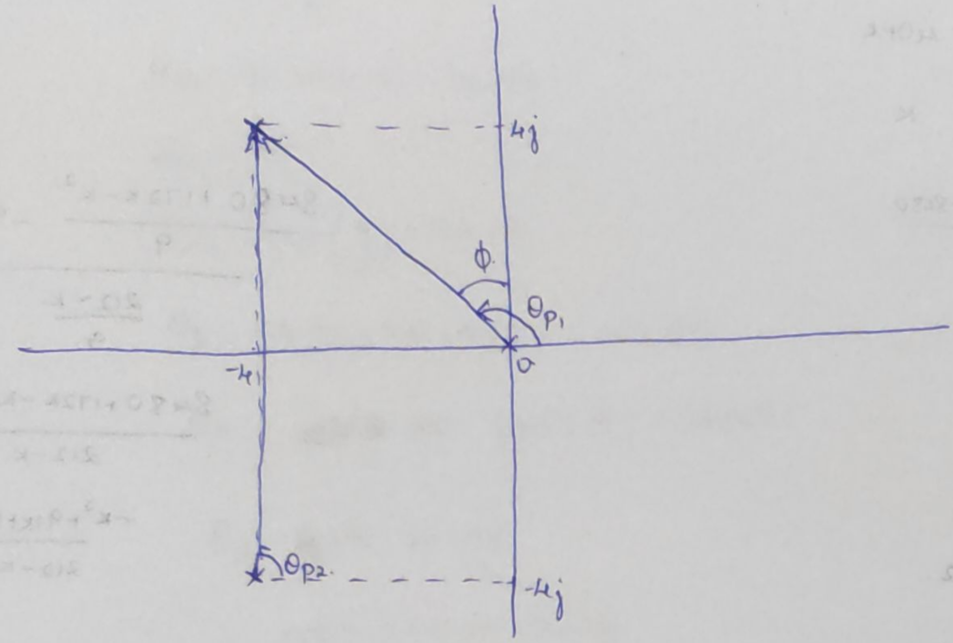
$$\omega=4.57$$

7. Construct the root locus of $L(s)=\frac{K}{s((s+4)^2+16)}$

$P-Z=3$

$\theta_A=60^\circ, 180^\circ, 300^\circ.$

$$\sigma=\frac{(-4-4)}{3}=\frac{-8}{3}=-2.26$$



$\theta_{p_1} = 90 + \phi$

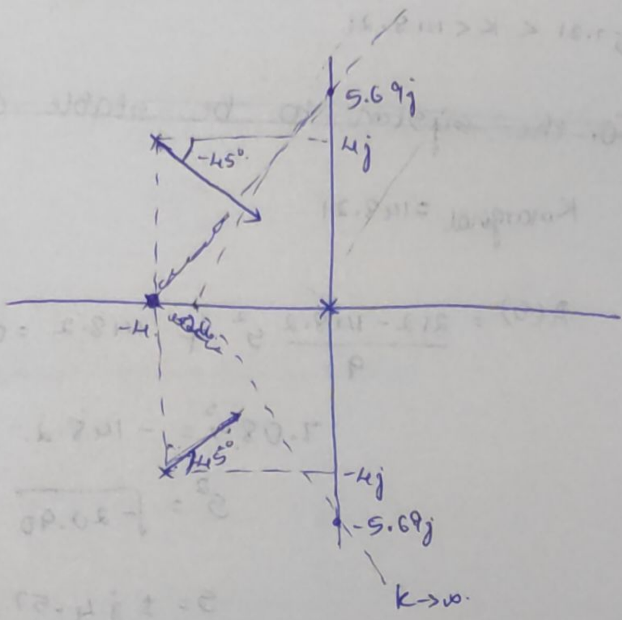
$\phi = \tan^{-1}(1) = 45°$

$\theta_{p_1} = 135°$

$\theta_{p_2} = 90°$

$\theta_p = 225°$

$\theta_d = 180 - 225 = -45°$



$1 + L(s) = 0$

$1 + \dfrac{K}{s\left((s+4)^2+16\right)} = 0$

$s(s^2+8s+32)+K=0$

$s^3+8s^2+32s+K=0$

s^3	1	32
s^2	8	K
s	$\frac{256-K}{8}$	
s^0	K	

$$\frac{256-K}{8} > 0$$

$$256 > K.$$

$$K_{marginal} = 256.$$

$$A(s) = 8s^2 + K = 0.$$

$$8s^2 + 256 = 0.$$

$$s^2 = -\frac{256}{8}$$

$$= -32$$

$$s = \pm j5.69.$$

$$\omega = 5.69.$$

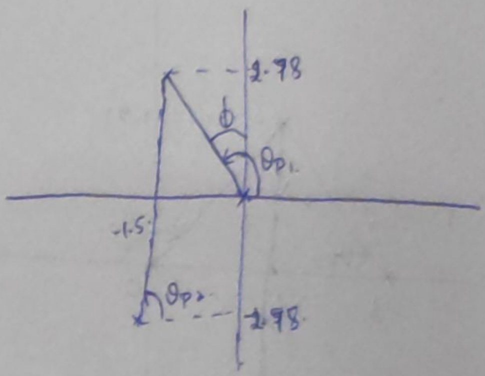
$L(s) = \dfrac{1}{s(s^2+3s+10)}$. Find angle of departure.

$p - z = 3.$

$\theta_A = 60°, 180°, 300°$

$\sigma = \dfrac{(-1.5 - 1.5)}{3.}$

$= -1.$



$\theta_d = 180 - \theta_p + \theta_z.$

$\theta_p = \theta_{p_1} + \theta_{p_2}.$

$\theta_{p_1} = 90 + \phi.$

$= 90 + \tan^{-1}\left(\dfrac{1.5}{2.78}\right)$

$= 118.34.$

$\theta_{p_2} = 90°$

$\theta_p = 208.34.$

$\theta_d = 180 - 208.34 = -28.34.$

2. $L(s) = \dfrac{1}{s^2+3s+10.}$. Find angle of departure.

Sketch the root locus $L(s) = \dfrac{1}{s(s+3)(s^2+3s+4.5)}$

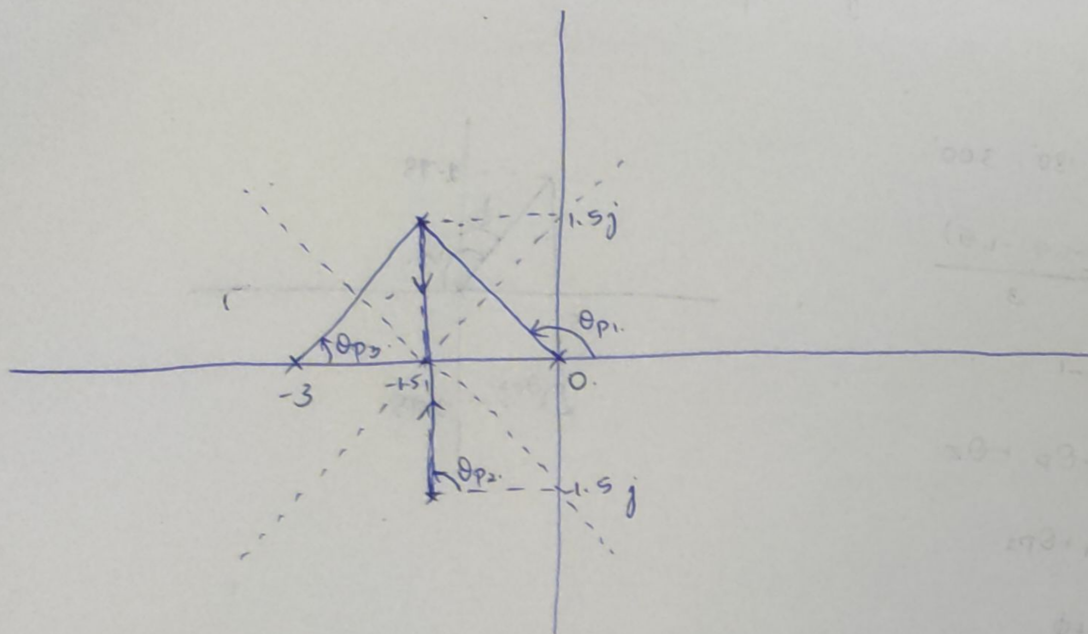
poles: $0, -3, -1.5+1.5i, -1.5-1.5i.$

centroid: $\sigma = \dfrac{(-3-1.5-1.5)}{4}$

$= -1.5$

$p - z = 4.$

$\theta_A = 45°, 135°, 225°, 315°$



BAP : $1 + KL(s) = 0$.

$$1 + \frac{K}{s(s+3)(s^2+3s+4.5)} = 0$$

$$K = -[(s^2+3s)(s^2+3s+4.5)]$$

$$= -[s^4 + 6s^3 + 13.5s^2 + 13.5s]$$

$$\frac{dk}{ds} = 0 \Rightarrow -[4s^3 + 18s^2 + 27s + 13.5] = 0$$

$$4s^3 + 18s^2 + 27s + 13.5 = 0$$

~~$s = -4.05, -0.973+0.403j, -0.973-0.403j$~~

$$s = -1.5, -1.5, -1.5.$$

$$s^4 + 6s^3 + 13.5s^2 + 13.5s + K = 0.$$

s^4	1	13.5	K.
s^3	6.	13.5.	
s^2	11.25.	K.	
s	$\frac{151.87-6K}{11.25}$.		
s^0	K.		

$151.87-6K > 0$.

$6K \leq 151.87$.

$K < 25.3125$.

$11.25s^2 + K = 0$.

$s^2 = -\frac{25.3125}{11.25}$.

$s = \pm 1.5j$

$\theta_{p_1} = 90+45 = 135°$

$\theta_{p_2} = 90°$

$\theta_{p_3} = 45°$

$\theta_d = 180 - \theta_p$.

