Orthogonal sets and bases:-

Consider a set $S = \{u_1, u_2, \dots u_n\}$ of non zero vectors in a inner product space V. S is called orthogonal if each pair of vectors in S are orthogonal and set S is called orthonormal if S is orthogonal and each vector in set S has unit length.

(i) orthogonal : $\langle u_i, u_j \rangle = 0$ for $i \neq j$

(ii) orthonormal: $\langle u_i, u_j \rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

Orthogonal basis and linear combination, Fourier coefficients.

Let set S consist of following 3 vectors in R^3.

$u_1 \ (1,2,1)$

$u_2 = (2,1,-4)$

$u_3 = (3,-2,1)$

write the above set as a linear combination with $v = (7,1,9)$ & find the unknowns x_1, x_2, x_3

$\rightarrow \quad v = u_1 x_1 + u_2 x_2 + u_3 x_3$

$(7,1,9) = (1,2,1)x_1 + (2,1,-4)x_2 + (3,-2,1)x_3$

$$\left.\begin{array}{ll} x_1 + 2x_2 + 3x_3 = 7 & \rightarrow x_1 = 3 \quad x_2 = -1, \ x_3 = 2 \\ 2x_1 + x_2 - 2x_3 = 1 & \\ x_1 - 4x_2 + x_3 = 9 & (7,1,9) = 3u_1 - u_2 + 2u_3 \end{array}\right\} \text{Method 1}$$

Method 2: If we take inner product of each vector in the set w.r.t given vector V. we can find the values for unknown using Fourier coefficient

$\langle v, u_i \rangle = \langle u_1 x_1 + u_2 x_2 + u_3 x_3, u_i \rangle$

$\langle v, u_i \rangle = x_i \langle u_i, u_i \rangle$

$$x_i = \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$$

To find x_i:-

$$x_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{7+2+9}{1+4+1} = 3.$$

$$x_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1$$

$$x_3 = \frac{\langle v, u_3\rangle}{\langle u_3, u_3\rangle} = \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2$$

Projections:

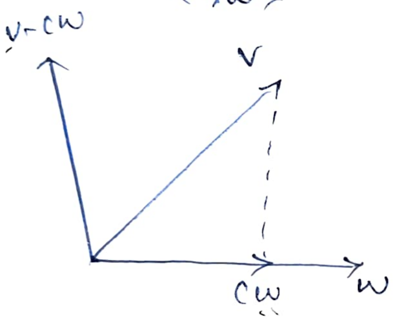
Let V be an inner product space. Suppose w is a given non zero vector in V and suppose p is another vector. We seek the projection of v along w which as indicated in below figure which will be the multiple cw of w such that $v' = v - cw$ is orthogonal to w.

$$\langle v, w\rangle = 0$$

$$\langle v - cw, w\rangle = 0$$

$$\langle v, w\rangle - c\langle w, w\rangle = 0$$

$$c = \frac{\langle v, w\rangle}{\langle w, w\rangle}$$



Accordingly the projection of v on w is denoted as

$$cw = \text{proj}\,\frac{\langle v, w\rangle}{\langle w, w\rangle}\langle v, w\rangle = \frac{\langle v, w\rangle}{\langle w, w\rangle}\cdot w$$

such a scalar c will be unique and it is called the Fourier coefficient of v w.r.t w as a component of v along w.

GRAM - SCHMIDT orthogonalization process:

$S = \{v_1, v_2 \ldots v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis given by $w_1, w_2 \ldots w_n$ of inner product space V as follows.

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle}\cdot w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle}\cdot w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle}\cdot w_2$$

$$\vdots$$

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle}\cdot w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle}\cdot w_2 \ldots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

Apply Gram - shmidt orthogonalization process to find an orthogonal basis & then an orthonormal basis for the subspace U of R^4 spanned by $v_1 = (1,1,1,1)$ $v_2 = (1,2,4,5)$ $v_3 = (1,-3,-4,-2)$

