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Fundamentals of Signals and Systems

Signal - definition, eg.

~~Types~~ Classification of signals

- a) Continuous time and discrete time
- b) Analog and digital
- c) Deterministic and random
- d) Even and odd
- e) Periodic and non periodic
- f) Energy and power

Basic operations on signals

- a) Amplitude Scaling
 $y(t) = A x(t)$
 - b) Addition
 $y(t) = x_1(t) + x_2(t)$
 - c) Multiplication
 $y(t) = x_1(t) \cdot x_2(t)$
- Operations
on dependent
Variable

operations on independent variable

a) Time Scaling

$$y_1(n) = x(n)$$

$$y_2(n) = x(2n)$$

$$y_3(n) = x(n/2)$$

b) Time Shifting

$$y_1(n) = x(n)$$

$$y_2(n) = x(n-2)$$

$$y_3(n) = x(n+2)$$

c) folding

$$y(n) = x(-n)$$

Prob:

1. A discrete time signal is represented as $x(n) = \{ \underset{n=1}{\text{1}}, \underset{n=2}{\text{2}}, \underset{n=3}{\text{3}}, \underset{n=4}{\text{4}}, \underset{n=5}{\text{5}} \}$

Sketch the following

a) $x(n-2)$

b) $x(n+2)$

c) $x(-n)$

d) $x(-n+2)$

e) $x(-n-2)$

Properties of Systems

- a) Linearity b) Causality
- c) Time invariance d) Memory
- e) Stability

LTI Systems

Convolution Sum

Let $x(n)$ and $h(n)$ be two finite duration sequences.
The convolution sum of $x(n) + h(n)$ is given by

$$\begin{aligned}y(n) &= x(n) * h(n) \\&= \sum_{k=-\infty}^{+\infty} x(k) h(n-k)\end{aligned}$$

Prob:

Let $x(n) = \{1, 2, 3, 4\}$ and

$h(n) = \{5, 6, 7, 8\}$

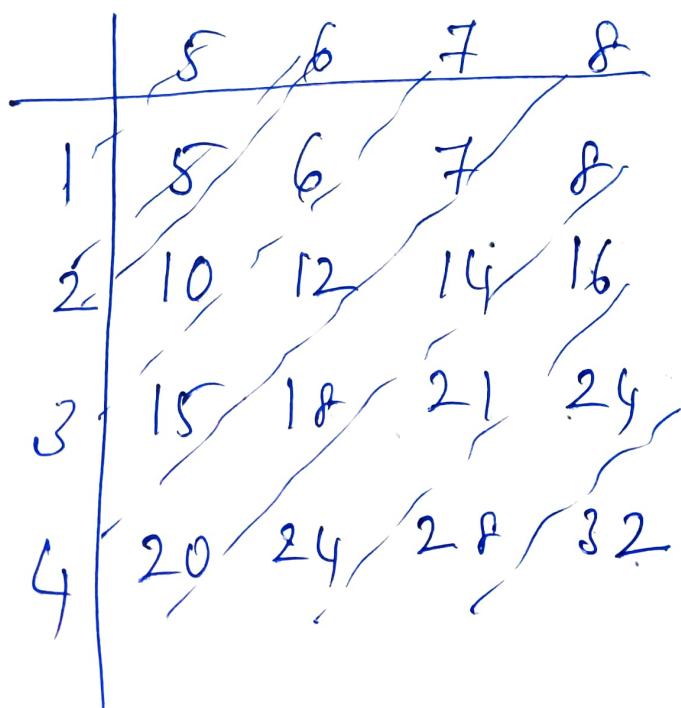
Find $y(n) = x(n) * h(n)$.

Soln

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k) h(n-k)$$

$x(n)$ & $h(n)$ are precisely causal

$\Rightarrow y(n)$ is also causal.



$$y(n) = \{5, 16, 34, 60, 61, 52, 32\}$$

$$L_1 = 4$$

$$L_2 = 4$$

4

$$L_1 + L_2 - 1 = 7$$

Convolution using Toeplitz Matrix

$$y(0) = x(0) h(0)$$

$$y(1) = x(0) h(1) + x(1) h(0)$$

$$y(2) = x(0) h(2) + x(1) h(1) + x(2) h(0)$$

|

|

|

Solution to convolution sum is nothing but solution to a system of linear equations,

$$y(6) = x(3) h(3)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \end{bmatrix} = \begin{bmatrix} x(0) & 0 & 0 & 0 \\ x(1) & x(0) & 0 & 0 \\ x(2) & x(1) & x(0) & 0 \\ x(3) & x(2) & x(1) & x(0) \\ 0 & x(3) & x(2) & x(1) \\ 0 & 0 & x(1) & x(2) \\ 0 & 0 & 0 & x(3) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

7x1

7x4

4x1

(5)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \\ 34 \\ 60 \\ 61 \\ 52 \\ 32 \end{bmatrix}$$

$$\boxed{A \ x = b}$$

Ex 2

Find Linear Convolution of

$$x(n) = \{-2, -1, 0, 1, 2\}$$

$$h(n) = \{-3, -2, -1, 0\}$$

$$\text{Ans } \{6, 7, 4, -2, -8, -5, -2, 0\}$$

$$L_1 = 5, \quad L_2 = 4$$

$$L_1 + L_2 - 1 = 8$$

(6)

De Convolution

$$Ax = b$$

- * A & b are known, we have to find x .
- * A is not square
- * Hence we have to find generalized inverse.

$$A^T A x = A^T b$$

$$x = \underbrace{[A^T A]^{-1}}_{\text{---}} A^T b$$

L - transform

①

Class 5

$$\mathcal{Z} \{x(n)\} = X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

The set of values of z for which summation converges is called ROC.

Eg:

$$1) x(n) = \{1, 2, 2, 1\}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

ROC $|z| > 0$

$$2) x(n) = \{1, 1, 2, \underset{\uparrow}{2}\}$$

$$X(z) = z^3 + z^2 + 2z + 2$$

ROC $|z| < \infty$

$$3) x(n) = \{2, 1, \underset{\uparrow}{1}, 2\}$$

$$X(z) = 2z^2 + z + 1 + 2z^{-1}$$

ROC $0 < |z| < \infty$

$$4) \quad x(n) = a^n u(n)$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

ROC ~~$|z| < \infty$~~ $|az^{-1}| < 1$
 $\Rightarrow |z| > |a|$

$$5) \quad x(n) = -b^n u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=1}^{\infty} (-b^{-1}z)^n$$

$$= \frac{-b^{-1}z}{1 - b^{-1}z} = \frac{z}{z - b}$$

ROC $|b^{-1}z| < 1 \Rightarrow |z| < |b|$

$$6) \quad x(n) = a^n u(n) - b^n u(-n-1)$$

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} - \sum_{n=-\infty}^{-1} b^n u(-n-1) z^{-n}$$

$$= \frac{z}{z - a} + \frac{z}{z - b}$$

ROC $|b| > |z| > |a|$

Properties of ROC

(3)

1. ROC cannot contain any poles
2. If $x(n)$ is finite and causal, ROC is entire z -plane except $z=0$
3. If $x(n)$ is finite and anti-causal ROC is entire z -plane except $z=\infty$
4. If $x(n)$ is double sided & finite, then ROC is entire z -plane except $z=0$ and $z=\infty$
5. If $x(n)$ is causal, infinite then, ROC is $|z| > r_{\max}$
Where r_{\max} is largest magnitude of poles
anti-causal & infinite
6. If $x(n)$ is anti-causal and infinite then ROC is $|z| < r_{\min}$
Where r_{\min} is smallest magnitude of poles
7. If $x(n)$ is two-sided and infinite then ROC is $r_1 < |z| < r_2$ where r_1 and r_2 are magnitude of poles
8. ROC of a stable LTI system contains unit circle in z -plane.

(4)

~~Problems~~

find inverse Z-transform of

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

for the following ROC's

a) $|z| > 1$ b) $|z| < \frac{1}{3}$ c) $\frac{1}{3} < |z| < 1$

Slnⁿ

$$\begin{aligned} \frac{X(z)}{z} &= \frac{1}{3z^2 - 4z + 1} = \frac{1}{3(z - \frac{1}{3})(z - 1)} \\ &= \frac{A}{z - \frac{1}{3}} + \frac{B}{z - 1} \end{aligned}$$

$$A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}$$

$$X(z) = \frac{-\frac{1}{2}z}{z - \frac{1}{3}} + \frac{\frac{1}{2}z}{z - 1}$$

a) Purely causal

$$x(n) = -\frac{1}{2} \left(\frac{1}{3}\right)^n u(n) + \frac{1}{2} u(n)$$

b) Purely anticausal

$$x(n) = \frac{1}{2} \left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2} u(-n-1)$$

c) $-\frac{1}{2} \left(\frac{1}{3}\right)^n u(n) - \frac{1}{2} u(-n-1)$

Fourier Transform

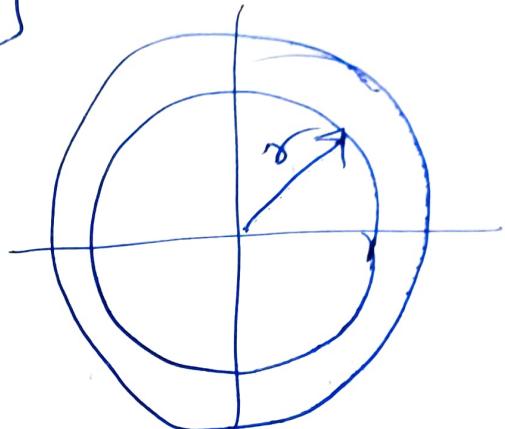
①

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$x(n)$ is finite duration Causal Signal

$$z = r e^{j\omega}$$

$$X(z) = \sum_{n=0}^{N-1} \{x(n) r^{-n}\} e^{-j\omega n}$$



$$\text{if } |r| = 1$$

$$X(z) \Big|_{|r|=1} = X(z) \Big|_{z=e^{j\omega}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\omega n} = X(\omega) = X(e^{j\omega})$$

Fourier transform (DTFT)

Samples on Unit Circle

- exists if $x(n)$ is absolutely summable

- freq. domain representation of the signal

Problems with FT

- $x(n)$ is discrete function of n
but $X(\omega)$ is continuous function of ω

Inverse FT

$$\rightarrow x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega.$$

— Need for DFT.

