Chapter 3

The DFT and FFT

3.1 Introduction

The sampled discrete-time Fourier transform (DTFT) of a finite length, discrete-time signal is known as the *discrete Fourier transform* (DFT). The DFT contains a finite number of samples, equal to the number of samples N in the given signal. Computationally efficient algorithms for implementing the DFT go by the generic name of *fast Fourier transforms* (FFTs). This chapter describes the DFT and its properties, and its relationship to DTFT. The chapter concludes with a discussion of FFT algorithms for computing DFT and its inverse.

3.2 Definition of DFT and its Inverse

Let us consider a discrete time signal $x(n)$ having a finite duration, say in the range $0 \leqslant n \leqslant N-1$. The DTFT of this signal is

$$X(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} \tag{3.1}$$

Let us sample $X(\omega)$ using a total of N equally spaced samples in the range: $\omega \in (0, 2\pi)$, so the sampling interval is $\frac{2\pi}{N}$. That is, we sample $X(\omega)$ using the frequencies

$$\omega = \omega_k = \frac{2\pi k}{N}, \quad 0 \leqslant k \leqslant N-1$$

The result is, by definition the DFT.

That is,

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}, \\ &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \end{aligned} \tag{3.2}$$

Equation (3.2) is known as N-point DFT analysis equation. Fig. 3.1 shows the Fourier transform of a discrete-time signal and its DFT samples.

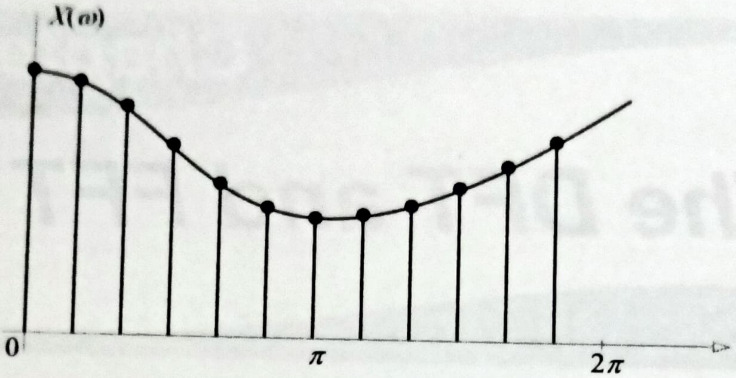

Fig. 3.1 Sampling of $X(\omega)$ to get $X(k)$. Solid line: $X(\omega)$; dots: DFT samples (shown for N=12).

While working with DFT, it is customary to introduce a complex quantity:

$$W_N = e^{-j\frac{2\pi}{N}}$$

Also, it is very common to represent the DFT operation for a sequence $x(n)$ of length N by DFT $\{x(n)\}$. As a consequence of this notation, we can rewrite equation (3.2) as

$$X(k) = \text{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad 0 \leqslant k \leqslant N-1$$

The complex quantity W_N is periodic with a period equal to N. That is, $W_N^{a+N} = e^{-j\frac{2\pi}{N}(a+N)} = e^{-j\frac{2\pi}{N}a} = W_N^a$, where a is any integer.

Figs. 3.2(a) and (b) shows the sequence W_N^n for $0 \leqslant n \leqslant N-1$ in the z-plane for N being even and odd respectively.

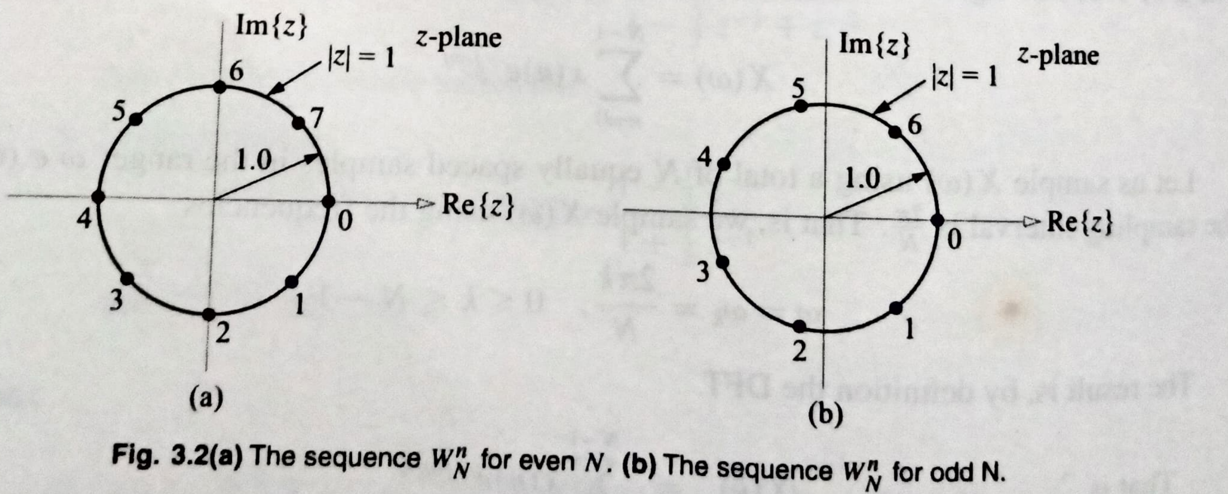

Fig. 3.2(a) The sequence W_N^n for even N. **(b)** The sequence W_N^n for odd N.

The sequence W_N^n for $0 \leqslant n \leqslant N-1$ lies on a circle of unit radius in the complex plane and the phases are equally spaced, beginning at zero.

The formula given in the lemma to follow is a useful tool in deriving and analysing various DFT oriented results.

