FIR filter design

Difference equation representation of FIR system is

$$y(n) = \sum_{k=0}^{N-1} b_k\, x(n-k)$$

$$\Rightarrow \quad Y(z) = \sum_{k=0}^{N-1} b_k\, z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k\, z^{-k}$$

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_{N-1} z^{N-1}$$

and $h(n) = \{b_0, b_1, \cdots b_{N-1}\}$

Linear phase FIR systems

FIR filter has linear phase if its impulse response satisfies the following conditions

$$h(n) = \pm h(N-1-n)$$

Properties of FIR filters

* Inherently stable
* Linear phase can be achieved
* All zero filters (NO poles)

Comparison of FIR and IIR filters

1) Length of h(n)
2) feed back from o/p
   — Recursive / non recursive
3) phase characteristics
   Linear / nonlinear
4) Stability of the filter
5) order of the filter for similar specifications
6) H/W requirement
7) Memory requirement
8) Applns.

FIR filter design methods

Windowing

$$H_d(\omega) \xrightarrow{F^{-1}} h_d(n) \xrightarrow{\cdot\, w(n)} h(n) \xrightarrow{Z} H(z)$$

Frequency Sampling

$$H_d(\omega) \xrightarrow{\text{sample}} H(k) \xrightarrow{IDFT} h(n) \xrightarrow{Z} H(z)$$

FIR filter design using windows

Different types of windows

☆ 1) Rectangular
2) Bartlett (triangular)
3) Blackmann
☆ 4) Hanning
☆ 5) Hamming
6) Kaiser

1) Rectangular window

$$w(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

frequency response

2) Hanning window

$$w(n) = \begin{cases} \frac{1}{2}\left(1 - \cos \frac{2\pi n}{N-1}\right) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

frequency response

3) Hamming window.

$$w(n) = \begin{cases} 0.54 - 0.46 \cos \frac{2\pi n}{N-1} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Attenuation and transition width of windows

Window	Transition width of Main lobe	Min. stopband attenuation	Relative amplitude of sidelobe
Rectangular	$\frac{4\pi}{N+1}$	-21 dB	-13 dB
Hanning	$\frac{8\pi}{N}$	-44 dB	-31 dB
Hamming	$\frac{8\pi}{N}$	-53 dB	-41 dB

Design of Linear phase FIR filters using windows

The desired frequency response of the filter is $H_d(\omega)$. $H_d(\omega)$ is given by

$$H_d(\omega) = \sum_{n=0}^{N-1} h_d(n)\, e^{-j\omega n}$$

and $h_d(n)$ is obtained by taking inverse fourier transform of $H_d(\omega)$

(6)

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega.$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

τ = Slope of Linear phase

$$\therefore h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{j\omega(n-\tau)} d\omega$$

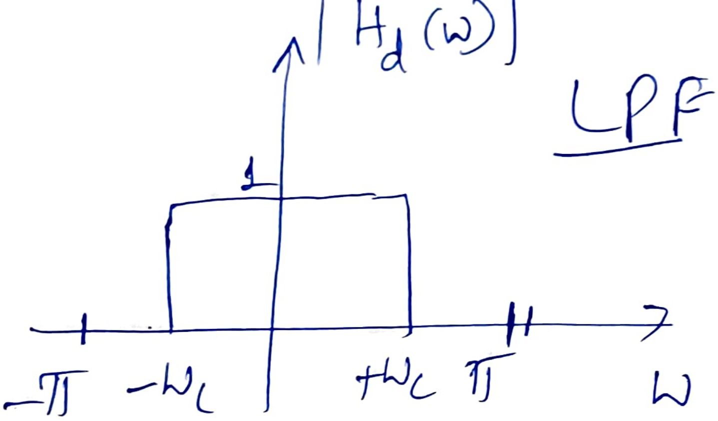
$$= \begin{cases} \dfrac{\sin \omega_c (n-\tau)}{\pi(n-\tau)} & \text{for } n \neq \tau \\ \dfrac{\omega_c}{\pi} & \text{for } n = \tau \end{cases}$$

