30/10/23

# Unit-3
## Digital Filters

Digital filter
Classification

- IIR ↓ Infinite Impulse response
- FIR ↓ Finite impulse response

Difference eqn representation of a system with input $x(n)$ & o/p $y(n)$

IIR:-

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

FIR:- $$\Sigma\, y(n) = \sum_{k=0}^{M-1} b_k\, x(n-k)$$

IIR systems $$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{\sum_{k=0}^{M-1} b_k Z^{-k}}{\sum_{k=0}^{N-1} a_k Z^{-k}}$$

31/10/23

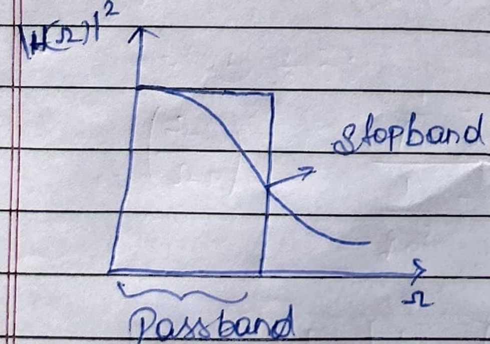
IIR filter design:

Step1: Obtain specifications of equivalent analog filter.

Step2: Design the analog filter in accordance with the specs.

Step3: Transform the analog filter to an equivalent digital filter

Analog filter design using Butterworth approximations



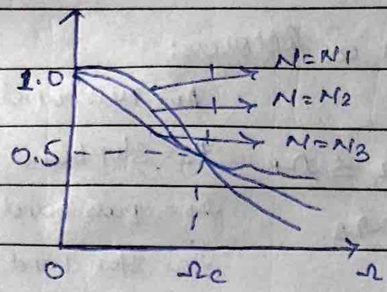

Magnitude squared response of butterworth filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where $N \rightarrow$ order of filter

$\Omega_c \Rightarrow$ 3dB cut-off freq.

## Observations:-



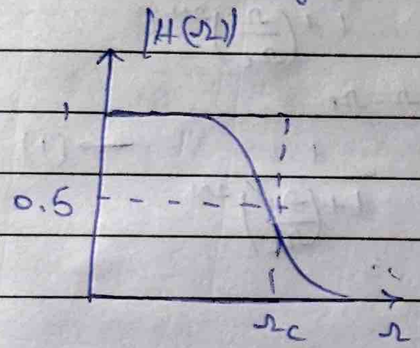

1. $|H(\Omega)|_{\Omega=0} = 1$ for all N
2. $|H(\Omega)| = 1/\sqrt{2}$ at $\Omega = \Omega_c$

$\Rightarrow 20\log_{10}|H(\Omega)| = -3.01dB$ @ $\Omega = \Omega_c$

3. $|H(\Omega)| \to 0$ as $\Omega \to \infty$
4. Maximally flat response in pass band

$$\left.\frac{d^n|H(\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0 \text{ for } n = 0, 1, \ldots, 2N-1$$

5. Monotonically decreasing in the stop band




6. @ $\Omega_c = 1 \Rightarrow$ normalized LPF

$$|H(\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$$

$$H(j\Omega) \cdot H(-j\Omega) = \frac{1}{1+\Omega^{2N}}$$

## Design of Analog lowpass Butterworth filter:-

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\ldots(s-s_{N+1})}$$

1. N
2. $\Omega_c$
3. Poles $s_0, s_1, \ldots, s_{N-1}$

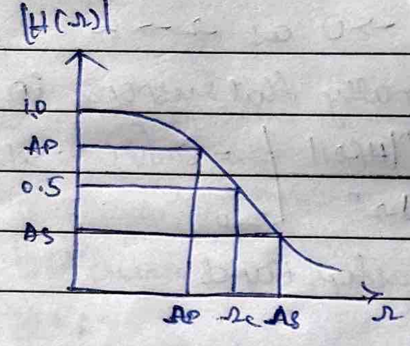
## Order of LP Butterworth filter.

The filter specifications are as follows:-

$A_P \le |H(\Omega)| \le 1$ $\quad 0 \le \Omega \le \Omega_P$

$|H(\Omega)| \le A_s$ $\quad \Omega \ge \Omega_s$

$A_P$ = pass band gain
$A_s$ = stop band gain
$\Omega_P$ = pass band edge freq
$\Omega_s$ = stop band edge freq




Pass band gain $\ge A_P$
Stop band gain $\le A_s$
To find minimum value of N

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

at $\Omega = \Omega_P$

$$A_P^2 = \frac{1}{1+\left(\frac{\Omega_P}{\Omega_c}\right)^{2N}} \rightarrow (1)$$

at $\Omega = \Omega_s$

$$A_s^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \rightarrow (2)$$

from eq$^n$ 1

$$\frac{1}{A_P^2} - 1 = \left(\frac{\Omega_P}{\Omega_c}\right)^{2N} \rightarrow (3)$$

from eq$^n$ 2

$$\frac{1}{A_s^2} - 1 = \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \rightarrow (4)$$

Eq$^n$ 4 $\div$ 3:

$$\left(\frac{\Omega_s}{\Omega_P}\right)^{2N} = \frac{\frac{1}{A_s^2} - 1}{\frac{1}{A_P^2} - 1}$$

$$N \ge \frac{\log\left[\frac{1}{A_s^2} - 1\right] / \left[\frac{1}{A_P^2} - 1\right]}{2 \log\left(\frac{\Omega_s}{\Omega_P}\right)}$$

To find $\Omega_c$

from eq$^n$ (3) $$\frac{\Omega_p}{\Omega_c} = \left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2N}} \rightarrow (5)$$

from eq$^n$ (4) $$\frac{\Omega_s}{\Omega_c} = \left[\frac{1}{A_s^2} - 1\right]^{1/2N} \rightarrow (6)$$

Rearranging terms in eq$^n$ (5)

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2N}}} \qquad \Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}$$

Actual cut-off

$$\boxed{\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2}}$$

Analog Butterworth filter design:

1. $|H(\Omega)|^2 = \dfrac{1}{1+16\Omega^4}$. Determine filter tr. fn $H(S)$.

$\rightarrow$ $|H(\Omega)|^2 = \dfrac{1}{1+\left(\frac{\Omega}{1/2}\right)^4}$ $\quad |H(\Omega)|^2 =$

$|H(\Omega)|^2 = \dfrac{1}{1+\left(\frac{\Omega}{1/2}\right)^{2\times2}}$

$N=2 \quad \Omega_c = 1/2$ rad/sec

Poles of Butterworth filter:

$P_K = \pm \Omega_c \, e^{j(2k+1+N)\frac{\pi}{2N}} \quad k=0,1,\ldots,N-1$

Let the design a normalized LPF $\Omega_c = 1$ r/s.

