3.2.6 LINEAR DEPENDENCE AND INDEPENDENCE OF ANY MATRIX

Consider a matrix of order $m \times n$, given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let the rank of A be r, then there exists at least one r-minor of A which is non-zero. If A_r be a square submatrix of order $r \times r$ such that $|A_r| \neq 0$, then r rows and columns of A_r are linearly independent, it follows that the matrix A has r rows and columns which are linearly independent. As the rank of A is r so that no set of $(r+1)$ rows and columns of A can be linearly independent. Hence the rank of a matrix A is defined to be the maximum number of linearly independent rows and columns of A.

Since on interchanging rows, the rank of A does not change so without loss of generality we may suppose that the first r rows of A are linearly independent. Let $x_1, x_2, x_3, \ldots, x_r$ denote the r independent vectors and let x_t be one of the remaining $(m-r)$ vectors, then the vectors $x_1, x_2, x_3, \ldots, x_r, x_t$ are linearly independent, therefore there exists scalars $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_r, \lambda_t$, not all zero such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r + \lambda_t x_t = O$$

Since $x_1, x_2, \ldots, x_r$ are linearly independent, so we take $\lambda_t \neq 0$, thus

$$x_t = \left(-\frac{\lambda_1}{\lambda_t}\right)x_1 + \left(-\frac{\lambda_2}{\lambda_t}\right)x_2 + \ldots + \left(-\frac{\lambda_2}{\lambda_t}\right)x_r$$

It follows that x_t is a linear combination of $x_1, x_2, \ldots, x_r$.

Hence if the rank of a matrix of order $m \times n$ is r, then it has a set of r linearly independent rows (or columns) and $(m-r)$ linearly dependent rows (or columns).

3.3 HOMOGENEOUS LINEAR EQUATIONS

Let us consider a system of linear homogeneous equations as follows

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

These equations are m equations in n unknowns. Any set of numbers $x_1, x_2, \ldots, x_n$ that satisfies all the equations (1) is called a solution of (1).

3.3.1 TRIVIAL SOLUTION

The solution $x_1 = 0, x_2 = 0, \ldots x_n = 0$ of the equations (1) is called *trivial* solution.

3.3.2 NON-TRIVIAL SOLUTION

Any other solutions, if exists, is called a *non-trivial* solution of equation (1).

Let the coefficient matrix be

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

and
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Then the system of equation (1) can also be written as

$$AX = O \qquad \ldots(2)$$

This equation (2) is called a *matrix equation*.

THEOREM 1. *If X_1 and X_2 are two non-trivial solutions of $AX = O$, then $k_1X_1 + k_2X_2$ is also a solution of $AX = O$, where k_1 and k_2 are any arbitrary numbers.*

Proof. We have

$AX = O$ and $AX_1 = O, AX_2 = O$ are given.

Now consider,

$$A(k_1X_1 + k_2X_2) = k_1(AX_1) + k_2(AX_2) = k_1(O) + k_2(O) = O$$

Hence $k_1X_1 + k_2X_2$ is the solution of $AX = O$.

THEOREM 2. *If the rank of A is r, then the number of linearly independent solutions of the equation $AX = O$ which is a system of m homogeneous linear equations in n unknowns is $(n-r)$.*

Proof. Since the equation is $AX = O$ ... (1)

where
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

and
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Since the rank of $A = r$, so A has r linearly independent columns. Suppose the matrix A can be written as

$$A = [c_1\, c_2 \ldots c_r \ldots c_n]_{1 \times n}$$

where $c_1, c_2, \ldots c_r, \ldots c_n$ are column vectors of the matrix A. Each $c_1, c_2, \ldots c_n$ has m vectors. Thus the equation (1) can be written as

$$x_1c_1 + x_2c_2 + \ldots + x_rc_r + \ldots + x_nc_n = 0 \qquad \ldots(2)$$

But each $c_{r+1}, c_{r+2}, \ldots c_n$ is a linear combination of $c_1, c_2, \ldots c_r$. Then

$$\left.\begin{aligned} c_{r+1} &= p_{11}c_1 + p_{12}c_2 + \ldots + p_{1r}c_r \\ c_{r+2} &= p_{21}c_1 + p_{22}c_2 + \ldots + p_{2r}c_r \\ &\cdots\cdots \\ c_n &= p_{k1}c_1 + p_{k2}c_2 + \ldots + p_{kr}c_r \end{aligned}\right\} \qquad \ldots(3)$$

where $\qquad k = (n-r)$

Now (3) can be written as

$$\left.\begin{aligned} p_{11}c_1 + p_{12}c_2 + \dots + p_{1r}c_r - 1.c_{r+1} + 0.c_{r+2} + \dots + 0.c_n &= 0 \\ p_{21}c_1 + p_{22}c_2 + \dots + p_{2r}c_r + 0.c_{r+1} + 1.c_{r+2} + \dots + 0.c_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_{k1}c_1 + p_{k2}c_2 + \dots + p_{kr}c_r + 0.c_{r+1} - 0.c_{r+2} - \dots + -1.c_n &= 0 \end{aligned}\right\} \quad \dots (4)$$

Thus equation (2) and (4) are same, so comparing we get

$$X_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} p_{21} \\ p_{22} \\ \vdots \\ p_{2r} \\ 0 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \dots X_{n-r} = \begin{bmatrix} p_{k1} \\ p_{k2} \\ \vdots \\ p_{kr} \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

where $\qquad k = (n-r)$.

Hence we obtained $(n-r)$ solutions of the equation $AX = O$. Next we have to show that X_1, X_2, X_{n-r} are linearly independent.

For this let us have

$$l_1X_1 + l_2X_2 + \dots + l_{n-r}X_{n-r} = O \qquad \dots (5)$$

Now comparing the $(r+1)^{th}$, $(r+2)^{th}$, n^{th} components on both sides of (5), we get

$$l_1 = 0 = l_2 = \dots = l_{n-r}$$

Hence $X_1, X_2, X_3, \dots, X_{n-r}$ are linearly independent. Finally we shall have that every solution of the equation $AX = O$ is a linear combination of $X_1, X_2, \dots, X_{n-r}$.

Suppose X is any solution of $AX = O$ with components $x_1, x_2, \dots x_n$. Then

$$X + x_{r+1}X_1 + x_{r+2}X_2 + \dots + x_nX_{n-r} \qquad \dots (6)$$

is also a solution of $AX = O$

Obviously, let $(n-r)$ components of the vector (6) be all equal to zero. Let $z_1, z_2, \dots z_r$ be the first r components of (6). Then $(z_1, z_2, \dots, z_r, 0, 0, \dots 0)$ is a solution of $AX = O$. Therefore from (2), we get

$$z_1c_1 + z_2c_2 + \dots + z_rc_r = O$$

This implies $z_1 = 0 = z_2 = \dots = z_r$ because $c_1, c_2, \dots c_r$ are linearly independent, and hence (6) comes out to be zero, then

$$X = -x_{r+1}X_1 - x_{r+2}X_2 - \dots - x_nX_{n-r}$$

This shows that every solution of $AX = O$ is a linear combination of $X_1, X_2, \dots X_{n-r}$.

3.4 NATURE OF THE SOLUTION OF THE EQUATION AX = O

Since $AX = O$ is a matrix equation of a system of m homogeneous linear equations in n unknowns and A is a coefficient matrix of order $m \times n$. Let the rank of A be r. Then obviously r cannot be greater than n. So that either r is n or r is less than n. Therefore these are some cases.

Case I. If $r = n$, then the equation $AX = O$, will have no linearly independent solution. So in this case only trivial solution will exist.

Case II. If $r < n$, then there will be $(n-r)$ linearly independent solution of $AX = O$ and thus in this case we shall have infinite solutions.

Case III. Suppose the number of equations is less than number of unknowns, *i.e.*, $m < n$ and since $r \leq m$, then obviously $r < n$. Thus in this case a non-zero solution will exist. Therefore, the equation $AX = O$ will have infinite solution.

WORKING RULE

In order to determine the solutions of the equation $AX = O$, we proceed to the following steps :

STEP 1. Reduce the matrix A to Echelon form by applying E-row transformations only. The Echelon form gives the rank of A.

STEP 2. Let A be matrix of order $m \times n$ and let $\rho(A) = r$. If $r = n$, then $AX = O$ will have zero solution only. If $r < n$, then we will assign $n-r$ arbitrarily chosen values to $n-r$ unknowns.

STEP 3. Let B be the Echelon form of A, then the equation $AX = O$ is equivalent to the equation $BX = O$. Reduce $BX = O$ to a system of equations and choose $n-r$ unknowns in this system of equations for assigning arbitrary values like $c_1, c_2, \dots, c_{n-r}$.

STEP 4. By back substitution of $(n-r)$ unknowns to the system of equations reduced from $BX = O$, we finally obtain the solutions. In case of $r < n$, we get infinite solutions.

Solved Examples

Example 1. *Find the non-trivial solutions of the equations:*

$$x + y - 6z = 0$$
$$-3x + y + 2z = 0$$
$$x - y + 2z = 0$$

Solution. The given system of equations can be written as

$$AX = O \qquad \dots (1)$$

where $\quad A = \begin{bmatrix} 1 & 1 & -6 \\ -3 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Reducing the matrix A into Echelon form, we have

Appling $R_2 \to R_2 + 3R_1, R_3 \to R_3 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -6 \\ 0 & 4 & -16 \\ 0 & -2 & 8 \end{bmatrix}$$

Again applying $R_2 \to \frac{1}{4}R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -6 \\ 0 & 1 & -4 \\ 0 & -2 & 8 \end{bmatrix}$$

Again applying $R_3 \to R_3 + 2R_2$ we get

$$A \sim \begin{bmatrix} 1 & 1 & -6 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix in Echelon form with two non-zero rows, therefore $\rho(A) = 2$

Thus the given system of equations is equivalent to

$$\begin{bmatrix} 1 & 1 & -6 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x+y-6z = 0 \qquad \text{... (2)}$$

$$y-4z = 0 \qquad \text{... (3)}$$

Let us put $z = c$ in (3), we get

$$y = 4c$$

Now putting $y = 4c$ and $z = c$ in (2), we get

$$x = 2c$$

Hence the non-trival solutions of the given system of equatons are $x=2c$, $y= 4c$, $z = c$, where c is a non-zero arbitrary number.

Example 2 *Show that the only real value of λ for which the following equations have non-zero solutions is 6:*

$$x+2y-3z = \lambda x,\ 3x+y+2z = \lambda y,\ 2x+3y+z = \lambda z$$

Solution The given system of equations can be rewritten as

$$(1-\lambda)x+2y-3z = 0 \qquad \text{...(1)}$$

$$3x+(1-\lambda)y+2z = 0 \qquad \text{...(2)}$$

$$2x+3y+(1-\lambda)z = 0 \qquad \text{...(3)}$$

This system of equations can be written as

$$AX = O \qquad \text{... (4)}$$

where

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For non-zero solutions, we must have $|A| = 0$

i.e.,

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(1-\lambda)(1-\lambda)+8+27-6(1-\lambda)-6(1-\lambda)-6(1-\lambda) = 0$$

$$\Rightarrow \quad 1-\lambda^3-3\lambda+3\lambda^2+35-18(1-\lambda) = 0$$

$$\Rightarrow \quad -\lambda^3+3\lambda^2+15\lambda+18 = 0$$

$$\Rightarrow \quad \lambda^3-3\lambda^2-15\lambda-18 = 0$$

$$\Rightarrow \quad (\lambda-6)(\lambda^2+3\lambda+3) = 0$$

Since $\lambda^2 + 3\lambda + 3 = 0$ given imaginary roots, therefore the only real value of λ for which the system of equations is to have a non-zero solution is 6.

Example 3. *Does the following system of equations possess a common non-zero solution:*

$$x+y+z=0$$
$$2x-y-3z=0$$
$$3x-5y+4z=0$$
$$x+17y+4z=0$$

Solution. The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$$

First reduce A into Echelon form.

Performing $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -8 & 1 \\ 0 & 16 & 3 \end{bmatrix}$$

performing $R_2 \to -\frac{1}{3}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} \\ 0 & -8 & 1 \\ 0 & 16 & 3 \end{bmatrix}$$

performing $R_3 \to R_3 + 8R_2, R_4 \to R_4 - 16R_4$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & \frac{43}{3} \\ 0 & 0 & \frac{71}{3} \end{bmatrix}$$

performing $R_3 \to \frac{3}{43}R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{71}{3} \end{bmatrix}$$

performing $R_4 \to R_4 + \frac{71}{3}R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is an Echleon form and having three non-zero rows so A has the rank 3. Since there are 3 number of unknown, hence a trival solution exists here, i.e., $x=1, y=0, z=0$.

Example 4. *Find all the solutions of the following system of linear homogeneous equations.*

$$x-2y+z-w=0$$
$$x+y-2z+3w=0$$
$$4x+y-5z+8w=0$$
$$5x-7y+2z-w=0$$

Solution. The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Change this matrix into Echelon form as follows:

performing $R_2 \to R_2 - R_1, R_3 \to R_3 - 4R_1$ and $R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

performing $R_2 \to \frac{1}{3}R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

performing $R_3 \to R_3 - 9R_2, R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is an Echelon form having two non-zero rows. Hence rank of $A=2$.

Therefore the given system of equation is equivalent to

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

or $\quad x-2y+z-w=0 \quad \ldots (1)$

$y-z+\frac{4}{3}w=0 \quad \ldots (2)$

Let $\quad z=c_1, w=c_2$

From (2) $\quad y=c_1-\frac{4}{3}c_2$

and from (1) $\quad x=c_1-\frac{5}{3}c_2$

Hence solution is $x=c_1-\frac{5}{3}c_2, y-c_1-\frac{4}{3}c_2, z=c_1, w=c_2$

where c_1 and c_2 are arbitrary numbers.

EXERCISE 3.1

Find the solution of the following system of linear homogeneous equations:

1. $x+2y+3z=0$, $3x+4y+4z=0$, $7x+10y+12z=0$
2. $x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$, $-4x+y-3z+w=0$
3. $x+y+z=0$, $2x+5y+7z=0$, $2x-5y+3z=0$
4. $3x+4y-z-6w=0$, $2x+3y+2z-3w=0$, $2x+y+4z-9w=0$, $x+3y+13z+3w=0$
5. $2x-3y+z=0$, $x+2y-3z=0$, $4x-y-2z=0$
6. $x+2y+3z=0$, $2x+3y+4z=0$, $7x+13y+19z=0$
7. $x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$
8. $2x-2y+5z+3w=0$, $4x-y+z+w=0$, $3x-2y+3z+4w=0$, $x-3y+7z+6w=0$

Hint to Selected Problems

1. Performing $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 7R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \Rightarrow |A| = 10$$

$\Rightarrow$ Rank of A is 3 which is equal to the number of unknown. Therefore, the only solution is $x=y=z=0$.

3. Do same as (1).

Answers

1. $x = 0 = y = z$ 2. $x = 0 = y = z = w$ 3. $x = 0 = y = z$

4. $x = 11c_1 + 6c_2, y = -8c_1 - 3c_2, z = c_1, w = c_2$ 5. $x = 0 = y = z$

6. $x = c, y = -2c, z = c$ 7. $x = -\frac{10}{7}c, y = \frac{8}{7}c, z = c$ 8. $x = \frac{5}{9}c, y = 4c, z = \frac{7}{9}c, w = c$

3.5 NON-HOMOGENEOUS EQUATIONS

Let us consider a system of equations which are non-homogeneous as follows:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \dots (1)$$

These are m equations in n unknowns. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, B = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

Then the system of equations (1) can also be written as

$$AX = B \quad \dots (2)$$

This equation is called a matrix equation. If $x_1, x_2, \dots x_n$ simultaneously satisfy the equation (2), then $(x_1, x_2, \dots x_n)$ is called the solution of (2).

3.5.1 CONSISTENCY AND INCONSISTENCY

When there exist one or more than one solution of the equation $AX = B$, then the equations are said to be consistent otherwise they are said to be inconsistent.

3.5.2 AUGMENTED MATRIX

The matrix of the type

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix of the equations.

3.6 CONDITION FOR CONSISTENCY

THEOREM **(Rouche's Theorem).** *The equation $AX = B$ is consistent if and only if the rank of A and the rank of the augmented matrix $[A|B]$ are same.*

Proof. Since the equation is $AX = B$

The matrix A can be written as

$$A = [c_1, c_2, \dots c_n] \quad \dots (1)$$

where $c_1, c_2, \dots c_n$ are column vectors. Then the equation (1) can be written as

$$[c_1, c_2, \dots, c_n] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = B$$

or
$$x_1c_1 + x_2c_2 + \dots + x_nc_n = B \quad \dots (2)$$

Suppose the rank of A is r, then A has r linearly independent columns. Let these columns be $c_1, c_2, \dots c_r$ and these $c_1, c_2, \dots c_r$ are linearly independent and remaining $(n - r)$ columns are linear combination of $c_1, c_2, \dots c_r$.

Necessary condition. Suppose the equations are consistent, there must exist $k_1, k_2, \dots, k_n$ such that

$$k_1c_1 + k_2c_2 + \dots + k_nc_n = B \quad \dots (3)$$

But $c_{r+1}, c_{r+2}, \dots c_n$ is a linear combination of $c_1, c_2, \dots c_r$ then from (2) it is obvious that B is also a linear combination of $c_1, c_2, \dots c_r$ and thus $[A|B]$ has the rank r. Hence the rank of A is same as the rank of $[A|B]$.

Sufficient condition. Suppose rank A = rank $[A|B] = r$. This implies that $[A|B]$ has r linearly independent columns. But $c_1, c_2, \dots c_r$ of $[A|B]$ are already linearly independent.

Thus B can be expressed as

$$B = k_1c_1 + K_2c_2 + \dots + k_r c_r \quad \dots (4)$$

where $k_1, k_2, \dots k_r$ are scalars.

Now, equation (4) becomes

$$B = k_1c_1 + K_2c_2 + \dots + k_rc_r + 0.c_{r+1} + \dots + 0.\, c_n \quad \dots (5)$$

Comparing (2) and (5), we get $x_1 = k_1, x_2 = k_2, \dots x_r = k_r, x_{r+1} = 0, \dots = x_n = 0$ and these values of $x_1, x_2, \dots x_n$ are the solution of $AX = B$. Hence the equations are consistent.

REMARKS

- The n equations in n unknowns have a unique solution.
- If rank of A < rank of $[A|B]$, then there is no solution.
- If $r = n$, then there will be a unique solution.
- If $r < n$, then $(n - r)$ variables can be assigned arbitrary values. Thus there will be infinite solutions and $(n - r + 1)$ solutions will be linearly independent.
- If $m < n$ and $r \le m \le n$, then equations will have infinite solutions.

WORKING RULE

In order to determine the solutions of the equation $AX = B$, we procced the following steps:

STEP 1. Reduce the augmented matrix $[A|B]$ to Echelon form by applying E-row transformations only. The Echelon form gives the rank of A and augmented matrix $[A|B]$.

STEP 2. (i) If the rank of A is not equal to the rank of $[A|B]$, then the system of equations has no solution, *i.e.*, equations are inconsistent.

(ii) If the rank of A is equal to the rank of $[A|B]$, then the equations are consistent and they will have unique solution if

rank of A = rank of $[A|B]$ = number of unknowns

and then will have infinite solutions if

rank of A = rank of $[A|B]$ = number of unknowns

STEP 3. Let $[A'|B']$ be the reduced Echelon form of $[A|B]$. Now reduce the equation $A'X = B'$ to a system of equations, after solving these equations we get the required solution.

Solved Examples

Example 1. *Show that the equations*

$$x + 2y - z = 3, 3x - y - 2z = 1, 2x - 2y + 3z = 2, x - y + z = -1$$

are consistent and solve them.

