

UNIT - 5

Canonical Forms

Similarity of Linear transformation

$$X \rightarrow V$$

$$T(X) \Rightarrow U \rightarrow V$$

$$T(X): V \rightarrow U$$

$$T^{-1} \text{ is present } T^{-1}(X): U \rightarrow V$$

Speciality of LT

$$T(X): V \rightarrow V$$

Invariant subspace

$$T: V \rightarrow V$$

W is subspace of V

$$X \in W$$

$$\forall X \in W \Rightarrow T(X) \in W$$

Theorem:- If W is a subspace under $T \in A(V)$, then T induces a linear transformation T_q on the co-efficient space V divided by W defined by $T_q(X+W) = T(X)+W$. Further if T satisfies the polynomial $q(X) \in F(X)$ then so does T_q thus the minimal polynomial of T_q divided by minimal polynomial of T

\Rightarrow

$$T_q(X+W) = T(X)+W$$

$$q(X) \in F(X)$$

$V/W \Rightarrow$ Quotient space

$W \rightarrow$ subspace.

in T_q

$$X+W \neq Y+W \in V/W$$

$$\alpha + \omega = \beta + \omega$$

$$\Rightarrow \alpha - \beta \in \omega$$

$$T(\alpha - \beta) = T(\alpha) - T(\beta) \in \omega$$

$$T(\alpha) + \omega = T(\beta) + \omega$$

$$T_q(\alpha + \omega) = T_q(\beta + \omega)$$

T_q is linear transformation

$$\alpha + \omega, \beta + \omega \in \omega$$

$$T_q\{(\alpha + \omega) + (\beta + \omega)\} = T_q(\alpha + \beta + \omega)$$

$$= T_q(\alpha + \beta) + \omega$$

$$= T(\alpha) + T(\beta) + \omega$$

$$= T_q(\alpha + \omega) + T_q(\beta + \omega)$$

$$T_q\{C(\alpha + \omega)\} = T_q[C(\alpha + \omega)]$$

$$= T(C(\alpha) + \omega) = C T(\alpha) + \omega$$

$$= C T_q(\alpha + \omega)$$

T_q is linear.

$$T_q^n = (T_q)^n \quad \forall n \geq 0$$

$$\alpha + \omega \in \forall \omega$$

$$T_q^2(\alpha + \omega) = T^2(\alpha) + \omega$$

$$= T_q(T_q(\alpha + \omega))$$

$$= T_q^2(\alpha + \omega)$$

$$(T_q^n) = (T_q)^n \quad \forall n \geq 0$$

$$q(x) \in P(x)$$

$$q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

$$q(T_q)(x+w) = q(T(x)) + w$$

$$= a_n T^n(x) + a_{n-1} T^{n-1}(x) + \dots + a_0 T(x) + w$$

$$= \sum x_i T_q^i(x) + w$$

$$= \sum x_i T^i(x) + w$$

$$= a_i \sum (T_q)^i(x+w) = q(T_q)(x+w)$$

T_q is root of $q(x) = 0$

Invariant Direct sum decomposition

$$T: V \rightarrow V$$

w_1, w_2, \dots, w_n is subspace in V .

$$V = w_1 \oplus w_2 \oplus \dots \oplus w_n$$

T

If $V = w_1 + w_2 + \dots + w_n$ where n_i is dimension of each subspace w_i and every subspace is invariant $T \in A(V)$ then the basis of V can be found so that the matrix of T is basis of the form

$$\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & A_n \end{bmatrix} \text{ where each } A_i \text{ is } n_i \times n_i \text{ matrix of}$$

linear transformation induced by T on w_i

proof:-

$$\text{Let } \{x_1^{(1)}, x_2^{(1)}, \dots, x_{n_1}^{(1)}\} \rightarrow w_1$$

$$\{x_1^{(2)}, x_2^{(2)}, \dots, x_{n_2}^{(2)}\} \rightarrow w_2 \quad w_1, w_2, \dots, w_n$$

$$\{x_1$$

$$q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

$$q(T_q)(x+w) = q(T(x)) + w$$

$$= a_n T^n(x) + a_{n-1} T^{n-1}(x) + \dots + a_0 T(x) + w$$

$$= \sum x_i T_q^i(x) + w$$

$$= \sum x_i T^i(x) + w$$

$$= \sum a_i (T_q)^i(x+w) = q(T_q)(x+w)$$

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Invariant Direct sum decomposition

