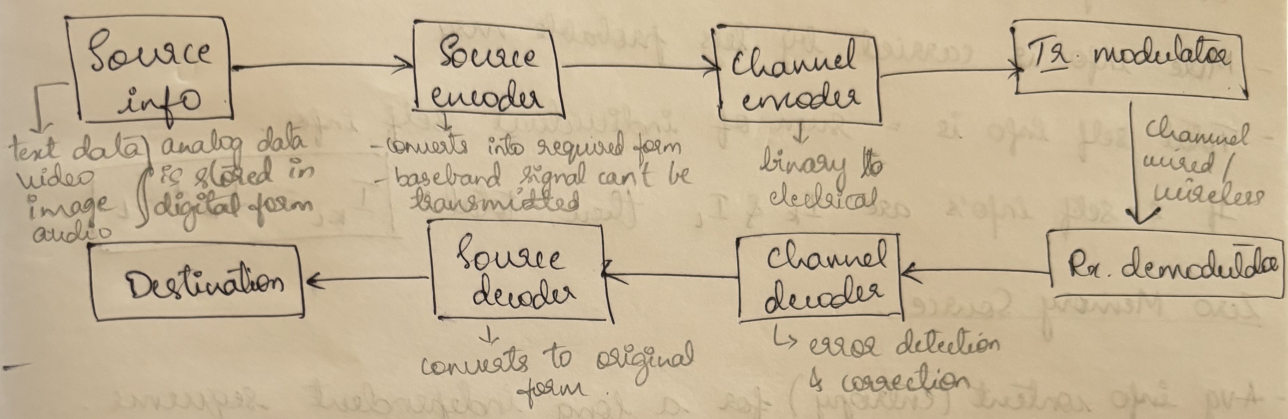
Information Theory

Digital comm. block diagram.



Measure of information / self information.

$S_k = \{S_1, S_2, --- S_q\}$ - source that produces info.

$P_k = \{P_1, P_2 --- P_q\}$ - probabilities.

Amount of info / self info, $\boxed{I_k = \log_2 \frac{1}{P_k}}$

- Info: sun rises in east $\rightarrow P_k = 1 \quad I_k = \log_2 \frac{1}{1} = 0$
  $\therefore$ not an info.
- Less probable (rare) $\rightarrow$ contains more info.

Problem - The binary symbols '0' & '1' are transmitted with probabilites $\frac{1}{4}$ & $\frac{3}{4}$ resp. Find corresponding self info.

$\rightarrow \quad I_k = \log_2 \frac{1}{P_k}$.

$0 \rightarrow I_k = \log_2 \frac{1}{1/4} = \log_2 4 = 2$ bits.

$1 \rightarrow I_k = \log_2 \frac{1}{3/4} = \log \frac{4}{3} = 0.415$ bits.

Observations on self info.

- It cannot be -ve. Each msg must contain certain info.
- Lowest probable self info is 0.
- More info is carried by less probable msg.
- Total self info is = sum of individual self info.

If 2 self info's are I_k & I_l then total :- $\boxed{I_{kl} = I_k + I_l}$

Zero Memory Source.

- Avg info content (entropy) for a long independent sequence.

$S = \{S_1, S_2 \text{ --- } S_q\}$ $\quad P = \{P_1, P_2 \text{ --- } P_q\}$.

Length of sequence = L symbols.

$P_1 L$ no. of messages of type S_1

$P_2 L$ " " " S_2

⋮

$P_q L$ " " " S_q

$I_k = \log \frac{1}{P_k}$

$P_1 L$ no. of message of type S_1 contain $P_1 L \log \frac{1}{P_1}$ bits of info.

$P_2 L$ " " " " " S_2 " $P_2 L \log \frac{1}{P_2}$ " "

⋮

$P_q L$ " " " " S_q " $P_q L \log \frac{1}{P_q}$ " "

$$I_{total} = P_1 L \log \frac{1}{P_1} + P_2 L \log \frac{1}{P_2} + \ldots + P_q L \log \frac{1}{P_q}$$

$$\boxed{I_{total} = L \sum_{i=1}^{q} P_i \log \frac{1}{P_i}}$$

Average info $= \frac{I_{total}}{L}$

Entropy $= \boxed{H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}}$ bits / msg symbol.

3) A discrete source emits one of 6^{th} symbols once every ms. The symbol prob. are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ & $\frac{1}{32}$ resp. Find source entropy & info rate.

$\rightarrow H(S) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

$= \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) + \frac{1}{16}\log_2(16) + \frac{1}{32}\log_2(32) + \frac{1}{32}\log_2(32)$

$= \frac{31}{16} = \underline{1.9375}$ bits / msg signal.

$R_s = r_s H(S)$ $\qquad r_s$ = msg symbol rate.

$= 1000 \times 1.9375$ $\qquad r_s = 1$ msg / ms. $= 1000$ sec.

$\underline{R_s = 1937.5 \text{ bps}}$

Find relation b/w Hartleys, nats & bits.