3.2.1 Lemma

$$\sum_{n=0}^{N-1} W_N^{kn} = N\,\delta(k) = \begin{cases} N, & k=0 \\ 0, & k \neq 0 \end{cases} \tag{3.3}$$

Proof:

We know that
$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}; \quad a \neq 1$$

Applying the above result to the left side of equation (3.3), we get

$$\begin{aligned}\sum_{n=0}^{N-1} (W_N^k)^n = \frac{1-W_N^{kN}}{1-W_N^k} &= \frac{1-e^{-j\frac{2\pi}{N}kN}}{1-e^{-j\frac{2\pi}{N}k}}; \quad k \neq 0 \\ &= \frac{1-1}{1-e^{-j\frac{2\pi k}{N}}} \\ &= 0, \quad k \neq 0\end{aligned}$$

when $k=0$, the left side of equation (3.3) becomes

$$\sum_{n=0}^{N-1} W_N^{0\times n} = \sum_{n=0}^{N-1} 1 = N$$

Hence, we may write

$$\begin{aligned}\sum_{n=0}^{N-1} W_N^{kn} &= \begin{cases} N, & k=0 \\ 0, & k \neq 0 \end{cases} \\ &= N\delta(k), \quad 0 \leqslant k \leqslant N-1\end{aligned}$$

3.2.2 Inverse DFT

The DFT values $(X(k),\ 0 \leqslant k \leqslant N-1)$, uniquely define the sequence $x(n)$ through the inverse DFT formula (IDFT):

$$x(n) = \text{IDFT}\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \tag{3.4}$$

The above equation is known as the synthesis equation.

Proof:

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} &= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m) W_N^{km}\right] W_N^{-kn} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{k=0}^{N-1} W_N^{-(n-m)k}\right]\end{aligned}$$

It can be shown that

$$\sum_{k=0}^{N-1} W_N^{-(n-m)k} = \begin{cases} N, & n = m \\ 0, & n \neq m \end{cases}$$
$$= N\delta(n-m)$$

Hence,

$$\frac{1}{N}\sum_{k=0}^{N-1} x(m)N\delta(n-m)$$
$$= \frac{1}{N} \times Nx(m)\Big|_{m=n} \qquad \text{(sifting property)}$$
$$= x(n)$$

3.2.3 Periodicity of $X(k)$ and $x(n)$

The N-point DFT and N-point IDFT are implicit periodic with period N. Eventhough $x(n)$ and $X(k)$ are sequences of length-N each, they can be shown to be periodic with a period N because the exponentials $W_N^{\pm kn}$ in the defining equations of DFT and IDFT are periodic with a period N. For this reason, $x(n)$ and $X(k)$ are called *implicit periodic sequences.* We reiterate the fact that for finite length sequences in DFT and IDFT analysis periodicity means implicit periodicity. This can be proved as follows:

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}$$

$$\Rightarrow \quad X(k+N) = \sum_{n=0}^{N-1} x(n)W_N^{(k+N)n}$$
$$= \sum_{n=0}^{N-1} x(n)W_N^{kn}\, W_N^{Nn}$$

Since, $W_N^{Nn} = e^{-j\frac{2\pi}{N}Nn} = e^{-j2\pi n} = 1$, we get

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$
$$= X(k)$$

Similarly,

$$x(n) \triangleq \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}$$

$$\Rightarrow \quad x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-k(n+N)}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}\, W_N^{-kN}$$

Since, $W_N^{-kN} = e^{j\frac{2\pi}{N}kN} = e^{j2\pi k} = 1$, we get

$$\begin{aligned} x(n+N) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \\ &= x(n) \end{aligned}$$

Since, DFT and its inverse are both periodic with period N, it is sufficient to compute the results for one period (0 to $N-1$). We want to emphasize that both $x(n)$ and $X(k)$ have a starting index of zero.

A very important implication of $x(n)$ being periodic is, if we wish to find DFT of a periodic signal, we extract one period of the periodic signal and then compute its DFT.

Example 3.1 Compute the 8-point DFT of the sequence $x(n)$ given below:

$$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$$

☐ Solution

The complex basis functions, W_8^n for $0 \leqslant n \leqslant 7$ lie on a circle of unit radius as shown in Fig. Ex.3.1.

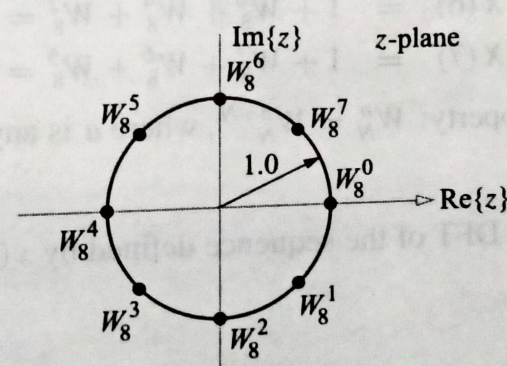

Fig. Ex.3.1 Sequence W_8^n for $0 \leqslant n \leqslant 8$.

Since $N = 8$, we get $W_8 = e^{-j\frac{2\pi}{8}}$.

Thus,

$$\begin{aligned} W_8^0 &= 1 \\ W_8^1 &= e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ W_8^2 &= e^{-j\frac{\pi}{2}} = -j \\ W_8^3 &= e^{-j\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ W_8^4 &= -W_8^0 = -1 \end{aligned}$$

$$
\begin{aligned}
W_8^5 &= -W_8^1 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
W_8^6 &= -W_8^2 = j \\
W_8^7 &= -W_8^3 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}
\end{aligned}
$$

By definition, the DFT of $x(n)$ is

$$
\begin{aligned}
X(k) &= \mathrm{DFT}\{x(n)\} \\
&= \sum_{n=0}^{7} x(n) W_8^{kn} \\
&= 1 + 1 \times W_8^k + 1 \times W_8^{2k} + 1 \times W_8^{3k} \\
&= 1 + W_8^k + W_8^{2k} + W_8^{3k}, \ k = 0, 1, \cdots 7 \\
X(0) &= 1 + 1 + 1 + 1 = 4 \\
X(1) &= 1 + W_8^1 + W_8^2 + W_8^3 = 1 - j2.414 \\
X(2) &= 1 + W_8^2 + W_8^4 + W_8^6 = 0 \\
X(3) &= 1 + W_8^3 + W_8^6 + W_8^1 = 1 - j0.414 \\
X(4) &= 1 + W_8^4 + W_8^0 + W_8^4 = 0 \\
X(5) &= 1 + W_8^5 + W_8^2 + W_8^7 = 1 + j0.414 \\
X(6) &= 1 + W_8^6 + W_8^4 + W_8^2 = 0 \\
X(7) &= 1 + W_8^7 + W_8^6 + W_8^5 = 1 + j2.414
\end{aligned}
$$

Please note the periodic property: $W_N^a = W_N^{a+N}$, where a is any integer.