$P_K = \pm \, e^{j(2K+1+N)\pi/2N}$

Put $k=0$

$P_0 = \pm e^{j(3\pi/4)}$ $\quad K=0,1$

$P_1 = \pm[-0.707 + j0.707]$

Put $k=1$

$P_1 = \pm e^{j(5\pi/4)}$

$P_1 = \pm[-0.707 - j0.707]$

To determine $H_1(S) = \dfrac{1}{(S-P_0)(S-P_1)}$

$= \dfrac{1}{(S-(-0.707+j0.707))(S-(-0.707+j0.707))}$

$= \dfrac{1}{(S+a-jb)(S+a+jb)}$

$= \dfrac{1}{(S+a)^2+b^2}$

$= \dfrac{1}{(S+0.707)^2+(0.707)^2}$

$= \dfrac{1}{(S^2+0.5+1.4142S)+0.5}$

$H_1(S) = \dfrac{1}{S^2+1.414S+1}$

Apply LP→LP → $s \to \frac{\Omega_P}{\Omega_{LP}} \cdot s$

$$H(s)\Big|_{s \to \frac{s}{0.5}} = \frac{1}{\left(\frac{s}{0.5}\right)^2 + 1.414\frac{s}{0.5} + 1}$$

$$= \frac{0.25}{s^2 + 0.707s + 0.25}$$

2 Determine the tf fn of a normalized BW filter with order

a. 2 b. 3 c. 4

→ $N = 2$,

$\Omega_c = 1$

$$P_K = \pm \Omega_c e^{j(2K+1+N)\frac{\pi}{2N}}, \quad K = 0, 1, \ldots, N-1$$

$$P_K = \pm e^{j(2K+1+N)\frac{\pi}{2N}}$$

$K = 0$

$$P_0 = \pm e^{3\pi/4} = \pm[-0.707 + 0.707j]$$

$$P_1 = \pm e^{5\pi/4} = \pm[-0.707 - j0.707]$$

$$H_1(s) = \frac{1}{(s+0.707)^2 + 0.707^2}$$

$$= \frac{1}{(s^2 + 1.414s + 0.5) + \underbrace{0.5}_{\Omega_{LP}}}$$

$$= \frac{1}{s^2 + 1.414s + 1}$$

LP → LP

$$s \to \frac{s}{0.5}$$

$$= \frac{1}{\left(\frac{s}{0.5}\right)^2 + 1.414\left(\frac{s}{0.5}\right) + 1}$$

$$= \frac{0.25}{s^2 + 0.707s + 0.25}$$

(b) $N=3$. $\Omega_c = 1$

$P_0 = \pm e^{j4\pi/6} = -0.5 + j0.86$

$P_1 = \pm e^{\pi} = \pm(-1)$

$P_2 = \pm e^{4\pi} = -0.5 - j0.866$

$$H_1(s) = \frac{1}{\cancel{(s-(-0.5+j0.86))(s-(-1))}\,((s+0.5)^2+(0.86)^2)(s+1)}$$

$$= \frac{1}{(s^2+0.251+\cancel{0.25}s)(s+1)}$$

$$= \frac{1}{s^3+0.25s+s^2+s^2+0.25+s}$$

$$= \frac{1}{\cancel{s^3+2s^2+1.25s+0.25}} \qquad \frac{1}{s^3+2s^2+2s+1}$$

LP → LP

$s \to \frac{s}{0.86} \to \frac{1}{\left(\frac{s}{0.86}\right)^3 + 2\left(\frac{s}{0.86}\right)^2 + 2\left(\frac{s}{0.86}\right) + 1}$

$$= \frac{1}{\frac{s^3}{0.65} + \frac{2s^2}{0.73} + \frac{2s}{0.86} + 1}$$

$$= \frac{0.649}{s^3 + 1.72s^2 + 1.47s + 0.649}$$

(c) $N=4$

$P_0 = \pm e^{j5\pi/8} = \pm[-0.3826 + j0.9238]$

$P_1 = \pm e^{7\pi/8} = \pm[-0.9238 + j0.3826]$

$P_2 = \pm e^{9\pi/8} = \pm[-0.924 + j0.3826]$

$P_3 = \pm e^{11\pi/8} = \pm[-0.3826 + j0.9238]$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)(s-P_3)}$$

$$= \frac{1}{((s+0.3826)^2 + (0.923872)((s+0.9238)^2 + 0.3826^2)}$$

$$= \frac{1}{(s^2 + 0.146 + 0.765 + 0.853)(s^2 + 0.853 + 1.84s + 0.14)}$$

$$= \frac{1}{(s^2+0.76s+1)(s^2+1.84s+1)}$$

$$= \frac{1}{s^4 + 1.84s^3 + s^2 + 0.76s^3 + 1.39s^2 + 0.76s + s^2 + 1.84s + 1)}$$

$$= \frac{1}{s^4 + 2.6s^3 + 3.39s^2 + 2.6s + 1}$$

3. Given $|H(\Omega)|^2 = \frac{1}{1+64\Omega^6} = \frac{1}{1+\left(\frac{\Omega}{0.5}\right)^6} \Rightarrow N=3$

$\Rightarrow$ $N=3$ $\Omega_c = 0.5$

$$P_K = \pm e^{j(2k+1+N)\frac{\pi}{2N}}$$

$$= \pm e^{j(2k+1+3)\frac{\pi}{6}}$$

$$P_0 = \pm e^{4\pi/6} = \pm(-0.5 + 0.866j)$$

$$P_1 = \pm e^{\pi} = \pm(-1)$$

$$P_2 = \pm e^{8\pi/6} = \pm(-0.5 - j0.866)$$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)}$$

$$= \frac{1}{(s+0.5)^2 + (0.866)^2(s+1)}$$

$$= \frac{1}{s^3 + 2s^2 + 2s + 1}$$

classmate

Date

Page

LP → LP

4. Design an analog Butterworth filter to meet following specification

$0.8 \le |H(\Omega)| \le 1 \quad 0 \le f \le 1000Hz$

$|H(\Omega)| \le 0.2 \quad f \ge 5000Hz$

$$\boxed{\begin{aligned} A_p \le |H(\Omega)| \le 1 \\ |H(\Omega)| \le A_s \end{aligned}}$$

$0 \le |\Omega| \le |\Omega_p|$

$\Omega \ge \Omega_s$

→ $A_p = 0.8 \quad A_s = 0.2 \quad \Omega_p = 2\pi f_p = 2\pi \times 1000 = 2000\pi$

$\Omega_s = 2\pi f_s = 2\pi \times 5000 = 10{,}000\pi$

$$\boxed{N \ge \frac{\log\left[\left(\frac{1}{A_s^2} - 1\right) \Big/ \left[\frac{1}{A_p^2} - 1\right]\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

$N \ge 1.167 = 2$

$$\boxed{\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2}}$$

$\Rightarrow \Omega_{cp} = \dfrac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2N}}} = 7255$ rad/s

$\Omega_c = 10724$

$\Omega_{cs} = \dfrac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}} = 14193$ rad/s.

Poles :-

$P_K = \pm \Omega_c \, e^{j(2K+1+N)\frac{\pi}{2N}}$

$P_0 = e^{3\pi/4} = \pm[-0.707 + j0.707]$

$P_1 = e^{5\pi/4} = \pm[-0.707 - j0.707]$

$$H_1(S) = \frac{1}{(S-P_0)(S-P_1)}$$

$$= \frac{1}{(S+0.707)^2 + 0.5}$$

$$= \frac{1}{S^2 + 1.414S + 1}$$

LP → LP

$$S \rightarrow \frac{S}{10729} \rightarrow \frac{1}{\left(\frac{S}{10729}\right)^2 + 1.414\left(\frac{S}{10729}\right) + 1}$$

$$= \frac{115111441}{S^2 + 15170.808 + 115111441}$$

4. Design an analog BW filter

$0.8 \le |H(\Omega)| \le 1 \qquad 0 \le \Omega \le 0.2\pi$

$|H(\Omega)| \le 0.2 \qquad 0.6\pi \le \Omega \le \pi$

→ $A_P = 0.8 \qquad \Omega_P = 0.2\pi$

$A_S = 0.2 \qquad \Omega_S = 0.6\pi$

$$N \ge \frac{\log\left(\left(\frac{1}{A_P^2}-1\right) / \left(\frac{1}{A_S^2}-1\right)\right)}{2\log\left(\frac{\Omega_S}{\Omega_P}\right)} = N \ge 1.708 \simeq 2$$

$$\Omega_C = \frac{\Omega_{CP} + \Omega_{CS}}{2} \qquad \rightarrow \Omega_{CP} = \frac{\Omega_P}{\left(\frac{1}{A_P^2}-1\right)^{1/4}} = \cancel{0.85}\ 0.7255$$

$$\Omega_{CS} = \frac{\Omega_S}{\left(\frac{1}{A_S^2}-1\right)^{1/4}} = \cancel{0.078}\ \cancel{0.7255}\ 0.85$$