Solution. The given equations can be written as:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \text{ i.e., } AX = B$$

Therefore, augmented matrix is

$$[A|B] = \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 3 & -1 & 2 & \vdots & 1 \\ 2 & -2 & 3 & \vdots & 2 \\ 1 & -1 & 1 & \vdots & -1 \end{bmatrix}$$

performing $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1 \to R_4 \to R_4 - R_1$

we get
$$[A|B] = \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -7 & 5 & \vdots & -8 \\ 0 & -6 & 5 & \vdots & -4 \\ 0 & -3 & 2 & \vdots & -4 \end{bmatrix}$$

performing $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & -6 & 5 & \vdots & -4 \\ 0 & -3 & 2 & \vdots & -4 \end{bmatrix}$$

performing $R_3 \to R_3 - 6R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & 0 & 5 & \vdots & 20 \\ 0 & 0 & 2 & \vdots & 8 \end{bmatrix}$$

performing $R_3 \to \frac{1}{5}R_3, R_4 \to \frac{1}{2}R_4$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 4 \\ 0 & 0 & 1 & \vdots & 4 \end{bmatrix}$$

performing $R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

This is an Echelon form and having three non-zero rows. Thus rank A = rank of $[A|B] = 3$. Therefore the equations are consistent.

and
$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad x + 2y - z = 3, -y = -4, z = 4$$

Hence the solution is $x = -1,\ y = 4, z = 4$

Example 2. *Solve the following equations by matrix method:*

$$x - 2y + 3z = 6$$
$$3x + y - 4z = -7$$
$$5x - 3y + 2z = 5$$

Solution. The given equations can be written as

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -4 \\ 5 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 5 \end{bmatrix}$$

i.e., $AX = B$

$\therefore$ Argumented matrix is

$$[A|B] = \begin{bmatrix} 1 & -2 & 3 & \vdots & 6 \\ 3 & 1 & -4 & \vdots & -7 \\ 5 & -3 & 2 & \vdots & 5 \end{bmatrix}$$

Performing $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 5R_1$, we get

$$[A|B] = \begin{bmatrix} 1 & -2 & 3 & \vdots & 6 \\ 0 & 0 & -13 & \vdots & -25 \\ 0 & 7 & -13 & \vdots & -25 \end{bmatrix}$$

performing $R_3 \to R_3 - R_2$

$$\sim\begin{bmatrix} 1 & -2 & 3 & : & 6 \\ 0 & 7 & -13 & : & -25 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

This is an Echelon form and having two non-zero rows and rank A = rank $[A|B]$ =2. Thus the equations are consistent.

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -25 \\ 5 \end{bmatrix}$$

i.e., $\quad x - 2y + 3z = 6$

$$7y - 13z = -25$$

Let $z = c$, then $\quad y = -\dfrac{25}{7} + \dfrac{13}{7}c$

$$x = -\frac{8}{7} + \frac{5}{7}c$$

Hence the solution is $x = -\dfrac{8}{7} + \dfrac{5}{7}c, y = -\dfrac{25}{7} + \dfrac{13}{7}c, z = c$

where c is an arbitrary constant.

Example 3. *Investigate for what values of λ, μ the simulataneous equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (i) no solution (ii) a unique solution (iii) infinite solution.*

Solution. The given equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

i.e., $\quad AX = B$

Therefore, augmented matrix is

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

performing $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

performing $R_3 \to R_3 - R_2$

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

If $\lambda \neq 3$, then rank A = rank $[A|B]$ = 3. Thus in this case a unique solution exists.
If $\lambda = 3$ and $\mu_0 \neq 10$, then rank A=2, rank $[A|B]$ is 3. Thus rank $A \neq$ rank $[A|B]$.

Hence in this case equations are inconsistent.
If $\lambda = 3$ and $\mu = 10$, then rank A = rank $[A|B]$= 2. Thus in this case solutions exist.

Example 4. *For what values of η the equations $x+y+z=1$, $x+2y+4z = \eta$, have a solution? Solve them completely in each case.*

Solution. The given system of equations can be written as

$$AX=B$$

where
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$$

Augmented matrix $[A|B]$ is given by

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \eta \\ 1 & 4 & 10 & : & \eta^2 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \eta - 1 \\ 0 & 3 & 9 & : & \eta^2 - 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we get

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \eta - 1 \\ 0 & 0 & 0 & : & \eta^2 - 3\eta + 2 \end{bmatrix}$$

This last equivalent matrix is in Echelon form. The given system of equations will have the solutions if

$$\text{rank of } A = \text{rank of } [A|B]$$

For Echelon form, the rank of A is 2 and the augmented matrix $[A|B]$ will have rank 2 if

$$\eta^2 - 3\eta + 2 = 0$$

i.e., if $\quad (\eta - 2)(\eta - 1) = 0$

i.e., if $\quad \eta = 1, 2$

The last equivalent matrix gives the system of equations as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta - 1 \\ \eta^2 - 3\eta + 2 \end{bmatrix} \quad \ldots (2)$$

$$\Rightarrow \quad \left.\begin{aligned} x + y + z &= 1 \\ y + 3z &= \eta - 1 \end{aligned}\right\}$$

Since rank of A = rank of $[A|B]$ if $\eta = 1$ and $\eta = 2$

Now we have two cases:

Case I: When $\eta = 1$

From (2), we have

$$\left.\begin{aligned} x+y+z&=1\\ y+3z&=0 \end{aligned}\right\} \quad \text{... (3)}$$

Since rank of A = rank of $[A|B]=2$ and number of unknowns is 3, therefore we will have $3-2=1$ unknown to be assigned.

Let us assign z to be c_1, therefore put $z=c_1$ in $y+3z=0$, we get $y=-3c_1$.

Again putting $y=-3c_1$ and $z=c_1$ in $x+y+z=1$, we get $x=1+2c_1$

Thus, in this case the solutions are

$$x=1+2c_1, y=-3c_1, z=c_1$$

where c_1 is an arbitrary number.

Case II : When $\eta = 2$

From (2), we have

$$\left.\begin{aligned} x+y+z&=1\\ y+3z&=1 \end{aligned}\right\} \quad \text{...(4)}$$

Let us assign z to be c_2, therefore, putting $z=c_2$ in $y+3z=1$, we get $y=1-3c_2$.

Again, putting $z=c_2$, $y=1-3c_2$ in $x+y+z=1$, we get $x=2c_2$.

Thus, in this case the solutions are

$$x=2c_2, y=1-3c_2, z=c_2$$

Where c_2 is an arbitrary number.

EXERCISE 3.2

1. Use matrix method to solve the equations
$2x-y+3z=9, x+y+z=6, x-y+z=2.$
2. Show that the equations $x-3y-8z+10=0$, $3x+y-4z=0$, $2x+5y+6z-13=0$ are consistent and solve them.
3. Examine if the system of equations
$x+y+4z=6, 3x+2y-2z=9,$
$5x+y+2z=13$
is consistent. Find also the solution if it exists.
4. For what values of λ will the following equtions fail to have a unique solution
$3x-y+\lambda z=1, 2x+y+z=2, x+2y-\lambda z=-1$
Will the equations have any solution for these values of λ?
5. Solve $2x+3y+z=9$, $x+2y+3z=6$, $3x+y+2z=8$.

Solve the following equations by matrix method:

6. $5x+3y+7z=4, 3x+26y-2z=9, 7x+2y+10z=5.$
7. $5x-6y+4z=15, 7x+4y-3z=19, 2x+y+6z=46.$
8. $x-y+2z=4, 3x+y+4z=6, x+y+z=1.$
9. $x+y+z=6, x+2y+3z=4, 3x+y-4z=0.$
10. $2x-y+3z=8, -x+2y+z=4, 3x+y-4z=0.$
11. Show that the following equations are inconsistent
$2x-y+z=4, 3x-y+z=6, 4x-y+2z=7,$
$-x+y-z=9.$
12. Show that the equations are inconsistent
$x-4y+7z=14, 3x+8y-2z=13,$
$7x-8y+26z=5.$
13. Prove that the following system of equations have a unique solution
$5x+3y+14z=4, y+2z=1,$
$x-y+2z=0.$
14. Solve the following equations by matrix mehod:
$x+y+z=9, 2x+5y+7z=52,$
$2x+y-z=0.$

Hint to Selected Problems

1. Consider the augmented matrix and perform the following opeartions sequentially
$R_1\leftrightarrow R_2, R_2\to R_2-2R_1, R_3\to R_3-R_1, R_3\to R_3-\frac{2}{3}R_2.$
2. Here, Rank $(A)=2$, which is less than the number of unknowns, therefore, given system of equations have infinite number of solutions.
3. The rank of augmented matrix is equal to the rank of (A). Therefore, the given system of equation is consistent.
4. The coefficient matrix A is non-singular if $\lambda \neq -\frac{7}{2}$. Thus the given system of equations have a unique solution if $\lambda \neq \frac{7}{2}$.
11. The rank of augmented matrix = 4
Rank of A is 3.
Hence the given system of equations is inconsistent.

Answers

1. $x=1, y=2, z=3$
2. $x=2c-1, y=3-2c, z=c$
3. Consistent; $x=2, y=2, z=\frac{1}{2}$
4. $\lambda \neq -\frac{7}{2}$ solution is unique; $\lambda = -\frac{7}{2}$, no solution.
5. $x=\frac{35}{18}, y=\frac{29}{18}, z=\frac{5}{18}$
6. $x=\frac{7}{11}, y=\frac{3}{11}, z=0$
7. $x=3, y=4, z=6$
8. $x=\frac{5}{2}-\frac{3}{2}c, y=-\frac{3}{2}+\frac{1}{2}c, z=c$
9. $x=c-2, y=8-2c, z=c$
10. $x=2, y=2, z=2$
14. $x=1, y=3, z=4$

3.7 GAUSS ELIMINATION METHOD

In this method, the variables from the system of linear equations are eliminated successively and the system of equations is therefore reduced to an upper triangular system from which the variable are determined by back substitution. This method is described as follows: Let us consider a system of linear equation

$$AX=B \quad \text{...(1)}$$

assuming $\det A \neq 0$. Equation (1) has the following form:

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=b_1\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=b_2\\ \ldots\quad \ldots\quad \ldots\quad &\ldots\quad \ldots\\ \ldots\quad \ldots\quad \ldots\quad &\ldots\quad \ldots\\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n&=b_n \end{aligned}\right\} \quad \text{...(2)}$$

Assuming $a_{11}\neq 0$ and divide the first equation by a_{11} and then we subtract this equation multiplied by $a_{21}, a_{31}, \ldots, a_{n1}$ from second, third ... nth equation of (2), we get

$$\left.\begin{aligned} x_1+a'_{12}x_2+\ldots+a'_{1n}x_n&=b'_1\\ a'_{22}x_2+\ldots+a'_{2n}x_n&=b'_2\\ \ldots\quad \ldots\quad \ldots\quad &\ldots\quad \ldots\\ \ldots\quad \ldots\quad \ldots\quad &\ldots\quad \ldots\\ a'_{n2}x_2+\ldots+a'_{n2}x_n&=b'_n \end{aligned}\right\} \quad \text{...(3)}$$

Next, we divide second equation of (3) by a'_{22} (assuming $a'_{22} \neq 0$) and subtract this equation multiplied by $a'_{32}, a'_{42}, \ldots, a'_{n2}$ from third, fourth ... nth equation of (3), we get

$$\left.\begin{aligned} x_1 + a'_{12}x_2 + \ldots + a'_{1n}x_n &= b'_1 \\ x_2 + a''_{23}x_3 + \ldots + a''_{2n}x_n &= b''_2 \\ a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad & \ldots \\ a''_{3n}x_3 + \ldots + a''_{nn}x_n &= b''_n \end{aligned}\right\} \quad \ldots(4)$$

Continuing in this way we get a system of equation as follows:

$$\left.\begin{aligned} x_1 + c_{12}x_2 + c_{13}x_3 + \ldots + c_{1n}x_n &= d_1 \\ x_2 + c_{23}x_3 + \ldots + c_{2n}x_n &= d_2 \\ &\vdots \\ &\vdots \\ c_{nn}x_n &= d_n \end{aligned}\right\} \quad \ldots(5)$$

This is a form of upper triangular system. From back substitution we can find the solution of the system of given equations.

WORKING RULE

Let us consider these equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \ldots(6)$$

STEP 1. First, eliminate x_1 from second and third equations. Assuming $a_{11} \neq 0$, now dividing first equation by a_{11} and then subtract from second and third after multiplied by a_{21} and respectively we get

$$\left.\begin{aligned} x_1 + a'_{12}x_2 + a'_{13}x_3 &= b'_1 \\ a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a'_{32}x_2 + a'_{33}x_3 &= b'_3 \end{aligned}\right\} \quad \ldots(7)$$

where $a'_{12} = \dfrac{a_{12}}{a_{11}}, a'_{13} = \dfrac{a_{13}}{a_{11}}, a'_{22} = a_{22} - a_{21}a'_{12}, a'_{23} = a_{23} - a_{21}a'_{13}$

$a'_{32} = a_{32} - a_{31}a'_{12}, a'_{33} = a_{33} - a_{31}a'_{13},$

$b'_1 = \dfrac{b_1}{a_{11}}, b'_2 = b_2 - a_{21}b'_1, b'_3 = b_3 - a_{31}b'_1$

STEP 2. Now eliminating x_2 from third equation in (7).

Again assuming $a_{22}' \neq 0$. Dividing second equation in (7) by a_{22}' and then subtract from third equation after multiplied by a_{32}' we get

$$\left.\begin{aligned} x_1 + a'_{12}x_2 + a'_{13}x_3 &= b'_1 \\ x_2 + a''_{23}x_3 &= b'_2 \\ a''_{33}x_3 &= b'_3 \end{aligned}\right\} \quad \ldots(8)$$

where $a''_{23} = \dfrac{a'_{23}}{a'_{22}}, a'_{33} = a'_{33} - a'_{32}a''_{23}, \; b''_2 = \dfrac{b'_2}{a'_{22}}, b''_3 = b_3' - a'_{32}b''_2.$

STEP 3. Evaluating x_1, x_2 and x_3 from (8) by back substitution.

REMARKS

- The coefficient a_{11}, a'_{22} and a''_{33} are called pivots.
- This method will fail if any one of the pivots a_{11}, a'_{22} and a''_{33} becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.
- From each of the procedure, the largest coefficient of x is chosen as pivot element.
- This method proposes a systematic astrology for reducing the system of equations to the upper triangular form using the forward elimination approach and then for obtaining values of unknowns using back substitution process.

Solved Examples

Example 1. *Solve the following equations by Gauss's elimination method*

$$6x + 3y + 2z = 6$$
$$6x + 4y + 3z = 0$$
$$20x + 15y + 12z = 0.$$

Solution. Here pivot element is 6. Now Divide first equation by 6, we get

$$x + \frac{1}{2}y + \frac{1}{3}z = 1 \quad \ldots(1)$$

Now eliminating x from second and third equation with the help of (1). Subtract (1) multiplied by 6 and 20 from second and third equation, respectively we get

$$y + z = -6 \quad \ldots(2)$$

$$5y + \frac{16}{3}z = -20 \quad \ldots(3)$$

Now eliminating y from (3) with the help of (2), we get

$$\left(\frac{16}{3} - 5\right)z = -20 + 30$$

$$\frac{1}{3}z = 10 \Rightarrow z = 30$$

Substitute the value of z into (2), we get

$$y = -6 - 30 = -36$$

and again substitute the values of y and z into (1), we get

$$x + \frac{1}{2}(-36) + \frac{1}{3}(30) = 1$$

$$x - 18 + 10 = 1 \Rightarrow x = 9$$

Hence the solution of the equations are

$$x = 9, y = -36, z = 30.$$

Example 2. *By Gauss's elimination method, solve the following equations*

$$5x - y - 2z = 142$$
$$x - 3y - z = -30$$
$$2x - y - 3z = -50$$

Solution. The largest coefficient in first equation is 5, which is pivot element. So divide first equation by 5, we get

$$x - \frac{1}{5}y - \frac{2}{5}z = \frac{142}{5} \quad \ldots(1)$$

Now eliminating x from second and third equation with help of (1) we get

$$-\frac{14}{5}y-\frac{3}{5}z=-\frac{292}{5} \quad ...(2)$$

$$-\frac{3}{5}y-\frac{11}{5}z=-\frac{309}{5} \quad ...(3)$$

Eliminating y from (2) and (3), we get

$$-\frac{145}{5}z=-\frac{3450}{5}$$

or $$z=\frac{3450}{145}=23.79$$

Substitute the value of z into (3) we get

$$-\frac{3}{5}y-\frac{11}{5}(23.79)=-\frac{309}{5}$$

$$-\frac{3}{5}y=-\frac{309}{5}+\frac{11(23.79)}{5}$$

$$-3y=-309+11(23.79)$$

$$-3y=-47.31$$

or $$y=15.77$$

Substitute the values of y and z into (1), we get

$$x-\frac{1}{5}(15.77)-\frac{2}{5}(23.79)=\frac{142}{5}$$

$$x=\frac{142}{5}+\frac{15.77}{5}+\frac{2(23.79)}{5}=\frac{205.35}{5}$$

or $$x=41.07$$

Hence the solution are given by

$$x=41.07, y=15.77, z=23.79.$$

Example 3. *Using Gauss's elimination method solve*

$$2x_1+4x_2+x_3=3$$
$$3x_1+2x_2-2x_3=2$$
$$x_1-x_2+x_3=6$$

Solution. Dividing first equation by 2, we get

$$x_1+2x_2+\frac{1}{2}x_3=\frac{3}{2} \quad ...(1)$$

Multiplying (1) by 3 and subtract from second and also subtract (1) from third of the given equation, we get

$$4x_2+\frac{7}{2}x_3=\frac{5}{2} \quad ...(2)$$

$$-3x_2+\frac{1}{2}x_3=\frac{9}{2} \quad ...(3)$$

Now dividing (2) by 4 and subtract after multiplies by –3 from (3), we get

$$25x_3=51$$

or $$x_3=\frac{51}{25}=2.04$$

Substitute the value of x_3 into (2), we get

$$+4x_2+\frac{7}{2}(2.04)=\frac{5}{2}$$

$$4x_2=\frac{5}{2}-\frac{7(2.04)}{2}$$

$$=\frac{5-14.28}{2}$$

$$\Rightarrow \quad x_2=-\frac{9.28}{8}=-1.16$$

Now substitute the value of x_2 and x_3 into (1), we get

$$x_1+2(-1.16)+\frac{1}{2}(2.04)=\frac{3}{2}$$

$$x_1=\frac{3}{2}+2(1.16)-\frac{1}{2}(2.04)$$

$$=\frac{3+4.64-2.04}{2}=\frac{5.6}{2}=2.8$$

or $$x_1=2.8$$

Hence the solutions are given by

$$x_1=2.8, x_2=-1.16, x_3=2.04.$$

Example 4. *Solve by the Gauss's elimination method.*

$$2x+y+4z=12$$
$$8x-3y+2z=23$$
$$4x+11y-z=33$$

Solution. Dividing first equation by 2, we get

$$x+\frac{1}{2}y+2z=6 \quad ...(1)$$

Now subtract (1) after multiplied by 8 and 4 respectively from second and third equation, we get

$$-7y-14x=-45 \quad ...(2)$$

$$9y-9z=9 \quad ...(3)$$

Now multiplying (4) by 9 and subtract from (3), we get

$$-27z=9-\frac{405}{7}$$

or $$-27z=-\frac{342}{7} \quad ...(5)$$

Hence the system of equations reduces to upper triangular form as follows:

$$\left.\begin{aligned} x+\frac{1}{2}y+2z&=6\\ y+2z&=\frac{45}{7}\\ -27z&=-\frac{342}{7}\end{aligned}\right\} \quad ...(6)$$

By back substitution, we get

$$z=\frac{342}{189}=1.81$$

and $$y+2(1.81)=\frac{45}{7}$$

$\Rightarrow$ $$y=\frac{45}{7}-2(1.81)=6.43-3.62$$
$$=2.81$$

and $$x+\frac{1}{2}(2.81)+2(1.81)=6$$

$\therefore$ $$x=6-\frac{1}{2}(2.81)-2(1.81)$$
$$=0.975$$

Hence the solution is $x = 0.975, y = 2.81, z = 1.81$.

Example 5. *Apply Gauss's elimination method to solve the equations*

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

Solution. Eliminating x from second and third equation with the help of first equation. Subtract first equation from second and after multiplied by 3 from third respectively, we get the system of equations as follows:

$$x + 4y - z = -5 \quad ...(1)$$
$$-3y - 5z = -7 \quad ...(2)$$
$$-13y + 2z = 19 \quad ...(3)$$

Elimination y from (3) with help of (2). Divide (2) by -3 and then this equation is subtracted after multiplies by -13 from (3), we get

$$\left.\begin{aligned} x+4y-z&=-5\\ y+\frac{5}{3}z&=\frac{7}{3}\\ \frac{71}{3}z&=\frac{148}{3}\end{aligned}\right\} \quad ...(4)$$

By back substitution from (4), we get

$$z=\frac{148}{71}$$

and $$y=\frac{7}{3}-\frac{5}{3}z=\frac{7}{3}-\frac{5}{3}\left(\frac{148}{71}\right)$$

$\Rightarrow$ $$y=-\frac{81}{71}$$

and $$x=-5-4y+z$$
$$=-5-4\left(-\frac{81}{71}\right)+\left(\frac{148}{71}\right)$$
$$=-5+\frac{472}{71}+\frac{117}{71}$$

Hence the solution are $x=\frac{117}{71}, y=-\frac{81}{71}, z=\frac{148}{71}$.