$I = \log_{10}\frac{1}{P}$ Hartleys, $I = \log_e \frac{1}{P}$ nats, $I = \log_2 \frac{1}{P}$ bits.

$1 \text{ Hartley} = \frac{I}{\log_{10}\frac{1}{P}} = \frac{\log_e \frac{1}{P}}{\log_{10}\frac{1}{P}} = \frac{-\log_e P}{-\log_{10} P}$ $\qquad \left[\log_a b = \frac{1}{\log_b a}\right]$

$1 \text{ Hartley} = \frac{\log_P 10}{\log_P e}$

1 Hartleys $= \log_e 10$ nats

$\boxed{1 \text{ Hartley} = 2.303 \text{ nats}}$

1 nats $= \log_2 e$ bits & $\log_2 10$ bits $\approx$ $\qquad$ 1 Hartley $= \log_2 10$ bits

$= 1.443$ bits $\qquad = 3.322$ bits.

4) A code is composed of dots & dashes. Assuming that a dash is 3 times as long as dot & has $\frac{1}{3}$ prob of occurrence. Calculate
a) Info in dot & dash $\qquad$ b) entropy of dot dash code
c) Avg rate of info if a dot lasts for 10ms & this time is allowed b/w symbols.

$\rightarrow P_{dot} + P_{dash} = 1$ $P_{dash} = \frac{1}{3} P_{dot}$

$P_{dash} = 1 - \frac{3}{4}$ $1 = P_{dot} + \frac{1}{3} P_{dot}$

$\boxed{P_{dash} = \frac{1}{4}}$ $\boxed{P_{dot} = \frac{3}{4}}$

a) $I_{dot} = \log \frac{1}{P_{dot}} = \underline{0.415 \text{ bit}}$ $I_{dash} = \log \frac{1}{P_{dash}} = \underline{2 \text{ bits}}$

b) $H(S) = \sum_{i=1}^{2} P_i \log_2 1/P_i$

duration of . = 10ms
" - " dash = 30ms

30 10 10 10

$= \frac{3}{4} \log_2 (\frac{4}{3}) + \frac{1}{4} \log_2 (4) = \underline{0.8113 \text{ bits/m. sym}}.$

c) $R_s = r_s H(S)$ $r_s = 4$ sym/100m sec.

$r_s = 40$ sym/sec.

$\boxed{R_s = 32.45 bps}$.

5) Find H(S) of source in nats/sym of a source that emits 1 out of 4 sym a, b, c & d resp. in a statistical independent sequence.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ & $\frac{1}{8}$

$\rightarrow H(S) = \sum P_i \log_2 \frac{1}{P_i} = \frac{1}{2} \log_2 (2) + \frac{1}{4} \log_2 (4) + \frac{1}{8} \log_2 (8) \times 2$

$\boxed{H(S) = 1.75 \text{ bps}}$

1 nats = 1.443 bits.
? = 1.75 bits.

1 Hartley = $\log_2 3.32$ bits
? = 1.75 bits

$H(S) = \frac{1.75}{1.443} = \underline{1.213 \text{ nats/sym}}$.

$\underline{H(S) = 0.52 \text{ Hartley/sym.}}$

Properties of Entropy.

$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{q} P_i I_i$

i) Prob. should vary b/w 0 & 1.

ii) Entropy fn is symmetric fn about its argument of entropy.

$H[P_k, (1-P_k)] = H[(1-P_k), P_k]$ $k = 1 \dots q$

value of H(S) remains same.

$P_A = \{P_1, P_2, P_3\}$ S_A $P_B = \{P_2, P_3, P_1\}$ S_B $P_C = \{P_3, P_1, P_2\}$ S_C.

$H(S_A) = H(S_B) = H(S_C)$

iii) Extremal property - finding out upper bound / max entropy.

lower bound will always be 0. $\log q - H(S)$ → no. of symbols.

$H(S)_{max} = \log_2 q$ bits/m. s.

Properties of Additivity

H'(S) - subgrouping → shouldn't decrease the entropy → $[H'(S) \geq H(S)]$.

Source efficiency, $\boxed{\eta_s = \frac{H(S)}{H(S)_{max}} \times 100}$

Source redundancy, $\boxed{R_{\eta s} = 1 - \eta_s} \times 100.$

Problem

1) A discrete msg source S, emits 2 indep. sym x & y with prob. 0.55 & 0.45 resp. Calculate η & $R_{\eta s}$.

$\rightarrow H(S) = 0.55 \log_2 \frac{1}{0.55} + 0.45 \log_2 \frac{1}{0.45} = \underline{0.9927 \text{ bits/m.s}}$

$H(S)_{max} = \log_2 (2) = \underline{1 \text{ bits/m-s.}}$

$\eta = \frac{0.9927}{1} = \underline{99.27\%}.$

$R_{\eta s} = 1 - 0.9927 = 0.0073 = \underline{0.73\%}.$

2) A pair of dice are tossed simult. The outcome of 1st dice is recorded as x_1 & that of 2nd dice is as x_2. Two events are defined as follows:

$A = \{(x_1, x_2) \text{ such that } x_1 + x_2 \leq 7\}$

$B = \{(x_1, x_2) \text{ such " } x_1 > x_2\}$

(11 12 13 14 15 16) (21 22 23 24 25)

which event conveys more info. Support your ans.

