Vector subspaces:

Theorem-1: The necessary and sufficient conditions for a non empty subset W of $V(F)$ to be a subspace are that : i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$.

ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

Proof: Suppose W is a subspace of $V(F)$. Then if

$$\beta \in W \Rightarrow -\beta \in W.$$

$$\alpha \in W, -\beta \in W \Rightarrow \alpha - \beta \in W.$$

$$a \in F, \alpha \in W \Rightarrow a\alpha \in W.$$

Conversely, suppose W is a subset of V and

i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$.

ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

We have to show that W is a subspace.

$$\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W$$

$0 \in W$. (identity exists)

$$0 \in W, \alpha \in W \Rightarrow 0 - \alpha \in W$$

$-\alpha \in W$ (inverse exists)

$\alpha \in W, -\beta \in W \Rightarrow \alpha + \beta \in W$. (closure property satisfies)

vector addition is always associative & commutative

Hence $(W, +)$ is an abelian group.

From ii) W is closed under multiplication.

V is a vectorspace and W is a subset of V all the other properties holds good.

Hence W is a subspace of vectorspace $V(F)$.

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Theorem-2: The necessary and sufficient condition for a non-empty subset of W of a vectorspace $V(F)$ to be a subspace of V is

$$a,b \in F,\ \alpha,\beta \in W \Rightarrow a\alpha + b\beta \in W.$$

Proof: Suppose W is a subspace of $V(F)$ then W is closed under vector addition and multiplication

$$a \in F,\ \alpha \in W \Rightarrow a\alpha \in W.$$

$$b \in F,\ \beta \in W \Rightarrow b\beta \in W.$$

$$a\alpha \in W,\ b\beta \in W \Rightarrow a\alpha + b\beta \in W$$

conversely, suppose W is a subset of $V(F)$ and given.

$$a, b \in F,\ \alpha,\beta \in W \Rightarrow a\alpha + b\beta \in W.$$

Put $a=0$, $b=0 \Rightarrow \cancel{a\alpha+b\beta}\ 0(\alpha) + 0(\beta) \in W.$

$0 \in W$ (identity)

Put $a=-1$, $b=0 \Rightarrow -1(\alpha) + 0(\beta) \in W$

$-\alpha \in W$ (inverse)

Put $a=1$, $b=1 \Rightarrow \alpha+\beta \in W$ (closed)

Further put $a=1$, $\beta=0 \Rightarrow a\alpha \in W$. (closed multiplication)

Since $W \subseteq V$ other properties hold.

$\therefore W$ is a vectorspace and a subspace of $V(F)$.

Algebra of subspaces:

Theorem-1: The intersection of any two subspaces of a vectorspace is a subspace.

Proof: $\cancel{\alpha,\beta}$ Let $\alpha,\beta \in W_1 \cap W_2 \Rightarrow \alpha,\beta \in W_1$ & $\alpha,\beta \in W_2$.

Since W_1 & W_2 are subspaces of V, we have

$$a,b \in F \ \&\ \alpha,\beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1.$$

$$\&\ a,b \in F \ \&\ \alpha,\beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2.$$

$$\Rightarrow a\alpha + b\beta \in W_1 \cap W_2 \Rightarrow \therefore W_1 \cap W_2 \text{ is a subspace.}$$

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Theorem-2: The intersection of an arbitrary collection of subspaces of a vectorspace is also a subspace.

Proof: Let $\{W_\lambda : \lambda \in X\}$ be an arbitrary collection of subspaces of V. We have to show $\cap\{W_\lambda : \lambda \in X\}$ is a subspace of V.

Let $\alpha, \beta \in \cap\{W_\lambda : \lambda \in X\}$

$\Rightarrow \alpha, \beta \in W_\lambda$ for each $\lambda \in X$.

Since W_λ is a subspace, for $a, b \in F$ we have

$a\alpha + b\beta \in W_\lambda$ for each $\lambda \in X$.

$a\alpha + b\beta \in \cap\{W_\lambda : \lambda \in X\}$.

