THEOREM 1. *If $S=\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the basis of a vector space $V(F)$, then each element of V is uniquely expressible as a linear combination of elements of S.*

Proof. Since S is the basis of a vector space $V(F)$, then by the definition of basis, each element of V is a linear combination of elements of S. Thus, we only show the uniqueness. Let there be two different sets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ of scalars corresponding to an element $\alpha \in V$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$$

and $$\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n \qquad \ldots(1)$$

$$\Rightarrow \quad a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$$

$$\Rightarrow \quad a_1\alpha_1 - b_1\alpha_1 + a_2\alpha_2 - b_2\alpha_2 + \ldots + a_n\alpha_n - b_n\alpha_n = 0$$

$$\Rightarrow \quad (a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + \ldots + (a_n - b_n)\alpha_n = 0$$

Since the set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent so that

$$a_1 - b_1 = 0,\ a_2 - b_2 = 0, \ldots, a_n - b_n = 0$$

$$\Rightarrow \quad a_1 = b_1, \quad a_2 = b_2, \ldots, \quad a_n = b_n.$$

Hence, the expression (1) is unique.

... vector space has a finite basis.

which is a contradiction, because we have taken $x \neq 0$.
Thus, the contradiction arises by assuming that $W_1 \cap W_2 \neq \{0\}$
Hence, $W_1 \cap W_2 = \{0\}$.
Consequently

$$V = W_1 \oplus W_2.$$

REMARK
- Here W_2 is the subspace complementary to the subspace W_2 of finite dimensional vector space V.

THEOREM 5. *If W_1 and W_2 are two finite dimensional subspaces of a vector space $V(F)$, then W_1+W_2 is finite dimensional and $\dim. W_1 + \dim. W_2 = \dim. (W_1 \cap W_2) + \dim. (W_1 + W_2)$.*

Proof. Since W_1 and W_2 are subspaces of V so that $W_1 \cap W_2$ will be a subspace of V and its dimension is finite. Let $\dim. W_1 = m$, $\dim. W_2 = n$ and $\dim. (W_1 \cap W_2) = r$.
Let $\{\alpha_1, \alpha_2, ..., \alpha_r\}$ be a basis of $W_1 \cap W_2$. Therefore, we can extend this basis to a basis of W_1 and also to a basis of W_2.
Let, $\quad S_1 = \{\alpha_1, \alpha_2, \dots \alpha_r, \beta_1, \beta_2, \dots \beta_{m-r}\}$
and $\quad S_2 = \{\alpha_1, \alpha_2, \dots \alpha_r, \gamma_1, \gamma_2, \dots \gamma_{n-r}\}$
be the basis of W_1 and W_2 respectively. Consider the set

$$S = \{\alpha_1, \alpha_2, \dots \alpha_r, \beta_1, \beta_2, \dots \beta_{m-r}, \gamma_1, \gamma_2, \dots \gamma_{n-r}\}$$

Now we have to show that S will form a basis for W_1+W_2. For this, we shall show that S is linearly independent and spans $W_1 + W_2$. For this, suppose

$$\sum a_i\alpha_i + \sum b_j\beta_j + \sum c_k\gamma_k = 0 \text{ for } a_i' s, b_j' s, c_k' s \in F.$$

Then $\sum c_k\gamma_k = \sum a_i\alpha_i + \sum b_j\beta_j \Rightarrow \sum c_k\gamma_k \in W_1$

Also, $\sum c_k\gamma_k \in W_2$. It follows that $\sum c_k\gamma_k \in W_1 \cap W_2$ and we have $\sum c_k\gamma_k \in d_i\alpha_i$ for some scalars $d_1, d_2, ..., d_r$. Since the set $\{\alpha_1, \alpha_2, \dots \alpha_r, \gamma_1, \gamma_2, \dots \gamma_{n-r}\}$ is linearly independent hence all the scalars $c_1 = 0 = c_2 = \dots = c_{n-r}$.

Thus $\sum a_i\alpha_i + \sum b_j\beta_j = 0$

and since the set $\{\alpha_1, \alpha_2, \dots \alpha_r, \beta_1, \beta_2, \dots \beta_{m-r}\}$ is also linearly independent,

then

$$a_1 = 0 = a_2 = \dots = a_r$$

and

$$b_1 = 0 = b_2 = \dots = b_{m-r}$$

Thus the set $S = \{\alpha_1, \alpha_2, \dots \alpha_r, \beta_1, \beta_2, \dots \beta_{m-r}, \gamma_1, \gamma_2, \dots \gamma_{n-r}\}$ is linearly independent.
Now we shall show that S spans $W_1 + W_2$.
Let α be an arbitrary element of $W_1 + W_2$ then it can be written as $\alpha = \beta + \gamma$ with $\beta \in W_1$ and $\gamma \in W_2$. Now S_1 and S_2 being the basis of W_1 and W_2 respectively, β and γ can be expressed uniquely in the form

$$\beta = \sum_{i=1}^{r} a_i\alpha_i + \sum_{j=1}^{n-r} b_j\beta_j, \text{ for some } a_i's \text{ and } b_j's.$$

and $\qquad \gamma = \sum_{i=1}^{r} e_i \alpha_i + \sum_{j=1}^{n-r} c_j \gamma_j$, for some e_i's and c_j's,

$\therefore \qquad \alpha = \beta + \gamma = \sum_{i=1}^{r} (a_i + e_i)\alpha_i + \sum_{j=1}^{n-r} b_j \beta_j + \sum_{j=1}^{n-r} c_j \gamma_j$

$\Rightarrow \qquad \alpha$ is a linear combination of elements of S.