→ $P(A) = \frac{21}{36} = \frac{7}{12} = 0.58$ $P(B) = \frac{15}{36} = \frac{5}{12} = 0.416$

$I_A = \log_2 \frac{1}{P_A}$ $\quad I_B = \log_2 \frac{1}{P_B}$ $\quad \boxed{I_B > I_A}$

$I_A = 0.777$ bits $\quad I_B = 1.263$ bits

3) Shortly before a horse race, a bookmaker believes that several horses entered in race have following prob of winning:

horse A = 0.04
B = 0.42
C = 0.31
D = 0.12
E = 0.11
} Probabilities.

He then receives a msg that ~~that~~ owing to minor injury one of the horses is not participating in race. Explain how would you access from info theory point of view, the info value of this msg:

a) If the horse in question is ~~wrong~~ known
b) " " is not ~~wrong~~ known

→ a) $I_A = \log_2 \frac{1}{0.04} = 4.643$ bits

$I_B = \log_2 \frac{1}{0.42} = 1.251$ bits

$I_C = \log_2 \frac{1}{0.31} = 1.68$ bits

$I_D = \log_2 \frac{1}{0.12} = 3.058$ bits

$I_E = \log_2 \frac{1}{0.11} = ~~0.134~~ = 3.184$

b) $H(S) = 0.04 \log_2 \frac{1}{0.04} + 0.42 \log_2 \frac{1}{0.42} + 0.31 \log_2 \frac{1}{0.31} + 0.12 \log_2 \frac{1}{0.12} + 0.11 \log_2 \frac{1}{0.11} = \underline{1.9173}$

$H(S)_{max} = \log_2 q = \underline{2.322}$

4) B & W TV picture consists of 525 lines of picture info. Assume that each line consists of 525 picture ele (pixel) & that each ele can have 256 brightness levels. Picture are repeated at the rate of 30 frame/sec. Calculate average rate of info conveyed by TV set to viewer.

→ 525×525 → lines × pixels.

$q = (256)^{525 \times 525} = 256 \times 525 \times 525 = 70560000$

$H(S)_{max} = \log_2 q = 26.07$

$R_s = r_s H(S) = 30 \times 26.07 = 782.1$

Extension of zero-memory source.

$S = \{S_1, S_2\}$ $\quad P = \{P_1, P_2\}$ $\quad P_1 + P_2 = 1$.

[no. of basic source symbol]^{extension} → $[2]^2 = 4$ $\quad$ 2nd extension.

S_1S_1 occurs with probabilities $P_1P_1 = P_1^2$
S_1S_2 " " " $P_1P_2 = P_1P_2$
S_2S_1 " " " $P_1P_2 \quad P_1P_2$
S_2S_2 " " " $P_2P_2 = P_2^2$

$P_1^2 + 2P_1P_2 + P_2^2 = 1$.

$H(S) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$ → $\boxed{H(S) = P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}}$

$\left[\log \frac{1}{ab} = b \log \frac{1}{a}\right]$

$H(S^2) = \sum_{i=1}^{4} P_i \log \frac{1}{P_i}$

$= P_1^2 \log \frac{1}{P_1^2} + P_1P_2 \log \frac{1}{P_1P_2} + P_1P_2 \log \frac{1}{P_1P_2} + P_2^2 \log \frac{1}{P_2^2}$

$= 2P_1^2 \log \frac{1}{P_1} + 2P_1P_2 \log \frac{1}{P_1P_2} + 2P_2^2 \log \frac{1}{P_2}$

$= 2P_1^2 \log \frac{1}{P_1} + 2P_1P_2 \log \frac{1}{P_1} + 2P_1P_2 \log \frac{1}{P_2} + 2P_2^2 \log \frac{1}{P_2}$

$= 2P_1(P_1+P_2) \log \frac{1}{P_1} + 2P_2(P_1+P_2) \log \frac{1}{P_2}$

$\therefore \boxed{P_1 + P_2 = 1}$ → $= 2P_1 \log \frac{1}{P_1} + 2P_2 \log \frac{1}{P_2}$

$= 2\left(P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}\right) = \underline{2H(S)}$.

3^{rd} extension $\Rightarrow [2]^3 = 8$.

$S_1 S_1 S_1$ occurs with probabilities $P_1 P_1 P_1 = P_1^3$
$S_1 S_1 S_2$ " " " $= P_1^2 P_2$
$S_1 S_2 S_2$ " " " $= P_1 P_2^2$
$S_1 S_2 S_1$ " " " $= P_1^2 P_2$
$S_2 S_2 S_2$ " " " $= P_2^3$
$S_2 S_2 S_1$ " " " $= P_2^2 P_1$
$S_2 S_1 S_1$ " " " $= P_1^2 P_2$
$S_2 S_1 S_2$ " " " $= P_1 P_2^2$

$H(S^3) = \sum_{i=1}^{8} P_i \log \frac{1}{P_i}$

Finally, $H(S^3) = 3H(S)$.