$= 180 - 270$.

$= -90$

2. $L(s) = \frac{s+1}{s(s-1)(s^2+5s+20)}$. Sketch the root locus.

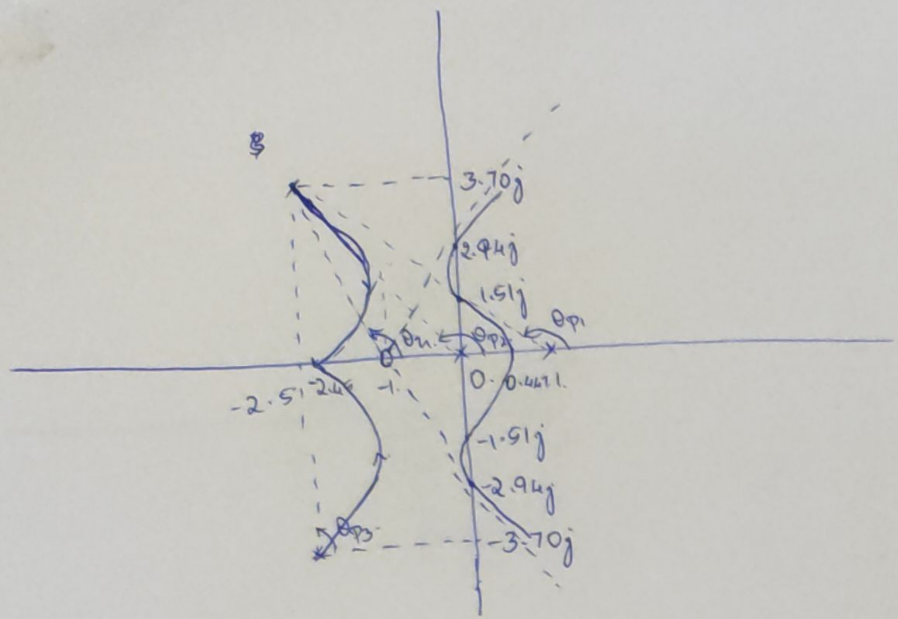
$P = 0, 1, -2.5+3.70j, -2.5-3.70j$

$z = -1$.

No of asymptotes = 3.

$\theta = 60°, 180°, 300°$

$\sigma = \frac{(1-2.5-2.5)+1}{3} = -1$.



BAP: $1+KL(s)=0$.

$$1+\frac{K(s+1)}{s(s-1)(s^2+5s+20)}=0.$$

$$K=-\frac{[(s^2-s)(s^2+5s+20)]}{s+1}.$$

$$=-\frac{[s^4+4s^3+15s^2-20s]}{s+1}.$$

$$\frac{dK}{ds}=0 \Rightarrow -\left[\frac{(s+1)(4s^3+12s^2+30s-20)-(s^4+4s^3+15s^2-20s)}{(s+1)^2}\right]=0$$

$$4s^4+16s^3+42s^2+10s-20-s^4-4s^3-15s^2+20s=0.$$

$$3s^4+12s^3+27s^2+30s-20=0.$$

$s=-2.45$.

$s=0.447$.

$s^4 + 4s^3 + 19s^2 - 20s + Ks + K = 0.$

$s^4 + 4s^3 + 15s^2 + (K-20)s + K = 0.$

s^4	1	15	K
s^3	4	K-20	
s^2	$\frac{-K+80}{4}$	K	
s	$\frac{-K^2+84-1600}{80-K}$		
s^0	K		

$$\frac{\frac{(K-20)(80-K)}{4} - 4K}{\frac{80-K}{4}}$$

$$\frac{-K^2+100K-1600-16K}{80-K}$$

$$\frac{-K^2+84K-1600}{80-K}$$

$-K^2+84K-1600 > 0.$

$K^2 - 84K + 1600 < 0.$

$29.19 < K < 54.80$

$\frac{-K+80}{4} > 0.$

$80 > K.$

$A(s) = \frac{80-K}{4} s^2 + K = 0.$

$12.70s^2 + 29.19 = 0.$

$s = \pm 1.51j$

$6.35s^2 + 54.8 = 0$

$s^2 = \pm 2.94j$

$\theta_{p_1} = 90 + 43 = 133.40.$

$\theta_{p_2} = 34.04 + 90 = 124.04.$

$\theta_{p_3} = 90.$

$\theta_{z_1} = 112.06.$

$\theta_d = 180 - \theta_p + \theta_z = -55.38.$

3. Sketch the root locus: $L(s) = \dfrac{s^2+4s+20}{s(s+3)}$

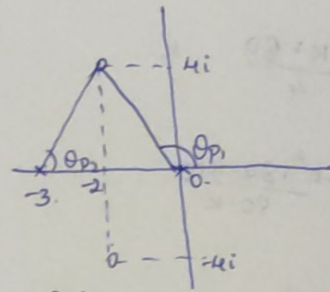
$P = 0, -3$

$z = -2+4i, -2-4i$

No of asymptotes $= 0$.

$\theta = 0$

$\sigma = \dfrac{-3-(-2-2)}{0} = \infty$



$\theta_d = \theta_z - \theta_p$

$= 90 - 116.56 - 75.96$

$= -102.52$

$\theta_a = 180 + \theta$

$\theta_a = 77.48°$

$1 + KL(s) = 0$

$1 + \dfrac{K(s^2+4s+20)}{s(s+3)} = 0$

$K = -\dfrac{(s^2+3s)}{s^2+4s+20}$

$\dfrac{dK}{ds} = 0 \Rightarrow \dfrac{(s^2+4s+20)(2s+3) - (s^2+3s)(2s+4)}{(s^2+4s+20)^2} = 0$

$2s^3 + 11s^2 + 52s + 60 - 2s^3 - 10s^2 - 12s = 0$

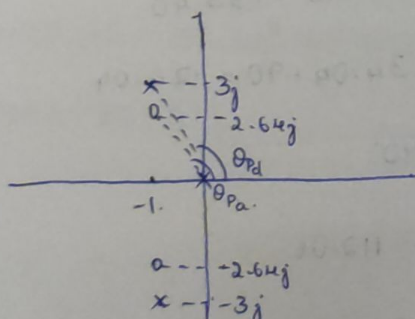
$\cancel{s^3} + s^2 + 40s + 60 = 0$

$s = -1.56, -38.43$

4. $L(s) = \dfrac{s^2+2s+8}{s(s^2+2s+10)}$

$P = 0, -1+3i, -1-3i$

$z = -1+2.64j, -1-2.64j$



$\theta_d = 180 - \theta_p + \theta_z$

$= 180 - (90+108.43) + 180$

$= 161.57°$

$\theta_a = 180 - (90+90) + 110.74 + 90$

$= 290.74°$

No of asymptotes = 1.

$\theta = 180^\circ$

$\sigma = \frac{(-1-1)-(-1-1)}{1} = 0.$

$1 + KL(s) = 0$

$1 + \frac{K(s^2+2s+8)}{s(s^2+2s+10)} = 0.$

$K = -\frac{s(s^2+2s+10)}{s^2+2s+8}$

$\frac{dK}{ds} = 0 \Rightarrow -\left[\frac{(s^2+2s+8)(3s^2+4s+10) - (s^3+2s^2+10s)(2s+2)}{(s^2+2s+8)^2}\right] = 0$

$3s^4 + 10s^3 + 42s^2 + 52s + 80 - 2s^4 - 6s^3 - 24s^2 - 20s = 0$

$s^4 + 4s^3 + 18s^2 + 32s + 80 = 0.$

5. Sketch the root locus: $L(s) = \frac{s^2+1}{s(s^2+4)}$.

$\alpha = 0$

$\theta = \pm 180^\circ$

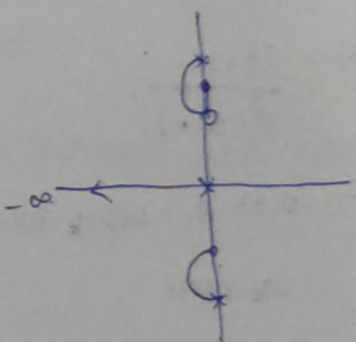
$\theta_d = \pm 180^\circ$

$\theta_a = \pm 180^\circ$

ω: none

$P = 0, \pm 2j$

$Z = \pm j$.



6. $L(s) = \frac{(s+1)^2}{s^3(s+4)}$.

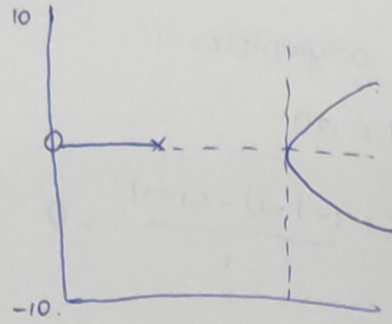
$P = 0, -4.$

$Z = -1, -1.$

$P - Z = 2.$

$\sigma = \frac{(0-4)-(-1-1)}{2} = 2.$

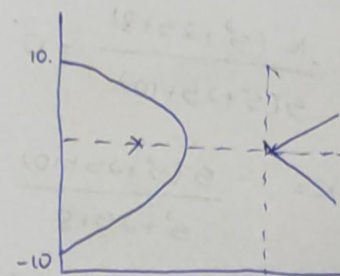
7. a) $L(s) = \dfrac{1}{s^2(s+8)}$



$\sigma = -2.67$, $\phi_i = \pm 60, \pm 180$

ω_0 = none.

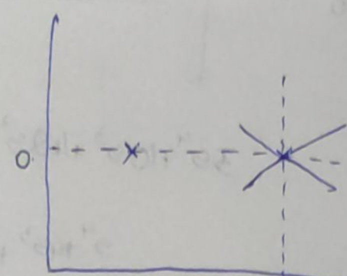
b) $L(s) = \dfrac{1}{s^3(s+8)}$



$\sigma = -2$, $\phi_i = \pm 45, \pm 135$.