$\rightarrow$ $w_1 = v_1 = (1,1,1,1)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle . w_1}{\langle w_1, w_1 \rangle} = \frac{1+2+4+5}{1+1+1+1} = \frac{12}{4} = 3 = (2,-1,1,2)$ $v_2 -$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle w_1}{\langle w_1, w_1 \rangle} - \frac{\langle v_3 w_2 \rangle w_2}{\langle w_2, w_2 \rangle} = v_3 - \frac{1-3-4-2}{1+9+16+4} \rightarrow \frac{-2+3-4-4}{4+1+1+4}$

$= v_3 - \frac{-8}{\cancel{30}\ 4} - \frac{-7}{10}$

$w_3 = (1,-3,-4,-2) + 2.7$

$w_3 = (3.7, -0.3, 1.3, 0.7)$

$= \cancel{\left(\frac{-37}{10}, \frac{+13}{10}, \frac{-13}{10}, \frac{14}{10}\right)}$ $\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}$

$=$ Normalization $= (16, -17, -13, 14)$

$|w_1| = \sqrt{4} \rightarrow \hat{w}_1 = \cancel{\frac{1}{\sqrt{4}}}\ \frac{1}{2}(1,1,1,1)$

$|w_2| = \sqrt{10} \rightarrow \hat{w}_2 = \frac{1}{\sqrt{10}}(-2,-1,1,2)$

$|w_3| = \sqrt{910} = \hat{w}_3 = \frac{1}{\sqrt{910}}(16,-17,-13,14)$

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Find the fourier coefficient c and the projection of $v = (1,-2,3,-4)$ along $w = (1,2,1,2)$ in R^4.

$\rightarrow$ $c = \frac{\langle v, w \rangle}{\langle w, w \rangle} = \frac{1-4+3-8}{1+4+1+4} = \frac{-8}{10} = -0.8\,(1,2,1,2)$

$(-0.8, -1.6, -0.8, -1.6)$

~~$v - cw = (1,-2,3,-4) + 0.8(1,2,1,2)$~~

~~$= (1.8, 0.8, 3.8, -1.2)$~~

2 Consider the subspace U of R^4 spanned by the vectors $v_1 = (1,1,1,1)$ $v_2 = (1,1,2,4)$ $v_3 = (1,2,-4,-3)$. Find

a. an orthogonal basis of U

b. An orthonormal basis of U.

$\rightarrow$ $w_1 = v_1 = (1,1,1,1)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle . w_1}{\langle w_1, w_1 \rangle} = (1,1,2,4) - \frac{1+1+2+4}{1+1+1+1}.(1,1,1,1) = (1,1,2,4) - (2,2,2,2)$

$= (-1,-1,0,2)$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle w_1}{\langle w_1, w_1 \rangle} - \frac{\langle v_3 w_2 \rangle . w_2}{\langle w_2, w_2 \rangle} = (1,2,-4,-3) - \frac{-4}{4}.(1,1,1,1) - \frac{-\cancel{9}\ 3}{\cancel{6}\ 2}(-1,-1,0,2)$

$= (2,3,-3,-2) + \frac{3}{2}(-1,-1,0,2)$

$w_3 = \cancel{\cos}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$

$\hat{w}_1 = \frac{1}{2}(1,1,1,1)$; $\hat{w}_2 = \frac{1}{\sqrt{6}}(-1,-1,0,2)$; $\hat{w}_3 = \frac{5}{\sqrt{2}}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$

3. Let $v_1 = (1,1,1,1)$ $v_2 = (0,1,1,1)$ $v_3 = (0,0,1,1)$. v_1, v_2, v_3 is clearly lin. independant & thus is basis of subspace w of $\mathbb{R}^4$. Construct an orthogonal basis for w & also find orthonormal

$\rightarrow$

$w_1 = v_1 = (1,1,1,1)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 = (0,1,1,1) - \frac{3}{4}(1,1,1,1) = \left(\frac{-3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2, w_2 \rangle} \cdot w_2$

$= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{\frac{1}{2}}{\left(\frac{9}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)}\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$= \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \frac{2}{3}\left(\frac{-3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$= \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}\right)$

$= \left(0, \frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$\hat{w}_1 = \frac{1}{2}(1,1,1,1)$; $\hat{w}_2 = \frac{\sqrt{3}}{2}\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$; $\hat{w}_3 = \frac{\sqrt{6}}{3}\left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

