Class-23

Filter design

Filter – selectively changes the waveshape of the signal in a desired manner.

main objective is

→ to improve the quality of the signal (eg. to remove noise)

→ to extract information from signals

Digital filter

→ Implemented on H/W or S/W

– operates on digital I/p to produce digital o/p.

– usually operates on digitized analog signals stored in computer memory

Adv

1) performance does not vary with environmental changes eg: thermal variations

2) frequency response can be more precisely adjusted using a programmable processor.

3) ~~Many~~ Several I/p signals can be filtered without replacing the hardware.

4) Digital filters can be designed to have linear phase.

5) Can be used at very low frequencies.

6) Digital filters are portable.

7) flexible.

Disadvantages

1) Speed limitations
— ADC and DAC are used
Speed depends on Conversion time of ADC, DAC and Speed of processor.

2) finite Word length effects

3) Long design and development time.

Digital filter Classification

Digital filter = Discrete time filter

- IIR (Infinite Impulse response)
- FIR (finite Impulse response)

Class-24

Difference Eqn representation of a system with I/p $x(n)$ and O/p $y(n)$

If y(n) is only retained in LHS and all are shifted to RHS, then current value of output is dependent on past values of o/p, current i/p and past values of i/p. Hence it is a recursive system(system whose present o/p depends on all 3 mentioned above).

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

It is causal system

$\longrightarrow$ IIR System.

FIR System

$$y(n) = \sum_{k=0}^{N-1} b_k\, x(n-k)$$

This is a non recursive system as there are no past o/p values.

It is causal system

IIR System

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}}$$

A system is said to be stable when its poles are inside unit-circle.

In FIR filter, it has no poles at all. Hence it is always stable whereas stability of IIR filter depends on roots of the denominator.

FIR filter is also called as moving average filter. It is all zero filert.

FIR filters

Adv & Disadv

IIR system has 2 types -
->All pole filter.
->Pole-zero filter.

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Poles are used to detect sudden spikes in any signal. IIR filters are used.
Ex- earthquakes.

IIR filters

Adv & Disadv.

Zeros are used while averaging of values is required. FIR filters are used.
Ex-Weather forecasting, stock market.

Class-25

IIR filter design

Steps

1) Obtain the specifications of equivalent analog filter.

2) Design the analog filter in accordance with the specifications

3) Transform the analog filter to an equivalent digital filter.

- analog filters
  - Butterworth
  - Chebyshev type-1(All pole filter)

- Low pass filter ⟶ prototype filter

- frequency transformations are used to get transfer functions of other filters

Adv of FIR:
-> It is all zero system, hence inherently stable.
-> Filter can be made adaptive in nature.
-> Linear phase can be easily established.

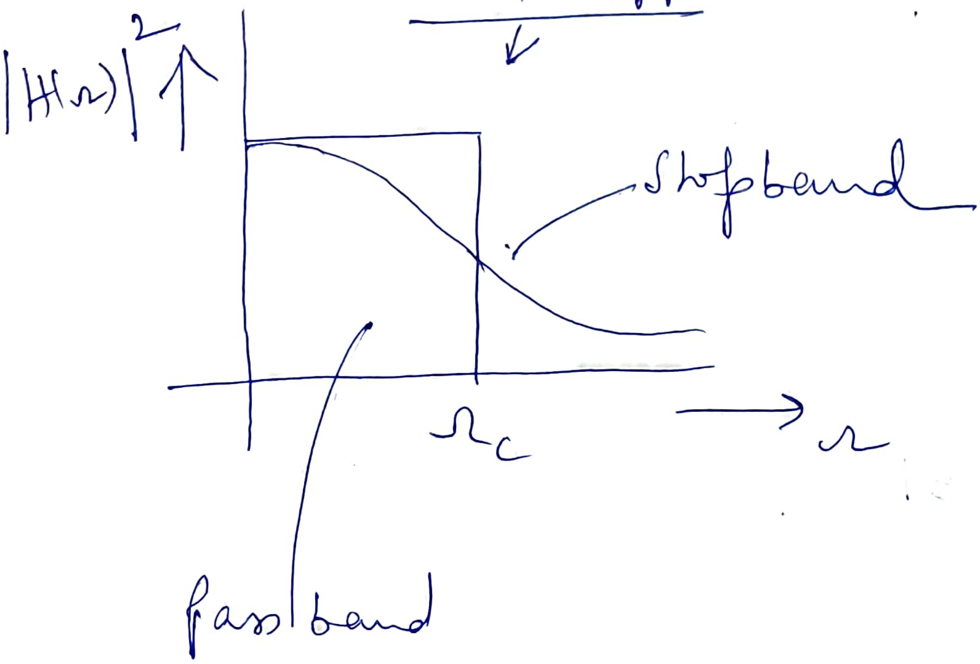
Disadv of FIR:
-> Order is very high to meet the required specs.

Adv of IIR:
-> Order is less.
->

Disadv of IIR:
-> It cannot be made adaptive as infinite number of weights cannot be made adaptive(we get infinite terms when numr is divided with denr).
-> Linear phase is impossible to achieve.
-> Since there are poles, implementation becomes complex.

Analog filter design using Butterworth Approximation

Ideal LPF



Magnitude squared frequency response of Butterworth filter is given by

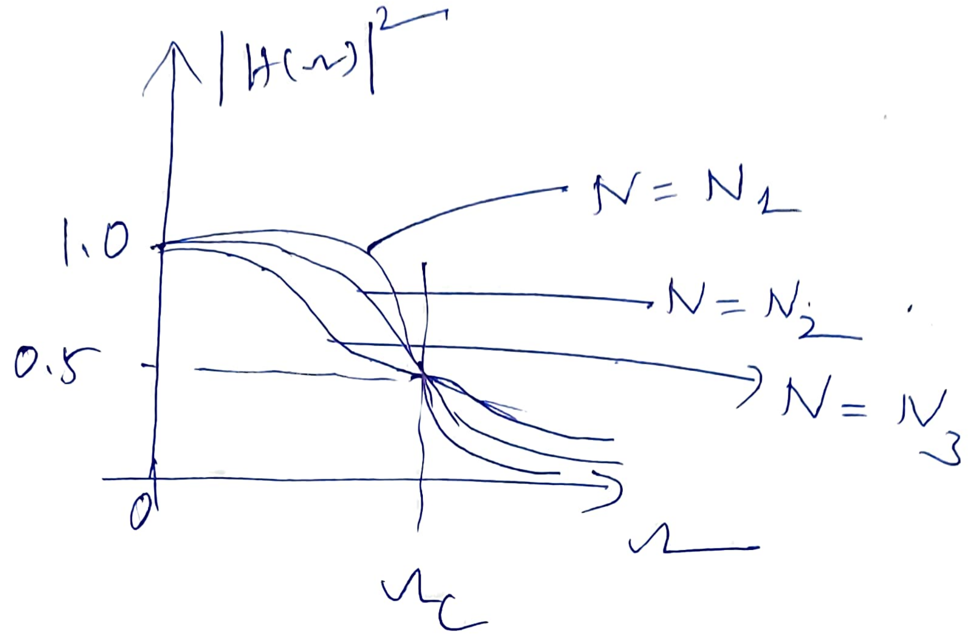
$$|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

N is the order

$\Omega_c \rightarrow$ 3 dB cutoff freq.

$\Omega \rightarrow$ Analog frequency

Capital omega



Observations

1) $|H(\Omega)|\Big|_{\Omega=0} = 1$ for all N

2) $|H(\Omega)| = \dfrac{1}{\sqrt{2}}$ at $\Omega = \Omega_c$

$\Rightarrow$ $20 \log_{10} |H(\Omega)| = -3.01\ dB$ at $\Omega = \Omega_c$

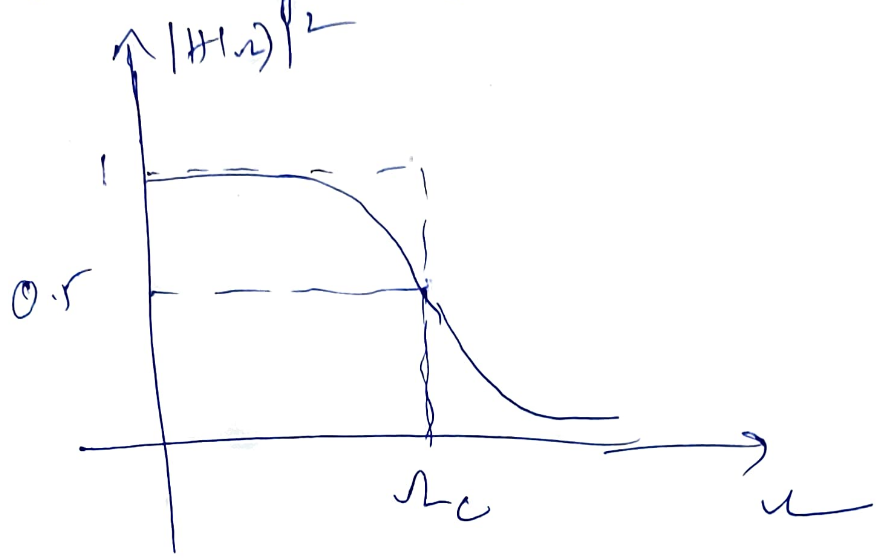
3) $|H(\Omega)| \to 0$ as $\Omega \to \infty$

4) Maximally flat responce in the pass band

$$\left.\frac{d^n |H(\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0. \text{ for } n = 0, 1, \dots 2N-1$$

(2)

5) Monotonically decreasing in the stopband



6) at $\Omega_c = 1 \Rightarrow$ Normalized LPF

$$|H(\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$$

$$H(j\Omega) \cdot H(-j\Omega) = \frac{1}{1+\Omega^{2N}}$$

Design of Analog Lowpass Butterworth filter

Gain of the filter.