$\Omega_C = \underline{0.788}$

$$P_K = \pm \Omega_c e^{\cancel{j(2k+1+N)\pi/2N}} = \pm e^{(2k+3)\pi/4}$$

$$P_0 = e^{3\pi/4}$$

$$P_1 = e^{5\pi/4}$$

$$H_1(S) = \frac{1}{S^2 + 1.414S + 1}$$

$$S \to \frac{S}{0.788} \qquad \frac{1}{\left(\frac{S}{0.788}\right)^2 + 1.414\left(\frac{S}{0.788}\right) + 1}$$

$$= \frac{0.864}{S^2 + \cancel{0.814}\, 1.11\,S + 0.864}$$

5. Design a HP BW filter for the given specifications -3dB Passband attenuation at a freq of $\Omega_P = 1000$ rad/sec and at least -15dB attenuation at 500 rad/sec

→ $A_P = -3dB$ $\qquad \Omega_P = 1000$ rad/s

$A_S = -15dB$ $\qquad \Omega_S = 500$ rad/s.

$$N \geq \frac{\log\left[\frac{10^{0.1A_S} - 1}{10^{0.1A_P} - 1}\right]}{2\log\left(\frac{\Omega_S}{\Omega_P}\right)} \qquad N = 3$$

1000.8

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2} \qquad \Rightarrow \Omega_{cp} = \frac{\Omega_P}{\left(10^{0.1A_P} - 1\right)^{1/6}} = 1000.8$$

$\Omega_c = 641.74$ rad/sec

$$\Rightarrow \Omega_{cs} = \frac{\Omega_S}{\left(10^{0.1A_S} - 1\right)^{1/6}} = 282.68$$

$$S \to \frac{641.74}{S} \qquad H(S) = \frac{1}{(S+1)(S^2+S+1)}$$

$$= \frac{1}{\left(\frac{641.74}{S} + 1\right)\left(\frac{641.74^2}{S^2} + \frac{641.74}{S} + 1\right)}$$

$$= \frac{-s^3}{\cancel{s^4 287930.3\,s^3 + 411830.28\,s^2+}}$$

$$= \frac{-s^3}{s^3 + 1283.48\,s^2 + 823660.458\,s + 26428\,7930.3}$$

6 Design an analog filter with maximally flat response in PB & an acceptable attenuation at 2dB at 20 rad/sec. The attn in SB shd be more than 10dB at 30 rad/sec.

→ $A_P = 2dB$ $\quad \Omega_P = 20$ rad/s

$A_S = 10dB$ $\quad \Omega_S = 30$ r/sec

$$N \geq \frac{\log \left[\frac{10^{0.1\times 2}-1}{10^{0.1\times 10}-1}\right]}{2\log\left(\frac{30}{20}\right)} = -3.31 = \underline{4}$$

$$\Omega_c > \frac{\Omega_{cp}+\Omega_{cs}}{2} \qquad \Omega_{cp} = \frac{\Omega_P}{(10^{0.1\times A_P}-1)^{1/8}} = 21.38$$

$$\Omega_c = \frac{21.38+22.79}{2} \qquad \Omega_{cs} = \frac{\Omega_S}{(10^{0.1\times A_S}-1)^{1/8}} = 22.79$$

$= \underline{22.085 \text{ rad/sec}}$

~~$s \to 22.085$~~

$$\boxed{\begin{array}{l}\text{Order } 4 \\ H(s) = \dfrac{1}{s^4 + 2.61s^3 + 3.413s^2 + 2.613s + 1}\end{array}}$$

Maximally flat response :-

$$s \to \frac{-s}{22.085} \qquad \frac{237897.35}{s^4 + 57.64s^3 + 1664.68s^2 + 28146.9s + 237897.35}$$

$P_k = \pm e^{\cancel{j(2k+1+N)\pi/2N}} \qquad e^{j(2k+5)\pi/10}$

$P_0 = \pm e^{5\pi/8} = -0.38 + 0.92j$

$P_1 = \pm e^{9\pi/10} = \cancel{-0.58 + 0.8j}\ -0.92 + j0.38$

$P_2 = \pm e^{9\pi/10} = \cancel{-0.95 + j0.3}$

$P_3 = \pm e^{9\pi/10} =$

7. A BW Analog HP filter that meets the following specs.

a. Max PB attn = 2dB

b. PB edge freq = 200 rad/sec

c. Min SB attn = 20dB

d. SB edge freq 100 rad/sec

→ $A_P = 2dB$ $\quad \Omega_P = 200$

$A_S = 20dB$ $\quad \Omega_S = 100$

$$N \geq \frac{\log\left(\frac{10^{0.1\times 20}-1}{10^{0.1\times 2}-1}\right)}{2\log\left(\frac{100}{200}\right)} = -3.7 \approx 4$$

$$\Omega_c = \frac{\Omega_{cP} + \Omega_{cS}}{2}$$

$$\Omega_{cP} = \frac{\Omega_P}{(10^{0.1\times A_P}-1)^{1/8}} = 213.86 \qquad \Omega_{cS} = \frac{\Omega_S}{(10^{0.1\times A_S}-1)^{1/8}} = 56.30$$

$\Omega_c = 135$

$$P_K = \pm \Omega_c e^{j(2K+1+N)\pi/2N}$$

$P_0 = \pm e^{5\pi/8} = -0.38 + 0.92j$

$P_1 = \pm e^{7\pi/8} = -0.92 + j0.38$

$P_2 = \pm e^{9\pi/8} = -0.92 - 0.38j$

$P_3 \quad \pm e^{11\pi/8} = -0.38 - 0.92j$

$$H(s) = \frac{1}{((s+0.38)^2+0.92^2)((s+0.92)^2+0.38^2)}$$

$$= \frac{1}{s^4 + 2.613s^3 + 3.41s^2 + 2.61s + 1}$$

$$s \to \frac{135}{s} = \frac{s^4}{s^4 + (2.61)(135)s^3 + (3.41)(135)^2 s^2 + 2.61(135)^3 s + (135)^4}$$

8/11/23

## $s \to z$ mapping :-

1. Approximation of derivatives
2. Impulse Invariance Transformation
3. Bilinear transformation
4. Pole-Zero mapping

(2 and 3: In syllabus.)