Example 3.2 Compute the DFT of the sequence defined by $x(n) = (-1)^n$ for

a. $N = 3$,

b. $N = 4$,

c. N odd,

d. N even.

☐ Solution

$$
\begin{aligned}
X(k) &= \mathrm{DFT}\{x(n)\} \\
&= \sum_{n=0}^{N-1} (-1)^n \, W_N^{nk} \\
&= \sum_{n=0}^{N-1} [-W_N^k]^n \\
&= \frac{1 - (-1)^N}{1 + W_N^k} \quad \text{for} \quad W_N^k \neq -1
\end{aligned}
$$

a. $N = 3$

$$X(k) = \frac{2}{1 + W_3^k} = \frac{2}{1 + \cos\left(\frac{2\pi k}{3}\right) - j\sin\left(\frac{2\pi k}{3}\right)}, \quad 0 \leqslant k \leqslant 2$$

b. $N = 4$

$$X(k) = 0 \quad \text{for} \quad W_4^k \neq -1 \quad \text{or} \quad k \neq 2$$

With $k = 2$, we get

$$\begin{aligned} X(2) &= \sum_{n=0}^{3} (-1)^n W_4^{2n} \\ &= 1 - W_4^2 + W_4^4 - W_4^6 \\ &= 1 - (-1) + (-1)^2 - (-1)^3 = 4 \end{aligned}$$

Hence, $$X(k) = 4\delta(k-2).$$

c. We know that

$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

If $$N = 2k, \quad \text{we get} \quad W_N^k = -1.$$

Since, N is odd no k exists. This means to say that $W_N^k \neq -1$ for all k from 0 to $N-1$. Therefore,

$$\begin{aligned} X(k) &= \frac{2}{1 + W_N^k}, \quad 0 \leqslant k \leqslant N-1 \\ &= \frac{2}{1 + \cos\frac{2\pi k}{N} - j\sin\frac{2\pi k}{N}} \end{aligned}$$

d. N even: $W_N^k = -1$, if $k = \frac{N}{2}$.

$$X(k) = 0 \quad \text{for} \quad k \neq \frac{N}{2}$$

and

$$\begin{aligned} X\left(\frac{N}{2}\right) &= \sum_{n=0}^{N-1} \left[-W_N^{\frac{N}{2}}\right]^n \\ &= \sum_{n=0}^{N-1} (1) = N \end{aligned}$$

Hence, $$X(k) = N\delta\left(k - \frac{N}{2}\right)$$

Example 3.3 Find the N-point DFT of the following sequences:

a. $x_1(n) = \delta(n)$

b. $x_2(n) = \delta(n - n_0)$

☐ **Solution**

a.

$$\begin{aligned} X_1(k) &= \text{DFT}\{x_1(n)\} \\ &= \sum_{n=0}^{N-1} x_1(n) W_N^{kn} \\ &= \sum_{n=0}^{N-1} \delta(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

Using sifting property, we get

$$X_1(k) = W_N^{kn}\big|_{n=0} = 1$$

b.

$$\begin{aligned} X_2(k) &= \text{DFT}\{x_2(n)\} \\ &= \sum_{n=0}^{N-1} x_2(n) W_N^{kn} \\ &= \sum_{n=0}^{N-1} \delta(n-n_0) W_N^{kn} \end{aligned}$$

Using sifting property, we get

$$\begin{aligned} X_2(k) &= W_N^{kn}\big|_{n=n_0} \\ &= W_N^{kn_0}, \qquad 0 \leqslant k \leqslant N-1 \\ &= e^{-j\frac{2\pi}{N}kn_0} \end{aligned}$$

Example 3.4 Find the N-point DFT of the sequence

$$x(n) = e^{j\omega m n}, \qquad 0 \leqslant n \leqslant N-1$$

☐ **Solution**

$$\begin{aligned} X(k) &= \text{DFT}\{x(n)\} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\ &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mn} W_N^{kn} \qquad \left(\because \omega = \frac{2\pi}{N}\right) \end{aligned}$$

$$= \sum_{n=0}^{N-1} W_N^{-mn} W_N^{kn}$$

$$= \sum_{n=0}^{N-1} W_N^{(k-m)n}$$

We know that $$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1}, \quad b \neq 1$$

Hence, $$X(k) = \frac{W_N^{(k-m)N} - 1}{W_N^{k-m} - 1}, \quad k \neq m$$

$$= \frac{W_N^{kN}\, W_N^{-mN} - 1}{W_N^{k-m} - 1}$$

$$= \frac{1 \times 1 - 1}{W_N^{k-m} - 1} = 0, \quad k \neq m$$

When, $k = m$,

$$X(m) = \sum_{n=0}^{N-1} 1 = N$$

Hence, $$X(k) = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases}$$

or $$X(k) = N\delta(k-m), \quad 0 \leqslant m \leqslant N-1$$

Example 3.5 Compute the N-point DFT of the sequence,

$$x(n) = a^n, \quad 0 \leqslant n \leqslant N-1$$

☐ Solution

$$X(k) = \text{DFT}\{x(n)\}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a^n W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a W_N^k)^n$$

We know that $$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1}, \quad b \neq 1$$

Hence,
$$\begin{aligned} X(k) &= \frac{a^N W_N^{kN} - 1}{a W_N^k - 1} \\ &= \frac{a^N - 1}{a W_N^k - 1}, \qquad 0 \leqslant k \leqslant N-1 \end{aligned}$$

Example 3.6 Compute the N-point DFT of the sequence,

$$x(n) = an, \quad 0 \leqslant n \leqslant N-1$$

□ Solution

We know that
$$\begin{aligned} X(k) &= \text{DFT}\{x(n)\} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \\ &= a \sum_{n=0}^{N-1} n W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \end{aligned} \tag{3.5}$$

Let, $$S = \sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1}, \quad b \neq 1$$

Differentiating both the sides of the above equation with respect to b, we get

$$\begin{aligned} \sum_{n=0}^{N-1} n b^{n-1} &= \frac{(b-1) N\, b^{N-1} - (b^N - 1) \times 1}{(b-1)^2} \\ \Rightarrow \quad \sum_{n=0}^{N-1} n b^n &= \frac{b\left(N b^N - N b^{N-1} - b^N + 1\right)}{(b-1)^2} \\ &= \frac{b\left(b^N (N-1) - N b^{N-1} + 1\right)}{(b-1)^2} \end{aligned}$$

