

Unit 2

Linear Transformations.

Let U & V be two vector spaces over the same field F . A mapping $D: U \rightarrow V$ is said to be a linear transformation from U to V which associates to each element α of U to a unique element $T(\alpha)$ of V such that $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ for all α and β in U & all scalars a, b in F .

Some Illustrations

1. If V is any vectorspace over F then identity transformation I defined by $I(\alpha) = \alpha$, $\alpha \in V$ is a linear transformation from V to V . Also zero transformation denoted by $O(\alpha) = 0$ is a linear transformation.

2. Let F be a field of real numbers and let V be the vector space of all polynomials. Then a mapping $D: V \rightarrow V$ given by $D[f(x)] = \frac{d}{dx} f(x)$, $f(x) \in V$ is a linear transformation.

$$\begin{aligned} \text{Since for any } f(x) \& g(x) \in V \text{ and } a, b \in F, D[a f(x) + b g(x)] = \frac{d}{dx} [af(x) + bg(x)] \\ &= \frac{d}{dx}[a f(x)] + \frac{d}{dx}[b g(x)] \\ &= a \left[D[f(x)] \right] + b \left[D[g(x)] \right] \end{aligned}$$

3. Let R be the field of real numbers and let V be the vector space of all functions from $R \times R$ which are continuous. Then a mapping $T: V \rightarrow V$ is given by $T[f(x)] = \int f(x) dt$

for any vector $f(x), g(x) \in V$ & $a, b \in R$

$$\begin{aligned} \therefore T[a f(x) + b g(x)] &= \int_0^x [a f(t) + b g(t)] dt \\ &= \int_0^x a f(t) dt + \int_0^x b g(t) dt \\ &= a T[f(x)] + b T[g(x)] \end{aligned}$$

4. Let V be the vectorspace of all $M \times N$ matrices over a field F & let Φ be a fixed $m \times n$ matrix and Ψ be a fixed matrix of order $m \times m$

Then a mapping $T: V \rightarrow V$ given by $T(A) = PAQ + AEV$ is a linear transformation.

$$\begin{aligned} \text{For any two matrices } A, B \in V \text{ & } a, b \in F. \quad T(aA+bB) &= P(aA+bB)Q \\ &= (aPA + bPB)Q \\ &= aPAQ + bPBQ \\ &= aT(A) + bT(B) \end{aligned}$$

Some other definitions:

1. Linear operator :- Let $V(F)$ be a vectorspace. Then a linear transformation from $V \times V$ to V is called a linear operator.
2. Zero transformation :- Let U and V be two vectorspaces over the same field F . Then the zero transformation of $U \times V$ is a mapping defined by $T(x) = 0 \forall x \in U$ where 0 is the 0 vector of V .
3. Identity transformation :- Let $V(F)$ be a vectorspace. Then a linear transformation $\tau: V \rightarrow V$ is said to be identity transformation defined by $\tau(x) \in x \forall x \in V$.
4. Negative of a linear transformation :- Let U and V be two vectorspaces over the same field F . Let T be a linear transformation of $U \rightarrow V$. Then a linear transformation $-T$ of $U \rightarrow V$ defined by $(-T)(x) = -[T(x)] \forall x \in U$ is called Negative of a linear transformation.

Properties of linear transformations

Let $U(F)$ and $V(F)$ be two vectorspaces and T be a linear transformation of $U \times V$. Then

- (i) $T(0) = 0$ where 0 on LHS is the vector of U and 0 on RHS is the zero vector V .
- (ii) $T(-x) = -T(x) \forall x \in U$
- (iii) $T(x_1 - x_2) = T(x_1) - T(x_2) \forall x_1, x_2 \in U$.
- (iv) $T[x_1 a_1 + x_2 a_2 + \dots + x_n a_n] = a_1 T(x_1) + a_2 T(x_2) + \dots + a_n T(x_n)$
 $\forall x_1, x_2, x_3, \dots, x_n \in U \text{ & } a_1, a_2, a_3, \dots, a_n \in F$

(i) proof: If $\alpha \in U$ then $T(\alpha) \in V$. Since V is the vector space, we have, $T(\alpha) + 0 = T(\alpha)$

$$T(\alpha+0) \rightarrow T(\alpha)+0 = T(\alpha+0)$$

$$T(\alpha)+0 = T(\alpha)+T(0)$$

$$0 = T(0)$$

(ii) we have $\alpha \in U$

$$\begin{aligned} T[\alpha + (-\alpha)] &= T(\alpha) + T(-\alpha) \\ T(0) &= T(\alpha) + T(-\alpha) \\ 0 &= T(\alpha) + T(-\alpha) \\ -T(\alpha) &= T(-\alpha) \end{aligned}$$

(iii) For all $\alpha, \beta \in U$

$$\begin{aligned} T(\alpha + -\beta) &= T(\alpha) + T(-\beta) \\ T(\alpha - \beta) &= T(\alpha) - T(\beta) \end{aligned}$$

(iv) Since $\alpha_1 Q_1 + \alpha_2 Q_2 + \dots + \alpha_n Q_n$ is a linear combination of vectors of U . Now we shall prove that the result by induction of n .

For $n=1$, $T(\alpha_1 Q_1) = \alpha_1 T(Q_1)$

For $n=2$, $T(\alpha_1 Q_1 + \alpha_2 Q_2) = \alpha_1 T(Q_1) + \alpha_2 T(Q_2)$

Suppose the result is true of $(n-1)$ values i.e.,

$$T(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_{n-1} \alpha_{n-1}) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_{n-1} T(\alpha_{n-1})$$

For n values,

$$T(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(\alpha_n)$$

Hence the result is proved by induction.

Theorem 81 Let U, V & W be vector spaces over F . Let T_1 be a linear transformation from U into V and T_2 be a d.t. from V into W . Then the composed fn $T_2 T_1$ be defined by

$$T_2 T_1(x) = T_2(T_1(x)) + \alpha \in W$$

Proof: For $\alpha, \beta \in U$ and $a, b \in F$ we have $T_2 T_1(a\alpha + b\beta) =$

$$\begin{aligned} & T_2 [T_1(a\alpha + b\beta)] \\ &= T_2 [aT_1(\alpha) + bT_1(\beta)] \\ &= a(T_2 T_1)(\alpha) + b(T_2 T_1)(\beta) \end{aligned}$$

$\therefore T_2 T_1$ is a lin. tr. from U into W .

Linear Operators:-

If V is a vectorspace over the field F , then a lin. tr. from $V \rightarrow V$ is called a linear operator.

Algebra of linear operators.

Let V be a vectorspace over the field F and let T_1, T_2, T_3 be linear operators on V , and let c be an element in F . Then

$$(i) \quad \mathbb{I}T = T\mathbb{I} = T, \quad \mathbb{I} \text{ being an identity operator}$$

$$(ii) \quad T_1(T_2 + T_3) = T_1 T_2 + T_1 T_3$$

$$(T_2 + T_3)T_1 = T_2 T_1 + T_3 T_1$$

$$(iii) \quad T_1(T_2 T_3) = (T_1 T_2)T_3$$

$$(iv) \quad c(T_1 T_2) = T_1(cT_2)$$

$$(v) \quad T0 = 0T = 0$$

Proof:

(i) For $x \in V$

$$\mathbb{I}T(x) = \mathbb{I}(T(x)) = T(x)$$

$$\mathbb{I}T = T$$

$$T\mathbb{I}(x) = T(\mathbb{I}(x)) = T(x)$$

$$T\mathbb{I} = T$$

$$\mathbb{I}T = T\mathbb{I} = T$$

(ii) $\forall \alpha \in V$

$$[T_1(T_2 + T_3)](\alpha) = T_1[(T_2 + T_3)(\alpha)]$$

$$= T_1[T_2(\alpha) + T_3(\alpha)]$$

$$= T_1T_2(\alpha) + T_1T_3(\alpha)$$

$$= [T_1T_2 + T_1T_3](\alpha)$$

$$\therefore [T_1(T_2 + T_3)] = T_1T_2 + T_1T_3$$

$$\text{Hence } [T_2 + T_3]T_1 = T_2T_1 + T_3T_1$$

(iii) $\forall \alpha \in V$

$$[T_1(T_2T_3)](\alpha) = [T_1(T_2T_3)(\alpha)]$$

$$= T_1[T_2(T_3(\alpha))]$$

$$= T_1T_2[T_3(\alpha)]$$

$$= [(T_1T_2)T_3](\alpha)$$

$$\therefore T_1(T_2T_3) = (T_1T_2)T_3$$

(iv) $\forall \alpha \in V, c \in F$

$$[c(T_1T_2)](\alpha) = c[T_1T_2](\alpha)$$

$$= c[T_1(T_2(\alpha))]$$

$$= c[T_1[T_2(\alpha)]]$$

$$= [(cT_1)T_2](\alpha)$$

$$\therefore c(T_1T_2) = (cT_1)T_2$$

$$\text{Hence } c(T_1T_2) = T_1(cT_2)$$

(v)

Problems

worth

1. Let T_1 & T_2 be linear operators on \mathbb{R}^2 defined as follows.

$$T_1(x_1, x_2) = (x_2, x_1)$$

$$T_2(x_1, x_2) = (x_1, 0)$$

$$T_1 T_2 \neq T_2 T_1$$

→ Let $\alpha = (x_1, x_2) \in \mathbb{R}^2$ then $(T_1 T_2)(\alpha) = T_1(T_2(\alpha))$

$$= T_1(T_2(x_1, x_2))$$

$$= T_1(x_1, 0)$$

$$[T_1 T_2(\alpha)] = (0, x_1) \text{ and } (T_2 T_1)(\alpha) = T_2(T_1(\alpha))$$

$$\Rightarrow T_2[T_1(x_1, x_2)]$$

$$= T_2(x_2, 0)$$

$$= (x_2, 0)$$

Therefore clearly we can see that $(T_1 T_2)(\alpha) \neq T_2 T_1(\alpha) \forall \alpha \in \mathbb{R}$

$$\therefore T_1 T_2 \neq T_2 T_1$$

2. Let $V(\mathbb{R})$ be the vector space of all polynomial functions in x with coefficients in the field \mathbb{R} of real numbers. Let D & T be two linear operators on V defined by $D(f(x)) = \frac{d}{dx} f(x)$ and

$$T(f(x)) = \int_0^x f(t) dt \text{ for every } f(x) \in V. \text{ Then } ST = I \text{ and } TD = I$$

→ Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots \in V$, where $a_0, a_1, a_2 \in \mathbb{R}$ then

$$DT(f(x)) = D[T(f(x))]$$

$$= D \left[\int_0^x (a_0 + a_1 t + a_2 t^2 + \dots) dt \right]$$

$$= D \left[\left(a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots \right) dx \right]$$

$$= \frac{d}{dx} \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots \right]$$

$$= a_0 + a_1 x + a_2 x^2 + \dots$$

$$= I[f(x)]$$

$$DT(f(x)) = I[f(x)] \forall f(x) \in V.$$

$$\begin{aligned}
 TD[f(x)] &= T \left[\frac{d}{dx} (a_0 + a_1 x + a_2 x^2 + \dots) \right] \\
 &= T \left[a_1 + 2a_2 x + 3a_3 x^2 + \dots \right] \\
 &= \int (a_1 + 2a_2 x + 3a_3 x^2 + \dots) dx \\
 &= a_1 x + a_2 x^2 + a_3 x^3 + \dots \neq f(x) + I[f(x)]
 \end{aligned}$$

$$TD \neq I.$$

Hence $DT \neq T.D.$

Let $V(R)$ be the vector space of all polynomials in x with co-efficients in the field R . Let D and T be two linear transformations on V defined by $D(f(x)) = \frac{d}{dx} f(x) + f(x) \in V$ $T(f(x)) = x(f(x)) + f(x) \in V$.

Then $ST, DT \in T.D$ also $DT - T.D = I$

\rightarrow Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots \in V$ where $a_0, a_1, a_2 \in R$ then

$$\begin{aligned}
 DT[f(x)] &= D[T f(x)] \\
 &= D[x f(x)] \\
 &= \frac{d}{dx} [x f(x)] \\
 &= x f'(x) + f(x).
 \end{aligned}$$

$$\begin{aligned}
 TD[f(x)] &= (T D)[f(x)] \\
 &= T[D f(x)] \\
 &= T[f'(x)] \\
 &= x f'(x).
 \end{aligned}$$

\therefore From the above eqn's we can say that $DT[f(x)] \neq TD[f(x)]$
 $DT \neq TD + f(x) \in V$

$$\begin{aligned}
 DT - TD &= x f'(x) + f(x) - x f'(x) \\
 &= f(x) \\
 &= I[f(x)]
 \end{aligned}$$

4. Range and Null space of linear transformation.

i. Range space: If T is a lin. fn from $V \rightarrow V$ then the range of T is a subspace of V . Let R_T be the range of T , i.e., the set of all vectors B in V such that $T(x) = B$ for some $x \in U$.

i.e., $R_T = \{B \in V : T(Q) = B \text{ for some } Q \in U\}$. If V is finite dimensional, then $\dim R_T$ is called rank of T and is denoted by $s(T)$.

2 Null space:

If T is a linear transformation from $U \rightarrow V$, then the null space of T denoted by $N(T)$ or N_T is the set of all vectors x in U such that $T(x) = 0$ where 0 is the 0 vector in V .

$$N_T = \{x \in U : T(x) = 0, 0 \in V\}$$

If U is finite dimensional then the finite dimension is denoted by nullity of T . $s(T) + n(T)$

Theorem: Let U & V be vectorspace over the field F and let T be a linear transformation from U onto V . Suppose U is finite dimensional then $\text{rank}(T) + \text{nullity}(T) = \dim U$ i.e., $s(T) + n(T) = \dim U$

Product of linear transformations:

Let $U(F)$, $V(F)$ & $W(F)$ be the vectorspace. Let T be a lin. tr. from $U \rightarrow V$ and S be a lin. tr. from $V \rightarrow W$. Then the composite tr. ST called the product of lin. tr. defined by $ST(x) = S[T(x)]$ is a lin. tr. from $U \rightarrow W$.

Since $T: U \rightarrow V$ is a linear tr. hence $T(x) = B$, $B \in V$ and $x \in U$. Also transformation S maps from $V \rightarrow W$ $\stackrel{S \text{ is a lin. tr.}}{\uparrow}$ Hence $S(B) \in W$ for $B \in V$.
 \therefore For $B = T(x) \in V$, $S[T(x)] \in W$

$$\text{Let } x_1, x_2 \in U \text{ & } a, b \in F \text{ then } (ST)(ax_1 + bx_2) = S[T(ax_1 + bx_2)]$$

$$\begin{aligned} &= S[aT(x_1) + bT(x_2)] \\ &= aS[T(x_1)] + bS[T(x_2)] \\ &= a(ST)(x_1) + b(ST)(x_2) \end{aligned}$$

$\therefore ST$ is a lin. tr.

Polynomials in a lin. operator:

Let T be a linear operator on a vectorspace $V(F)$, then T, T is a lin. operator on U . Since the product of lin. operators is an associative operation, if n is a +ve integer, then we define $T^n = T^2$ $\therefore TT \cdots T = T^n$

In general, T^n is a lin. operator on V . Also $T^0 = E$.
 If $m \in n$ are non-ve integers then we see that $T^m T^n = T^{m+n}$
 $(T^m)^n = T^{mn}$

Invertible lin. fr.

A linear fr. T from a vectorspace $U(F) \rightarrow V(F)$ is called invertible or regular if there exists a unique lin. fr. T^{-1} from $V(F) \rightarrow U(F)$ such that T^{-1} is the identity fr. on V & TT^{-1} is the identity fr. on V . Furthermore T is invertible if &

(i) T is one to one

(ii) T is onto

$$\text{i.e., } R(T) = V.$$

Theorem 1: Let U & V be vectorspaces over the same field F & let T be lin. fr. from $U \rightarrow V$. If T is invertible then T^{-1} is a lin. fr. from $V \rightarrow U$.

Proof: Since T is invertible hence for each $B \in V$, there is a unique $\alpha \in U$ such that $T(\alpha) = B$ i.e. $T^{-1}(B) = \alpha$. $T(\alpha) = B \Leftrightarrow T^{-1}(B) = \alpha$

Now we shall show T^{-1} is lin.

Let $a, b \in U$ & $a, b \in F$

$$T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$$

$\therefore T^{-1}$ is lin.

But for B_1, B_2 in V , there are unique α_1, α_2 in U respectively such that $T(\alpha_1) = B_1 \Leftrightarrow T^{-1}(B_1) = \alpha_1$; $T(\alpha_2) = B_2 \Leftrightarrow T^{-1}(B_2) = \alpha_2$

Thus we have

$$T(a\alpha_1 + b\alpha_2) = aB_1 + bB_2$$

$$a\alpha_1 + b\alpha_2 = T^{-1}(aB_1 + bB_2)$$

$$aT^{-1}(B_1) + bT^{-1}(B_2) = T^{-1}(aB_1 + bB_2)$$

Hence T^{-1} is a lin. fr.

Theorem 2: Let T_1 be an invertible lin. fr. from $U(F) \rightarrow V(F)$ & T_2 be an invertible lin. fr. from $V(F) \rightarrow W(F)$. Then $T_1 \circ T_2$ is invertible and $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

Proof To show $T_2 T_1$ is invertible, we shall show it mapping is one-one and onto.

If $\alpha_1, \alpha_2 \in U$ such that $(T_2 T_1)(\alpha_1) = (T_2 T_1)(\alpha_2)$ then

$$T_2[T_1(\alpha_1)] = T_2[T_1(\alpha_2)]$$

$$T_1(\alpha_1) = T_2(\alpha_2)$$

$$\alpha_1 = \alpha_2$$

Thus $T_2 T_1$ is one to one mapping. Also T_1 and T_2 being onto mapping then for each $B \in V$, there exists a unique $\alpha \in U$ such that $T_1(\alpha) = B$. Then for each $V \in W$ there exists a unique $B \in V$ such that $T_2(B) = V$.

\therefore There exists $B \in V$ for V obtained by $T_2(B)$ there exists $\alpha \in U$ such that $V = T_2(T_1(\alpha)) \quad \therefore V = (T_2 T_1)\alpha$

$\therefore T_2 T_1$ is onto mapping. Hence $T_2 T_1$ is invertible.

$$\text{Also } (T_2 T_1)(T_1^{-1} T_2^{-1}) = T_2(T_1 T_1^{-1}) T_2^{-1}$$

$$= (T_2 I) T_2^{-1}$$

$$= T_2 T_2^{-1}$$

$$= I$$

$$T_1^{-1} T_1 = I$$

$$\text{Hence } (T_2 T_1)^{-1} = T_1^{-1} T_2^{-1}$$

Non-singular lin. tr.

Let U & V be vector spaces over the field F . Then a lin. tr T from $U \rightarrow V$ is called non singular if the null space of T is zero.

Thus if T is non-singular then $T(\alpha) = 0$ i.e., $\alpha = 0$

Also when T is non-singular & $\alpha, \beta \in U$, $T(\alpha) = T(\beta) \Rightarrow T(\alpha) - T(\beta) = 0$

$$T(\alpha - \beta) = 0$$

$$\alpha - \beta = 0$$

$$\alpha = \beta$$

Hence T is non singular and it implies one-to-one mapping.

Theorem 1: Let T be a lin. tr from $U(F) \rightarrow V(F)$. Then T is non singular iff T carries each lin. independent subset of U into a lin. independent subset of V .

Proof: Let us first prove that T is non-singular.

