classmate

Date
Page

Theorem 2: Let u & v be two finite dimensional vector spaces over the same field F & let $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be an ordered basis for u and let $\{B_1, B_2 \ldots B_n\}$ be an ordered set in v. Then there is precisely one linear transformation T from u onto v such that $T(\alpha_j) = B_j$, $j = 1, 2 \ldots n$

Proof:- Since the set $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a basis of $u(F)$, then for each $\alpha \in U$ there are some scalars $a_1, a_2, \ldots a_n$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = \sum_{i=1}^{n} a_i\alpha_i$

For this vector α we define a transformation $T: U \to V$

$T(\alpha) = a_1 B_1 + a_2 B_2 + \cdots + a_n B_n = \sum_{i=1}^{n} a_i B_i$

From the above equation it is clear that $T(\alpha_j) = B_j$ for each j.

Now we shall st T is linear. For this $\alpha = \sum_{i=1}^{n} a_i\alpha_i$ and $B = \sum_{i=1}^{n} b_i B_i$ are any two vectors in u. Then $\forall\, a, b \in F$ we have

$$
\begin{aligned}
T[a\alpha + bB] &= T\left[a \sum_{i=1}^{n} a_i\alpha_i + b \sum_{i=1}^{n} b_i B_i\right] \\
&= T\left[\sum_{i=1}^{n} a a_i\alpha_i + \sum_{i=1}^{n} b b_i B_i\right] \\
&= T\left[\sum_{i=1}^{n} a a_i + \sum_{i=1}^{n} b b_i\right] \alpha_i \\
&= \sum_{i=1}^{n} (a a_i + b b_i) B_i \\
&= a \sum_{i=1}^{n} a_i B_i + b \sum_{i=1}^{n} b_i B_i \\
&= a\, T\left[\sum_{i=1}^{n} a_i\alpha_i\right] + b\, T\left[\sum_{i=1}^{n} b_i\alpha_i\right] \\
&= a T(\alpha) + b T(B).
\end{aligned}
$$

Algebra of Linear Transformations:-

Theorem 1: Let u & v be two vector spaces over the field F. Let T_1 and T_2 be two linear transformations from U into V, then the fn $(T_1 + T_2)$ defined by $(T_1 + T_2)\alpha = T_1(\alpha) + T_2(\alpha)$ $\forall\, \alpha \in u$ is a linear transformation from u into v.

If c is any element of F then the fn ~~CT~~ $C.T$ is defined by $(C.T)\alpha = CT(\alpha)$ is a linear transformation from U into V. The set of all transformations $L(U, V)$

together with addition and scalar multiplication defined above is a vector space over field F.

Proof: For $\alpha, \beta \in U$ and $a, b \in F$ we have $(T_1+T_2)(a\alpha+b\beta) =$

$T_1(a\alpha+b\beta) + T_2(a\alpha+b\beta)$

$= [aT_1(\alpha) + bT_1(\beta)] + [aT_2(\alpha) + bT_2(\beta)]$

$= aT_1(\alpha) + aT_2(\alpha) + bT_1(\beta) + bT_2(\beta)$

$= a[T_1+T_2]\alpha + b[T_1+T_2]\beta$

where T_1+T_2 is linear transformation

Again T is linear transformation and c is any scalar then for $\alpha, \beta \in U$ & $a, b \in F$ we have $(cT)(a\alpha+b\beta) = c[T(a\alpha)+T(b\beta)]$

$= c[aT(\alpha)] + c[bT(\beta)]$

$= ac\,T(\alpha) + bc\,T(\beta)$

$= a\,cT(\alpha) + b(cT)(\beta)$

where $c.T$ is linear transformation

Now we shall show that the set of all linear transformations $L(U,V)$ from U into V forms a vector space w.r.t above defined compositions.

First we show that $\{L(U,V), +\}$ is an abelian group.

(i) Closure property:- If $T_1, T_2 \in L(U,V)$ then we have already proved that T_1+T_2 is linear transformation Associative

(ii) Associative prop :- $\forall\, T_1, T_2, T_3 \in L(U,V)$ and $\forall\, \alpha \in U$ we have $[(T_1+T_2)+T_3]\alpha = (T_1+T_2)\alpha + T_3(\alpha)$

$= T_1(\alpha) + T_2(\alpha) + T_3(\alpha)$

$= T_1(\alpha) + [T_2(\alpha) + T_3(\alpha)]$

$= T_1(\alpha) + (T_2+T_3)(\alpha)$

$= [T_1 + (T_2+T_3)]\alpha$

$[(T_1+T_2)+T_3] = [T_1 + (T_2+T_3)]$

classmate

Date

Page

(iii) Commutative property: $\forall T_1, T_2 \in L(U,V)$ & $\forall \alpha \in U$ we have