Example 6. *Solve the following system by Gauss's elimination method :*

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 4$$

Solution. We have

$$2x + y + z = 10 \quad ...(1)$$
$$3x + 2y + 3z = 18 \quad ...(2)$$
$$x + 4y + 9z = 4 \quad ...(3)$$

Divide (1) and 2 and subtract after multiplied by 3 from (2) then subtract from (3), we get

$$x+\frac{1}{2}y+\frac{1}{2}z=5 \quad ...(4)$$
$$\frac{1}{2}y+\frac{3}{2}z=3 \quad ...(5)$$
$$\frac{7}{2}y+\frac{17}{2}z=11 \quad ...(6)$$

Now divide (5) by $\frac{1}{2}$ and then subtract after multiplied by $\frac{7}{2}$ from (6) we get,

$$x+\frac{1}{2}y+\frac{1}{2}z=5 \quad ...(7)$$
$$y + 3z = 6 \quad ...(8)$$
$$-2z = -10 \quad ...(9)$$

From back substitution in (9), (8) and (7) we get

$$z = 5$$

and $$y + 3z = 6$$
$$y + 3(5) = 6$$
$$y = 6 - 15 \Rightarrow y = -9$$
$$y = -9$$

and $$x+\frac{1}{2}(-9)+\frac{1}{2}(5)=5$$
$$x=5+\frac{9}{2}-\frac{5}{2}$$

Hence the solution is $x = 7, y = -9, z = 5$.

Example 7. *By Gauss's elimination method, solve*

$$4x + 11y - z = 33$$
$$x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$

Solution. Given equation are

$$x + y + 4z = 12 \quad ...(1)$$
$$4x + 11y - z = 33 \quad ...(2)$$
$$8x - 3y + 2z = 20 \quad ...(3)$$

Eliminating x from (2) and (3) so subtract (1) after multiplied by 4 and 8 from (2) and (3), we get

$$x + y + 4z = 12$$
$$7y - 17z = -15 \quad \ldots(4)$$
$$-11y - 30z = -76 \quad \ldots(5)$$
$$\ldots(6)$$

Now divide (5) by 7 and then subtract after multiplied by -11 from (6), we get

$$x + y + 4z = 12 \quad \ldots(7)$$
$$y - \frac{17}{7}z = 12 \quad \ldots(8)$$
$$-\frac{397}{7}z = -\frac{697}{7} \quad \ldots(9)$$

By back substitution in (9), (8) and (7) we get

$$z = \frac{697}{397} = 1.756$$

From (8)

$$y = \frac{-15}{7} + \frac{17}{7}z = -\frac{15}{7} + \frac{17}{7}\left(\frac{697}{397}\right)$$
$$= \frac{1}{7}\left(\frac{5894}{397}\right) = \frac{5894}{2779} = 2.121$$

From (7)

$$x + y + z = 12$$
$$\Rightarrow \quad x + \frac{5894}{2779} + 4\left(\frac{697}{397}\right) = 12$$
$$\Rightarrow \quad x = 12 - \frac{5894}{2779} + 4\left(\frac{697}{397}\right) = \frac{7938}{2779} = 2.856$$

Hence the solution is $x = 2.856$, $y = 2.121$, $z = 1.756$.

Example 8 *Solve by Gauss's elimination method*

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

Solution Given equation are

$$x + 2y + z = 3 \quad \ldots(1)$$
$$2x + 3y + 3z = 10 \quad \ldots(2)$$
$$3x - y + 2z = 13 \quad \ldots(3)$$

Here pivot element of (1) is 1. Now eliminating x from (2) and (3) by subtracting (1) after multiplied by 2 and 3 respectively from (2) and (3), we get

$$x + 2y + z = 3 \quad \ldots(4)$$
$$-y + z = 4 \quad \ldots(5)$$
$$-7y - z = 4 \quad \ldots(6)$$

Now eliminating y from (6) with the help of (5) by subtracting (5) after multiplied

by -7 from (6), we get

$$x + 2y + z = 3 \quad \ldots(7)$$
$$-y + z = 4 \quad \ldots(8)$$
$$6z = 32 \quad \ldots(9)$$

By back substitution from (7), (8) and (9), we get

From (9) $\quad z = \frac{32}{6} = \frac{16}{3}$

From (8)

$$-y = 4 - z$$
$$-y = 4 - \frac{32}{6} = -\frac{8}{6}$$
$$\therefore \quad y = \frac{8}{6} = \frac{4}{3}$$

From (7)

$$x + 2y + z = 3$$
$$x + 2\left(\frac{8}{6}\right) + \frac{32}{6} = 3$$
$$x = 3 - \frac{32}{6} - \frac{16}{6}$$
$$= \frac{18 - 48}{6} = -\frac{30}{6}$$
$$x = -5$$

Hence the solution is $x = -5, y = \frac{4}{3}, z = \frac{16}{3}$.

3.8 LU DECOMPOSITION METHOD

This method is based on the fact that every square matrix A can be expressed as the form LU where L is a unit lower triangular matrix while U is upper triangular matrix and provided all the principal minors of A are non-singular.

That is, If $A = [a_{ij}]_{n \times n}$, then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ and so on.}$$

For simplicity and understanding this method, let us consider a system of three equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These equations can be written in matrix form as follows:

$$AX = B$$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \ldots(1)$$

Let

$$A = LU$$

where $$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \ldots (2)$$

Then from (2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

or $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Comparing the two matrices, we get

(i) $$u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13}$$

(ii) $$l_{21}u_{11} = a_{21}$$

or $$l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}}$$

(iii) $$l_{31}u_{11} = a_{31}$$

or $$l_{31} = \frac{a_{31}}{u_{11}} = \frac{a_{31}}{a_{11}}$$

(iv) $$l_{21}u_{12} + u_{22} = a_{22}$$

or $$l_{21} = \frac{a_{22} - u_{22}}{u_{12}}$$

or $$u_{22} = a_{22} - l_{21}u_{12} = a_{22} - \frac{a_{21}}{a_{11}}.a_{12}$$

(v) $$l_{21}u_{13} + u_{23} = a_{23}$$

$$u_{23} = a_{23} - l_{21}u_{13}$$

$$u_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$

(vi) $$l_{31}u_{12} + l_{32}u_{22} = a_{32}$$

or $$l_{32} = \frac{1}{u_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right] = \frac{\left[a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right]}{\left[a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right]}$$

(vii) $$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$$

or $$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

$$= a_{33} - \frac{a_{31}}{a_{11}}.a_{13} - \frac{1}{u_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}}.a_{12}\right]\left[a_{23} - \frac{a_{21}}{a_{11}}.a_{13}\right]$$

$$= a_{33} - \frac{a_{31}a_{13}}{a_{11}} \frac{\left[a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right]\left[a_{23} - \frac{a_{21}}{a_{11}}.a_{13}\right]}{\left[a_{22} - \frac{a_{21}.a_{12}}{a_{11}}\right]}$$

Thus from above we can find the elements of L and U. Now L and U are therefore obtained.

Replacing A by LU in (1), we get

$$LUX = B \quad \ldots (3)$$

Now let $$UX = Y \quad \ldots (4)$$

where $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From (3) and (4), we get

$$LY = B$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

From this equation, we get

$$\left.\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - l_{21}y_1 \\ y_3 &= b_3 - l_{31}b_1 - l_{32}b_2 \end{aligned}\right\} \quad \ldots (5)$$

Now from (4)

$$UX = Y$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From this equation, we get

$$x_3 = \frac{y_3}{u_{33}},$$

$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}}$$

and $$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}.$$

With the help of L and U, x_1, x_2, x_3 can be calculated.

Remarks

- LU decomposition method is superior to Gauss's elimination method.
- This method is applicable if the coefficient matrix can be expressed as the product of lower and upper triangular matrix.
- This method can also be renamed as Method of factorization.

Solved Examples

Example 1. *Solve the following equation by decomposition method.*

$$2x_1 + x_2 + x_3 = 2$$
$$x_1 + 3x_2 + 2x_3 = 2$$
$$3x_1 + x_2 + 2x_3 = 2$$

Solution. Given equation are

$$2x_1 + x_2 + x_3 = 2 \quad \ldots (1)$$
$$x_1 + 3x_2 + 2x_3 = 2 \quad \ldots (2)$$
$$3x_1 + x_2 + 2x_3 = 2 \quad \ldots (3)$$

Here, the coefficient matrix A is given by

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

and $$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore$ $$AX = B$$

Now $$A = LU$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \ldots (4)$$

On comparing two matrices, we get

$$u_{11} = 2, u_{12} = 1, u_{13} = 1$$

$$l_{21}u_{11} = 1 \text{ or } l_{21} = \frac{1}{u_{11}} = \frac{1}{2}$$

$$l_{31}u_{11} = 3 \text{ or } l_{31} = \frac{3}{u_{11}} = \frac{3}{2}$$

$$l_{21}u_{12} + u_{22} = 3$$

$$u_{22} = 3 - l_{21}u_{12} = 3 - \frac{1}{2}(1) = 3 - \frac{1}{2} = \frac{5}{2}$$

$$l_{21}u_{13} + u_{23} = 2$$

$$u_{23} = 2 - l_{21}u_{13} = 2 - \frac{1}{2}(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$l_{31}u_{12} + l_{32}u_{22} = 1$$

$$l_{32} = \frac{1 - l_{31}u_{12}}{u_{22}} = \frac{1 - \frac{3}{2}(1)}{\frac{5}{2}} = -\frac{1}{5}$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$$

$$u_{33} = 2 - l_{31}u_{13} - l_{32}u_{23} = 2 - \frac{3}{2}(1) - \left(-\frac{1}{5}\right)\left(\frac{3}{2}\right)$$

$$= 2 - \frac{3}{2} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

Thus $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -\frac{1}{5} & 1 \end{bmatrix}$$

and $$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & \frac{4}{5} \end{bmatrix}$$

Since $$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -\frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

From these equation , we get

$$y_1 = 2$$

$$\frac{1}{2}y_1 + y_2 = 2$$

$$\frac{3}{2}y_1 - \frac{1}{5}y_2 + y_3 = 2$$

On solving, we get

$$y_1 = 2, y_2 = 1, y_3 = -\frac{4}{5}$$

and since, we have $$UX = Y$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -\frac{4}{5} \end{bmatrix}$$

$$\therefore \quad 2x_1 + x_2 + x_3 = 2$$
$$\frac{5}{2}x_2 + \frac{3}{2}x_3 = 1$$
$$\frac{4}{5}x_3 = -\frac{4}{5}$$

By back substitution, we get

$$x_3 = -1, x_2 = 1, x_1 = 1$$

Hence solution is $x_1 = 1, x_2 = 1, x_3 = -1$.

Example 2 *Solve the following equations by LU decomposition method*

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Solution Above equation can be written as follows.

$$AX = B \quad \ldots (1)$$

Where $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Let $A = LU$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Comparing, we get

$$u_{11} = 2, u_{12} = 3, u_{13} = 1$$

and $\quad l_{21}u_{11} = 1$ or $l_{21} = \frac{1}{2}$

$$l_{21}u_{12} + u_{22} = 2$$
$$u_{22} = 2 - l_{21}u_{12}$$
$$= 2 - \frac{1}{2}(3) = 2 - \frac{3}{2} = \frac{1}{2}$$
$$l_{21}u_{13} + u_{23} = 3$$

or $\quad u_{23} = 3 - l_{21}u_{13} = 3 - \frac{1}{2}(1) = \frac{5}{2}$

$$l_{31}u_{11} = 3$$

$$l_{31} = \frac{3}{u_{11}} = \frac{3}{2}$$

and $\quad l_{31}u_{12} + l_{32}u_{22} = 1$

$$l_{32} = \frac{1 - l_{31}u_{12}}{u_{22}} = \frac{1 - \frac{3}{2}(3)}{1/2} = \frac{2 - 9}{1} = -7$$

Now, $\quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$

$$u_{33} = 2 - l_{31}u_{13} - l_{32}u_{23}$$
$$= 2 - \frac{3}{2}(1) - (-7)\left(\frac{5}{2}\right)$$
$$= 2 - \frac{3}{2} + \frac{35}{2} = \frac{36}{2} = 18$$

Thus L and U are given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

Since, $\quad LY = B$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

From above equation, we get

$$y_1 = 9$$
$$\frac{1}{2}y_1 + y_2 = 6$$
$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8$$

Solving these equation , we get

$$y_1 = 9, y_2 = \frac{3}{2}, y_3 = 5$$

Further since $\quad UX = Y$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$\therefore \quad 2x+3y+z=9$$

$$\frac{1}{2}y+\frac{5}{2}z=\frac{3}{2}$$

$$\Rightarrow \quad 18z=5$$

By back substitution, we get

$$z=\frac{5}{18}$$

and
$$\frac{1}{2}y+\frac{5}{2}\left(\frac{5}{18}\right)=\frac{3}{2}$$

$$\Rightarrow \quad y=3-\frac{25}{18}=\frac{29}{18}$$

and
$$2x+3\left(\frac{29}{18}\right)+\left(\frac{5}{18}\right)=9$$

$$2x=9-\frac{29}{6}-\frac{5}{18}=9-\frac{92}{18}$$

$$\Rightarrow \quad 2x=\frac{70}{18} \Rightarrow x=\frac{35}{18}$$

Hence the solution is

$$x=\frac{35}{18}, y=\frac{29}{18}, z=\frac{5}{18}.$$

EXERCISE 3.3

1. *Solve the following equations by Gauss's ellimination method :*

(i) $x_1+x_2+2x_3=4$
$3x_1+x_2-3x_3=-4$
$2x_1-3x_2-5x_3=-5$

(ii) $2x_1+x_2-4x_3=12$
$8x_1-3x_2+2x_3=20$
$4x_1+11x_2-x_3=33$

(iii) $x_1+x_2+x_3=10$
$2x_1+x_2+2x_3=17$
$3x_1+2x_2+x_3=17$

(iv) $2x+3y-z=5$
$4x+4y-3z=3$
$2x-3y+2z=2$

(v) $2x+y+z=10$
$3x+2y+3z=18$
$x+4y+9z=16$

(v) $2x_1+4x_2+x_3=2$
$3x_1+2x_2-2x_3=-2$
$x_1-x_2+x_3=6$

2. *Solve the following equations by LU decomposition method :*

(i) $2x+y+2z=2$
$x+5y+3z=4$
$x+y-z=0$

(ii) $x+y+3z=10$
$3x+2y+4z=20$

$3x+5y-z=30$

(iii) $2x-3y+10z=3$
$-x+4y+2z=20$
$5x+2y+z=-12$

Answers

1. (i) $x_1=1, x_2=-1, x_3=2$ (ii) $x_1=3, x_2=2, x_3=1$ (iii) $x_1=2, x_2=3, x_3=5$
(iv) $x=1, y=2, z=3$ (v) $x=7, y=-9, z=5$ (vi) $x_1=2, x_2=-1, x_3=3$

2. (i) $x=\frac{1}{5}, y=\frac{2}{5}, z=\frac{3}{5}$ (ii) $x=2, y=5, z=1$ (iii) $x=-4, y=3, z=2$

Self Assessment Test

1. Find the eigenvalue of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

2. Show that the characteristic vector corresponding to characteristic root λ of matrix A is also a characteristic vector of every matrix $f(A)$ where $f(x)$ is any scalar polynomial and the corresponding root for $f(A)$ is $f(\lambda)$.

   In general show that if $g(x) = \dfrac{f_1(x)}{f_2(x)}$ and $|f_2(A)| \neq 0$ and $g(\lambda)$ is a characteristic root of $g(A) = \dfrac{f_1(A)}{f_2(A)}$.

3. Show that any two eigenvectors corresponding to two distinct characteristic roots of a
   (i) Hermitian
   (ii) real symmetric
   (iii) unitary
   matrix are orthogonal.

4. Find the characteristic roots and characteristic vectors for the matrix $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ and show that the matrix A satisfies its characteristic equation.

5. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$. Hence, evaluate A^{-1}.

6. Prove that the eigenvalues of the matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$ are a_1, b_2, c_3.

7. Find all eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

8. Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence, find A^4.

9. If the characteristic roots of $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ are λ_1 and λ_2, show that the characteristic roots of $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ are $\dfrac{1}{\lambda_1}$ and $\dfrac{1}{\lambda_2}$.

10. If A is any matrix which satisfy $A^3 - A^2 + A - I = 0$ and $A_{3\times 3}$ then show that $A^4 = A = I$.

11. Let $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigenvalues of A are 4 and 8 then show that $x = -4$ and $y = 10$.

12. Let A be an $n \times n$ complex matrices whose characteristic polynomial is given by $f(t) = t^n + c_{n-1}t^{n-1} + \ldots + c_1 t + c_0$ then show that $\det(A) = c_0$.

13. If the eigenvalues of a 3×3 matrix A are 1,2 and -3 then show that $A^{-1} = \dfrac{1}{6}\left[7I - A^2\right]$.

14. If a square matrix of order 10 has exactly 4 distinct eigenvalues, then show that the degree of its minimal polynomial is at least 4.

15. Show that the matrix $M = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ has both positive and negative real eigenvalues.

16. Show that the characteristic polynomial of $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is $\det|xI - A| = (x-2)^2(x-1)$.

■■■

5 VECTOR SPACES

5.1 INTRODUCTION

We are familiar with the concept of semigroup, a group and a ring. A group was obtained from a semigroup by imposing certain restrictions on the composition of a semigroup. A ring was obtained by defining a certain composition on a group structure and by giving rules that connected the group composition with the new composition. We shall now discuss another algebraic structure called a vector space or linear space which is going to involve a group structure, a ring structure and an operation connecting the elements of these two structures.

In order to discuss a vector space we need two basic things. One of them is the set of vectors and the other is the set of scalars. Therefore, to define a vector space we need a field F. The elements of field F are called the scalars. In addition, we need two binary operations. One of them is internal composition and the other is external composition. Now we distinguish the internal and external compositions as follows:

5.1.1 INTERNAL COMPOSITION

Let R be any set. If $a * b \in$ R for all $a, b \in$ R and $a*b$ is a unique, then '*' is known as the internal composition. That is, a binary operation defined over the vectors is called internal composition (vector addition).

5.1.2 EXTERNAL COMPOSITION

Let V be the set of vectors and F be a field. Then a binary operation defined between the vectors and scalars is called external composition. That is, if $a \circ \alpha \in$ V for all $\alpha \in$ V and $a \in F$ and $a \circ \alpha$ is unique, then o is called an external composition or scalar multiplication.

5.2 PROLOGUE TO VECTOR SPACE

We have already settled that a non-empty set with a binary composition is a groupoid, an associative groupoid is a semigroup; a semigroup with an identity is a monoid, a monoid in which each element has an inverse is known as a group. Each link in this chain imposes one additional condition on the binary composition defined on the set. Let us put down all these conditions together and look at the *group structure* a fresh.

"Let G be a non-empty set and * be a binary operation defined on it, then the structure $(G, *)$ is said to be a group if the following axioms are satisfied."

(i) **Closure property.** $a * b \in G\ \forall\ a, b \in G$

(ii) **Associativity.** The operation * is associative on G, *i.e.*,

$$a*(b*c) = (a*b)*c\ \forall\ a, b, c \in G$$

(iii) **Existence of identity.** There exists an element $e \in G$, such that

$$a^*e = e^*a = a \ \forall \ a \in G$$

e is called identity of $*$ in G.

(iv) **Existence of inverse.** For each element $a \in G$, there exists an element $b \in G$ such that

$$a * b = b^*a = e$$

The element b is called the inverse of element a with respect to $*$ and we write

$$b = a^{-1}$$

Remarks

- When we say $*$ is a binary operation defined on a non-empty set G, it implies that G is closed under the binary operation $*$, i.e.,

$$a \in G, b \in G \Rightarrow a * b \in G \ \forall \ a, b \in G$$

- A group is not simply a set, but it is an *algebraic structure.*
- Because of the associativity, the paranthesis can be dropped in products of more than two elements of a group and instead of writing $a^*(b^*c)$ or $(a^*b)^*c$ we may simply write a^*b^*c. The associative law can be extended to any finite number of elements.