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})}$$

1) N
2) Ω_c
3) poles $s_0, s_1 \cdots s_{N-1}$

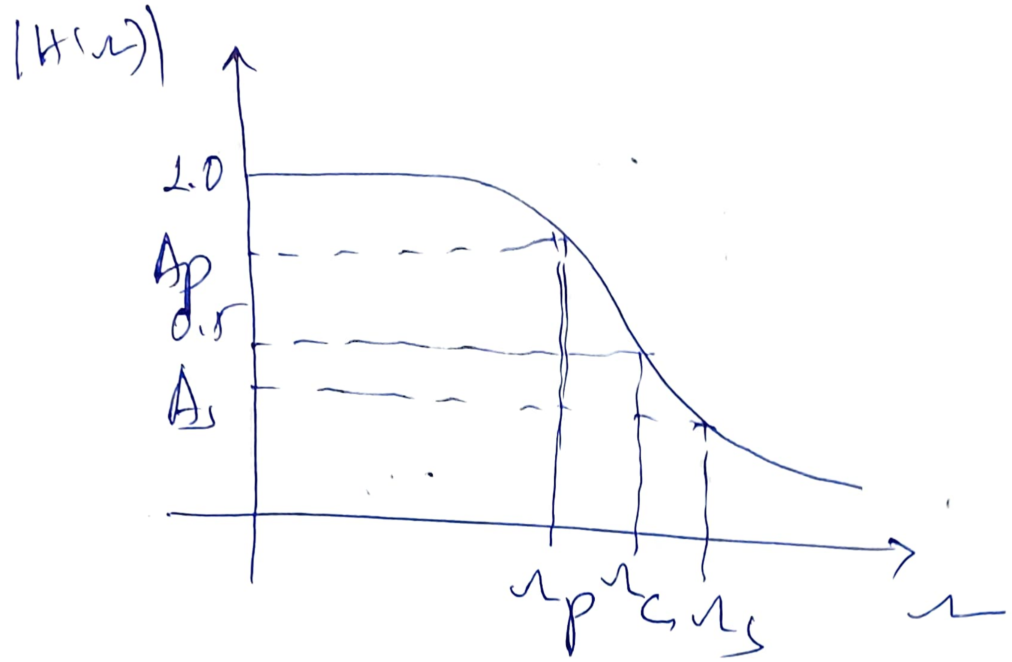
Order of Low pass Butterworth filter

The filter specifications are as follows

$$A_p \le |H(\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_p$$

$$|H(\Omega)| \le A_s \qquad \Omega \ge \Omega_s$$

A_p = passband gain
A_s = stopband gain
Ω_p = passband edge freq
Ω_s = stopband edge freq



pass band gain $\geq A_p$

Practically, omega,p != omega.s. There exists transition band b/w them.

Stop band gain $\leq A_s$

To find Minimum value of N

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

at $\Omega = \Omega_p$,

$$A_p^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \longrightarrow (1)$$

and $\Omega = \Omega_s$

$$A_s^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \longrightarrow (2)$$

from ①

$$\frac{1}{A_p^2} - 1 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \rightarrow ③$$

and from ②

$$\frac{1}{A_s^2} - 1 = \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} — ④$$

④ ÷ ③ gives

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{1/A_s^2 - 1}{1/A_p^2 - 1}$$

$$\boxed{N \geq \frac{\log\left[\left(1/A_s^2 - 1\right) / \left(1/A_p^2 - 1\right)\right]}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

N is always approximated to higher integer value.

To find Ω_c

from ③ $\quad \frac{\Omega_p}{\Omega_c} = \left[1/A_p^2 - 1\right]^{1/2N} \rightarrow ⑤$

from (4)

$$\frac{\Omega_s}{\Omega_c} = \left[\frac{1}{A_s^2} - 1\right]^{1/2N} \quad \text{--- (6)}$$

Rearranging terms in (5)

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{1/2N}}$$

and from (6)

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}$$

Actual cut off freq is avg of Ω_{cp} and Ω_{cs}

i.e. $$\boxed{\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2}}$$

To determine poles of $H(s)$

at $s = j\Omega$,

$$|H(s)| = |H(-s)|$$

$$s = j\Omega \Rightarrow s^2 = -\Omega^2.$$

$$|H(\Omega)|^2 = H(s).H(-s)$$

$$= \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^N}$$

poles of $H(s).H(-s)$ is obtained by equating denominator to zero.

$$1+\left(\frac{-s^2}{\Omega_c^2}\right)^N = 0$$

$$\left(\frac{-s^2}{\Omega_c^2}\right)^N = -1$$

$$\left(\frac{-s^2}{\Omega_c^2}\right) = (-1)^{1/N} \quad \text{—} (1)$$

Now,

$$e^{j(2k+1)\pi} = -1$$

using this in 1

$$\frac{-s^2}{\Omega_c^2} = e^{j(2k+1)\pi/N}$$

$$s^2 = -\Omega_c^2 \, e^{j(2k+1)\pi/N}$$

$$s = \pm j \, \Omega_c \, e^{j(2k+1)\pi/2N}$$

Now, $e^{j\pi/2} = j$

$$s = \pm \Omega_c \, e^{j(2k+1)\pi/2N + j\pi/2}$$

$$s = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$$k = 0, 1, \ldots N-1$$

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})}$$

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$$N \geq \frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{2\log\frac{\Omega_s}{\Omega_p}}$$

If A_p and A_s are in dB

$$A_s \text{ in dB} = -20\log_{10} A_s$$

$$\log_{10} A_s = -\frac{A_s \text{ in dB}}{20}$$

$$A_s = 10^{-\frac{A_s \text{ in dB}}{20}}$$

$$\frac{1}{A_s} = 10^{\frac{A_s \text{ in dB}}{20}}$$

$$\frac{1}{A_s^2} = 10^{\frac{A_s \text{ in dB}}{10}} = 10^{0.1 A_s \text{ in dB}}$$

$$N \geq \frac{\log\left[\frac{10^{0.1 A_s dB}-1}{10^{0.1 A_p dB}-1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Frequency Transformations

Analog frequency Transformations

1) Low pass to Low pass

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s$$

where Ω_{LP} is pass band edge frequency of the desired filter.

2) Low pass to High pass

$$s \rightarrow \frac{\Omega_p \Omega_{HP}}{s}$$

Ω_{HP} is the pass band edge frequency of the high pass filter.

3) Low pass to Band pass

$$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

where Ω_l = lower band edge frequency
Ω_u = upper band edge frequency.

4) Low pass to Band stop.

$$s \rightarrow \Omega_p \frac{(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

Ω_l = lower band edge frequency

Ω_u = upper band edge frequency

Prob

1) Let $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$ represent transfer function of a low pass filter with cut off freq 1 rad/sec. Use frequency transformations to find the transfer functions of the following analog filters (a) Low pass filter with passband of 10 rad/sec (b) High pass filter with cut off frequency of 10 rad/sec.

Soln

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} ; \quad \Omega_p = 1 \text{ r/s}$$

a) LP – LP transformation

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s$$

$$\Omega_p = 1 \text{ r/s} \quad \Omega_{LP} = 10 \text{ r/s}$$

$$s \rightarrow \frac{s}{10}$$

(18)

$$H_1(s) = H(s)\Big/_{s = s/10}$$

$$= \frac{1}{\frac{s^2}{100} + \sqrt{2}\frac{s}{10} + 1} = \frac{100}{s^2 + 10\sqrt{2}s + 100}$$

b) LP → HP transformation

$$s \to \frac{\Omega_p \Omega_{Hp}}{s}$$

$$s \to \frac{10}{s}$$

$$H_2(s) = H(s)\Big/_{s = \frac{10}{s}} = \frac{1}{\frac{100}{s^2} + \frac{10\sqrt{2}}{s} + 1}$$

$$= \frac{s^2}{s^2 + 10\sqrt{2}s + 100}$$

2) Let $H(s) = \frac{1}{s^2+s+1}$ represent the transfer function of a lowpass filter with pass band of 1 r/s. Use frequency transformation to find the transfer function of the following analog filters

a) High pass filter with cut off frequency 10 rad/sec.

b) Band pass filter with a pass band of 10 r/s and Centre freq. 100 r/s.