$(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)$

$= T_2(\alpha) + T_1(\alpha)$

$= (T_2+T_1)(\alpha)$

$\Rightarrow T_1+T_2 = T_2+T_1$

(iv) Existance of Identity,

The zero transformation denoted by $\hat{0}$ & defined by $\hat{0}.\alpha = 0 \;\; \forall \alpha \in U$ is a linear transformation

Also if $T \in L(U,V)$ then $(T+\hat{0}) = \hat{0}+T = T \;\forall T$

$\therefore \hat{0} \in L(U,V)$

(v) Existense of inverse: For each $T \in L(U,V)$ there exists $-T \in L(U,V)$ defined by $(-T)(\alpha) = -T(\alpha) \;\forall \alpha \in U$. $-T$ is linear and $T+(-T) = (-T)+(T) = \hat{0}$.

$\therefore -T$ is additive inverse.

(vi) Distributive property: $\forall T_1, T_2 \in L(U,V)$, $\alpha \in U$ and $a \in F$

then $a[T_1+T_2](\alpha) = a(T_1+T_2)(\alpha)$

$= a[T_1(\alpha) + T_2(\alpha)]$

$= aT_1(\alpha) + aT_2(\alpha)$

$= [aT_1 + aT_2](\alpha)$

$a[T_1+T_2](\alpha) = [aT_1+aT_2](\alpha)$

$a[T_1+T_2] = aT_1 + aT_2$

$T \in L(U,V)$, $a, b \in F$

$[(a+b)T](\alpha) = (a+b)T(\alpha) = aT(\alpha) + bT(\alpha)$

$(a+b)T = aT + bT$

(vii) $T \in L(U,V)$ $\alpha \in U$ & $a, b \in F$

$[(ab)T](\alpha) = (ab)T(\alpha) = a[bT(\alpha)]$

$= a[bT](\alpha)$

$(ab)T = a(bT)$

(viii) If 1 is the unity $\in F$ then $\forall$ $L(U,V)$ and $\alpha \in U$, $[1.T](\alpha)$
$= 1.T(\alpha)$
$= T(\alpha)$

$\therefore$ Above property proves $L(U,V)$ is also a LT of F

Test

<u>Theorem 2</u>: Let U be an m dimensional vector space over the field F, and let V an n dimensional vector space over F. Then the vector space $L(U,V)$ is finite dimensional and has dimension mn

Proof:- Since U & V both are finite dimensional vector spaces of dimensions m & n respectively. Therefore let $B = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ & $B' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be the ordered basis of U and V respectively. For each pair of integers i, j with $1 \le i \le m$, $1 \le j \le n$ we define a linear transformation T_{ij} from U into V by

$$T_{ij}(\alpha_k) = \begin{cases} 0 ; & \text{if } k \neq j \\ \beta_i ; & \text{if } k = j \end{cases}$$

The existance and uniqueness of above LTs follows the previous theorem.

We claim that this $m \times n$ transformations form a basis of $L(U,V)$

(i) For m,n scalars a_{ij} we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij} = 0$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_k) = 0(\alpha_k) \; \forall \; \alpha_k \in U$$

$1 \le k \le n$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_k) = 0$$

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_k)\right] = 0$$

$$\sum_{i=1}^{n} \left[a_{i1} T_{i1}(\alpha_k) + a_{i2} T_{i2}(\alpha_k) + \ldots + a_{im} T_{im}(\alpha_k)\right] = 0$$

$$\left[\sum_{i=1}^{n} a_{i1} T_{i1}\alpha_k + \sum_{i=1}^{n} a_{i2} T_{i2}(\alpha_k) + \cdots + \sum_{i=1}^{n} a_{im} T_{im}(\alpha_k)\right] = 0$$

$$a_{11} T_{11}\alpha_k + a_{12} T_{12}\alpha_k + \cdots + a_{im} T_{im}(\alpha_k) + a_{21} T_{21}\alpha_k + a_{22} T_{22}(\alpha_k) +$$
$$a_{2m} T_{2m}(\alpha_k) + \cdots + = 0$$

Since $B' = \beta_1, \beta_2 \ldots \beta_n$ is a basis of V, therefore it is linearly independant so that $a_{11} = 0 = a_{21} = a_{n1}$

$a_{12} = 0 = a_{22} = a_{n2}$

$\vdots$

$a_{1m} = 0 = a_{2m} = a_{mn}$

Thus $\{T_{ij} : 1 \le i \le m \text{ and } 1 \le j \le n\}$ spans $L(U,V)$ is linearly independant.