ω_0 = none.

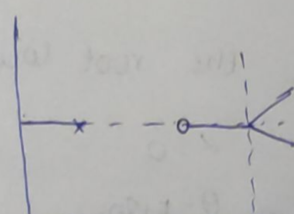
c) $L(s) = \dfrac{1}{s^4(s+8)}$



$\sigma = -1.6$ $\phi_i = \pm 36, \pm 108$,

ω_0 = none

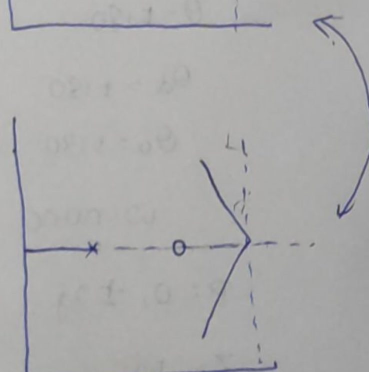
d) $L(s) = \dfrac{s+3}{s^2(s+8)}$



$\sigma = -2.5$, $\phi_i = \pm 90$.

ω_0 = none.

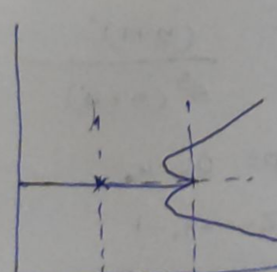
e) $L(s) = \dfrac{(s+3)}{s^3(s+4)}$



$\sigma = -0.33$, $\phi_i = \pm 60, \pm 180$.

ω_0 = none.

f) $L(s) = \dfrac{(s+1)^2}{s^3(s+10)^2}$



$\sigma = -6$, $\phi_i = \pm 60, \pm 180^{\circ}$.

$\omega_0 = \pm 1.31, \pm 7.63$.

Design of compensators.

Controllers:

1. P - controller
2. PI controller $\longrightarrow$ Lag compensator.
3. PID controller $\longrightarrow$ Lead compensator.
4. PID controller $\longrightarrow$ Lag-lead compensator.

The controller gains K_p, K_I and K_d are selected to obtain the desired steady state error. $\Rightarrow$ controller improves steady state response.

$$e_{ss}(\text{skp}) = \frac{1}{1+k_p} \longrightarrow \text{controller gain.}$$

$e_{ss}\downarrow \Rightarrow k_p\uparrow$.

If gain is increased to decrease the steady state error transient response will be poor $\Rightarrow$ effects M_p, t_r.

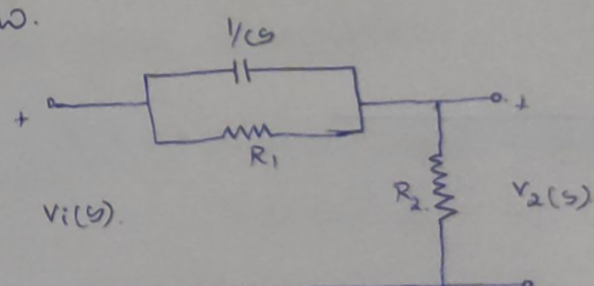
Add poles and zeros, to improve steady state response along with transient response.

Design of compensator.

- Time domain method. $\downarrow$ Root locus
- Frequency domain method. $\downarrow$ Bode plot.

Design of lead compensator using RL technique: Lead compensator adds +ve phase angle to the system phase angle.

$\Rightarrow$ lead n/w.



$$Z_1 = R_1 \parallel 1/cs$$

$$= \frac{R_1 \cdot 1/cs}{R_1 + 1/cs}$$

$$= \frac{R_1}{1+R_1 cs}$$

$Z_2 = R_2$.

$$V_2(s) = \frac{Z_2}{Z_1 + Z_2} V_1(s).$$

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{\frac{R_1}{1+R_1Cs} + R_2}$$

$$= \frac{R_2(1+R_1Cs)}{R_1 + R_2(1+R_1Cs)}$$

$$= \frac{R_2(1+R_1Cs)}{(R_1+R_2) + R_1R_2Cs}$$

$$= \underbrace{\frac{R_2}{R_1+R_2}}_{\alpha} \quad \frac{1+R_1Cs}{1+\underbrace{\frac{R_2}{R_1+R_2}}_{\alpha} R_1Cs}$$

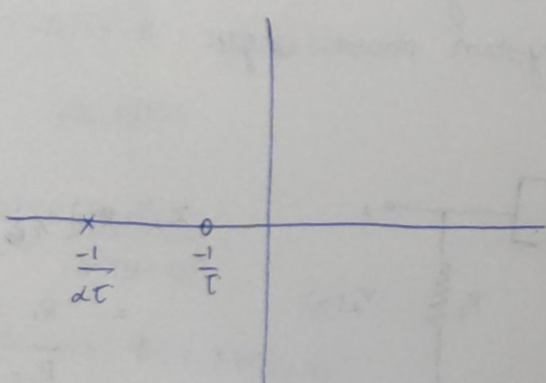
$$\alpha = \frac{R_2}{R_1+R_2} < 1.$$

$$\frac{V_2(s)}{V_1(s)} = \alpha \frac{1+\tau s}{1+\alpha\tau s}.$$

Zeros: $1+\tau s = 0 \Rightarrow s = \frac{-1}{\tau} = s_1$

Poles: $1+\alpha\tau s = 0 \Rightarrow s = -\frac{1}{\alpha\tau} = s_2.$

$$|s_2| > |s_1|$$



Frequency response: Sinusoidal i/p

$s = j\omega$

$$\frac{V_2(j\omega)}{V_1(j\omega)} = \alpha \frac{1+j\omega\tau}{1+j\omega\alpha\tau}$$

$$\angle \frac{V_2(j\omega)}{V_1(j\omega)} = \underset{(+ve)}{\phi} = \angle Nr - \angle Dr$$

$$= \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$$

$$\tan^{-1}(\omega\tau) > \tan^{-1}(\alpha\omega\tau)$$

$$\omega\tau > \alpha\omega\tau$$

$$\tau > \alpha\tau$$

$$\frac{1}{\tau} < \frac{1}{\alpha\tau}$$

$$\boxed{\frac{1}{\alpha\tau} > \frac{1}{\tau}}$$

1. The plant transfer function of unity f/b control system is given by $G(s) = \frac{k}{s(s+1)}$. Design a lead compensator using root locus technique to meet the following specifications.

i) $\xi = 0.7$.

ii) $t_s = 1.4s$ (2 % tolerance)

iii) $K_v \geq 2\ sec^{-1}$

Step-1: Using the specifications calculate the desired pole locations to meet the given specifications.

$\xi < 0.7 \Rightarrow$ underdamped response

$\Rightarrow$ desired poles will be complex conjugates.

$$s_d = -\xi\omega_n \pm j\omega_d$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$t_s = \frac{3.91}{\xi\omega_n}$$

$$1.4 = \frac{3.91}{0.7 \times \omega_n}$$

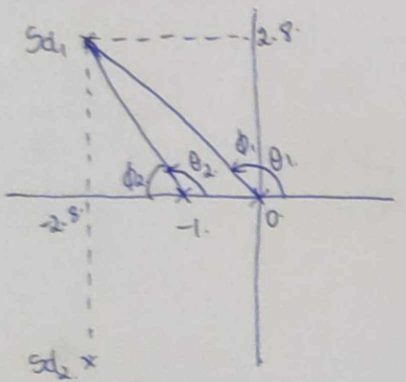
$\omega_n = 3.98 = 4$ rad/s

$S_d = -0.7 \times 4 \pm j\omega_d$

$= -2.8 \pm j4\sqrt{1-\xi^2}$

$= -2.8 \pm j2.85$

Step-2: Angle requirement from lead compensator.



To obtain the desired specifications Sd1 & Sd2 must lie on Root locus

RL satisfies $1 + L(s) = 0$

$1 + G(s) = 0$

$G(s) = -1 = 1\angle -180°$

For every point on root locus $|G| = 1$, $\angle G = -180°$.

$\phi_1 = 45°$; $\theta_1 = 90 + 45° = 135°$

$\phi_2 = 57.26°$, $\theta_2 = 122.73$

$\theta_1 + \theta_2 = -257.73°$

$\frac{\angle Nr}{\angle Dr} = \underset{Z}{\angle Nr} - \underset{P}{\angle Dr}$ → angle contributed by Pole is -ve always

$-257.73 + \phi = -180°$

$\phi = 77.73°$ (angle requirement from lead compensator)

+ve (lead compensator).

The standard form of lead controller is given by $D(s) = \frac{s+z}{s+p}$

$s + z = 0$

$s = -z$ (zero of compensator)

$s + p = 0 \Rightarrow s = -p$ (pole of compensator) where to fix pole & zero of compensator?

$$Z = \frac{\omega_n \sin r}{\sin(\theta + r)}$$

$\theta = \cos^{-1}\xi = 45.57°$

$$P = \frac{\omega_n \sin(r+\phi)}{\sin(\theta + r + \phi)}.$$

$r = 28.365°$

$z = 1.977$

$P = 8.09.$

$$r = \frac{1}{2}[180 - \theta - \phi].$$

Transfer function of lead compensator.

$$D(s) = \frac{s+z}{s+p} = \frac{s+1.977}{s+8.09}$$

Required gain at P $|D(s)\,G(s)|_{\text{at } p} = 1$

$$\left|\frac{s+z}{s+p} \cdot \frac{k}{s(s+1)}\right|_{s=-2.8+j2.8} = 1.$$

$$|(-0.036 - 4.855)k| = 1.$$

$$|k| = \cancel{\quad}\,27.06.$$

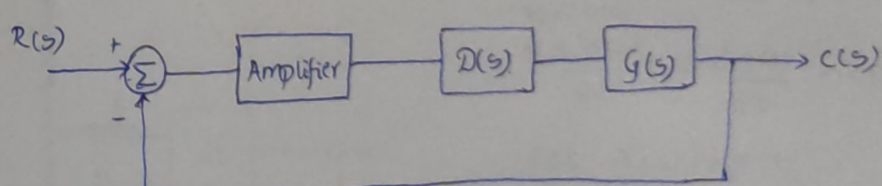
Gain requirement based on k_v

$$K_v = \lim_{s \to 0} s\,G(s) = 2 \quad \text{(Given)}.$$

Additional gain can be adjusted using amplifier.

