13/9/23

Laplace Transform

Definition: $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$

$s = \sigma + j\omega$

$\delta(t) = 1$

$u(t) = 1/s$

$t = 1/s^2$

$t^2 = 2/s^3$

$t^3 = 6/s^4$

$t^n = \frac{n!}{s^{n+1}}$

$e^{-\alpha t} = 1/s+\alpha$

$e^{\alpha t} = 1/s-\alpha$

$t e^{-\alpha t} = \frac{1}{(s+\alpha)^2}$

$t e^{\alpha t} = \frac{1}{(s-\alpha)^2}$

$t^n e^{-\alpha t} = n!/(s+\alpha)^{n+1}$

$t^n e^{\alpha t} = n!/(s-\alpha)^{n+1}$

$\cos\omega t = s/s^2+\omega^2$

$\sin\omega t = \omega/s^2+\omega^2$

$e^{-\alpha t}\sin\omega t = \omega/(s+\alpha)^2+\omega^2$

$e^{-\alpha t}\cos\omega t = s/(s+\alpha)^2+\omega^2$

$\sinh\alpha t = \alpha/s^2-\alpha^2$

$\cosh\alpha t = s/s^2-\alpha^2$

Properties of LT :-

1. Linearity property :-

   Let C_1, C_2 be constants

   $f(t)$ & $g(t)$ are functions w.r.t time t

   then $L\{C_1 f(t) + C_2 g(t)\} = L\{C_1 f(t)\} + L\{C_2 g(t)\}$

2. First shifting theorem :-

   If $L\{f(t)\} = F(s)$ then

   $L\{e^{+at} f(t)\} = F(s-a)$

3. Change of scale:-

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$, freq. scaling

& $L\{f(t/a)\} = aF(sa)$, time scaling

4. Differentiation:-

If $L\{f(t)\} = F(s)$ then $L\frac{d^n f(t)}{dt^n} = s^n L\{f(t) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots f^{n-1}(0)$

For $n=1$ $\quad L\frac{d}{dt}f(t) = sL\{f(t)\} - f(0)\}$

For $n=2$ $\quad L\frac{d^2 f(t)}{dt^2} = s^2 L\{f(t) - sf(0)\} - f'(0)\}$

For $n=3$ $\quad L\frac{d^3 f(t)}{dt^3} = s^3 L\{f(t) - s^2 f(0) - sf'(0) - f''(0)\}$

5. Integration:-

If $L\{f(t)\} = F(s)$ then.

$$L\left[\iiint \cdots \int f(t)\,d^n\right] = \frac{1}{s^n}L f(t) + \frac{s^{n-1}f(0)}{s^n} + \cdots + \frac{f(0)}{s}$$

6. Time shifting:-

$L\{f(t-T)u(t-T)\} = e^{-st}F(s)$ when $u(t-T)$ is unit step.

Solve using Laplace transformer

i. $f''(t) + 3f'(t) + 2f(t) = 0$ where $f(0)=1$, $f'(0)=0$

$\rightarrow$ $L\{f''(t) + 3f'(t) + 2f(t)\}$

$= Lf''(t) + 3Lf'(t) + 2Lf(t)$.

$= s^2 L\{f(t)\} - sf(0) - f'(0)\} + 3sL\{f(t)\} - f(0)\} + 2F(s)$

$= s^2F(s) - s + 3s[F(s) - 1] + 2F(s) = 0$

$= F(s)[s^2 + 3s + 2] - s - 3s = 0$.

$\Rightarrow F(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$ $\qquad \frac{-3 \pm \sqrt{9-8}}{2a} = \frac{-3 \pm 1}{2} = -1, -2$

$$\frac{S+3}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}.$$

$$S+3 = A(S+2) + B(S+1)$$

$s=-1 \Rightarrow 2 = A \Rightarrow \boxed{A=2}$

$s=-2 \Rightarrow 1 = -B \Rightarrow \boxed{B=-1}$

$$\Rightarrow \frac{S+3}{(S+1)(S+2)} = \frac{2}{S+1} + \frac{-1}{S+2} \Rightarrow 2e^{-t}\, u(t) - e^{-2t}\, u(t)$$

HW

a. $\frac{3}{s^2+4} = \frac{3}{2}\sin 2t$

b. $\frac{2s+1}{s^2-2s+2} =$

c. $\frac{2s+2}{s^2+2s+5}$

Solve the differential eqn $\frac{dx}{dt} + 9x = 12$ where $x(0^+) = 1$

A system is represented by the relation $X(S) = R(S)\frac{120}{S^2+5S+10}$ where $R(S)$ is the LT of unit step fn. Find the value of $x(t)$ at $t \to \infty$

Hint: use initial value theorem.

$$\rightarrow X(S) = \frac{1}{s}\,\frac{120}{s^2+5s+10}$$

$\frac{-5 \pm \sqrt{25-4\ldots}}{2}$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

Laplace transform | Fourier transform.