$P_1^3 + 3P_1^2 P_2 + 3P_1 P_2^2 + P_2^3 = 1$

$\therefore \boxed{H(S^n) = nH(S)}$

Problem.

1) A ZMS has a source alphabet $S = S_1 S_2 S_3$ with prob. $P = \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$. Find $H(S)$ also determine $H(S^2)$ & verify that $H(S^2) = 2H(S)$

$[3]^2 = 9$

$\rightarrow H(S) = P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2} + P_3 \log \frac{1}{P_3}$

$H(S) = \frac{3}{2}$

$S_1 S_1 = P_1^2$
$S_1 S_2 = P_1 P_2$
$S_1 S_3 = P_1 P_3$
$S_2 S_1 = P_2 P_1$
$S_2 S_2 = P_2^2$
$S_2 S_3 = P_2 P_3$
$S_3 S_1 = P_3 P_1$
$S_3 S_2 = P_3 P_2$
$S_3 S_3 = P_3^2$

$H(S^2) = P_1^2 + 2P_1 P_2 + 2P_1 P_3 + 2P_2 P_3 + P_2^2 + P_3^2$

$= \sum_{i=1}^{9} P_i \log \frac{1}{P_i}$

$= P_1^2 \log \frac{1}{P_1^2} + 2P_1 P_2 \log \frac{1}{P_1 P_2} + 2P_1 P_3 \log \frac{1}{P_1 P_3}$
$+ 2P_2 P_3 \log \frac{1}{P_2 P_3} + P_2^2 \log \frac{1}{P_2^2} + P_3^2 \log \frac{1}{P_3^2}$

$H(S^2) = 3 = 2H(S) = 2\left(\frac{3}{2}\right) = 3$

2) A source emits one of the 4 probable msgs m_1, m_2, m_3 & m_4, with prob $7/16$, $5/16$, $1/8$, $1/8$ resp. Find $H(S)$. List all elements for 2^{nd} extension of source. Hence S.T $H(S^2) = 2H(S)$.

$\rightarrow H(S) = \sum_{i=1}^{4} P_i \log_2 \frac{1}{P_i} = \underline{1.796}$.

$[4]^2 = 16$.

$m_1 m_1 = P_1^2$	$m_2 m_1 = P_1 P_2$	$m_3 m_1 = P_1 P_3$	$m_4 m_1 = P_1 P_4$
$m_1 m_2 = P_1 P_2$	$m_2 m_2 = P_2^2$	$m_3 m_2 = P_2 P_3$	$m_4 m_2 = P_2 P_4$
$m_1 m_3 = P_1 P_3$	$m_2 m_3 = P_2 P_3$	$m_3 m_3 = P_3^2$	$m_4 m_3 = P_3 P_4$
$m_1 m_4 = P_1 P_4$	$m_2 m_4 = P_2 P_4$	$m_3 m_4 = P_4 P_3$	$m_4 m_4 = P_4^2$

$H(S^2) = \sum_{i=1}^{16} P_i \log \frac{1}{P_i}$.

$= P_1^2 \log \frac{1}{P_1^2} + P_1 P_2 \log \frac{1}{P_1 P_2} + P_1 P_3 \log \frac{1}{P_1 P_3} + P_1 P_4 \log \frac{1}{P_1 P_4}$
$+ P_1 P_2 \log \frac{1}{P_1 P_2} + P_2^2 \log \frac{1}{P_2^2} + P_2 P_3 \log \frac{1}{P_2 P_3} + P_2 P_4 \log \frac{1}{P_2 P_4}$
$+ P_1 P_3 \log \frac{1}{P_1 P_3} + P_2 P_3 \log \frac{1}{P_2 P_3} + P_3^2 \log \frac{1}{P_3^2} + P_4 P_3 \log \frac{1}{P_4 P_3}$
$+ P_1 P_4 \log \frac{1}{P_1 P_4} + P_2 P_4 \log \frac{1}{P_2 P_4} + P_3 P_4 \log \frac{1}{P_3 P_4} + P_4^2 \log \frac{1}{P_4^2}$

$= 3.59$ bits/m.s.

$H(S^2) = 2 \times H(S) = 2 \times 1.796 = \underline{3.59}$

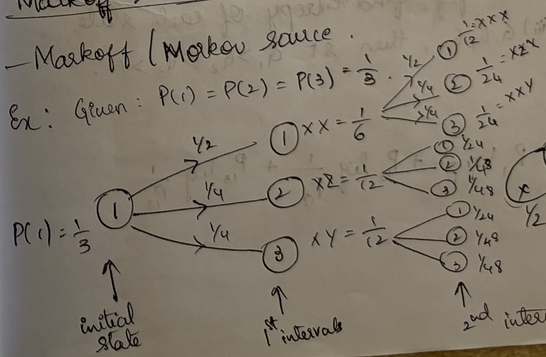
Average info content of symbol in a long dependent sequences.