Eg Determine the Convolution of the Sequences $x_1(n) = x_2(n) = \{1, 1, 1\}$

By $x_3(n) = x_1(n) * x_2(n)$

$$X_3(\omega) = X_1(\omega) \cdot X_2(\omega)$$

$$\begin{aligned} X_1(\omega) &= \sum_{n=-1}^{+1} x_1(n) e^{-j\omega n} \\ &= e^{j\omega} + 1 + e^{-j\omega} = 1 + 2 \cos \omega \end{aligned}$$

$$\begin{aligned} X_3(\omega) &= (1 + 2 \cos \omega)^2 = 1 + 4 \cos^2 \omega + 4 \cos \omega \\ &= 3 + 4 \cos \omega + 2 \cos 2\omega \end{aligned}$$

$$= 3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega})$$

$$X_3(\omega) = e^{j2\omega} + 2e^{-j\omega} + 3 + 2e^{j\omega} + e^{-j2\omega}$$

$$x_3(n) = \{1, 2, 3, 2, 1\}$$

①

$$\frac{\text{DFT}}{N-1} X(n) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

FT equation

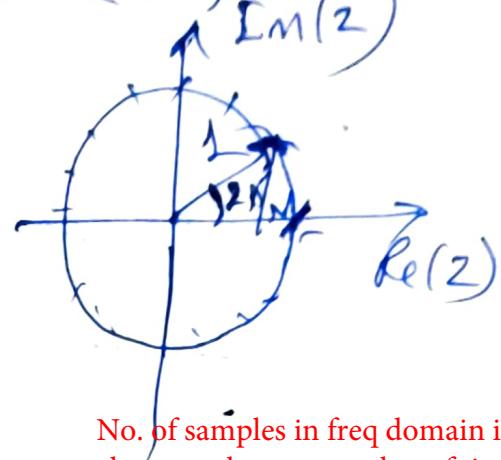
$x(n)$ is a finite duration sequence
in the range $0 \leq n \leq N-1$

→ $X(\omega)$ is sampled.

→ N equally spaced samples are chosen

→ Sampling Interval $\frac{2\pi}{N}$

$$\Rightarrow \omega = \frac{2\pi}{N} k, \quad 0 \leq k \leq N-1$$



No. of samples in freq domain is chosen to be same as that of time domain for convenience purpose.
It is not a hard and fast rule.

$$X(\omega) \Big|_{\omega = \frac{2\pi}{N} k} = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} kn}$$

$$\Rightarrow X(k) = \text{DFT} \{x(n)\} = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

W_N is a complex quantity.

$$W_N = e^{-j \frac{2\pi}{N}}$$

— Twiddle factor

ω_N is periodic with period N (2)

$$\omega_N^{kn+N} = \omega_N^{kn} \cdot \omega_N^N$$

$$\omega_N^N = e^{-j \frac{2\pi}{N} \cdot N} = e^{-j 2\pi} = 1.$$

∴ $\omega_N^{kn+N} = \omega_N^{kn}$

Lemme

imp

$$\sum_{n=0}^{N-1} \omega_N^{kn} = N f(k) = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases}$$

Proof

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \text{ if } a \neq 1$$

$$\sum_{n=0}^{N-1} \omega_N^{kn} = \frac{1-\omega_N^{kN}}{1-\omega_N^k} = \frac{(-)}{1-\omega_N^k} = 0 \text{ if } k \neq 0$$

$$\sum_{n=0}^{N-1} \omega_N^0 = \sum_{n=0}^{N-1} 1 = \underline{\underline{N}}$$

Inverse DFT

(3)

$$x(n) = \text{IDFT} \{X(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn}$$

$$0 \leq n \leq N-1$$

Periodicity of $X(k)$ and $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) w_N^{kn}$$

Periodic function becomes non-periodic when transformed to other domain. Vice-versa also holds good.

Hence $x(n)$ and $X(k)$ are implicitly periodic (only due to the definitions of DFT and IDFT).

$$\Rightarrow X(k+N) = \sum_{n=0}^{N-1} x(n) w_N^{(k+N)n}$$

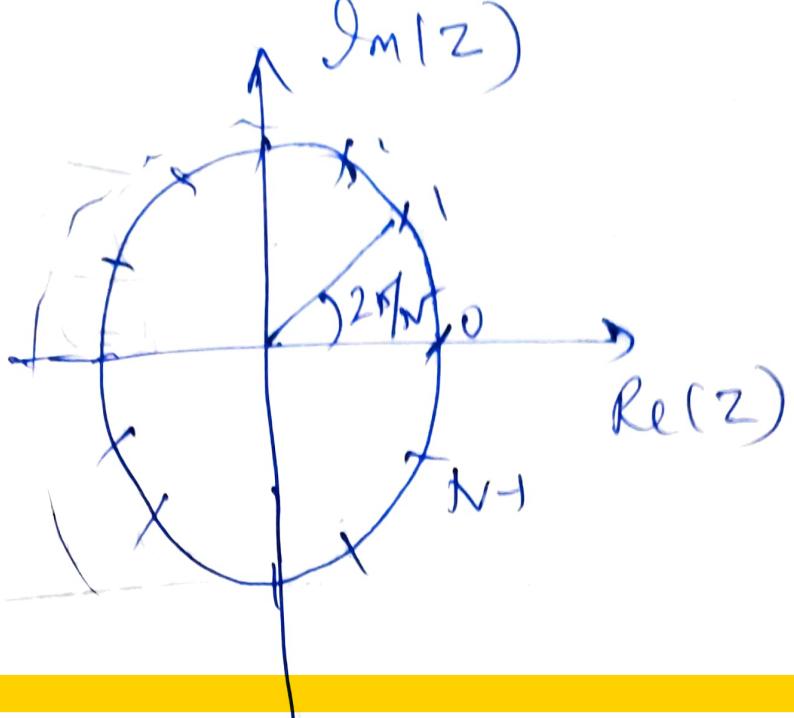
$$= \sum_{n=0}^{N-1} x(n) w_N^{kn} \cdot \cancel{w_N^{Nn}} \quad \left| \begin{array}{l} w_N^{nN} \\ = e^{\frac{j2\pi}{N} nN} \\ = 1 \end{array} \right.$$

$$= X(k) \quad \text{--- A}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn}$$

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-k(n+N)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn} \cdot \cancel{w_N^{-kN}} = x(n) \quad \text{--- B}$$



④

Prob:

1) Compute 4 point DFT of the sequence $x(n) = \{1, 2, 3, 4\}$

$\text{Purely causal sequence}$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(0) = \sum_{n=0}^3 x(n) = 1 + 2 + 3 + 4 = 10$$

$$\begin{aligned} X(1) &= \sum_{n=0}^3 x(n) W_4^n = x(0) + x(1) W_4^1 + \\ &\quad x(2) W_4^2 + x(3) W_4^3 \\ &= 1 + 2xj + \cancel{3x-1} + 4xj \end{aligned}$$

$$\begin{aligned} X(2) &= \sum_{n=0}^3 x(n) W_4^{2n} = x(0) + x(1) W_4^2 + \\ &\quad x(2) W_4^4 + x(3) W_4^6 \end{aligned}$$

$$\omega_4^0 = 1 \rightarrow \omega_4^1 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$\omega_4^2 = e^{-j\frac{2\pi}{4} \times 2} = e^{-j\pi} = -1$$

$$\omega_4^3 = e^{-j\frac{2\pi}{4} \times 3} = +j$$

1) Periodicity property of ω_N^{kn}

$$\omega_N^{kn+n} = \omega_N^{kn}$$

$$\text{LHS} = \omega_N^{kn} \cdot \omega_N^n = \omega_N^{kn} e^{-j\frac{2\pi}{N} \cdot N} = \omega_N^{kn}$$

2) Symmetry property of ω_N^{kn}

$$\omega_N^{kn+N/2} = -\omega_N^{kn}$$

$$\text{LHS} : \omega_N^{kn} \cdot \omega_N^{N/2} = \omega_N^{kn} e^{-j\frac{2\pi}{N} \times \frac{N}{2}} = -\omega_N^{kn}$$

$$\text{if } N=4$$

$$\omega_4^0 = 1$$

$$\omega_4^1 = -j$$

$$\omega_4^2 = -1$$

$$\omega_4^3 = j$$

$$\omega_4^4 = \omega_4^0 = 1$$

$$\begin{aligned} \omega_4^6 &= \omega_4^2 \\ \omega_4^5 &= \omega_4^7 = \omega_4^1 \end{aligned}$$

(6)

$$X(2) = 1 + 2x^{-1} + 3x^1 + 4x^{-1}$$

$$= 1 - 2 + 3 - 4 = -2$$

$$X(3) = \sum_{n=0}^3 x(n) W_3^{3n} = x(0) + x(1) W_3^3$$

$$+ x(2) W_3^6 + x(3) W_3^9$$

$$= 1 + 2x^j + 3x^{-1} + 4x^{-j}$$

$$= 1 + 2j - 3 - 4j$$

$$= -2 - 2j$$

$$\therefore X(k) = \{ 1, -2 + 2j, -2, -2 - 2j \}$$

Class-7

- 2) Compute 4 point DFT of the
sequence $x(n) = \{ 1, 2, 5, 4 \}$

Unless specified, we assume that length of sequence in time domain is same as that of in frequency domain. Hence "compute N point DFT" is same as "Compute DFT". Unless mentioned, we also assume causal sequence.