Hence $\cap\{W_\lambda : \lambda \in X\}$ is a subspace of V.

Theorem-3: The union of 2 subspaces of a vectorspace is not necessarily a subspace.

Proof: Let W_1, W_2 be 2 subspaces of $V(F)$.

Let $W_1 = \{(a_1, a_2, 0) : a_1, a_2 \in F\}$.

$W_2 = \{(a_1, 0, a_3) : a_1, a_3 \in F\}$

So $W_1 \cup W_2$ is a set of all 3-tuples of the form $(a_1, a_2, 0)$ & $(a_1, 0, a_3)$.

Now let $\alpha = (1,2,0)$, $\beta = (3,0,5)$ of $W_1 \cup W_2$.

for $a = 1$, $b = 2$,

$a\alpha + b\beta = 1(1,2,0) + 2(3,0,5)$

$= (7, 2, 10) \notin W_1 \cup W_2$

~~which may not be~~

Thus if $\alpha \in W_1 \cup W_2$ & $\beta \in W_1 \cup W_2$ then $a\alpha + b\beta \in W_1 \cup W_2$ need

Theorem-4: The union of 2 subspaces of a vectorspace is a subspace iff one is contained in the other

Proof: Let ~~V(F)~~ W_1, W_2 be 2 subspaces of $V(F)$.

Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then we have to show that $W_1 \cup W_2$ is a subspace of V.

$W_1 \cup W_2 = W_2$ if $W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2$ is a subspace

$W_1 \cup W_2 = W_1$ if $W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2$ is a subspace.

Conversely, let $W_1 \cup W_2$ is a subspace of V, we have to prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Let W_1 is not a subset of W_2 and W_2 is not a subset of W_1.

There exists an α in W_1 which is not present in W_2 & there exists a β in W_2 which is not in W_1.

~~$\alpha \in W_1$~~ But we have $\alpha \in W_1 \cup W_2$ & $\beta \in W_1 \cup W_2$.

Since $W_1 \cup W_2$ is a subspace,

$a, b \in F$, $\alpha, \beta \in W_1 \cup W_2 \Rightarrow a\alpha + b\beta \in W_1 \cup W_2$.

Taking $a = 1$, $b = 1$.

$\alpha + \beta \in W_1 \cup W_2$.

Let $\alpha + \beta \in W_1$. WKT $\alpha \in W_1$

$\Rightarrow (\alpha + \beta) + (-\alpha) \in W_1$

$\beta \in W_1$, which is a contradiction.

IIIly let $\alpha + \beta \in W_2$, WKT $\beta \in W_2$.

$(\alpha + \beta) + (-\beta) \in W_2$.

$\alpha \in W_2$ which is a contradiction.

Hence $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

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Linear sum of 2 subspaces:

Theorem-1: The linear sum of 2 subspaces of a vectorspace is also a subspace.

Proof: Let W_1 & W_2 be 2 subspaces of $V(F)$. We have to show that W_1+W_2 is a subspace of $V(F)$.

Let α, β be any 2 arbitrary elements of W_1+W_2.

$\alpha, \beta \in W_1+W_2$.

$\alpha = \alpha_1+\alpha_2$ & $\beta = \beta_1+\beta_2$ where $\alpha_1, \beta_1 \in W_1$ & $\alpha_2, \beta_2 \in W_2$.

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in V$. $\Rightarrow W_1+W_2 \subseteq V$

W_1 is a subspace

$\Rightarrow \alpha_1, \beta_1 \in W_1 \Rightarrow a\alpha_1+b\beta_1 \in W_1$.

W_2 is a subspace.

$\alpha_2, \beta_2 \in W_2 \Rightarrow a\alpha_2+b\beta_2 \in W_2$.

$(a\alpha_1+b\beta_1)+(a\alpha_2+b\beta_2) \in (W_1+W_2)$

$a(\alpha_1+\alpha_2)+b(\beta_1+\beta_2) \in W_1+W_2$.