Let us consider $S = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be an arbitrarily lin. independent subset of U . Then we have to show that $S_1 = \{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_k)\}$ is linearly independent subset of V . For scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ we have $\{a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_k T(\alpha_k)\} = 0$

$$\{T(a_1 \alpha_1) + T(a_2 \alpha_2) + \dots + T(a_k \alpha_k)\} = 0$$

$$T\{a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_k \alpha_k\} = 0$$

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_k \alpha_k = 0$$

$$a_1 = 0 = a_2 = \dots = a_k$$

Hence S_1 is linearly independent subset of V .

Conversely, suppose that T carries each lin. independent subset of U into a lin. indep. subset of V . Let α be a non-zero vector in U then α is lin. independent. Also $T(\alpha) \neq 0 \because$ the set consisting of 0 vector alone is dependent which indicates nullspace of T is zero space. Hence T is non-singular.

Theorem 2: Let U & V be finite dimensional v.spaces over the field F such that $\dim U = \dim V$. If T is lin. tr from $U \rightarrow V$ then the following are equivalent.

- (i) T is invertible
- (ii) T is non singular.

(iii) T is onto mapping i.e., range of T is V .

(iv) If $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is a basis of U then $\{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)\}$ is a basis of V .

\rightarrow Proofs (i) & (ii)

Since T is invertible, it's one-to-one & onto mapping. Therefore T is non singular.

(ii) & (iii)

Let T be non-singular & let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the basis of U . Then the set $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)$ is linearly independent subset of V but $\dim U = \dim V$.

$\therefore T(\alpha_1), \dots, T(\alpha_n)$ is a basis for V .

For any $B \in V$, $a_1, a_2, \dots, a_n \in F$ then we have $B = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(\alpha_n)$

$\therefore a_n T(\alpha_n)$

$$B = T\{a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n\}$$

where $B \in R_T$

Thus $V \subseteq R_T$; $V = R_T$; $V = T$

(ii) & (iv)

Suppose range of T equal to V $R_T = V$ & let the set $\alpha_1, \alpha_2, \dots, \alpha_n$ be a basis of U so that an arbitrary element $x \in U$ is expressible as lin. combination of $\alpha_1, \alpha_2, \dots, \alpha_n$.

$$x = b_1 \alpha_1 + b_2 \alpha_2 + \dots + b_n \alpha_n \quad b_1, b_2, \dots, b_n \in F$$

$$T(x) = T(b_1 \alpha_1 + b_2 \alpha_2 + \dots + b_n \alpha_n)$$

$$\Rightarrow b_1 T(\alpha_1) + b_2 T(\alpha_2) + \dots + b_n T(\alpha_n)$$

This shows that each element of range of T is expressible as lin. combination of $\{T(\alpha_1) + T(\alpha_2) + \dots + T(\alpha_n)\}$. Thus the set spans range of T w.r.t.

$R_T = V$ also w.r.t $\dim U = \dim V = n$. Hence to fn applied to element of U i.e., x from which we obtain $\{T(\alpha_1) + T(\alpha_2) + \dots + T(\alpha_n)\}$ forms the basis of V .

(iv) & (i)

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of U such that $\{T(\alpha_1) + T(\alpha_2) + \dots + T(\alpha_n)\}$ be the basis of V . Let us consider x which is an arbitrary element of U , then we have $b_1 \alpha_1 + b_2 \alpha_2 + \dots + b_n \alpha_n \quad b_1, b_2, \dots, b_n \in F$.

$$T(x) = 0$$

$$T(b_1 \alpha_1 + b_2 \alpha_2 + \dots + b_n \alpha_n) = 0$$

$$b_1 T(x_1) + b_2 T(x_2) + \dots + b_n T(x_n) = 0$$

$$b_1 = 0 = b_2 = \dots = b_n$$

$$\therefore x = 0$$

Hence T is non singular & $\therefore T$ is one to one mapping
Consequently T is also invertible.

problems!

Let T be a linear operator on a vector space $V(F)$. If $T^2 = 0$ what can you say about the relation of the range of T to the null space of T . Give an example of a linear operator on $V_2(\mathbb{R})$ such that $T^2 = 0$.

But $T \neq 0$

Since $T^2 = 0$, then for $x \in V$ $T^2(x) = 0(x) \Rightarrow T[T(x)] = 0$

$$T(x) \in N(T)$$

$$T(x) \in R(T) \forall x \in V$$

$$R(T) \subset N(T)$$

Hence when $T^2 = 0$ the range of T is contained in $N(T)$.

(B2) Let T be a linear map from $V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ such that $T(a,b) = (0,a)$
 $a(b,0) \in V_2(\mathbb{R})$

$$T \neq 0.$$

$$\text{Also } T^2(a,b) = T[T(a,b)] = T[(0,a)] = (0,0)$$

$$\therefore T^2 = 0$$

2. If $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear operator defined by $T(x,y,z) = (x+z, x-z, y)$. Show that T is invertible and find $T^{-1}(2,4,6)$

$$\rightarrow \text{Let } T(x,y,z) = (0,0,0)$$

$$(x+z, x-z, y) = (0,0,0)$$

$$\Rightarrow x+z=0, x-z=0, y=0$$

By solving the above eqns we get

$$x=0, y=0, z=0$$

$$\therefore \text{for } x \in \mathbb{R}^3, T(x) = 0 \Rightarrow x = 0$$

Thus T is non singular and hence it is invertible

$$\text{Now } T(x,y,z) = (p,q,r)$$

$$x+z=p, x-z=q, y=r$$

$$x = \frac{p+q}{2}$$

$$z = \frac{p-q}{2}$$

$$T^{-1}(p,q,r) = x, y, z$$

$$= \frac{p+q}{2}, \frac{p-q}{2}, \frac{r}{2}$$

$$T(2,4,6) = (3,6,-1)$$

3. Find two linear operators T & S on $V_2(\mathbb{R})$ such that $T(S)$ but $S \cdot T \neq 0$

$$\rightarrow T(a,b) = (0,a) \in V_2(\mathbb{R}) \text{ & } a,b \in \mathbb{R}$$

$$S(a,b) = (a,0) \in V_2(\mathbb{R}) \text{ & } a,b \in \mathbb{R}$$

$$TS(a,b) = T[S(a,b)]$$

$$= T[(0,0)]$$

$$= (0,0) + (0,b) \in V_2(\mathbb{R})$$

$$ST(a,b) = S[T(a,b)]$$

$$= S[$$

Co-ordinate vector:

Let V be a finite dimensional v. space over a field F and let $\dim V = N$ then $B = \{x_1, x_2, \dots, x_N\}$ is a basis of V . Then $\forall x \in V$ $x = a_1x_1 + a_2x_2 + \dots + a_Nx_N$ for $a_i \in F$.

Then the co-ordinate vector of x relative to B which we write as column vector unless otherwise specified as $[x]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$

Matrix representation of linear transformation:

Let U be an m dimensional v. space & V be an n dim v. space
 Let $B = [x_1, x_2, \dots, x_m]$ & $B' = [B_1, B_2, \dots, B_n]$ be the basis of U and V respectively.

Let T be a lintra from U onto V . Since B' is basis of V so that each $T(x_i)$ is a lin-comb. of elements of B'

'For $\alpha_{ij} \in F$ where $1 \leq i \leq m$ & $1 \leq j \leq n$

$$T(x_i) = \alpha_{i1}B_1 + \alpha_{i2}B_2 + \dots + \alpha_{in}B_n$$

$$T(x_1) = \alpha_{11}B_1 + \alpha_{12}B_2 + \dots + \alpha_{1n}B_n$$

$$T(x_m) = \alpha_{m1}B_1 + \alpha_{m2}B_2 + \dots + \alpha_{mn}B_n$$

The transpose of the above matrix of co-efficients denoted by $[T]_B$ is called the matrix representation of T , relative to the ordered basis B .

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

Let V be the n -space of polynomials in t over the field of reals \mathbb{R} of degree less than or equal to 3, and let $D: V \rightarrow V$ be the differential operator defined by diff of polynomial $D[P(t)] = \frac{d}{dt}[P(t)]$.

→ Let us consider the polynomials for the vector space \mathbb{F}^V to compute the matrix of D in basis $B = [1, t, t^2, t^3]$.

Now apply the fn.

$$D[1] = 0 = 0 + 0t + 0t^2 + 0t^3$$

$$D[t] = 1 = 1 + 0t + 0t^2 + 0t^3$$

$$D[t^2] = 2t = 0 + 2t + 0t^2 + 0t^3$$

$$D[t^3] = 3t^2 = 0 + 0t + 3t^2 + 0t^3$$

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thm: Let U be an m -dim v-space over the field F and V an n -dim v-space

Let B be an ordered basis for U and B' an ordered basis for V . Let T be any lin. ft from $U \rightarrow V$. Then for any vector $x \in U$ $[T]_B[x]_B = [T(x)]_{B'}$

Proof: Let $B = \{x_1, x_2, \dots, x_m\}$ be an ordered basis for U and $B' = \{B_1, B_2, \dots, B_n\}$ be the ordered basis for V . T is the lin. ft from $U \rightarrow V$,

Sol Then T is determined by its action on x_i , $1 \leq i \leq m$.

Each of m vectors $T(x_i)$ is uniquely expressible as a lin. comb of elements of B'

$$T(x_i) = \sum_{j=1}^n a_{ij} B_j \quad \text{where } a_{11}, a_{12}, \dots, a_{1n} \text{ are the co-ordinates.}$$

If x be any vector in U , $x = a_1 x_1 + a_2 x_2 + \dots + a_m x_m$

$$[x]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

$$\begin{aligned}
 \text{Now, } T(x) &= T[a_1x_1 + a_2x_2 + \dots + a_nx_n] \\
 &= a_1[T(x_1)] + a_2[T(x_2)] + \dots + a_n[T(x_n)] \\
 &= a_1a_{11}B_1 + a_1a_{12}B_2 + \dots + a_1a_{1n}B_n + a_2a_{21}B_1 + a_2a_{22}B_2 + \\
 &\quad \dots + a_ma_{m1}B_1 + a_ma_{m2}B_2 + \dots + a_ma_{mn}B_n
 \end{aligned}$$

$$[T(x)]_B = \begin{bmatrix} a_1a_{11} & a_2a_{21} & \dots & a_ma_{m1} \\ a_1a_{12} & a_2a_{22} & \dots & a_ma_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1a_{1n} & a_2a_{2n} & \dots & a_ma_{mn} \end{bmatrix}_{n \times m} \xrightarrow{\text{Transpose}} \text{Transpose}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m \times 1}$$

$$[T(x)]_B = [T]_B [x]_B$$

Change of Basis:-

In this section we will see how the representation of a matrix of lin. fr changes if we take another basis.