$\Rightarrow \qquad S$ spans $W_1 + W_2$.

Hence, S is basis of $W_1 + W_2$ so that $W_1 + W_2$ is finite dimensional dimensional $(m+n-r)$.

Finally,

$$\dim W_1 + \dim W_2 = m+n = r + (m+n-r)$$
$$= \dim (W_1 \cap W_2) + \dim (W_1 + W_2).$$

THEOREM 6. *If a finite dimensional vector space $V(F)$ be the direct sum of its two subspaces W_1 and W_2, then $\dim V = \dim W_1 + \dim W_2$.*

Proof. Since V is finite dimensional, therefore W_1 and W_2 are also finite dimensional. Let

$$\dim W_1 = m, \dim W_2 = n$$

Also $\qquad V = W_1 \oplus W_2$, implying that

(i) $V = W_1 + W_2$

(ii) $W_1 \cap W_2 = \{0\}$.

Let $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis of W_1 and the set $S_2 = \{\beta_1, \beta_2, ..., \beta_n\}$ is a basis of W_2.

Now consider a set

$$S_3 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n\}$$

We claim that S_3 forms a basis of V.

For some scalars $a_1, a_2, ..., a_m, b_1, b_2, ..., b_n \in F$, we have

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n = 0$$

$\Rightarrow \qquad a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = -(b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n)$

$\Rightarrow \qquad a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m \in W_1$ as $b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n \in W_2$

and $\qquad b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n \in W_2$ as $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m \in W_1$

$\Rightarrow \qquad a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m \in W_1 \cap W_2$

$\Rightarrow \qquad b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n \in W_1 \cap W_2$

But from(i) $W_1 \cap W_2 = \{0\}$.

$\Rightarrow \qquad a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = 0$

and $\qquad b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n = 0$

Since S_1 and S_2 both are linearly independent, therefore

$$a_1 = 0 = a_2 = ... = a_m, \; b_1 = 0 = b_2 = ... = b_n.$$

$\Rightarrow \quad S_1$ is linearly independent.

Next, let γ be an arbitrary element of V, then

$$\gamma = \alpha + \beta, \alpha \in W_1, \beta \in W_2 \qquad [\because V = W_1 + W_2]$$

Since $\qquad \alpha \in W_1 \Rightarrow \alpha \in a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m$ for some a_i's $\in F$

and $\qquad \beta \in W_2 \Rightarrow \beta \in b_1\beta_1 + b_2\beta_2 + ... + b_m\beta_m$ for some b_i's $\in F$

$$\therefore \quad \gamma = \alpha+\beta = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \ldots\ldots + b_n\beta_n$$

$\Rightarrow$ S_3 generates V, thus S_3 forms a basis of V.

Accordingly, $\dim. V = m+n = \dim. W_1 + \dim. W_2$.

REMARK

- It should be noted that in case of finite dimensional spaces, since a basis is a maximal linearly independent subset of the vector space, so the dimension of a finite dimensional vector spaces may be regarded as the maximum of numbers of elements in all linearly independent subsets. If this is adopted as definition of the dimension of a vector space, is n iff it has a basis consisting of n elements.

V containing less than n vectors can span V.

5.14 DIMENSION OF SUBSPACE OF A VECTOR SPACE

THEOREM 1. *Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.*

Proof. Let $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a linearly independent subset of V. Then we shall show that the set

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$$S_1 = \{\beta, \alpha_1, \alpha_2, ,\ldots\alpha_n\}$$

obtained by adjoining β to S is also linearly independent where $\beta \in V$, but not in the subspace of V which is spanned by S.

Since $\quad \alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct vectors in S such that

$$a_1\,\alpha_1 + a_2\alpha_2 + \ldots. + a_n\alpha_n + b\beta = 0$$

where, all a's. are zero. ...(1)

We actually show that $b = 0$. Let, if possible, $b \neq 0$. Then from (1), we have

$$\beta = \left(-\frac{a_1}{b}\right)\alpha_1 + \left(-\frac{a_2}{b}\right)\alpha_2 + \ldots + \left(-\frac{a_n}{b}\right)\alpha_n$$

$\Rightarrow$ β is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$\Rightarrow$ β is in the subspace of V spanned by $\alpha_1, \alpha_2, \ldots, \alpha_n$.

But it is contradictory to the hypothesis that β is not in the subspace spanned by S.

Hence $b = 0$. Consequently, the set S_1 is linearly independent.