4. Consider the v. space $\mathbb{P}(T)$ with inner product $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$. Apply Gram Schmidt algorithm for the set $\{1, t, t^2\}$ to obtain an orthogonal set f_0, f_1, f_2 with integer coefficients

$\rightarrow$ ~~w_1~~ $f_0 = 1$

$f_1 = t - \frac{\int_0^1 1 \cdot t \cdot dt}{\int_0^1 1 \, dt} = t - \frac{\frac{1}{2}}{1} = t - 1/2$

$f_2 = t^2 - \frac{\frac{1}{3}}{1} t - \frac{\frac{1}{4}}{1/3} = t^2 - \frac{\cancel{8}}{12}\frac{t}{3} - \frac{3}{4}$ $\rightarrow$ $t^2 - \frac{3}{4}t - \frac{2}{3}$ $\rightarrow$ wrong answer.

5. Suppose $v = (1,3,5,7)$. Find the projection of v onto w or in other words, find $w \in W$ that minimizes $\|v - w\|$ where w is the subspace of $\mathbb{R}^4$ spanned by a) $u_1 = (1,1,1,1)$ $u_2 = (1,-3,4,-2)$. (b) $u_1 = (1,1,1,1)$ $u_2 = (1,2,3,2)$

$\rightarrow c_1 = \frac{16}{4} = 4$ $\quad c_2 = \frac{-2}{30} = \frac{-1}{15}$ $\quad$ [$\because u_1$ & u_2 are orthogonal
$1-3+4-2=0$

$\quad\quad\rightarrow \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} \quad\quad \hookrightarrow \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle}$

Then $w = \text{proj}(v, W)$

$= c_1 u_1 + c_2 u_2$

$= (4,4,4,4) - \left(\frac{1}{15}, \frac{-1}{5}, \frac{4}{15}, \frac{-2}{15}\right)$

$= \left(\frac{59}{15}, \frac{68}{15}, \frac{56}{15}, \frac{62}{15}\right)$

$\quad\quad\quad \hookrightarrow \frac{63}{15}$

b. $\langle u_1, u_2\rangle \neq 0$

$\Rightarrow$ use Gram schmidt to find orthogonal basis.

Since u_1 & u_2 are not orthogonal to each other, find w_1 & w_2

$\rightarrow w_1 = u_1 = (1,1,1,1)$

$w_2 = u_2 - \frac{\langle u_2 w_1\rangle}{\langle w_1, w_1\rangle} \cdot w_1 = (1,2,3,2) - \frac{8}{4}(1,1,1,1) = (-1,0,1,0)$.

$c_1 = \frac{16}{4} = 4$ $\quad\quad c_2 = \frac{4}{2} = 2.$

$\frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle}$

$\text{proj}(v, W)$

$= c_1 w_1 + c_2 w_2$

$= 4(1,1,1,1) + 2(-1,0,+1,0)$

$= \cancel{(2,0,6,0)}\ (2,4,6,4)$

Orthogonal matrices:

1. Find an orthogonal matrix P whose first row is $u_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\rightarrow$ First find a non-zero vector $w_2 = (x,y,z)$ which is orthogonal to u_1

i.e., $0 = \langle u_1, w_2\rangle \Rightarrow \left(\frac{x}{3} + \frac{2y}{3} + \frac{2z}{3}\right) = 0 \rightarrow x + 2y + 2z = 0$
$(0, 1, -1) = 0$

- One such soln is $w_2 = (0,1,-1)$. Now, normalize w_2 to find second row of P

$u_2 = \frac{w}{\|w\|} = \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

Next find $w_3 = (x,y,z)$. that is orthogonal to both u_1 & u_2

$0 = \langle u, w_3\rangle = \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0$

$0 = \langle u_2, w_3\rangle = 0 + \frac{y}{\sqrt{2}}, -\frac{z}{\sqrt{2}} = 0$

} Assume $x = 4$, $y = -1$, $z = -1$

$w_3 = (4,-1,-1)$

$u_3 = \left(\frac{4}{\sqrt{18}}, \frac{-1}{\sqrt{18}}, \frac{-1}{\sqrt{18}}\right) \rightarrow \frac{w_3}{\|w_3\|}$

$\Rightarrow$

$$P = \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{4}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} \end{bmatrix}$$