- The LT of fn $x(t)$ can be represented as a continuous sum of exponential damped wave of form e^{st}

$f^{-1}(0)$

- The FT of fn $x(t)$ can be represented by a continuous sum of exp fns of the form of $e^{j\omega t}$

- LT is applied for solving DEs that relate i/p & o/p of the system
- FT is applied for solving DEs that relate i/p & o/p of the sys.
- LT can be used to analyze unstable system
- FT cannot be used to analyze unstable systems.
- Does not require if a fn defined for a set of negative numbers
- Only defined for fn that are defined for all real numbers.
- LT exists for every fn with a FT
- It is not always true that every fn with a LT has a FT.
- LT is widely used for solving DE. since LT exists even for signals for which FT does not exist
- It is rarely used for solving DE since FT does not exist for many signals

Ex: $x(t)$ as it is not absolutely integrable

Properties of FT:-

- Linearity:

$$F[\alpha f_1(t) + \beta f_2(t)] = \alpha F[f_1(t)] + \beta F[f_2(t)]$$

Time scaling:-

If $F(\omega) = F[f(t)]$ then $F[f(at)] = \frac{1}{|a|} F\left[\frac{\omega}{a}\right]$ $\quad F\left[f\left(\frac{t}{a}\right)\right] = |a| F(a\omega)$

Frequency shifting:-

If $F(\omega) = F[f(t)]$ then $F\left[f(t)\, e^{j\omega_0 t}\right] = F(\omega - \omega_0)$

Time differentiation:

If $F(\omega) = F[f(t)]$ then $F[f'(t)] = j\omega F(\omega)$

In general, $F[f^n(t)] = (j\omega)^n F(\omega)$

Time shifting:

If $F(\omega) = F[f(t)]$ then $F[f(t - t_0)] = e^{-j\omega t_0} F(\omega)$

Time Integration:-

If $F(\omega) = F[f(t)]$ then $F\left[\int_{-\infty}^{\infty} f(t)\,dt\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\, \delta(\omega)$

7. Reversal

If $F(\omega) = F[f(t)]$

$F[f(-t)] = F(-\omega)$

$= F^*(\omega)$

Complex conjugate

8. Duality :-

$F[f(t)] = F(\omega)$

$F[F(t)] = 2\pi f(-\omega)$

9. Convolutions

$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$

$f(t)$	$F(\omega)$
$\delta(t)$	1
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$\frac{2\sin\omega\tau}{\omega}$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
$e^{-\alpha t}u(t)$	$\frac{1}{\alpha+j\omega}$
$e^{\alpha t}u(-t)$	$\frac{1}{\alpha-j\omega}$
$t^n e^{-\alpha t}u(t)$	$\frac{n!}{(\alpha+j\omega)^{n+1}}$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$e^{-\alpha t}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(\alpha+j\omega)^2+\omega_0^2}$
$e^{-\alpha t}\cos\omega_0 t\, u(t)$	$\frac{\alpha+j\omega}{(\alpha+j\omega)^2+\omega_0^2}$
1	$2\pi\delta(\omega)$

1. $f''(t) = \delta(t+1) - 2\delta(t) + \delta(t-1)$

$\rightarrow (j\omega)^2 F(\omega) = e^{j\omega} - 2 + e^{-j\omega}$

$$= \frac{2\cos\omega - 2}{(j\omega)^2}$$

$$= \frac{2(1-\cos\omega)}{\omega^2}$$

2. Find IFT of $F(\omega) = \dfrac{10j\omega+4}{(j\omega)^2+6j\omega+8} = \dfrac{10s+4}{s^2+6s+8}$

$j\omega \rightarrow s \qquad = \dfrac{10s+4}{(s+2)(s+4)}$

$\rightarrow 10s+4 = A(s+4) + B(s+2)$

$s=-2 \Rightarrow -16 = 2A \Rightarrow \boxed{A=-8}$

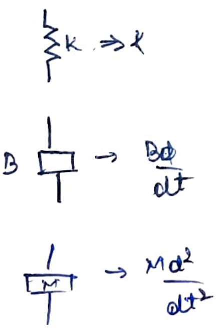
$s=-4 \Rightarrow -36 = -2B \Rightarrow \boxed{B=18}$

$$\frac{10s+4}{s^2+6s+8} = \frac{-8}{(s+2)} + \frac{18}{(s+4)} = \left[-8e^{-2t} + 18e^{-4t}\right]u(t)$$