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linear transformation induced by T on w_i

proof:-

$$\text{Let } \{x_1^{(1)}, x_2^{(1)}, \dots, x_{n_1}^{(1)}\} \rightarrow w_1$$

$$\{x_1^{(2)}, x_2^{(2)}, \dots, x_{n_2}^{(2)}\} \rightarrow w_2 \quad w_1, w_2, \dots, w_n$$

$$\{x_1$$

Since $v = w_1 \oplus w_2 \oplus \dots \oplus w_n$ and w_i is T -invariant

$$T(\alpha_j) \in w_0$$

$$T(\alpha_j)^{(i)} \in w_0 = a_1^{(i)} \alpha_1^{(i)} + a_2^{(i)} \alpha_2^{(i)}$$

\uparrow matrix with respect to basis v is

$$\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_r \end{bmatrix}$$

Important
Similarity of two linear transformation Canonical form for checking

1) Normal form:-

$$A \rightarrow \text{Normal form } A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$I_r \rightarrow$ Square identity matrix order r .
 $r \rightarrow$ rank of matrix.

Theorem:-

Let T be the LT from $U \rightarrow V$ and rank of $T = r$, then there exists a basis of U and V such that the matrix representation of T has the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:-

Let $\dim U = m$, $\dim V = n$ and w be the kernel space of T

Rank of T is r

$$m = (r) + (N)$$

\hookrightarrow rank \rightarrow Dimension of null space

$$N = m - r$$

vector space basis $\rightarrow \{v_1, v_2, v_3, \dots, v_n, \alpha_1, \alpha_2, \dots, \alpha_{m-r}\}$

$T(v_i) = u_i$ (u_1, \dots, u_m) basis of image (T)

$$T(\alpha_i) \neq 0$$

$$\{u_1, u_2, \dots, u_n\} \rightarrow T(v_1) = u_1$$

$$T(v_2) = u_2$$

$$T(v_3) = u_3$$

$$T(\alpha_1) = u_{r+1}$$

$$T(\alpha_2) = u_{r+2}$$

$$T(\alpha_3) = u_{r+3}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

2) Triangular form:

$$T: V \rightarrow V$$

$$T \in A(V)$$

Theorem: If T belongs to $A(V)$ as all χ char root in F then there is a basis of V in which matrix representation V is triangular.

Proof: $\dim V = 1$

\dim of basis 1×1

Trivially triangular

functions

vector space $(n-1)$

$$\dim V = n > 1$$

$$\lambda_i \in F$$

$T(\alpha) \rightarrow$ characteristic eqⁿ

$$T(\alpha_1) = a_{11}\alpha_1$$

$$T(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2$$

\vdots

$$T(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{nn}\alpha_n$$

ω -subspace $\rightarrow V$

quotient space $T_q = V/\omega$

$$\dim T_q = \dim V - \dim W = n-1$$

T induces transformation T_q on V_q

minimal polynomial

$$\text{char } T_q = (\lambda-1)(\lambda-2)^2$$

$$\lambda \rightarrow A$$

$$(A-1)(A-2) \neq 0 \Rightarrow \text{Not minimal.}$$

$$(A-1)(A-2)^2 = 0 \Rightarrow \text{minimal}$$

$$\dim V_q = n-1$$

$$B \text{ min } \{ \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \dots, \bar{\alpha}_n \}$$

$$T_q(\bar{\alpha}_2) = a_{22} \bar{\alpha}_2$$

$$T_q(\bar{\alpha}_3) = a_{22} \bar{\alpha}_2 + a_{33} \bar{\alpha}_3$$

$$T_q(\bar{\alpha}_n) = a_{n2} \bar{\alpha}_2 + \dots + a_{nn} \bar{\alpha}_n$$

$$T_q(\bar{\alpha}_1) = a_{22} \bar{\alpha}_2$$

$$T_q(\alpha_2 + \omega) = a_{22}(\alpha_2 + \omega)$$

$$T(\alpha_1) = a_{11} \alpha_1$$

$$T(\alpha_n) = a_{n1} \alpha_1 + \dots + a_{nn} \alpha_n$$

Theorem

If $\dim V = n$ and $T \in A(V)$ has all its char roots in F then T satisfies a polynomial degree of n over F