for $n = \tau$

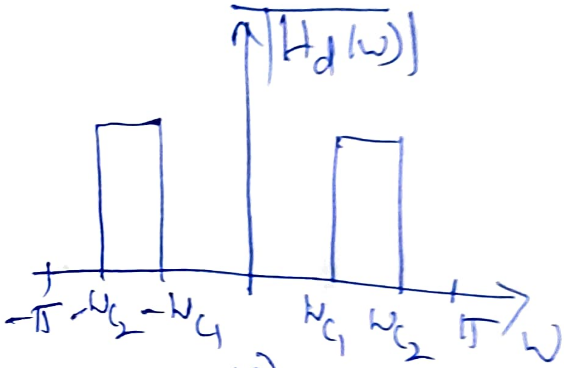
$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{0}\, d\omega = \frac{\omega_c}{\pi}$$

$$h(n) = h_d(n) \cdot \omega(n).$$

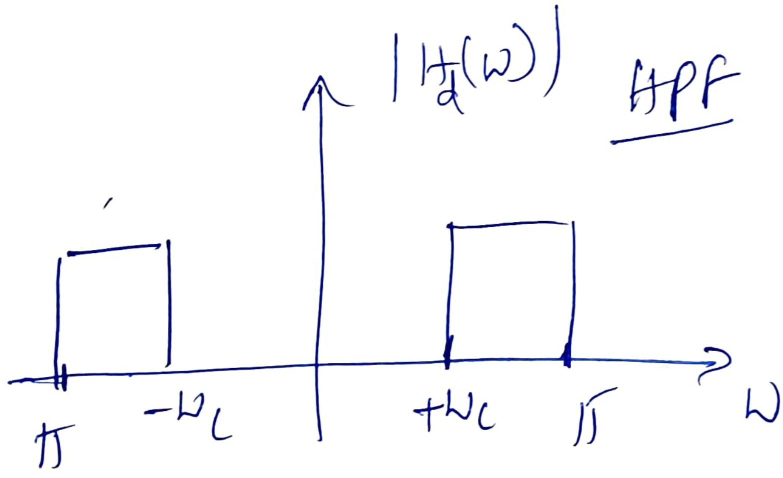
LPF



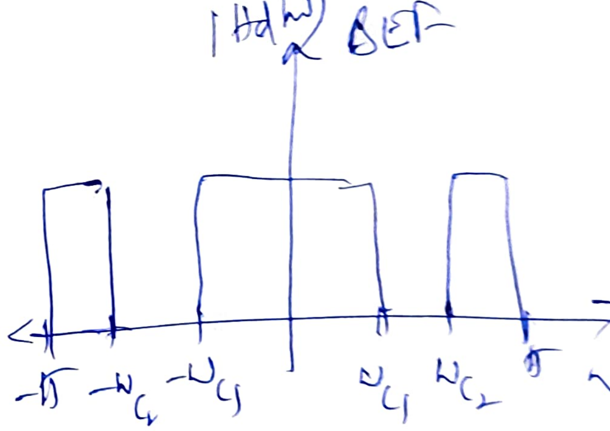
BPF



HPF



BEF



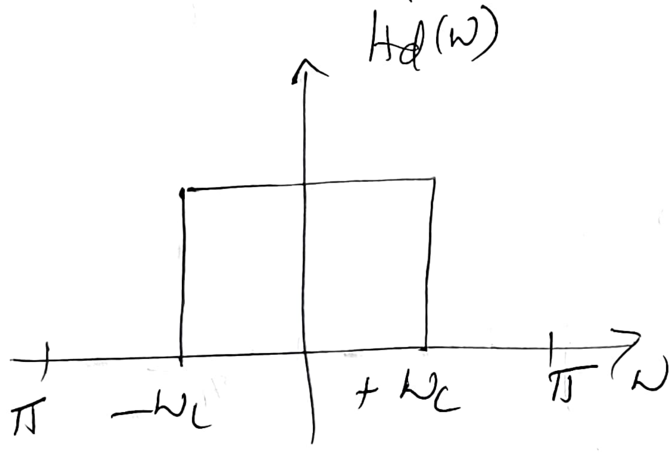
Gibbs phenomenon

Examples of FIR filter design using windows

To design a low pass FIR filter with cut off frequency ω_c and phase delay τ.

The desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$



$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{j\omega(n-\tau)} \, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{+j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{j(n-\tau)} \right]$$

$$= \frac{1}{\pi(n-\tau)} \cdot \left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j} \right]$$

$$\underline{h_d(n) = \frac{1}{\pi(n-\tau)} \left[\sin \omega_c (n-\tau) \right]} \quad \text{if } n \neq \tau$$

for $n = \tau$,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} d\omega = \underline{\frac{\omega_c}{\pi}}$$

To achieve linear phase symmetry condition has to be satisfied

ie $\cancel{h(n) = \pm h(N-1-n)}$

$$h(n) = \pm h(N-1-n)$$

$$\Rightarrow h_d(n) w(n) = \pm h_d(N-1-n) w(n)$$

$$\Rightarrow h_d(n) = \pm h_d(N-1-n)$$

$$\Rightarrow \frac{\sin w_c(n-\tau)}{\pi(n-\tau)} = \pm \frac{\sin w_c(N-1-n-\tau)}{\pi(N-1-n-\tau)}$$

$$\Rightarrow -(N-1-n-\tau) = n-\tau$$

$$\Rightarrow \boxed{\tau = \frac{N-1}{2}}$$

Prob:

1) Design a Linear phase/Symmetric FIR Low pass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Length of the filter should be 7. and $\omega_c = 1$ rad/sample. Use

a) Rectangular window

b) Hanning window

c) Hamming window.

Soln: a). The desired impulse response is given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } -1 < \omega < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-1}^{+1} e^{-j\omega\tau} e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^{+1} e^{j\omega(n-\tau)} \, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)} \right]_{-1}^{+1}$$

$$h_d(n) = \frac{\sin(n-\tau)}{\pi(n-\tau)} \quad \text{for } n \neq \tau$$

$$h_d(n) = \frac{1}{2\pi} \int_{-1}^{+1} d\omega = \frac{1}{\pi} \quad \text{for } n = \tau.$$

$$h_d(n) = \begin{cases} \dfrac{\sin(n-\tau)}{\pi(n-\tau)} & \text{for } n \neq \tau \\ \dfrac{1}{\pi} & \text{for } n = \tau \end{cases}$$

The filter should have Linear phase

$$\therefore \quad h_d(n) = \pm h_d(N-1-n)$$

$$\Rightarrow \quad h_d(n) w(n) = \pm h_d(N-1-n) \cdot w(n)$$

$$\Rightarrow \quad h_d(n) = \pm h_d(N-1-n)$$

$$\Rightarrow \frac{\sin(n-\tau)}{\pi(n-\tau)} = \pm \frac{\sin(N-1-n-\tau)}{\pi(N-1-n-\tau)}$$

$$\Rightarrow -(n-\tau) = N-1-n-\tau$$

$$\Rightarrow \boxed{\tau = \frac{N-1}{2}}$$

$$\therefore \tau = \frac{7-1}{2} = 3$$

$$\therefore h_d(n) = \begin{cases} \frac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \frac{1}{\pi} & \text{for } n = 3 \end{cases}$$

$$w(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

n	$h_d(n)$	$w(n)$	$h(n) = h_d(n) \cdot w(n)$
0	0.01497	1	0.01497
1	0.14472	1	0.14472
2	0.26785	1	0.26785
3	0.31831	1	0.31831
4	0.26785	1	0.26785
5	0.14472	1	0.14472
6	0.1497	1	0.01497