5.2.1 ABELIAN OR COMMUTATIVE GROUP

A group $(G, *)$ is said to be *abelian* or *commutative* if $a^*b = b^*a \ \forall \ a, b \in G$. The group which are not abelian, called non-abelian or non-commutative.

Remarks

- An abelian group under addition is sometimes called a '*module*'.
- The commutative group is also known as *Abelian group* after the name of famous mathematician Abel.
- The smallest group for a given composition is the set $\{e\}$, containing identity element.
- A group consisting of the identity element only, is called a trivial group, others are called non-trivial groups.

5.2.2 FINITE AND INFINITE GROUP

If a group contains a finite number of elements, called a finite group.

If the number of elements in a group is infinite, then it is called an *infinite group*.

5.2.3 ORDER OF A GROUP

The number of elements in a finite group is called the order of the group. It is denoted by $o(G)$.

An infinite group is called a group of infinite order.

Illustrations

(1) The set Z of integers is an infinite abelian group with respect to the operation addition but Z is not a group with respect to the multiplication.

(2) Let $G = \{1\}$, then G is an abelian group of order 1 with respect to multiplication.

5.2.4 FIELD

Let F be any non-empty set equipped with two binary operations addition $(+)$ and multiplication $(.)$, i.e., for all $a, b \in F$, $a + b \in F$ and $a \,.\, b \in F$. Then the algebraic structure $(F, +, .)$ is said to be a field if it satisfies the following conditions :

(i) Addition is associative, i.e., $(a+b)+c = a+(b+c) \ \forall \ a, b, c \in F$.

(ii) Addition is commutative, i.e., $a + b = b + a \ \forall \ a, b \in F$

(iii) $\exists$ an identity element 0 in F such that $a + 0 = 0 + a = a \ \forall \ a \in F$.

(iv) To each element $a \in F \ \exists \ -a \in F$ such that $a + (-a) = 0$

(v) Multiplication is commutative, i.e., $a.b = b.a \ \forall a, b \in F$.

(vi) Multiplication is associative, i.e., $(a.\,b).c = a.\,(b.\,c) \ \forall a,b, c \in F$.

(vii) There exists a non-zero element 1 (one) in F such that $a.1 = a \ \forall a \in F$

(viii) To every non-zero element $a \in F$ there corresponds an element a^{-1} in F such that $a.\,a^{-1} = 1$

(ix) Multiplication is distributive over addition i.e.,

$$a.(b+c) = a.b + a.c \ \forall a,b,c \in F$$

5.2.5 SUBFIELD

Let F be a field. A non-empty subset K of F is said to be subfield of F if K is closed w.r.t. addition and multiplication in F and K itself is a field.

Illustrations

(1) The set Q of rational numbers is a field.

(2) The set R of real numbers is a field.

(3) Q is a subfield of R.

(4) C is a field.

(5) R is a subfield of C.

5.3 VECTOR SPACES

Let V be a non-empty set of vectors and F be a field. Then an algebraic structure $(V, +, .)$ together with two binary operations vectors addition and scalar multiplication is said to be vector space over F if this structure satisfies the following conditions.

(i) $(V, +)$ is an abelian group.

(ii) $a(\alpha + \beta) = a\alpha + a\beta, \ \forall \ \alpha, \beta \in V$ and $\forall \ a \in F$

(iii) $(a + b)\alpha = a\alpha + b\alpha, \ \ \forall \ \alpha \in V$ and $\forall \ a,b \in F$

(iv) $(ab)\alpha = a(b\alpha), \ \forall \ \alpha \in V$ and $\forall \ a,b \in F$

(v) $1\alpha = \alpha, \ \forall \ \alpha \in V$ and $1 \in F$

This vector space V over F is denoted by $V(F)$.

For example, if $F = R$, the field of real numbers, then $V(R)$ is a vector space and it is called a real vector space.

Illustrations

(1) Let $R^2 = \{(a_1, a_2): a_1 \in R, a_2 \in R)$. The set R^2 is a vector space over R with addition and scalar multiplication defined as follows :

$$(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$$

$$c(a_1, a_2) = (ca_1, ca_2), \ \forall \ a_1, a_2, b_1, b_2, c \in R$$

(2) Vector in 3-dimensional space form a vector space over R with respect to addition and scalar multiplication of vectors.

(3) Let R^n be the set of n-tuples of real numbers, i.e.,

$$R^n = \{(a_1, a_2, ..., a_n): a_i \in R$$

Then R^n is a vector space over R with pointwise addition and scalar multiplicati as defined in (1).

(4) Let C^n be the set of all ordered n-tuples of complex numbers. Then C^n is a vecto space over C with addition and scalar multiplication.

Solved Examples

Example 1. *Show that a field K can be regarded as a vector space over any subfield F of K.*

Solution. Since $F \subset K$, then we have to prove that K is a vector space over F. Now K is set of vectors and the elements of F are the scalars. Addition of vectors is th addition composition in the field K so that $(K, +)$ forms an abelian group. Al the composition of scalar multiplication is the multiplication composition in th field k, then

$$a\alpha \in K \;\forall\; a \in F \text{ and } \alpha \in K$$

If 1 is the unity element of K, then 1 is also the unity element of subfield F. No we shall make the following observations :

(i) $a(\alpha+\beta) = a\alpha + a\beta \;\forall\; a \in F$ and $\alpha, \beta \in K$ [By left distributive law in K]

(ii) $(a + b)\alpha = a\alpha + b\alpha \;\forall\; a, b \in F$ and $\forall\; \alpha \in K$ [By right distributive law in K]

(iii) $(ab)\alpha = a(b\alpha) \;\forall\; a, b \in F$ and $\forall\; \alpha \in K$ [By associativity of multiplication in K]

(iv) $1\alpha = \alpha \;\forall\; \alpha \in K$, $1 \in F$. Since 1 is also the unity element of the field K, so tha

$$1\alpha = \alpha \;\forall\; \alpha \in K.$$

Hence K is a vector space over the field F, which is denoted by $K\,(F)$.

REMARKS

- As the field F is a subfield of itself so that F (F) is a vector space, therefore, in particular R(R) and C (C) are vector spaces with respect to usual addition and multiplication.
- Since R is a subfield of C so that C(R) is a vector space. But R(C) is not a vector space because R is not closed with respect to scalar multiplication, for example.

$$1 + i \in C \text{ and } 3 \in R$$

$$3(1 + i) = 3 + 3i \notin R$$

- Similarly R(Q) is a vector space but Q(R) is not a vector space.

Example 2. *Show that the set of all ordered n-tuples forms a vector space over a field F.*

Solution. If $a_1, a_2, a_3, \ldots, a_n$ are n elements of a field F, then an ordered set $\alpha = (a_1, a_2, \ldots, a_n)$ is called an n-tuple over F.

Let V be the set of all ordered n-tuples, then

$$V = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots a_n \in F\}$$

Now we define equality of two n-tuples, addition of two n-tuples and multiplication of an n-tuple with a scalar as follows:

Equality of two n-tuples:

Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuple of V. Then they are said to be equal if and only if $a_i = b_i$ for $i = 1, 2, 3, \ldots, n$.

Addition of two n-tuples:

Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuple of V, then

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Scalar multiplication of an n-tuple : Let $(a_1, a_2, \ldots, a_n)$ be an n-tuple and $a \in F$, then

$$a(a_1, a_2, \ldots, a_n) = (aa_1, aa_2, \ldots aa_n).$$

Now we shall show that V is a vector space with respect to addition composition and scalar multiplication.

(i) Closure property: For all $\alpha = (a_1, a_2, \ldots, a_n) \in V$ and $\beta = (b_1, b_2, \ldots, b_n) \in V$

$$\alpha + \beta = (a_1 + b_1,\; a_2 + b_2, \ldots, a_n + b_n)$$

Since $a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n$ are all elements of F so that

$$\alpha + \beta \in V \;\forall\; \alpha, \beta \in V$$

Hence V is closed for addition of n-tuples.

(ii) Associativity of addition in V: For all $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$ and $\gamma = (c_1, c_2, \ldots, c_n)$ of V

$$\begin{aligned}
\alpha + (\beta + \gamma) &= (a_1, a_2, \ldots, a_n) + [(b_1, b_2, \ldots, b_n) + (c_1, c_2, \ldots, c_n)] \\
&= (a_1, a_2, \ldots, a_n) + [(b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n) \\
&= (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots, a_n + (b_n + c_n)) \\
&= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots, (a_n + b_n) + c_n) \\
&\qquad [\because F \text{ is associative for addition.}] \\
&= ((a_1 + b_1), (a_2 + b_2), \ldots, (a_n + b_n)) + (c_1, c_2, \ldots, c_n) \\
&= [(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)] + (c_1, c_2, \ldots, c_n) \\
&= (\alpha + \beta) + \gamma
\end{aligned}$$

(iii) Existence of additive identity in V. Since $0 \in F$ so that $0 = (0, 0, \ldots, 0) \in V$.

Also if $\alpha = (a_1, a_2, \ldots, a_n)$ is any element of V, then

$$\begin{aligned}
\alpha + 0 &= (a_1, a_2, \ldots, a_n) + (0, 0, \ldots, 0) \\
&= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\
&= (a_1, a_2, \ldots, a_n) = \alpha
\end{aligned}$$

$\therefore$ $\quad \alpha + 0 = \alpha \;\forall\; \alpha \in V.$

Similarly $\quad 0 + \alpha = \alpha \;\forall\; \alpha \in V$

Hence $\quad 0 = (0, 0, \ldots, 0)$ is the additive identity in V.

(iv) Existence of additive inverse in V. If $\alpha = (a_1, a_2, \ldots, a_n)$ is any element of V, then $-\alpha = (-a_1, -a_2, \ldots, -a_n) \in V$ because $-a_1, -a_2, \ldots, -a_n \in F$.

Also, we have

$$\begin{aligned}
-\alpha + \alpha &= (-a_1, -a_2, \ldots, -a_n) + (a_1, a_2, \ldots a_n) \\
&= (-a_1 + a_1, -a_2 + a_2, \ldots, -a_n + a_n) \\
&= (0, 0, \ldots, 0) = 0
\end{aligned}$$

$\therefore$ $\quad -\alpha + \alpha = 0 \;\forall\; \alpha \in V$

Similarly, $\quad \alpha + (-\alpha) = 0 \;\forall\; \alpha \in V$

Hence $(-a_1, -a_2, \ldots, -a_n)$ is the additive inverse of $(a_1, a_2, \ldots, a_n)$.

(v) **Commutativity of addition in V.** For all $\alpha = (a_1, a_2, ..., a_n)$ and $\beta = (b_1, b_2, ..., b_n) \in V$, we have

$$\alpha + \beta = (a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n)$$
$$= (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$$
$$= (b_1 + a_1, b_2 + a_2, ..., b_n + a_n)$$
$$= (b_1, b_2, ..., b_n) + (a_1 + a_2, ..., a_n)$$
$$= \beta + \alpha$$

$\therefore$ $\alpha + \beta = \beta + \alpha \ \forall \alpha, \beta \in V$

Hence, V is an abelian group under addition of n-tuples.

Now we observe that

(i) For all $\alpha = (a_1, a_2, ..., a_n)$ and $\beta = (b_1, b_2, ..., b_n)$ and $a \in F$

$$a(\alpha + \beta) = a(a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$$
$$= (a(a_1 + b_1), a(a_2 + b_2), ..., a(a_n + b_n))$$
$$= (aa_1 + ab_1, aa_2 + ab_2, ..., aa_n + ab_n)$$
$$= (aa_1, aa_2, ..., aa_n) + (ab_1, ab_2, ..., ab_n)$$
$$= a(a_1, a_2, ..., a_n) + a(b_1, b_2, ..., b_n)$$
$$= a\alpha + a\beta$$

(ii) For all $a, b \in F$ and $\alpha = (a_1, a_2, ..., a_n) \in V$

$$(a + b)\alpha = (a + b)(a_1, a_2, ..., a_n)$$
$$= ((a + b)a_1, (a + b)a_2, ..., (a + b)a_n)$$
$$= (aa_1 + ba_1, aa_2 + ba_2, ..., aa_n + ba_n)$$
$$= (aa_1, aa_2, ..., aa_n) + (ba_1, ba_2, ..., ba_n)$$
$$= a(a_1, a_2, ..., a_n) + b(a_1, a_2, ..., a_n)$$
$$= a\alpha + b\beta$$

(iii) For all $a, b \in F$ and $\alpha = (a_1, a_2, ..., a_n) \in V$

$$(ab)\alpha = (ab)(a_1, a_2, ..., a_n)$$
$$= ((ab)a_1, (ab)a_2, ..., (ab)a_n)$$
$$= (a(ba_1), a(ba_2), ..., a(ba_n))$$
$$= a(ba_1, ba_2, ..., ba_n)$$
$$= a[b(a_1, a_2, ..., ba_n)]$$
$$= a(b\alpha)$$

(iv) If 1 is the unity element of F and $\alpha = (a_1, a_2, ..., a_n) \in V$ then

$$1\alpha = 1(a_1, a_2, ..., a_1) = (1a_1, 1a_2, ..., 1a_n) = (a_1, a_2, ..., a_n) = \alpha.$$

Hence, V is a vector space over the field F. This vector space is denoted by $V_n(F)$. Sometimes, it is also denoted by F^n or $F^{(n)}$ or $F^n(F)$.

Remark

- $V_2(F) = \{(a_1, a_2) : a_1, a_2, \in F\}$ is a vector space of all ordered pairs over F. Similarly $V_3(F) = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in F\}$ forms a vector space of all ordered triads over F.

Example 3. *Show that the set of all $m \times n$ matrices with their elements as real numbers is a vector space over the field F of real numbers with respect to addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication.*

Solution. Let $M_{mn} = \{A, B, C, ...\}$ be the set of all $m \times n$ matrices. We shall show that $M_{mn}(F)$ will form abelian group under addition.

(i) **Closure property.** For all $A, B \in M_{mn}$, we have $A + B \in M_{mn}$. Hence M_{mn} is closed under addition of matrices.

(ii) **Associativity.** For all $A, B, C \in M_{mn}$, we have

$$A + (B + C) = (A + B) + C$$

(iii) **Existence of identity.** If O be the null matrix of order $m \times n$, then $O \in M_{mn}$. Also for all $A \in M_{mn}$, we have

$$A + O = A = O + A$$

Hence, O is additive identity in M_{mn}.

(iv) **Existence of inverse.** If $A \in M_{mn}$, then $-A \in M_{mn}$. Also for all $A \in M_{mn}$, we have

$$(-A) + A = O = A + (-A)$$

Hence, $-A$ is the additive inverse of A.

(v) **Commutativity.** For all $A, B \in M_{mn}$, we have

$$A + B = B + A$$

Hence, M_{mn} is an abelian group under addition.

If $a \in F$ and $A = [a_{ij}]_{m \times n} \in M_{mn}$ then

$$aA = a[a_{ij}]_{m \times n} = [aa_{ij}]_{m \times n} \in M_{mn}$$

Now we observe that:

(i) For all $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$ in M_{mn} and $a \in F$

Then, $a(A + B) = a([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) = a([a_{ij} + b_{ij}]_{m \times n})$

$$= [a(a_{ij} + b_{ij})]_{m \times n} = [(aa_{ij} + ab_{ij})]_{m \times n}$$
$$= [aa_{ij}]_{m \times n} + [ab_{ij}]_{m \times n} = a[a_{ij}]_{m \times n} + a[b_{ij}]_{m \times n}$$
$$= aA + aB$$

(ii) For all $a, b \in F$ and $A = [a_{ij}]_{m \times n}$

$$(a + b)A = (a + b)[a_{ij}]m \times n = [(a + b)a_{ij}]_{m \times n}$$
$$= [(aa_{ij} + ba_{ij})]_{m \times n} = [aa_{ij}]_{m \times n} + [ba_{ij}]_{m \times n}$$
$$= a[a_{ij}]_{m \times n} + b[a_{ij}]_{m \times n}$$
$$= aA + bA$$

(iii) For all $a, b \in F$ and $A = [a_{ij}]_{m\times n} \in M_{mn}$

$$(ab)A = [ab][a_{ij}]_{m\times n} = [(ab)\, a_{ij}]_{m\times n}$$
$$= [a(ba_{ij})]_{m\times n} = a[ba_{ij}]_{m\times n}$$
$$= a(b[a_{ij}]_{m\times n}) = a(bA)$$

(iv) Since $1 \in F$ and $A = [a_{ij}]_{m\times n} \in M_{mn}$, then

$$1A = 1[a_{ij}]_{m\times n} = [1a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = A$$

Hence, M_{mn} is a vector space over F.

REMARK

If M_{mn} is a set of all $m \times n$ matrices with their elements as rational numbers and F is the field of real numbers, then M_{mn} (F) will not form a vector space because $\sqrt{2} \in F$ and if $A \in M_{mn}$ (F), then $\sqrt{2}A \notin M_{mn}$ (F).

Example 4. *Show that the set of all polynomials over a field F is a vector space.*

Solution. If $(F, +, .)$ be the field, then $F[x]$ denotes the set of all polynomials in the indeterminate x with coefficient from F.

Now we define the addition of polynomials and scalar multiplication of polynomial as follows :

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots$

$g(x) = b_0 + b_1x + b_2x^2 + \ldots$

be any two polynomials of $F(x)$, where $a_0, a_1, a_2, \ldots \in F$ and $b_0, b_1, b_2, \ldots \in F$.

Addition of polynomial in $F[x]$.

For $f(x)$ and $g(x) \in F[x]$, we have

$$f(x) + g(x) = (a_0 + a_1x + a_2x^2 \ldots) + (b_0 + b_1x + b_2x^2 + \ldots)$$
$$= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \ldots$$

Scalar multiplication in $F[x]$ *by an element of* F.

If $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in F[x]$ and $h \in F$ then

$$kf(x) = k(a_0 + a_1x + a_2x^2 + \ldots)$$
$$= (ka_0) + (ka_1)x + (ka_2)x^2 + \ldots$$

Now we shall prove that $F[x]$ is a vector space over F for the above compositions.

(i) **Closure property.** For all $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in F[x]$ and

$g(x) = b_0 + b_1x + b_2x^2 + \ldots \in F[x]$, then we have

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots$$

Since $a_0+b_0, a_1+b_1, a_2+b_2, \ldots \in F$, therefore $f(x) + g(x) \in F[x]$.

(ii) **Associativity.** For all $f(x), g(x), h(x) \in F[x]$ where

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots$$
$$g(x) = b_0 + b_1x + b_2x^2 + \ldots$$
$$h(x) = c_0 + c_1x + c_2x^2 + \ldots$$

we have

$$f(x) + [g(x) + h(x)] = (a_0 + a_1x + a_2x^2 + \ldots)$$
$$+[(b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 + \ldots]$$
$$= [a_0 + (b_0 + c_0)] + [a_1 + (b_1 + c_1)]x + [a_2 + (b_2 + c_2)]x^2 + \ldots$$
$$= [(a_0 + b_0) + c_0] + [(a_1 + b_1) + c_1)]x + [(a_2 + b_2) + c_2)]x^2 + \ldots$$
$$= [(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \ldots] + [c_0 + c_1x + c_2x^2 + \ldots]$$
$$= [f(x) + g(x)] + h(x)$$

(iii) **Existence of additive identity.**

If $0(x)$ denotes the zero polynomial over F, then

$$0(x) = 0 + 0x + 0x^2 + \ldots$$

$\therefore$ $0(x) \in F[x]$

Also, for all $f(x) = a_0 + a_1x + a_2x^2 + \ldots$, we have

$$0(x) + f(x) = (0 + 0x + 0x^2 + \ldots) + (a_0 + a_1x + a_2x^2 + \ldots)$$
$$= (0 + a_0) + (0 + a_1)x + (0 + a_2)x^2 + \ldots$$
$$= a_0 + a_1x + a_2x^2 + \ldots$$
$$= f(x)$$

Similarly, $f(x) + 0(x) = f(x)$

Hence $0(x)$ is the additive identity in $F[x]$.