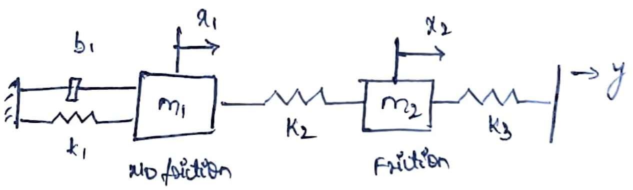
3. $G(\omega) = \dfrac{\omega^2+21}{\omega^2+9} = \dfrac{\cancel{A\omega+B}}{\cancel{\omega^2+9}} \quad \dfrac{\omega^2+9+12}{\omega^2+9} = 1 + \dfrac{12}{\omega^2+9}$

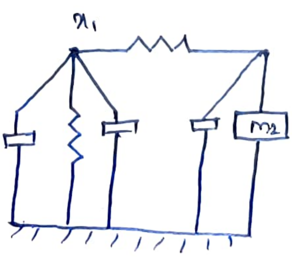
$= \delta(t) + 2e^{-3|t|}$

25/9/23



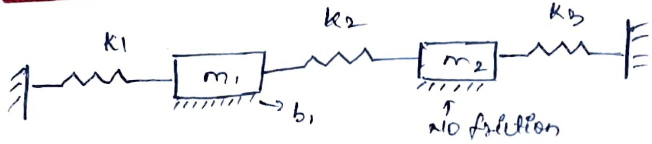
a)

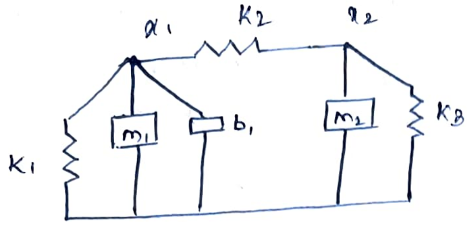




$\Sigma in = \Sigma out$

b)

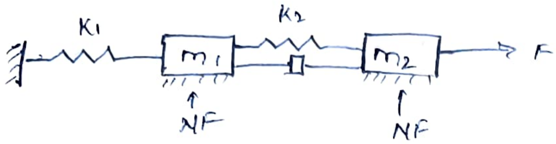




$$m_1\frac{d^2x_1}{dt^2} = -K_1x_1 - K_2(x_1-x_2) - b_1\frac{dx_1}{dt}$$

$$m_2\frac{d^2x_2}{dt^2} = -K_2(x_2-x_1) - K_3x_2$$

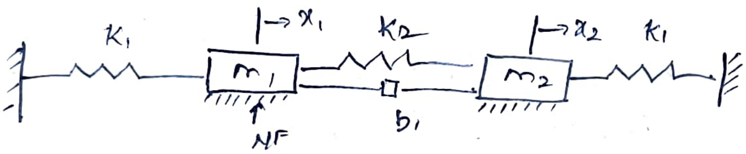
c)

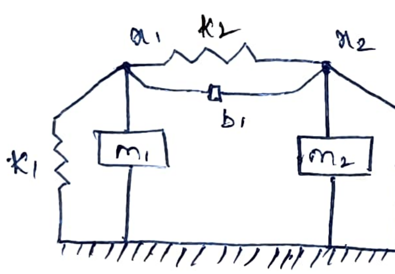


$$m_1\frac{d^2x_1}{dt^2} = -K_1x_1 - K_2(x_1-x_2) - b_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$

$$m_2\frac{d^2x_2}{dt^2} = F - K_2(x_2-x_1) - b_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

2. Write the DE for the mechanical sys shown below. Given that there are no initial conditions for both masses.





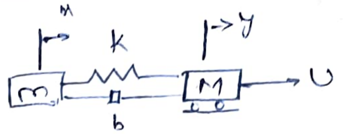
$$m_1\frac{d^2x_1}{dt^2} = -K_1x_1 - K_2(x_1-x_2) - b_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$