(ii) Now we show that $\{T_{ij} : 1 \le i \le m \text{ \& } 1 \le j \le n\}$ spans $L(U,V)$ for this let T be an arbitrary linear transformation from U into V i.e., $T \in L(U,V)$.

$$T(\alpha_j) = \sum_{i=1}^{n} a_{ij} \beta_i = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_j)$$

$$T = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}$$

Hence $\{T_{ij} : 1 \le i \le m \text{ \& } 1 \le j \le n\}$ generates $L(U,V)$ and is a basis of $L(U,V)$ & its a finite dimensional with dimension of $L(U,V) = mn$.

Classmate
Date
Page

2 Let mat $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$ Determine whether or not the rows of A are orthogonal (b) A is an orthogonal matrix (c). The columns of A are orthogonal

→ a). $\langle u_1, u_2 \rangle = 1+3-4=0$

$\langle u_1, u_3 \rangle = 7-5-2=0$ } rows are orthogonal

$\langle u_2, u_3 \rangle = 7-15+8=0$

b. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix} = 1+1+1+1+9+16+49+25+4$

$= 107 \neq 0$

∴ Mat A is not orthogonal

c. $\langle c_1, c_2 \rangle = 1+3-35 \neq 0$

$\langle c_1, c_3 \rangle = -1+4+14 \neq 0$ } Columns are not orthogonal

$\langle c_2, c_2 \rangle = -1+12-10 \neq 0$

3 Let B be the mat obtained by normalizing each row in mat A given in previous question a) Find mat B b). Are the rows of mat B orthogonal c) Is B orthogonal

→ a. $B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & \frac{-5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{bmatrix}$

b. $\langle r_1, r_2 \rangle = 0$

$\langle r_1, r_3 \rangle = 0$ } rows are orthogonal

$\langle r_2, r_3 \rangle = 0$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ → ~~rows are~~ unit vectors

∴ B is orthogonal

c. $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{26} + \frac{9}{26} + \frac{16}{26} + \frac{49}{78} + \frac{25}{78} + \frac{4}{78} = 3 \neq 0$

mat ~~A~~ is ~~not~~ orthogonal

If both rows and columns are orthogonal, then the matrix is orthogonal

d. $\langle c_1, c_2 \rangle = \frac{1}{3} + \frac{3}{26} - \frac{35}{78} = 0$

$\langle c_1, c_3 \rangle = \frac{-1}{3} + \frac{4}{26} + \frac{14}{78} = 0$ } columns are orthogonal

$\langle c_2, c_3 \rangle = \frac{-1}{3} + \frac{12}{26} - \frac{10}{26} = 0$

QR factorization:

If A is m×n matrix with lin. independant column then mat A can be factored as A = QR where Q is an m×n matrix whose columns form an orthonormal basis for column A and R is an m×n uppertriangular invertible matrix with +ve entries on its diagonal

1. Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
$\quad V_1 \quad V_2 \quad V_3$

→ $w_1 = v_1 = (1,1,1,1)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = (0,1,1,1) - \frac{3}{4}(1,1,1,1) = \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle w_1}{\langle w_1, w_1 \rangle} - \frac{\langle v_3 w_2 \rangle w_2}{\langle w_2, w_2 \rangle}$

$= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{\frac{1}{2}}{\frac{3}{4}}\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$= (0,0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

$= \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = (0,-2,1,1)$

$\hat{w}_1 = \frac{1}{2}(1,1,1,1)$

$\hat{w}_2 = \frac{1}{\sqrt{12}}(-3,1,1,1)$

$\hat{w}_3 = \frac{1}{\sqrt{6}}(0,-2,1,1)$

$Q = \begin{bmatrix} 1/2 & \frac{-3}{\sqrt{12}} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$

To find R, observe that $Q^T Q = I$ because columns of Q are orthonormal

Hence $Q^T A = Q^T(QR) = IR = R$

$Q^T A = R = \begin{bmatrix} 2 & 1.5 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$

classmate
Date
Page

Least square problems:

For the given set of linear equations expressed in the form of $Ax=b$ that have no sol^n due to inconsistent sys; the best one can do is to find an x that makes Ax as close as possible to b.

Think of Ax as an approximation to b. The smaller the distance b/w b and Ax given by $\|b-Ax\|$, the better the approximation. The general least square problem is to find an x that makes $\|b-Ax\|$ as small as possible.

The least squares arises from the fact that $\|b-Ax\|$ is the square root of a sum of squares.