(iv) **Existence of additive inverse.**

If $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in F[x]$, then define

$$-f(x) = -a_0 + (-a_1)x + (-a_2)x^2 + \ldots$$

Since $-a_0, -a_1, -a_2, \ldots \in F$, therefore $-f(x) \in F[x]$

Also, we have

$$-f(x) + f(x) = (-a_0 + (-a_1)x + (-a_2)x^2 + \ldots) + (a_0 + a_1x + a_2x^2 + \ldots)$$
$$= (-a_0 + a_0) + (-a_1 + a_1)x + (-a_2 + a_2)x^2 + \ldots$$
$$= 0 + 0x + 0x^2 + \ldots$$
$$= 0(x)$$

Similarly, $f(x) + f(-f(x)) = 0(x)$

Hence, $-f(x)$ is an additive inverse of $f(x)$.

(v) **Commutativity under addition.**

For all $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots$ of $F[x]$,

we have

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots$$

$$= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 + \ldots$$
$$= g(x) + f(x)$$

Hence $F(x)$ is an abelian group with respect to addition of polynomials.

Further, we observe that

(i) If $k \in F$ and $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots$ are in $F(x)$, then

$$k[f(x) + g(x)] = k[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots]$$
$$= k(a_0 + b_0) + k(a_1 + b_1)x + k(a_2 + b_2)x^2 + \ldots$$
$$= (ka_0 + kb_0) + (ka_1 + kb_1)x + (ka_2 + kb_2)x^2 + \ldots$$
$$= [(ka_0) + (ka_1)x + (ka_2)x^2 + \ldots]$$
$$+[(kb_0) + (kb_1)x + (kb_2)x^2 + \ldots]$$
$$= k[a_0 + a_1x + a_2x^2 + \ldots] + k[b_0 + b_1x + b_2x^2 + \ldots]$$
$$= k\,f(x) + k\,g(x)$$

(ii) If $k_1, k_2 \in F$ and $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in F[x]$, then

$$(k_1 + k_2)f(x) = (k_1 + k_2)[a_0 + a_1x + a_2x^2 + \ldots]$$
$$= (k_1 + k_2)a_0 + (k_1 + k_2)a_1x + (k_1 + k_2)a_2x^2 + \ldots$$
$$= (k_1a_0 + k_2a_0) + (k_1a_1 + k_2a_1)x + (k_1a_2 + k_2a_2)x^2 + \ldots$$
$$= [(k_1a_0) + (k_1a_1)x + (k_1a_2)x^2 + \ldots]$$
$$+[(k_2a_0) + (k_2a_1)x + (k_2a_2)x^2 + \ldots]$$
$$= k_1(a_0 + a_1x + a_2x^2 + \ldots) + k_2(a_0 + a_1x + a_2x^2 + \ldots)$$
$$= k_1 f(x) + k_2 f(x)$$

(iii) If 1 is the unity element in F and $f(x) \in F(x)$ where $f(x) = a_0 + a_1x + a_2x^2 + \ldots$, then

$$1.\,f(x) = (1a_0) + (1a_1)x + (1a_2)x^2 + \ldots$$
$$= a_0 + a_1x + a_2x^2 + \ldots$$
$$= f(x)$$

Hence, $F[x]$ is a vector space over F with respect to the addition and scalar multiplication of polynomials.

Example 5. *Show that the set of all convergent sequences is a vector space over the field of real numbers.*

Solution. Let V denote the set of all convergent sequences of real numbers.

Let
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = (\alpha_n)$$

$$\beta = (\beta_1, \beta_2, \ldots, \beta_n, \ldots) = (\beta_n)$$
$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots) = (\gamma_n) \text{ , etc.}$$

any members of V.

Now we shall show that $(V, +)$ is an abelian group.

(i) Closure property. For all convergent sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ in V, we have

$$\alpha + \beta = \langle\alpha_n\rangle + \langle\beta_n\rangle = \langle\alpha_n + \beta_n\rangle.$$

$\therefore$ $\alpha + \beta$ is also convergent.

Hence $\alpha + \beta \in V \; \forall \alpha, \beta \in V$.

(ii) Associativity. For all convergent sequences $\alpha = \langle\alpha_n\rangle, \beta = \langle\beta_n\rangle$ and $\gamma = \langle\gamma_n\rangle$ in V, we have

$$\alpha + (\beta + \gamma) = \langle\alpha_n\rangle + \langle\langle\beta_n\rangle + \langle\gamma_n\rangle\rangle = \langle\alpha_n\rangle + \langle\langle\beta_n + \gamma_n\rangle\rangle$$
$$= \langle\alpha_n + (\beta_n + \gamma_n)\rangle = \langle(\alpha_n + \beta_n) + \gamma_n\rangle$$
$$= \langle(\alpha_n + \beta_n)\rangle + \langle\gamma_n\rangle = (\langle\alpha_n\rangle + \langle\beta_n\rangle) + \langle\gamma_n\rangle$$
$$= (\alpha + \beta) + \gamma$$

(iii) Existence of identity in V. Since $\langle 0\rangle = \langle 0, 0, 0, \ldots, 0, \ldots\rangle$ is a convergent sequence so that $\langle 0\rangle \in V$.

$$\text{If } \alpha = \langle\alpha_n\rangle \in V, \text{ then } \langle 0\rangle + \alpha = \langle 0\rangle + \langle\alpha_n\rangle$$
$$= \langle 0 + \alpha_n\rangle = \langle\alpha_n\rangle = \alpha$$

Similarly, $\alpha + \langle 0\rangle = \alpha$

Hence $\langle 0\rangle$ is an identity in V.

(iv) Existence of inverse in V. If $\alpha = \langle\alpha_n\rangle \in V$, then $-\alpha = \langle -\alpha_n\rangle \in V$.

Also, $-\alpha + \alpha = \langle -\alpha_n\rangle + \langle\alpha_n\rangle$
$$= \langle -\alpha_n + \alpha_n\rangle = \langle 0\rangle$$

Similarly, $\alpha + (-\alpha) = \langle 0\rangle$

Hence, $-\alpha$ is an additive inverse of α.

(v) Commutativity in V. For all $\alpha = \langle\alpha_n\rangle$ and $\beta = \langle\beta_n\rangle$ of V, we have

$$\alpha + \beta = \langle\alpha_n\rangle + \langle\beta_n\rangle = \langle\alpha_n + \beta_n\rangle$$
$$= \langle\beta_n + \alpha_n\rangle = \langle\beta_n\rangle + \langle\alpha_n\rangle$$
$$= \beta + \alpha$$

Hence, $(V, +)$ is an abelian group.

Next, if $a \in \mathbb{R}$ (set of all real numbers) and $\alpha = \langle\alpha_n\rangle \in V$, then $a\alpha = a\langle\alpha_n\rangle = \langle a\alpha_n\rangle$ which is also convergent.

$\therefore\ a\alpha \in V\ \forall a \in R$ and $\alpha \in V$.

Now we observe that

(i) If $\alpha = \langle \alpha_n \rangle, \beta = \langle \beta_n \rangle \in V$ and $a \in R$, then

$$a(\alpha+\beta) = a(\langle \alpha_n \rangle + \langle \beta_n \rangle) = a\langle(\alpha_n + \beta_n)\rangle$$
$$= \langle a(\alpha_n + \beta_n)\rangle = \langle a\alpha_n \rangle + \langle a\beta_n \rangle = a\langle \alpha_n \rangle + a\langle \beta_n \rangle$$
$$= a\alpha + a\beta$$

(ii) If $a, b \in R$ and $\alpha = \langle \alpha_n \rangle \in V$, then

$$(a+b)\alpha = (a+b)\langle \alpha_n \rangle = \langle (a+b)\alpha_n \rangle$$
$$= \langle (a\alpha_n + b\alpha_n)\rangle = \langle a\alpha_n \rangle + \langle b\alpha_n \rangle$$
$$= a\langle \alpha_n \rangle + b\langle \alpha_n \rangle$$
$$= a\alpha + b\alpha$$

(iii) If $a, b \in R$ and $\alpha = \langle \alpha_n \rangle \in V$, then

$$(ab)\alpha = (ab)\langle \alpha_n \rangle$$
$$= \langle (ab)\alpha_n \rangle = \langle a(b\alpha_n)\rangle = a\langle b\alpha_n \rangle$$
$$= a(b\langle \alpha_n \rangle) = a(b\alpha)$$

(iv) Since 1 is the unity of element in R and $\alpha = \langle \alpha_n \rangle \in V$, then

$$1\alpha = 1\langle \alpha_n \rangle = \langle 1\alpha_n \rangle = \langle \alpha_n \rangle = \alpha$$

Hence V is a vector space over the field of real numbers.

Example 6. *Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define*

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$

and

$$c(x, y) = (cx, 0)$$

Is V, with these operations, a vector space over the field of real number?

Solution. Clearly, for the operation of addition, the identity element does not exist. For, suppose that the ordered pair $(x_1, y_1) \in V$ is the identity element for addition. Then we must have

$$(x, y) + (x_1, y_1) = (x, y) = (x_1, y_1) + (x, y)\ \forall\ x, y \in R \quad \text{... (1)}$$

But by definition of addition, we have

$$(x, y) + (x_1, y_1) = (x, y) = (x + x_1, 0)$$

$\Rightarrow$

$$(x, y) = (x + x_1, 0) \quad \text{[Using (1)]}$$

If $y \neq 0$, then we cannot have

$$(x, y) = (x + x_1, 0)$$

Therefore, there does not exist an element $(x_1, y_1) \in V$ such that

$$(x, y) + (x_1, y_1) = (x, y)\ \forall (x, y) \in V$$

Hence, the additive identity in V does not exist and hence V is not a vector space over the field of real numbers.

Example 7. *Let V be the set of all pairs (x, y) of real numbers and let F be the field of real numbers. Define*

$$(x, y) + (x_1, y_1) = (3y + 3y_1, -x - x_1)$$
$$c(x, y) = (3cy, -cx)$$

Verify that V, with these operations, is not a vector space over the field of real numbers.

Solution. Clearly, V is closed under addition, but it is not associative under addition.

As,
$$(x, y) + [(x_1, y_1) + (x_2, y_2)]$$
$$= (x, y) + [(3y_1 + 3y_2, -x_1 - x_2)]$$
$$= [3y + 9y_1 + 9y_2, -x - (-x_1 - x_2)]$$
$$= (3y + 9y_1 + 9y_2, -x + x_1 + x_2)$$

and
$$[(x, y) + (x_1, y_1)] + (x_2, y_2)$$
$$= [(3y + 3y_1, -x - x_1)] + (x_2, y_2)$$
$$= (9y + 9y_1 + 3y_2, -x - x_1 - x_2)$$

so that $(x, y) + [(x_1, y_1) + (x_2, y_2)] \neq [(x, y) + (x_1, y_1)] + (x_2, y_2)$

Therefore, V is not associative under addition.

Hence, V is not a vector space.

Example 8. *Show that the set of all real valued continuous functions defined on [0, 1] is a vector space over field of reals.*

Solution. Let V be the set of all real valued continuous functions defined on [0, 1]. Now we have to show that V is a vector space over R (field of real numbers) under vector addition and scalar multiplication which is defined as follows:

$$(f + g)(x) = f(x) + g(x), \forall f, g \in V$$

and
$$(af)(x) = af(x), \forall f \in V \text{ and } a \in R$$

First we show that $(V, +)$ is an abelian group.

Let $f, g \in V$, then

$$(f + g)(x) = f(x) + g(x), \forall f, g \in V$$

$\therefore f + g \in V$. Thus V is closed under vector addition.

Now let $0(x) \in V$, Then we have

$$f(x) + 0(x) = (f + 0)x = f(x), \forall f \in V$$

$\therefore$ $0(x)$ is the additive identity in V

Let $-f \in V$, then we have

$$-f(x) + f(x) = (-f + f)x = 0(x), \forall f \in V$$

$\therefore$ $-f$ is an additive inverse of f in V.

Since vector addition is always associative as well as commutative, consequently $(V, +)$ is an abelian group.

Further, since V is closed under scalar multiplication therefore, af is a real valued continuous function defined on [0, 1].

(i) If $a \in R$ and $f, g \in V$, then we have

$$a[(f+g)\,x] = a[f(x) + g(x)]$$

$$= af(x) + ag(x) = (af + ag)(x)$$

$\therefore \quad a(f+g) = af + ag$

(ii) If $a, b \in R$ and $f \in V$, then we have

$$[(a+b)f](x) = (a+b)f(x) = af(x) + bf(x) = (af+bf)(x)$$

$\therefore \quad (a+b)f = af + bf$

(iii) If $a, b \in R$ and $f \in V$, then we have

$$[(ab)f](x) = (ab)f(x) = a(bf(x)) = [a(bf)](x)$$

$\therefore \quad (ab)f = a(bf)$

(iv) If $1 \in R$ and $f \in V$, then we have

$$(1f)(x) = 1f(x) = f(x)$$

$\therefore \quad 1f = f, \forall f \in V$

Hence V is a vector space over R.

Example 10. *Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Examine in each of the following cases whether V is a vector space over the field of real numbers or not?*

(i) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$; $c(x, y) = (|c|x, |c|y)$

(ii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$; $c(x, y) = (0, cy)$

(iii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$; $c(x, y) = (c^2x, c^2y)$

Solution. (i) In this case, we shall show that

$(a+b)\alpha \neq a\alpha + b\alpha \, \forall a, b \in F$ and $\alpha \in V$

Let $a, b \in F$ and $\alpha = (x, y) \in V$

$$(a+b)\alpha = (a+b)(x, y)$$

$$= (|a+b|x, |a+b|y) \text{ (By addition of scalar multiplication)}$$

Also, $\quad a\alpha + b\alpha = a(x, y) + b(x, y)$

$$= (|a|x, |a|y) + (|b|x, |b|y)$$

$$= |(|a| + |b|)x, (|a| + |b|)y|$$

$\therefore \quad (a+b)\alpha \neq a\alpha + b\alpha \, \forall a, b \in F$ and $\alpha \in V$.

Hence, V is not a vector space.

(ii) In this case we shall show that $1\alpha \neq \alpha \forall \alpha \in V$.

Let $\quad \alpha = (x, y) \in V$ and $1 \in F$

$$1\alpha = 1(x, y) = (0, 1y) = (0, y)$$

If $x \neq 0$, then $\alpha \neq (0, y)$

$\therefore \quad 1\alpha \neq \alpha \, \forall \alpha \in V$

Hence V is not a vector space.

(iii) In this case we shall show that

$$(a+b)\alpha \neq a\alpha + b\alpha \, \forall a, b \in F \text{ and } \forall \alpha \in V$$

Let $\quad \alpha = (x, y) \in V$ and $a, b \in F$

$$(a+b)\alpha = (a+b)(x, y)$$

$$= ((a+b)^2 x, (a+b)^2 y)$$

(By definition of scalar multiplication given by (iii))

Also, $\quad a\alpha + b\alpha = a(x, y) + b(x, y)$

$$= (a^2x, a^2y) + (b^2x, b^2y)$$

$$= (a^2x + b^2x, a^2y + b^2y)$$

$$= ((a^2 + b^2)x, (a^2 + b^2)y)$$

Since $\quad (a+b)^2 = a^2 + b^2 + 2ab \neq a^2 + b^2$

$\therefore \quad (a+b)\alpha \neq a\alpha + b\alpha \, \forall a, b \in F$ and $\forall \alpha \in V$.

Hence V is not a vector space.

Example 11. *How many elements are there in the vector space of polynomials of degree at most n in which the coefficients are the elements of the field $Z(p)$, the integer modulo p over the field $Z(p)$, p being a prime number?*

Solution. Since $Z(p)$ is a field under addition and multiplication modulo p

i.e., $\quad I(p) = ((0, 1, 2, 3, ..., p-1) +_p, \times_p)$

Clearly, the number of distinct element in $I(p)$ is p.

Let $f(x)$ be a polynomial of degree at most n over the field $I(p)$.

Then $\quad f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

where $a_0, a_1, a_2, ..., a_n \in Z(p)$

In $f(x)$, there are $n+1$ terms and each term has a coefficient from $Z(p)$. But $I(p)$ has p distinct elements, therefore we must have $p \times p \times p \times ... \times p$ upto $(n+1)$ times, *i.e.*, p^{n+1} distinct polynomials of degree atmost n over the field $I(p)$.

Hence, if P_n is the vector space of polynomials of degree at most n over the field $Z(p)$, then P_n has p^{n+1} distinct elements.

Example 12. *Let $K = Z_3$, the integers modulo 3. How many elements are in the vector space $V = K^4$?*

Solution. There are three choices 0, 1 or 2 for each of the four components of a vector in V. Hence, V has $3 \times 3 \times 3 \times 3 = 3^4 = 81$ elements.

Example 13. *Is Z_7 a vector space over Z_5?*

Solution. We know that Z_n = set of integers modulo n.

Then, $Z_5 = \{0, 1, 2, 3, 4\}$ and $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

Now Z_5 is not a subfield of Z_7 as $2+3 = 0$ in Z_6 but $2+3 \neq 0$ in Z_7. Hence Z_7 is not a vector space over Z_5.

Example 14. *Show that the set $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in R\}$ of all polynomials of degree 2 over R is a vector space over R w.r.t. the composition*

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad ...(1)$$

and $$a(a_0 + a_1x + a_2x^2) = aa_0 + aa_1x + aa_2x^2 \quad ...(2)$$

Solution We can easily verify that $(V, +)$ is an abelian group with additive identity $0 = 0 + 0.x + 0.x^2$ and the additive inverse of $a_0 + a_1x + a_2x^2 \in V$ is $(-a_0) + (-a_1)x + (-a_2)x^2 \in V$.

Let $\alpha, \beta \in R, f, g \in V$ by (2) $\alpha f \in V$ and

(i) $\alpha(f + g) = \alpha f + \alpha g$ (ii) $(\alpha + \beta)f = \alpha f + \beta f$

(iii) $\alpha(\beta f) = (\alpha\beta)f$ (iv) $1.f = f$

To verify the first property let $f = a_0 + a_1x + a_2x^2, g = b_0 + b_1x + b_2x^2 \in V$

Then $\alpha(f + g) = \alpha[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2]$

$= \alpha(a_0 + b_0) + \alpha(a_1 + b_1)x + \alpha(a_2 + b_2)x^2$ [By (1)]

$= (\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2) + (\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2)$ [By (2)]

$= \alpha f + \alpha g$

In a similar way, we can verify the remaining properties. Hence V is a vector space over R.

Hence W is not a subspace of V.

EXERCISE 5.1

1. Show that the complex field **C** is a vector space over the field R of reals.
2. Prove that the set of all vectors in a plane is vector space over the field of real numbers.
3. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$
$$c(x, y) = (cx, y)$$

Show that with these operations V is not a vector space over the field of real numbers.

4. Let V be the set of all ordered pairs (x, y) of reals and let F be the field of real numbers, then show that V is not a vector space over F with respect to addition and multiplication defined as follows:

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$
$$c(x, y) = (cx + cy)$$
$$\forall (x, y), (x_1, y_1) \in V \text{ and } c \in F$$

5. Let V be the set of ordered pairs (z_1, z_2) of complex numbers. Show that V is a vector space over the real field R with addition in V and scalar multiplication on V defined by

$$(z_1, z_2) + (w_1, w_2) = (z_1 + w_1, z_2 + w_2)$$

and $a(z_1, z_2) = (az_1, az_2)$

$\forall z_1, z_2, w_1, w_2 \in C$ and $a \in R$

6. Let S be a set and V be the set of all subsets of S. Define vector addition and scalar multiplication as follows:

$$A + B = A \cup B \; \forall A, B \in V$$
$$cA = A \; \forall c \in R$$

Is V a vector space over R with these operations.

7. Find which of the following for a given matrix addition and scalar multiplication over the given field form a vector space:

(i) V = set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ over R

(ii) V = set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ over R

(iii) V = set of all 2×2 matrices of the form $\begin{bmatrix} a & a+b \\ a+b & a \end{bmatrix}$ over R

8. Let S be any non-empty set and F be any field. Let V be the set of all functions from S to F i.e., let

$$V = \{f : f : S \to F\}$$

Let us define sum of two elements f and g in V as follows:

$$(f + g)(x) = f(x) + g(x) \forall x \in S$$

Also, let us define scalar multiplication of an element f in V by an element c in F as follows:

$$(cf)(x) = cf(x) \forall x \in S$$

Then $V(F)$ is a vector space.