Let $B = \{x_1, x_2, \dots, x_n\}$ be a bases of V and let, $B' = \{B_1, B_2, \dots, B_n\}$ be another basis of V and suppose

$$B_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$B_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$B_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

Then the transpose of the co-efficient matrix of the above eqn is called transition matrix from bases B & B' and it is written as

$$P \in P = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

The matrix P is invertible and its P^{-1} is the transition matrix from new basis to old basis.

Ex Let $\{(-1, 0), (0, 1)\}$ and $\{(1, 0), (0, 1)\}$ be two bases of \mathbb{R}^2 then

$$(-1, 0) = 1(1, 0) + 1(0, 1) \text{ and } (0, 1) = -1(1, 0) + 0(0, 1)$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

~~Ex Imp~~

Theorem 1: Let P be the transition matrix from basis B to $\phi(B')$ in a V -space V . Then for any vector $\alpha \in V$, $[P(\alpha)]_{B'} = [\alpha]_B$ and $[\alpha]_{B'} = P^{-1}[\alpha]_B$.

Let V be an n -dim V -space and let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \dots, \beta_n\}$ be two bases of V and let P be the transition mat from B to B' , then we have

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{bmatrix}$$

Now suppose $\alpha \in V$ such that $\alpha = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n$. Substituting for β 's from lin fr eqⁿ we get $\alpha = b_1\alpha_{11}\alpha_1 + b_1\alpha_{12}\alpha_2 + \dots + b_1\alpha_{1n}\alpha_n + b_2\alpha_{21}\alpha_1 + b_2\alpha_{22}\alpha_2 + \dots + b_2\alpha_{2n}\alpha_n + \dots + b_n\alpha_{n1}\alpha_1 + b_n\alpha_{n2}\alpha_2 + \dots + b_n\alpha_{nn}\alpha_n$.

$$[\alpha]_{B'} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ and } [\alpha]_B = \begin{bmatrix} b_1\alpha_{11} + b_1\alpha_{12} + \dots + b_1\alpha_{1n} \\ b_2\alpha_{21} + b_2\alpha_{22} + \dots + b_2\alpha_{2n} \\ \vdots \\ b_n\alpha_{n1} + b_n\alpha_{n2} + \dots + b_n\alpha_{nn} \end{bmatrix}$$

$$P[\alpha]_{B'} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\text{Since } P[\alpha]_{B'} = [\alpha]_B \rightarrow P^{-1}P[\alpha]_{B'} = P^{-1}[\alpha]_B$$

$$I[\alpha]_B = P^{-1}[\alpha]_B$$

$$[\alpha]_{B'} = P^{-1}[\alpha]_B$$

Trace of a matrix

Let there be a square matrix of order n over F then $\text{Tr}(A)$ is the sum of the elements of A lying along the principle diagonal of the matrix.

$$\text{Let } A = [a_{ij}]_{n \times n} \quad \text{Tr}(A) = a_{11} + a_{22} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$$

Theorem

Let A and B be two square matrices of order n over a field F and $\lambda \in F$. Then

$$(i) \quad \text{Tr}(\lambda A) = \lambda \text{Tr}(A)$$

$$(ii) \quad \text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$$

$$(iii) \quad \text{Tr}(AB) = \text{Tr}(BA)$$

Proof: Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and $\lambda \in F$ then

$$\lambda A = [\lambda a_{ij}]_{n \times n}$$

$$\begin{aligned} (i) \quad \text{Tr}(\lambda A) &= \sum_{i=1}^n \lambda a_{ii} \\ &= \lambda \sum_{i=1}^n a_{ii} \\ &= \lambda \text{Tr}(A) \end{aligned}$$

$$\begin{aligned} (ii) \quad A+B &= [a_{ij} + b_{ij}]_{n \times n} \\ \text{Tr}(A+B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \end{aligned}$$

$$\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$$

$$(iii) \quad AB = [c_{ij}]_{n \times n}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

and

$$BA = [d_{ij}]_{n \times n}$$

$$d_{ij} = \sum_{k=1}^n b_{ik} a_{kj}$$

$$\text{Tr}(AB) = \sum_{i=1}^n c_{ii}$$

$$\text{Tr}(AB) = \sum_{i=1}^n \left[\sum_{k=1}^n a_{ik} b_{kj} \right]$$

$$= \sum_{k=1}^n \left[\sum_{i=1}^n a_{ik} b_{ki} \right]$$

$$\sum_{k=1}^n \left[\sum_{i=1}^n b_{ki} a_{ik} \right]$$

$$= \sum_{k=1}^n d_{kk}$$

$$\text{Tr}(AB) = \text{Tr}(BA)$$

Theorem 2: Similar matrices have the same trace
Proof: Let A and B be two similar matrices. Then there exists an invertible matrix C such that $B = C^{-1}AC$.

$$\text{Let } D = C^{-1}A$$

$$B = DC$$

$$\text{tr}(B) = \text{tr}(DC)$$

$$\text{tr}(B) = \text{tr}(CD)$$

$$\text{tr}(B) = \text{tr}(CC^{-1}A)$$

$$\text{tr}(B) = \text{tr}(IA)$$

$$\underline{\text{tr}(B) = \text{tr}(A)}.$$

Problem:

1. Let $T: R^2(R) \rightarrow R^2(R)$, where for any $(x, y) \in R^2$, $T(x, y) = (2x, \frac{1}{2}y)$.

Find the matrix associated with T w.r.t the ordered basis $\{(1, 0), (0, 1)\}$

→ Let $B = \{(1, 0), (0, 1)\}$ be an ordered basis of $R^2(R)$ and $T(x, y) = (2x, \frac{1}{2}y)$
 $(a, b) = 2(1, 0) + y(0, 1)$
 $(a, b) = (x+y)$

Let us apply the tr fn. $T(1, 0) = (2, 0) = 2(1, 0) + 0(0, 1)$

$$T(0, 1) = (0, \frac{1}{2}) = 0(1, 0) + \frac{1}{2}(0, 1)$$

$$\therefore [T]_B = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

2. Find the matrix of the lin. fn T on $V_3(R)$ defined as $(a, b, c) = (2b+c, a-4b, 3a)$, w.r.t to the ordered basis B & also w.r.t ordered basis B' where (i) $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, (ii) $B' = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

→ we have, (i) $T(a, b, c) = (2b+c, a-4b, 3a)$

$$\text{then } T(1, 0, 0) = (0, 1, 3) = 0(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$$

$$T(0, 1, 0) = (2, -4, 0) = 2(1, 0, 0) + (-4)(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 0, 1) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$$

$$[T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Let (a, b, c) be any element of $V_3(\mathbb{R})$, then there exists $(x, y, z) \in \mathbb{R}^3$ such that $(a, b, c) = x(1, 1, 1) + y(1, 1, 0) + z(1, 0, 0)$

$$(a, b, c) = (x+y, z, x+y, z)$$

$$c = x$$

$$b = x+y$$

$$a = x+y+z$$

$$\Rightarrow x = c \quad y = b - x \quad z = a - b \rightarrow ①$$

$$T(a, b, c) = (2b+c, a-4b, 3a)$$

$$T(1, 1, 1) = (3, -3, 3)$$

$$T(1, 1, 0) = (2, -3, 3)$$

$$T(1, 0, 0) = (0, 1, 3)$$

$$(a, b, c) = c(1, 1, 1) + (b-x)(1, 1, 0) + (a-b)(1, 0, 0)$$

$$\text{Therefore } T(1, 1, 1) = (3, -3, 3) = 3(1, 1, 1) - 6(1, 1, 0) + 6(1, 0, 0)$$

$$T(1, 1, 0) = (2, -3, 3) = 3(1, 1, 1) - 6(1, 1, 0) + 5(1, 0, 0)$$

$$T(1, 0, 0) = (0, 1, 3) = 3(1, 1, 1) - 2(1, 1, 0) - 1(1, 0, 0)$$

$$[T]_{B'} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 3 & 5 & -1 \end{bmatrix}$$

3 Let T be a linear operator on \mathbb{R}^2 defined by $T(x, y) = (xy, 3x-y)$.

Find the matrix representation of T relative to the basis $\{(1, 3), (2, 5)\}$

→ Let (x, y) be an element of \mathbb{R}^2 then there exists $a, b \in \mathbb{R}$ such that

$$(x, y) = a(1, 3) + b(2, 5)$$

$$x = a+2b \quad x \cdot 3 \rightarrow 3a+6b = 3x$$

$$y = 3a+5b$$

$$\begin{aligned} 3a+6b &= y \\ \hline b &= 3x-y \end{aligned}$$

$$a = x - 6x + 2y$$

$$a = -5x + 2y$$

$$T(x,y) = \begin{pmatrix} 2y \\ 3x-y \end{pmatrix}$$

$$T(1,3) = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$T(2,5) = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$T(1,3) = -30(1,3) + 18(2,5)$$

$$T(2,5) = -48(1,3) + 29(2,5)$$

$$\begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$$

31/10/23
the vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$ & $\alpha_3 = (0,-3,2)$ form a basis for \mathbb{R}^3 . Express each of the std basis vectors as a lin. comb of $\alpha_1, \alpha_2, \alpha_3$.