Soln

a) LP → HP

$$s \rightarrow \frac{\Omega_p \Omega_{HP}}{s} = \frac{10}{s}$$

$$H_1(s) = H(s)\Big|_{s=\frac{10}{s}} = \frac{1}{\left(\frac{10}{s}\right)^2 + \frac{10}{s} + 1}$$

$$= \frac{s^2}{s^2+10s+100}$$

b) Centre freq. $\Omega_0 = \sqrt{\Omega_l \Omega_u}$ 20

$$\Omega_0 = 100 \text{ r/s}$$

$$\therefore \Omega_l \Omega_u = 100^2 \longrightarrow (1)$$

Pass band $= \Omega_u - \Omega_l = 10 \longrightarrow (2)$

$$LP \longrightarrow BP$$

$$s \longrightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 100^2}{s \times 10}.$$

$$= \frac{s^2 + 10000}{10s}$$

$$H_2(s) = H(s) \Big|_{s = \frac{s^2+10000}{10s}}$$

$$= \frac{1}{\left(\frac{s^2+10000}{10s}\right)^2 + \left(\frac{s^2+10000}{10s}\right) + 1}$$

$$= \frac{100 s^2}{s^4 + 10s^3 + 20100 s^2 + 10^5 s + 10^8}.$$

Analog Butterworth filter design

Prob

1) Given that $|H(\Omega)|^2 = \dfrac{1}{1+16\Omega^4}$,

determine the analog filter transfer function $H(s)$.

Solun

$$|H(\Omega)|^2 = \frac{1}{1+16\Omega^4}$$

$$= \frac{1}{1+\left(\frac{\Omega}{1/2}\right)^4} = \frac{1}{1+\left(\frac{\Omega}{1/2}\right)^{2\times 2}}$$

$$= \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$\therefore$ $\underline{N = 2}$ and $\underline{\Omega_c = 1/2 \text{ r/s}}$

Poles of Butterworth filter

$$P_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$$k = 0, 1, \ldots N-1$$

Let us design a normalized filter first

i.e., $\Omega_c = 1$ r/s

$$P_k = \pm e^{j(2k+1+N)\pi/2N}$$

$$k = 0, 1$$

for $k=0$

$$P_0 = \pm e^{j3\pi/4}$$

$$= \pm [-0.707 + j0.707]$$

for $k=1$

$$P_1 = \pm e^{j5\pi/4}$$

$$= \pm [-0.707 - j0.707]$$

To determine $H_1(s)$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)}$$

$$= \frac{1}{[s-(-0.707+j0.707)][s-(-0.707-j0.707)]}$$

$$= \frac{1}{(s+a-jb)(s+a+jb)}$$

$$= \frac{1}{\left((s+a)^2+b^2\right)}$$

$$= \frac{1}{\left[(s+0.707)^2+(0.707)^2\right]}$$

$$= \frac{1}{(s^2+0.5+1.414s)+0.5}$$

$$H_1(s) = \frac{1}{s^2+1.414s+1}$$

Applying LP → LP transformation

$$H(s)\Big/_{s \to \frac{s}{0.5}} = \frac{1}{\left(\frac{s}{0.5}\right)^2+1.414\times\frac{s}{0.5}+1}$$

$$= \frac{(0.5)^2}{s^2+(0.5)\times1.414s+(0.5)^2}$$

$$H(s) = \frac{0.25}{s^2+0.707s+0.25}$$

Prob Determine the transfer function of Normalized Butterworth filter with order a) 2 b) 3 c) 4. (24)

a) $$P_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$k = 0, 1$

$\Omega_c = 1 \text{ r/s}$

$N = 2$

$$P_k = \pm e^{j(2k+3)\pi/4}$$

Put $k = 0$

$$P_0 = \pm e^{j3\pi/4}$$
$$= \pm(-0.707 + j0.707)$$

Put $k = 1$

$$P_1 = \pm e^{j5\pi/4}$$
$$= \pm(-0.707 - j0.707)$$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2 + 1.414s + 1}$$

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b) $N = 3$

$$P_k = \pm \Omega_c e^{j(2k+1+N)\pi/2N}$$

$k = 0, 1, 2$

$\Omega_c = 1$ r/s and $N = 3$

$$P_k = \pm e^{j(2k+4)\pi/6}$$

Put $k = 0$,

$$P_0 = \pm e^{j4\pi/6}$$

$$= \pm(-0.5 + j0.866)$$

Put $k = 1$

$$P_1 = \pm e^{j6\pi/6}$$

$$= \pm(-1)$$

Put $k = 2$

$$P_2 = \pm e^{j8\pi/6}$$

$$= \pm(-0.5 - j0.866)$$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)}$$

$$H(s) = \frac{1}{(s+1)\left[(s+0.5-j0.866)(s+0.5+j0.866)\right]}$$

$$= \frac{1}{(s+1)\left[(s+0.5)^2+(0.866)^2\right]}$$

$$= \frac{1}{(s+1)(s^2+s+1)}$$

Prob

1) Given $|H(\Omega)|^2 = \dfrac{1}{1+64\Omega^6}$,

determine the analog filter system function $H(s)$

Examples

1. Design an analog Butterworth filter to meet the following specifications

$$0.8 \le |H(\Omega)| \le 1 \qquad 0 \le f \le 1000\,HZ$$

$$|H(\Omega)| \le 0.2 \qquad f \ge 5000\,HZ$$

Sol^n

$$A_p = 0.8 \qquad A_s = 0.2$$

$$\Omega_p = 2\pi f_p = 2\pi \times 1000 = 2000\pi \text{ r/s}$$

$$\Omega_s = 2\pi f_s = 2\pi \times 5000 = 10000\pi \text{ r/s}$$

Step 1

To find the order of the filter

$$N \ge \frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\frac{1}{A_p^2} = 1.56$$

$$\frac{1}{A_s^2} = 25$$

$$\ge 1.167$$

$$\boxed{N = 2}$$

Step 2

To find Ω_c

(28)

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{1/2N}}$$

$$= 7263 \text{ r/s}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}$$

$$= 14196 \text{ r/s}$$

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2} = \underline{\underline{10729}} \text{ r/s}$$

$$= 3415\pi \text{ r/s}$$

<u>Step 3</u>

To design a normalised 2^{nd} order filter.

$$P_0 = \pm e^{j(2\times 0 + 1 + 2)\pi/2\times 2}$$

$$= \pm(-0.707 - j0.707)$$

$$P_1 = \pm e^{j(2\times 1 + 1 + 2)\pi/4}$$

$$= \pm(-0.707 + j0.707)$$

$$H_1(s) = \frac{1}{(s+0.707-j0.707)(s+0.707+j0.707)}$$

$$= \frac{1}{(s+0.707)^2+(0.707)^2}$$

$$H_1(s) = \frac{1}{s^2+1.414s+1}$$

<u>Step 4</u>

To design the filter for required Ω_c.

$LP \rightarrow LP$

$$s \rightarrow \frac{s}{10729}$$

$$H(s) = \frac{1}{\left(\frac{s}{10729}\right)^2 + 1.414\left(\frac{s}{10729}\right)+1}$$

$$= \frac{(10729)^2}{s^2+1.414 \times 10729\, s + (10729)^2}$$

Prob

Design an analog butterworth filter for the following specifications

$$0.8 \le |H(\Omega)| \le 1 \qquad 0 \le \Omega \le 0.2\pi$$

$$|H(\Omega)| \le 0.2 \qquad 0.6\pi \le \Omega \le \pi$$

Soln

$A_p = 0.8$ $\qquad$ $A_s = 0.2$

$\Omega_p = 0.2\pi$ r/s $\qquad$ $\Omega_s = 0.6\pi$ r/s

Prob: Design a high pass Butterworth filter for the given specifications. (31)

−3 dB passband attenuation at a frequency of $\Omega_p = 1000$ r/s. and at least −15 dB attenuation at 500 r/s.

Soln: Specifications of high pass filter to be designed

$A_p = -3\,dB$ $\quad A_s = -15\,dB$

$\Omega_p = 1000$ r/s $\quad \Omega_s = 500$ r/s

Step 1

Design a normalized LPF.

for Equivalent LPF $\Omega_s = 1000$ r/s $\quad \Omega_p = 500$ r/s

order of the filter

$$N \geq \frac{\log\left[\frac{10^{0.1 A_s dB} - 1}{10^{0.1 A_p dB} - 1}\right]}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{\log[30.9]}{2\log(2)} \geq 2.49$$

$\boxed{N = 3}$

Transfer of function of normalised 3rd order Butterworth LPF is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

∴ Transfer function of 3rd order HPF

with Ω_c

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}}\right]$$

$$= \frac{1}{2}\left[\frac{\Omega_p}{\left[10^{0.1 A_p dB}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[10^{0.1 A_s dB}-1\right]^{1/2N}}\right]$$

$\simeq 1000$

frequency transformation is

$$s \rightarrow \frac{1000}{s}$$

$$H(s) = \frac{s^3}{(s+1000)(s^3+1000s+1000^2)}$$

Prob;

Design an analog filter with maximally flat response in the passband and an acceptable attenuation of -2 dB at 20 r/s. The attenuation in the stopband should be more than 10 dB beyond 30 r/s.