9. Let U and W be vector spaces over a field F. Let V be the set of ordered pairs, *i.e.*,

$$V = \{(u, w) : u \in U, w \in W\}$$

Show that U is a vector space over F with addition in V and scalar multiplication on V defined by

$$(u, w) + (u_1, w_1) = (u + u_1, w + w_1)$$

and $k(u, w) = (ku, kw)$

for all $u, u_1 \in U$, $w, w_1 \in W$ and $k \in F$.

10. Let V be the set of ordered pairs (x, y) of real numbers. Show that V is not a vector space over R with addition in V and scalar multiplication on V defined by:

(i) $(x, y) + (x_1, y_1) = (x, y)$ and $c(x, y) = (cx, cy)$

(ii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ and $c(x, y) = (c^2x, c^2y)$

(iii) $(x, y) + (x_1, y_1) = (0, 0)$ and $c(x, y) = (cx, cy)$

(iv) $(x, y) + (x_1, y_1) = (xx_1, yy_1)$ and $c(x, y) = (cx, cy)$

(v) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ and $c(x, y) = c(x, 0)$

11. Let $V = \{<a_n> : a_n \in R\}$, *i.e.*, V is the set of all real sequences. Prove that V is a vector space over R.

12. Let F be a field and

$$F^n = \{(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n) : \alpha_i \in F, 1 \le i \le n\}$$

Show that F^n is a vector space over F under the composition

$$\alpha(a_0 + a_1x + a_2x^2 + ... + a_mx^m)$$
$$= \alpha a_0 + \alpha a_1 x + ... + \alpha a_m x^m \quad (m < n)$$

Answers

6. V is not vector space. 7. (i) No. (ii) Yes (iii) Yes

5.4 ELEMENTARY PROPERTIES OF VECTOR SPACES

THEOREM 1. *Let $V(F)$ be a vector space over a field F and 0 be the zero (null) vector of V. Then prove that*

(i) $a0 = 0, \forall a \in F$

(ii) $0\alpha = 0, \forall \alpha \in V$

(iii) $a(-\alpha) = -(a\alpha), \forall a \in F, \alpha \in V$

(iv) $(-a)\alpha = -(a\alpha) \; \forall a \in F, \alpha \in V$

(v) $a(\alpha - \beta) = a\alpha - a\beta, \forall a \in F, \alpha, \beta \in V$

(vi) $a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

(vii) $a\alpha = a\beta \Rightarrow \alpha = \beta, \forall a \in F, \alpha, \beta \in V, a \ne 0$

(viii) $a\alpha = b\alpha \Rightarrow a = b \; \forall a, b \in F, \alpha \in V$ and $\alpha \ne 0$.

Proof. (i) $a0 = 0$

we have $a0 = a(0 + 0)$ [$\because 0 + 0 = 0$]

$= a0 + a0$ [By property of vector space]

or $0 + a0 = a0 + a0$ [$\because 0 + a0 = a0$]

or $0 = a0$

(ii) $0\alpha = 0$

We have $0\alpha = (0 + 0)\alpha$ [$\because 0 + 0 = 0, 0 \in F$]

$= 0\alpha + 0\alpha$ [By the definition of $V(F)$]

or $\quad 0 + 0\alpha = 0\alpha + 0\alpha$ $\quad [\because 0\alpha \in V \therefore 0 + 0\alpha = 0\alpha]$

or $\quad 0 = 0\alpha$

(iii) $\quad a(-\alpha) = -(a\alpha)$

We have,

$a0 = 0$ $\quad$ [From (i)]

or $\quad a(-\alpha + \alpha) = 0$ $\quad [\because -\alpha + \alpha = 0]$

or $\quad a(-\alpha) + a\alpha = 0$ $\quad$ [By the definition of $V(F)$]

(iv) $\quad (-a)\alpha = -(a\alpha)$

Since, we have

$0\alpha = 0$ $\quad$ [From (ii)]

or $\quad (-a + a)\alpha = 0$ $\quad [\because -a + a = 0, a \in F]$

or $\quad (-a)\alpha + a\alpha = 0$ $\quad$ [By the definition of $V(F)$]

or $\quad (-a)\alpha = -(a\alpha)$

(v) $\quad a(\alpha - \beta) = a\alpha - a\beta$

We have,

$$a(\alpha - \beta) = a[\alpha + (-\beta)]$$

$$= a\alpha + a(-\beta) \quad \text{[By definition of } V(F)]$$

$\therefore \quad a(\alpha - \beta) = a\alpha - a\beta$ $\quad [\because a(-\beta) = -(a\beta)]$

(vi) $\quad a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

Suppose $0 \neq a \in F$, then a^{-1} exists in F.

Now $\quad a\alpha = 0$ $\quad$ [given]

$\Rightarrow \quad a^{-1}a\alpha = a^{-1}0$

$\Rightarrow \quad (a^{-1}a)\alpha = 0$ $\quad [\because a^{-1}0 = 0]$

$\Rightarrow \quad 1\alpha = 0$ $\quad [aa^{-1} = 1]$

$\Rightarrow \quad \alpha = 0$ $\quad [\because 1\alpha = \alpha]$

Suppose $\alpha \neq 0$, then to prove $a = 0$, let us assume that $a \neq 0$, then a^{-1} exists. Since we have

$a\alpha = 0$

$\Rightarrow \quad a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow \quad (a^{-1}a)\alpha = 0$

$\Rightarrow \quad 1\alpha = 0$ $\quad [\because a^{-1}a = 1]$

$\Rightarrow \quad \alpha = 0$

$\Rightarrow$ This gives a contradiction because we have taken $\alpha \neq 0$. $\quad [\because 1\alpha = \alpha]$

Hence, $\quad a = 0$

(vii) We have,

$a\alpha = a\beta \Rightarrow a\alpha - a\beta = 0$

$\Rightarrow \quad a(\alpha - \beta) = 0$

$\Rightarrow \quad \alpha - \beta = 0$ $\quad [\because a \neq 0$ and from (vi)]

$\Rightarrow \quad \alpha = \beta$

Hence, $\quad a\alpha = a\beta \Rightarrow$ for all $a \neq 0 \in F$ and $\alpha, \beta \in V$

(viii) We have,

$a\alpha = b\alpha \Rightarrow a\alpha - b\alpha = 0$

$\Rightarrow \quad (a-b)\alpha = 0$

$\Rightarrow \quad a - b = 0$ $\quad [\because \alpha \neq 0$ and from (vi)]

$\Rightarrow \quad a = b$

Hence, $\quad a\alpha = b\alpha \Rightarrow a = b.$

5.5 VECTOR SUBSPACES: VECTOR SPACE WITHIN VECTOR SPACE

Just like a subgroup and a subring, we do have the concept of a vector subspace which is generally addresses as a subspace.

Definition. *Let W be a non-empty subset of V, where V is a vector space over a field F. Then W is said to be a vector subspace of V(F) if W is itself a vector space aver F with respect to the same operations as defined on V.*

For Example: The set $W = \{ (a, 0, b) : a, b \in R\}$ is a subspace of R^3(R).

5.6 ELEMENTARY PROPERTIES OF VECTOR SUBSPACES

THEOREM 1. *The necessary and sufficient conditions for a non-empty subset W of V(F) to be a subspace are that:*

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Proof. Suppose W is a subspace of a vector space $V(F)$. Then if

$$\beta \in W \Rightarrow -\beta \in W$$

$\therefore \quad \alpha \in W, -\beta \in W \Rightarrow \alpha + (-\beta) \in W$

[$\because$ W is closed under vector addition.]

$\Rightarrow \quad \alpha - \beta \in W$

and $\quad a \in F, \alpha \in W \Rightarrow a\alpha \in W$ $\quad$ [$\because$ W is closed under scalar multiplication.]

Conversely, Suppose W is a subset of V and

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Now we have to show that W is a subspace. For this purpose we proceed as follows:

$\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W.$ $\quad$ [From (i)]

$\Rightarrow 0 \in W \Rightarrow$ identity exists.

and $\quad 0 \in W, \alpha \in W \Rightarrow 0 - \alpha \in W.$ $\quad$ [From (i)]

$\Rightarrow -\alpha \in W \Rightarrow$ inverse exists.

Now, $\quad \alpha \in W, -\beta \in W \Rightarrow \alpha - (-\beta) \in W.$

$\Rightarrow \quad \alpha + \beta \in W$ $\quad$ [From (i)]

$\therefore \quad W$ is closed under vector addition.

Also vector addition is always associative and commutative. Thereore $(W,+)$ is an abelian group.

From (ii) it is obvious that W is closed under multiplication. Space V is a vector space over F therefore remaining properties will also hold in W. Hence W is a vector space and Hence W is itself a subspace.

THEOREM 2: *The necessary and sufficient condition for a non-empty subset of W of a vector space $V(F)$ to be a subspace of V is*

$$a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$$

Proof: Suppose W is a subspace of a vector space $V(F)$. Then W is closed under vector addition and multiplication, therefore we have

$$a \in F, \alpha \in W \Rightarrow a\alpha \in W$$

and $$b \in F, \beta \in W \Rightarrow b\beta \in W$$

$$\therefore \quad a\alpha \in W, b\beta \in W \Rightarrow a\alpha + b\beta \in W$$

Conversely, Suppose W is a subset of $V(F)$ and

$$a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W \text{ is given.}$$

Then we have to show that W is a subset of $V(F)$.

Now taking $a=1$, $b=1$, then

$$1 \in F; \alpha, \beta \in W \Rightarrow 1\alpha + 1\beta \in W$$

$$\Rightarrow \quad \alpha + \beta \in W \qquad [1\alpha = \alpha, 1\beta = \beta]$$

$\therefore$ W is closed under vector addition.

And taking $a=0$, $b=-1$, we have

$$0\alpha + (-1)\beta \in W$$

$$\Rightarrow \quad 0 + (-\beta) \in W \qquad [\because (-1)\beta = -\beta]$$

$$\Rightarrow \quad -\beta \in W$$

$\therefore$ Additive inverse exists in W.

Again, taking $a=0, b=0$, we have

$$0\alpha + 0\beta \in W$$

$$\Rightarrow \quad 0 \in W \qquad [\because 0\alpha = 0, \text{ similarly}, 0\beta = 0]$$

Since $W \subseteq V$, therefore vector addition is associative and commutative.

Thus, W is an abelian group under vector addition.

Further, taking $\beta = 0$, we have

$$a\alpha + b0 \in W$$

$$\Rightarrow \quad a\alpha \in W \qquad [\because b0 = 0 \text{ and } a\alpha + 0 = a\alpha]$$

$\therefore$ W is closed under scalar multiplication.

The rest properties will hold in W because $W \subseteq V$ and these properties hold in V.

Hence W is a vector space and consequently W is a subspace of $V(F)$.

5.7 ALGEBRA OF SUBSPACES

THEOREM 1. *The intersection of any two subspaces of a vector space is a subspace.*

Proof: Let $V(F)$ be a vector space over F and W_1, W_2 be two subspaces of $V(F)$. Then, we have to show that $W_1 \cap W_2$ is a subspace of $V(F)$

Let $$\alpha, \beta \in W_1 \cap W_2 \Rightarrow \alpha, \beta \in W_1 \text{ and } \alpha, \beta \in W_2$$

Since, W_1 and W_2 are subspaces of V, so we have

$$a, b \in W \text{ and } \alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1 \qquad ...(1)$$

and, $$a, b \in F \text{ and } \alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2 \qquad ...(2)$$

From (1) and (2), we get

if $$a, b \in F \text{ and } \alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$$

Hence, $W_1 \cap W_2$ is a subspace of V.

THEOREM 2. *The intersection of an arbitrary collection of a subspaces of a vector space is also a subspace.*

Proof: Let $\{W_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of subspaces of a vector space V(say). Then we have to show that $\cap \{W_\lambda : \lambda \in \Lambda\}$ is a subspace of V.

Let $$\alpha, \beta \in \cap \{W_\lambda : \lambda \in \Lambda\}$$

$$\Rightarrow \quad \alpha, \beta \in W_\lambda \text{ for each } \lambda \in \Lambda.$$

Since, each W_λ is a subspace of V, then for any two scalars $a, b \in F$, we have

$$a\alpha + b\beta \in W_\lambda \text{ for each } \lambda \in \Lambda$$

$$\Rightarrow \quad a\alpha + b\beta \in \cap \{W_\lambda : \lambda \in \Lambda\}$$

Hence, $\cap \{W_\lambda : \lambda \in \Lambda\}$ is a subspace of V.

THEOREM 3. *The union of two subspaces of a vector space is not necessarily a subspace.*

Proof: Let W_1, W_2 be two subspaces of a vector space V and suppose that

$$W_1 = \{(a_1, a_2, 0) : a_1, a_2 \in F\}$$

and $$W_2 = \{(a_1, 0, a_3) : a_1, a_3 \in F\}$$

Obviously, W_1 and W_2 are subspaces of $R^3(R)$. By definition of W_1 and W_2, we have $W_1 \cup W_2$ containing all triads, *i.e.*, 3-tuples of the form $(a_1, a_2, 0)$ and those of the form $(a_1, 0, a_3)$.

Now, if we consider the elements $\alpha = (1,2,0)$ and $\beta = (3,0,5)$ of $W_1 \cup W_2$, then for scalars $a=1$ and $b=2$.

$$a\alpha + b\beta = 1(1,2,0) + 2(3,0,5) = (1,2,0) + (6,0,10)$$

$$= (7,2,10) \notin W_1 \cup W_2$$

Thus, if $\alpha \in W_1 \cup W_2$ and $\beta \in W_1 \cup W_2$, then it is not necessarily implied that $a\alpha + b\beta \in W_1 \cup W_2$ for some $a, b \in F$.

Hence, $W_1 \cup W_2$ is a subspace of $R^3(R)$.

THEOREM 4. *The union of two subspaces of a vector space is a subspace iff one is contained in the other.*

Proof: Let $V(F)$ be a vector space and W_1, W_2 be two subspaces of V. Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Then we have to show that $W_1 \cup W_2$ is a subspace of V.

Now $$W_1 \cup W_2 = W_2 \text{ if } W_1 \subseteq W_2$$ and W_2 is a subspace, therefore $W_1 \cup W_2$ is subspace.

Also, $W_1 \cup W_2 = W_1$ if $W_1 \subseteq W_2$ and since W_1 is a subspace, therefore $W_1 \subseteq W_2$ is a subspace V.

Conversely, Suppose $W_1 \cup W_2$ is a subspace of V. Then we have to show that

$$W_1 \subseteq W_2 \text{ or } W_2 \subseteq W_1.$$

Let us assume that W_1 is not a subset of W_2 and W_2 is not a subset of W_1.

Now, W_1 is not a subset of W_2, this implies that there exists an element α in W_1

which is not in W_2.

Also, W_2 is not a subset of W_1, therefore there exists an element β in W_2 which is not in W_1. But we have $\alpha \in W_1 \cup W_2$ and $\beta \in W_1 \cup W_2$ and since $W_1 \cup W_2$ is a subspace of V, we have.

$$a,b \in F, \alpha,\beta \in W_1 \cup W_2 \Rightarrow a\alpha + b\beta \in W_1 \cup W_2$$

Now taking $a=1, b=1$, we have $\quad [\because 1\alpha \in W_1 \cup W_2 \subseteq V \therefore 1\alpha = \alpha]$

$$1\alpha + 1\beta \in W_1 \cup W_2$$

$$\Rightarrow \quad \alpha + \beta \in W_1 \cup W_2$$

$$\Rightarrow \quad \alpha + \beta \in W_1 \text{ or } \alpha + \beta \in W_2$$

Suppose $\alpha + \beta \in W_1$ and $\alpha \in W_1$, then $(\alpha + \beta) - \alpha \in W_1$, because W_1 is a subspace of V. Therefore $\beta \in W_1$ and this gives a contradiction.

Now, suppose $\alpha + \beta \in W_2$ and $\beta \in W_2$, then

$$(\alpha + \beta) - \beta \in W_2 \qquad [\because W_2 \text{ is a subspace.}]$$

$$\Rightarrow \quad \alpha \in W_2$$

Hence, either $\quad W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Solved Examples

Based on the following Results

- The necessary and sufficient conditions for a non-empty subset W of $V(F)$ to be a subspace are that
  (i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$
  (ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$
  or equivalently $\quad a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$.
- The intersection of two subspaces of a vector space is a subspace.
- The union of two subspaces of a vector space is a subspace if and only if one is contained in the other.

Example 1. *Show that the set $W = \{(a, b, c) : a - 3b + 4c = 0\}$ is a subspace of the 3-tuple space R^3 (R).*

Solution. Let $\alpha = (a_1, b_1, c_1)$ and $\beta = (a_2, b_2, c_2)$ be any two elements of W, such that

$$a_1 - 3b_1 + 4c_1 = 0 \text{ and } a_2 - 3b_2 + 4c_2 = 0$$

For $a, b \in R$, we have

$$a\alpha + b\beta = a(a_1, b_1, c_1) + b(a_2, b_2, c_2)$$
$$= (aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)$$
$$= (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$$

Now $(aa_1 + ba_2) - 3(ab_1 + bb_2) + 4(ac_1 + bc_2)$
$$= (aa_1 - 3ab_1 + 4ac_1) + (ba_2 - 3bb_2 + 4bc_2)$$
$$= a(a_1 - 3b_1 + 4c_1) + b(a_2 - 3b_2 + 4c_2)$$
$$= a.0 + b.0 = 0$$

So, $\quad a\alpha + b\beta \in W$

Thus, $\alpha \in W, \beta \in W \Rightarrow a\alpha + b\beta \in W \; \forall a, b \in R$.

Hence, W is a subspace of R^3 (R).

Example 2. *Show that the set $W = \{(a_1, a_2, 0)\} : a_1, a_2 \in F\}$ is a subspace of $V_3(F)$.*

Solution. We have $W = \{(a_1, a_2, 0) : a_1, a_2 \in F\}$

Clearly, $(0, 0, 0) \in W \Rightarrow W$ is non-empty.

Let $\alpha, \beta \in W$. Then $\alpha = (a_1, a_2, 0)$ and $\beta = (b_1, b_2, 0)$ for $a_1, a_2, b_1, b_2 \in F$,

Now, for any $a, b \in F$,

$$a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$$
$$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, 0)$$

Since $aa_1 + bb_1, aa_2 + bb_2 \in F$, therefore $a\alpha + b\beta \in W$

Hence W is a subspace of V_3 (F).

Example 3. *Let W be the collection of all elements from the space $M_2(F)$ of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Show that W is a subspace of M_2 (F).*

Solution. We have $W = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in F\right\}$

Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W \Rightarrow W$ is non-empty.

Let $\alpha, \beta \in W$. Then $\alpha = \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix}$ and $\beta = \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$ for all $a_1, b_1, a_2, b_2 \in F$.

If a, b are any element of F, then

$$a\alpha + b\beta = a\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} + b\begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$$
$$= \begin{bmatrix} aa_1 & ab_1 \\ -ab_1 & aa_1 \end{bmatrix} + \begin{bmatrix} ba_2 & bb_2 \\ -bb_2 & ba_2 \end{bmatrix}$$
$$= \begin{bmatrix} aa_1 + ba_2 & ab_1 + bb_2 \\ -(ab_1 + bb_2) & aa_1 + ba_2 \end{bmatrix}$$

Since $aa_1 + ba_2, ab_1 + bb_2 \in F$, therefore $a\alpha + b\beta \in W$.

Hence W is a subspace of M_2 (F).

Example 4. *Which of the following sets of vectors $\alpha = (a_1, a_2, \ldots a_n) \in R^n$ are subspaces of $R^n (n \geq 3)$?*

(i) *all α such that $a_1 \leq 0$*
(ii) *all α such that a_3 is an integer*
(iii) *all α such that $a_2 + 4a_3 = 0$*
(iv) *all α such that $a_1 + a_2 + \ldots + a_n = k$ (a given constant)*

Solution. (i) Let $W = \{(a_1, a_2, ..., a_n) : a_1 \le 0\}$.

Clearly, $(-1, a_2, a_3, ..., a_n) \in W \Rightarrow W$ is non-empty.

Let $\alpha, \beta \in W$. Then we have

$$\alpha = (a_1, a_2, ..., a_n) \text{ and } \beta = (b_1, b_2, ..., b_n)$$

with $a_1 \le 0$ and $b_1 \le 0$.