Let $a, b, c \in \mathbb{R}$ such that $a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$

$$a(1,0,-1) + b(1,2,1) + c(0,-3,2) = 0.$$

$$a+b=0$$

$$2b-3c=0$$

$$-a+b+2c=0$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} = \begin{matrix} 1(7) - 1(-3) \\ 7 + 3 \\ = 10 \end{matrix}$$

$$P(A) = 3$$

The given values are lin. independent & hence it forms a basis for the given V.SPACE
a, b, c will have zero sol?

Let $B = \{(0,0), (0,1,0), (0,0,1)\}$ be the std basis for \mathbb{R}^3

$$\alpha_1 = (1,0,-1) = 1e_1 + 0e_2 - 1e_3$$

$$\alpha_2 = (1,2,1) = 1e_1 + 2e_2 + 1e_3$$

$$\alpha_3 = (0,-3,2) = 0e_1 - 3e_2 + 2e_3$$

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}$$

Let P be the fr. matrix from the bases B to B'

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \Rightarrow |P| = 10 \quad P^{-1} = \frac{1}{|P|} \text{adj}(P)$$

Co-factors of elements

$$A_{11} = 7 \quad \text{if } 2 \text{ row of } P$$

$$A_{12} = 3 \quad \text{are}$$

$$A_{13} = 2$$

$$A_{21} = -2$$

$$A_{22} = 2$$

$$A_{23} = -2$$

$$A_{31} = -3$$

$$A_{32} = 3$$

$$A_{33} = 2$$

$$\therefore \text{adj}(P) = \begin{bmatrix} 9 & -2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & 2 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \frac{1}{10} \begin{bmatrix} 9 & -2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & 2 \end{bmatrix}$$

Now $e_1 = 1e_1 + 0e_2 + 0e_3 \therefore$ Co-ordinate matrix of e_1 related to $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
 Obtain the co-ordinate matrix of e_1 related to the change
 matrix B'

$$[e_1]_{B'} = P^{-1}[e_1]_B = \frac{1}{10} \begin{bmatrix} 9 & -2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 3/10 \\ 2/10 \end{bmatrix}$$

$$\therefore e_1 = \frac{1}{10}e_1 + \frac{3}{10}e_2 + \frac{2}{10}e_3$$

$$[e_2]_{B'} = P^{-1}[e_2]_B = \begin{bmatrix} -2/10 \\ 2/10 \\ -2/10 \end{bmatrix} \quad e_2 = -\frac{2}{10}e_1 + \frac{2}{10}e_2 - \frac{2}{10}e_3$$

$$[e_3]_{B'} = P^{-1}[e_3]_B = \begin{bmatrix} -3/10 \\ 3/10 \\ 2/10 \end{bmatrix} \quad e_3 = -\frac{3}{10}e_1 + \frac{3}{10}e_2 + \frac{2}{10}e_3$$

5. Consider the following lin. op & on \mathbb{R}^2 & a basis S . $G(x, y) = (2x - 7y, 4x + 3y)$ $S = \{u_1, u_2\} = \{(1, 3), (2, 5)\}$

a. Find the mat. representation $G(S)$ of $[G]_S$

b. Verify $[G]_S [V]_S = [G(V)]_S$ for the vector $V = (4, -3)$ in \mathbb{R}^3 .

$$\Rightarrow G(u_1) = (2(1) - 7(3), 4(1) + 3(3)) \\ = (-19, 13)$$

$$G(u_2) = (-31, 23)$$

Now find co-ordinates of arbitrary vector $\mathbf{v} = (a, b)$ in \mathbb{R}^2 related to basis S .

$$\begin{bmatrix} a \\ b \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{aligned} x+2y &= a \\ 3x+5y &= b \end{aligned} \quad \begin{array}{l} \uparrow 3 \\ \begin{array}{l} 3x+6y=3a \\ 3x+5y=b \end{array} \end{array} \quad \begin{array}{l} x+6a-2b=a \\ x=\underline{-5a+2b} \end{array}$$

$$\therefore (a, b) = (-5a+2b)u_1 + \frac{y=3a-b}{(3a-b)u_2}$$

$$V = [-5a+2b, 3a-b] \quad \rightarrow \text{sub. } G(u_1) \& G(u_2)$$

$$[G]_S = \begin{bmatrix} 121u_1 & -40u_2 \\ 201u_1 & -116u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 121 & 201 \\ -40 & -116 \end{bmatrix} \rightarrow [G]_S.$$

use the formula $(a, b) = (-5a+2b)u_1 + (3a-b)u_2$ to get $V = (4, -3) = -26u_1 + 15u_2$

$$[V]_S = \begin{bmatrix} -26 \\ 15 \end{bmatrix}$$

$$[G]_S [V]_S = \begin{bmatrix} 121 & 201 \\ -40 & -116 \end{bmatrix} \begin{bmatrix} -26 \\ 15 \end{bmatrix} = \begin{bmatrix} -131 \\ 80 \end{bmatrix}$$

$$G(V) = G(4, -3) = \underbrace{(29, 7)}_{\text{Sub in } f_1 \& f_2} \rightarrow \text{Sub in } V = [-5a+2b, 3a-b]$$

$$V = [-131, 80]$$

c. Consider the following basis of \mathbb{R}^2 . $E = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ and $S = \{u_1, u_2\} = \{(1, 3), (2, 4)\}$

a. Find change of basis matrix P from usual basis $[E]_S$

b. Find the change of basis matrix Q from S back to E

c. Find the "co-ordinate" vector $[v]_S$ of $v = (5, -3)$ related to S

\rightarrow a.

$$A = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}_{P} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad e_1$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$b. (a, b) = 2(1, 3) + 3(1, 4)$$

$$\begin{aligned} x+y &= a \Rightarrow x = 4a-b \\ 3x+4y &= b \quad y = -3a+b. \end{aligned}$$

$$\text{or } Q = P^{-1}$$

$$V = (a, b) = (4a-b)(1, 3) + (-3a+b)(1, 4)$$

$$V = (4a-b)u_1 + (-3a+b)u_2$$

P_i

$$Q = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

Sub
(5, -3)

$$c. [V]_s = \begin{bmatrix} 23 \\ -18 \end{bmatrix}$$

Unit-3

Part-A

Diagonalization:

Suppose an n^2 matrix A is given, the mat A is said to be diagonalizable, if there exists a non singular matrix P such that

$$B = P^{-1}AP \text{ where } B \text{ is diagonal}$$

Suppose a lin. operator T which maps $T: V \rightarrow V$ is given. The lin. operator T is said to be diagonalizable if there exists a basis \mathcal{B} of V such that the mat representation of T relative to the basis \mathcal{B} is a diagonal matrix.

We concentrate mainly on condition under which the lin. operator T is diagonalizable.

Polynomials of matrices:-

Consider a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$ over a field K . Consider a sq. mat A we define $f(A)$ as follows.

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$$

where I is the identity matrix.

In particular we say that mat A is a root of $f(t)$ if $f(A) = 0 \therefore$ zero matrix

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$. Let $f(t) = 2t^2 - 3t + 5$ & $g(t) = t^2 - 5t - 2$. Find $f(A)$ & $g(A)$

$$\rightarrow f(t) = 2t^2 - 3t + 5$$

$$\begin{aligned} f(A) &= 2A^2 - 3A + 5 \\ &= 2\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 14 \\ 21 & 37 \end{bmatrix} \end{aligned}$$

$$g(t) = t^2 - 5t - 2$$

$$g(A) = A^2 - 5A - 2$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore g(A) \text{ is a root of } g(t).$$

2. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ find $f(A)$ where a. $f(t) = t^2 - 3t + 7$ b. $f(t) = t^2 - 6t + 13$

$$\rightarrow A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1-8 & 4 \\ -2-10 & -8+25 \end{bmatrix} = \begin{bmatrix} -7 & 24 \\ -12 & 17 \end{bmatrix}$$

a. $f(t) = t^2 - 3t + 7$.

$$f(A) = \begin{bmatrix} -7 & 24 \\ -12 & 17 \end{bmatrix} - \begin{bmatrix} 3 & 12 \\ -6 & 15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -6 & 9 \end{bmatrix}$$

b. $f(t) = t^2 - 6t + 13$

$$f(A) = \begin{bmatrix} -7 & 24 \\ -12 & 17 \end{bmatrix} - \begin{bmatrix} 6 & 24 \\ -12 & 30 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$f(A)$ forms a root of $f(t)$

Note: Let f and g be polynomials. For any sq. mat A & scalar k .

$$(i) (f+g)A = f(A) + g(A) \quad (ii) (fg)A = f(A) \cdot g(A) \quad (iii) (kf)A = k f(A)$$

$$(iv) f(A) \cdot g(A) = g(A) \cdot f(A)$$

Characteristic polynomial and Cayley-Hamilton theorem.

Let $A = [a_{ij}]$ be a sq. matrix. The mat $M = A - tI_n$, where I_n is the n^2 identity matrix and t is an indeterminate. This may be obtained by subtracting t down the diag. of A . The -ve of M is the mat $-M = tI_n - A$ & its determinant is given by

$$\Delta t = \det(tI_n - A)$$

$$= (-1)^n \det(A - tI_n)$$

which is a polynomial in t of degree n is called the characteristic polynomial of t .

$$\begin{aligned} \Delta(t) &= \det(tI_n - A) \\ &= (t - a_{11})(t - a_{22}) \dots (t - a_{nn}) \end{aligned}$$

Examples

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ its characteristic eqn is given by $\Delta(t) = tI_2 - A$

$$\begin{aligned} \Delta(t) &= (tI_2 - A) \\ &= \begin{vmatrix} t-1 & 3 \\ 4 & t-5 \end{vmatrix} = (t-1)(t-5) - 12 \\ &\quad + 2 - 5t - t + 5 - 12 \\ \Delta(t) &= \underline{\underline{t^2 - 6t - 7}} \end{aligned}$$

Cayley-Hamilton theorem

Every mat A is a root of its characteristic polynomial

Proof: Suppose $A = [a_{ij}]$ is a triangular mat. Then $tI_n - A$ is a triangular mat with diagonal entries $t - a_{ii}$ and hence

$$\begin{aligned} \Delta(t) &= \det(tI_n - A) \\ &= (t - a_{11})(t - a_{22}) \dots (t - a_{nn}) \end{aligned}$$

Observe that the roots of $\Delta(t)$ are, diagonal elements of A .