Letting $b = W_N^k$ in the above expression, we get

$$\begin{aligned} \sum_{n=0}^{N-1} n W_N^{kn} &= \frac{W_N^k \left[W_N^{kN}(N-1) - N W_N^{k(N-1)} + 1\right]}{\left[W_N^k - 1\right]^2} \\ &= \frac{W_N^k \left[N - 1 - N W_N^{-k} + 1\right]}{\left[W_N^k - 1\right]^2} \qquad \left(\because\ W_N^{kN} = \left(e^{-j\frac{2\pi}{N}kN}\right) = e^{-j2\pi k} = 1\right) \\ &= \frac{N\left[W_N^k - 1\right]}{\left[W_N^k - 1\right]^2} = \frac{N}{W_N^k - 1}, \quad k \neq 0 \end{aligned}$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

When $k = 0$, equation (3.5) becomes

$$X(0) = a\sum_{n=0}^{N-1} n = \frac{aN(N-1)}{2}$$

Hence,

$$X(k) = \begin{cases} \frac{aN(N-1)}{2}, & k = 0 \\ \frac{aN}{W_N^k - 1}, & k \neq 0 \end{cases}$$

Example 3.7 a. Compute the N-point DFT of the sequence, $x(n) = 1,\ 0 \leqslant n \leqslant N-1$.

b. For $N = 5$, compute the DFT of $x_1(n) = (1, 1, 1, 0, 0)$ and compare the result with the DFT of $x_2(n) = (1, 1, 1)$ for $N = 3$.

☐ Solution

a.

$$\begin{aligned} X(k) &= \mathrm{DFT}\{x(n)\} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\ &= \sum_{n=0}^{N-1} 1 \times (W_N^k)^n \end{aligned}$$

We know that

$$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b - 1}, \quad b \neq 1.$$

Hence,

$$X(k) = \frac{W_N^{kN} - 1}{W_N^k - 1} = \frac{1-1}{W_N^k - 1} = 0, \quad k \neq 0.$$

When $k = 0$, we get

$$X(0) = \sum_{n=0}^{N-1} 1 \times 1 = N$$

Hence,

$$X(k) = \begin{cases} N, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

or

$$X(k) = N\delta(k)$$

b. Given,

$$x_1(n) = (1, 1, 1, 0, 0)$$

$$N = 5$$

In the present context, $W_5 = e^{-j\frac{2\pi}{5}}$

$$
\begin{aligned}
\Rightarrow \qquad W_5^0 &= 1 \\
W_5^1 &= 0.309 - j0.951 \\
W_5^2 &= -0.809 - j0.587 \\
W_5^3 &= -0.809 + j0.587 \\
W_5^4 &= 0.309 + j0.951
\end{aligned}
$$

Hence,

$$
\begin{aligned}
X_1(k) = \mathrm{DFT}\{x_1(n)\} &= \sum_{n=0}^{4} x_1(n) W_5^{kn}, \quad 0 \leqslant k \leqslant 4 \\
&= 1 + W_5^k + W_5^{2k} \\
\Rightarrow \qquad X_1(0) &= 1 + 1 + 1 = 3 \\
X_1(1) &= 1 + W_5^1 + W_5^2 = 0.5 - j1.538 \\
X_1(2) &= 1 + W_5^2 + W_5^4 = 0.5 + j0.364 \\
X_1(3) &= 1 + W_5^3 + W_5^1 = 0.5 - j0.364 \\
X_1(4) &= 1 + W_5^4 + W_5^3 = 0.5 + j1.538
\end{aligned}
$$

Also, given $\qquad x_2(n) = (1, 1, 1), \quad N = 3$

Now,

$$
\begin{aligned}
W_N &= e^{-j\frac{2\pi}{N}} \\
W_3 &= e^{-j\frac{2\pi}{3}} \\
\Rightarrow \qquad W_3^0 &= 1 \\
W_3^1 &= -0.5 - j0.866 \\
W_3^2 &= -0.5 + j0.866
\end{aligned}
$$

Hence,

$$
\begin{aligned}
X_2(k) &= \mathrm{DFT}\{x_2(n)\} \\
&= \sum_{n=0}^{2} x_2(n) W_N^{kn}, \qquad 0 \leqslant k \leqslant 2 \\
&= W_3^{0k} + W_3^k + W_3^{2k} \\
&= 1 + W_3^k + W_3^{2k} \\
\Rightarrow \qquad X_2(0) &= 1 + 1 + 1 = 3 \\
X_2(1) &= 1 + W_3^1 + W_3^2 = 0 \\
X_2(3) &= 1 + W_3^2 + W_3^1 = 0
\end{aligned}
$$

Thus, we find that $X_1(k) \neq X_2(k)$.

Example 3.8 Compute the N-point DFT of the sequence:

$$x(n) = \cos(n\omega_0), \quad \omega_0 = \frac{2\pi}{N}k_0, \qquad 0 \leqslant n \leqslant N-1.$$

Solution

Given,
$$x(n) = \cos(n\omega_0)$$
$$\Rightarrow \quad x(n) = \frac{1}{2}e^{jn\omega_0} + \frac{1}{2}e^{-jn\omega_0}$$
$$= \frac{1}{2}e^{jn\frac{2\pi}{N}k_0} + \frac{1}{2}e^{-jn\frac{2\pi}{N}k_0}$$
$$= \frac{1}{2}e^{-j\frac{2\pi}{N}(-k_0 n)} + \frac{1}{2}e^{-j\frac{2\pi}{N}(k_0 n)}$$
$$= \frac{1}{2}W_N^{-k_0 n} + \frac{1}{2}W_N^{k_0 n}$$

Hence,
$$X(k) = \text{DFT}\{x(n)\}, \quad 0 \leqslant k \leqslant N-1$$
$$= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k-k_0)n} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k+k_0)n}$$
$$= \frac{1}{2}S_1 + \frac{1}{2}S_2 \tag{3.6}$$