## IIT :-

[Analog filter] $H(s) \xrightarrow{L^{-1}} h(t) \xrightarrow{\text{uniform samples}} h(nT) \xrightarrow{Z\text{-transform}} H(z)$ Digital

Let $H(s)$ be the system fn of analog filter, using partial fraction expansion

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k} \rightarrow ①$$

Taking Inverse LT

$$h(t) = \sum_{k=1}^{N} C_k e^{P_k t} \quad \text{for } t \geq 0$$
$$= \sum_{k=1}^{N} C_k e^{P_k t} u(t).$$

Sampling $h(t)$ uniformly

$$h(nT) = h(n) = \sum_{k=1}^{N} C_k e^{P_k nT} u(nT) \rightarrow ②$$

Taking Z-transform,

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}} \rightarrow ③$$

from ① & ③ we can write mapping as

$$\frac{1}{s - P_k} \longrightarrow \frac{1}{1 - e^{P_k T} z^{-1}}$$

$$\boxed{H(s) = \frac{1}{s - P_k}}$$

$H(s)$ has poles at $s = P_k$ & $H(z)$ has poles at $z = e^{P_k T}$

analog poles at $s = P_k$ is mapped to digital pole at $z = e^{P_k T}$

classmate

Date
Page

$Z = e^{P_k T}$

$Z = e^{sT}$

In pole fn $Z = re^{j\omega}$ & $S = \sigma + j\Omega$

$re^{j\omega} = e^{(\sigma + j\Omega)T}$

$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$

Equating real & img parts.

$$\boxed{\begin{array}{c} r = e^{\sigma T} \\ \omega = \Omega T \end{array}} \rightarrow (4)$$

Mapping summary :-

A. From eq$^n$ (4), if $\sigma < 0$ then $0 < r < 1$

$\sigma < 0 \Rightarrow$ LHS of s plane

$0 < r < 1 \Rightarrow$ area inside unit circle in Z-plane

LHS of s plane is mapped inside unit circle in Z-plane

B. From eq$^n$ (4), if $\sigma > 0$ then $r > 1$

RHS of S-plane is mapped outside the unit circle in Z-plane

C. if $\sigma = 0$ then $r = 1$

$\therefore$ $j\Omega$ axis is mapped to unit circle in Z-plane

Bi-linear Transformation:-

BLT is obtained by using trapezoidal formula for numerical integration

Consider analog system

$$y(t) = \frac{dx(t)}{dt} \rightarrow (1)$$

$$x(t) \rightarrow \boxed{S} \rightarrow y(t) = \frac{dx(t)}{dt}$$

$$H(S) = \frac{Y(S)}{X(S)} = S \rightarrow (2)$$

① is now integrated b/w the limits

$nT-T$ & $nT$

$$\int_{nT-T}^{nT} y(t).dt = \int_{nT-T}^{nT} \frac{dx(t)}{dt}.dt$$

$$\int_{nT-T}^{nT} y(t).dt = x(nT) - x(nT-T) \rightarrow ③$$

using trapezoidal rule on LHS of ③.

$$\frac{T\left(y(nT) + y(nT-T)\right)}{2} = x(nT) - x(nT-T) \rightarrow ④$$

using $x(nT) = x(n)$ in ④

$$\frac{T}{2}\left[y(n) - y(n-1)\right] = x(n) - x(n-1) \rightarrow ⑤.$$

Applying Z-transform

$$\frac{T}{2}\left[Y(Z) + Z^{-1}Y(Z)\right] = X(Z) - Z^{-1}X(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{2}{T}\left[\frac{(1-Z^{-1})}{(1+Z^{-1})}\right] \rightarrow ⑥$$

Comparing ② & ⑥.

$$S \rightarrow \frac{2}{T}\left[\frac{(1-Z^{-1})}{(1+Z^{-1})}\right]$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

$$S = \frac{2}{T}\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{Z-1}{Z+1}\right] = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]$$

$$= \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega) + 1}\right]$$

$$\alpha + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos w} + j\frac{2r\sin w}{1+r^2+2r\cos w}\right]$$

$$\left.\begin{aligned}\alpha &= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos w}\right]\\ \Omega &= \frac{2}{T}\left[\frac{2r\sin w}{1+r^2+2r\cos w}\right]\end{aligned}\right\} \circledast$$

7/11/23

1. The system fns of analog filter is given by $H(s) = \frac{1}{(s+1)(s+2)}$ Find H(z) using IIT method Take sampling freq as 5 samples per second

→ $\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$

$1 = A(s+2) + B(s+1)$

$s = -1 \Rightarrow \boxed{1 = A}$

$s = -2 \Rightarrow 1 = -B \Rightarrow \boxed{B = -1}$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

using IIT mapping.

$$\frac{1}{s-p_k} \rightarrow \boxed{\frac{1}{1-e^{p_k T}z^{-1}}}$$

$F = 5$

$T = 1/F = 0.2$ sec

$$\frac{1}{s+1} \rightarrow \frac{1}{1-e^{-0.2}z^{-1}},$$

$$\frac{1}{s+2} \rightarrow \frac{1}{1-e^{-0.4}z^{-1}}$$

$$\frac{0.818z^{-1} - 0.67z^{-1}}{1-(-0.67-0.818)z^{-1}+0.548z^{-2}}$$

$$H(z) = \frac{1}{1-0.818z^{-1}} - \frac{1}{1-0.67z^{-1}} = \frac{0.148z^{-1}}{1-1.48z^{-1}+0.548z^{-2}}$$

2 Find $H(z)$ if $H(s) = \frac{b}{(s+a)^2+b^2}$. Using this result find $H(z)$ when

$$H(s) = \frac{1}{s^2+2s+2}$$

$\rightarrow$ $$\frac{b}{(s+a)^2+b^2} = \frac{b}{(s+a+jb)(s+a-jb)}$$

$$s_1 = -a+jb$$
$$s_2 = -a-jb$$

$$H(s) = \frac{A}{(s+a+jb)} + \frac{B}{(s+a-jb)}$$

$$b = A(s+a-jb) \cancel{(s+a} + B(s+a+jb)$$

$s = -a-jb \Rightarrow b = -2jbA$ $\boxed{A = 0.5j}$

$s = -a+jb \Rightarrow b = 2jbB$ $\boxed{B = -0.5j}$

$$H(s) = \frac{0.5j}{(s+a+jb)} - \frac{0.5j}{(s+a-jb)}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_kT}z^{-1}} = \sum_{k=1}^{N} \frac{C_k z}{z-e^{P_kT}}$$

$$= 0.5j \left[ \frac{z}{z-e^{-(a-jb)T}} - \frac{z}{z-e^{-(a+jb)T}} \right]$$

$$= \frac{z e^{-aT} \sin bT}{z^2 - 2e^{-aT}\cos bT z + e^{-2aT}} \longrightarrow \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

$$= \frac{z^{-1} e^{-aT} \sin bT}{1 - 2e^{-aT}\cos bT z^{-1} + e^{-2aT} z^{-2}}$$

$$H(s) = \frac{1}{s^2+2s+2} \rightarrow H(z) = \frac{e^{-T}\sin T z^{-1}}{1-2e^{-T}\cos T z^{-1} + e^{-2T} z^{-2}}$$

$$= \frac{1}{(s+1)^2+1^2}$$

classmate

Date ______

Page ______

H.W

3. Find H(z) using BLT. Assume

1. $H(s) = \frac{2}{(s+1)(s+2)}$ 2. $H(s) = \frac{10}{s^2+7s+10}$

BLT:

1. Convert the analog filter $H(s) = \frac{2}{(s+1)(s+3)}$ into a digital filter using BLT. taking $T = 0.1 sec$

→ Mapping $s \to z$

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{2}{\left(20\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right)\left(20\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+3\right)}$$

$$= \frac{2}{\left(\frac{20\times 2}{1+z^{-1}}\right)\left(\frac{20\times 4+2z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2}{\left(\frac{21-19z^{-1}}{1+z^{-1}}\right)\left(\frac{23-17z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2(1+z^{-2}+2z^{-1})}{(21-19z^{-1})(23-17z^{-1})}$$

$$= \frac{2(1+2z^{-1}+z^{-2})}{483-357z^{-1}-437z^{-1}+323z^{-2}}$$

$$= \frac{2(1+2z^{-1}+z^{-2})}{483-794z^{-1}+323z^{-2}}$$

2. Convert second order normalized filter to digital filter using BLT. take $T = 1 sec$

→ $H(s) = \frac{1}{s^2+1.414s+1}$ $s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$$= \frac{1}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 2\times 1.414\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1}$$

$$= \frac{1}{4\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + 2.828\frac{(1-z^{-1})}{(1+z^{-1})} + 1}$$

$$= \frac{(1+z^{-1})^2}{4(1-z^{-1})^2 + 2.828(1-z^{-1})(1+z^{-1}) + (1+z^{-1})^2}$$

$$= \frac{(1+z^{-1})^2}{4+4z^{-2}-8z^{-1}+2.828-2.828z^{-2}+1+z^{-2}+2z^{-1}}$$

$$= \frac{(1+z^{-1})^2}{7.828-6z^{-1}+2.172z^{-2}}$$

$$= \frac{0.127(1+z^{-1})^2}{1-0.766z^{-1}+0.277z^{-2}}$$

3. Transform analog filter $H(s) = \frac{s+0.1}{(s+0.1)^2+9}$ into digital filter using BLT. The digital filter shd have $\omega_r = \pi/4$.

→

$(s+0.1)^2+9 = (s+0.1-j3)(s+0.1+j3)$

$s = -0.1 \pm 3j$ $\qquad \Omega = \frac{2}{T}\tan\frac{\omega}{2}$

$= \sigma \pm j\Omega$

$\Omega = 3$ $\qquad T = \frac{2}{\Omega}\tan\frac{\omega}{2}$

$\omega = \frac{\pi}{4}$

$T = 0.2761$ sec

$\frac{2}{T} = 7.24.$

$$H(z) = \frac{7.24\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left(7.24\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right)^2+9.}$$

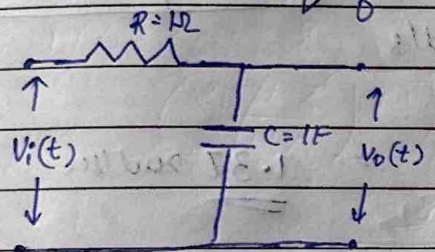
$$= \frac{\frac{7.24-7.24z^{-1}+0.1+0.1z^{-1}}{(1+z^{-1})}}{52.41\frac{(1-z^{-1})^2}{(1+z^{-1})^2}+0.01+1.448\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+9}$$

$$= \frac{7.24 - 7.24z^{-1} + 0.1 + 0.1z^{-1} / (1+z^{-1})}{52.41(1-z^{-1})^2 + 0.01(1+z^{-1})^2 + 1.448(1-z^{-1})(1+z^{-1}) + 9(1-z^{-1})^2} \over (1+z^{-1})^2$$

$$= \frac{(7.25 - 7.23z^{-1})(1+z^{-1})}{}$$

4. Obtain digital filter equivalent of analog filter shown in fig using a. IIT b. BLT. Assume sampling freq $F_s = 8F_c$ where $F_c$ is cutoff freq of the filter

→



$$V_o(t) = \frac{X_C}{R+X_C} V_i(t)$$

$$\frac{V_o(t)}{V_i(t)} = \frac{X_C}{R+X_C}$$

Taking Laplace transform:

$$H(s) = \frac{1/s}{1+1/s} = \frac{1}{s+1}$$

$$F_c = \frac{8}{2\pi RC} = \frac{8}{2\pi} = \frac{4}{\pi} \quad T = \frac{\pi}{4} = 0.785$$

a. $$\frac{1}{s-p_k} = \frac{1}{1-e^{p_kT}z^{-1}}$$

$$H(z) = \frac{1}{1-e^{-0.785}z^{-1}} = \frac{1}{1-0.45z^{-1}}$$

b. $$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{2}{0.785}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{1}{2.55\frac{(1-z^{-1})}{(1+z^{-1})}} = \frac{0.282(1+z^{-1})}{1-0.45z^{-1}}$$

classmate

Date ______
Page ______

## Digital Butterworth filter design:-

1. Design a BW filter using BLT for the following specifications

→ $0.8 \leq |H(\omega)| \leq 1 \qquad 0 \leq \omega \leq 0.2\pi$

$|H(\omega)| \leq 0.2 \qquad 0.6\pi \leq \omega \leq \pi$

→ $A_P = 0.8 \qquad \omega_P = 0.2\pi$

$A_S = 0.2 \qquad \omega_S = 0.6\pi$

Step 1: To obtain specs of corresponding analog filter :-

$\Omega_P = \frac{2}{T} \tan \frac{\omega_P}{2}$

$\boxed{\text{If not specified take } \frac{2}{T} = 1}$

$\Omega_P = 0.32 \text{ rad/s}$

$\Omega_S = 2/T \tan \frac{\omega_S}{2} = \underline{1.37 \text{ rad/sec}}$

Step 2: To determine order $N$ of the filter

$$N \geq \frac{\log \left(\frac{1}{A_S^2} - 1\right) / \left(\frac{1}{A_P^2} - 1\right)}{2 \log \frac{\Omega_S}{\Omega_P}}$$

$N \geq 1.29 \quad = \underline{2}$

Step 3: Find $\Omega_c$

$$\Omega_{CP} = \frac{\Omega_P}{\left(\frac{1}{A_P^2} - 1\right)^{1/2N}} = \cancel{0.725} \ \underline{0.369}$$

$$\Omega_{CS} = \frac{\Omega_S}{\left(\frac{1}{A_S^2} - 1\right)^{1/4}} = 0.618$$

$$\Omega_c = \frac{\Omega_{CP} + \Omega_{CS}}{2} = \underline{0.493}$$

Step 4: Finding system fn of normalized LPF

$P_K = \pm \Omega_c e^{\cancel{j(2K+1+N)\pi/2N}} \Rightarrow P_K = e^{j(2K+3)\frac{\pi}{4}}$

$P_0 = \pm(-0.707 + j0.707)$

$P_1 = \pm(-0.707 - j0.707)$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2+1.414s+1}$$

Step 5: To find the required $H(s)$ using freq transformation $LP \to LP$

$$s \to \frac{s}{0.5} \to \Omega_c$$

$$H(s) = H(s)\Big|_{s \to \frac{s}{0.5}} = \frac{0.25}{s^2+0.707s+0.25}$$

Step 6: Finding $H(z)$ using BLT

$$s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow 2/T = 1$$

$$H(z) = \frac{0.25}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.707\frac{(1-z^{-1})}{(1+z^{-1})} + 0.25}$$

$$H(z) = \frac{0.127(1+2z^{-1}+z^{-2})}{1-0.766z^{-1}+0.277z^{-2}}$$

HW.