Soln

$A_p = -2$ dB $A_s = -10$ dB

$\Omega_p = 20$ r/s $\Omega_s = 30$ r/s

$$N \geq \frac{\log\left[\frac{10^{0.1 A_s dB} - 1}{10^{0.1 A_p dB} - 1}\right]}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$N \geq 3.37$

$$\boxed{N = 4}$$

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[10^{0.1A_p\,dB}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[10^{0.1A_s\,dB}-1\right]^{1/2N}}\right]$$

$$\Omega_c = 22 \text{ r/s}$$

Normalised fourth order Butterworth filter

$$H_1(s) = \frac{1}{\left[(s+0.382)^2+(0.923)^2\right]\left[(s+0.923)^2+(0.382)^2\right]}$$

$$= \frac{1}{(s^2+0.764s+1)(s^2+1.846s+1)}$$

LP → LP transformation

$$s \rightarrow \frac{s}{22}$$

$$H(s) = \frac{(22)^4}{(s^2+370s+484)(s^2+894s+484)}$$

$S \rightarrow Z$ mapping

1) Approximation of derivatives
2) Impulse Invariance Transformation (IIT)
3) Bilinear transformation
4) pole - zero mapping

Impulse Invariance transformation

$$\begin{array}{c}\text{[Analog filter]} \\ H(s)\end{array} \xrightarrow{L^{-1}} h(t) \xrightarrow[\text{sampling}]{\text{Unifrom}} h(nT)$$

$$h(nT) \xrightarrow{\text{Z-trans}} \begin{array}{c} H(z) \\ \text{(Digital filter)}\end{array}$$

Let $H(s)$ be the system function of analog filter

Using partial fraction expansion,

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s - p_k} \rightarrow (1)$$

Taking inverse Laplace transform,

$$h(t) = \sum_{k=1}^{N} C_k e^{p_k t} \quad \text{for } t \geq 0$$

$$\Rightarrow \sum_{k=1}^{N} C_k e^{p_k t} u(t)$$

Sampling $h(t)$ uniformly gives

$$h(nT) = h(n) = \sum_{k=1}^{N} C_k e^{p_k nT} u(nT)$$

Taking Z- transform

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}} \rightarrow (3)$$

from (1) & (3) we can write mapping as

$$\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$H(s) = \frac{1}{s - p_k}$$

$\therefore$ $H(s)$ has pole at $s = p_k$.

and $H(z)$ has pole at $z = e^{p_k T}$.

analog pole at $s = p_k$ is mapped to digital pole at $z = e^{p_k T}$

$$z = e^{p_k T}$$

$$\Rightarrow z = e^{sT}$$

In polar form $z = re^{j\omega}$ and

$$s = \sigma + j\Omega$$

$$\therefore re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

$\therefore$ Equating real and imaginary parts,

$$\boxed{\begin{aligned} r &= e^{\sigma T} \\ \omega &= \Omega T \end{aligned}} \longrightarrow (4)$$

Mapping Summary

A) from (4) if $\sigma < 0$ then $0 < r < 1$

$\sigma < 0 \Rightarrow$ LHS of s plane

$0 < r < 1 \Rightarrow$ area inside unit circle in Z-plane

$\therefore$ LHS of s plane is mapped inside unit circle in Z-plane

B) from (4) if $\sigma > 0$, then $r > 1$

$\therefore$ RHS of s plane is mapped outside unit circle in Z-plane.

C) ~~of~~ from (4) if $\sigma = 0$ then $r = 1$

$\therefore$ $j\Omega$ axis is mapped to unit circle in Z-plane.

Limitations of IIT

Range of ω is $-\pi \le \omega \le \pi$

$\omega = \Omega T$

$\therefore -\pi \le \Omega T \le \pi$

$-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \rightarrow$ gets mapped to

$-\pi \le \omega \le \pi$

$\therefore$ In general we can write

$$\frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$$

is mapped to $-\pi \le \omega \le \pi$

$\Rightarrow$ multiple segments of $j\Omega$ axis are mapped repeatedly on ~~____~~ in the range $-\pi \le \omega \le \pi$

$\Rightarrow$ Causes **Aliasing**.

Prob

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1) The system function of analog filter is given by $H(s) = \dfrac{1}{(s+1)(s+2)}$

Find $H(z)$ using IIT method.

Take sampling frequency as 5 samples/sec.

Soln

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = 1, \quad B = -1$$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

using IIT mapping

$$\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$T = \frac{1}{f} = \frac{1}{5} = 0.2 \text{ sec}$$

$$\frac{1}{s+1} \longrightarrow \frac{1}{1 - e^{-1 \times 0.2} z^{-1}}$$

$$\frac{1}{s+2} \longrightarrow \frac{1}{1 - e^{-2 \times 0.2} z^{-1}}$$

$$H(z) = \frac{1}{1 - 0.818 z^{-1}} - \frac{1}{1 - 0.67 z^{-1}}$$

$$= \frac{0.148 z^{-1}}{1 - 1.488 z^{-1} + 0.548}$$

Prob find $H(z)$ if $H(s) = \dfrac{b}{(s+a)^2 + b^2}$

using this result find $H(z)$ when

$$H(s) = \frac{1}{s^2 + 2s + 2}$$

Soln

$$(s+a)^2 + b^2 = 0$$

$$s = -a \pm jb$$

$$s_1 = -a + jb \qquad s_2 = -a - jb$$

$$H(s) = \frac{b}{(s+a-jb)(s+a+jb)}$$

$$= \frac{A}{(s+a-jb)} + \frac{B}{(s+a+jb)} \quad \text{—(1)}$$

$$A = 1/j2 \qquad B = -1/j2$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

$$= \sum_{k=1}^{N} \frac{C_k z}{z - e^{P_k T}}$$

using this in (1)

$$\Rightarrow H(z) = \frac{1}{j2}\left[\frac{z}{z - e^{(-a+jb)T}} - \frac{z}{z - e^{(-a-jb)T}}\right]$$

$$= \left[\frac{z e^{-aT} \sin bT}{z^2 - 2e^{-aT} \cos bT\, z + e^{-2aT}}\right]$$

$$= \frac{e^{-aT} \sin bT\, z^{-1}}{1 - 2e^{-aT} \cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

Given $H(s) = \dfrac{1}{s^2 + 2s + 2}$

$$H(s) = \frac{1}{(s+1)^2 + 1^2}$$

$$\Rightarrow \quad H(z) = \frac{e^{-T} \sin T z^{-1}}{1 - 2e^{-T} \cos T z^{-1} + e^{-2T} z^{-2}}$$

Prob

find $H(z)$ using IIT. Assume $T = 1$

1) $H(s) = \dfrac{2}{(s+1)(s+2)}$

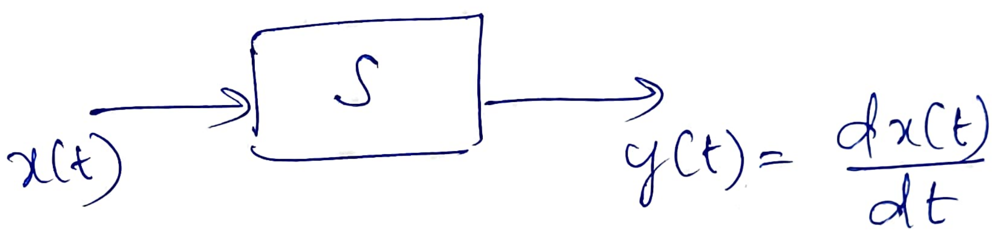
2) $H(s) = \dfrac{10}{s^2 + 7s + 10}$

Bilinear transformation (BLT)

Bilinear transformation is obtained by using trapezoidal formula for numerical integration.