①

1) Compute N point DFT of the sequence $x(n) = a^n \quad 0 \leq n \leq N-1$

$$\text{Solu}^y \quad X(k) = \sum_{n=0}^{N-1} x(n) \omega_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a \omega_N^k)^n \quad \left| \begin{array}{l} \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \\ \text{if } a \neq 1 \end{array} \right.$$

$$= \frac{1 - a^N}{1 - a \omega_N^k}$$

$$= \frac{1 - a^N}{1 - a \omega_N^k}$$

2) Compute N point DFT of the sequence $x(n) = 1 \quad 0 \leq n \leq N-1$

$$\text{Solu}^y \quad X(k) = \sum_{n=0}^{N-1} x(n) \omega_N^{kn} = \sum_{n=0}^{N-1} 1 \cdot \omega_N^{kn}$$

$$= \frac{1 - \omega_N^{kN}}{1 - \omega_N^k} = 0 \quad \text{if } k \neq 0$$

(2)

when $k = 0$

$$X(0) = \sum_{n=0}^{N-1} |x| = N$$

$$\therefore X(k) = \begin{cases} N & \text{for } k=0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$$= N \delta(k)$$

3) Compute Inverse DFT of
the 4-dimensions $X(k) = \{2, 1+j, 0, 1-j\}$

Sln

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn} \quad 0 \leq n \leq N-1$$

$$x(n) = \frac{1}{4} \sum_{k=0}^3 X(k) w_N^{-kn} \quad 0 \leq n \leq 3$$

$$x(0) = \frac{1}{4} [2 + (1+j) + (1-j)] = 1$$

$$x(1) = 0$$

$$x(2) = 0$$

$$x(3) = 1$$

$$x(n) = \{1, 0, 0, 1\}$$

$$\begin{cases} w_4^{-1} = e^{-j \frac{2\pi}{4} \times 1} \\ w_4^{-2} = j \\ w_4^{-3} = -j \\ w_4^{-4} = j \end{cases}$$

dsp 5

①

Matrix Relation for Computing DFT

i) Compute 4 point DFT of the sequence

$$x(n) = \{1, 2, 1, 0\}$$

Soluⁿ

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(k) = \sum_{n=0}^3 x(n) W_4^{kn}$$

$$X(0) = \sum_{n=0}^3 x(n) = x(0) + x(1) + x(2) + x(3)$$

$$X(1) = \sum_{n=0}^3 x(n) W_4^n = x(0) W_4^0 + x(1) W_4^1 \\ + x(2) W_4^2 + x(3) W_4^3$$

$$X(2) = \sum_{n=0}^3 x(n) W_4^{2n} = x(0) W_4^0 + x(1) W_4^2 \\ + x(2) W_4^4 + x(3) W_4^6$$

$$X(3) = \sum_{n=0}^3 x(n) W_4^{3n} = x(0) W_4^0 + x(1) W_4^3 \\ + x(2) W_4^6 + x(3) W_4^9$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \quad (2)$$

$$\omega_4^1 = e^{-j\frac{2\pi}{4}x1} = e^{-j\frac{\pi}{2}} = -j$$

$$\omega_4^0 = 1$$

$$\omega_4^2 = -\omega_4^0 = -1$$

$$\omega_4^3 = -\omega_4^1 = +j$$

$$\omega_4^4 = \omega_4^0 = 1$$

$$\omega_4^6 = \omega_4^2 = -1$$

$$\omega_4^9 = \omega_4^5 = \omega_4^1 = -j$$

* Using symmetry
and periodicity

Property of ω_N^{kn}

$$\omega_N^{kn+N} = \omega_N^{kn}$$

$$\omega_N^{kn+N/2} = -\omega_N^{kn}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j^2 \\ 0 \\ j^2 \end{bmatrix}$$

$$\therefore X(k) = \{ 4, -j^2, 0, j^2 \}$$

Eg. 2 find 4 point DFT of the sequence ③

$$x(n) = \{1, 2, 3, 1\}$$

Soln

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k=0, 1, -N$$

$$X(k) = \sum_{n=0}^3 x(n) W_4^{kn} \quad k=0, 1, -3$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$\therefore X(k) = \{7, -2-j, 1, -2+j\}$$

(4)

It helps to compute IDFT
using DFT code itself
which saves memory

IDFT using DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad n = 0, 1, \dots, N-1$$

$$\begin{aligned} x^*(n) &= \frac{1}{N} \left[\sum_{k=0}^{N-1} x(k) W_N^{-kn} \right]^* \\ &= \frac{1}{N} \sum_{k=0}^{N-1} x^*(k) W_N^{kn} \end{aligned}$$

$$= \frac{1}{N} \text{DFT} [x^*(k)]$$

$$\therefore x(n) = \frac{1}{N} \left\{ \text{DFT} [x^*(k)] \right\}^*$$

Ex. Find IDFT of 4 point sequence

$$X(k) = \{ 4, -j^2, 0, j^2 \}$$

- using defining equation of IDFT
- using DFT

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$$a) \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) \omega_N^{-kn} \quad n = 0, 1, \dots, N-1$$

$$x(n) = \frac{1}{4} \sum_{k=0}^3 x(k) \omega_4^{-kn} \quad n = 0, 1, 2, 3$$

$$x(0) = \frac{1}{4} [x(0) + x(1) + x(2) + x(3)]$$

$$x(1) = \frac{1}{4} \sum_{k=0}^3 x(k) \omega_4^{-k} \stackrel{=} 1$$

$$= \frac{1}{4} [x(0) + x(1) \omega_4^{-1} + x(2) \omega_4^{-2} + x(3) \omega_4^{-3}]$$

$$x(2) = \frac{1}{4} \sum_{k=0}^3 x(k) \omega_4^{-2k} \stackrel{=} 2$$

$$= \frac{1}{4} [x(0) + x(1) \omega_4^{-2} + x(2) \omega_4^{-4} + x(3) \omega_4^{-6}]$$

$$x(3) = \frac{1}{4} \sum_{k=0}^3 x(k) \omega_4^{-3k} \stackrel{=} 1$$

$$= \frac{1}{4} [x(0) + x(1) \omega_4^{-3} + x(2) \omega_4^{-6} + x(3) \omega_4^{-9}]$$

$$= 0$$

(6)

$$b) X^*(k) = \{ 4, j2, 0, -j2 \}$$

$$\text{DFT} [X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & j & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ j \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{aligned} x(n) &= \text{IDFT} [X(k)] = \frac{1}{N} \left[\text{DFT} \{X^*(k)\} \right]^* \\ &= \frac{1}{4} \begin{bmatrix} 4 \\ j \\ 0 \\ 4 \end{bmatrix}^* = \{ 1, 2, 1, 0 \} \quad \uparrow \end{aligned}$$

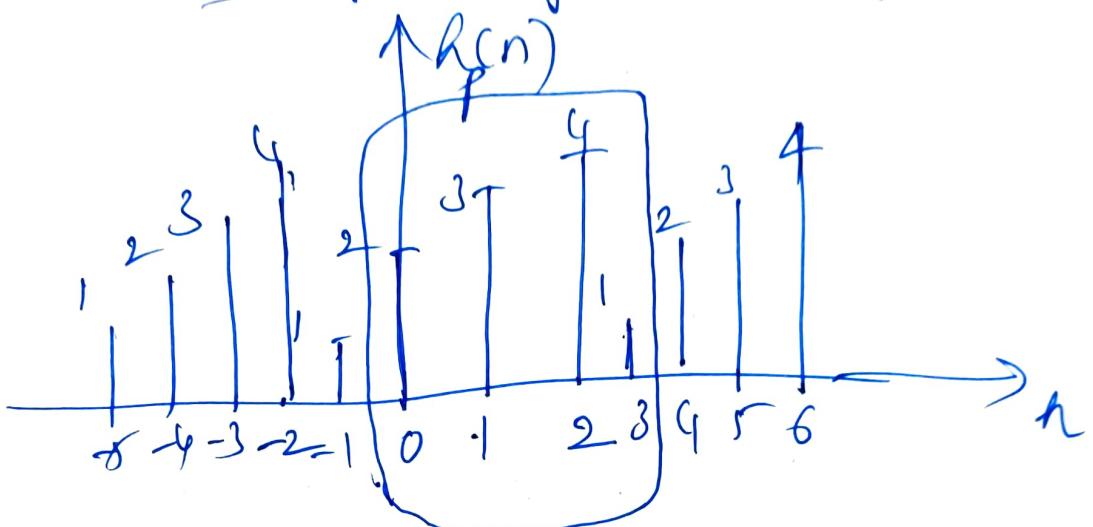
Prob: find IDFT of $\{ 7, -2j, 1, -2j \}$

- using
- Defining equation of IDFT
 - using DFT

DFT of a non-causal sequence

Q 1) Determine 4 point DFT of the sequence $h(n) = \{1, 2, 3, 4\}$

Using Implicit periodicity property



$$h'(n) = \{2, 3, 4, 1\}$$

$$H(k) = \sum_{n=0}^3 h(n) W_4^{kn} \quad 0 \leq k \leq 3$$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

(2)

DFT of a non-causal sequence

Ex

Determine DFT of the sequence

$$h(n) = \begin{cases} \frac{1}{2} & -2 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

plot the magnitude and phase response.

Soluⁿ $h(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right\}$

$$h'(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2} \right\}$$

$$H(k) = \left\{ \frac{5}{2}, 1.207, -\frac{1}{2}, -0.207, \frac{1}{2}, -0.207, -\frac{1}{2}, 1.207 \right\}$$

Dont use matrix method if length is greater than 4.

We try to make the length in powers of 2 so that computations becomes easier by being able to use symmetry and periodicity property efficiently.