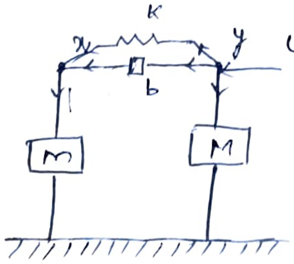
$$m_2\frac{d^2x_2}{dt^2} = K_2(x_1-x_2) - b_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) - K_1x_2$$

3. In ~~a~~ any mechanical sys. there is flexibility b/w one part of the sys and another. fig Q3 depicts such a situation, where a force u is applied to the mass M and another mass m is connected to it. The coupling b/w the objects is often modelled by spring const. k with a damping coeff. b although the actual situation is usually much more complicated than this.

a. Write the eqns of motion governing this sys

b. Find the t.f. for b/w the control i/p u & o/p y.





$$U = M\frac{d^2y}{dt^2} + k(y-x) + b\left(\frac{dy}{dt} - \frac{dx}{dt}\right)$$

$$k(x-y) + b\frac{d}{dt}(x-y) = m\frac{d^2x}{dt^2}$$

Taking Laplace Transform :-

$$U(s) = M s^2 Y(s) + k(Y(s) - X(s)) + bs(y-x)$$

$$k(X(s) - Y(s)) + b.s(x-y) = m s^2 X(s)$$

$$(-k - bs)X(s) + (Ms^2 + k + bs)Y(s) = U(s) \quad \text{①}$$

$$(ms^2 - k - bs)X(s) + (k + bs)Y(s) = 0 \quad \text{②}$$

$$\begin{bmatrix} ms^2+bs+k & -(k+bs) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}$$

Cramer's rule:

$$\begin{bmatrix} ms^2+bs+k & -(k+bs) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix} \qquad \begin{bmatrix} 0 & -(k+bs) \\ -(bs+k)U & ms^2+bs+k \end{bmatrix}$$

$\cancel{(Ms^2+bs+k)}$

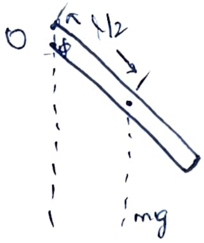
$\cancel{(ms^2+bs+k)^2 - (bs+k)^2 = mMs^4 + Mbs^3 + Ms^2k + bms^3 + b^2s^3 + bsk + mks^2 + bks + k^2 - (b^2s^2 + k^2 + 2bsk)}$

$$Y = \frac{(ms^2+bs+k)U}{(ms^2+bs+k)(Ms^2+bs+k) - (k+bs)^2}$$

$$\frac{Y}{U} = \frac{(ms^2+bs+k)}{(ms^2+bs+k)(Ms^2+bs+k) - (k+bs)^2}$$

4. Write the eqns of a motion of a pendulum consisting of a 4kg stick of length L suspended from pivot. How long should the rod be in order for the period to be exactly 2secs.

~~HINT~~ NOTE: The inertia I of a thin stick about an end point is $\frac{1}{3}ML^2$. Assume θ is small enough that $\sin\theta \approx \theta$



$F = I\alpha$

$M = -mg\sin\theta \cdot \frac{l}{2} = I_0 \frac{d^2\theta}{dt^2} = \frac{1}{3} ml^2 \frac{d^2\theta}{dt^2}$

$-\cancel{m}g\frac{l}{2}\sin\theta = \frac{1}{3}\cancel{m}l^2 \frac{d^2\theta}{dt^2}$

$\frac{d^2\theta}{dt^2} + \frac{3}{2}\frac{g}{l}\sin\theta = 0$

$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{3}{2}\frac{g}{l}\theta = 0$

$T = 2$ secs.