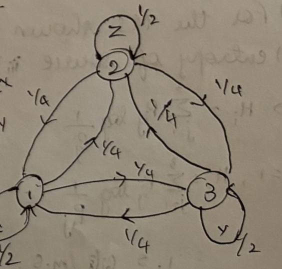
- Depends on previous.

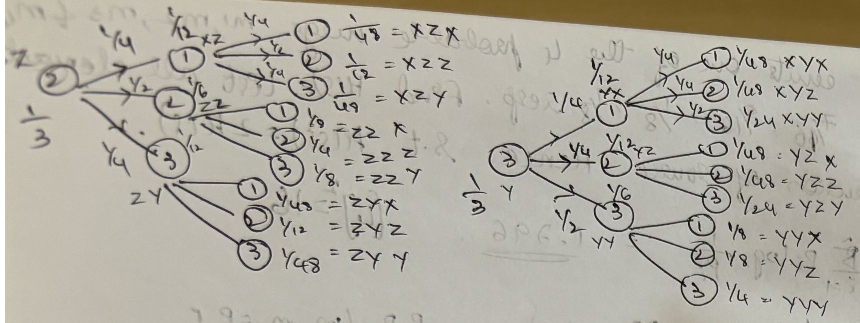
Markoff statical model for info source.

- Markoff / Markov source.

Ex: Given: $P(1) = P(2) = P(3) = \frac{1}{3}$.







Entropy & info source using Markov source.

Entropy of each state, $H_i = \sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}}$

Average entropy, $H = \sum_{i=1}^{n} P_i H_i$

$$H = \sum_{i=1}^{n} P_i \left[\sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}} \right] \text{ bits/m.s}$$

Info. rate : $R_s = r_s H$ bits/sec.

Theorem :- $G_N = \frac{1}{N} \sum_i P(m_i) \log \frac{1}{P(m_i)}$

$$G_N = \frac{1}{N} H(\bar{S}^N)$$

Info rate : $\lim_{N \to \infty} G_N = \cancel{H_{bps}}\ H_{bps}$

Problem.

1) For the M.S. shown in fig, find i) entropy of each state ii) entropy of source iii) $G_1 G_2 G_3$, then st $G_1 > G_2 > G_3 > H$.

→ $H_i = \sum_{j=1}^{3} P_{ij} \log \frac{1}{P_{ij}}$

$i = 1$, $H_1 = \sum_{j=1}^{3} P_{1j} \log \frac{1}{P_{1j}} = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}} + P_{13} \log \frac{1}{P_{13}}$

$= 1.5$ bits/m.s.

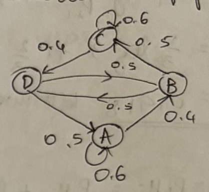
$H(S) = P_i H_i = P_1 H_1 + P_2 H_2 + P_3 H_3 = \frac{1}{3}(1.5 \times 3) = 1.5$.

2) Consider a state diagram of MS as shown in fig.

a) compute state probabilities.

b) Find H(S) of each state

c) " " of source.



→ $P(A) = 0.6P(A) + 0.5P(D)$

$P(B) = 0.4P(B) + 0.5P(D)$

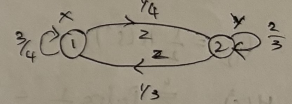
$P(C) = 0.6P(C) + 0.5P(B)$

$P(D) = 0.4P(C) + 0.5P(B)$.

3) The given state diagram ~~of a~~ for MS shown in fig,

i) Find H(S) of the source.

ii) " G_1, G_2. Hence st $G_1 > G_2 > H$.



→ $\left[H_i = \sum P_{ij} \log \frac{1}{P_{ij}} \right]$

$P(1) = \frac{3}{4} P(1) + \frac{1}{3} P(2)$ $\qquad$ $P(2) = \frac{2}{3} P(2) + \frac{1}{4} P(1)$

$\frac{1}{3} P(2) = P(1) - \frac{3}{4} P(1)$

$P(2) = \frac{3}{4} P(1)$

$P(1) + P(2) = 1$

$P(1) + \frac{3}{4} P(1) = 1$

$\boxed{P(1) = \frac{4}{7}}$

$P(2) = \frac{3}{4} \times \frac{4}{7}$

$\boxed{P(2) = \frac{12}{28} = \frac{3}{7}}$

$$H_1 = \sum_{j=1}^{2} P_{ij} \log \frac{1}{P_{ij}} = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}}$$

$$= \frac{3}{4} \log \frac{4}{3} + \frac{1}{4} \log 4$$

$$H_1 = \underline{0.8113} \text{ bits/m. sym.}$$

$$H_2 = P_{21} \log \frac{1}{P_{21}} + P_{22} \log \frac{1}{P_{22}}$$

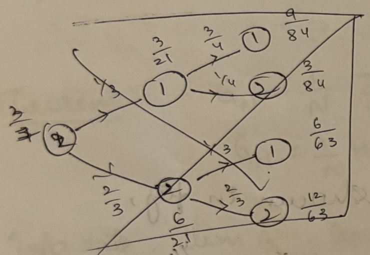
$$= \frac{1}{3} \log 3 + \frac{2}{3} \log \frac{3}{2}$$