Properties of DFT

1. Conjugate Symmetry

If $x(n)$ is real and then its DFT

$$X(k) = X^*(N-k)$$

Proof: $X(k) = \text{DFT} \{x(n)\}$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0, 1, \dots, N-1$$

$$X^*(k) = \left\{ \sum_{n=0}^{N-1} x(n) W_N^{kn} \right\}^*$$

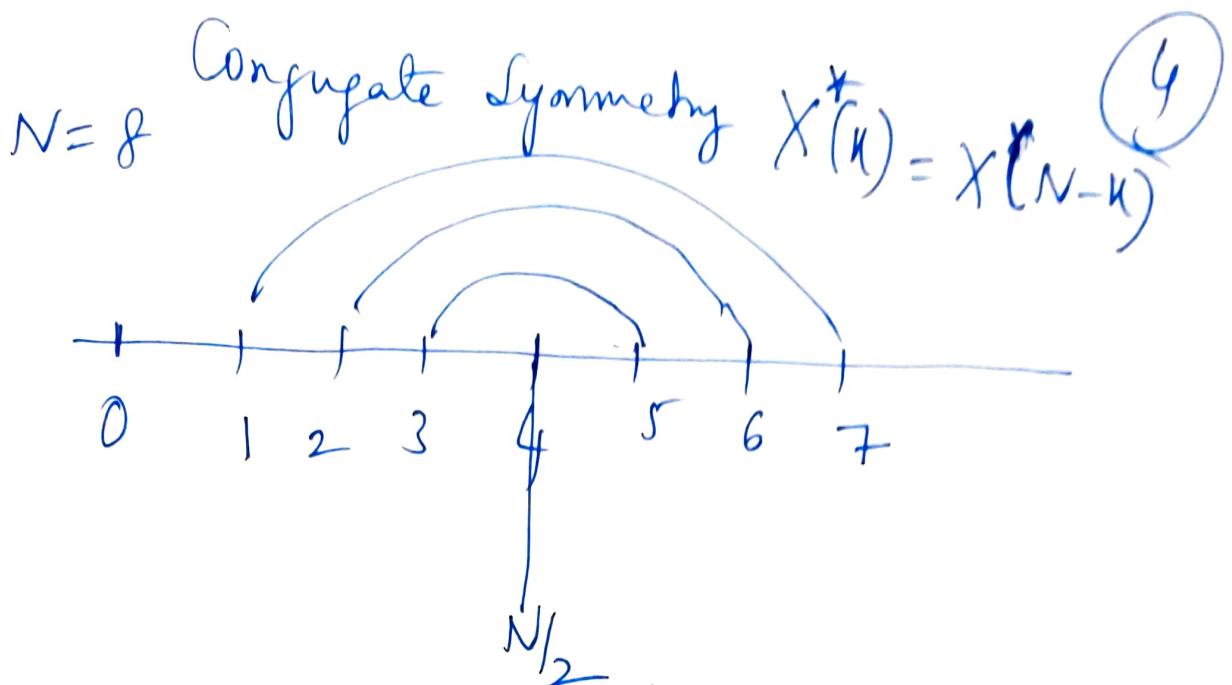
$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \cdot W_N^{Nn} \quad \begin{cases} \text{since} \\ W_N^{nN} = 1 \end{cases}$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{(N-k)n} \quad \begin{cases} \text{since} \\ x(n) \text{ is real} \\ x^*(n) = x(n) \end{cases}$$

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n} = X(N-k)$$

proved



Prob: Compute 5 point DFT of the

Sequence $x(n) = \{1, 0, 1, 0, 1\}$ and

Hence Verify symmetry property.

Soln)

$$\omega_N = e^{-j \frac{2\pi}{N}} ; N = 5 ; \omega_5 = e^{-j \frac{2\pi}{5}}$$

$$\omega_5^0 = 1$$

$$\omega_5^1 = e^{-j \frac{2\pi}{5}}$$

$$\omega_5^2 = e^{-j \frac{2\pi}{5} \times 2} = 0.309 - j 0.951$$

$$\omega_5^3 = e^{-j \frac{2\pi}{5} \times 3} = -0.809 - j 0.587$$

$$\omega_5^4 = e^{-j \frac{2\pi}{5} \times 4} = -0.809 + j 0.587$$

$$= 0.309 + j 0.951$$

$$X(k) = \sum_{n=0}^4 x(n) \omega_5^{kn}$$

$$k = 0, 1, 2, 3, 4$$

$$= x(0) + x(1) \omega_5^k + x(2) \omega_5^{2k} + x(3) \omega_5^{3k} + x(4) \omega_5^{4k}$$

(5)

$$X(k) = 1 + \omega_5^{2k} + \omega_5^{4k}$$

$$X(0) = 1 + 1 + 1 = 3$$

$$X(1) = 1 + \omega_5^2 + \omega_5^4 = 0.5 + j0.364$$

$$\begin{aligned} X(2) &= 1 + \omega_5^4 + \omega_5^8 = 1 + \omega_5^4 + \omega_5^3 \\ &= 0.5 + j1.538 \end{aligned}$$

$$\begin{aligned} X(3) &= 1 + \omega_5^6 + \omega_5^{12} = 1 + \omega_5^1 + \omega_5^2 \\ &= 0.5 - j1.538 \end{aligned}$$

$$\begin{aligned} X(4) &= 1 + \omega_5^8 + \omega_5^{16} = 1 + \omega_5^3 + \omega_5^1 \\ &= 0.5 - j0.364 \end{aligned}$$

Verification

Congnate Symmetry property

$$X^*(k) = X(n-k)$$

$$X^*(k) = X(5-k)$$

$$\therefore X^*(1) = X(4) \text{ and}$$

$$X^*(2) = X(3)$$

Hence verified.

Prob 2 ^{HW} Compute 5 point DFT of the
Sequence $x(n) = \{1, 2, 3, 4, 5\}$

and hence verify Conjugate Symmetry
property.

Properties of DFT2) Linearity

of DFT $\{x_1(n)\} \rightarrow X_1(k)$

and DFT $\{x_2(n)\} \rightarrow X_2(k)$

then DFT $\{a x_1(n) + b x_2(n)\} \rightarrow a X_1(k) + b X_2(k)$

Example

1) find 4 point DFT of the sequence

$$x(n) = \cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{4}n\right)$$

using Linearity property.

Solution Let $x(n) = x_1(n) + x_2(n)$

$$x_1(n) = \cos\left(\frac{\pi}{4}n\right) = \left\{ 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\}$$

$$x_2(n) = \sin\left(\frac{\pi}{4}n\right) = \left\{ 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}} \right\}$$

$$\mathcal{W}_4^0 = 1 \quad \mathcal{W}_4^1 = -j$$

$$X_1(k) = \text{DFT } \{x_1(n)\}$$

$$X_2(k) = \text{DFT } \{x_2(n)\}$$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(2)

$$X_1(k) = \left\{ \begin{array}{l} 1 \\ \uparrow \\ 1-j1.414 \\ 1 \\ 1+j1.414 \end{array} \right\}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

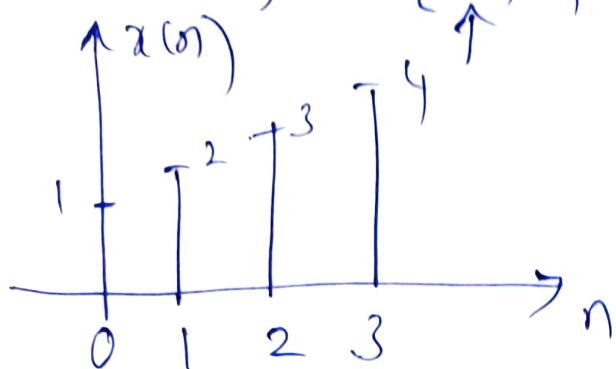
$$X_2(k) = \left\{ \begin{array}{l} 2 \cdot 1.414 \\ \uparrow \\ -1 \\ -0 \cdot 414 \\ -1 \end{array} \right\}$$

$$\begin{aligned} X(k) &= \text{DFT} \{ x_1(n) \} + \text{DFT} \{ x_2(n) \} \\ &= X_1(k) + X_2(k) \\ &= \left\{ \begin{array}{l} 3 \cdot 1.414 \\ \uparrow \\ -j1.414 \\ 0.586 \\ j1.414 \end{array} \right\} \end{aligned}$$

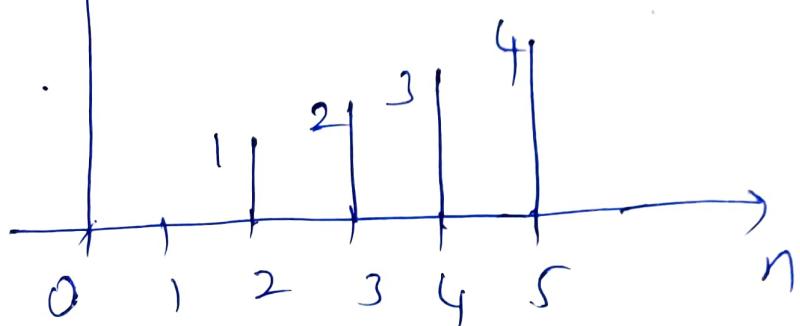
Concept of Circular Shift

③

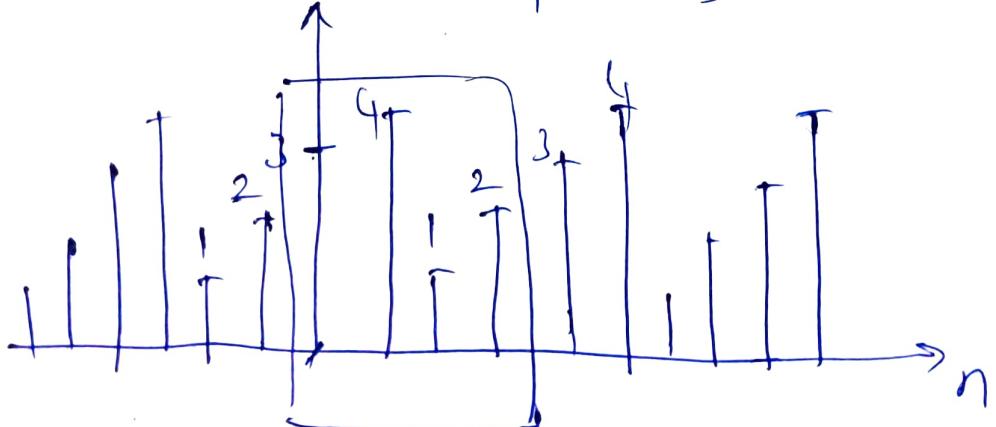
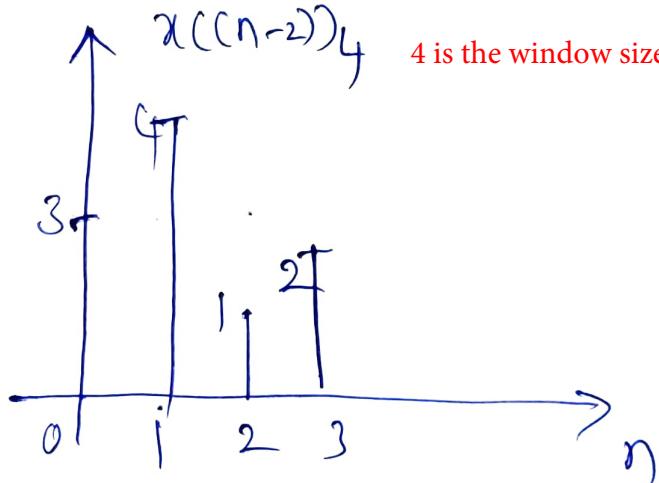
Let $x(n) = \{1, 2, 3, 4\}$



$x(n-2) \Rightarrow$



$x((n-2))_4$ 4 is the window size



(4)

Eg: 2) Let $x(n)$ be an 8 point sequence
~~defined as~~ defined as $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

- a) Illustrate $x(n)$ and $x(n-2)$ and $x((n-2))_8$ graphically.
- b) find $X(k)$
- c) find DFT $\{x((n-2))_8\}$ using time shift property.

i) Let $x(n)$ be a 4 point sequence
defined as $x(n) = \{4, 5, -6, 7\}$

Represent the following graphically

- (a) $x(n)$
- (b) $x(n-3)$
- (c) $x((n-3))_4$
- (d) find $X(k)$.