Example: Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ & its characteristic eqn is $t^2 - 6t + 7$. Now according to Ch theorem A is the root of $\Delta(t)$
 $\Delta(A) = A^2 - 6A + 7I$.

$$A^2 = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+12 & 3+15 \\ 4+20 & 12+25 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 24 & 37 \end{bmatrix}$$

$$\Delta(A) = \begin{bmatrix} 13 & 18 \\ 24 & 37 \end{bmatrix} - \begin{bmatrix} 6 & 18 \\ 24 & 30 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

NOW, suppose A & B are similar matrices $B = P^{-1}AP$ where P is invertible. Then both matrix A & B have same characteristic polynomial.

Proof: Let $t\mathbb{I} = P^{-1}t\mathbb{I}P$, we have $\Delta_B(t) = \det(t\mathbb{I} - B)$

$$\begin{aligned} &= \det(P^{-1}t\mathbb{I}P - B) \\ &= \det(P^{-1}t\mathbb{I}P - P^{-1}AP) \\ &= \det[P^{-1}(t\mathbb{I} - A)P] \\ &= \det(P^{-1}) \cdot \det(t\mathbb{I} - A) \cdot \det(P) \end{aligned}$$

Since $\det(P^{-1}) \cdot \det(P) = 1$.

$$\Rightarrow \Delta_B(t) = \det(t\mathbb{I} - A)$$

$$\Delta_B(t) = \Delta_A(t)$$

Characteristic polynomial of degree 2 & 3:

a. Suppose, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then characteristic polynomial $\Delta(t) = t^2 - (a_{11} + a_{22})t + \det(A)$

$$\Delta(t) = t^2 - tr(A)t + \det(A)$$

b. Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then characteristic polynomial

$$\Delta(t) = t^3 - tr(A)t^2 + (a_{11} + a_{22} + a_{33})t - \det(A)$$

$a_{11} + a_{22} + a_{33}$ are co-factors of the given matrix

1. Find the characteristic polynomial for each of the polynomials

Q. $A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix} \rightarrow \Delta(t) = t^2 - 15t + 44$

$$b) B = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix} \quad \Delta(t) = t^2 - 9t + 20$$

$$c) C = \begin{bmatrix} 5 & -2 \\ 4 & -4 \end{bmatrix} \quad \Delta(t) = t^2 - t - 12$$

d) Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$

$$\rightarrow A_{11} = -27 - 6 = 21$$

$$A_{22} = 9 - 2 = 7$$

$$A_{33} = 3$$

$$\begin{aligned} |\Delta| &= 1(21) - 1(-2) + 2(-3) \\ &= 17 \end{aligned}$$

$$\underline{\Delta(t) = t^3 - 13t^2 + 31t - 17}$$

Diagonalization Eigen values and Eigen vectors

Let A be any square matrix, then A can be represented by a diagonal matrix $D = \text{diag}(k_1, k_2 \dots k_n)$ iff there exists a basis \mathbf{v} consisting of column vectors such that $v_1, v_2 \dots v_n$ such that

$$Av_1 = k_1 v_1$$

$$Av_2 = k_2 v_2$$

$$Av_n = k_n v_n$$

In such a case A is said to be diagonalizable Furthermore $D = P^{-1}AP$ where P is the non-singular matrix whose columns are respectively the basis vectors $v_1, v_2 \dots v_n$

Definitions

Let A be any sq. mat. A scalar λ is called an eigenvalue of A if there exists a non-zero column vector v such that $Av = \lambda v$.

Any vector satisfying this relation is called an eigen vector of A belonging to the eigen value λ .

Example: Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ and let $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Find diag mat D & obtain mat A from diag mat D

\rightarrow Let us find the eigen vector vectors.

$$AV_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = V_1 \quad \lambda_1 = 1$$

$$AV_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = V_2 \quad \lambda_2 = 4$$

Therefore V_1 & V_2 are eigen vectors of A belonging respectively to the eigen values $\lambda_1 = 1$ & $\lambda_2 = 4$. Observe that V_1 & V_2 are linearly independent & hence form a basis of \mathbb{R}^2 . Accordingly A is diagonalizable by using transition matrix P . $D = P^{-1}AP$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \frac{1}{|P|} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

As expected the diagonal elements 1 & 4 are eigen values respectively with eigen vectors V_1 & V_2 which are the column of P .

To factorize A from D :

$$A = PDP^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A^m = PD^mP^{-1} \Rightarrow D^4 = \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix}$$

$$A^4 = P D^4 P^{-1} = \begin{bmatrix} 171 & 85 \\ 170 & 86 \end{bmatrix}$$

$$\text{Suppose } f(t) = t^3 - 5t^2 + 3t + 6$$

$$f(1) = 1 - 5 + 3 + 6 = 5$$

$$f(-1) = 64 - 80 + 12 + 6 = 2$$

$$f(A)_2 = P D P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ -10 & 2 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

Computing eigen values & eigen vectors diagonalizing matrices

Step 1: Find the char. polynomial $\Delta(t)$ of A

Step 2: Find the roots of $\Delta(t)$ to obtain the eigen values of A .

Step 3: Repeat step 1 & step 2 for each eigen value $\lambda(A)$. From the matrix

$$M = A - \lambda I \text{ by subtracting } \lambda \text{ down the diagonal of } A$$

Step 4: Find a basis for the soln space of the homogeneous system $Mx=0$
(we obtain eigen vectors from this.)

Step 5: Consider the collection $S = \{v_1, v_2, \dots, v_n\}$ of all eigen vectors
obtained in step 4. If $M \neq n$, then A is not diagonalizable.
If $M=n$ then A is, diagonalizable using $D = P^{-1}AP$.

1. Find the diagonal elements for the given matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$

$$\Rightarrow \Delta(t) = t^2 - (3)t + -10$$

$$= t^2 - 3t - 10$$

$$= (t+2)(t-5)$$

$$\lambda_1 = 5 \quad \lambda_2 = -2$$

$$M = A - \lambda I.$$

$$= \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \rightarrow \lambda_1 = 5$$

$$\lambda_2 = -2$$

$$M = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Mx = 0$$

$$\begin{cases} -x + 2y = 0 \\ 3x - 6y = 0 \end{cases} \rightarrow -x + 2y = 0$$

y → free variable

x → pivot element

$$Mx = 0$$

$$\begin{cases} 6x + 2y = 0 \\ 3x + y = 0 \end{cases} \rightarrow 3x + y = 0$$

$$y = 3 \rightarrow x = -1$$

$$-x + 2y = 0$$

$$y = 1 \rightarrow x = 2$$

$$v_1 = (2, 1)$$

$$v_2 = (-1, 3)$$

Let P be the mat whose cols are v_1 & v_2

$$\Rightarrow P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{|P|} \text{adj}^T P = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$= \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

2. Find D for given mat $B = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$

$$\rightarrow \text{Char eqn} = \Delta(t) = t^2 - 8t + 16.$$

$$\lambda_1 = \lambda_2 = 4. = (t-4)^2$$

$$\frac{\sqrt{64-64}}{2} = 0.$$

$$M = A - \lambda I$$

$$\rightarrow \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$Mx = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \begin{cases} x-y=0 \\ x-y=0 \end{cases} \quad \begin{cases} x-y=0 \\ x-y=0 \end{cases}$$

$$y=1 \Rightarrow x=1 \quad V = (1, 1)$$

Ex1

The system has only one independent soln i.e. $x=1, y=1$. Thus $V = (1, 1)$ & its multiples are the only eigen vectors of V .

Accordingly B is not diagonalizable since there does not exist a basis consisting of eigen vectors of B .

$$3. \text{ Find } D \text{ for } A = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$$

$$\rightarrow \Delta(t) = t^2 - 10t + 11 = t^2 + 1$$

We consider 2 cases.

1. A is matrix over real field R. $\Delta(t)$ has no real roots. Thus A has no eigen values and no eigen vectors. A is not diagonalizable.

2. A is a mat over the complex field C. $\Delta(t) = (t-i)(t+i)$ has two roots i.e. i which forms the eigen values and A has two independent eigen vectors. $\therefore A$ is diagonalizable over field C.

4. Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ as a real symmetric mat. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

$$\rightarrow \Delta(t) = t^2 - 7t + 6.$$

$$(t-6)(t-1)$$

$$\frac{7+\sqrt{49-24}}{2}$$

$$M_2 = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$$

$$V_2 \in M_2 X = \begin{cases} -4x - 2y = 0 \\ -2x - y = 0 \end{cases} \Rightarrow \begin{cases} -2x - y = 0 \\ x - 2y = 0 \end{cases} \Rightarrow y = 2 \Rightarrow x = -1 \\ V_2 = (-1, 2)$$

$$M_2 \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \rightarrow \begin{cases} x - 2y = 0 \\ -2x + 4y = 0 \end{cases} \Rightarrow \begin{cases} x - 2y = 0 \\ x - 2y = 0 \end{cases} \Rightarrow y = 1, x = 2 \\ V_2 = (2, 1).$$

Since V_1 & V_2 are orthogonal, normalizing V_1 & V_2 yields orthonormal vectors.