proof for Let $\{ \lambda_1, \lambda_2, \dots, \lambda_p \} \in F \rightarrow$ char roots of T

$$\{ \alpha_1, \alpha_2, \dots, \alpha_n \} \text{ basis } \in V$$

$$T(\alpha_1) = \lambda_1 \alpha_1$$

$$T(\alpha_2) = a_{21} \alpha_1 + a_{32} \alpha_2 + \lambda_3 \alpha_3$$

$$T(\alpha_n) = a_{n1} \alpha_1 + a_{n2} \alpha_2 + \dots + \lambda_n \alpha_n$$

$$(T - \lambda_1 I)\alpha_1 = 0$$

$$(T - \lambda_2 I)\alpha_2 = 0$$

$$(T - \lambda_n I)\alpha_n = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{n(n-1)}\alpha_{(n-1)}$$

$$(T - \lambda_2 I)(T - \lambda_1 I)\alpha_2 = 0$$

$$(T - \lambda_1 I)(T - \lambda_{n-1} I)(T - \lambda_{n-2} I) \dots = 0$$

Annihilator of basis of V

$$S=0 \Rightarrow (T - \lambda_n I)(T - \lambda_{n-1} I) \dots = 0$$

Theorem: - problems:-

1) W is an invariant subspace under $S: V \rightarrow V$ and $T: V \rightarrow V$. Show that W is invariant under $S+T$ & ST

\Rightarrow As W is S invariant and T invariant subspace

$$\text{Let } \alpha \in W \text{ then } (S+T)\alpha = S(\alpha) + T(\alpha)$$

$$\text{where } S(\alpha), T(\alpha) \in W$$

$$S(\alpha) + T(\alpha) \in W$$

$$(S+T)(\alpha) \in W$$

Also

$$ST(\alpha) = S(T(\alpha))$$

$T(\alpha) \in W$ is invariant under S

$$S(T(\alpha)) \in W$$

$$ST(\alpha) \in W$$

Nilpotent transformation

$$T: V \rightarrow V$$

$$T(\alpha) \in V.$$

A linear transformation $T: V \rightarrow V$ is said to be nilpotent if $T^n = 0$ for some least +ve integer n .

T = (Addition) or (Multiplication)

$$T^n \rightarrow T(\alpha) = 0$$

$$n \in \mathbb{R} \quad T^n = 0$$

$$T^{n-1} \neq 0$$

n - index of nilpotency.

ex: $T^3 \neq 0$
[n=3]

Jordan canonical form

The matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & & \\ 0 & \lambda & 1 & \\ & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \text{ is called Jordan block matrix belonging to } \lambda.$$

In this matrix the λ 's are on diagonal and 1's are on the super diagonal and other elements are equal to zero.

Theorem

Let $T: V \rightarrow V$ is a linear operator whose char and minimal polynomial are respectively given by.

$$\Delta(x) = (x - \lambda_1)^{n_1} (x - \lambda_2)^{n_2} \dots (x - \lambda_r)^{n_r}$$

$$m(x) = (x - \lambda_1)^{m_1} (x - \lambda_2)^{m_2} \dots (x - \lambda_r)^{m_r}$$

where λ_i are different scalars the T has a block

diagonal matrix representation

$$T = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_r \end{bmatrix}$$

for each λ_i the corresponding J block have the following properties.