3) Circular time shift:

If $DFT \{x(n)\} \rightarrow X(k)$ then,

$$DFT \{x((n-m))_N\} \rightarrow W_N^{mk} X(k)$$

Proof: Using the defining equation of IDFT,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$x((n-m))_N = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-k(n-m)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left\{ X(k) W_N^{km} \right\} W_N^{-kn}$$

$$x((n-m))_N = IDFT \left\{ X(k) W_N^{km} \right\}$$

$$\Rightarrow DFT \left\{ x((n-m))_N \right\} \rightarrow X(k) W_N^{km}$$

(6)

Q: find 4 point DFT of the sequence

$x(n) = \{1, 2, 3, 1\}$. Using circular time shift

property, find the DFT of the sequence

$$y(n) = x((n-2))_4$$

Sol: $X(k) = \text{DFT} \{x(n)\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1+j & \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

Circular time shift property,

$$\text{DFT} \{x((n-m))_N\} = \omega_N^{mX} X(k)$$

$$\therefore \text{DFT} \{x((n-2))_4\} = \omega_4^{2K} X(k) = Y(k)$$

$$Y(0) = \omega_4^0 X(0) = 7.$$

$$Y(1) = \omega_4^{2 \times 1} X(1) = \omega_4^2 X(1) = +2+j$$

$$Y(2) = \omega_4^{2 \times 2} X(2) = \omega_4^4 X(2) = 1$$

$$Y(3) = \omega_4^{3 \times 2} X(3) = \omega_4^6 X(3) = 2-j$$

$$\therefore Y(k) = \{7, 2+j, 1, 2-j\}$$

4) Circular frequency shift

7

If DFT $\{x(n)\} = X(k)$, then
 IDFT $\{X((k-l))_N\} = \sum_N^{ln} x(n)$

Proof:

$$\text{DFT } \{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$0 \leq k \leq N-1$$

$$\begin{aligned} X((k-l))_N &= \sum_{n=0}^{N-1} x(n) W_N^{(k-l)n} \\ &= \sum_{n=0}^{N-1} \{x(n) W_N^{-ln}\} W_N^{kn} \end{aligned}$$

$$X((k-l))_N \doteq \text{DFT } \{x(n) \cdot W_N^{-ln}\}$$

(i) find DFT of the sequence

$x(n) = \{1, 2, 1, 0\}$. Using circular

frequency shift property, find $y(n)$ if

$$y(k) = X((k-2))_4$$

Solution

$$X(k) = \text{DFT} \{ x(n) \}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & j & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j^2 \\ 0 \\ j^2 \end{bmatrix}$$

$$Y(k) = X((k-2))_2$$

$$= \{ \underset{\uparrow}{0}, j^2, 4, -j^2 \}$$

$$y(n) = \sum_N^{-\ln} x(n) = \sum_4^{-2n} x(n)$$

$$y(0) = \sum_4^0 x(0) = 1 \times 1 = 1$$

$$y(1) = \sum_4^{-2} x(1) = -1 \times 2 = -2$$

$$y(2) = \sum_4^{-4} x(2) = 1 \times 1 = 1$$

$$y(3) = \sum_4^{-6} x(3) = -1 \times 0 = 0$$

$$y(n) = \{ 1, -2, 1, 0 \}$$

IDFT {Y(k)}

5) DFT of a Complex Conjugate Sequence

Let $x(n)$ be a complex sequence
of DFT $\{x(n)\} = X(k)$ then

$$\text{DFT } \{x^*(n)\} = X^*(-k)_N = X^*(N-k)$$

Proof: $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$ $0 \leq k \leq N-1$

$$X^*(k) = \left\{ \sum_{n=0}^{N-1} x(n) W_N^{kn} \right\}^*$$

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \quad \text{--- (A)}$$

Changing k to $-k$, we get

$$X^*(-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{kn}$$

$$\underline{X^*(-k) = \text{DFT } \{x^*(n)\}}$$

Changing k to $N-k$ in (A), we get

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n}$$

②

$$= \sum_{n=0}^{N-1} x^*(n) e^{-j\frac{2\pi}{N} kn}$$

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n) e^{j\frac{2\pi}{N} kn}$$

$\therefore X^*(N-k) = \text{DFT} \{x^*(n)\} = X^*(-k)$

Prob: 5 point DFT of a Complex sequence

$x(n)$ is given by

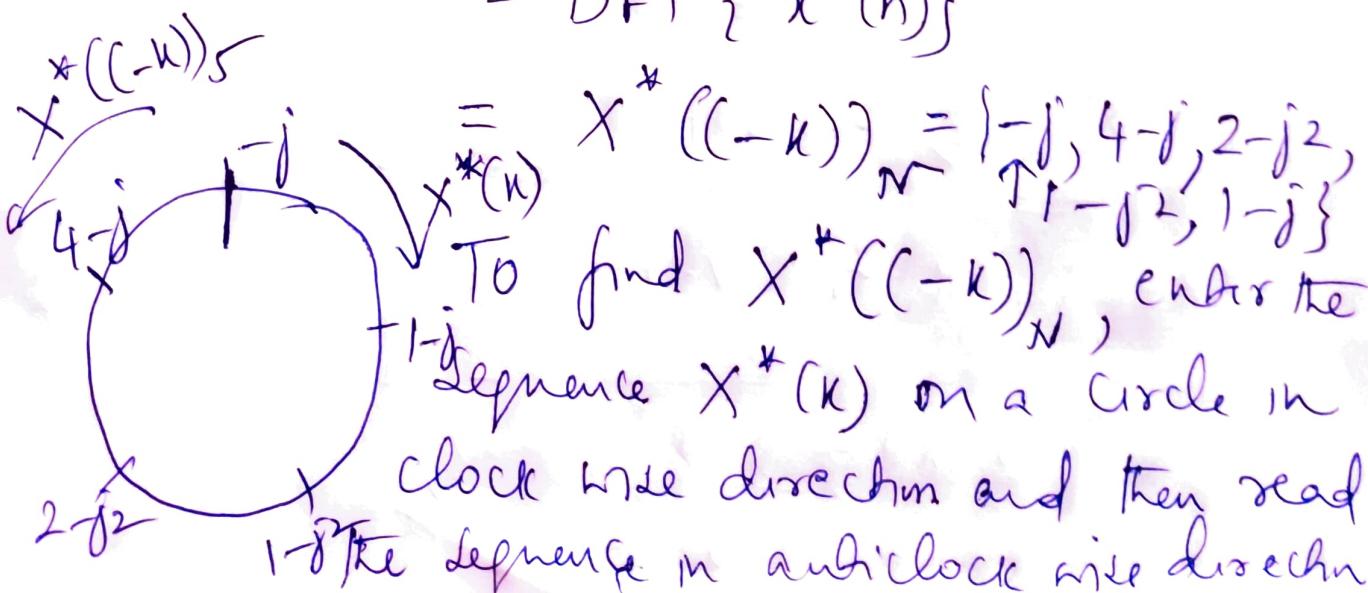
$$X(k) = \left\{ j, 1+j, 1+j^2, 2+j^2, 4+j \right\}$$

Find $Y(k)$ if $y(n) = x^*(n)$.

Soln

$$Y(k) = \text{DFT} \{y(n)\}$$

$$= \text{DFT} \{x^*(n)\}$$



6) Circular Convolution in time

(3)

$$y(n) = x(n) \otimes h(n)$$

$$= x(n) \circledast_N h(n)$$

$$= x(n) \otimes_N h(n)$$

$$y(n) = \sum_{k=0}^{N-1} x((n-k))_N h((n-k))_N$$

$$k_N = 0$$

$$\Rightarrow y(k) = x(k) H(k)$$

Proof:

$$y(n) = \text{IDFT} \{ y(k) \}$$

$$= \text{IDFT} \{ x(k) \cdot H(k) \}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x(k) H(k) w_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{l=0}^{N-1} x(l) w_N^{lk} \right\} \left\{ \sum_{p=0}^{N-1} h(p) w_N^{pk} \right\}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} x(l) \sum_{p=0}^{N-1} h(p) \sum_{k=0}^{N-1} w_N^{(l+p-k)k}$$

$$= \sum_{l=0}^{N-1} x(l) h(N-l)$$

(9)

$$\int_{k=0}^{N-1} (l+p-k) k = N$$

if $l+p-k = 0$

$\therefore p = k - l$

$$= \frac{1}{N} \sum_{l=0}^{N-1} x(l) h(N-l) \cdot N$$

$$y(n) = \sum_{l=0}^{N-1} x(l) h(N-l)$$

$$= x(n) \circledast h(n)$$

(5)

Prob Find $x_1(n) \otimes x_2(n)$ if

$$x_1(n) = \{1, 2, 3, 1\} \text{ and}$$

$$x_2(n) = \{4, 3, 2, 2\}$$

Length of the result will be equal to the length of the larger input sequence among the two inputs.