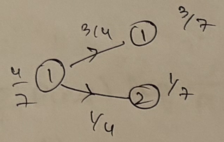
$$H_2 = 0.918 \text{ bits/msg. sym}$$

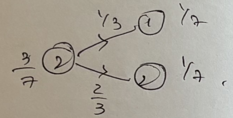
$$H = \sum P_i H_i = P_1 H_1 + P_2 H_2 = \frac{4}{7}(0.8113) + \frac{3}{7}(0.918) = \underline{0.857} \text{ b/ms}$$

$$G_N = \frac{1}{N} H(\bar{S}^N)$$

$$G_1 = H(\bar{S}) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i} = \frac{4}{7} \log \frac{7}{4} + \frac{3}{7} \log \frac{7}{3} = \underline{0.985}.$$







$$G_2 = \frac{1}{2} H(\bar{S}^2)$$

$$H(\bar{S}^2) = \sum_{i=1}^{4} P_{ij} \log \frac{1}{P_{ij}} = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}} + P_{21} \log \frac{1}{P_{21}} + P_{22} \log \frac{1}{P_{22}}$$

$$\boxed{G_2 = 0.921}$$

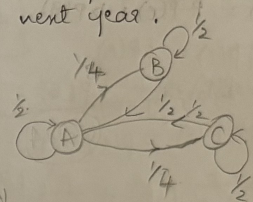
$$G_1 > G_2 > H.$$

Q) Design an info. s/m which gives info every year for about 200 students passing out with BE EC from university. The students can get into one of the 3 fields given below.

a) Go abroad for higher studies — A
b) Join civil services/MBA — B
c) " industries in India — C

Based on data given below, construct the model for the source & find source entropy.

- On the avg. 100 students are going abroad.
- Out of 100 going abroad this year, 50 were reported going abroad next year while 25 each went to MBA/C.S or joined industries in India.
- Out of 100 remaining in India this year, 50 continued to do so while 50 went abroad next year.



$$P(A) = \frac{1}{2}P(A) + \frac{1}{2}P(B) + \frac{1}{2}P(C).$$

$$P(B) = \frac{1}{2}P(B) + \frac{1}{4}P(A) \Rightarrow 1 - \frac{1}{2}P(B) = \frac{1}{4}P(A)$$

$$2P(B) = P(A)$$

$$P(B) = \frac{1}{2}P(A)$$

$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

$$P(C) = \frac{1}{2}P(C) + \frac{1}{4}P(A)$$

$$\frac{1}{4}P(A) = \left[1 - \frac{1}{2}\right]P(C)$$

$$P(A) = 2P(C)$$

$$P(C) = \frac{1}{2}P(A) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(A) + P(B) + P(C) = 1$$

$$\frac{1}{2}P(A) + \frac{1}{2} \times \frac{1}{2}P(A) + \frac{1}{2} \times \frac{1}{2}P(A) + \frac{1}{2}P(A) + \frac{1}{2}P(A) = 1$$

$$\frac{1}{2}P(A) + \frac{1}{4}P(A) + \frac{1}{4}P(A) + \frac{1}{2}P(A) \times 2 = 1$$

$$2P(A) = 1$$

$$P(A) = 1/2$$

P(A)
P(B)
P(C)

H_A
H_B H.
H_C

$$H_A = P_{A1} \log \frac{1}{P_A} + P_{A2} \log \frac{1}{P_{A2}} + P_{A3} \log \frac{1}{P_{A3}}$$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$$

$$= \frac{3}{2} = \underline{1.5}$$

$$H_B = P_{B1} \log \frac{1}{P_B} + P_{B2} \log \frac{1}{P_{B2}} + P_{B3} \log \frac{1}{P_{B3}}$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = \underline{1}$$

$$H_C = P_{C1} \log \frac{1}{P_{C1}} + P_{C2} \log \frac{1}{P_{C2}} + P_{C3} \log \frac{1}{P_{C3}}$$

$$= \frac{1}{2}\log 2 + 0 + \frac{1}{2}\log 2$$

$$= \underline{1}$$

$$H = P_A H_A + P_B H_B + P_C H_C$$

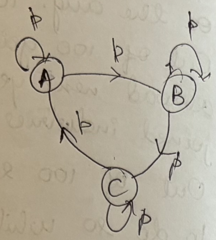
$$= \frac{1}{2} \times \frac{3}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{5}{4}$$

$$\underline{H = 1.25}$$

5) For the 1^{st} order MS, the source alphabets $S = \{A, B, C\}$ shown in fig, a) compute probs of states

b) Find $H(\bar{S})$ & $H(\bar{S}^2)$



$\rightarrow P(A) = pP(A) + pP(C)$

$P(B) = pP(B) + pP(A) \Rightarrow (1-p)P(B) = pP(A)$

$P(C) = pP(C) + pP(B)$ $\qquad P(B) = \left[\frac{p}{1-p}\right]P(A)$

$(1-p)P(C) = pP(B)$

$P(C) = \left(\frac{p}{1-p}\right)P(B) = \left(\frac{p}{1-p}\right)\left(\frac{p}{1-p}\right)P(A) = \frac{p^2}{(1-p)^2}P(A)$