Consider an analog system.

$$y(t) = \frac{dx(t)}{dt} \longrightarrow (1)$$



$$H(s) = \frac{Y(s)}{X(s)} = s \longrightarrow (2)$$

(1) is now integrated b/w the limits $(nT-T)$ and nT.

$$\int_{nT-T}^{nT} y(t)\,dt = \int_{nT-T}^{nT} \frac{dx(t)}{dt}\,dt.$$

$$\int_{nT-T}^{nT} y(t)\,dt = x(nT) - x(nT-T) \longrightarrow (3)$$

using trepezoidal rule on LHS of (45) (3), we get

$$\frac{T\left[y(nT)+y(nT-T)\right]}{2} = x(nT)-x(nT-T) \quad \text{—(4)}$$

using $x(nT) = x(n)$ in (4) we get

$$\frac{T}{2}\left[y(n)+y(n-1)\right] = x(n)-x(n-1) \quad \text{—(5)}$$

Applying Z- transform to (5), we get

$$\frac{T}{2}\left[Y(z)+z^{-1}Y(z)\right] = X(z)-z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{T}\left[\frac{(1-z^{-1})}{(1+z^{-1})}\right] \longrightarrow (6)$$

Comparing (2) and (6), we get ~~s to~~ z mapping as

$$\boxed{s \rightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

To obtain the relationship b/w analog and digital frequencies

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right] \text{ --- } \star$$

Evaluating $\star$ at $s = j\Omega$ and $z = e^{j\omega}$, we get,

$$j\Omega = \frac{2}{T}\left[\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right]$$

$$= \frac{2}{T}\left[\frac{e^{+j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}\right]$$

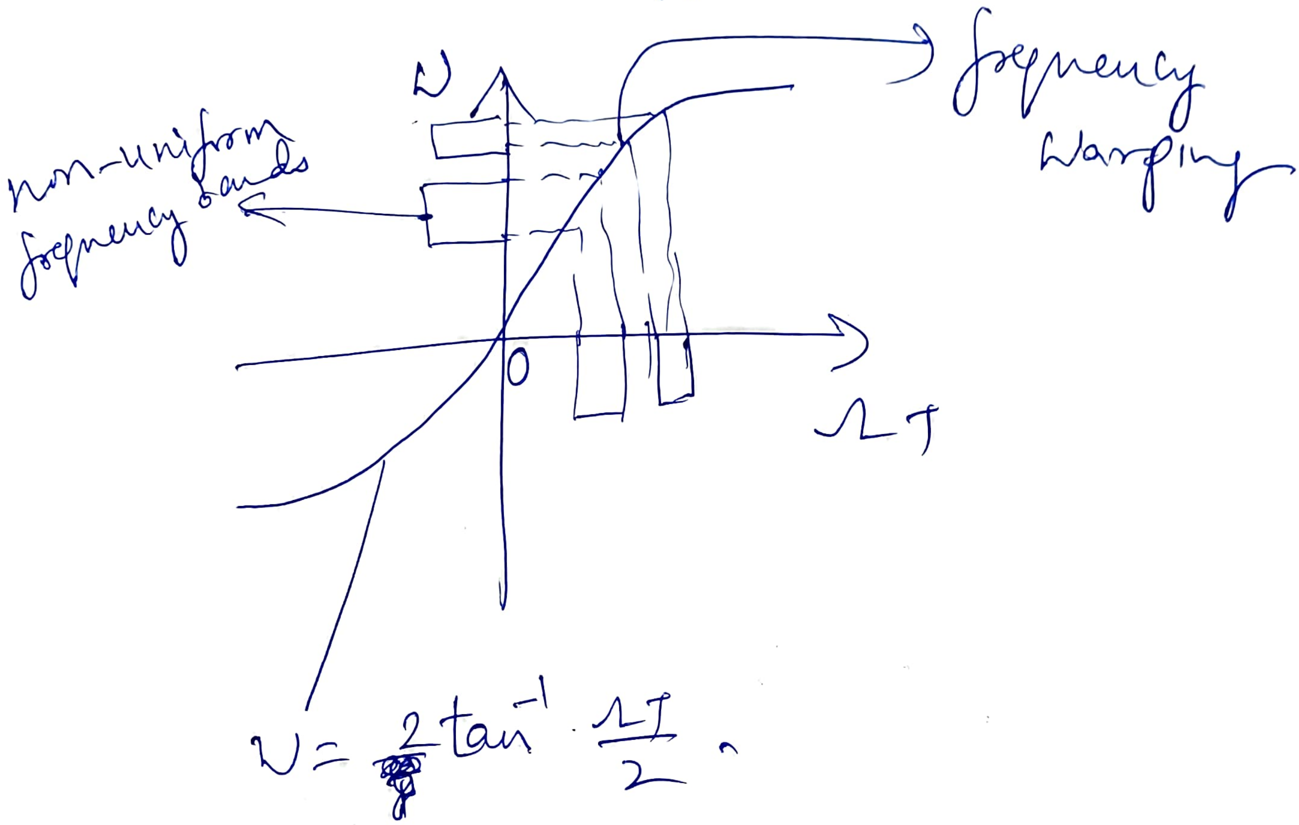
$$= \frac{2}{T}\,\frac{2j\sin\omega/2}{2\cos\omega/2}$$

$$j\Omega = j\frac{2}{T}\tan\omega/2$$

$$\Rightarrow \boxed{\Omega = \frac{2}{T}\tan\omega/2}$$

8) $\cancel{\omega} \; \omega = 2\tan^{-1}\frac{\Omega T}{2}$

$\Rightarrow$ relationship b/w analog and digital frequencies is highly non linear.



* The non linear relationship b/w ω and Ω is called frequency warping
* because of frequency warping, the evenly spaced bands of analog filter are mapped to unevenly spaced bands of digital filter
* Lower frequency bands are expanded and higher frequency bands are compressed.

S plane to z-plane mapping

(4A)

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\cancel{\sigma + j\Omega = \frac{2}{T}\left[\frac{1-(re^{j\omega})^{-1}}{1+}\right]}$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$= \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]$$

$$= \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega)-1}{r(\cos\omega + j\sin\omega)+1}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\Rightarrow \quad \left.\begin{aligned} \sigma &= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] \\ \Omega &= \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right] \end{aligned}\right\} \text{— (A)}$$

Mapping Summary

* from Eqn ① of Ⓐ, if $r=1$ then $\sigma = 0 \Rightarrow j\Omega$ axis in s plane is mapped to unit circle in z-plane
* from Eqn ① of Ⓐ, if $r > 1$, then $\sigma > 0 \Rightarrow$ RHS of s plane is mapped to outside the unit circle
* from Eqn ① of Ⓐ if $r < 1$, then $\sigma < 0 \Rightarrow$ LHS of s plane is mapped to inside the unit circle.

$\Rightarrow$ A stable analog filter is converted to a stable digital filter

Adv. of BLT

* Mapping of frequency is one-one
  NO Aliasing
* Stable analog filter is converted to a stable digital filter

disadv of BLT

* Warping effect.

Prob:

1) Convert the analog filter

$$H(s) = \frac{2}{(s+1)(s+3)}$$ into a digital

filter using BLT. Take $T = 0.1$ sec.

Soln: Mapping from s to z in BLT is

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = \frac{2}{\left\{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+1\right\}\left\{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+3\right\}}$$

$T = 0.1$ sec

$$\frac{2}{T} = 20.$$

$$\therefore H(z) = \frac{2(1+z^{-1})^2}{483 - 794z^{-1} + 323z^{-2}}$$

2) Convert 2nd order normalized filter to digital filter using BLT. Take $T = 1$ sec

Soln

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = \frac{1}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\}^2 + \sqrt{2}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\} + 1}$$

$$= \frac{0.127(1+2z^{-1}+z^{-2})}{1-0.766z^{-1}+0.277z^{-2}}$$

Ω

3) Transform analog filter

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$ into digital

filter using BLT. The digital filter should have resonant frequency

$\omega_r = \pi/4$

soln'

$$(s+0.1)^2+9 = (s+0.1-j3)(s+0.1+j3)$$

$$\Rightarrow s = -0.1 \pm j3$$

$$= \sigma \pm j\Omega$$

$$\Rightarrow \underline{\Omega = 3} \text{ r/s}$$

$$\omega_r = \pi/4$$

$$\Omega = \frac{2}{T} \tan\frac{\omega}{2}$$

$$\Rightarrow T = \frac{2}{\Omega} \tan\frac{\omega}{2}$$

$$T = 0.2761 \text{ sec}$$

$$\& \quad \frac{2}{T} = 3.621$$

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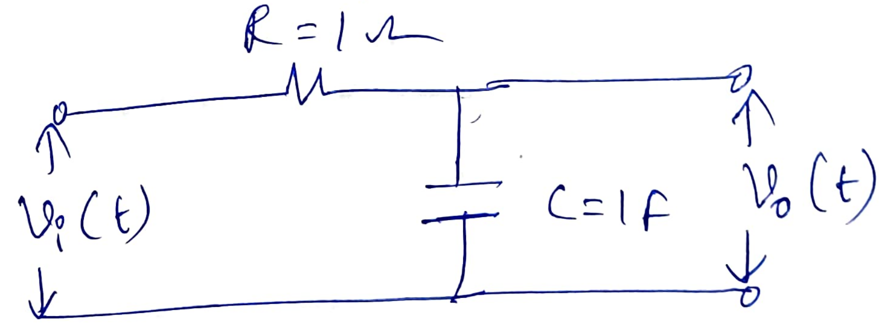
$$H(z) = \frac{3.621\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{\left[3.621\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1\right]^2 + 9}$$

$$= \frac{0.1629\left(1 - 0.053z^{-1} - 0.95z^{-2}\right)}{1 - 0.36z^{-1} + 0.94z^{-2}}$$