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= 0.01497 + 0.14472\, z^{-1} + 0.26785\, z^{-2} + 0.31831\, z^{-3} + 0.26785\, z^{-4} + 0.14472\, z^{-5} + 0.01497\, z^{-6}$$

$$H(z) = 0.01497\left[1 + z^{-6}\right] + 0.14472\left(z^{-1} + z^{-5}\right) + 0.26785\left(z^{-2} + z^{-4}\right) + 0.31831\, z^{-3}$$

Frequency response $H(w) = H(z)\Big/_{z = e^{jw}}$

$$H(w) = 0.1497\left(1 + e^{-j6w}\right) + 0.14472\left(e^{-jw} + e^{-j5w}\right) + 0.26785\left(e^{-j2w} + e^{-j4w}\right) + 0.31831\, e^{-j3w}$$

$$= e^{-j3w}\Big[0.1497\left(e^{j3w} + e^{-j3w}\right) + 0.14472\left(e^{j2w} + e^{-j2w}\right) + 0.26785\left(e^{+jw} + e^{-jw}\right) + 0.31831\Big]$$

$$= e^{-jw3}\left[0.31831 + 2\times 0.267850\cos w + 2\times 0.14472\cos 2w + 2\times 0.1497\cos 3w\right]$$

$$|H(\omega)| = 0.31831 + 2\times 0.26785\cos\omega + 2\times 0.14472\cos 2\omega + 2\times 0.1497\cos 3\omega$$

$$\phi(\omega) = -3\omega.$$

b) Hanning window.

$$W(n) = \left\{ \frac{1}{2}\left(1 - \cos\frac{2\pi n}{N-1}\right)\right\} \quad 0 \le n \le 6$$

$$= \{0, 0.25, 0.75, 1, 0.75, 0.25, 0\}$$

$$h(n) = h_d(n)\,w(n)$$

$$= \{0, 0.03618, 0.20085, 0.31831, 0.20085, 0.03618, 0\}$$

Prob 2. Design a bandpass linear phase FIR filter having cut off frequencies $\omega_{c_1} = 1$ r/s and $\omega_{c_2} = 2$ r/s. Obtain unit sample response using the following window

$$W(n) = \begin{cases} 1 & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Also find the magnitude of frequency response

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & \text{otherwise} \end{cases}$$

$\tau = \frac{N-1}{2}$ for Linear phase FIR filter

To find $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{-j\omega\tau} e^{j\omega n}\, d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{-j\omega\tau} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega(n-\tau)}\, d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega(n-\tau)}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_{c_2}}^{-\omega_{c_1}} + \left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{\omega_{c_1}}^{\omega_{c_2}}\right]$$

$$h_d(n) = \frac{\sin\omega_{c_2}(n-\tau) - \sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)} \quad \text{for } n \ne \tau$$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} d\omega\right] \quad \text{for } n=\tau$$

$$h_d(n) = \frac{\omega_{c_2}-\omega_{c_1}}{\pi}$$

$$\tau = \frac{N-1}{2} = 3$$

$$\therefore h_d(n) = \begin{cases} \dfrac{\sin 2(n-3) - \sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases}$$

$$h(n) = h_d(n) \cdot w(n)$$

$$= \{-0.04462, -0.26517, 0.02159, 0.31831, 0.2159, -0.26517, -0.04462\}$$

$$|H(\omega)| = h(3) + 2\left[h(0)\cos 3\omega + h(1)\cos 2\omega + h(2)\cos\omega\right]$$

$$= 0.31831 + 2\left[-0.04462\cos 3\omega - 0.26517\cos 2\omega + 0.2159\cos\omega\right]$$

Prob Design a normalized Linear phase FIR filter having the phase delay $\tau = 4$ and atleast 40 dB attenuation in the stop band. Also obtain the magnitude of frequency response of the filter.

Soln normalized filter

$$\omega_c = 1 \text{ rad/ sample}$$

$$\tau = \frac{N-1}{2} \Rightarrow 4 = \frac{N-1}{2}$$

$$\underline{N = 9}$$

Atleast 40 dB attenuation in the stop band

$\Rightarrow$ Hanning window is to be used.