$27.06 = 2 \times$ amplifier gain.

amplifier gain $= 13.5.$



Calculate actual k_v

$$k_v = \lim_{s\to 0} s \times \text{amplifier gain} \times D(s) \times G(s).$$

$$= \lim_{s\to 0} 13.5 \times \frac{s+z}{s+p} \times \frac{2}{s(s+1)} \times s$$

$$= 6.6.$$

2. Design a lead controller system with unity feedback and having

$G(s) = \dfrac{k}{s(s+1)(s+4)}$ to meet the specifications: $\xi = 0.5$.

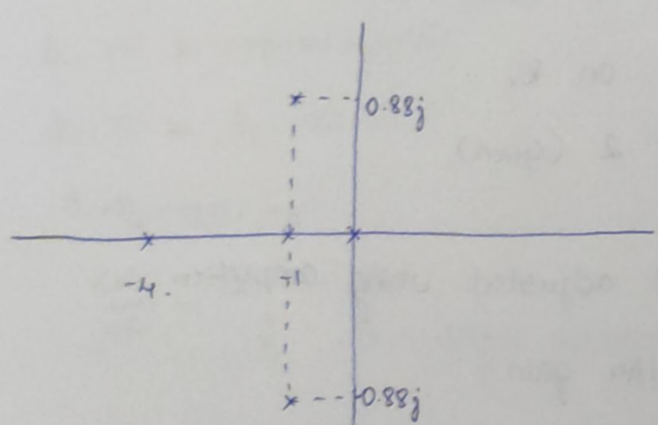
$\omega_n = 2$ rad/s.

$k_v \geq 1.5$.

$$S_d = -\xi\omega_n \pm j\omega_d.$$

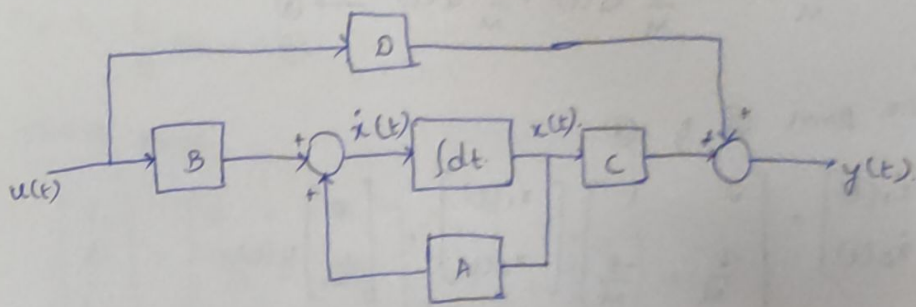
$$= -1 \pm j\sqrt{1-\xi^2}$$

$$= -1 \pm j0.88$$



State space

The state of a system defined as the minimum number of interconnection that must be specified at any intial time t_0. So that the complete dynamic behaviour of the system at any time. $t > t_0$ is determined with the input $u(t)$ is known.



$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow ①$$

$$y(t) = Cx(t) + Du(t) \longrightarrow ②$$

$x(t) =$ state vector $\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1}$ $\quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix}_{r\times 1}$

$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}_{p\times 1}$

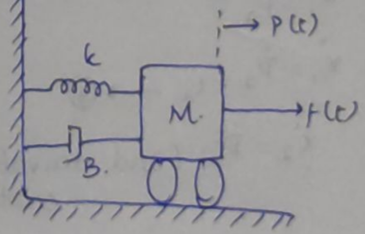
A : State matrix $n\times n$

B : i/p matrix $n\times r$.

C : o/p matrix $p\times n$

D : direct transmission matrix $p\times r$.

State equation for mechanical system: The system is described by 2nd order differential equation & consists of 2 state variables.



$$M\frac{d^2p(t)}{dt^2} + B\frac{dp(t)}{dt} + kp(t) = f(t) \longrightarrow ①$$

Let $x_1(t) = p(t)$

$$\dot{x}_1(t) = \frac{dp(t)}{dt} \longrightarrow ②$$

Let $x_2(t) = \frac{dp(t)}{dt} = \dot{x}_1(t)$

from ① $\dot{x}_2(t) = \frac{d^2p(t)}{dt^2}$

$M\dot{x}_2(t) + Bx_2(t) + kx_1(t) = u(t)$.

$\dot{x}_2(t) = -\frac{k}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{1}{M}u(t) \longrightarrow ③$.

In matrix form ② & ③.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t).$$

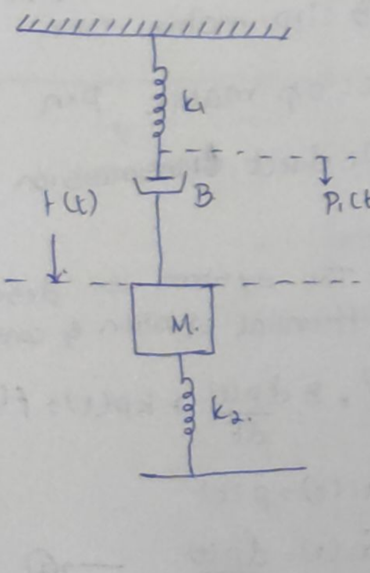
$$\dot{x}(t) = Ax(t) + Bu(t).$$

& $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{B}{M} \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1/M \end{bmatrix}$

$$y(t) = [1 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Cx(t).$$

$$C = [1 \quad 1].$$

Problem:



$M\ddot{P}_2 + B(\dot{P}_2 - \dot{P}_1) + k_2P_2 = f(t) \longrightarrow ①$

$B(\dot{P}_1 - \dot{P}_2) + k_1P_1 = 0 \longrightarrow ②$

2nd Order DE $\Rightarrow$ 2 state variables

$x_1 = P_2$.

$\dot{x}_1 = \dot{P}_2 = x_2 \longrightarrow ③$

$\dot{x}_2 = \ddot{P}_2$ (2nd derivative of P_2 exists)

$x_3 = P_1$

$M\dot{x}_2 + k_1 x_3 + k_2 x_1 = u(t)$

$$\dot{x}_2 = -\frac{k_2}{M}x_1 - \frac{k_1}{M}x_3 + \frac{1}{M}u(t) \longrightarrow (4).$$

$B\dot{x}_3 - Bx_2 + k_1 x_3 = 0$

$$\dot{x}_3 = x_2 - \frac{k_1}{B}x_3 \longrightarrow (5).$$

In matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ -\frac{k_2}{m} & 0 & -\frac{k_1}{m}\\ 0 & 1 & -\frac{k_1}{B}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 1/M\\ 0\end{bmatrix}}_{B}u(t).$$

$$y = \underbrace{\begin{bmatrix}1 & 1 & 0\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \qquad D = 0.$$

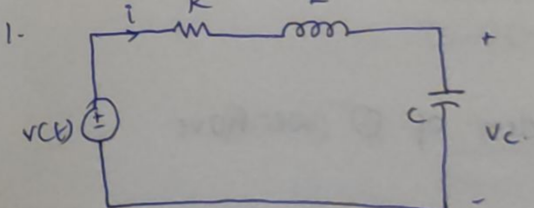
State equations for electrical circuits:

L & C are energy storing elements.

Inductor : $\frac{1}{2}LI^2$ choose inductor current as state variable.

Capacitor : $\frac{1}{2}CV^2$ choose capacitor voltage as state variable.

1.



$x_1 = i$ ind current

$x_2 = v_c$ cap vtg.

$$L\frac{di}{dt} + Ri + V_c = V(t). \text{ from KVL}$$

$$L\dot{x}_1 + Rx_1 + x_2 = u(t).$$

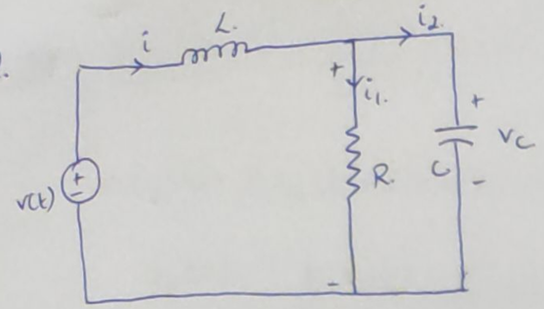
$$\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u(t) \longrightarrow (1)$$

$$x_2 = \frac{1}{C}\int i\,dt = \frac{1}{C}\int x_1\,dt$$

$$\dot{x}_2 = \frac{1}{C}x_1 \longrightarrow (2)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-R/L & -1/L\\ 1/C & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1/L\\ 0\end{bmatrix}u(t).$$

2.



$x_1 = i = i_1 + i_2$

$x_2 = V_c$

$\cancel{L\frac{di}{dt} + Ri_1 + \frac{1}{C}\int i_2}$

$L\frac{di}{dt} + \cancel{R} V_c = u(t)$

$L\dot{x}_1 + x_2 = u(t)$

$\dot{x}_1 = -\frac{1}{L}x_2 + \frac{1}{L}u(t)$

$x_2 = \frac{1}{C}\int i_2\,dt$

$x_2 = \frac{1}{C}\int (i - i_1)\,dt$

$= \frac{1}{C}\int \left(x_1 - \frac{V_c}{R}\right)dt$

$\dot{x}_2 = \frac{1}{C}\left[x_1 - \frac{x_2}{R}\right]$

$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{RC}x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t)$$

Transfer function from state equation:

$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow ①$

$y(t) = Cx(t) + Du(t) \longrightarrow ②$

$\dot{x}(t) = \frac{dx(t)}{dt}$

Taking laplace transform on both sides of ① we have

$sx(s) - x(0) = Ax(s) + Bu(s)$.