If a, b be any element of R, then

$$a\alpha + b\beta = a(a_1, a_2, ..., a_n) + b(b_1, b_2, ..., b_n)$$
$$= (aa_1, aa_2, ..., aa_n) + (bb_1, bb_2, ..., bb_n)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, ..., aa_n + bb_n)$$

Since $a_1 \le 0$ and $b_1 \le 0$

If $a<0$ and $b<0$, then $aa_1>0$ and $bb_1>0$ so that $aa_1+bb_1>0$. Thus $a\alpha + b\beta \notin W$. Hence W is not a subspace.

(ii) Let $W = \{(a_1, a_2, ..., a_n) : a_3 \text{ is an integer}\}$

Clearly, $(a_1, a_2, 1, a_4, ..., a_n) \in W \Rightarrow W$ is non-empty.

Let $\alpha = (a_1, a_2, 2, a_4, ..., a_n)$ be in W, $a = \frac{1}{3}$ an element of R, then

$$a\alpha = \frac{1}{3}(a_1, a_2, 2, a_4, ..., a_n)$$
$$= \left(\frac{a_1}{3}, \frac{a_2}{3}, \frac{2}{3}, \frac{a_4}{3} ... \frac{a_n}{3}\right)$$

$\Rightarrow$ $a\alpha \notin W$

Hence W is not a subspace of R^n.

(iii) Let $W = \{(a_1, a_2, a_3, ..., a_n) : a_2 + 4a_3 = 0\}$

Clearly, $(a_1, -4, 1, ..., a_n) \in W \Rightarrow W$ is non-empty.

Let $\alpha = (a_1, a_2, a_3, ... a_n)$ and $\beta = (b_1, b_2, b_3, ... b_n)$ be any two elements in W such that $a_2 + 4a_3 = 0$ and $b_2 + 4b_3 = 0$

Let a, b be any elements of R, then

$$a\alpha + b\beta = a(a_1, a_2, a_3, ..., a_n) + b(b_1, b_2, b_3, ..., b_n)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3, ..., aa_n + bb_n)$$

Now, $(aa_2 + bb_2) + 4(aa_3 + bb_3) = a(a_2 + 4a_3) + b(b_2 + 4b_3)$
$$= a.0 + b.0 = 0$$

Since $aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3, ..., aa_n + bb_n \in R$,

therefore $a\alpha + b\beta \in W$.

Hence W is a subspace of R^n.

(iv) Let $W = \{(a_1, a_2, ..., a_n) : a_1 + a_2 + ... + a_n = k(\text{given})\}$

If $k=0$, then W is a subspace of R^n, but if $k \ne 0$, then W is not a subspace.

Example 5. *If a_1, a_2, a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of elements of field F, such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a subspace of $V_3(F)$.*

Solution. We have

$$W = \{(x_1, x_2, x_3) : a_1x_1 + a_2x_2 + a_3x_3 = 0, a_1, a_2, a_3 \text{ are fixed}\}.$$

Clearly, $(0, 0, 0) \in W \Rightarrow W$ is non-empty.

Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$ be any two elements of W, then $x_1, x_2, x_3, y_1, y_2, y_3$ are elements of F such that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$
$$a_1y_1 + a_2y_2 + a_3y_3 = 0$$

If a, b be any two elements of F then

$$a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$$
$$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$$

Now $a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$
$$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$$
$$= a.0 + b.0 = 0.$$

Since, $ax_1 + by_1, ax_2 + by_2, ax_3 + by_3 \in F$, therefore, $a\alpha + b\beta \in W$.

Hence W is a subspace of $V_3(F)$.

Example 6. *Let R be the field of real numbers. Which of the following are subspaces of V_3 (R)?*

(i) $W_1 = \{(x, x, x) : x \in R\}$

(ii) $W_2 = \{(x, y, z) : x, y, z \text{ are rational numbers}\}$

Solution. (i) Since $(0, 0, 0) \in W_1 \Rightarrow W_1$ is non-empty.

Let $\alpha = (x, x, x)$ and $\beta = (y, y, y)$ be any two elements of W_1.

If a, b are any two elements of R, then

$$a\alpha + b\beta = a(x, x, x) + b(y, y, y)$$
$$= (ax + by, ax + by, ax + by)$$

Since $ax + by \in R$, therefore $a\alpha + b\beta \in W_1$

Hence W_1 is a subspace of V_3 (R).

(ii) Since $(0, 0, 0) \in W_2 \Rightarrow W_2$ is non-empty.

Let $\alpha = (x, y, z)$ be an element of W_2, then x, y, z are rational numbers.

If $a = \sqrt{5}$ be an element of R, then

$$a\alpha = \sqrt{5}(x, y, z) = (\sqrt{5}x, \sqrt{5}y, \sqrt{5}z)$$

But $\sqrt{5}x, \sqrt{5}y, \sqrt{5}z$ are not rationals, therefore $a\alpha \notin W_2$.

Hence W_2 is not a subspace of V_3 (R).

Example 7. *Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where W contains all 2×2 matrices with zero determinant.*

Solution. Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W \Rightarrow W$ is non-empty.

Let $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in R$ and $a \neq 0, b \neq 0$. Then

$$|A| = 0 = |B| \Rightarrow A, B \in W$$

Now
$$A + B = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and
$$|A + B| \neq 0 \Rightarrow A + B \in W$$

Hence, W is not a subspace of V.

Example 8. *Let $M_n(F)$ be the vector space of all $n \times n$ matrices over the field F. Let W be the subset of $M_n(F)$ consisting of all symmetric matrices. Show that W is a subspace of $M_n(F)$.*

Solution. Let
$$W = \{[a_{ij}]_{n\times n} : a_{ij} = a_{ji}\}.$$

Clearly, $O \in W$, where O is the null matrix of order $n \times n$. Thus W is non-empty.

Let $A = [a_{ij}]_{n\times n}$ and $A = [b_{ij}]_{n\times n}$ be any two elements of W, then

$$a_{ij} = a_{ji} \text{ and } b_{ij} = b_{ji}$$

If a, b be any two elements of F, then

$$aA + bB = a[a_{ij}]_{n\times n} + b[b_{ij}]_{n\times n}$$
$$= [aa_{ij} + bb_{ij}]_{n\times n}$$
$$= [aa_{ji} + bb_{ji}]_{n\times n} \qquad [\because a_{ij} = a_{ji} \text{ and } b_{ij} = b_{ji}]$$

Therefore, $aA+bB$ is a symmetric matrix so that $aA+bB \in W$.

Hence, W is a subspace of $M_n(F)$.

Example 9. *Let V(F) be the vector space of all $n \times 1$ matrices over the field F. Let A be an $m \times n$ matrix over F. Then the set W of all $n \times 1$ matrices X over F such that $AX = O$ is a subspace of V, here O is a null matrix of the type $m \times 1$.*

Solution. Clearly, $O_{n\times 1} \in W \Rightarrow W$ is non-empty.

Let X, Y be any elements of W. Then X and Y are of order $n \times 1$ matrices over F such that

$$AX = O, AY = O$$

If $a \in F$, then $aX+Y$ is also an $n \times 1$ matrix over F.

Now
$$A(aX+Y) = A(aX) + AY = a(AX) + AY$$
$$= a.O + O = O + O = O$$
$$\Rightarrow A(aX+Y) \in W$$

Hence, W is a subspace of V.

Example 10. *Let V be the vector space of all polynomials in an indeterminate x over a field F, i.e., $V = F[x]$. Let W be a subset of V consisting of all polynomials of degree $\leq n$. Then W is a subspace of V.*

Solution. Let α, β be any two elements of W.

Let
$$\alpha = a_0 + a_1 x + \ldots + a_r x^r, r \leq n$$
and
$$\beta = b_0 + b_1 x + \ldots + b_s x^s, s \leq n$$
where
$$a_0, a_1, \ldots, a_r, b_0, b_1, b_2, \ldots b_s \in F$$

If $a, b \in F$, then
$$a\alpha + b\beta = a(a_0 + a_1 x + \ldots + a_r x^r) + b(b_0 + b_1 x + \ldots + b_s x^s)$$
$$= (aa_0 + aa_1 x + \ldots + aa_r x^r) + (bb_0 + bb_1 + \ldots + bb_s x^s).$$

If $s > r$ and setting $a_k = 0$ for all $r+1 \leq k \leq s$, then
$$a\alpha + b\beta = (aa_0 + bb_0) + (aa_1 + bb_1)x \ldots + (aa_s + bb_s)x^s$$

Since $aa_0 + bb_0, aa_1 + bb_1, \ldots, aa_s + bb_s \in F$, then $a\alpha + b\beta$ is a polynomial of degree $\leq n$, therefore $a\alpha + b\beta \in W$.

Hence W is a subspace of V.

Example 11. *Prove that the set of all solutions (a, b, c) of the equation $a+b+2c = 0$ is a subspace of vector space V_3(R).*

Solution. Let
$$W = \{(a, b, c) : a+b+2c = 0\}$$

Clearly, $(0, 0, 0) \in W \Rightarrow W$ is non-empty.

Let $\alpha = (a_1, b_1, c_1)$ and $\beta = (a_2, b_2, c_2)$ be any two elements of W, then

$$a_1 + b_1 + 2c_1 = 0$$
and
$$a_2 + b_2 + 2c_2 = 0$$

If a, b be any two elements of R, then
$$a\alpha + b\beta = a(a_1, b_1, c_1) + b(a_2, b_2, c_2)$$
$$= (aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)$$
$$= (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$$

Now,
$$aa_1 + ba_2 + ab_1 + bb_2 + 2(ac_1 + bc_2)$$
$$= a(a_1 + b_1 + 2c_1) + b(a_2 + b_2 + 2c_2)$$
$$= a.0 + b.0$$
$$= 0 + 0 \quad = 0$$

Since, $aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2 \in R$ then $a\alpha + b\beta \in W$

Hence W is a subspace of V_3 (R).

Example 12. *Let $V = R^3(R)$ the real vector space and let $W_1 = \{(0, y, z) : y, z \in R\}$, $W_2 = \{(x, y, 0) : x, y \in R\}$. What is $W_1 \cap W_2$? Is it subspace of V? Is $W_1 \cup W_2$ subspace of V?*

Solution. Clearly, W_1 and W_2 are subspaces of V.

From the definition of W_1 and W_2, we define $W_1 \cap W_2$ as follows:
$$W_1 \cap W_2 = \{(0, y, 0) : y \in R\}.$$

Let $\alpha = (0, y_1, 0)$ and $\beta = (0, y_2, 0)$ be any two elements of $W_1 \cap W_2$.

If a, b be any elements of R, then

$$a\alpha + b\beta = a(0, y_1, 0) + b(0, y_2, 0)$$
$$= (0, ay_1, 0) + (0, by_2, 0)$$
$$= (0, ay_1 + by_2, 0)$$

Since, $ay_1 + by_2 \in R$ therefore $a\alpha + b\beta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of V.

Next, let $\alpha = (0, y_1, z_1) \in W_1$ and $\beta = (x_1', y_1', 0) \in W_2$ where $y_1, z_1, x_1', y_1' \in R$.

Now α and β are both elements of $W_1 \cup W_2$, then for $a, b \in$R, we have

$$a\alpha + b\beta = a(0, y_1, z_1) + b(x_1', y_1', 0)$$
$$= (bx_1', ay_1 + by_1', az_1)$$

Since, $(bx_1', ay_1 + by_1', az_1)$ belongs neither to W_1 nor to W_2, therefore $(bx_1', ay_1 + by_1', az_1) \in W_1 \cup W_2$ or $a\alpha + b\beta \notin W_1 \cup W_2$.

Hence $W_1 \cup W_2$ is not a subspace of V.

Example 13 *Let V be the (real) vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?*

(i) all f such that $f(x^2) = [f(x)]^2$

(ii) all f which are continuous.

Solution (i) Let $W = \{f : f(x^2) = [f(x)]^2\}$.

Clearly, $O(x) \in W \Rightarrow W$ is non-empty.

Let f and g be any two elements of W, then

$f(x)^2 = [f(x)]^2$ and $g(x)^2 = [g(x)]^2$.

If a and b are any two elements of R, then

$$(af + bg)(x^2) = (af)(x^2) + (bg)(x^2) = af(x^2) + bg(x^2)$$
$$= a[f(x)]^2 + b[g(x)]^2$$

Now $[(af + bg)(x)]^2 = [af(x) + bg(x)]^2$

$$= a^2[f(x)]^2 + b^2[g(x)]^2 + 2abf(x)g(x)$$

$\therefore \quad (af + bg)(x^2) \neq [(af + bg)(x)]^2$

$\Rightarrow \quad af + bg \in W$

Hence W is not a subspace of V.

(ii) Let $W = \{f : f$ is continuous$\}$

Since f is continuous so that af is also continuous for any $a \in R$.

Clearly, $O(x) \in W \Rightarrow W$ is non-empty.

Let f and g be any two elements of W, then f and g are continuous.

If a, b are any two elements of W, then $af + bg$ is also continuous, therefore $af + bg \in W$.

Hence W is a subspace of V.

EXERCISE 5.2

1. Show that the set $W = \{(a, 0, 0) : a \in R\}$ is a subspace of V_3 (R).
2. Show that the subset $W = \{(a,b,c) : a+b+c=0\}$ of R^3 is a subspace of R^3.
3. Show that the set W of the elements of the vector space V_3 (R) of the form $(x+2y, y, -x+3y)$ where $x, y \in R$, is a subspace of V_3(R).
4. Let $V = R^3$. Show that the set $W = \{(a, b, c) : a, b, c \in Q\}$ of V is not a subspace of V.
5. Let F be the field of integers modulo 2, V be the set of all 2×2 matrices over F. Show that $V(F)$ is a finite vector space. Give two non-trivial subspaces of this vector space.
6. Let V be a vector space of all $n\times n$ matrices. Prove that the set W consisting of all $n\times n$ real matrices which commute with a given matrix T of V forms a subspace of V.
7. Let C be the field of complex numbers and let n be a positive integer $(n \geq 2)$. Let V be the vector space of all $n\times n$ matrices over C. Which of the following sets of matrices A in V are subspaces of V?
   (i) All invertible A;
   (ii) All non-invertible A;
   (iii) All A such that $AB=BA$, where B is some fixed matrix in V.
8. Which of the following sets of vectors $\alpha = (a_1, a_2, \ldots, a_n)$ in R^n are subspaces of R^n $(n \geq 3)$?
   (i) All α such that $a_1 \geq 0$
   (ii) All α such that $a_1 + 3a_2 = a_3$
   (iii) All α such that $a_2 = a_1^2$
   (iv) all α such that $a_1 a_2 = 0$
   (v) all α such that a_2 is rational.
9. Determine whether or not W is a subspace of R^3 if W consists of those vectors $(a, b, c) \in R^3$ for which:
   (i) $a = 2b$ (ii) $ab = 0$
   (iii) $a \leq b \leq c$ (iv) $a = b^2$
   (v) $k_1 a + k_2 b + k_3 c = 0, k_1, k_2, k_3 \in R$
10. Let V be the vector space of all polynomials over the field R. Determine whether or not W is a subspace of V where
    (i) W contains all polynomials with integral coefficients.
    (ii) W consists of all polynomials $a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$, i.e., polynomials with only even powers of x.
11. Let V be the (real vector) space of all functions f from R into R. Which of the following sets of functions are subspaces of V?
    (i) all f such that $f(0) = f(1)$
    (ii) all f such that $f(3) = 1 + f(-5)$
    (iii) all f such that $f(-1) = 0$
    (iv) all f which are bounded.
12. Let $AX = B$ be a non-homogeneous system of linear equations in n unknowns over a field F. Show that the solution set W of the system is not a subspace of F^n.
13. Let V be the vector space of all $n\times n$ square matrices over a field F. Show that W is a subspace of V if W consists of all matrices which are:
    (i) skew-symmetric
    (ii) upper triangular
    (iii) diagonal
    (iv) scalar
14. Let V be the vector space of infinite sequences $(a_1, a_2, \ldots, a_n, \ldots)$ in a field F. Show that W is a subspace of V if :
    (i) W consists of all sequences with 0 as the first component;
    (ii) W consists of all sequences with only a finite number of non-zero components.
15. Let W be the set of all vectors of the form $(x, 2x, -3x, x) x \in R$, then prove that W is subspace of V_4 (R).
16. Let W be the set of all five-tuples of real numbers of the form $(x, 2x, -3x, 5x, x)$, $x \in R$. Show that W is a subspace of V_5 (R).
17. If R is the field of real numbers and
    $$W_1 = \{(x, 0, 0) : x \in R\}$$
    $$W_2 = \{(0, y, 0) : y \in R\}$$
    are two subspaces of $V_3(R)$. What is $W_1 \cap W_2$: Is it a subspace of V? Is $W_1 \cup W_2$ a subspace of V?
18. Give an example of a subset W of a vector space $V = R^2$ which is not a subspace of V but for which :
    (i) $W + W = W$
    (ii) $W + W \subset W$

...e vector space of $n\times n$ matrices F. Let W_1 and W_2 be the subspaces ... triangular matrices and lower triangular matrices respectively, find:

(i) $W_1 + W_2$

(ii) $W_1 \cap W_2$

Answers

5. $W_1 = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in F\right\}$ and $W_2 = \left\{\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} : a, b \in F\right\}$ are two non-trivial subspaces.

7. (i) not a subspace (ii) not a subspace (iii) a subspace

8. (i) not a subspace (ii) a subspace (iii) not a subspace (iv) not a subspace (v) not a subspace

9. (i) a subspace (ii) not a subspace (iii) not a subspace (iv) not a subspace (v) a subspace

10. (i) not a subspace (ii) a subspace

11. (i) a subspace (ii) not a subspace (iii) a subspace (iv) a subspace

17. (i) $W_1 \cup W_2 = \{(0, 0, 0)\}$ (ii) $W_1 \cup W_2 = \{(x, y, 0) : x, y \in R\}$

18. (i) $W = \{(0, 0)\ (0, 1)\ (0, 2)\ (0, 3), \ldots\}$ (ii) $W = \{(0, 8), (0, 6), (0, 7), \ldots\}$

19. (i) $W_1 + W_2 = V$ (ii) $W_1 \cap W_2$ = set of all diagonal matrices.

5.8 LINEAR SUM OF TWO SUBSPACES

Let W_1 and W_2 be two subspaces of a vector $V(F)$. Then the linear sum of W_1 and W_2 is the set of all those elements each one of which is expressible as the sum of an element of W_1 and an element of W_2. The linear sum of W_1 and W_2 can be written as $W_1 + W_2$. That is

$$W_1 + W_2 = \{\alpha + \beta : \alpha \in W_1, \beta \in W_2\}$$

Remark

- If $a \in W_1$, then $a = a + 0$ with $0 \in W_1$ and $0 \in W_2$ so $W_1 \subseteq W_1 + W_2$. Similarly, $W_2 \subseteq W_1 + W_2$.

THEOREM 1. *The linear sum of two subspaces of a vector space is also a subspace.*

Proof Let W_1 and W_2 be two subspaces of a vector space $V(F)$. Then we have to show that $W_1 + W_2$ is a subspace of $V(F)$.

Let α, β be any two arbitrary elements of $W_1 + W_2$,

Then, $\alpha, \beta \in W_1 + W_2$

$\Rightarrow$ $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$, where $\alpha_1, \beta_1 \in W_1$ and $\alpha_2, \beta_2 \in W_2$

Since, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in V$

$\therefore$ $W_1 + W_2 \subseteq V$.

Since, W_1 and W_2 are subspaces of V. Then

$\alpha_1, \beta_1 \in W_1 \Rightarrow a\alpha_1 + b\beta_1 \in W_1$ for some $a, b \in F$

and $\alpha_2, \beta_2 \in W_2 \Rightarrow a\alpha_2 + b\beta_2 \in W_2$

Now, $a\alpha_1 + b\beta_1 \in W_1$ and $a\alpha_2 + b\beta_2 \in W_2$

$\Rightarrow (a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2) \in W_1 + W_2$

$\Rightarrow a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) \in W_1 + W_2$

$\Rightarrow a\alpha + b\beta \in W_1 + W_2$

$\therefore$ $\alpha, \beta \in W_1 + W_2,\ a, b \in F \Rightarrow a\alpha + b\beta \in W_1 + W_2$

Hence, $W_1 + W_2$ is a subspace.