1. $H(S) = \dfrac{2}{(S+1)(S+2)}$

$\Rightarrow \dfrac{2}{(S+1)(S+2)} = \dfrac{A}{(S+1)} + \dfrac{B}{S+2}$

$2 = A(S+2) + B(S+1)$

$S=-1 \Rightarrow \boxed{2 = A}$

$S=-2 \Rightarrow 2 = -B \rightarrow \boxed{B = -2}$

$H(S) = \dfrac{2}{S+1} - \dfrac{2}{(S+2)}$

$\dfrac{1}{S-P_K} = \dfrac{1}{1+e^{P_K T} Z^{-1}}$

$H(Z) = \dfrac{2}{1+e^{-T} Z^{-1}} - \dfrac{2}{1+e^{-2T} Z^{-1}}$

$= \dfrac{2(1+e^{-2T}Z^{-1} - 1 - e^{-T}Z^{-1})}{(1+e^{-2T}Z^{-1}) + e^{-T}Z^{-1} + e^{-3T}Z^{-2}}$

$= \dfrac{2Z^{-1}(e^{-2T} - e^{-T})}{1+(e^{-2T}+e^{-T})Z^{-1} + e^{-3T}Z^{-2}} = \dfrac{0.464 Z^{-1}}{1-0.502Z^{-1} + 0.048Z^{-2}}$

2 $H(S) = \dfrac{10}{s^2+7s+10} = \dfrac{10}{(S+2)(S+5)}$ $\quad \dfrac{-7 \pm \sqrt{49-40}}{2} = \dfrac{-7\pm3}{2}$

$\dfrac{-4}{2} = -2$

$-7 \quad -6$

$\dfrac{10}{(S+2)(S+5)} = \dfrac{A}{(S+2)} + \dfrac{B}{(S+5)}$

$10 = A(S+5) + B(S+2)$

$S=-2 \quad 10 = 3A \Rightarrow A = 3.33$

$S=-5 \quad 10 = -3B \Rightarrow B = -3.33$

$H(S) = 3.33\left[\dfrac{1}{S+2} + \dfrac{1}{S+5}\right]$

$\dfrac{1}{S-P_K} = \dfrac{1}{1+e^{+P_K T} Z^{-1}}$

$H(Z) = 3.33\left[\dfrac{1}{1+e^{-2T}Z^{-1}} - \dfrac{1}{1+e^{-5T}Z^{-1}}\right]$

$$= 3.33\left[\frac{e^{-5T}z^{-1} - e^{-2T}z^{-1}}{1+e^{-5T}z^{-1} + e^{-2T}z^{-1} + e^{-7T}z^{-2}}\right]$$

$$= \frac{3.33\, z^{-1}\left(e^{-5T} - e^{-2T}\right)}{1 + \left(e^{-5T}+e^{-2T}\right)z^{-1} + e^{-7T}z^{-2}}$$

$$= \frac{0.4272 z^{-1}}{1 - 0.1417 z^{-1} + 9.085\times 10^{-4} z^{-2}}$$

1.B A digital LPF is req. to meet the following specifications. An acceptable PB attn = −1.9328dB & PB edge freq. of $\omega_P = 0.2\pi$ rad & SB attn of −30.9794 dB or higher behind $0.6\pi$ rad. The filter must have maximally flat freq. response in the PB. Find $H(z)$ using IIT.

→ Step 1: To obtain specs of corresponding analog filter. Take $T = 1$ sec

$A_P = -1.9328\,dB$ $\quad \omega_P = 0.2\pi$ $\quad \Omega_P = \frac{\omega_P}{T} = 0.2\pi$

$A_S = -30.9794\,dB$ $\quad \omega_S = 0.6\pi$ $\quad \Omega_S = \frac{\omega_S}{T} = 0.6\pi$

Step 2: To find order N

$$N \geq \frac{\log\left(\frac{10^{-0.1A_S}-1}{10^{-0.1A_P}-1}\right)}{2\log\frac{\Omega_S}{\Omega_P}} = 1.709 \simeq 2$$

Step 3: To find $\Omega_c$

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2}$$ $\quad 0.788$

$$\Omega_{cp} = \frac{\Omega_P}{\left(10^{-0.1A_P}-1\right)^{1/2N}} = \frac{0.2\pi}{10^{0.1\times -1.9328}-1} = 0.726$$

$$\Omega_{cs} = \frac{\Omega_S}{\left(10^{-0.1A_S}-1\right)^{1/4}} = \frac{0.6\pi}{\left(10^{0.1\times 30.9794}-1\right)^{1/4}} = 0.316$$

Step4: To determine

$$P_K = \pm \Omega_c \, e^{j(2K+1+N)\pi/2N}$$

$$P_K = e^{j(2K+3)\pi/4}$$

$$P_0 = \pm(-0.707 + j0.707)$$

$$P_1 = \pm(-0.707 - j0.707)$$

$$H_1(S) = \frac{1}{(S-P_0)(S-P_1)} = \frac{1}{S^2+1.414S+1}$$

Step5: LP → LP Transformation

$$S \to \frac{S}{0.726}$$

$$H(S) = \frac{0.526}{S^2+1.02S+0.526} = \frac{1.02 \times 0.513}{(S+0.513)^2+(0.513)^2}$$

Step6: Find H(Z) by using IIT.

$$\frac{b}{(s+a)^2+b^2} = \frac{e^{-aT}\sin bT\, Z^{-1}}{1-2e^{-aT}\cos bT\, Z^{-1}+e^{-2aT}Z^{-2}}$$

T = 1sec. $a = 0.513$ $b = 0.513$

$$= \frac{0.299Z^{-1}}{1-1.04Z^{-1}+0.358Z^{-2}}$$

2 Apply IIT to the analog ft fn $H(S) = \frac{S^2+4.525}{S^2+0.6923S+0.5046}$ with T=1sec

→

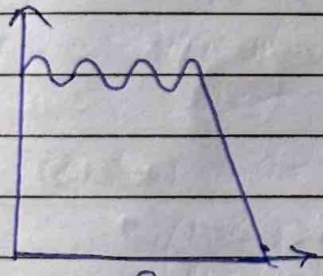
Chebyshev filter design:

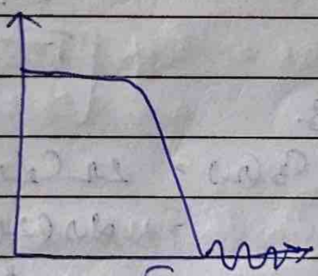
Two types:

1. Chebyshev - I
2. Chebyshev - II

1. Chebyshev - I:
   - All poles filter, usually called Chebyshev filter
   - Ripples in PB
   - Steeper roll-off than Butterworth

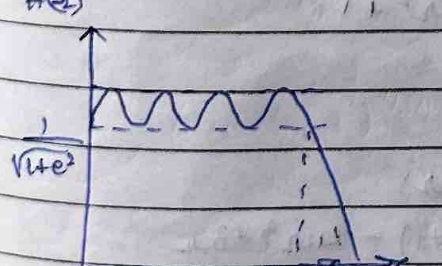
2. Chebyshev II:
   - pole zero filter.
   - Does not roll off fast as Chebyshev I
   - usually called inverse Chebyshev filter.



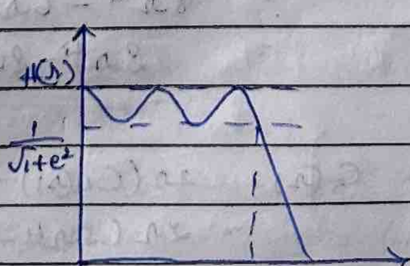




Chebyshev I filter:-

- These filters are ~~all~~ pole filters.
- In the PB, these filters show equi-ripple behaviour & have monotonic characteristics in the SB.