- a) using time domain approach
- b) frequency domain approach.

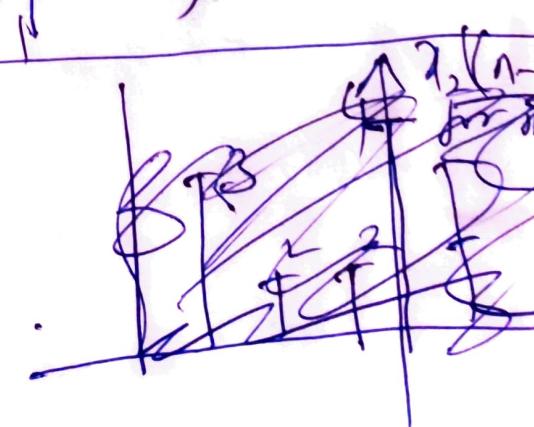
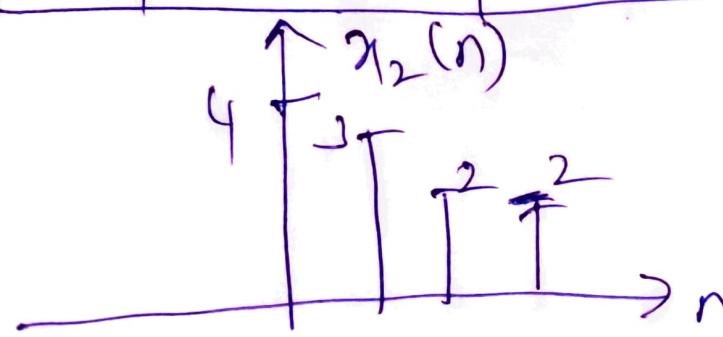
Soln

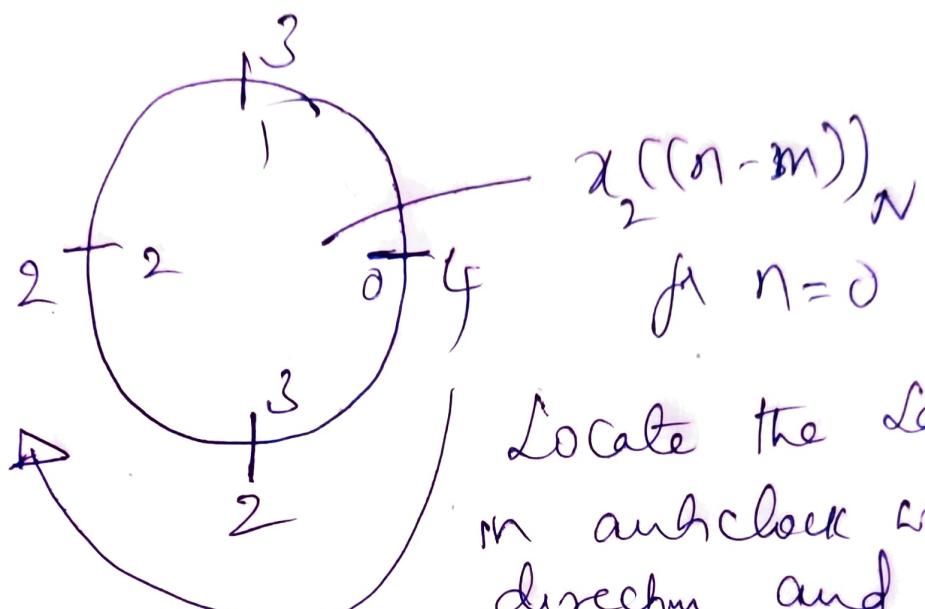
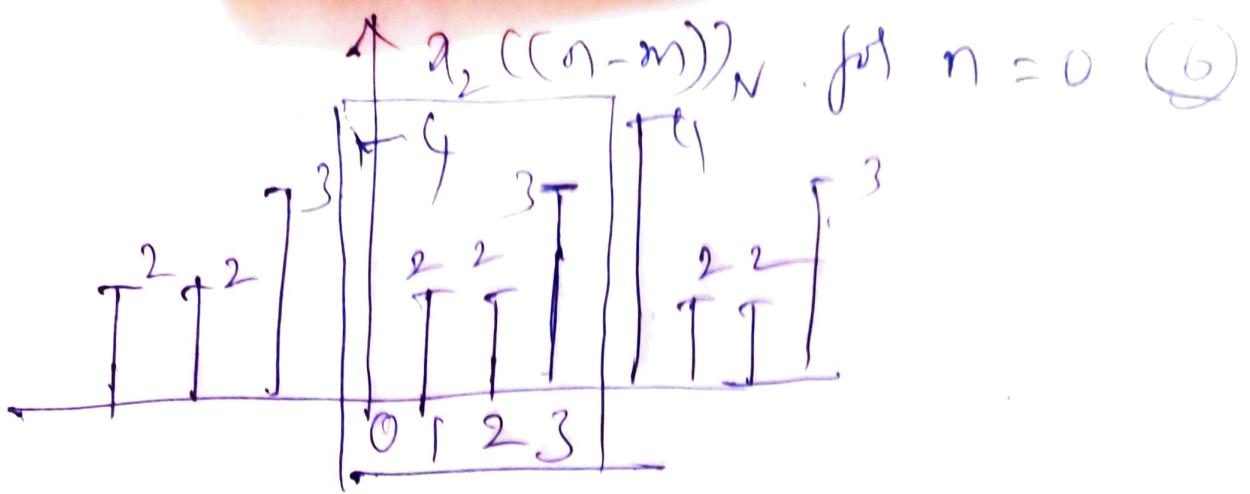
a) $y(n) = \sum_{m=0}^{N-1} x_1((m))_N x_2((n-m))_N$

$$0 \leq n \leq N-1$$

(ii) Tabular approach

n	$x_1((m))_N$	$x_2((n-m))_N$	$y(n)$
0	(1 2 3 1)	(4 2 2 3)	17
1	(1 2 3 1)	(3 4 2 2)	19
2	(1 2 3 1)	(2 3 4 2)	22
3	(1 2 3 1)	(2 2 3 4)	19





Locate the samples
in anticlockwise
direction and read
in clockwise direction.

Class-12

(ii) Matrix approach

Circulant matrix.

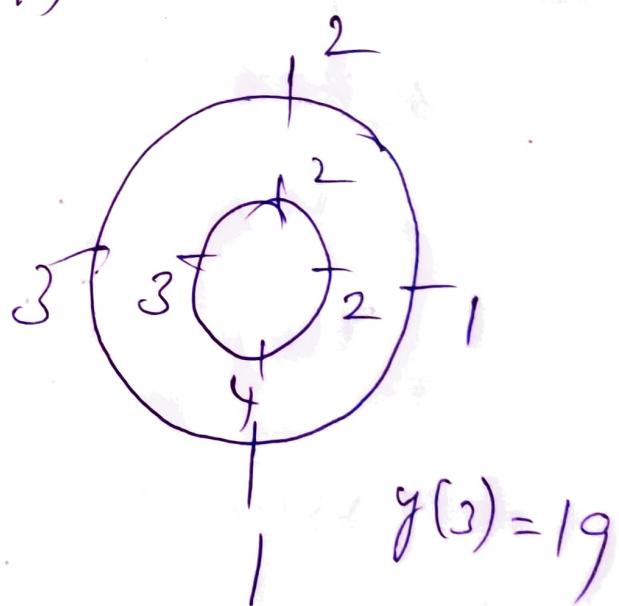
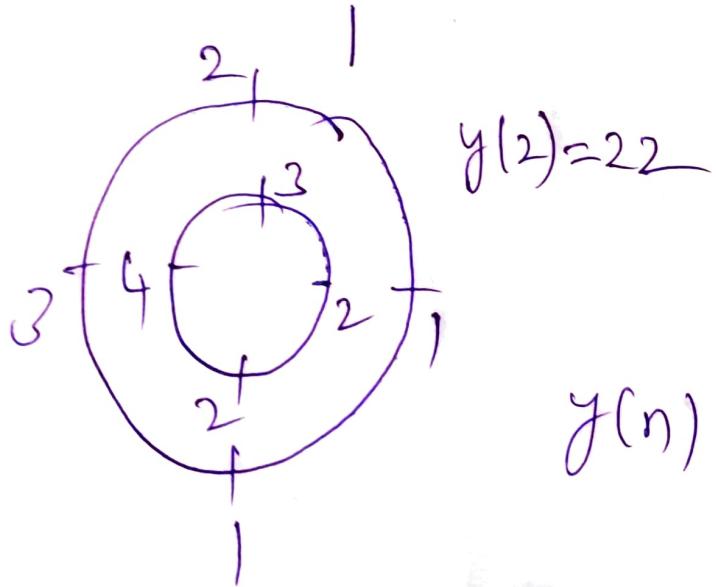
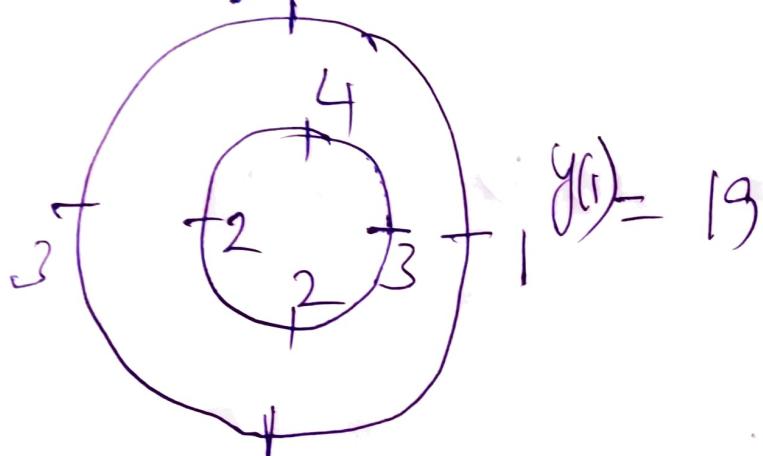
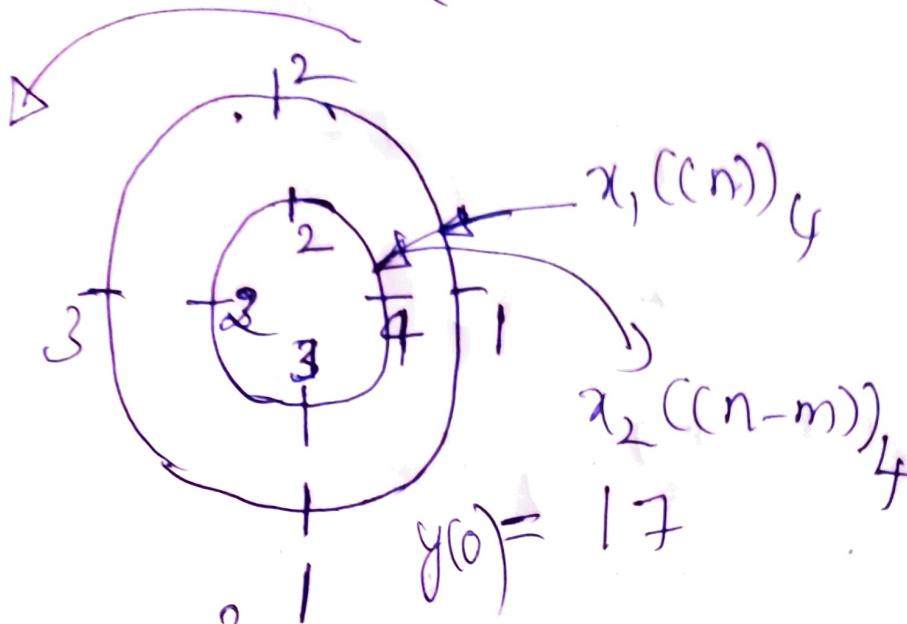
$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