$$\hat{V}_1 = \frac{V_1}{\sqrt{V_1^2}} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \quad \hat{V}_2 = \frac{V_2}{\sqrt{V_2^2}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \rightarrow P^{-1} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \quad -\frac{1}{5} - \frac{4}{5} = -\frac{5}{5} = -1$$

$$D = P^{-1} A P$$

$$\Rightarrow \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

5. Consider Q. eqn $Q(x, y) = 2x^2 - 4xy + 5y^2$. Diagonalize the given eqn
As per earlier problem, we obtain $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

glut23

1. Find the char. polynomial of each

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix}$$

$$\rightarrow a. \Delta(t) = t^2 - 3t - 18$$

$$b. \Delta(t) = t^2 + 5t + 1$$

$$c. \Delta(t) = t^2 + 9,$$

2. Find the char. polynomial. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 6 & 4 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 3 & -4 \end{bmatrix}$

$$\rightarrow a. t^3 - 6t^2 + (-16 - 13 - 6)t - 38$$

$$t^3 - 6t^2 - 35t - 38$$

$$b. t^3 - t^2 + (-8 - 4 + 20)t + 62$$

$$\underline{t^3 - t^2 + 8t + 62}$$

$$+ (-8) - 6(12) - 2(-9)$$

$$- 8 - 72 + 18$$

3. Find char. polynomial

$$A = \begin{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} & 1 & 1 \\ 0 & \begin{bmatrix} 6 & -5 \\ 2 & 3 \end{bmatrix} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta_1(t) = t^2 - 6t + 3 \quad \rightarrow A(t) = \Delta_1(t) \cdot \Delta_2(t)$$

$$\Delta_2(t) = t^2 - 9t + 28 \quad = (t^2 - 6t + 3)(t^2 - 9t + 28)$$

$$= t^4 - 9t^3 + 28t^2 - 6t^3 + 54t^2 - 168t$$

$$+ 3t^2 - 27t + 84$$

$$= \underline{\underline{t^4 - 15t^3 + 85t^2 - 195t + 84}}$$

$\Rightarrow \Delta_1(t)$

$$B = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} & 2 & 2 \\ 0 & \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta_1(t) = t^2 - 4t + 3$$

$$\Delta_2(t) = t^2 - 11t + 30$$

$$\Delta(t) = (t^2 - 4t + 3)(t^2 - 11t + 30)$$

$$= t^4 - 11t^3 + 30t^2 - 4t^3 + 44t^2 - 120t$$

$$+ 3t^2 - 33t + 90$$

$$= \underline{\underline{t^4 - 15t^3 + 77t^2 - 153t + 90}}$$

4. Find char poly. of each of the following lin. operators,

a. $F: R^2 \rightarrow R^2$ defined by $F(x, y) = (3x + 5y, 2x - 7y)$

b. $D: V \rightarrow V$ defined by $D(t) = df/dt$, $S = \{ \sin t, \cos t \}$

\rightarrow a) $A = \begin{bmatrix} 3 & 5 \\ 2 & -7 \end{bmatrix} \quad \rightarrow D(t) = t^2 + 4t - 31$

b) The matrix A representing the differential operator relating to the basis S we get

$$D(\sin t) = \cos t \rightarrow 0 \sin t + 1 \cos t$$

$$D(\cos t) = -\sin t \rightarrow -1 \sin t + 0 \cos t$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$d(t) = t^2 + 1$$

5. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ a. Find all eigen values and corresponding eigen vectors

b. Find a non-singular mat P such that $D = P^{-1}AP$ is

c. Find A^6 and $f(A)$ where $t^4 - 3t^3 - 6t^2 + 7t + 8$ d. Find a real cube root of A i.e., a mat B such that $B^3 = A$ & B has real eigen values.

$$\Delta(t) = t^2 - 5t + 4.$$

$$\lambda_1 = 4, \lambda_2 = 1.$$

$$\frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2} = 4, 1$$

$$M = A - \lambda_1 I$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$M = A - \lambda_2 I$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$MX = 0$$

$$\begin{cases} -2x + 2y = 0 \\ x - y = 0 \end{cases} \quad \begin{cases} x - y = 0 \\ x + 2y = 0 \end{cases}$$

$$y = 1, x = 1$$

$$v_1 = (1, 1)$$

$$MX = 0$$

$$\begin{cases} x + 2y = 0 \\ x + 2y = 0 \end{cases} \quad \begin{cases} x + 2y = 0 \\ y = 1, x = -2 \end{cases}$$

$$v_2 = (-2, 1)$$

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \cdot \text{adj}^c P \Rightarrow \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow D^6 = \begin{bmatrix} 4096 & 0 \\ 0 & 1 \end{bmatrix}$$

c.

$$A^6 = P D^6 P^{-1}$$

$$= \begin{bmatrix} 1366 & 2730 \\ 1365 & 2731 \end{bmatrix}$$

$$f(\lambda_1) = 4^4 - 3(4)^3 - 6(4)^2 + 7(4) + 3 = -1$$

$$f(\lambda_2) = 1^4 - 3(1)^3 - 6(1)^2 + 7 + 3 = 2$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

d. The real cube root of A can be found by finding cube root of D which is equal to.

$$D = \begin{bmatrix} \sqrt[3]{4} & 0 \\ 0 & \sqrt[3]{1} \end{bmatrix}$$

Therefore cube root of A is nothing but mat B as given by

$$B = P^{-1}DP^{-1}$$

$$B = P^{-1} \begin{bmatrix} 1.195 & 0.3916 \\ 0.3916 & 1 \end{bmatrix} P$$

6. Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$
- Find all eigenvalues & eigenvectors.
 - Is A diagonalizable. If yes then find P such that $D = P^{-1}AP$ is diag.

$$\rightarrow \Delta(t) = t^3 - 11t^2 + \left(\frac{12+9+18}{39}\right)t - 45$$

$$4(12) - 1(6) - 1(-3) = 48 - 6 + 3 = 45$$

$$\lambda_1 = 5 \quad \lambda_2 = 3 \quad \lambda_3 = 3.$$

$$M = A - \lambda I$$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 - 3 \end{bmatrix}$$

$$MX = 0$$

$$x + y - z = 0$$

$$2x - 2z = 0$$

$$x + y - 3z = 0$$

Assume free variable as $z = 1$

$$2y - 4z = 0 \quad z = 1 \Rightarrow y = 2 \quad x = 1$$

$$y = 2z$$

$$x = z$$

$$\rightarrow v_1 = (1, 2, 1)$$

$$M = A - \lambda I$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{array}{l} x+y-z=0 \\ 2x+2y-2z=0 \\ x+y-z=0 \end{array} \quad v_2, v_3 \text{ are free variables}$$

$$\Rightarrow y = -1, z = 0.$$

$$v_2 = (1, -1, 0)$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow A_{11} = \dots$$

$$A_{12} =$$

$$t^3 - 11t^2 + 39t - 45$$

$$A^3 - 11A^2 + 39A - 45I$$

$$\Rightarrow A^3 - 11A^2 + 39A - 45AA'$$

$$A(A^2 - 11A + 39I - 45A') = 0.$$

$$P^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 1 & 0 & -1 \\ -0.5 & -0.5 & 1.5 \end{bmatrix} \quad A' = \frac{1}{45} (A^2 - 11A + 39I)$$

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore \text{Let mat } B = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix} \quad 3(-10+6) + 1(14-6) + 1(-58+30) \\ -12 + 8 - 28$$

$$\rightarrow \Delta(t) = t^3 - 0t^2 + \underbrace{(-4+0+ -8)}_{-12} t + 16$$

$$\lambda_1 = -4, \lambda_2 = 2, \lambda_3 = -8-2i$$

$$\lambda_1 = 1, 2, -4$$

$$g \begin{vmatrix} 1 & 0 & -12 & 16 \\ 0 & 2 & 4 & -16 \\ 1 & 2 & -8 & 0 \end{vmatrix}$$

$$M_2 = A - \lambda I$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 7 & -6 & 1 \\ 6 & -6 & 6 \end{bmatrix}$$

$$\begin{aligned} 2x - y + z &= 0 \\ 7x - 6y + z &= 0 \\ 6x - 6y + 6z &= 0 \end{aligned}$$

$$2x - y + z = 0$$

$$7x - 6y + z = 0$$

$$M_2 = \begin{bmatrix} 1 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{bmatrix}$$

$$V_1 = (1, 1, 0)$$

$$7x - y + z = 0$$

$$V_2 = (0, 1, 1)$$

$$7x - y + z = 0$$

$$6x - 6y + 6z = 0$$

$$z = 1$$

$$M_2 = \begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix}$$

$$x - y + z = 0$$

$$-x - 7y + z = 0$$

$$6x - 6y = 0 \Rightarrow x = y, \\ z = 0$$

$$V_1 = (1, 1, 0)$$

~~Ans~~

$$V_1 = (1, 1, 0) \quad V_2 = (0, 1, 1)$$

$$P_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

These basis

The basis for $\mathcal{S} = V_1, V_2$ which has only 2 independent eigenvectors. But we require 3 basis vectors to match the dim of D .

\therefore mat B is not diagonalizable

$$\text{Let mat } B = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

$$-44 - 16$$

$$-11 - 64$$

$$\Delta(t) = t^3 - 6t^2 + (0 + \underbrace{-60 + -75}_{-135})t + -400$$

$$11(4 - 4) + 8(32 + 8) + 4(16 + 4)$$

$$320 + 80 \rightarrow 400$$

$$\lambda_1 = -5 \quad \lambda_2 = 16$$

$$\begin{array}{r|rrrr} -5 & 1 & -6 & -135 & -400 \\ & 0 & -5 & 55 & +400 \\ & & 1 & -11 & -80 \\ & & & & 0 \end{array}$$

$$t^2 - 11t - 80$$

$$\frac{11 \pm \sqrt{121 - 120}}{2}$$

$$\frac{11 + 1}{2} = 6.5$$

$$M = \begin{bmatrix} 16 & -8 & 4 \\ -8 & 4 & -2 \\ 4 & -2 & 1 \end{bmatrix} \quad \begin{array}{l} 16x - 8y + 4z = 0 \\ -8x + 4y - 2z = 0 \\ 4x - 2y + z = 0 \end{array}$$

$$M = \begin{bmatrix} -5 & -8 & 4 \\ -8 & -17 & 2 \\ 4 & -2 & -20 \end{bmatrix} \rightarrow \begin{array}{l} -5x - 8y + 4z = 0 \\ -8x - 17y - 2z = 0 \\ 4x - 2y - 20z = 0 \end{array}$$

$$D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Q. Find the char. eqn of the mat A_F

$$\rightarrow |A - \lambda I| = 0$$

Verify Cayley Hamilton theorem for the given matrix hence find its inverse

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = (-\lambda)[(2-\lambda)(3-\lambda)] + 2[-4+2\lambda] \\ = (1-\lambda)(6-2\lambda-3\lambda+\lambda^2) - 8+4\lambda \\ = 6-5\lambda+\lambda^2-6\lambda+5\lambda^2-\lambda^3-8+4\lambda \\ = -\lambda^3+6\lambda^2-7\lambda+2 \\ \lambda^3-6\lambda^2+7\lambda+2=0.$$

$$A(t) = t^3 - 6t^2 + (6-1+2)t + 2$$

$$A(t) = t^3 - 6t^2 + 7t + 2$$

To verify Cayley Hamilton theorem we substitute the given matrix to the char. eqn.

$$\rightarrow A^3 - 6A^2 + 7A + 2I \quad \rightarrow ①$$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 28 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 18 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0$$

Since determinant for given matrix is non zero, therefore Inverse exists.