For obtaining tfr function initial conditions are zero.

$\Rightarrow x(0) = 0$

$\Rightarrow sX(s) = Ax(s) + Bu(s)$

$x(s)[sI - A] = Bu(s)$

$x(s) = \cancel{\frac{B}{sI-A}u(s)}\ [sI - A]^{-1} B\, u(s) \longrightarrow ③$

I = Identity matrix

from ②

$$y(s) = Cx(s) + Du(s)$$

$$= C[sI-A]^{-1}B\,u(s) + Du(s).$$

$$= [C[sI-A]^{-1}B + D]\,u(s).$$

$$\frac{y(s)}{u(s)} = C[sI-A]^{-1}B + D$$

1. consider the following state equation

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u. \qquad y = [0 \quad 1]\,x$$

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = [0 \quad 1] \quad D = 0$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

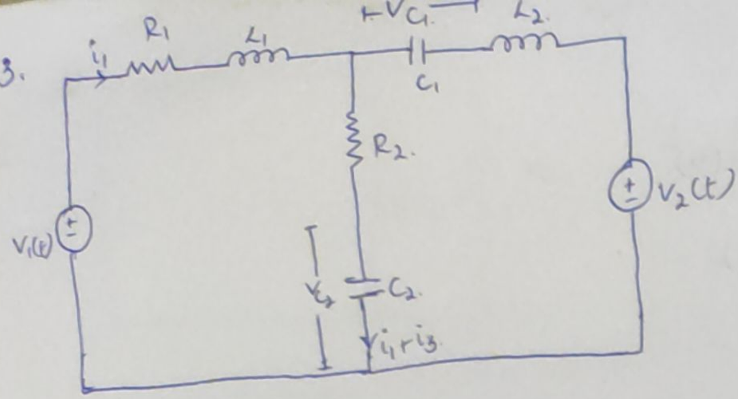
$$[sI-A]^{-1}B = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s \\ -2 \end{bmatrix}$$

$$C[sI-A]^{-1}B = \frac{1}{(s+1)(s+2)}[0 \quad 1]\begin{bmatrix} s \\ -2 \end{bmatrix}$$

$$= \frac{-2}{(s+1)(s+2)}$$

3.



$x_1 = i_1$

$x_2 = i_3$

$x_3 = V_{C_1}$

$x_4 = V_{C_2}$

Solution of state equation:

Solving for $x_1(t), x_2(t)$ ---- and o/p $y(t)$.

methods — LT method (freq domain method.
— Time domain method.

State equations:

$\dot{x}(t) = Ax(t) + Bu(t)$.

$sX(s) - x(0) = AX(s) + BU(s)$.

$sX(s) - AX(s) = x(0) + BU(s)$.

$X(s)\{sI - A\} = x(0) + BU(s)$.

$[sI-A]^{-1} \cdot X(s)[sI-A] = [sI-A]^{-1}x(0) + [sI-A]^{-1}BU(s)$

$X(s) = [sI-A]^{-1}x(0) + [sI-A]^{-1}BU(s)$

Taking inverse laplace transform on both sides

$$x(t) = \underbrace{L^{-1}\{[sI-A]^{-1}x(0)\}}_{\text{matrix}} + \underbrace{L^{-1}\{[sI-A]^{-1}BU(s)\}}_{\text{matrix}}$$

$\downarrow$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \end{bmatrix}$$

Depends on initial condition $x(0)$.
↓
Free response
or
natural response
or
homogeneous solution.
$x_n(t)$

Depends on i/p $x(t)$.
↓
Forcing function.
↓
Forced response.
∴ $x_f(t)$.

$x(t) = x_n(t) + x_f(t)$

$\downarrow$

Total response

or

complete solution

or

Solution or response.

$x_n(t) = \mathcal{L}^{-1}\{[sI - A]^{-1} x(0)\}$

$x_f(t) = \mathcal{L}^{-1}\{[sI - A]^{-1} B u(s)\}$

$y(t) = Cx(t) + Du(t)$.

$\phi(s) = [sI - A]^{-1} \longrightarrow$ state transition matrix (freq domain)

$\mathcal{L}^{-1}\{\phi(s)\} = \phi(t) \longrightarrow$ state transition matrix (time domain)

1. A system is described by $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

u = unit step i/p. The initial conditions are $x_1(0) = 1$, $x_2(0) = -1$. Find

a) state transition matrix.

b) Solution $x(t)$.

c) The output $y(t)$, $C = [0 \ 1]$.

a) $\phi(s) = [sI - A]^{-1}$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

b) $x_n(t) = \mathcal{L}^{-1}\{\phi(s) x(0)\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)} \begin{bmatrix} s-1 \\ -(s+5) \end{bmatrix}\right\}$$

$$x_n(t) = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{s-1}{(s+1)(s+2)} \\ \dfrac{-(s+5)}{(s+1)(s+2)} \end{bmatrix}\right\}$$

$$\frac{s-1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$s-1 = A(s+2) + B(s+1)$$

$s=-1$. $\quad s=-2$.

$\Rightarrow -2 = A$. $\quad -3 = B(-1)$

$B = 3$.

$$\frac{-(s+5)}{(s+1)(s+2)} = \frac{C}{s+1} + \frac{D}{s+2}$$

$$-(s+5) = C(s+2) + D(s+1)$$

$s=-1$. $\quad s=-2$.

$\Rightarrow -4 = C$. $\quad \Rightarrow D = 3$

$$x_n(t) = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{-2}{s+1} + \dfrac{3}{s+2} \\ \dfrac{-4}{s+1} + \dfrac{3}{s+2} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} -2e^{-t} + 3e^{-2t} \\ -4e^{-t} + 3e^{-2t} \end{bmatrix}$$

$$x_f(t) = \mathcal{L}^{-1}\left\{[sI-A]^{-1}BU(s)\right\}$$

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad u = \begin{cases} 1, & t \geq 0. \\ 0 & \text{elsewhere.} \end{cases}$ $\quad U(s) = \dfrac{1}{s}$.

$$x_f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\begin{bmatrix} s \\ -2 \end{bmatrix}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{\cancel{s}}{\cancel{s}(s+1)(s+2)} \\ \dfrac{-2}{s(s+1)(s+2)} \end{bmatrix}\right\}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

$s=-1 \Rightarrow 1 = A$. $\quad s=-2 \Rightarrow B=-1$.

$$\frac{-2}{s(s+1)(s+2)} = \frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+2}$$

$$-2 = C(s+1)(s+2) + Ds(s+2) + Es(s+1)$$

$s=0 \Rightarrow C=-1$. $\quad s=-1 \Rightarrow D=2$ $\quad s=-2 \Rightarrow E=-1$

$$x_f(t) = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-1}{s} + \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^{-t} - e^{-2t} \\ -1 + 2e^{-t} - e^{-2t} \end{bmatrix}$$

$$x(t) = x_n(t) + x_f(t).$$

$$= \begin{bmatrix} -e^{-t} + 2e^{-2t} \\ -1 - 2e^{-t} + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x_1(t) = -e^{-t} + 2e^{-2t}$$

$$x_2(t) = -1 - 2e^{-t} + 2e^{-2t}.$$

c). $y(t) = Cx(t) + \cancel{D}^{0} u(t).$

$$y(t) = Cx(t)$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -e^{-t} + 2e^{-2t} \\ -1 - 2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$= -1 - 2e^{-t} + 2e^{-2t}.$$

2. Determine the response for a system described by $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_1 - x_2 + u(t)$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Input is unit step. Also find $y(t)$ if $C = \begin{bmatrix} 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

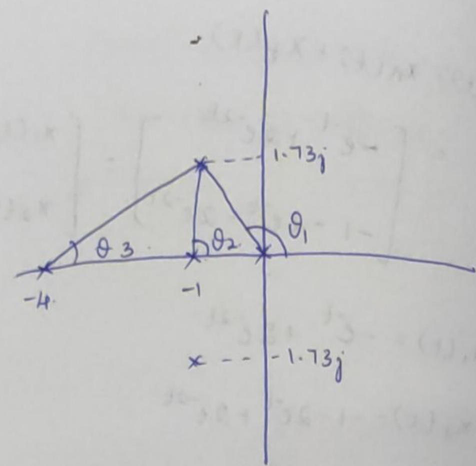
$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Lead compensator:

1. Design a suitable lead compensator for a system with unity f/b and having plant t/f function $G(s) = \frac{k}{s(s+1)(s+4)}$ to meet the following specifications. $\xi = 0.5$

$\omega_n = 2$ rad/s.

$k_v \geq 1.5$



$S_d = -\xi\omega_n \pm j\omega_d$

$= -1 \pm 1.73j$

$\theta_1 = 120.02 = 120°$

$\theta_2 = 90°$

$\theta_3 = 29.97 = 30°$

$\theta = -[120+90+30]$

$= -240°$

Required angle by the compensator : $\phi - (\theta_1+\theta_2+\theta_3) = -180°$

$\phi = 60°$

$\gamma = \frac{1}{2}[180-\theta-\phi]$

$Z = \frac{\omega_N \sin(\gamma)}{\sin(\theta+\gamma)}$

$P = \frac{\omega_N \sin(\gamma+\phi)}{\sin(\theta+\gamma+\phi)}$

$\theta = \cos^{-1}\xi = 60°$

$\gamma = \frac{1}{2}[180-60-60]$

$= 30°$

$Z = \frac{2\sin 30°}{\sin(60+30)} = 1$

$P = \frac{2\sin 90°}{\sin(60+30+60)} = 4$

$|D(s)\,G(s)| = 1.$

$$\left| \frac{s+1}{s+4} \cdot \frac{k}{s(s+1)(s+4)} \right|_{s=-1\pm 1.93j} = 1.$$

$$\left| \frac{k}{-23.96 + 0.012j} \right| = 1.$$

$k = 24.$

$K_v = 1.5.$

$$K_v = \lim_{s \to 0} s \cdot \frac{k}{s(s+1)(s+4)}.$$

$= 6$

Amplifier gain: $\dfrac{k}{K_v} = \dfrac{24}{6} = 4.$

2. $G(s) = \dfrac{k}{s(s+1)(s+5)}$, $\% M_p = 25\%$, $t_s = 5s$ (2% tolerance)

Find ξ, ω_n.

$$M_p = e^{-\pi \xi / \sqrt{1-\xi^2}}.$$

$$t_s = \frac{3.91}{\xi \omega_n}.$$

$\xi = 0.4$, $\omega_n = 2$ rad/s , $s_d = -0.8 \pm j1.83$. $z = 1.5$ $p = 3$. $k = 26.5$ (amp)
$K_v = 0.2$ (uncap)
K_v (cap) $= 2.65$.