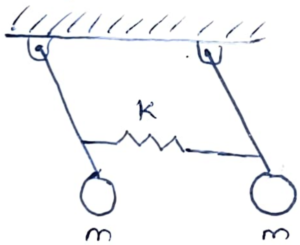
$T = \frac{2\pi}{\omega}$

$\Rightarrow \omega = \frac{2\pi}{T} = \pi$

$\omega^2 = \frac{3}{2}\frac{g}{l}$ $\Rightarrow l = \frac{3}{2}\frac{g}{\omega^2} = \frac{1.5 \times 9.8}{\pi^2} = 1.489 \approx \underline{1.5m}$

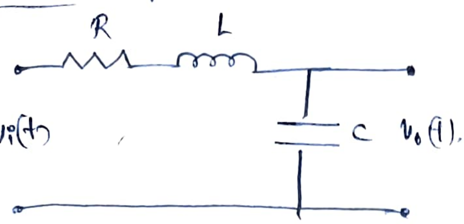
$\omega = \sqrt{\frac{3}{2}\frac{g}{l}}$

Q. write the eqⁿs of motion for double pendulum system shown in fig. Assume that displacement angles of the pendulum are small enough to ensure that spin is always horizontal. The pendulum rods are taken to be massless of the length l, & the spring attached 3/4th of the way down



6. ~~For~~ write the eqⁿ of motion for a body of mass m suspended from a fixed point ~~with~~ by a spring with a constant k. Carefully define where the body displacement is zero

Second Order Systems:-



$$\frac{V_0(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{\frac{RCs + s^2LC + 1}{Cs}}$$

$$= \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\Rightarrow \omega_n = \frac{1}{\sqrt{LC}}$$

$$2\xi\omega_n = \frac{R}{L}$$

$$\xi = \frac{R}{2L\omega_n} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$T_r = \frac{\pi - \theta}{\omega_d}$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \ s.$$

$$\% M/P = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100$$

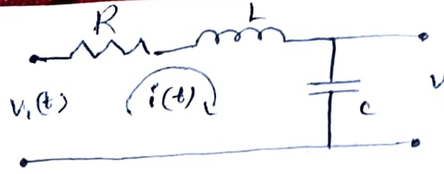
$$T_s = \frac{4}{\xi\omega_n}$$

Types of inputs

$u(t)$, $\delta(t)$, $r(t)$
↓ ↓ ↓
$1/s$, 1, $1/s^2$

---

1. For the electrical ckt shown in fig Q1. Find the following a. Time domain eqⁿ relation $i(t)$ & $v_1(t)$ b) Time domain eqⁿ relation equating $i(t)$ & $v_2(t)$ c) Assuming all IC are zero, tf fn $V_2(s)/V_1(s)$ & damping ratio ξ and undamped natural frequency ω_n of the system d) The values of R that will result in $v_2(t)$ having an overshoot of not more than 25%, assuming $v_1(t)$ is a unit step, $L = 10mH$ & $C = 4\mu F$.



Apply KVL.

a) $v_1(t) = R\,i(t) + L\frac{d}{dt}i(t) + \cancel{v_2(t)}\ \frac{1}{C}\int i(t)\,dt$

b) $v_2(t) = \frac{1}{C}\int i(t).dt$

$C\frac{d}{dt}v_2(t) = i(t)$

Apply LT,

$$V_1(t) = RC\frac{d}{dt}V_2(t) + \cancel{R}LC\frac{d^2}{dt^2}V_2(t) + V_2(t)$$

$$V_1(s) = RCsV_2(s) + RLs^2V_2(s) + V_2(s)$$

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{\cancel{CR}Ls^2 + RCs + 1} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$$

%) $$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\ln M_p = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

$$1.38\sqrt{1-\xi^2} = \pi\xi$$

$$1.904(1-\xi^2) = \pi^2\xi^2$$

$$1.904 - 1.904\xi^2 = \pi^2\xi^2$$

$$1.904 = \xi^2[\pi^2 + 1.904]$$

$$\underline{\xi = 0.4}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$$

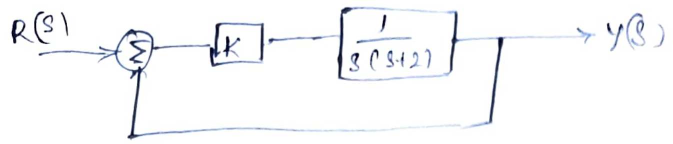
$$R = 2\xi\sqrt{\frac{L}{C}}$$

$$= 2 \times 0.4 \times \sqrt{\frac{10\times10^{-3}}{4\times10^{-6}}}$$

$$\underline{R = 40\Omega}$$

For the unity f/b sys shown in fig. ... specify the gain K of the proportional controller so that the o/p $y(t)$ has an overshoot of not more than 10% in response to a unit step.