(iii) Circles method

7

$$x_1(n) = \{1, 2, 3, 1\}$$

$$x_2(n) = \{4, 3, 2, 2\}$$



$$y(n) = \{17, 19, 22, 19\}$$

b) frequency domain approach

$$X_1(k) = \sum_{n=0}^3 x_1(n) w_4^{kn} = 1 + 2w_4^k + 3w_4^{2k} + w_4^{3k}$$

$$X_2(k) = 4 + 3w_4^k + 2w_4^{2k} + 2w_4^{3k}$$

$$Y(k) = X_1(k) X_2(k)$$

We get 16 terms, group them to get 4 terms.

$$= 17 + 19w_4^k + 22w_4^{2k} + 19w_4^{3k}$$

$$y(n) = \{ \underset{\uparrow}{17}, 19, 22, 19 \}$$

HW Find $y(n) = x(n) \otimes h(n)$ using

a) time domain approach

b) frequency domain approach

$$x(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$$

$$h(n) = \{ \underset{\uparrow}{5}, 6, 7, 8 \}$$

(1)

dip 9

1) Find Circular Convolution of the following sequences using

a) time domain approach

b) frequency domain approach

$$x_1(n) = \{ \underset{\uparrow}{2}, 3, 1, 1 \} \text{ and } x_2(n) = \{ \underset{\uparrow}{1}, 3, 5, 3 \}$$

2) find Circular Convolution of the following sequences

$$x_1(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$$

$$x_2(n) = \{ \underset{\uparrow}{5}, 6, 7 \}$$

Make length equal by adding one 0.

Soln Make the lengths equal

$$x_1(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$$

$$x_2(n) = \{ \underset{\uparrow}{5}, 6, 7, 0 \}$$

(2)

3) for the sequences $x_1(n) = \{ \underset{\uparrow}{2}, 1, 1, 2 \}$
 and $x_2(n) = \{ \underset{\uparrow}{1}, -1, 1, -1 \}$

- Compute Circular Convolution $[x_3(n)]$
- Compute linear convolution $[x_4(n)]$
- Compute linear convolution using
Circular Convolution $[x_5(n)]$
- Compute Circular Convolution using
linear convolution $[x_6(n)]$

Soln) a)

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

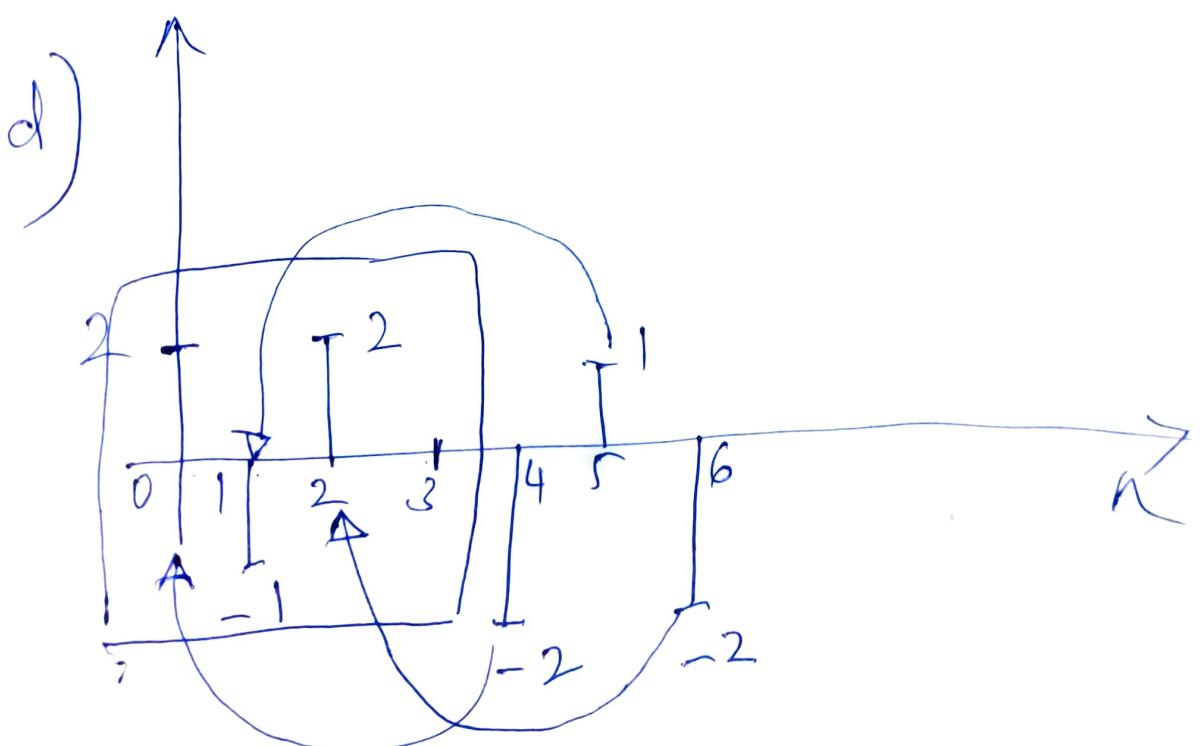
$$b) \begin{bmatrix} x_4(0) \\ x_4(1) \\ x_4(2) \\ x_4(3) \\ x_4(4) \\ x_4(5) \\ x_4(6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \\ -2 \\ 1 \\ -2 \end{bmatrix}$$

Circulant matrix. It requires same length which is equal to the required no. of samples for linear convolution(which is 7)

$$c) \begin{bmatrix} x_5(0) \\ x_5(1) \\ x_5(2) \\ x_5(3) \\ x_5(4) \\ x_5(5) \\ x_5(6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$x_1(n) = \{2, 1, 1, 2, 0, 0, 0\}$$

$$x_2(n) = \{1, -1, 1, -1, 0, 0, 0\}$$



$$x_6(n) = \{2+2, -1+1, 2-2, 0\} \\ = \{0, 0, 0, 0\}$$

HD Repeat (3) for the following
sequences

$$x_1(n) = \{ \underset{\uparrow}{1}, 1, 2, 5 \}$$

$$x_2(n) = \{ \underset{\uparrow}{3}, 2, 4 \}$$

1) For the sequences $x_1(n) = \cos\left(\frac{2\pi n}{N}\right)$

and $x_2(n) = \sin\left(\frac{2\pi n}{N}\right) \quad 0 \leq n \leq N-1$.

find the N point Circular Convolution

$$x_3(n) = x_1(n) \circledast x_2(n).$$

We can't use circulant matrix as we don't know 'N'. It is easier to solve such a problem in frequency domain.

Soln

$$x_3(n) = \sum_{m=0}^{N-1} x_1((m)) \frac{x_2((n-m))}{N}$$

$$\Rightarrow X_3(k) = X_1(k) X_2(k)$$

$$X_3(k) = \text{IDFT} \{ X_3(k) \}$$

$$\begin{aligned} x_1(n) &= \cos\left(\frac{2\pi n}{N}\right) \\ &= \frac{1}{2} \left[e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}} \right] \end{aligned}$$

$$= \frac{1}{2} \left[W_N^{-n} + W_N^n \right]$$

$$\begin{aligned}
 x_2(n) &= \sin \frac{2\pi n}{N} \\
 &= \frac{1}{2j} \left[e^{j \frac{2\pi n}{N}} - e^{-j \frac{2\pi n}{N}} \right] \\
 &= \frac{1}{2j} \left[w_N^{-n} - w_N^n \right]
 \end{aligned}$$

$$\begin{aligned}
 X_1(k) &= \sum_{n=0}^{N-1} x_1(n) w_N^{kn} \\
 &= \frac{1}{2} \left\{ \sum_{n=0}^{N-1} w_N^{-n} w_N^{kn} + \sum_{n=0}^{N-1} w_N^n w_N^{kn} \right\} \\
 &= \frac{1}{2} \left\{ \sum_{n=0}^{N-1} w_N^{(k-1)n} + \sum_{n=0}^{N-1} w_N^{(k+1)n} \right\} \\
 &= \frac{1}{2} \left\{ N \delta(k-1) + N \delta(k+1) \right\} \quad \text{By using Lemma} \\
 X_2(k) &= \frac{1}{2j} \left\{ N \delta(k-1) - N \delta(k+1) \right\} \quad \text{--- (2)}
 \end{aligned}$$

$$X_3(k) = X_1(k) \cdot X_2(k)$$

$$= \frac{N^2}{4j} \left\{ \delta(k-1) - \delta(k+1) \right\} \quad \text{--- (3)}$$