3. Let us consider an example of a unity f/b type-II system where open loop tfr function $G(s) = \dfrac{k}{s^2}$. ~~It can be seen~~ It is desired to compensate the system so as to meet the following transient specification: $t_s \le 4s$ & $M_p = 20\%$.

$$M_p = e^{-\pi \xi / \sqrt{1-\xi^2}}.$$

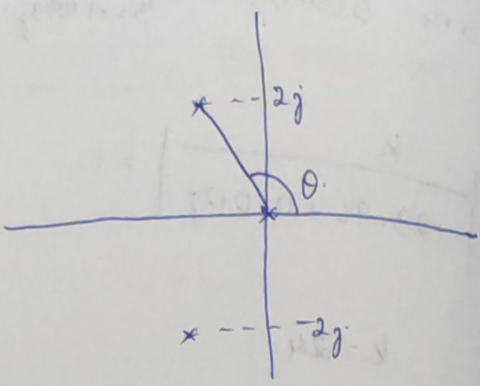
$$0.2 = e^{-\pi \xi / \sqrt{1-\xi^2}}.$$

$$\ln(0.2) = \frac{-\pi \xi}{\sqrt{1-\xi^2}}.$$

$2.6 - 2.6\xi^2 = 9.86\xi^2$

$2.6 = 12.46\xi^2$

$\xi = 0.45$



$t_p = \dfrac{3.91}{\xi\omega_N}$

$4 = \dfrac{3.91}{\xi\omega_N}$

$\xi\omega_N = 1$

$\omega_N = 2.22$

$s_d = -\xi\omega_N \pm j\omega_N\sqrt{1-\xi^2}$

$= -1 \pm j2$

$\theta = 116^\circ \times 2 = 232^\circ$

$\phi - 232 = -180$

$\phi = 52^\circ$

$\gamma = \frac{1}{2}[180 - 63.25 - 52]$

$= 32.37^\circ$

$z = 1.19$

$P = 4.12$

$D(s) = \dfrac{s+1.19}{s+4.12}$

$|D(s) \cdot G(s)| = 1$

$\left| \dfrac{s+1.19}{s+4.12} \cdot \dfrac{k}{s^2} \right|_{s \to -1+2j} = 1$

$k = 9.22$

canonical forms: Minimal components for the realization of transfer function.

Controllable canonical form

Observable canonical form.

Controllability
Observability } control system.

If a system is fully controllable it can be realized in controllable canonical form.

If a system is fully observable it can be realized in observable canonical form.

Controllable canonical form (CCF):

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{1 \cdot s^3 + a_2 s^2 + a_1 s + a_0}$$

$\downarrow$

$a_3 = 1.$

Numerator degree = Dr degree.

$$\frac{Y(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{[b_3 s^3 + b_2 s^2 + b_1 s + b_0]}{1} \cdot \frac{1}{[s^3 + a_2 s^2 + a_1 s + a_0]}$$

$$\frac{Y(s)}{X_1(s)} = b_3 s^3 + b_2 s^2 + b_1 s + b_0 \longrightarrow (1)$$

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \longrightarrow (2).$$

from (1).

$$Y(s) = X_1(s)\,[b_3 s^3 + b_2 s^2 + b_1 s + b_0].$$

$$Y(s) = b_3 s^3 X_1(s) + b_2 s^2 X_1(s) + b_1 s X_1(s) + b_0 X_1(s).$$

Time domain	Freq domain (LT)
$y(t) = y$.	$Y(s) = Y$.
$x(t) = x$.	$X(s) = X$.
$\frac{d}{dt} \xleftrightarrow{LT} s$.	$\frac{dx}{dt} = \dot{x}$
$\frac{d^2}{dt^2} \longleftrightarrow s^2$.	$\frac{d^2x}{dt^2} = \ddot{x}$
$\frac{d^4}{dt^4} \longleftrightarrow s^4$.	$\frac{d^3x}{dt^3} = \dddot{x}$

$y = b_3\dddot{x}_1 + b_2\ddot{x}_1 + b_1\dot{x}_1 + b_0x_1$. ($x_1$ = state variable).

order of the system = 3

$\Rightarrow$ 3 state variables $\Rightarrow$ x_1

$x_2 = \dot{x}_1$

$x_3 = \dot{x}_2 = \ddot{x}_1$

$\dot{x}_3 = \dddot{x}_1$

$y = b_3\dot{x}_3 + b_2x_3 + b_1x_2 + b_0x_1 \longrightarrow$ (3) (o/p equation)

from (2),

$u(s) = x_1(s)[s^3 + a_2s^2 + a_1s + a_0]$.

$u(s) = s^3x_1(s) + a_2s^2x_1(s) + a_1sx_1(s) + a_0x_1(s)$.

Inverse LT.

$u = \dot{x}_3 + a_2x_3 + a_1x_2 + a_0x_1$.

$\dot{x}_3 = -a_0x_1 - a_1x_2 - a_2x_3 + u \longrightarrow$ (4)

Substituting (4) in (3).

$y = b_3(-a_0x_1 - a_1x_2 - a_2x_3 + u) + b_2x_3 + b_1x_2 + b_0x_1$.

$= -a_0b_3x_1 - a_1b_3x_2 - a_2b_3x_3 + b_3u + b_2x_3 + b_1x_2 + b_0x_1$.

$= x_1(b_0 - a_0b_3) + x_2(b_1 - a_1b_3) + x_3(b_2 - a_2b_3) + b_3u$.

$\bar{b}_0 = b_0 - a_0 b_3$

$\bar{b}_1 = b_1 - a_1 b_3$

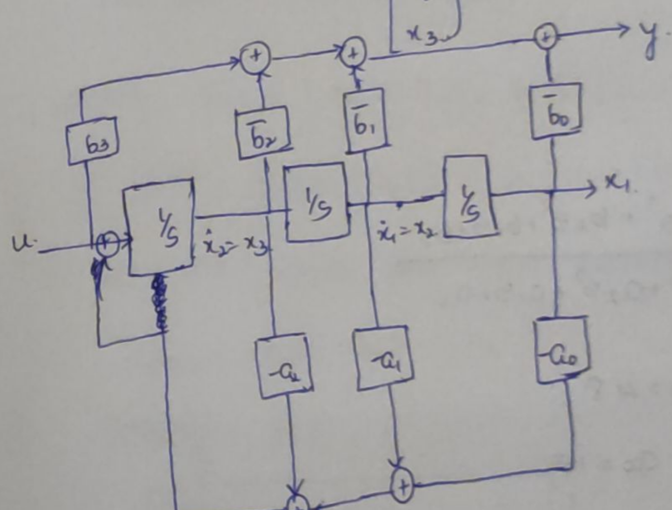
$\bar{b}_2 = b_2 - a_2 b_3$

$$\boxed{\begin{aligned} y &= \bar{b}_0 x_1 + \bar{b}_1 x_2 + \bar{b}_2 x_3 + b_3 u \rightarrow ⑤ \\ \dot{x}_1 &= x_2 \rightarrow ⑥ \\ \dot{x}_2 &= x_3 \rightarrow ⑦ \\ \dot{x}_3 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + u \rightarrow ⑧ \end{aligned}} \rightarrow \text{state equations}$$

Let us represent the state equations in matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_3 \end{bmatrix}_{3\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\bar{b}_0 \;\; \bar{b}_1 \;\; \bar{b}_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$



1. $\frac{Y(s)}{U(s)} = \frac{s^3 + 12s^2 + 44s + 48}{s^3 + 9s^2 + 23s + 15}$. a) Write the state equations in controllable canonical form.

b) Show the corresponding realization.

a) $\frac{Y(s)}{U(s)} = \frac{s^3 + 12s^2 + 44s + 48}{s^3 + 9s^2 + 23s + 15} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_2 s + a_0}$

$a_3 = 1$ $\quad a_1 = 23$

$a_2 = 9$ $\quad a_0 = 15$

$b_3=1 \quad b_2=12 \quad b_1=44 \quad b_0=48$

$\bar{b}_0 = b_0 - a_0 b_3 = 33$

$\bar{b}_1 = b_1 - a_1 b_3 = 21$

$\bar{b}_2 = b_2 - a_2 b_3 = 3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 33 & 21 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -15x_1 - 23x_2 - 9x_3 + u$

$y = 33x_1 + 21x_2 + 3x_3 + u$

2. $$\frac{Y(s)}{U(s)} = \frac{12s^2+44s+48}{s^3+9s^2+23s+15} = \frac{b_3s^3+b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$$

$b_3=0 \quad b_2=12 \quad b_1=44 \quad b_0=48$

$a_3=1 \quad a_2=9 \quad a_1=23 \quad a_0=15$

$\bar{b}_0 = 48$

$\bar{b}_1 = 21$

$\bar{b}_2 = 3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 48 & 21 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observable canonical form:

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{1.s^3 + a_2 s^2 + a_1 s + a_0}$$