Multiply eqn ② by \tilde{A}^{-1}

$$A^2 - 6A + 7I + 2\tilde{A}^{-1} = 0.$$

$$\tilde{A}^{-1} = \frac{1}{2}[A^2 + 6A - 7I]$$

$$= \frac{1}{2} \left[- \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} + 6 \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \right]$$

$$= \begin{pmatrix} -3 & 0 & 2 \\ -1 & 0.5 & 0.5 \\ 2 & 0 & -1 \end{pmatrix}$$

10. Consider the matrix $A' = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. Also find D .

$$\rightarrow |A - \lambda I| = 0$$

$$-2\lambda + \lambda^2$$

$$\begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} \quad (3-\lambda)(2-\lambda) - 2(4-2\lambda) = 0 \\ = -6\lambda + 3\lambda^2 - 8 + 4\lambda = 0. \\ + 2\lambda^2 - \lambda^3 \\ = -\lambda^3 + 5\lambda^2 - 2\lambda - 8 = 0$$

$$\xrightarrow{\text{R1} - R2} \begin{vmatrix} 1 & -5 & 2 & 8 \\ 0 & -1 & 6 & -8 \\ 1 & -6 & 8 & 0 \end{vmatrix}$$

$$\rightarrow \lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$$

$$\xrightarrow{\text{M2} - A - \lambda I}$$

$$\lambda_1 = -1, 2, 4$$

$$\frac{6 \pm \sqrt{36 - 32}}{2}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\frac{6+8}{2}, \underline{4, 2}$$

$$5x + 2y = 0$$

$$-2x + y = 0$$

$$2x + y = 0$$

$$x + 3z = 0$$

$$x + y + z = 0$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\rightarrow R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 2 & -12 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 + 3x_3 = 0$$

$$x_2 - 6x_3 = 0$$

$$x_3 = 1$$

$$V_1 = (-3, 6, 1)$$

$$x_1 = -3$$

$$x_2 = 6$$

$$M = A - \lambda I$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + 2x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

$$x_1 = 0$$

NO x_3 .

$$V_2 = (0, 0, 1)$$

$\rightarrow x_1 = 0, x_2 = 0$ Assume $x_3 = 1$

$$M = A - \lambda I$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$-x_1 + 2x_2 = 0$$

$$2x_1 - 4x_2 = 0$$

$$x_1 - 2x_3 = 0$$

$$x_1 - 2x_2 = 0$$

$$x_1 - 2x_3 = 0$$

$$V_3 = (2, 1, 1)$$

$$x_3 = 1 \Rightarrow x_1 = 2$$

$$P_2 = \begin{bmatrix} -3 & 0 & 2 \\ 6 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{15} & \frac{2}{15} & 0 \\ -\frac{1}{3} & \frac{1}{3} & 1 \\ \frac{2}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

Do in adj form
or sub P in chal.
eqn

$$D = P^{-1} A P$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Unit-3

Part - B

Inner product spaces:- and orthogonality

Definition: Let V be a real n -space. Suppose to each pair of vectors $(u, v) \in V$ there is assigned a real no. denoted by $\langle u, v \rangle$. This function is called inner product of V if it satisfies the following axioms.

$[E_1]$ (Linear property): $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$

$[S_2]$ (Symmetric property): $\angle U, UV = \angle V, UW$

[83] (positive definite property): $\langle u_1, u_2 \rangle > 0$ and $\langle u, u \rangle = 0$ if $u = 0$.

Example: Let V be a real inner product space. Then by linearity property, $\langle 3u, -4u_2, 2v, -5v_2 + 6v_3 \rangle = 6\langle u, v \rangle - 15\langle u, v_2 \rangle +$

$$-8 \langle u_2, v_1 \rangle + 20 \langle u_2, v_2 \rangle - 24 \langle u_2, v_3 \rangle$$

Example 21 Let $u = (1, 3, -4, 2)$, $v = (4, -2, 2, 1)$, $w = (5, -1, -2, 6)$ in \mathbb{R}^4

$$a. \& T \langle 3u - 2v, w \rangle = 3\langle u, w \rangle - 2\langle v, w \rangle$$

$$\langle u, w \rangle = 5 - 3 + 8 + 12 = 22$$

$$\angle v, w \gamma = 20 + 2 - 4 - 16 = 24$$

$$3(0-20) = (3, 9, -12, 6) - (8, -4, 4, 2) = (-5) \begin{pmatrix} 15 \\ -18 \\ 8 \end{pmatrix}$$

$$\langle 3u-2v, w \rangle = -25 - 5 + 16 + 48 = 18$$

$$3\angle U, w_4 - 2\angle V, w_4 = 18$$

12
 3. Consider vectors $U = (1, 2, 4)$, $V = (2, -3, 5)$, $W = (4, 2, -3)$ in \mathbb{R}^3 . Find
 a. $U \cdot V$ b. $U \cdot W$ c. $V \cdot W$ d. $(U+V) \cdot W$ e. $\|U\|$ f. $\|V\|$

$$\rightarrow a. U \cdot V = 2 - 6 + 20 = 16$$

$$b. U \cdot W = 8 - 6 - 15 = -13$$

$$c. V \cdot W = 8 - 6 - 15 = -13$$

$$d. (U+V) \cdot W = 12 - 2 - 27 = -17$$

$$e. \|U\| = \sqrt{1+4+16} = \sqrt{21}$$

$$f. \|V\| = \sqrt{4+9+25} = \sqrt{38}$$

4. Consider the following polynomials in $P(t)$ with inner pr. $\langle f, g \rangle$

$$= \int_0^1 f(t) \cdot g(t) dt \text{ where } f(t) = t+2, g(t) = 3t-2, h(t) = t^2-3t-3$$

a. find $\langle f, g \rangle$ & $\langle f, h \rangle$

b. find $\|f\|$, $\|g\|$

c. normalize f & g

$$\rightarrow a. \langle f, g \rangle = \int_0^1 (t+2) \cdot 3t-2 dt = \int_0^1 3t^2 - 2t + 6t - 4 dt$$

$$= \left[t^3 + 2t^2 - 4t \right]_0^1$$

$$= (1+2-4) \leftarrow 0$$

$$= \underline{\underline{-1}}$$

$$\langle f, h \rangle = \int_0^1 (t+2) t^2 - 3t - 3 dt$$

$$= \int_0^1 t^3 - t^2 - 9t - 6 dt$$

$$= \left[\frac{t^4}{4} - \frac{t^3}{3} - \frac{9t^2}{2} - 6t \right]_0^1$$

$$= -10.583$$

$$\begin{aligned}
 b) \|f\| &= \langle f, f \rangle = \sqrt{\int_0^1 f(t) \cdot f(t) \cdot dt} \\
 &= \sqrt{\int_0^1 t^2 + 4 + 4t \cdot dt} \\
 &= \sqrt{6.33} \\
 &= \underline{\underline{2.51}}
 \end{aligned}$$

$$\begin{aligned}
 \|g\| &= \langle g, g \rangle = \sqrt{\int_0^1 9t^2 + 4 - 6t \cdot dt} \quad (3t-2)(3) \\
 &= 1
 \end{aligned}$$

$$c) \hat{f} = \frac{f}{\|f\|} = \frac{t+2}{2.51} = 0.39t + 0.79$$

$$\hat{g} = \frac{g}{\|g\|} = \frac{3t-2}{1} = 3t-2$$

Orthogonality:

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal & u is said to be orthogonal to v if $\langle u, v \rangle = 0$. The relation is clearly symmetric i.e. if u is orthogonal to v then v is orthogonal to u .

Examples Find the orthogonality for the vectors $u = (1, 1, 1)$ $v = (1, 2, -3)$ $w = (1, -4, 3)$ in R^3 . Then find / verify whether $\langle u, v \rangle$, $\langle u, w \rangle$, $\langle v, w \rangle$ is orthogonal to each other or not.

$$\rightarrow \langle u, v \rangle = 1 + 2 - 3 = 0 \quad \left. \right\} \text{orthogonal}$$

$$\langle u, w \rangle = 1 - 4 + 3 = 0$$

$$\langle v, w \rangle = 1 - 8 - 9 = -16 \rightarrow \text{not orthogonal}$$

Q Verify whether the functions $\sin t$ & $\cos t$ in vector space $C[-\pi, \pi]$ of continuous fns on the closed interval is orthogonal or not

$$\begin{aligned}
 \rightarrow \langle \sin t, \cos t \rangle &= \int_{-\pi}^{\pi} \sin t \cdot \cos t \cdot dt \\
 &= 0
 \end{aligned}$$

12
3. Find k so that $u = (1, 2, k, 3)$ $v = (3, k, 7, -5)$ in \mathbb{R}^4 are
orthogonal

$$\Rightarrow u \cdot v = 3 + 2k + 7k - 15 = 0$$

$$9k = 12$$

$$k = \frac{12}{9} = \underline{\underline{\frac{4}{3}}}$$