$a\alpha+b\beta \in W_1+W_2$.

$\Rightarrow W_1+W_2$ is a subspace.

Direct sum of vectorspaces:

Theorem-1: The necessary and sufficient condition for a vectorspace V to be the direct sum of two subspaces W_1 & W_2 are:

i) $V = W_1+W_2$.

ii) $W_1 \cap W_2 = \{0\}$.

Linear combination of vectors:

Theorem-1: The linear span $L(S)$ of a non-empty subset S of a vector space $V(F)$ is the smallest subspace of V containing S.

Proof: We have $L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n : a_i \in V\}$

For any $\alpha \in S$, $1 \in F$ then $\alpha = 1 \cdot \alpha$

$\Rightarrow \alpha \in L(S)$.

$\Rightarrow S \subseteq L(S)$

Now to show $L(S)$ is a subspace,

let α, β be any 2 arbitrary elements of $L(S)$

then $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ for $\alpha_1, \alpha_2, \dots \alpha_n \in S$ & $a_1, a_2, \dots a_n \in F$.

& $\beta = b_1\beta_1 + b_2\beta_2 + \dots + b_m\beta_m$ for $\beta_1, \beta_2, \dots \beta_m \in S$ & $b_1, b_2, \dots b_m \in F$.

For all $a, b \in F$.

$a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) + b(b_1\beta_1 + b_2\beta_2 + \dots b_m\beta_m)$

$= aa_1\alpha_1 + aa_2\alpha_2 + \dots + aa_n\alpha_n + bb_1\beta_1 + bb_2\beta_2 + \dots + bb_m\beta_m$

$= (aa_1)\alpha_1 + (aa_2)\alpha_2 + \dots (aa_n)\alpha_n + (bb_1)\beta_1 + (bb_2)\beta_2 + \dots + (bb_m)\beta_m$

$a\alpha + b\beta$ is a linear combination of elements of S.

$a\alpha + b\beta \in L(S)$. $\Rightarrow L(S)$ is a subspace of V.

Let W is a subspace of ~~V~~ containing S. Let ~~$\alpha_1, \alpha_2, \dots \alpha_n, \beta_1, \beta_2, \dots \beta_m$~~ $\alpha_1, \alpha_2, \dots \alpha_t \in S \subset W$ & W being a subspace then

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_t\alpha_t \in W, \forall a_t \in F$.

This implies that W contains all linear combinations of S, therefore $L(S) \in W$.

Theorem-2: If S, T are two subsets of a vectorspace V then

i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

ii) $L(S \cup T) = L(S) + L(T)$

iii) $L[L(S)] = L(S)$

Proof: $\alpha \in L(S) \Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$, $\alpha_1, \alpha_2, \dots \alpha_n \in S$
$a_1, a_2, \dots a_n \in F$.

i) $S \subseteq T$ so $\alpha_1, \alpha_2, \dots \alpha_n \in T \Rightarrow \alpha \in L(T)$.

Thus $\alpha \in L(S) \Rightarrow \alpha \in L(T)$

$L(S) \subseteq L(T)$ if $S \subseteq T$.

ii) $S \subseteq S \cup T$ & $T \subseteq S \cup T$.

$L(S) \subseteq L(S \cup T)$

$L(T) \subseteq L(S \cup T)$

$L(S) + L(T) \subseteq L(S \cup T)$.

Let α be an arbitrary element of $L(S \cup T)$ then α is linear combination of elements of $S \cup T$. Some of $\alpha_i \in S$ and some of $\alpha_i \in T$.

$\therefore \alpha \in L(S) + L(T)$

$L(S \cup T) \subseteq L(S) + L(T)$

so $L(S \cup T) = L(S) + L(T)$.

iii) $S \subseteq L(S) \Rightarrow L(S) \subseteq L[L(S)]$.

Let α be any arbitrary element of $L[L(S)]$ then α is LC of elements of $L(S)$.

$\alpha = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n$.

where β_i is LC of elements of S.