Definition) If mat. A is $m \times n$ and b is in R^m, a least square sol^n of $Ax=b$ is an $\hat{x}$ in R^n such that $\|b-A\hat{x}\| \leq \|b-Ax\| \ \forall x$ in R^n.

Sol^n of general least square problem) :

Given mat A and constant b and apply best approx theorem to get $\hat{b} = proj_{colA} b$.

Because $\hat{b}$ is the column space of A, the eq^n $Ax = \hat{b}$ is inconsistent and there is an $\hat{x}$ in R^n such that $A\hat{x} = \hat{b}$

Since $\hat{b}$ is the closest point in column A to b, a vector $\hat{x}$ is a least squares solution of $Ax=b$

$$\hat{b} - A\hat{x} = 0$$

$$A^T(b - A\hat{x}) = 0$$

$$A^Tb - A^TA\hat{x} = 0$$

$$\cancel{A^Tb \ \ \hat{x}}$$

$$\boxed{A^TA\hat{x} = A^Tb}$$

classmate

Date

Page

1. Find a least square sol^n of the inconsistent system $Ax = b$ for $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

$\rightarrow$ $A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$

$A^T b = \cancel{\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}} \cdot \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$

$A^T A \hat{x} = A^T b$

$\hat{x} = (A^T A)^{-1} A^T b$

$= \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. For the given A and b in the above example determine the least square error in the least square solution of $Ax = b$.

$b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ $A\hat{x} = b$

$A\hat{x} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$ $\|b - A\hat{x}\| = \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix} = \sqrt{84}$

$\Rightarrow \sqrt{2^2 + 4^2 + 8^2}$

3. Find a least square solution for $Ax = b$ $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$

classmate
Date
Page

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$\hat{x} = \underbrace{(A^T A)^{-1}}_{A^{-1}} \underbrace{A^T b}_{B}.$$

$$M = \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{array}\right]$$

$R_2 \to R_2 - \frac{1}{3}R_1$; $R_3 \to R_3 - \frac{1}{3}R_1$; $R_4 \to R_4 - \frac{1}{3}R_1$

$$M = \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4/3 & -\frac{2}{3} & -\frac{2}{3} & \frac{-13}{3} \\ 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 2/3 \\ 0 & -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 14/3 \end{array}\right] \longrightarrow M = \left[\begin{array}{cccc:c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_4 = 3$
$x_2 - x_4 = -5$
$x_3 - x_4 = -2$

$x_1 = 3 - x_4$
$x_2 = -5 + x_4$
$x_3 = -2 + x_4$

$$\hat{x} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

24/11/23

Application for inner products:-

1. Find the equation $y = \beta_0 + \beta_1 x$ of the least squared line that best fits the data points $(2,1)$ $(5,2)$ $(7,3)$, $(8,3)$

$Ax = B$

$\rightarrow$ $X\beta = y$.

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

Least square method

$$X^T X \beta = X^T y$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\beta = (X^T X)^{-1} X^T y$$

$$= \begin{bmatrix} \frac{2}{7} & \frac{5}{14} \end{bmatrix}$$

$$y = \frac{2}{7} + \frac{5}{14} x$$

2. Weighted least squares:-

Find the least squares line $y = \beta_0 + \beta_1 x$ that best fits the data $(-2,3)$, $(-1,5)$, $(0,5)$, $(1,4)$, $(2,3)$. Suppose the errors in measuring the y values of the last two data points are greater than for the other points. Weight this data as half much as the rest of the data.

classmate
Date
Page

→ $(wx)^T wx\beta = (wx)^T wy$

$$X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad Y = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 4 \\ 3 \end{bmatrix}$$

For a weighing matrix, choose w with diagonal entries as $(2,2,2,1,1)$ left multiplication by w scales the rows of x and y as

$$wx = \begin{bmatrix} 2 & -4 \\ 2 & -2 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad w.y = \begin{bmatrix} 6 \\ 10 \\ 10 \\ 4 \\ 3 \end{bmatrix}$$

we get:

$$\beta = (wx)^{T^{-1}} (wx)^{-1} (wx)^T wy.$$

$$= \cancel{\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ -4 & -2 & 0 & 1 & 2 \end{bmatrix}^{-1}}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ -4 & -2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 2 & -2 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -9 \\ -9 & 25 \end{bmatrix}$$

$$\beta = ((wx)^T wx)^{-1} (wx)^T y.$$

$$(wx)^T wy = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ -4 & -2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 10 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 59 \\ -34 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 4.34 \\ 0.20 \end{bmatrix}$$

$$y = 4.34 + 0.20x$$