N. ODD




N = EVEN

Q-620-69

classmate
Date
Page

Chebyshev polynomials:-

$C_N(\Omega) = \cos(N \cos^{-1}(\Omega))$ for $|\Omega| \leq 1$

$C_N(\Omega) = \cosh(N \cosh^{-1}(\Omega))$ for $|\Omega| > 1$

For $N=0 \Rightarrow C_0(\Omega) = \cos(0) = 1$

$N=1 \rightarrow C_1(\Omega) = \cos(\cos^{-1}\Omega) = \Omega.$

Higher order chebyshev polynomials are obtained using the recursive formula

$$C_N(\Omega) = 2\Omega C_{N-1}(\Omega) - C_{N-2}(\Omega)$$

1. Find chebyshev polynomials for $N = 2, 3, 4, 4, 5$

→ $N=2$

$C_2(\Omega) = 2\Omega C_1(\Omega) - C_0(\Omega)$

$= 2\Omega^2 - 1$

$N=3$

$C_3(\Omega) = 2\Omega C_2(\Omega) - C_1(\Omega)$

$= 2\Omega(2\Omega^2 - 1) - \Omega$

$= 4\Omega^3 - 2\Omega - \Omega$

$= 4\Omega^3 - 3\Omega$

$N=4$,

$C_4(\Omega) = 2\Omega C_3(\Omega) - C_2(\Omega)$

$= 2\Omega(4\Omega^3 - 3\Omega) - 2\Omega^2 + 1$

$= 8\Omega^4 - 6\Omega^2 - 2\Omega^2 + 1$

$= 8\Omega^4 - 8\Omega^2 + 1$

$N=5$ $C_5(\Omega) = 2\Omega(C_4(\Omega)) - C_3(\Omega)$

$= 2\Omega(8\Omega^4 - 8\Omega^2 + 1) - 4\Omega^3 + 3\Omega$

$= 16\Omega^5 - 16\Omega^3 + 2\Omega - 4\Omega^3 + 3\Omega$

$= 16\Omega^5 - 20\Omega^3 + 5\Omega$

Observations:

- For $\Omega >> 1$ first term dominate

$C_N(\Omega) \approx 2^{N-1} \Omega^N$

Magnitude function of chebyshev filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega)} \rightarrow \text{(A)}$$

where $\epsilon$ = ripple factor

$C_N(\Omega)$ = Chebyshev polynomial of order $N$

for normalized filter $\Omega_P = \Omega_C = 1$ rad/sec

Observations:

1. for $|\Omega| \le 1$

ripple in PB $= 1 - \frac{1}{\sqrt{1+\epsilon^2}}$

2. at $\Omega = 1$

$C_N^2(\Omega) = 1$ [ $C_N(1) = 1$ always ]

at $\Omega = 1$

(A) can be written as

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2}$$

$$|H(\Omega)| = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$\therefore A_P = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$\epsilon = \sqrt{\frac{1}{A_P^2} - 1} \rightarrow \text{(B)}$$

3. for $\Omega >> 1$ $\epsilon^2 C_N^2(\Omega) >> 1$

~~$1+|H(\Omega)|^2$~~ $|H(\Omega)|^2 = \frac{1}{\epsilon^2 C_N^2(\Omega)}$

$$H(\Omega) = \frac{1}{\epsilon C_N(\Omega)}$$

$$|H(\Omega)| \text{ in dB} = 20\log_{10} 1 - 20\log_{10}(\epsilon C_N(\Omega))$$

$$= 0 - 20\log[\epsilon 2^{N-1} \Omega^N]$$

$$= -20\log\epsilon - 20(N-1)\log_{10} 2 - 20N\log\Omega$$

$\Omega >> 1 \Rightarrow$ Stop band $\therefore \Omega = \Omega_s$

$$|H(\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 20(N-1)\log_{10} 2 - 20\log N \log\Omega_s'$$

$$= -20\log_{10}\epsilon - 6(N-1) - 20N\log_{10}\Omega_s'$$

$\Omega_s'$ = Normalized stop band edge freq.

poles of $H(S)$

$$S_K = \sigma_K + j\Omega_K$$

$$\sigma_K = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2K-1}{2N}\right)\pi$$

$$\Omega_K = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2K-1}{2N}\right)\pi$$

$K = 1, 2, 3 \ldots N$

System fn $H(S)$ of C. filter

$$H(S) = \frac{K}{(S-S_1)(S-S_2)\ldots(S-S_N)}$$

$$= \frac{K}{S^N + b_{N-1}S^{N-1} + b_{N-2}S^{N-2} + \ldots b_0}$$

Constant $K$ is given by

$$K = \begin{cases} b_0 & \text{for } N \text{ odd} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & \text{for } N \text{ even} \end{cases}$$

Date
Page

1. Design analog chebyshev filter to meet the following specifications & PB ripple -1dB $0 \le \Omega \le 10$ r/s
-60dB $\Omega \ge 50$ r/s

Step 1: Normalize specifications. $A_p = -1$ $\Omega_p = 10$

$\Omega_p' = \frac{\Omega_p}{\Omega_p} = \frac{10}{10} = 1$ r/s. $A_s = -60$ $\Omega_s =$

$\Omega_s' = \frac{\Omega_s}{\Omega_p} = \frac{50}{10} = 5$ r/s.

Step 2: To determine $\epsilon$.

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1} = \sqrt{10^{0.1 \times A_p} - 1} = 0.5$$

Step 3: To determine order N.

$$|H(\Omega)| \text{ in dB} = -20\log_{10} \epsilon - \cancel{20}\,\epsilon(N-1)\log_{10} - 20N\log \Omega_s'$$

$$-60 = -20\log_{10}(0.5) - \epsilon(N-1) - 20N\log 5.$$

$N = 3.6$

Step 4:

$$H_1(S) = \frac{K}{(S-S_1)(S-S_2)(S-S_3)(S-S_4)}$$

$S_k = \sigma_k + j\Omega_k$

$$\sigma_k = -\sinh\left[\frac{1}{4}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{4}\sinh^{-1}\left(\frac{1}{0.508}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$S_1 = \sigma_1 + j\Omega_1$
$= -0.14 + j\,0.98$

$S_2 = -0.337 + j\,0.407$

$S_3 = -0.337 + j\,-0.407$

$S_4 = -0.88 + j\,-0.98$

$$H_1(S) = \frac{K}{(S+0.139+0.98j)(S+0.337+j0.407)(S+0.337-0.407j)(S+0.139-0.98j)}$$

$$H_1(S) = \frac{K}{((S+0.139)^2+(0.983)^2)((S+0.337)^2+(0.407)^2}$$

$$= \frac{K}{(S^2+0.278S+0.985)(S^2+0.670S+0.279)}$$

$b_0 = 0.985 \times 0.279$
$= 0.2748$

$$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.2748}{\sqrt{1+0.508^2}} = 0.244$$

$$H_1(S) = \frac{0.244}{(S^2+0.278S+0.985)(S^2+0.670S+0.279)}$$

Step 5: Apply freq transformation

$$H(S) = H_1(S)\Big|_{LP \to LP} \quad S \to \frac{S}{10} \qquad \frac{\Omega_P S}{\Omega_{LP}}$$

$$H(S) = \frac{0.244 \times 10^4}{(S^2+0.278S+0.985)(S^2+0.670S+0.279)}$$

2. Design an analog Chebyshev to meet the following specifications $A_P = 2.5dB$ $\Omega_P = 20 r/s$
$A_S = -30dB$ $\Omega_S = 50 r/s$