Prob

4) Obtain digital filter equivalent of the analog filter shown in fig using a) IIT b) BLT. Assume sampling frequency $F_s = 8f_c$ where f_c is cut off frequency of the filter



Soln

$$V_o(t) = \frac{X_C}{R + X_C} V_i(t)$$

$$\frac{V_o(t)}{V_i(t)} = \frac{X_C}{R + X_C}$$

Taking Laplace Transforms, 59

$$H(s) = \frac{1/s}{1+1/s} = \frac{1}{s+1}$$

Cut off frequency $f_c = \frac{1}{2\pi RC}$

$$f_s = 8f_c = \frac{8}{2\pi RC} = \frac{8}{2\pi \times 1 \times 1}$$

$$f_s = 1.273 \text{ Hz}$$

$$T = 1/f_s = 0.7855 \text{ sec}$$

a) $$\frac{1}{s - p_k} \rightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$\frac{1}{s+1} = H(s)$$
$$\Rightarrow s = -1$$
$$\Rightarrow p_k = -1$$

$$H(z) = \frac{1}{1 - e^{-1 \times 0.7855} z^{-1}} = \frac{1}{1 - 0.456 z^{-1}}$$

b) $\cancel{\frac{1}{s+1}}$ $$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{2}{0.7855}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = \frac{1}{2.55\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1} = \frac{0.282(1+z^{-1})}{1-0.44z^{-1}}$$

Digital Butterworth filter design

Prob 1) Design a Butterworth filter using BLT for the following specifications:

$$0.8 \leq |H(\omega)| \leq 1 \qquad 0 \leq \omega \leq 0.2\pi$$

$$|H(\omega)| \leq 0.2 \qquad 0.6\pi \leq \omega \leq \pi$$

Solu^n:

$A_p = 0.8 \qquad A_s = 0.2$

$\omega_p = 0.2\pi \qquad \omega_s = 0.6\pi$

Step 1. To obtain specifications of corresponding analog filter:

$A_p = 0.8 \qquad A_s = 0.2$

$$\boxed{\text{Take } \frac{2}{T} = 1}$$

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}$$

$$= 0.325 \text{ r/s}$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2}$$

$$= 1.376 \text{ r/s}$$

Step 2

To determine order N of the filter.

$$N = \frac{\log\left[\frac{\left(1/A_s^2 - 1\right)}{\left(1/A_p^2 - 1\right)}\right]}{2 \log \frac{\Omega_s}{\Omega_p}}$$

$= 1.3$

Take $\boxed{N = 2}$

Step 3 To determine Ω_c.

$$\Omega_{cp} = \frac{\Omega_p}{\left(1/A_p^2 - 1\right)^{1/2N}}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left(1/A_s^2 - 1\right)^{1/2N}}$$

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2} = 0.5 \text{ r/s}$$

Step 4

finding System function of normalised LPF

$$P_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$$N = 2 \qquad \Omega_c = 1 \text{ r/s}$$

$$P_0 = \pm(-0.707 + j0.707)$$

$$P_1 = \pm(-0.707 - j0.707)$$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2 + 1.414s + 1}$$

Step 5

To find the required H(s) using frequency transformation

$$LP \rightarrow LP \; ; \; s \rightarrow \frac{\Omega_p}{\Omega_{Lp}} s = \frac{s}{0.5}$$

$$H(s) = H_1(s)\Big/_{s \rightarrow \frac{s}{0.5}} = \frac{0.25}{s^2 + 0.707s + 0.25}$$

Step 6

finding $H(z)$ by applying BLT.

$$s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\frac{2}{T} = 1$$

$$s \rightarrow \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = H(s)\Big/_{s=\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.25}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.707\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.25}$$

$$H(z) = \frac{0.127\left(1+2z^{-1}+z^{-2}\right)}{1-0.766z^{-1}+0.277z^{-2}}$$

Prob 2:

59

A digital LPF is required to meet the following specifications:

a) An acceptable passband attenuation of -1.9328 dB

b) Passband edge frequency of $\omega_p = 0.2\pi$ rad

c) Stopband attenuation of -13.9794 dB or higher beyond ~~...~~ 0.6π rad.

The filter must have maximally flat frequency response in the passband.

Find $H(z)$ using IIT?

Soln

$A_p = -1.9328$ dB $\quad \omega_p = 0.2\pi$

$A_s = -13.9794$ dB $\quad \omega_s = 0.6\pi$

Step 1: To obtain specifications of corresponding analog filter.

Take $T = 1$ sec.

$A_p = -1.9328$ dB

$A_s = -13.9794$ dB

$$\Omega = \frac{\omega}{T} \;;\; T = 1\ \text{sec}$$

$$\Omega_p = \frac{\omega_p}{T} = 0.2\pi$$

$$\Omega_s = \frac{\omega_s}{T} = 0.6\pi$$

Step 2

To determine order N of the filter

$$N = \frac{\log\left[\dfrac{10^{0.1A_s dB}-1}{10^{0.1A_p dB}-1}\right]}{2\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$N = 1.7$$

Take $\boxed{N=2}$

Step 3

To determine Ω_c

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[10^{0.1A_p dB}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[10^{0.1A_s dB}-1\right]^{1/2N}}\right]$$

$$\Omega_c = 0.726\ r/s$$

Step 4

To determine the system function of normalized LPF

$$p_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$$N = 2 \; ; \; \Omega_c = 1 \text{ r/s}$$

$$p_0 = \pm (-0.707 + j0.707)$$

$$p_1 = \pm (-0.707 - j0.707)$$

$$H_1(s) = \frac{1}{(s-p_0)(s-p_1)} = \frac{1}{s^2 + 1.414s + 1}$$

Step 5

To find the required H(s) using frequency transformation

LP → LP

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{0.726}$$

$$H(s) = H_1(s) \Big/_{s = \frac{s}{0.726}} = \frac{0.526}{s^2 + 1.03s + 0.526}$$

$$= \frac{1.026 \times 0.513}{(s+0.513)^2 + (0.513)^2}$$

<u>Step 6</u>

Finding $H(z)$ by applying IIT

$$\frac{b}{(s+a)^2 + b^2} = \frac{e^{-aT} \sin bT z^{-1}}{1 - 2e^{-aT} \cos bT z^{-1} + e^{-2aT} z^{-2}}$$

$T = 1$ sec $\quad a = 0.513 \quad b = 0.513$

$$H(z) = \frac{0.302 z^{-1}}{1 - 1.043 z^{-1} + 0.36 z^{-2}}$$

———o———

Chebyshev filter design

Two types

a) Cheby – I

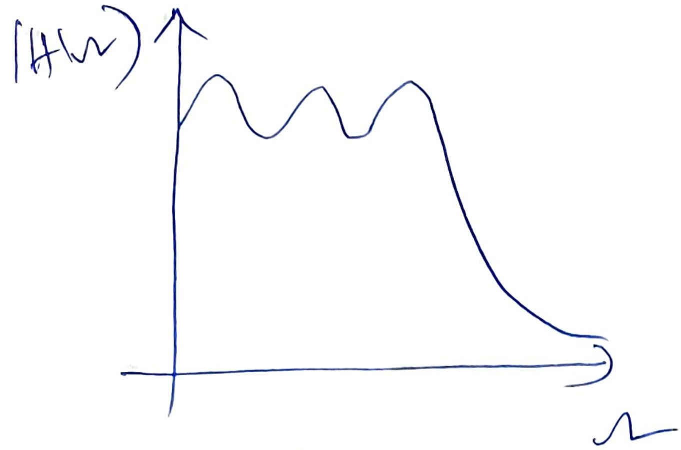
b) Cheby – II

a) Cheby – I

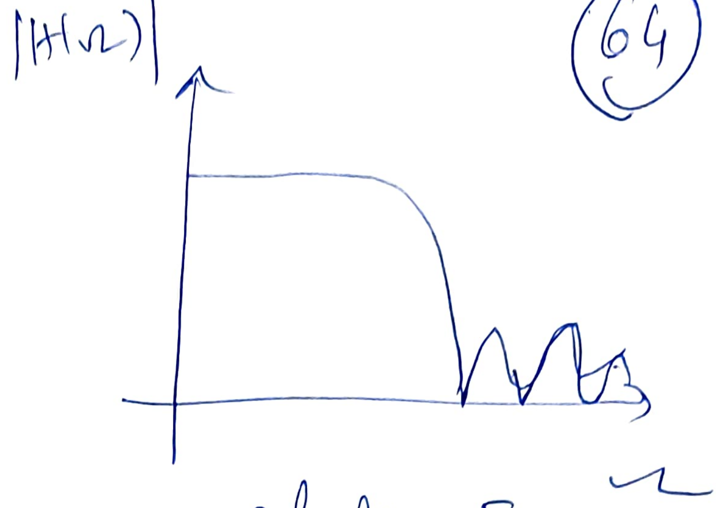
- All pole filters
- usually called as chebyshev filters
- ripples in the pass band
- steeper roll off than Butterworth filters

b) Cheby – II

- pole - zero filter
- does not roll off so fast as Cheby – I
- usually called Inverse chebyshev filter
- ripples in the stop band