(3)

$$X_3(k) = \frac{N}{2} \left\{ \frac{N}{2} \right\} \left[\delta(k-1) - \delta(k+1) \right] \\ - (4)$$

$$\therefore x_3(n) = \frac{N}{2} \sin \left(\frac{2\pi n}{N} \right) \quad 0 \leq n \leq N-1$$

Multiplication in time

It is complex convolution

$$\text{DFT } \{x_1(n), x_2(n)\} = \frac{1}{N} \left[X_1(k) \otimes_N X_2(k) \right]$$

Proof:

$$\text{DFT } \{x_1(n), x_2(n)\} = \sum_{n=0}^{N-1} \{x_1(n), x_2(n)\} e^{-j \frac{2\pi}{N} kn}$$

$$= \sum_{n=0}^{N-1} x_1(n) \left[\frac{1}{N} \sum_{l=0}^{N-1} \underbrace{X_2(l)}_{\text{IDFT of } X_2(k)} e^{-j \frac{2\pi}{N} ln} \right] e^{-j \frac{2\pi}{N} kn}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \left\{ \sum_{n=0}^{N-1} x_1(n) e^{-j \frac{2\pi}{N} (k-l)n} \right\}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2((l)) X_1((k-l))_N = \frac{1}{N} \{X_1(k) \otimes X_2(k)\}$$

(4)

E: find $X_3(k)$, if $\underline{x}(n) = x_1(n) \cdot x_2(n)$.

where $x_1(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

and $x_2(n) = \cos\left(\frac{\pi}{4}n\right) \quad 0 \leq n \leq 7$.

Sln

$$X_1(k) = \sum_{n=0}^7 x_1(n) w_8^{kn}$$

$$n=0$$

$$= \sum_{n=0}^7 1 \cdot w_8^{kn} = \sum_{n=0}^7 w_8^{kn}$$

$$= \begin{cases} 0 & k \neq 0 \\ 8 & k = 0 \end{cases} \quad \left| \sum_{n=0}^{N-1} w_N^{kn} = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases} \right.$$

$$X_1(k) = \left\{ \underbrace{1, 0, 0, 0, 0, 0, 0, 0}_k \right\}$$

$$X_2(k) = \sum_{n=0}^7 x_2(n) w_8^{kn} = \sum_{n=0}^7 \cos\left(\frac{\pi}{4}n\right) w_8^{kn}$$

$$= \frac{1}{2} \sum_{n=0}^7 \left[e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} \right] w_8^{kn}$$

(5)

$$= \frac{1}{2} \sum_{n=0}^7 [w_8^{-n} + w_8^n] w_p^{kn}$$

$$\begin{aligned} &= e^{j\frac{\pi}{4}n} \\ &= e^{j\frac{2\pi n}{8}} \end{aligned}$$

$$= \frac{1}{2} \sum_{n=0}^7 \{ w_8^{(k-1)n} + w_8^{(k+1)n} \}$$

$$= \frac{1}{2} \left\{ \sum_{n=0}^7 w_8^{(k-1)n} + \sum_{n=0}^7 w_8^{(k+1)n} \right\}$$

$$= \frac{1}{2} \left[8 \delta(k-1) + 8 \delta(k+1) \right]$$

$$\sum_{n=0}^{N-1} w_N^{(k-k_0)n}$$

$$= 4 \delta(k-1) + 4 \delta(k+1)$$

$$= N \delta(k-k_0)$$

$$= \begin{cases} 4 & k = 1 \\ 4 & k = -1 \text{ or } -1 + 8 = 7 \\ 0 & \text{otherwise} \end{cases}$$

$$X_2(k) = \{0, 4, 0, 0, 0, 0, 0, 4\}$$

(6)

Now, DFT $\{x_1(n) x_2(n)\}$

$$X_3(k) = \frac{1}{N} [X_1(k) \otimes X_2(k)]$$

$$\begin{bmatrix} X_3(0) \\ X_3(1) \\ X_3(2) \\ X_3(3) \\ X_3(4) \\ X_3(5) \\ X_3(6) \\ X_3(7) \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\therefore X_3(k) = \{0, 4, 0, 0, 0, 0, 0, 4\}$$

↑

DSP IIParseval's theorem

(1) It says that energy content of the signal does not change even if the domain of representation changes (time and frequency domain).

$$\text{time domain energy} \quad \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \quad \text{frequency domain energy}$$

Proof: $n=0$

$$\begin{aligned} \text{(LHS)} \quad & \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x^*(n) x(n) \\ & = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} \right\} x(n) \end{aligned}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \right\} x(n)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \left\{ \sum_{n=0}^{N-1} x(n) W_N^{nk} \right\}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) X(k) = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

RHS

(2)

Prob

1. find the energy of the 4 point
 sequence a) using time domain approach
 b) using frequency domain approach

$$x(n) = \sin\left(\frac{2\pi}{N}n\right) \quad 0 \leq n \leq 3$$

Soln) a) $x(n) = \sin\left(\frac{2\pi}{4}n\right) = \sin\left(\frac{\pi}{2}n\right)$
 $0 \leq n \leq 3$
 $= \{0, 1, 0, -1\}$

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = 1^2 + 1^2 = 2 \text{ Joules}$$

b) $X(k) = \text{DFT}\{x(n)\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2j \\ 0 \\ +2j \end{bmatrix}$$

$$E = \frac{1}{4} \sum_{k=0}^3 |X(k)|^2 = \frac{1}{4} [+4+4] \\ = 2 \text{ Joules}$$

2) Verify Parseval's theorem for the dependence $x(n) = \{ \underset{\uparrow}{1}, 3, 5, 3 \}$ ③

Solution $x(n) = \{ \underset{\uparrow}{1}, 3, 5, 3 \}$

$$E = \sum_{n=0}^3 |x(n)|^2 = 1^2 + 3^2 + 5^2 + 3^2 = 1 + 9 + 25 + 9 = 44 \text{ Joules} \quad - \textcircled{1}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1+j & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \\ 0 \\ -4 \end{bmatrix}$$

$$E = \frac{1}{4} \sum_{k=0}^3 |x(k)|^2 = \frac{1}{4} \left[12^2 + (-4)^2 + 0 + (-4)^2 \right]$$

$$= \frac{1}{4} [144 + 16 + 0 + 16]$$

$$= \frac{176}{4} = 44 \text{ Joules} \quad - \textcircled{2}$$

Hence verified

④

$$\textcircled{3} \quad \text{If } x(n) = \{1, 2, 0, 3, -2, 4, 7, 5\}$$

find the following

Should be solved without solving $X(k)$

$$\text{a) } X(0) \quad \text{b) } X(4) \quad \text{c) } \sum_{k=0}^7 X(k)$$

$$\text{d) } \sum_{k=0}^7 |x(k)|^2$$

$$\underline{\text{Solve}} \quad \text{a) } X(k) = \sum_{n=0}^7 x(n) \omega_N^{kn}$$

$$\text{Put } k=0$$

$$X(0) = \sum_{n=0}^7 x(n) = 1 + 2 + 0 + 3 - 2 + 4 + 7 + 5 = 20$$

$$\text{b) } \text{Put } k = N/2$$

$$X(N/2) = \sum_{n=0}^7 x(n) \omega_N^{n \cdot N/2}$$

$$= \sum_{n=0}^7 (-1)^n x(n)$$

$$\therefore X(4) = \sum_{n=0}^7 (-1)^n x(n)$$

$$\begin{aligned} \omega_N^{nN/2} &= e^{j2\pi n \frac{k}{N}} \\ &= e^{-j\pi n} \\ &= (-1)^n \end{aligned}$$

$$\begin{aligned}
 X(4) &= x(0) - x(1) + x(2) - x(3) + x(4) \\
 &\quad - x(5) + x(6) - x(7) \\
 &= 1 - 2 + 0 - 3 - 2 - 4 + 7 - 5 \\
 &= \underline{\underline{-8}}.
 \end{aligned} \tag{5}$$

c) Using definition of IDFT,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-j\frac{2\pi}{N}kn}$$

$$0 \leq n \leq N-1$$

for $n=0$

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$\therefore \sum_{k=0}^{7} X(k) = 8 \times x(0) = 8 \times 1 = \underline{\underline{8}}$$

$$d) * \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\sum_{k=0}^{7} |X(k)|^2 = 8 \sum_{n=0}^{7} |x(n)|^2$$

$$= 8(1+4+0+9+4+16+49+25) = 864$$

Appln of DFT in Linear filtering (6)

Q 1) An FIR filter has an impulse response of $h(n) = \{1, 2\}$. Determine the response of the filter to the I/P sequence $x(n) = \{1, 2, 3\}$ using DFT and IDFT. Verify the result by direct computation of linear convolution.

$$\text{Soln} \quad L_1 = 2 \quad L_2 = 3$$

$$\text{Length of o/p sequence } 2+3-1 = 4$$

$$h_1(n) = \{1, 2, 0, 0\}$$

$$x_1(n) = \{1, 2, 3, 0\}$$

$$X_1(k) = \sum_{n=0}^3 x_1(n) e^{-j\frac{2\pi}{4}kn}$$

$$= 1 + 2e^{-j\frac{\pi}{2}} + 3e^{-j\frac{3\pi}{2}}$$

$$Y_1(k) = \sum_{n=0}^3 h_1(n) e^{-j\frac{2\pi}{4}kn}$$

$$= 1 + 2e^{-j\frac{\pi}{2}}$$

$$y(n) = x_1(n) \cdot h_1(n) \quad (7)$$

$$= 1 + 4n_y^1 + 7n_y^{2k} + 6n_y^{3k}$$

$$\therefore y(n) = \{1, 4, 7, 6\}$$

Direct Computation

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix}$$

Ques: Determine the O/P $y(n)$ for a system

W/H $h(n) = \{1, 2\}$ as an input

$x(n) = \{1, 2, 0, 4\}$ using a) Circular

Convolution b) DFT and IDFT

c) Verify the result by direct Computation