To find S_1 and S_2:
$$S_1 = \sum_{n=0}^{N-1} W_N^{(k-k_0)n}$$
$$= \frac{W_N^{(k-k_0)N} - 1}{W_N^{(k-k_0)} - 1} = \frac{W_N^{kN}\,W_N^{-k_0 N} - 1}{W_N^{(k-k_0)} - 1}$$
$$= \frac{1 \times 1 - 1}{W_N^{(k-k_0)} - 1} = 0, \quad k \neq k_0$$

When $k = k_0$, we get
$$S_1 = \sum_{n=0}^{N-1} (1)^n = N$$

Hence,
$$S_1 = \sum_{n=0}^{N-1} W_N^{(k-k_0)n} = \begin{cases} 0, & k \neq k_0 \\ N, & k = k_0 \end{cases}$$

or
$$S_1 = N\delta(k - k_0)$$

Similarly,
$$S_2 = N\delta(k + k_0)$$
$$= N\delta[k - (N - k_0)]$$

Thus,
$$X(k) = \frac{1}{2}S_1 + \frac{1}{2}S_2$$
$$= \frac{N}{2}\delta(k - k_0) + \frac{N}{2}\delta[k - (N - k_0)]$$

Example 3.9 Compute the inverse DFT of the sequence,

$$X(k) = (2, 1+j, 0, 1-j)$$

☐ **Solution**

$$\begin{aligned} x(n) &= \text{IDFT}\{X(k)\} \\ &\triangleq \frac{1}{N}\sum_{n=0}^{N-1} x(n)W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \end{aligned}$$

Please note that:

$$W_N^{-kn} = \left[W_N^{kn}\right]^*$$

Since, $N = 4$, we get

$$\begin{aligned} x(n) &= \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-kn}, \quad 0 \leqslant n \leqslant 3 \\ &= \frac{1}{4}\left[X(0)W_4^{-0\times n} + X(1)W_4^{-n} + X(2)W_4^{-2n} + X(3)W_4^{-3n}\right] \\ &= \frac{1}{4}\left[2 + (1+j)W_4^{-n} + 0 + (1-j)W_4^{-3n}\right] \end{aligned}$$

Hence,

$$\begin{aligned} x(0) &= \frac{1}{4}[2 + (1+j) + (1-j)] = 1 \\ x(1) &= \frac{1}{4}\left[2 + (1+j)W_4^{-1} + (1-j)W_4^{-3}\right] \\ &= \frac{1}{4}[2 + (1+j)j + (1-j)(-j)] = 0 \\ x(2) &= \frac{1}{4}\left[2 + (1+j)W_4^{-2} + (1-j)W_4^{-6}\right] \end{aligned}$$

Because of periodicity, $W_4^{-6} = W_4^{-2}$.

Hence,

$$\begin{aligned} x(2) &= \frac{1}{4}[2 + (1+j)(-1) + (1-j)(-1)] = 0 \\ x(3) &= \frac{1}{4}\left[2 + (1+j)W_4^{-3} + (1-j)W_4^{-9}\right] \end{aligned}$$

Because of periodicity, $W_4^{-9} = W_4^{-5} = W_4^{-1}$.

Hence,

$$\begin{aligned} x(3) &= \frac{1}{4}\left[2 + (1+j)W_4^{-3} + (1-j)W_4^{-1}\right] \\ &= \frac{1}{4}[2 + (1+j)(-j) + (1-j)(j)] = 1 \end{aligned}$$

Hence,

$$x(n) = (1, 0, 0, 1)$$

3.3 Matrix Relation for Computing DFT

The defining relation for DFT of a finite length sequence $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Let us evaluate $X(k)$ for different values of k in the range $(0, N-1)$ as given below:

$$\begin{aligned}
X(0) &= W_N^0 x(0) + W_N^0 x(1) + \cdots + W_N^0 x(N-1) \\
X(1) &= W_N^0 x(0) + W_N^1 x(1) + \cdots + W_N^{(N-1)} x(N-1) \\
X(2) &= W_N^0 x(0) + W_N^2 x(1) + \cdots + W_N^{2(N-1)} x(N-1) \\
\vdots & \qquad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\
X(N-1) &= W_N^0 x(0) + W_N^{(N-1)} x(1) + \cdots + W_N^{(N-1)(N-1)} x(N-1)
\end{aligned}$$

Putting the N DFT equations in N unknowns in the matrix form, we get

$$\mathbf{X} = \mathbf{W}_N \mathbf{x} \tag{3.7}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $(N \times 1)$ matrices, and W_N is an $(N \times N)$ square matrix called the *DFT matrix*. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

The elements W_N^{kn} of $\mathbf{W}_N$ are called *complex basis functions or twiddle factors.*

Example 3.10 Compute the 4-point DFT of the sequence, $x(n) = (1, 2, 1, 0)$.

□ **Solution**

With $N = 4$, $W_4 = e^{-j\frac{2\pi}{4}} = -j$.

We know that $\mathbf{X} = \mathbf{W}_N \mathbf{x}$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Exploting the periodic property: $W_N^a = W_N^{a+N}$, where a is any integer the above matrix relation becomes

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

Hence, $X(k) = (4, -j2, 0, j2)$

3.4 Matrix Relation for Computing IDFT

We know that $\mathbf{X} = \mathbf{W}_N \mathbf{x}$

Premultiplying both the sides of the above equation by $\mathbf{W}_N^{-1}$, we get

$$\mathbf{W}_N^{-1}\mathbf{X} = \mathbf{W}_N^{-1}\mathbf{W}_N \mathbf{x}$$

$$\Rightarrow \mathbf{W}_N^{-1}\mathbf{X} = \mathbf{x}$$

or

$$\mathbf{x} = \mathbf{W}_N^{-1}\mathbf{X} \tag{3.8}$$

In the above equation (3.8), $\mathbf{W}_N^{-1}$ is called *IDFT matrix*.