$P(A) + P(B) + P(C) = 1$

$P(A) + \left(\frac{p}{1-p}\right)P(A) + \frac{p^2}{(1-p)^2}P(A) = 1$

$P(A)\left[1 + \frac{p}{1-p} + \frac{p^2}{(1-p)^2}\right] = 1$

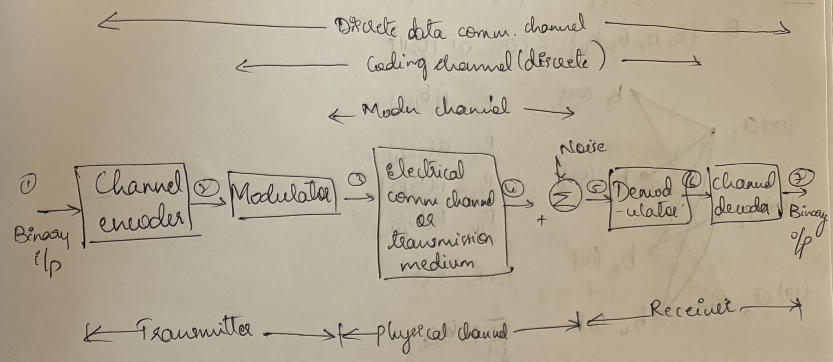
$P(A)\left[(1-p)^2 + p(1-p) + p^2\right] = (1-p)^2$

$P(A)\left[p^2 + p - \cancel{p^2} + 1 + p^2 - 2p\right] = 1 + p^2 - 2p$

$P(A) = \frac{(1-p)^2}{p^2-p+1}$

$P(B) = \left(\frac{p}{\cancel{1-p}}\right)\left(\frac{(1-p)^{\cancel{2}}}{p^2-p+1}\right) = \frac{p-p^2}{p^2-p+1}$

$P(C) = \left(\frac{p^2}{\cancel{(1-p)^2}}\right)\frac{\cancel{(1-p)^2}}{p^2-p+1} = \frac{p^2}{p^2-p+1}$



Discrete comm. channel.

Representation of channel.

i/p 'A' with 'r' symbols. $A = \{a_1, a_2, a_3 \ldots a_r\}$

o/p 'B' " 's' " $B = \{b_1, b_2, b_3 \ldots b_s\}$

Conditional probabilities $P(b_j/a_i)$

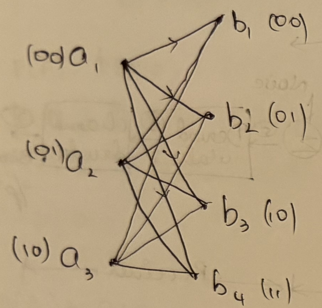
$$A\left\{\begin{matrix}a_1\\a_2\\\vdots\\a_r\end{matrix}\right. \rightarrow \boxed{P(b_j/a_i)} \rightarrow \left.\begin{matrix}b_1\\b_2\\\vdots\\b_s\end{matrix}\right\}B$$ — rep of channel.

$$P(B/A) = P(b_j/a_i) = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r \end{array}\begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) & \cdots & P(b_s/a_1) \\ P(b_1/a_2) & \cdot & \cdot & \cdots & \cdot \\ P(b_1/a_3) & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) & \cdots & P(b_s/a_r) \end{bmatrix}_{r \times s}$$

Rep of channel matrix / Noise matrix

<u>Ex</u>: $A = \{a_1 a_2 a_3\} = \{00, 01, 10\}$

$B = \{b_1 b_2 b_3 b_4\} = \{00, 01, 10, 11\}$



$P_{11} = P(b_1/a_1)$

$P_{12} = P(b_2/a_1)$

$P_{13} = P(b_3/a_1)$

$P_{14} = P(b_4/a_1)$

$P_{11} + P_{12} + P_{13} + P_{14} = 1$

$\sum_{j=1}^{s} P(b_j/a_i) = 1$

$P(a_1) + P(a_2) + P(a_3) + \dots P(a_r) = 1$ $\Rightarrow$ Theorem of total probabilities