$\hookrightarrow a_3 = 1.$

$$Y(s)[s^3 + a_2 s^2 + a_1 s + a_0] = U(s)[b_3 s^3 + b_2 s^2 + b_1 s + b_0]$$

$Y(s) \equiv Y. \quad U(s) \equiv U.$

$$s^3 Y + a_2 s^2 Y + a_1 s Y + a_0 Y = b_3 s^3 U + b_2 s^2 U + b_1 s U + b_0 U.$$

Keep highest power of s, associated with Y. on LHS

$$s^3 Y = b_3 s^3 U + b_2 s^2 U + b_1 s U + b_0 U - a_2 s^2 Y - a_1 s Y - a_0 Y$$

$$s^3 Y = b_3 s^3 U + [b_2 U - a_2 Y] s^2 + [b_1 U - a_1 Y] s + [b_0 U - a_0 Y]$$

$$Y = b_3 U + [b_2 U - a_2 Y] s^{-1} + [b_1 U - a_1 Y] s^{-2} + [b_0 U - a_0 Y] s^{-3}$$

Let us write $Y = b_3 U + x_1 \longrightarrow ① \xRightarrow{ILT} \boxed{y = b_3 u + x_1}$ *

where. $x_1 = s^{-1}[b_2 U - a_2 Y] + s^{-2}[b_1 U - a_1 Y] + s^{-3}[b_0 U - a_0 Y]$

$$s x_1 = [b_2 U - a_2 Y] + s^{-1}[b_1 U - a_1 Y] + s^{-2}[b_0 U - a_0 Y].$$

Let us write the above equation as.

$$s x_1 = b_2 U - a_2 Y + x_2.$$

Taking ILT.

$$\boxed{\dot{x}_1 = b_2 u - a_2 y + x_2}\ *$$

where $x_2 = s^{-1}[b_1 U - a_1 Y] + s^{-2}[b_0 U - a_0 Y].$

$$s x_2 = [b_1 U - a_1 Y] + s^{-1}[b_0 U - a_0 Y].$$

~~Taking ILT~~

$$s x_2 = b_1 U - a_1 Y + x_3.$$

Taking ILT.

$$\boxed{\dot{x}_2 = b_1 u - a_1 y + x_3}\ *$$

where $x_3 = s^{-1}[b_0 u - a_0 y]$.

$s x_3 = b_0 u - a_0 y$.

Taking ILT

$\dot{x}_3 = b_0 u - a_0 y$.

Now consider

$\dot{x} = Ax + Bu$.

can contain x & u but not y.

$\dot{x}_1 = b_2 u - a_2 y + x_2$.

$\dot{x}_1 = b_2 u - a_2 [b_3 u + x_1] + x_2$

$$\boxed{\dot{x}_1 = -a_2 x_1 + x_2 + [b_2 - a_2 b_3] u}$$

Now consider.

$\dot{x}_2 = b_1 u - a_1 y + x_3$.

$\dot{x}_2 = b_1 u - a_1 [b_3 u + x_1] + x_3$.

$$\boxed{\dot{x}_2 = -a_1 x_1 + x_3 + [b_1 - a_1 b_3] u}$$

Similarly

$\dot{x}_3 = b_0 u - a_0 y$.

$= b_0 u - a_0 [b_3 u + x_1]$.

$$\boxed{\dot{x}_3 = -a_0 x_1 + [b_0 - a_0 b_3] y}$$

Let us represent these equations in matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 - a_2 b_3 \\ b_1 - a_1 b_3 \\ b_0 - a_0 b_3 \end{bmatrix} u.$$

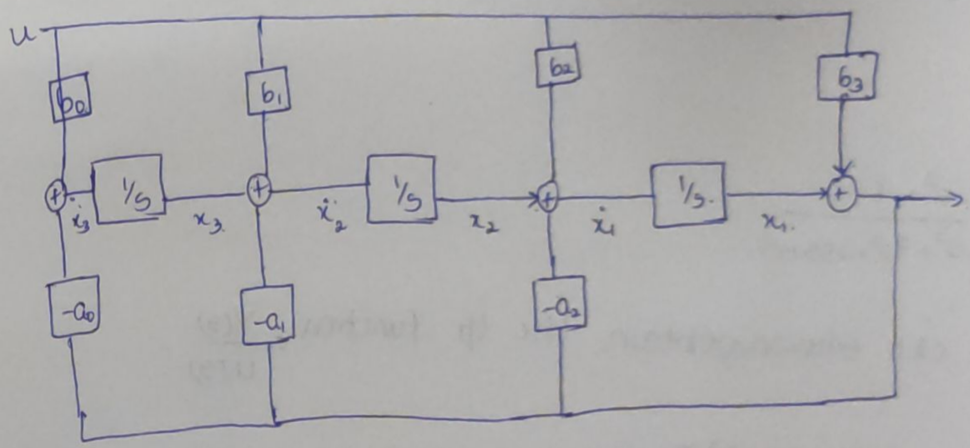
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{b}_2 \\ \bar{b}_1 \\ \bar{b}_0 \end{bmatrix}}_{B} u.$$

$Y = x_1 + b_3 u.$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{D}{\underset{\downarrow}{b_3 u.}}$$

$$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \bar{b}_2 \\ \bar{b}_1 \\ \bar{b}_0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad D = b_3$$

Realization:



i) Repeat the procedure by taking $b_3 = 0$, $D = 0$, $\bar{b}_2 = b_2$, $\bar{b}_1 = b_1$, $\bar{b}_0 = b_0$

Given that $\dfrac{Y(s)}{U(s)} = \dfrac{s^3 + 14s^2 + 50s + 55}{s^3 + 12s^2 + 30s + 15}$

$b_3 = 1$, $b_2 = 14$, $b_1 = 50$, $b_0 = 55$, $a_3 = 1$, $a_2 = 12$, $a_1 = 30$, $a_0 = 15$.

$\bar{b}_2 = 2$ $\quad \bar{b}_1 = 20$ $\quad \bar{b}_0 = 40.$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -12 & 1 & 0 \\ -30 & 0 & 1 \\ -15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 20 \\ 40 \end{bmatrix} u.$$

$\dot{x}_1 = -12x_1 + x_2 + 2u.$

$\dot{x}_2 = -30x_1 + x_3 - 20u.$

$\dot{x}_3 = -15x_1 + 40u.$

$y = x_1 + u.$

Example.

2. $\frac{Y(s)}{U(s)} = \frac{14s^2+50s+55}{s^3+12s^2+30s+15}$

$b_3 = 0, \quad b_2 = 14, \quad b_1 = 50, \quad b_0 = 55.$

$a_3 = 1, \quad a_2 = 12, \quad a_1 = 30, \quad a_1 = 15.$

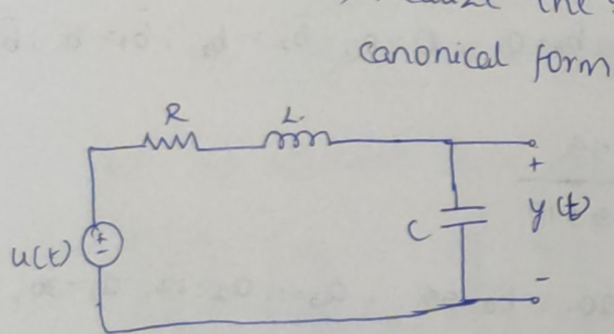
$\bar{b}_2 = 14, \quad \bar{b}_1 = 50, \quad \bar{b}_0 = 55.$

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = [$

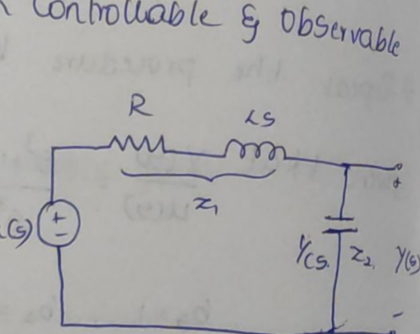
3. $\frac{Y(s)}{U(s)} = \frac{s^2+6}{s^3+9s^2+23s+15}$

4. For the ckt shown a) obtain the tfr function $\frac{Y(s)}{U(s)}$

b) Realize the system in controllable & observable canonical form.



$\Longrightarrow$ transformed ckt.



$Y(s) = \frac{U(s) Z_2}{Z_1+Z_2}$

$\frac{Y(s)}{U(s)} = \frac{Z_2}{Z_1+Z_2}$

$= \frac{1/Cs}{R+Ls+1/Cs}.$

$$= \frac{1/C}{RS + LS^2 + 1/C}$$

$$= \frac{1/LC}{S^2 + \frac{R}{L}S + \frac{1}{LC}}$$

$$\frac{Y(S)}{U(S)} = \frac{b_2S^2 + b_1S + b_0}{S^2 + a_1S + a_0}$$

$b_2 = 0 \qquad b_1 = 0 \qquad b_0 = 1/LC$

$a_2 = 1 \qquad a_1 = \frac{R}{L} \qquad a_0 = \frac{1}{LC}$

State model of electromechanical system:

DC motor.

$$\text{vtg} \rightarrow \boxed{\text{DCM}} \rightarrow \text{angular speed}$$

$$\frac{d\theta}{dt} = \omega$$

Obtain the state model of DC motor taking Armature current and motor speed as state variables.