The defining equation for finding IDFT of a sequence $X(k)$ is

$$\begin{aligned} x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[W_N^{kn}\right]^* \end{aligned}$$

The first set of N IDFT equations in N unknowns may be expressed in the matrix form as

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N^*\mathbf{X} \tag{3.9}$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of $\mathbf{W}_N$. Comparison of equations (3.8) and (3.9) leads us to conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^*$$

This very important result shows that $\mathbf{W}_N^{-1}$ requires only conjugation of $\mathbf{W}_N$ multiplied by $\frac{1}{N}$, an obvious computational advantage. **The matrix relations (3.7) and (3.9) together define DFT as a linear transformation.**

3.5 Using the DFT to Find the IDFT

We know that

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad n = 0, 1, \cdots N-1$$

Taking complex conjugates on both the sides of the above equation, we get

$$x^*(n) = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}\right]^*$$

$$\Rightarrow \quad x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{kn} \tag{3.10}$$

The right-hand side of equation (3.10) is recognized as the DFT of $X^*(k)$, so we can rewrite equation (3.10) as follows:

$$x^*(n) = \frac{1}{N}\,\mathrm{DFT}\{X^*(k)\} \tag{3.11}$$

Taking complex conjugates on both the sides of equation (3.11), we get

$$x(n) = \frac{1}{N}\left[\mathrm{DFT}\{X^*(k)\}\right]^*$$

The above result suggests that DFT algorithm itself can be used to find IDFT. In practice, this is indeed what is done.

Example 3.11 Find the IDFT of 4-point sequence,

$$X(k) = (4, -j2, 0, j2)$$

using the DFT.

☐ Solution

Let us first conjugate the sequence $X(k)$ to get $X^*(k) = (4, j2, 0, -j2)$. As a second step, we find the DFT of $X^*(k)$.

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} X^*(0) \\ X^*(1) \\ X^*(2) \\ X^*(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

Finally, we get the conjugate of the above result and divide it by $N = 4$ to get IDFT of $X(k)$ as

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}(4, 8, 4, 0) = (1, 2, 1, 0)$$

Example 3.12 Consider a signal of length equal to 4 defined by

$$x(n) = (1, 2, 3, 1)$$

a. Compute the 4-point DFT by solving explicitly the 4-by-4 system of linear equations defined by the inverse DFT formula.

b. Verify the result of part (a) by finding $X(k)$ using the defining equation for DFT.

☐ Solution

We have,

$$\text{IDFT}\{X(k)\} = x(n)$$

$$\triangleq \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}nk}$$

$$\Rightarrow \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}nk} = N\,x(n), \quad 0 \leqslant n \leqslant N-1$$

Since $N = 4$, we get

$$\sum_{k=0}^{3} X(k)e^{j\frac{2\pi}{4}nk} = 4x(n), \quad n = 0, 1, 2, 3$$

Hence, we get the following linear equations:

$$\begin{aligned}
X(0) + X(1) + X(2) + X(3) &= 4x(0) = 4 \\
X(0) + X(1)\,e^{j\frac{\pi}{2}} + X(2)\,e^{j\pi} + X(3)\,e^{j\frac{3\pi}{2}} &= 4x(1) = 8 \\
X(0) + X(1)\,e^{j\pi} + X(2)\,e^{j2\pi} + X(3)\,e^{j3\pi} &= 4x(2) = 12 \\
X(0) + X(1)\,e^{j\frac{3\pi}{2}} + X(2)\,e^{j3\pi} + X(3)\,e^{j\frac{9\pi}{2}} &= 2x(3) = 4
\end{aligned}$$

Putting the above set of linear equations in the matrix form, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

b.

$$\begin{aligned}
\text{DFT}\,\{x(n)\} &= X(k) \\
&\triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1
\end{aligned}$$

In the present context,

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3 \\
&= x(0) + x(1) W_4^{k} + x(2) W_4^{2k} + x(3) W_4^{3k} \\
\Rightarrow \quad X(k) &= 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k}, \quad 0 \leqslant k \leqslant 3
\end{aligned}
$$

Evaluation of $X(k)$ needs the following complex basis functions:

$$
W_4^0 = \left[e^{-j\frac{2\pi}{4}}\right]^0 = 1,\ W_4^1 = \left[e^{-j\frac{2\pi}{4}}\right]^1 = -j,\ W_4^2 = \left[e^{-j\frac{2\pi}{4}}\right]^2 = -1,\ W_4^3 = \left[e^{-j\frac{2\pi}{4}}\right]^3 = j.
$$

Also,

$$W_N^a = W_N^{a+N}, \quad \text{where } a \text{ is any integer}$$

Hence,

$$
\begin{aligned}
X(0) &= 1 + 2 + 3 + 1 = 7 \\
X(1) &= 1 + 2W_4^1 + 3W_4^2 + 3W_4^3 = -2 - j \\
X(2) &= 1 + 2W_4^2 + 3W_4^4 + W_4^6 \\
&= 1 + 2W_4^2 + 3W_4^0 + W_4^2 = 1 \\
X(3) &= 1 + 2W_4^3 + 3W_4^6 + W_4^9 \\
&= 1 + 2W_4^3 + 3W_4^2 + W_4^1 = -2 + j
\end{aligned}
$$

3.6 Concept of Circular Shift and Circular Symmetry

Let us consider a sequence $x(n)$ defined for all n, the translated version of $x(n)$ is written as $x(n - n_0)$, where n_0 represents the number of indices that the sequence $x(n)$ is translated to right. For a finite length sequence defined for $0 \leqslant n \leqslant N - 1$, if a regular shift is employed, parts of the sequence would fall outside the defined range for n and the first part of the sequence would be undefined. It was proved in the earlier discussions that although $x(n)$ is defined for $0 \leqslant n \leqslant N - 1$, $x(n)$ is implicit periodic with a period equal to N. Because of this implied periodic nature of $x(n)$, the fundamental form of sequence translation, useful in a mathematical sense, is a circular translation.