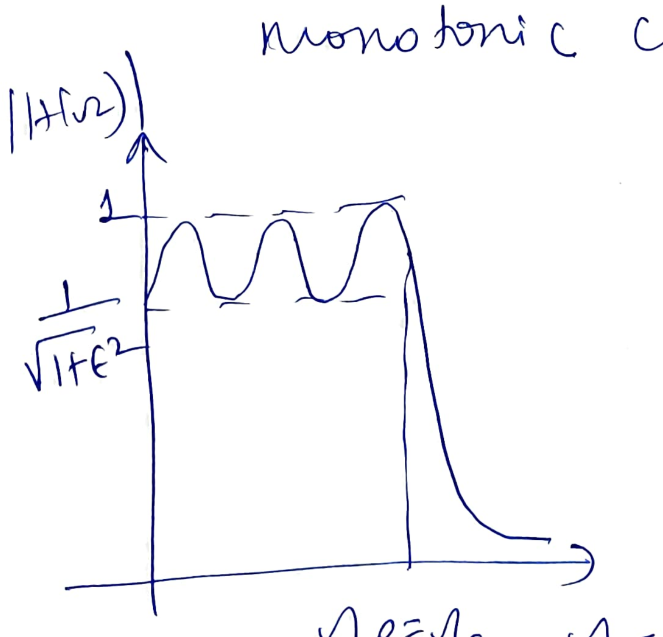
Cheby-I



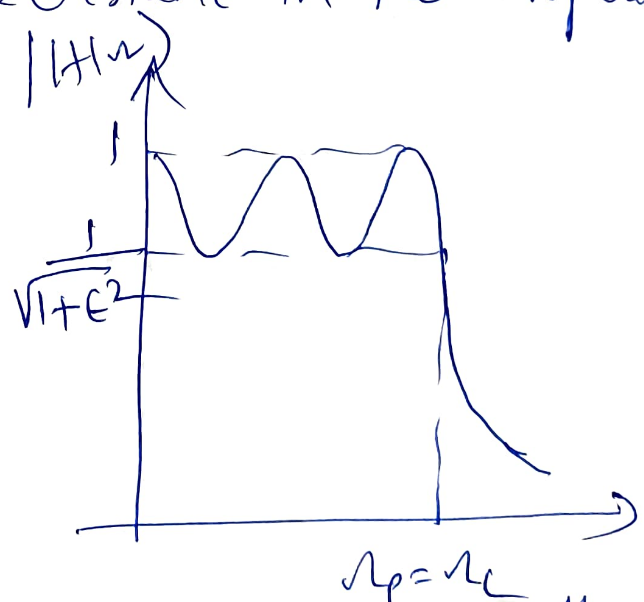
Cheby-II

Chebyshev-I filters

- These filters are all pole filters
- In the passband, these filters show equiripple behaviour and have monotonic characteristic in the stopband



N odd



N Even

Comparison b/w Butterworth and Chebyshev filters.

a) frequency response
b) order
c) Transition band
d) phase response
e) poles of H(s) - location.

Chebyshev polynomials

$$C_N(\Omega) = \cos(N \cos^{-1}(\Omega)) \text{ for } |\Omega| \leq 1$$
$$= \cosh(N \cosh^{-1}(\Omega)) \text{ for } |\Omega| > 1$$

for $N=0 \Rightarrow C_0(\Omega) = \cos 0 = 1$

for $N=1 \Rightarrow C_1(\Omega) = \cos[\cos^{-1}\Omega] = \Omega$

Higher order Chebyshev polynomials are obtained using the recursive formula

$$\boxed{C_N(\Omega) = 2\Omega\, C_{N-1}(\Omega) - C_{N-2}(\Omega)}$$

Prob:

Find Chebyshev polynomials for $N = 2, 3, 4, 5$.

Soln

$$C_N(x) = 2x\, C_{N-1}(x) - C_{N-2}(x)$$

$N=2$ $\quad C_2(x) = 2x\, C_1(x) - C_0(x)$

$$C_2(x) = 2x^2 - 1$$

$N=3$ $\quad C_3(x) = 2x\, C_2(x) - C_1(x)$

$$= 4x^3 - 3x$$

$N=4$ $\quad C_4(x) = 8x^4 - 8x^2 + 1$

$N=5$ $\quad C_5(x) = 16x^5 - 20x^3 + 5x$

Observation

for $x >> 1$, first term dominates

and $\boxed{C_N(x) = 2^{N-1} x^N}$

Magnitude function of Chebyshev filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega)} \quad \text{— (A)}$$

where ϵ = ripple factor

$C_N(\Omega)$ = Chebyshev polynomial of order N.

for normalized filter $\Omega_p = \Omega_c = 1$ r/s

Observations

a) for $|\Omega| \leq 1$,

ripple in the passband $= 1 - \dfrac{1}{\sqrt{1+\epsilon^2}}$

b) at $\Omega = 1$,

$C_N^2(\Omega) = 1$. [$C_N(1) = 1$ always]

at $\Omega = 1$, (A) can be written as

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2} \quad \text{or} \quad |H(\Omega)| = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$\therefore \quad A_p = \frac{1}{\sqrt{1+\epsilon^2}}$$

or $\cancel{\quad} = \cancel{\quad}$

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1} \quad \text{—— (B)}$$

for $\Omega >> 1$, $\epsilon^2 C_N^2(\Omega) >> 1$

$$\therefore \quad |H(\Omega)|^2 = \frac{1}{\epsilon^2 C_N^2(\Omega)}$$

$$|H(\Omega)| = \frac{1}{\epsilon\, C_N(\Omega)}$$

$$|H(\Omega)| \text{ in dB} = 20 \log_{10} 1 - 20 \log_{10}(\epsilon\, C_N(\Omega))$$

$$= 0 - 20 \log_{10}\left[\epsilon\, 2^{N-1} \Omega^N\right]$$

$$= -20 \log \epsilon - 20 (N-1) \log_{10} 2$$

$$- 20 N \log \Omega$$

$\Omega >> 1 \Rightarrow$ stop band. $\therefore \Omega = \Omega_s$

$$\therefore |H(\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 20(N-1)\log_{10}2 - 20N\log \Omega_s'$$

$$= -20\log_{10}\epsilon - 6(N-1) - 20N\log_{10}\Omega_s'$$

Ω_s' = Normalized stopband Edge freq.

<u>poles of H(s)</u>

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$$k = 1, 2, 3 \ldots N$$

System function $H(s)$ of Chebyshev filter

$$H(s) = \frac{k}{(s-s_1)(s-s_2)\cdots(s-s_N)}$$

$$= \frac{k}{s^N + b_{N-1}s^{N-1} + b_{N-2}s^{N-2} + \cdots + b_0}$$

Constant k is given by

$$k = \begin{cases} b_0 & \text{for } N \text{ odd} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & \text{for } N \text{ even} \end{cases}$$

Prob:

1) Design analog Chebyshev filter to meet the following specifications.

passband ripple 1 dB $\quad 0 \le \Omega \le 10$ r/s

Stopband attenuation -60 dB $\quad \Omega \geq 50$ r/s

Soln (71)

$\Omega_p = 10$ r/s $\quad A_p = -1$ dB

$\Omega_s = 50$ r/s $\quad A_s = -60$ dB

Step 1:

Normalised Specifications.

$$\Omega_p' = \frac{10}{10} = 1 \text{ r/s} \qquad A_p = -1\text{ dB}$$

$$\Omega_s' = \frac{\Omega_s}{\Omega_p} = \frac{50}{10} = 5 \text{ r/s} \qquad A_s = -60\text{ dB}$$

Step 2

To determine ϵ.