- i) There is atleast one J_i of order m and all other J_i of order $\leq m$.
- ii) Sum of the orders of J_i is n .
- iii) the number of J_i is equals the Geometric multiplicity of λ_i .
- iv) The no of J_i of each possible order is uniquely determined by T .

proof:- primary decomposition.

$$T = T_1 \oplus T_2 \oplus T_3$$

$(x - \lambda_i)^{m_i} \rightarrow$ minimal polynomial

$$(T_i - \lambda_i I)^{m_i} = 0 \quad i = 1, 2, 3, \dots$$

$$N_i = T_i - \lambda_i I$$

$$T_i = N_i + \lambda_i I \Rightarrow N_i^{m_i} = 0$$

canonical form as

$$\begin{bmatrix} \begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} & \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \end{bmatrix}$$

$T_i = N_i + \lambda_i I \rightarrow$ Reduced to different blocks of different size.

Rational Canonical form

minimal polynomial can be represented as product of linear polynomial.

$$T: V \rightarrow V$$

$$b_1(x) = q_1(x)^{l_1} q_2(x)^{l_2} \dots q_k(x)^{l_k}$$

$q_k(x) \Rightarrow$ Distinct monic polynomial

$$\begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & C_3 & \\ & & & C_x \end{bmatrix} \quad C_1 = C_2 = \text{Companion matrix.}$$

$$V = V_1 \oplus V_2 \oplus V_3$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$C_1 \quad C_2 \quad C_3$$

3

Problems

Determine all invariant subspaces of A where $A = \begin{bmatrix} 2 & -5 \\ 1 & 2 \end{bmatrix}$ viewed as an operator on \mathbb{R}^2

$(\mathbb{R}^2) \setminus \{0\} \Rightarrow$ Basic invariant subspaces consider

$$\Delta(x) = |xI - A|$$

$$= (x-2)(x+2) + 5$$

$$= x^2 + 1$$

Clearly A has no eigen values in \mathbb{R} so A has no eigen vectors in \mathbb{R}^2 . Hence $\mathbb{R}^2 + \{0\}$ are the only subspace invariant under A .

2) Let matrix A given by $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Show that it is nilpotent and find its index of nilpotency.

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3) Show that it is nilpotent and find its index of nilpotency also find nilpotent matrix m in canonical form which is similar to A

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

~~$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$~~

A is nilpotent matrix of index 2.

Diagonal matrix.

Nullity of matrix = 3

$$A = \begin{bmatrix} m_2 & & \\ & m_2 & \\ & & m_1 \end{bmatrix}$$

$$m_2 = 2 \times 2$$

$$m_1 = 1 \times 1$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine all possible Jordan canonical forms for a linear operator whose char polynomial is

$$\Delta x = (x-2)^3 (x-5)^2$$

(*) Find all possible rational canonical form for 6×6 matrices with minimal polynomial $m(x) = (x+1)^3$

Soln: $V = 6$

(i) ~~$C(x-2)^3 \oplus C(x-5)^2$~~ $C(x+1)^3 \oplus C(x+1)^3$

(ii) $C(x+1)^3 \oplus C(x+1)^2 \oplus C(x+1)$

(iii) $C(x+1)^3 \oplus C(x+1) \oplus C(x+1) \oplus C(x+1)$

(iv) ~~$C(x+1)^3$~~ $C(1+x^3) = C[x^3 + 3x^2 + 3x + 1] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$

$C(1+x^2) = C(x^2 + 2x + 1) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

$C(1+x) = [-1]$

(i)
$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array} \right]$$

(ii)
$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

iii)

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right]$$

2) Let A be a 4×4 matrix with minimal polynomial $m(x) = (x^2+1)(x^3-3)$. Find the rational canonical form for matrix A if A is a matrix over
 (i) Rational field \mathbb{F} (ii) Real field \mathbb{R} (iii) a complex field \mathbb{C}

Soln

$$m(x) = (x^2+1)(x^3-3)$$

(i) ^{Rational} ~~Rational~~ field \mathbb{F}

$$C(x^2+1) \oplus C(x^3-3)$$

$$C(x^2+1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$C(x^3-3) = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

ii) Real field \mathbb{R}

$$C(x^2+1) \oplus C(x+\sqrt{3}) \oplus C(x+\sqrt{3})$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{array} \right]$$

iii) complex field

$$C(x-i) \oplus C(x+i) \oplus C(x-\sqrt{3}) \oplus C(x+\sqrt{3})$$

$$\left[\begin{array}{cc|cc} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ \hline 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{array} \right]$$