$\rightarrow$



$$T(S) = \frac{\frac{K}{S(S+2)}}{1+\frac{K}{S(S+2)}} = \frac{\frac{K}{S(S+2)}}{\frac{K+S(S+2)}{S(S+2)}} = \frac{K}{K+S(S+2)} = \frac{K}{S^2+2S+K}$$

$\omega_n = \sqrt{K}$

$2\xi\omega_n = 2$

$\xi = \frac{1}{\omega_n} = \frac{1}{\sqrt{K}}$

$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

$\Rightarrow \ln(0.1) = \frac{-\pi\xi'}{\sqrt{1-\xi'^2}}$

$2.302^2(1-\xi'^2) = \pi^2\xi'^2$

$2.302 = \xi'^2(\pi^2+2.302)$

$\xi' = \sqrt{\frac{2.302}{\pi^2+2.302}}$

$\xi' = \cancel{0.434}\ 0.591$

$\Rightarrow (\sqrt{K})^2 = \frac{1}{\xi'^2} = 2.86.$

$0 \le K \le 2.86$

3. The open loop transfer fn of a unity f/b sys is $G(S) = \frac{K}{S(S+2)}$. The desired system response to a unit step i/p is specified as $T_p = 1$ sec and overshoot $M_p = 5\%$. a. Determine whether both specification can be met simultaneously in selecting the right value of K. b. Select the region in S where both specifications are met. Indicate what root positions are possible for some likely values of K.

c. Relax the specification in a by the same factor & pick a suitable value of K.

$T(s) = \dfrac{K}{s^2+2s+K}$

$\gamma = 2.21$

$M_P = 0.11$

$T_P = 2.21s.$

$M_P = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

$(\ln(0.15))^2 = \dfrac{\pi^2\xi'^2}{1-\xi'^2}$

$8.980 = (\pi^2 + 8.980)\xi'^2$

$\xi' = 0.69. \Rightarrow K = 2.1$

~~$w_n =$ ~~ 4.4kr ~~rad/s.~~

$T_P = \dfrac{\pi}{w_n\sqrt{1-\xi^2}} \rightarrow w_n = \dfrac{\pi}{\sqrt{1-0.69^2}} = 4.338$

$\gamma \rightarrow$ relaxation factor

$M_P^* = \gamma\, 0.05$

$t_P^* = \dfrac{\gamma\pi}{w_n\sqrt{1-\xi'^2}}$

$w_n\sqrt{1-\xi^2} = \gamma\pi$

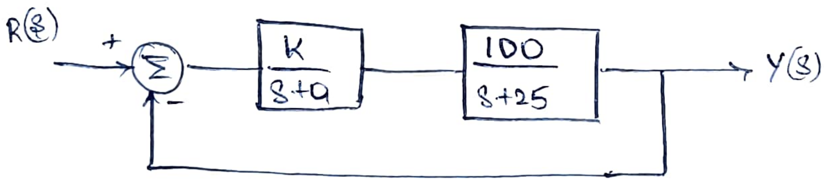
$\gamma = \dfrac{4.338\times\sqrt{1-0.69^2}}{\pi}$

$\dfrac{\pi}{w_n\sqrt{1-\xi^2}} = \dfrac{\pi}{\sqrt{K}\times\sqrt{1-\left(\frac{1}{\sqrt{K}}\right)^2}}$

$M_P = e^{\frac{-\pi\, 1/\sqrt{K}}{\sqrt{1-\left(\frac{1}{\sqrt{K}}\right)^2}}}$

$\gamma\times 0.05 = e^{-\gamma}$

4. For the unity f/b sys shown in fig; specify the gain & the pole location in the compensator so that overall closed loop response to a unit step i/p has an overshoot of no more than 25% & 1% settling time of no more than 0.1 sec.



$\xi \geq 0.4037$

$w_n = 114.07$ rad/s

$K \simeq 113.34$

$a = 67.10$

$\dfrac{Y(s)}{R(s)} = \dfrac{100K}{s^2+(25+a)s+25a+100K}$

$w_n = \cancel{10\sqrt{K}}\ \sqrt{25a+100K}$

$2\xi w_n = 25+a.$

$\xi w_n = \dfrac{(25+a)}{2}$

$t_s = \dfrac{4}{\xi w_n} \quad \dfrac{8}{25+a}$

$25+a = 80$

$a = 80-25$

$S_1, S_2 = -\xi w_n \pm j w_n\sqrt{1-\xi^2}$

$S_1 = -46.05 + 104.36j$

$S_2 = -46.05 - 104.36j$