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1} = \sqrt{10^{0.1 A_p dB} - 1}$$

$$= 0.508$$

Step 3

To determine order N

$$|H(\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 6(N-1) - 20N\log \Omega_s'$$

$$-60 = -20\log(0.5) - 6(N-1) - 20N\log 5$$

$$\boxed{N = 3.9 = 4}$$

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$$H_1(s) = \frac{k}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$$

$$s_1 = -0.139 + j0.983$$

$$s_2 = -0.337 + j0.407$$

$$s_3 = -0.337 - j0.407$$

$$s_4 = -0.139 - j0.983$$

$$H_1(s) = \frac{k}{(s+0.139-j0.983)(s+0.139+j0.983)(s+0.337-j0.407)(s+0.337+j0.407)}$$

$$= \frac{k}{\left[(s+0.139)^2+(0.983)^2\right]\left[(s+0.337)^2+(0.407)^2\right]}$$

$$H_1(s) = \frac{k}{(s^2+0.278s+0.985)(s^2+0.674s+0.279)}$$

$$b_0 = 0.985 \times 0.279$$

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$$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.274}{\sqrt{1+(0.508)^2}}$$

$$K = 0.244$$

$$H_1(s) = \frac{0.244}{(s^2+0.278s+0.985)(s^2+0.674s+0.279)}$$

Step 4

Applying frequency transformation

LP — LP

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{10}$$

$$H(s) = H_1(s)\Big/_{s=\frac{s}{10}} = \frac{0.244}{\left(\left(\frac{s}{10}\right)^2 + 0.278 \times \frac{s}{10} + 0.985\right)\left(\left(\frac{s}{10}\right)^2 + 0.674 \times \frac{s}{10} \times 0.279\right)}$$

$$H(s) = \frac{0.244 \times 10^4}{(s^2+2.78s+98.5)(s^2+6.74s+27.9)}$$

Prob2 Design an analog chebyshev filter to meet the following specifications (73)

$A_p = 2.5\ dB$ $\qquad \Omega_p = 20\ r/s$

$A_s = 30\ dB$ $\qquad \Omega_s = 50\ r/s.$

Soln

$$\epsilon = \sqrt{10^{0.1A_p} - 1} = 0.882$$

$$\Omega_p' = 1 \qquad \Omega_s' = \frac{50}{20} = 2.5\ r/s$$

$$N = 3.$$

Design of Digital Chebyshev filters

1) Design a digital low pass Chebyshev filter using Bilinear Transformation to meet the following specifications.

$$0.75 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le 0.25\pi$$

$$|H(\omega)| \le 0.23 \qquad 0.63\pi \le \omega \le \pi$$

$\boxed{\text{BLT} \rightarrow \text{Assume } T = 2\,\text{sec}}$

Soln

$A_p = 0.75 \qquad \omega_p = 0.25\pi$ rad

$A_s = 0.23 \qquad \omega_s = 0.63\pi$ rad

Step 1

To obtain specifications of corresponding analog filter

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = 0.413 \text{ r/s}$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = 1.52 \text{ r/s}$$

$$A_p \text{ in dB} = -20 \log_{10} A_p = 2.5 \text{ dB}$$

$$A_s \text{ in dB} = -20 \log_{10} A_s = 12.76 \text{ dB}$$

Step2 To find Normalized frequency values (11)

$$\Omega_p' = \frac{0.413}{0.413} = 1 \text{ r/s}$$

$$\Omega_s' = \frac{1.52}{0.413} = 3.68 \text{ r/s}$$

Step3

To find ϵ

$$\epsilon = \left[10^{0.1 A_p dB} - 1\right]^{1/2}$$

$$= 0.88$$

Step4 To find order N

$$|H(\Omega)| \text{ in dB} = -20 \log_{10} \epsilon - 6(N-1) - 20 N \log \Omega_s'$$

$$\Rightarrow N = 1.146$$

$$\boxed{N = 2}$$

Step 3 To find $H_1(s)$

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = +\cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$$s_1 = -0.357 + j0.792$$

$$s_2 = -0.357 - j0.792$$

$$H_1(s) = \frac{k}{(s-s_1)(s-s_2)}$$

$$= \frac{k}{(s+0.357+j0.792)(s+0.357-j0.792)}$$

$$= \frac{k}{\left[(s+0.357)^2+(0.792)^2\right]} = \frac{k}{s^2+0.714s+0.754}$$

$$k = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.754}{\sqrt{1+(0.88)^2}} = 0.566$$

$$H_1(s) = \frac{0.566}{s^2 + 0.7145 + 0.754}$$

<u>Step 4</u>: <u>Apply frequency transformation</u>

$$s \rightarrow \frac{\Omega_p}{\Omega_{Lp}} s = \frac{s}{0.413}$$

$$H(s) = \frac{0.566}{\left(\frac{s}{0.413}\right)^2 + 0.714 \times \left(\frac{s}{0.413}\right) + 0.754}$$

$$= \frac{0.096}{s^2 + 0.293s + 0.128}$$

<u>Step 5</u> <u>To find $H(z)$</u>

Mapping from s to z is

$$s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\frac{2}{T} = 1$$

$$\therefore s = \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = H(s) \Big/ s = \frac{1-z^{-1}}{1+z^{-1}} = \frac{0.096\left(1+z^{-1}\right)^2}{(1-z^{-1})^2 + 0.293(1-z^{-2}) + 0.128(1+z^{-1})^2}$$

$$\therefore H(z) = \frac{0.096\left(1+z^{-1}\right)^2}{\left(1+z^{-2}-2z^{-1}\right)\cancel{6}+0.293-0.293z^{-2}+0.128\left(1+z^{-2}+2z^{-1}\right)}$$

$$= \frac{0.096\left(1+z^{-1}\right)^2}{1.421-1.744z^{-1}+0.835z^{-2}}$$

$$H(z) = \frac{0.063\left(1+z^{-1}\right)^2}{1-1.22z^{-1}+0.587z^{-2}}$$

Prob

2) Design a low pass Chebyshev filter using Impulse Invariance transformation for satisfying the following constraints.

$\omega_p = 0.162$ rad $\quad \omega_s = 1.63$ rad

pass band ripple = 3 dB

stop band attenuation = 30 dB.

$\boxed{\text{IIT} \rightarrow \text{Assume } T = 1 \text{ sec}}$

Soln

$\omega_p = 0.162$ rad $\quad A_p = -3$ dB

$\omega_s = 1.163$ rad $\quad A_s = -30$ dB

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Step 1 To obtain specifications of equivalent analog filter

$$\Omega_p = \frac{\omega_p}{T} = 0.162 \text{ r/s}$$

$$\Omega_s = \frac{\omega_s}{T} = 1.63 \text{ r/s}$$

$A_p = -3$ dB, $\quad A_s = -30$ dB

Step 2 Normalizing frequency values

$$\Omega_p' = \frac{0.162}{0.162} = 1 \text{ r/s}$$

$$\Omega_s' = \frac{1.63}{0.162} = 10.06 \text{ r/s}$$

Step 3 To find ϵ

$$\epsilon = \left[10^{0.1 A_p dB} - 1\right]^{1/2} = 0.997$$

Step 4 To find N

$|H(\Omega)|$ in dB $= -20\log_{10}\epsilon - 6(N-1)$

$- 20N\log_{10}\Omega_s'$

$N \geq 1.38$

$\boxed{N=2}$

Step 5 To find $H_1(s)$

$$H_1(s) = \frac{K}{(s-s_1)(s-s_2)}$$

$s_k = \sigma_k + j\Omega_k$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$s_1 = -0.322 + j0.777$

$s_2 = -0.322 - j0.777$

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$$H_1(s) = \frac{k}{(s+0.322-j0.777)(s+0.322+j0.777)}$$

$$= \frac{k}{\left[(s+0.322)^2+(0.777)^2\right]}$$

$$= \frac{k}{s^2+0.644s+0.707}$$

$$k = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.707}{\sqrt{1+(0.997)^2}}$$

$$k = 0.5$$

$$\therefore H_1(s) = \frac{0.5}{s^2+0.644s+0.707}$$

Step 6 Applying frequency transformation

$LP \to LP$

$$s \to \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{0.162}$$

$$H(s) = H_1(s)\Big/_{s=\frac{s}{0.162}}$$

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$$H(s) = \frac{0.5}{\left(\frac{s}{0.162}\right)^2 + 0.644\left(\frac{s}{0.162}\right) + 0.707}$$

$$= \frac{0.013}{s^2 + 0.104\,s + 0.018} = \frac{0.052 \times 0.25}{\left[(s+0.52)^2 + (0.25)^2\right]}$$

<u>Step 7</u> To find $H(z)$ using IIT

Mapping from $s \to z$

$$\frac{b}{(s+a)^2 + b^2} = \frac{e^{-aT} \sin bT\, z^{-1}}{1 - 2e^{-aT} \cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

$$s^2 + 0.104s + 0.018 = (s + 0.52 - j0.25)(s + 0.52 + j0.25)$$

$$= \left[(s+0.52)^2 + (0.25)^2\right]$$

$a = 0.52, \quad b = 0.25, \quad T = 1$

$$\therefore H(z) = \frac{0.594 \times 0.25\, z^{-1}}{1 - 2 \times 0.594 \times 0.97\, z^{-1} + 0.35\, z^{-2}}$$

$$H(z) = \frac{0.15\, z^{-1}}{1 - 1.15\, z^{-1} + 0.35\, z^{-2}}$$