Armature ckt:

$$e_a = L\frac{di_a}{dt} + i_a R + e_b \rightarrow \text{back emf} \longrightarrow ①$$

$$e_b = k_b \frac{d\theta}{dt} = k_b \omega \longrightarrow ②$$

O/p side:

$$T_a = k_t i_a \longrightarrow ③$$

$$T_a = J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt}$$

$$T_a = J\frac{d\omega}{dt} + b\omega \longrightarrow ④$$

State model of electromechanical system.

$$x_1 = \omega \Rightarrow \dot{x}_1 = \frac{d\omega}{dt}$$

$$x_2 = i_a \Rightarrow \dot{x}_2 = \frac{di_a}{dt}$$

$$e_a = L\dot{x}_2 + x_2 R + k_b x_1$$

$$\dot{x}_2 = -\frac{k_b}{L}x_1 - \frac{R}{L}x_2 + \frac{e_a}{L}$$

Consider ④

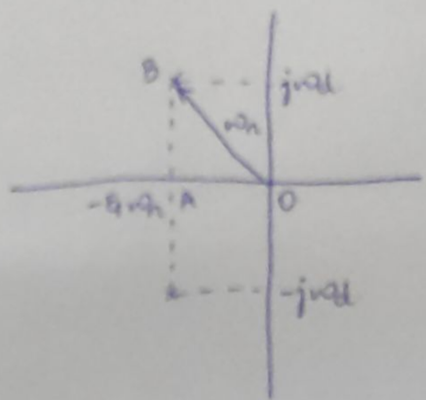
$$k_t x_2 = J\dot{x}_1 + b x_1$$

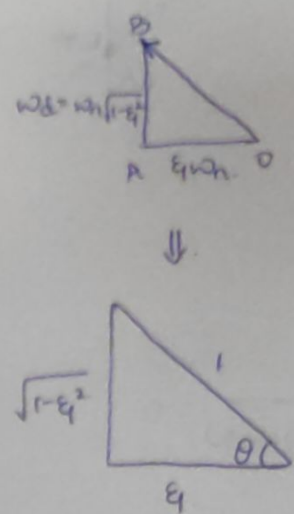
$$\dot{x}_1 = -\frac{b}{J}x_1 + \frac{k_t}{J}x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{-b}{J} & \frac{k_t}{J} \\ \frac{-k_b}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e_a$$

Design of proportional controller using root locus method:

1. $L(s) = \frac{1}{(s+3)(s+4)(s+5)}$. Design a proportional controller so that $\xi = 0.5$, $\omega_n = 3.5$ r/s. Use root locus method.





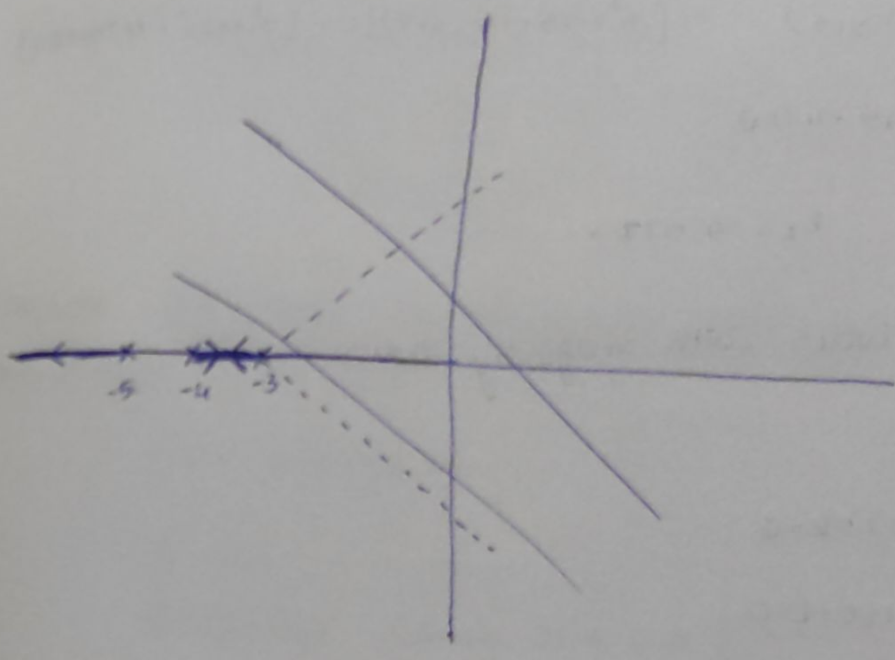
$OB = \sqrt{\omega_d^2 + (\xi\omega_n)^2}$

$OB = \sqrt{\omega_n^2(1-\xi^2) + \xi^2\omega_n^2}$

$= \omega_n$.

$\cos\theta = \xi$

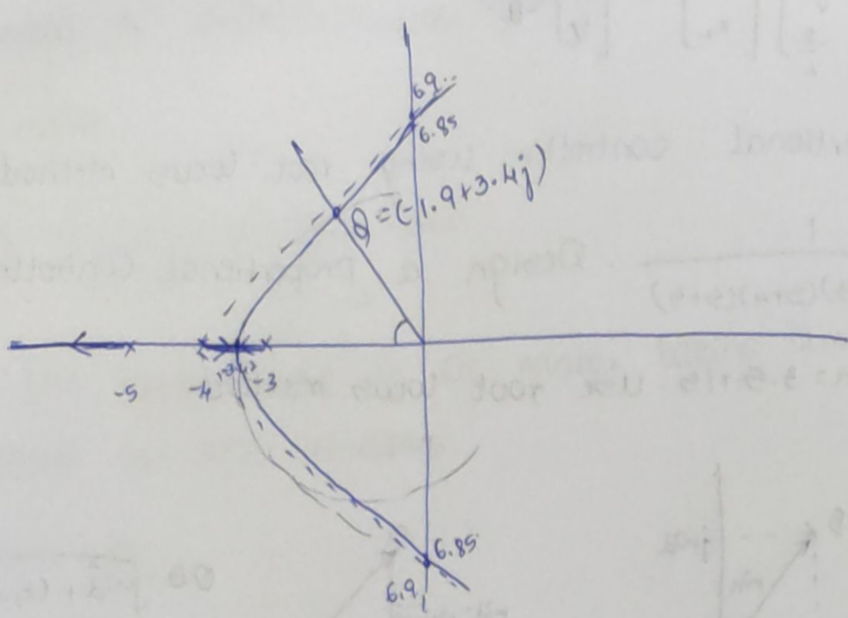
$\theta = \cos^{-1}(\xi)$.



$P - Z = 3$

angle of asymptotes: $60^\circ, 180^\circ, 300^\circ$.

$\sigma = \frac{-3-4-5-0}{3} = -4$.



BAP: $\frac{dk}{ds} = 0$

$1 + L(s) = 0$

$1 + \frac{k}{(s+3)(s+4)(s+5)} = 0$

$k = -(s+3)(s+4)(s+5) = -[(s^2+7s+12)(s+5)] = -[s^3+12s^2+47s+60]$

$\frac{dk}{ds} = 0 \Rightarrow 3s^2+24s+47 = 0$

$k_1 = -3.42$. $k_2 = -4.577$. x

Intersection of root locus with imaginary axis.

$1 + L(s) = 0$

$(s+3)(s+4)(s+5) + k = 0$

$s^3 + 12s^2 + 47s + 60 + k = 0$

s^3	1	47
s^2	12	$60+k$
s	$\frac{564-60-k}{12}$	
s^0	$60+k$	

$$\frac{504-k}{12} \geq 0.$$

$k < 504.$

$60+k > 0.$

$k \geq -60. \Rightarrow k > 0.$ (k is always +ve).

$0 < k < 504.$

When $k = 504$,

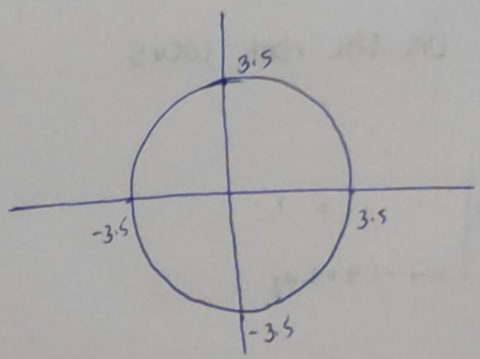
$12 s^2 + (60+k) = 0.$

$12 s^2 + 564 = 0$

$s^2 = \frac{-564}{12}.$

$s = \pm j\, 6.85.$

$\omega_n = 3.5$ as the radius, draw a circle with origin as centre.



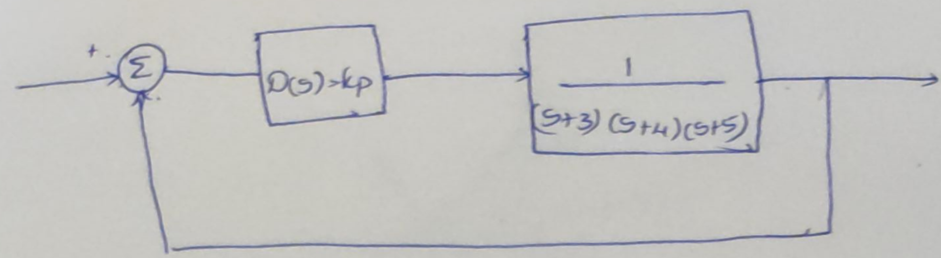
Locate the point of intersection with the root locus. Take the point as P.

$P = -2.5 + j2.25$

$\xi = 0.5.$

$\theta = \cos^{-1}(\xi)$ (Draw and find Q)

For the system to satisfy the requirements, $\xi = 0.5$ and $\omega_n = 3.5$, the root locus of the compensated system should pass through the points P & Q.



Open loop tfr function:

$$L_1(s) = DG = \frac{k_p}{(s+3)(s+4)(s+5)}$$

$$\left| \frac{k_p}{(s+3)(s+4)(s+5)} \right|_{s \to -2.5 + j2.25} = 1.$$ (condition for any point to be on root locus)

$$k_p = 21 = k_{p1}.$$

For point $\emptyset$ to be on the root locus.

$$\left| \frac{k_p}{(s+3)(s+4)(s+5)} \right|_{s \to -1.9 + 3.4j} = 1.$$

$$k_p = 65.7 \Rightarrow k_{p2}.$$

$\therefore$ we can choose the controller gain in the range

$$k_{p1} < k_p < k_{p2}.$$

$$\boxed{21 < k_p < 65.7.}$$