→ Normalize

$$\Omega_P' = \frac{\Omega_P}{\Omega_P} = \frac{20}{20} = 1 r/s$$

$$\Omega_S' = \frac{\Omega_S}{\Omega_P} = \frac{50}{20} = 2.5 r/s$$

$$\epsilon = \sqrt{\frac{1}{A_P^2} - 1} = \sqrt{10^{0.1 \times A_P} - 1} = 0.88$$

$$|H(\Omega)|_{dB} = -20\log\epsilon - 6(N-1) + 20N\log\Omega_s$$

$$-30 = -20\log(0.88) - 0.88(N-1) - 20N\log(2.5)$$

$$-31.11 = -0.88N + 0.88 + 7.95N$$

$$-31.99 = 7.07N$$

$$\boxed{N = 3}$$

$$H_1(s) = \frac{K}{(s-s_1)(s-s_2)(s-s_3)}$$

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$$s_1 = -0.165 + j0.976$$

$$s_2 = -0.33$$

$$s_3 = -0.165 - j0.911$$

$$H_1(s) = \frac{K}{((s+0.165)^2 + (0.971)^2)(s+0.33)}$$

$$= \frac{K}{(s^2 + 0.33s + 0.857)(s+0.33)}$$

$$b_0 = 0.857 \times 0.33$$

$$b_0 = 0.282$$

$$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.282}{\sqrt{1+0.882^2}} = 0.211$$

$$H_1(s) = \frac{0.211}{(s^2+0.33s+0.857)(s+0.33)}$$

$$H(s) = H_1(s)\Big|_{LP\to HP}\ \frac{s}{20}$$

$$= \frac{0.211}{\left(\left(\frac{s}{20}\right)^2 + 0.33\left(\frac{s}{20}\right) + 0.857\right)\left(\frac{s}{20} + 0.33\right)}$$

$$H(s) = \frac{0.211 \times (20)^3}{(s^2 + 6.6s + 342.8)(s + 6.6)}$$

$$= \frac{1688}{s^3 + 6.6s^2 + 342.8s + 6.6s^2 + 43.56s + 2262.48}$$

$$= \frac{1688}{s^3 + 13.2s^2 + 386.36s + 2262.48}$$

2 Design a digital LP Chebyshev filter using BLT to meet the following specifications

$0.75 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le 0.25\pi$

$|H(\omega)| \le 0.23 \qquad 0.63\pi \le \omega \le \pi$

$T = 2$ sec

→ $A_P = 0.75$ $\quad \omega_P = 0.25\pi \quad \Omega_P = \cancel{0.37}\ 0.414$

$A_S = 0.23 \quad \omega_S = 0.63\pi \quad \Omega_S = \cancel{0.78}\ 1.522$

2.49, 12.76

$\Omega_P' = 1$

$\Omega_S' = 3.67$

$$\epsilon = \sqrt{\frac{1}{A_P^2} - 1} = \sqrt{\frac{1}{0.75^2} - 1} = 0.882$$

$|H(\Omega)|_{dB} = -20\log\epsilon - 6(N-1) - 20N\log\Omega_S'$

~~$-6.23 = -20\log(0.882) - 6N + 6 - 20N \times 0.50$~~

~~$-6.23 = 1.09 - 6N + 6 - 20N$~~

$12.76 = -20\log(0.882) - 6N + 6 + 11.29N$

$N = 2$

$S_k = \sigma_k + j\Omega_k$

$S_1 = -0.35 + j0.79$

$S_2 = -0.35 - j0.79$

$$H(s) = \frac{K}{(s+0.35)^2 + (0.791)^2}$$

$$= \frac{K}{s^2 + 0.7s + 0.7466}$$

$b_0 = 0.7466$

$$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = 0.56$$

$$H_1(s) = \frac{0.56}{s^2 + 0.7s + 0.7466}$$

$$H(s) = H(s)\Big|_{s \to \frac{s}{0.414}} \rightarrow \Omega_P$$

$$H(s) = \frac{0.56}{\left(\frac{s}{0.414}\right)^2 + 0.7\left(\frac{s}{0.414}\right) + 0.7466}$$

$$= \frac{0.56 \times (0.414)^2}{s^2 + 0.2898s + 0.127}$$

$$= \frac{0.096}{s^2 + 0.2898s + 0.127}$$

BLT:

$$s \longrightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.096}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.2898\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.127}$$

$$= \frac{0.096\,(1+z^{-1})^2}{(1-z^{-1})^2 + 0.2898\,(1-z^{-1})(1+z^{-1}) + 0.127\,(1+z^{-1})^2}$$

$$= \frac{0.096\,(1+z^{-1})^2}{1 - 2z^{-1} + z^{-2} + 0.2898\,(1-z^{-2}) + 0.127\,(1+2z^{-1}+z^{-2})}$$

$$= \frac{0.096\,(1+z^{-1})^2}{-1.16z^{-2} - 1.746z^{-1} + 1.41}$$

$$= \frac{0.063\,(1+z^{-1})^2}{1 - 1.238z^{-1} + 0.587z^{-2}}$$

3. Design a LP Chebyshev filter using IIT for satisfying the following constraints.

$w_p = 0.162$ rad

$w_s = 1.63$ rad

$A_p = 3$ dB  $T = 1$ sec

$A_s = 30$ dB

→ $\Omega_p = \cancel{\frac{2}{T}\tan\frac{w_p}{2}} = \frac{w_p}{T} = 0.162$ rad/sec.

$\Omega_s = \frac{w_s}{T} = \frac{1.63}{1}$ rad/sec

$\Omega_p' = 1$

$\Omega_s' = \frac{1.63}{0.162} = 10.06$

$\epsilon = \sqrt{10^{0.1 \times A_P} - 1} = 0.997$

$|H(j\Omega)| = -20\log\epsilon - 6(N-1) - 20N\log\Omega_s'$

$30 - 0.026 - 6 = -6N$

$\boxed{N = 2}$

$S_k = \sigma_k + j\Omega_k$

$S_1 = -0.32 + j0.776$

$S_2 = -0.32 - j0.776$

$H_1(s) = \frac{0.5}{s^2 + 0.644s + 0.707}$

$b_0 = 0.707$

$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = 0.5$

$H(s) = H_1(s)|_{s \to \frac{s}{0.162}}$

$H(s) = \frac{(0.5) \times 0.162^2}{s^2 + (0.162)(0.644)s + 0.707 \times (0.162)^2}$

$= \frac{0.131}{s^2 + 0.104s + 0.018}$

$$H(s) = \frac{0.131}{(s+0.052)^2 + (0.123)^2}$$

$$= \frac{0.131}{(s+0.052+j0.123)(s+0.052-j0.123)}$$

$$= \frac{0.131}{(1-e^{-(0.052+j0.123)}z^{-1})(1-e^{-0.052-j0.123}z^{-1})}$$

$$= \frac{1.065 \times e^{-0.052} \sin(0.123) z^{-1}}{1 - 2e^{-0.052}\cos(0.123)z^{-1} + e^{-2\times 0.052} z^{-2}}$$

$$= \frac{0.12z^{-1}}{1 - 1.88z^{-1} + 0.901z^{-2}}$$

4. A digital IIR LP filter is required to meet the foll. freq domain specification.

$A_P \le 1dB$

$\omega_P = 0.33\pi$ rad

$A_S \ge 40dB$

$\omega_s \le 0.5\pi$ rad.

→ BLT :-

$\Omega_P = \frac{2}{T} \tan\frac{\omega_P}{2} = 1.14.$ $\quad \Omega_P' = 1$

$\Omega_s = \frac{2}{T} \tan\frac{\omega_s}{2} = 2$ $\quad \Omega_s' = 1.75$

BW $N \ge \frac{\log\left(\frac{1}{A_s^2}-1\right) / \left(\frac{1}{A_{P}^2}-1\right)}{2\log\left(\frac{2}{1.14}\right)} = 9.39 \simeq 10$

$\varepsilon = \sqrt{10^{0.1\times A_P}-1} = 0.508$

$H(\Omega) = -20\log\varepsilon - 6(N-1) - 20N\log\Omega_s'$

$N = 6$