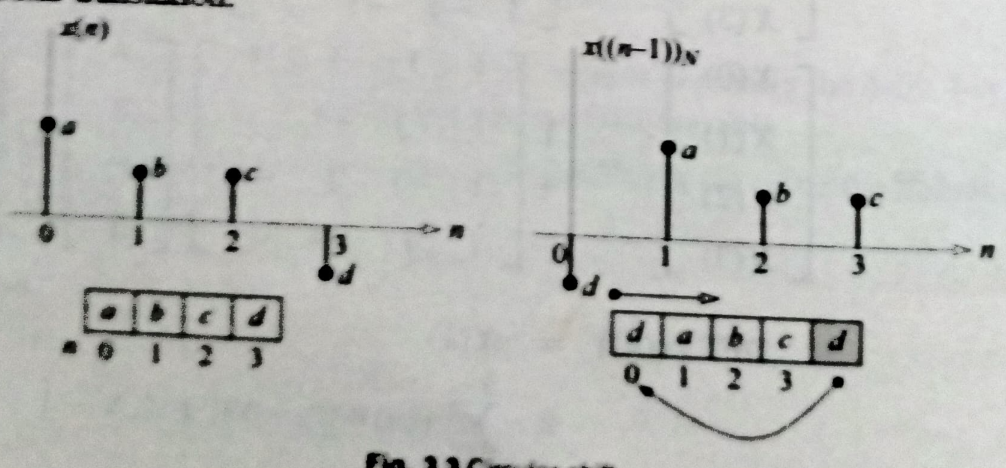

Fig. 3.3 Circular shift

The circular shift or translation of $x(n)$ with $0 \leqslant n \leqslant N-1$ by an amount n_0 to right is denoted by $x((n-n_0))_N$. This operation tantamounts, wrapping the first part of the sequence that falls outside the range for n around to the first part of the sequence, or just a straight translation of the periodic extension outside of 0 to $N-1$ of the given sequence. In otherwords, any value that falls off the interval $(0, \cdots N-1)$ after the shift, it comes back from the other side as shown in Fig. 3.3.

Thus, we follow the guidelines given below for generating circularly shifted signal.

- To generate $x((n-n_0))_N$: Move the last n_0 samples of $x(n)$ to the beginning.
- To generate $x((n+n_0))_N$: Move the first n_0 samples of $x(n)$ to the end.

Circular folding generates the signal $x((-n))_N$ from $x(n)$. We fold $x(n)$, create the periodic extension of the folded signal and pick N samples of the periodic extension over $(0, N-1)$. In order to understand circular folding, let us consider a finite duration sequence,

$$x(n) = (\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4), \quad 0 \leqslant n \leqslant 3$$

Then,

$$x(-n) = (4\ 3\ 2\ \underset{\substack{\uparrow \\ n=0}}{1})$$

The periodic extension of $x(-n)$ is shown below:

$$\cdots \ \vdots\ 4\,3\,2\,1\ \vdots\ 4\,3\,2\,\underset{\substack{\uparrow \\ n=0}}{1}\ \vdots\ 4\,3\,2\,1\ \vdots\ \cdots$$

Let us now, pick 4 samples starting from $n = 0$. This results in a circularly folded sequence is given by

$$x((-n))_N = (1, 4, 3, 2)$$

Because of the implied periodicity of $x(n)$, it may be noted that $x((-n))_N = x(N-n)$. Thus, for the example considered,

$$\begin{aligned} x((-n))_N &= x(4-n), \qquad 0 \leqslant n \leqslant 3 \\ &= (x(4), x(3), x(2), x(1)) \end{aligned}$$

Since, $\quad x(n+N) = x(n)$, we have $x(4) = x(0)$.

Hence,

$$\begin{aligned} x((-n))_N &= (x(0), x(3), x(2), x(1)) \\ &= (\underset{\substack{\uparrow \\ n=0}}{1}, 4, 3, 2) \end{aligned}$$

Even symmetry of $x(n)$ with $0 \leqslant n \leqslant N-1$ requires that $x(n) = x((-n))_N$. Similarly, the odd symmetry of $x(n)$ with $0 \leqslant n \leqslant N-1$ requires that $x(n) = -x((-n))_N$.

3.7 Properties of DFT

In the following section, we shall discuss some of the important properties of the DFT. They are strikingly similar to other frequency-domain transforms, but must always be used in keeping with implied periodicity (of both DFT and IDFT) in time and frequency-domains.

3.7.1 Linearity

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k), \qquad k = 0, 1, \cdots N-1$$

with $X_1(k)$ and $X_2(k)$ are the DFTs of the sequences $x_1(n)$ and $x_2(n)$, respectively, both of lengths N.

Proof:

$$\underset{n=0,1,\ldots N-1}{x(n)} \longrightarrow \boxed{\text{DFT}\{\cdot\}} \longrightarrow \underset{k=0,1,\ldots N-1}{X(k)}$$

Fig. 3.4 DFT operation viewed as a system represented by an operator DFT$\{\cdot\}$.

In Fig. 3.4, we have represented the DFT operation by an operator DFT $\{\cdot\}$. This figure always reminds us that the input, $x(n)$ and output, $X(k)$ are of same length, N. Hence, N is known as the transform length for the DFT operation.

We know that, $$\text{DFT}\{x(n)\} \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Letting $x(n) = ax_1(n) + bx_2(n)$, we get

$$\begin{aligned} \text{DFT}\{ax_1(n) + bx_2(n)\} &= \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)]\, W_N^{kn} \\ &= a\sum_{n=0}^{N-1} x_1(n) W_N^{kn} + b\sum_{n=0}^{N-1} x_2(n) W_N^{kn} \\ &= aX_1(k) + bX_2(k), \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

Sometimes, we represent the linearity property as given below:

$$a\, x_1(n) + b\, x_2(n) \overset{\text{DFT}}{\longleftrightarrow} a\, X_1(k) + b\, X_2(k)$$

Example 3.13 Find the 4-point DFT of the sequence,

$$x(n) = \cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{4}n\right)$$

Use linearity property.

☐ Solution

Given $N = 4$.