Prob The frequency response of a linear phase FIR filter is given by

$$H(w) = e^{j3w}\left[2 + 1.8\cos 3w + 1.2\cos 2w + 0.5\cos w\right]$$

Find the impulse response of the filter.

Solⁿ

$$\cancel{H(w) = e^{-jw\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{\frac{N-3}{2}} h(n)\cos w\left(n-\frac{N-1}{2}\right)\right]}$$

$$H(w) = e^{-jw\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{\frac{N-3}{2}} h(n)\cos w\left(n-\frac{N-1}{2}\right)\right]$$

$\therefore$ $h(0) = 0.9$, $h(1) = 0.6$, $h(2) = 0.25$

$h(3) = 2$

filter has linear phase

$\therefore h(n) = h(N-1-n)$

$h(4) = h(6-4) = h(2)$

$h(5) = h(6-5) = h(1)$

$h(6) = h(6-6) = h(0)$

$\therefore h(n) = \{0.9, 0.6, 0.25, 2, 0.25, 0.6, 0.9\}$

Design of Linear phase FIR filters using frequency Sampling method

Let Desired frequency response of the FIR filter to be designed be $H_d(\omega)$

Frequency response is sampled uniformly, at M points,

Frequency samples are given by

$$\omega_k = \frac{2\pi}{M}k \qquad k = 0, 1, \cdots M-1$$

Sampled frequency response is DFT and is denoted by $H(k)$

i.e, $H(k) = H_d(\omega)\Big/_{\omega = \omega_k = \frac{2\pi}{M}k}$

$H(k)$ is M point DFT.

By taking IDFT of $H(k)$ we get $h(n)$

$$\therefore h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k)\, e^{j\frac{2\pi nk}{M}} \qquad n = 0, 1 \cdots M-1$$

$\rightarrow$ 1

* $h(n)$ is impulse response of the FIR filter of length M.

* For the FIR filter to be realizable, the coefficients $h(n)$ should be real.

→ This is possible if all the complex terms appear in conjugate pairs.

i.e., $H(K) = H^*(M-K)$ → 2

Using 2 in 1 we can write

$$h(n) = \frac{1}{M}\left\{H(0) + 2\sum_{K=1}^{P} Re\left[H(K)\, e^{j\frac{2\pi K n}{M}}\right]\right\}$$

where $P = \begin{cases} \frac{M-1}{2} & \text{if } M \text{ is odd} \\ \frac{M}{2}-1 & \text{if } M \text{ is even} \end{cases}$

Prob Design a lowpass FIR ~~FIR~~ filter using frequency sampling technique having cut off frequency of $\frac{\pi}{2}$ rad/sample. The filter should have linear phase and length 17. (3)

Sol^n The desired frequency response of a linear phase FIR lowpass filter is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{N-1}{2}\right)} & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Considering only positive range of frequencies, we can write

$$\Rightarrow H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{N-1}{2}\right)} & \text{for } 0 \leq \omega \leq \omega_c \\ 0 & \text{for } \omega_c \leq \omega \leq \pi \end{cases}$$

$$N = 17, \Rightarrow \frac{N-1}{2} = 8.$$

and $\omega_c = \pi/2$

$$\therefore H_d(\omega) = \begin{cases} e^{-j\omega 8} & \text{for } 0 \leq \omega \leq \pi/2 \\ 0 & \text{for } \pi/2 \leq \omega \leq \pi \end{cases}$$

To sample $H_d(\omega)$,

put, $\omega = \frac{2\pi k}{N}$ $\quad k = 0, 1, \ldots N-1$

$= \frac{2\pi k}{17}$ $\quad k = 0, 1 \ldots 16$

$\therefore H(k) = H_d(\omega) \Big|_{\omega = \frac{2\pi k}{17}}$

$$= \begin{cases} e^{-j\frac{2\pi k}{17} \times 8} & \text{for } 0 \leq \frac{2\pi k}{17} \leq \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \leq \frac{2\pi k}{17} \leq \pi \end{cases}$$

$$= \begin{cases} e^{-j\frac{16\pi k}{17}} & \text{for } 0 \leq k \leq \frac{17}{4} \\ 0 & \text{for } \frac{17}{4} \leq k \leq \frac{17}{2} \end{cases}$$