$\sum_{i=1}^{r} P(a_i) = 1$

$P(b_1) = P(b_1/a_1) P(a_1) + \dots + P(b_1/a_r) P(a_r)$

$P(b_2) = P(b_2/a_1) P(a_1) + \dots + P(b_2/a_r) P(a_r)$

$\vdots$

$P(b_s) = P(b_s/a_1) P(a_1) + \dots + P(b_s/a_r) P(a_r)$

I/p prob $P(a_i)$

O/p " $P(b_j)$

Condition " $P(b_j/a_i)$

I/p " " $P(a_i/b_j)$

$P(a_i/b_j) P(b_j) = P(b_j/a_i) P(a_i)$

$$P(a_i/b_j) = \frac{P(b_j/a_i) P(a_i)}{P(b_j)}$$

<u>Joint probability</u>

$P(a_i, b_j) = P(a_i \cap b_j) = P(b_j/a_i) P(a_i) = P(a_i/b_j) P(b_j)$

$$P(b_j/a_i) P(a_i) = \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} \begin{bmatrix} P(b_1/a_1) P(a_1) & P(b_2/a_1) P(a_1) & \dots & P(b_s/a_1) P(a_1) \\ P(b_1/a_2) P(a_2) & P(b_2/a_2) P(a_2) & \dots & P(b_s/a_2) P(a_2) \\ \vdots & & & \\ P(b_1/a_r) P(a_r) & \dots & \dots & P(b_s/a_r) P(a_r) \end{bmatrix}$$

$$P(a_i, b_j) = \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} \begin{bmatrix} P(a_1, b_1) & P(a_1, b_2) & \dots & P(a_1 b_s) \\ P(a_2, b_1) & \dots & \dots & P(a_2 b_s) \\ \vdots & & & \\ P(a_r b_1) & \dots & \dots & P(a_r b_s) \end{bmatrix}$$

$P(A, B)$

<u>Property 1</u>: $P(b_1) = P(b_1/a_1) P(a_1) + P(b_1/a_2) P(a_2) + \dots + P(b_1/a_r) P(a_r)$

$P(b_1) = P(a_1, b_1) + P(a_2, b_1) + \dots P(a_r, b_1)$

$P(b_2) = P(a_1 b_2) + \dots + P(a_r b_2)$

$\vdots$

$P(b_s) = P(a_1 b_s) + \dots + P(a_r b_s)$

$\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)$

<u>Property 2</u> - $P(a_1) = P(b_1/a_1) P(a_1) + P(b_2/a_1) P(a_1) + \dots + P(b_s/a_1) P(a_1)$

$P(a_1) = P(a_1, b_1) + P(a_1, b_2) + \dots P(a_1, b_3)$

$P(a_2) = P(a_2, b_1) + \dots P(a_2 b_3)$

$\vdots$

$P(a_r) = P(a_r, b_1) + \dots P(a_r, b_s)$

$\sum_{i=1}^{s} P(a_i, b_j) = P(a_i)$

<u>Property 3</u> - $P(a_1) + P(a_2) + \dots P(a_r) = P(a_1 b_1) + P(a_1 b_2) + \dots P(a_1 b_s) +$
$P(a_2 b_1) + P(a_2 b_2) + \dots P(a_2 b_s) +$
$P(a_r b_1) + P(a_r b_2) + \dots P(a_r b_s) = 1$

<u>Problem</u> - In a comm. system, a transmitter has 3 i/p symbols i.e., $A = \{a_1 a_2 a_3\}$ & receiver also has 3 symbols i.e., $B = \{b_1 b_2 b_3\}$. The matrix given below shows JPM.

a) Find missing prob in Table.

b) Find $P(b_3/a_1)$ & $P(a_1/b_3)$.

c) Are the events a_1 & b_1 statistically independent? Why?.

a_i \ b_j	b_1	b_2	b_3
a_1	$\frac{1}{12}$	*	$\frac{5}{36}$
a_2	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{5}{36}$
a_3	*	$\frac{1}{6}$	*
$P(b_j)$	$\frac{1}{3}$	$\frac{14}{36}$	*

$P(b_1) + P(b_2) + P(b_3) = 1$

$\frac{1}{3} + \frac{14}{36} + P(b_3) = 1$

$$\boxed{P(b_3) = \frac{5}{18}}$$

$P(a_1, b_1) + P(a_2 b_1) + P(a_3, b_1) = P(b_1)$

$\frac{1}{12} + \frac{5}{36} + P(a_3, b_1) = \frac{1}{3}$

$P(a_3, b_1) = \frac{1}{3}$

$P(a_1 b_2) + P(a_2, b_2) + P(a_3, b_2) = \frac{14}{36}$

$P(a_1, b_2) = \frac{1}{9}$

$P(a_1 b_3) + P(a_2, b_3) + P(a_3 b_3) = P(b_3)$

$P(a_3, b_3) = 0$

$P(a_1, b_3) = P(b_3/a_1)\, P(a_1)$

$P(b_3/a_1) = \frac{P(a_1 b_3)}{P(a_1)} = \frac{\frac{5}{36}}{\frac{1}{3}} \to \left(\frac{1}{12} + \frac{1}{9} + \frac{5}{36}\right) = 5/12$

~~$P(a_1/b_3) = P($~~

$P(b_3, a_1) = P(a_1/b_3)\, P(b_3)$

$P(a_1/b_3) = \frac{P(b_3 a_1)}{P(b_3)} = \frac{\frac{5}{36}}{\frac{5}{18}} = \frac{1}{2}$

$P(a_1 \cap b_1) = P(a_1, b_1) = P(a_i)\, P(b_j)$ *

$\frac{1}{12} \neq \frac{1}{3} \times \frac{1}{3}$