We know that, $W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_4 = e^{-j\frac{\pi}{2}}$

Hence,

$$\begin{aligned} W_4^0 &= 1 \\ W_4^1 &= e^{-j\frac{\pi}{2}} = -j \\ W_4^2 &= e^{-j\pi} = -1 \\ W_4^3 &= e^{-j\frac{3\pi}{2}} = j \end{aligned}$$

Let $$x_1(n) = \cos\left(\frac{\pi}{4}n\right)$$

and $$x_2(n) = \sin\left(\frac{\pi}{4}n\right)$$

Then, the values of $x_1(n)$ and $x_2(n)$ for $0 \leqslant n \leqslant 3$ are tabulated below:

n	$x_1(n) = \cos\left(\frac{\pi}{4}n\right)$	$x_2(n) = \sin\left(\frac{\pi}{4}n\right)$
0	1	0
1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
2	0	1
3	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

As the next step in the problem solving, we compute the 4-point DFTs, $X_1(k)$ and $X_2(k)$.

$$\begin{aligned} X_1(k) &= \text{DFT}\{x_1(n)\} \\ &\triangleq \sum_{n=0}^{3} x_1(n) W_4^{kn}, \quad k = 0, 1, 2, 3 \end{aligned}$$

$$\Rightarrow \quad X_1(k) = 1 + \frac{1}{\sqrt{2}} W_4^k + 0 - \frac{1}{\sqrt{2}} W_4^{3k}$$

Hence,

$$\begin{aligned} X_1(0) &= 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 \\ X_1(1) &= 1 + \frac{1}{\sqrt{2}} W_4^1 - \frac{1}{\sqrt{2}} W_4^3 = 1 - j1.414 \\ X_1(2) &= 1 + \frac{1}{\sqrt{2}} W_4^2 - \frac{1}{\sqrt{2}} W_4^6 \\ &= 1 + \frac{1}{\sqrt{2}} W_4^2 - \frac{1}{\sqrt{2}} W_4^2 = 1 \end{aligned}$$

$$
\begin{aligned}
X_1(3) &= 1 + \frac{1}{\sqrt{2}}W_4^3 - \frac{1}{\sqrt{2}}W_4^9 \\
&= 1 + \frac{1}{\sqrt{2}}W_4^3 - \frac{1}{\sqrt{2}}W_4^1 \\
&= 1 + j1.414
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
X_2(k) &= \mathrm{DFT}\{x_2(n)\} \\
&\triangleq \sum_{n=0}^{3} x_2(n) W_4^{kn}
\end{aligned}
$$

$$
\Rightarrow \quad X_2(k) = \frac{1}{\sqrt{2}}W_4^k + W_4^{2k} + \frac{1}{\sqrt{2}}W_4^{3k}
$$

Hence,

$$
\begin{aligned}
X_2(0) &= \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 2.414 \\
X_2(1) &= \frac{1}{\sqrt{2}}W_4^1 + W_4^2 + \frac{1}{\sqrt{2}}W_4^3 = -1 \\
X_2(2) &= \frac{1}{\sqrt{2}}W_4^2 + W_4^0 + \frac{1}{\sqrt{2}}W_4^6 \\
&= \frac{1}{\sqrt{2}}W_4^2 + W_4^0 + \frac{1}{\sqrt{2}}W_4^2 = -0.414 \\
X_2(3) &= \frac{1}{\sqrt{2}}W_4^3 + W_4^6 + \frac{1}{\sqrt{2}}W_4^9 \\
&= \frac{1}{\sqrt{2}}W_4^3 + W_4^2 + \frac{1}{\sqrt{2}}W_4^1 = -1
\end{aligned}
$$

Finally, applying the linearity property, we get

$$
\begin{aligned}
X(k) &= \mathrm{DFT}\{x_1(n) + x_2(n)\} \\
&= X_1(k) + X_2(k) \\
&= (X_1(0) + X_2(0),\ X_1(1) + X_2(1),\ X_1(2) + X_2(2),\ X_1(3) + X_2(3)) \\
&= (\underset{\underset{k=0}{\uparrow}}{3.414},\ -j1.414,\ 0.586,\ j1.414)
\end{aligned}
$$

It may be noted that the arrow, $\uparrow$ explicitly represents the position index of $k = 0$ or $n = 0$ of a given sequence. The absence of this arrow also implicitly means that the first element in a sequence always has the index $k = 0$ or $n = 0$.

Example 3.14 Compute DFT $\{x(n)\}$ of the sequence given below using the linearity property.

$$
x(n) = \cosh an, \qquad 0 \leqslant n \leqslant N - 1
$$

□ Solution

Given $$x(n) = \cosh an, \quad 0 \leqslant n \leqslant N-1$$

Then the N-point DFT of the sequence $x(n)$ is

$$\begin{aligned} X(k) = \text{DFT}\{x(n)\} &= \text{DFT}\{\cosh an\} \\ &= \text{DFT}\left\{\frac{1}{2}e^{an} + \frac{1}{2}e^{-an}\right\} \end{aligned}$$

Applying linearity property, we get

$$X(k) = \frac{1}{2}\text{DFT}\left\{e^{an}\right\} + \frac{1}{2}\text{DFT}\left\{e^{-an}\right\}, \quad 0 \leqslant n \leqslant N-1$$

We know from Example 3.5, that

$$\text{DFT}\{b^n\} = \frac{b^N - 1}{bW_N^k - 1}, \quad 0 \leqslant k \leqslant N-1$$

Hence,

$$\begin{aligned} X(k) &= \frac{1}{2}\left[\frac{e^{aN} - 1}{e^a W_N^k - 1} + \frac{e^{-aN} - 1}{e^{-a} W_N^k - 1}\right] \\ &= \frac{W_N^k\left[e^{a(N-1)} + e^{-a(N-1)} - e^{-a} - e^{-a}\right] - e^{-aN} - e^{-aN} + 2}{2\left[1 - W_N^k\,(e^a - e^{-a}) + W_N^k\right]} \\ &= \frac{1 - \cosh Na + W_N^k\,[\cosh(N-1)a - \cosh a]}{1 - 2W_N^k \cosh a + W_N^k}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

3.7.2 Circular time shift

If $$\text{DFT}\{x(n)\} = X(k),$$

then $$\text{DFT}\{x((n-m))_N\} = W_N^{mk}\, X(k), \quad 0 \leqslant k \leqslant N-1$$

Proof:

$$\underset{n=0,\,1,\,\cdots\,N-1}{x(n)} \longrightarrow \boxed{\text{DFT}\{\cdot\}} \longrightarrow \underset{k=0,\,1,\,\cdots\,N-1}{X(k)}$$

Fig. 3.5 DFT viewed as an operator.

Fig. 3.5 time and again reminds that $x(n)$ and $X(k)$ are of the same length, N.
From the definition of inverse DFT, we have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$\Rightarrow \quad x(n-m) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-k(n-m)}$$