Since k is always an integer, range is considered as

$0 \leq k \leq 4$ and $5 \leq k \leq 8$

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$$\therefore\ H(k) = \begin{cases} e^{-j\frac{16\pi k}{17}} & \text{for } 0 \le k \le 4 \\ 0 & \text{for } 5 \le k \le 8 \end{cases}$$

Now, $h(n) = \text{IDFT}\{H(k)\}$

$$= \sum_{k=0}^{16} H(k)\, e^{j\frac{2\pi kn}{17}} \longrightarrow *$$

Since $h(n)$ is real,

$$H(k) = H^*(N-k)$$

using this in $*$ we can write

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)\, e^{j\frac{2\pi kn}{17}}\right]\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{k=1}^{8} Re\left[e^{-j\frac{16\pi k}{17}} e^{j\frac{2\pi kn}{17}}\right]\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{k=1}^{8} Re\ e^{j\frac{2\pi k}{17}\{n-8\}}\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{k=1}^{8} \cos\left[\frac{2\pi k}{17}(8-n)\right]\right] \quad n = 0, 1 \ldots 16$$

6

$h(n) =$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{16} h(n)\, z^{-n}.$$

Prob Determine the impulse response $h(n)$ of the Linear phase FIR filter having desired frequency response

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{N-1}{2}\right)} & \text{for } 0 \le \omega \le \pi/2 \\ 0 & \text{for } \pi/2 \le \omega \le \pi \end{cases}$$

Take $N = 7$. Use frequency sampling approach.

Soln

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & 0 \le \omega \le \pi/2 \\ 0 & \pi/2 \le \omega \le \pi \end{cases}$$

$$H(k) = H_d(\omega) \Big/_{\omega = \frac{2\pi k}{N} = \frac{2\pi k}{7}}$$

$$= \begin{cases} e^{-j\frac{2\pi k}{7} \times 3} & \text{for } 0 \le \frac{2\pi k}{7} \le \frac{\pi}{2} \\ 0 & \text{for } \pi/2 \le \frac{2\pi k}{7} \le \pi \end{cases}$$

$$= \begin{cases} e^{-j\frac{6\pi k}{7}} & \text{for } 0 \le k \le \frac{7}{4} \\ 0 & \text{for } \frac{7}{4} \le k \le \frac{7}{2} \end{cases}$$

(8)

$$H(k) = \begin{cases} e^{-j\frac{6\pi k}{7}} & \text{for } 0 \le k \le 2 \\ 0 & \text{for } 2 \le k \le 4 \end{cases}$$

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j\frac{2\pi nk}{N}} \longrightarrow (1)$$

$h(n)$ is real $\Rightarrow$ $H(k) = H^*(N-k)$ $\longrightarrow (2)$

using (2) in (1)

$$h(n) = \frac{1}{N} \left[H(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} Re \left\{ H(k) e^{j\frac{2\pi nk}{N}} \right\} \right]$$

$N = 7 \Rightarrow \frac{N-1}{2} = 3$

$$\therefore h(n) = \frac{1}{7} \left[H(0) + 2 \sum_{k=1}^{3} Re \left[e^{-j\frac{6\pi k}{7}} e^{j\frac{2\pi nk}{7}} \right] \right]$$

$$= \frac{1}{7} \left[H(0) + 2 \sum_{k=1}^{3} Re \left\{ e^{j\frac{2\pi k}{7}(n-3)} \right\} \right]$$

$$h(n) = \frac{1}{7} \left[1 + 2 \sum_{k=1}^{3} \cos \left[\frac{2\pi k}{7}(n-3) \right] \right]$$

$n = 0, 1, \ldots 6$