$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1m}\alpha_m$.

$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + \dots a_{2m}\alpha_m$

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On substituting β values to α

we get α is a linear combination of elements of $S \Rightarrow \alpha \in L(S)$.

$L[L(S)] \subseteq L(S)$.

$\Rightarrow L[L(S)] = L(S)$.

Theorem-3: The linear sum of 2 subspaces W_1 & W_2 of a vectorspace $V(F)$ is generated by their union.

$W_1 + W_2 = L(W_1 \cup W_2)$.

Proof: WKT, linear sum of 2 subspaces is also a subspace and linear span of a subset of a vector space is also a subspace.

$\Rightarrow W_1 + W_2$ & $L(W_1 \cup W_2)$ are subspaces of $V(F)$.

Let α be an arbitrary element of $W_1 + W_2$.

$\alpha \in W_1 + W_2$.

$\alpha = \alpha_1 + \alpha_2$ for $\alpha_1 \in W_1$, $\alpha_2 \in W_2$.

since $\alpha_1 \in W_1$ & $\alpha_2 \in W_2 \Rightarrow \alpha_1 + \alpha_2 \in W_1 + W_2$.

Also, $\alpha = \alpha_1 + \alpha_2 = 1 \cdot \alpha_1 + 1 \cdot \alpha_2 \Rightarrow \alpha$ is a linear combination of elements of $W_1 \cup W_2$.

$\Rightarrow \alpha \in L(W_1 \cup W_2)$.

$\alpha \in W_1 + W_2 \Rightarrow \alpha \in L(W_1 \cup W_2)$

$W_1 + W_2 \subseteq L(W_1 \cup W_2)$.

But $L(W_1 \cup W_2)$ is the smallest subspace of V containing $W_1 \cup W_2$. Since $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$.

$L(W_1 \cup W_2) \subseteq W_1 + W_2$.

$W_1 + W_2 = L(W_1 \cup W_2)$

Finite dimension vector space:

Theorem-1: If $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is a basis of vector space $V(F)$ then each element of V is uniquely expressible as a LC of elements of S.

Proof: Since S is a basis of $V(F)$ then by definition of basis, each element of V is a linear combination of elements of S. So we should show the uniqueness.

Let there be 2 different sets $\{a_1, a_2, \dots a_n\}$ & $\{b_1, b_2, \dots b_n\}$ of scalars corresponding to an element $\alpha \in V$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

& $$\alpha = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n.$$

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n.$$

$$(a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + \dots + (a_n - b_n)\alpha_n = 0.$$

$S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is LID

$a_1 - b_1 = 0 \quad a_2 - b_2 = 0 \quad \dots \quad a_n - b_n = 0$

$a_1 = b_1 \quad a_2 = b_2 \quad \dots \quad a_n = b_n.$

Dimension of subspace of a vectorspace:

Theorem-1: Let S be a linearly independent subset of a vectorspace V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.

Proof: Let $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a LID subset of V. We shall show that $S_1 = \{\beta, \alpha_1, \alpha_2, \dots \alpha_n\}$ is also LID where $\beta \in V$ but not in the subspace of V spanned by S.

since $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$

$a_1\alpha_1 + a_2\alpha_2 + \cdots a_n\alpha_n + b\beta = 0$. where $a_i = 0$.

This is possible only if $b = 0$.

if $b \neq 0$,

$$\beta = \left(\frac{-a_1}{b}\right)\alpha_1 + \left(\frac{-a_2}{b}\right)\alpha_2 + \cdots + \left(\frac{-a_n}{b}\right)\alpha_n$$

β is a linear combination of $\alpha_1, \alpha_2, \cdots \alpha_n$.

β is in the subspace of V spanned by $\alpha_1, \alpha_2, \cdots \alpha_n$.
which is contradiction to β not in the subspace of V spanned by $\alpha_1, \alpha_2, \cdots \alpha_n$.

$\therefore b = 0$. $\Rightarrow$ Set S_1 is also linearly independent.