5.9 DIRECT SUM OF VECTOR SUBSPACES

Let W_1 and W_2 be two subspaces of a vector space V. Then V is said to be the direct sum of W_1 and W_2 if each element of V can be uniquely expressed as the sum of an element of W_1 and an element of W_2. If V is direct sum of W_1 and W_2, then it can be written as $V = W_1 \oplus W_2$.

In general, if V is the direct sum of $W_1, W_2, \ldots\ldots W_n$, then

$$V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_n$$

Here $W_1, W_2, \ldots, W_n$ *are called* complementary spaces.

THEOREM 1. *The necessary and sufficient condition for a vector space V to be the direct sum of two of its subspaces W_1 and W_2 are :*

(i) $V = W_1 + W_2$

(ii) $W_1 \cap W_2 = \{0\}$

Proof **Condition is necessary:**

Suppose V is the direct sum of W_1 and W_2, then each element of V can be uniquely expressed as sum of an element of W_1 and an element of W_2, so in particular each element of V is expressible as the sum of an element of W_1 and element of W_2, this concludes that

$$V = W_1 + W_2$$

Next, we shall show that $W_1 \cap W_2 = \{0\}$, for this let, if possible, there be a non-zero vector in $W_1 \cap W_2$ and let it be $\alpha \in W_1 \cap W_2$. Then we may write

$\alpha = \alpha + 0$ with $\alpha \in W_1$ and $0 \in W_2$.

and $\alpha = 0 + \alpha$ with $0 \in W_1$ and $\alpha \in W_2$.

Since $W_1 + W_2 \in V$, so $\alpha \in V$ and $V = W_1 \oplus W_2$ therefore, α can be uniquely expressed as the sum of an element of W_1 and an element of W_2. Thus contains only zero vector. This implies $W_1 \cap W_2 = \{0\}$.

Condition is Sufficient:

Suppose the conditions:

(i) $V = W_1 + W_2$ (ii) $W_1 \cap W_2 = \{0\}$

hold then we shall show that $V = W_1 \oplus W_2$.

From (i) we conclude that each element of V can be expressed as the sum of an element of W_1 and an element of W_2. Therefore, we shall only show that this representation is unique.

Let, if possible, an element $\alpha \in V$ has two representations, that is,

$\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

and $\alpha' = \alpha'_1 + \alpha'_2$ with $\alpha'_1 \in W_1$ and $\alpha'_2 \in W_2$.

$\Rightarrow$ $\alpha_1 + \alpha_2 = \alpha'_1 + \alpha'_2$

$$\alpha_1-\alpha'_1=\alpha'_2-\alpha_2$$

$\Rightarrow$ $\alpha_1,\alpha'_1\in W_1\Rightarrow\alpha_1-\alpha'_1\in W_1$

$\alpha_2,\alpha'_2\in W_2\Rightarrow\alpha'_2-\alpha_2\in W_2$ [$\because$ W_1 and W_2 are subspaces.]

and $\alpha_1-\alpha'_1=\alpha'_2-\alpha_2\in W_1\cap W_2$.

$\therefore$ $W_1\cap W_2=\{0\}$, this implies

But $\alpha_1-\alpha'_1=0=\alpha'_2-\alpha_2$

$\Rightarrow$ $\alpha_1=\alpha'_1$ and $\alpha'_2=\alpha_2$.

This shows that each element of V can be uniquely expressed as the sum of an element of W_1 and an element of W_2. Hence, $V=W_1\oplus W_2$, *i.e.*, V is the direct sum of W_1 and W_2.

Solved Examples

Based on the following Results

- The necessary and sufficient conditions for a non-empty subset W of $V(F)$ to be a subspace is that $a,b\in F,\ \alpha,\beta\in W\Rightarrow a\alpha+b\beta\in W$
- $W_1+W_2=\{\alpha+\beta:\alpha\in W_1,\beta\in W_2\}$
- The necessary and sufficient condition for a vector space V to be the direct sum of two of its subspaces W_1 and W_2 are
  (i) $V=W_1+W_2$ (ii) $W_1\cap W_2=\{0\}$

Example 1. *In $V=R^3$. Let W_1 be the xy-plane and let W_2 be the z-plane given by*

$W_1=\{(x,y,0):x,y\in R\}$ *and* $W_2=\{(0,y,z):y,z\in R\}$

Show that $V=\{W_1\oplus W_2\}$

Solution. Let $(x,y,z)\in V$, then this element can be written as the sum of an element of W_1 and an element of W_2 on one and only one way *i.e.*,

$$(x,y,z)=(x,y,0)+(0,0,z)$$

Accordingly, V is the direct sum of W_1 and W_2, that is $V=\{W_1\oplus W_2\}$

Example 2. *In $V=R^3$ and W_1 be the xy-plane and let W_2 be the yz-plane:*

$W_1=\{(x,y,0):x,y\in R\}$ *and* $W_2=\{(0,y,z):y,z\in R\}$

then show that V is not the direct sum of W_1 and W_2.

Solution. Let (a,b,c) be any element of R^3, then

$$(a,b,c)=(x,y,0)+(0,y,z)=(x,2y,z)$$

$\Rightarrow$ $a=x,\ b=2y,\ c=z$

$\Rightarrow$ $a,b,c\in R$ [$\because$ $x,y,z\in R$]

$\Rightarrow$ every element of V can be written as the sum of an element of W_1 and an element of W_2.

But such sums are not unique, for example,

Let $(3,5,7)\in R^3$, then

$(3,5,7)=(3,2,0)+(0,3,7)$, $(3,2,0)\in W_1$ and $(0,3,7)\in W_2$.

Also $(3,5,7)=(3,1,0)+(0,4,7)$, $(3,1,0)\in W_1$ and $(0,4,7)\in W_2$.

Hence, V is not the direct sum of W_1 and W_2.

Example 3. *If $V_3(R)$ is a vector space and $W_1=\{(a,0,c):a,c\in R\}$ and $W_2=\{(0,b,c):b,c\in R\}$ are two subspaces of $V_3(R)$, then show that $V=W_1+W_2$ and $V\neq W_1\oplus W_2$.*

Solution. Let (x,y,z) be an arbitrary element of $V_3(R)$, then

$$(x,y,z)=(a,0,c)+(0,b,c)=(a,b,2c)$$

$\Rightarrow$ $x=a,\ y=b,\ z=2c$

$\Rightarrow$ $x,y,z\in R$ [$\because$ $a,b,c\in R$]

$\Rightarrow$ every element of $V_3(R)$ can be written as the sum of an element of W_1 and an element of W_2.

$\Rightarrow$ $V=W_1+W_2$

But such representations is not unique, for example,

Let $(3,5,6)\in V_3(R)$, then

$(3,5,6)=(3,0,2)+(0,5,4)$; $(3,0,2)\in W_1$ and $(0,5,4)\in W_2$.

Also $(3,5,6)=(3,0,5)+(0,5,1)$; $(3,0,5)\in W_1$ and $(0,5,1)\in W_2$.

So here $(3,5,6)$ can be written as the sum of elements of W_1 and W_2 in two ways.

Hence $V\neq W_1\oplus W_2$.

Example 4. *Let V be a vector space of all functions from R into R; let V_e be the subset of even functions, such that $f(-x)=f(x)$; let V_0 be the subset of odd functions $f(-x)=-f(x)$. Prove that*

(a) V_e and V_0 are subspaces of V.

(b) $V_e+V_0=V$

(c) $V_e\cap V_0=\{0\}$

Solution. (a) Since, V is a vector space of all functions, therefore,

$$V_e\subseteq V,\ V_0\subseteq V.$$

Let $f(x),g(x)\in V_e$, so that $f(-x)=f(x)$ and $g(-x)=g(x)$ and let $a,b\in R$, then consider,

$$h(x)=af(x)+bg(x)$$

Now $h(-x)=af(-x)+bg(-x)$

or $h(-x)=af(x)+bg(x)$ $[f(-x)=f(x),\ g(-x)=g(x)]$

$=h(x)$

$\Rightarrow$ $h(x)\in V_e$

$\Rightarrow$ $af(x)+bg(x)\in V_e$

Consequently, if $f(x),g(x)\in V_e$, then $af(x)+bg(x)\in V_e$. Hence V_e is a subspace of V.

Similarly, we can prove that V_0 is a subspace of V.

(b) Let $f(x)$ be any element of V.

Consider $f(x)=\frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]$

Let $\alpha(x)=\frac{1}{2}[f(x)+f(-x)]$ and $\beta(x)=\frac{1}{2}[f(x)-f(-x)]$

$\therefore$ $f(x)=\alpha(x)+\beta(x)$.

Also $\alpha(-x)=\frac{1}{2}[f(-x)+f(x)]=\alpha(x)$

$\therefore \quad \alpha(x) = V_e$

and $\quad \beta(-x) = \frac{1}{2}[f(-x) - f(x)] = \frac{1}{2}[f(x) - f(-x)] = \beta(x)$

$\therefore \quad \beta(x) = V_0.$

Consequently, $f(x) = \alpha(x) + \beta(x)$ where, $\alpha(x) = V_e$ and $\beta(x) = V_0$. Hence every element of V can be expressed as the sum of an element of V_e and an element of V_0. That is,

$$V = V_e + V_0.$$

(c) $V_e \cap V_0 = \{0\}$

Let if possible, there exists a non-zero function $f(x)$, which belongs to $V_e \cap V_0$. Therefore, if $f(x) \in V_e$, then $f(-x) = f(x)$ and $f(x) \in V_0$, then $f(-x) = -f(x)$. So that,

$$f(x) = -f(x)$$

$$\Rightarrow \quad 2f(x) = 0 \Rightarrow f(x) = 0$$

This gives a contradiction, because $f(x)$ is assumed to be non-zero function. Hence every function of $V_e \cap V_0$ is a zero function. Consequently $V_e \cap V_0 = \{0\}$.

Example 5. *If W_1 and W_2 are subspaces of a vector space $V(F)$, then show that W_1+W_2 is also a subspace of $V(F)$.*

Solution. Let us define

$$W_1+W_2 = \{\alpha_1+\alpha_2 : \alpha_1 \in W_1, \alpha_2 \in W_2\}.$$

We have to show that W_1+W_2 is a subspace of V.

Let $\alpha \in W_1+W_2 \Rightarrow \alpha = \alpha_1+\alpha_2$ for some $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$

$\Rightarrow \alpha = \alpha_1+\alpha_2$ for some $\alpha_1, \alpha_2 \in V \Rightarrow \alpha \in V$

$\therefore \quad W_1+W_2 \subseteq V$

Now, let $\alpha_1+\alpha_2 \in W_1+W_2$ and $\beta_1+\beta_2 \in W_1+W_2$, where $\alpha_1+\alpha_2 \in W_1$ and $\beta_1+\beta_2 \in W_2$.

Since, W_1 and W_2 are subspaces of V, then

$$a, b \in F, \alpha_1, \beta_1 \in W_1 \Rightarrow a\alpha_1+b\beta_1 \in W_1$$

$$a, b \in F, \alpha_2, \beta_2 \in W_2 \Rightarrow a\alpha_2+b\beta_2 \in W_2$$

$$a\alpha_1+b\beta_1 \in W_1, a\alpha_2+b\beta_2 \in W_2$$

$$\Rightarrow (a\alpha_1+b\beta_1)+(a\alpha_2+b\beta_2) \in W_1+W_2$$

$$\Rightarrow a(\alpha_1+\alpha_2)+b(\beta_1+\beta_2) \in W_1+W_2$$

Since, $\alpha_1+\alpha_2 \in W_1+W_2$ and $\beta_1+\beta_2 \in W_1+W_2$.

Thus we have

$$a, b \in F, \alpha_1+\alpha_2, \beta_1+\beta_2 \in W_1+W_2$$

$$\Rightarrow a(\alpha_1+\alpha_2)+b(\beta_1+\beta_2) \in W_1+W_2$$

Hence, W_1+W_2 is subspace of $V(F)$.

Example 6. *Let R be the field of real numbers, show that the set $W = \{(x, 2y, 3z) : x, y, z \in R\}$ is a subspace of $V_3(R)$.*

Solution. Since, $W = \{(x, 2y, 3z) : x, y, z \in R\}$.

Let $\alpha, \beta \in W$, where $\alpha = \{x_1, 2y_1, 3z_1\}$ and $\beta = \{x_2, 2y_2, 3z_2\}$ and $x_1, y_1, z_1, x_2, y_2, z_2 \in R$,

If $a, b \in R$, then

$$a\alpha+b\beta = a(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2)$$

$$= (ax_1, 2ay_1, 3az_1) + (bx_2, 2by_2, 3bz_2)$$

$$= (ax_1+bx_2), 2(ay_1+by_2), 3(az_1+bz_2)$$

$$\therefore \quad a\alpha+b\beta \in W,$$

Because $ax_1+bx_2, ay_1+by_2, az_1+bz_2 \in R$.

Hence, W is a subspace of $V_3(R)$.

Example 7. *Show that the set of all real valued continuous functions defined on $[0,1]$ is a vector space over field of reals.*

Solution. Let V be the set of all real valued continuous functions defined on $[0,1]$. Now we have to show that V is a vector space over R (field of real numbers) under vector addition and scalar multiplication which is defined as follows:

$$(f+g)(x) = f(x)+g(x), \forall f, g \in V$$

and $$(af)(x) = af(x), \forall f \in V \text{ and } a \in R.$$

First we shall show that $(V,+)$ is an abelian group.

Let $f, g \in V$, then

$$(f+g)(x) = f(x)+g(x), \forall f, g \in V$$

$\therefore \quad f+g \in V$. Thus V is closed under vector addition.

Now let $0(x) \in V$, we have

$$f(x)+0(x) = (f+0)(x) = f(x), \forall f \in V$$

$\therefore \quad 0(x)$ is the additive identity in V.

Let $-f \in V$, then we have

$$-f(x)+f(x) = (-f+f)(x) = 0(x), \forall f \in V$$

$\therefore \quad -f$ is the additive inverse of V.

Since vector addition is always associative as well as commutative, consequently $(V,+)$ is an abelian group.

Further, since V is closed under scalar multiplication therefore af is a real valued continuous function defined on $[0,1]$.

(i) If $a \in R$ and $f, g \in V$, then we have

$$a[(f+g)x] = a[f(x)+g(x)]$$

$$= af(x)+ag(x) = (af+ag)(x)$$

$$\therefore \quad a(f+g) = af+ag.$$

(ii) If $a,b \in R$ and $f \in V$, then we have

$$[(a+b)f](x)=(a+b)f(x)=af(x)+bf(x)=(af+bf)(x)$$

$\therefore$ $(a+b)f = af+bf.$

(iii) If $a,b \in R$ and $f \in V$, then we have

$$[(ab)f](x)=(ab)f(x)=a[bf(x)]=[a(bf)](x).$$

$\therefore$ $(ab)f = a(bf).$

(iv) If $1 \in R$ and $f \in V$, then we have

$$(1f)(x)=1f(x)=f(x)$$

$\therefore$ $1f = f, \ \forall f \in V$

Hence V is a vector space over R.

Example 8. *Show that $R^2(R)$ is not a vector space when addition and scalar multiplication composition are defined by*

$$(a_1,a_2)+(b_1,b_2)=(a_1+b_1, a_2+b_2)$$

and $$a(a_1,a_2)=(aa_1,a_2), \ \forall a,a_1,a_2,b_1,b_2 \in R.$$

Solution. Suppose $a=1$ and $b=2$ and $(a_1,a_2)=(3,4)$, then using the given compositions, we have

$$(a+b)(a_1,a_2) = (1+2).(3+4)$$
$$= 3.(3,4) = (3.3,4) = (9,4)$$

and $$a.(a_1,a_2)+b.(a_1,a_2) = 1.(3,4)+2.(3,4)$$
$$= (3.1,4)+(3.2,4)$$
$$= (3,4)+(6,4)$$
$$= (3+6,4+4) = (9,8)$$

$\therefore$ $(a+b).(a_1,a_2) \neq a.(a_1,a_2)+b.(a_1,a_2).$

Hence, R^2 (R) is not a vector space.

Example 9. *Prove the solution set W of the differential equation*

$$2\frac{d^2y}{dx^2}-9\frac{dy}{dx}+2y=0$$

is a subspace of vector space of all real valued functions of R.

Solution. Let $W=\left\{y: 2\frac{d^2y}{dx^2}-9\frac{dy}{dx}+2y=0\right\}$, be the set of all solutions of the given differential equation where, $y=f(x)$.

Now if we define a real valued function denoted by 0 on R by $0(x)=0, \forall x \in R$, then $0(x)$ satisfies the given differential equation, so that $0(x) \in W$.

Let $y_1=f(x)$ and $y_2=g(x)$ be any two elements of W, then we have

$$2\frac{d^2f(x)}{dx^2}-\frac{df(x)}{dx}+2f(x)=0 \quad ....(1)$$

and $$2\frac{d^2g(x)}{dx^2}-\frac{dg(x)}{dx}+2g(x)=0 \quad ....(2)$$

Let a, b be any two scalars.

Now, multiplying (1) by a and (2) by b and then adding, we get

$$2\frac{d^2}{dx^2}[af(x)+bg(x)]-9\frac{d}{dx}[af(x)+bg(x)]+2[af(x)+bg(x)]=0$$

which shows that $af(x)+bg(x)$ is also the solution of the given differential equation.

So that $[af(x)+bg(x)] \in W.$

$\therefore$ $f(x) \in W, g(x) \in W \Rightarrow af(x)+bg(x) \in W \ \forall a, b \in R.$

Hence W is a subspace of a vector space of all real valued functions of R.

Example 10. *Let V be the vector space of all functions from the real field R into R. Show that the set $W=\{f: f(7)=2+f(1)\}$ is not a subspace of V.*

Solution. Let f and g be any two elements of W, i.e.,

$$f(7)=2+f(1) \text{ and } g(7)=2+g(1).$$

Then $$(f+g)(7)=f(7)+g(7)$$
$$=2+f(1)+2+g(1)$$
$$=4+f(1)+g(1)$$
$$=4+(f+g)(1)\neq 2+(f+g)(1).$$

Hence $f+g \neq W$, and so W is not a subspace of V.

Example 11. *Let V be the vector space of all square $n \times n$ matrices over a field of reals R. Show that W is a subspace of V, where*

(i) W consists of the symmetric matrices.

(ii) W consists of all matrices which commute with a given matrix M, i.e.,

$$W=\{A \in V : AM = MA\}.$$

Solution. (i) The null matrix $0 \in W$, as all its entries being zero and it is symmetric.

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be any two elements of W.

Then $a_{ij}=a_{ji}$ and $b_{ij}=b_{ji}$.

For any scalars a, $b \in R$, we have

$$aA+bB=a[a_{ij}]+b[b_{ij}]=[aa_{ij}+bb_{ij}]=[c_{ij}]=C$$

where, $c_{ij}=aa_{ij}+bb_{ij}$

Now, $c_{ij}=aa_{ij}+bb_{ij}=aa_{ji}+bb_{ji}=c_{ji}$

Thus $aA+bB$ is symmetric so it belongs to W. Hence W is subspace of V.

(ii) The null matrix $0 \in W$ as $OM=MO$.

Now suppose A and B be any two elements of W then, $AM=MA$ and $BM=MB$.

For $a,b \in R$, we have

$$(aA+bB)M=(aA)M+(bB)M=a(AM)+b(BM)$$
$$=a(MA)+b(MB)=M(aA+bB)$$

$\therefore$ $aA+bB \in W, \ A, B \in W$ and $\forall a, b \in R.$

Hence W is a subspace of V.

Example 12. *let $V=R^3$. Show the set $W=\{(a, b, c): a^2+b^2+c^2 \leq 1\}$ is not a subspace of V.*

Solution. Let $\alpha=(1,0,0) \in W$, $\beta=(0,1,0) \in W$. But we have

$$\alpha+\beta=(1,0,0)+(0,1,0)$$
$$=(1,1,0) \notin W \text{ as } 1^2+1^2+0^0=2>1.$$

Hence W is not a subspace of V.

Example 13 *Let V be the vector space of all 2×2 matrices over the real field R. Show that W is not a subspace of V, where*

(i) W consists of all matrices with zero determinant.

(ii) W consists of all matrices A from which $A^2 = A$.

Solution (i) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be two elements of W. But,

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin W \text{ as } |A + B| = 1 \neq 0$$

$\therefore$ W is not a subspace of V.

(ii) The unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in W$ as $I^2 = I$.

But $2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \notin W$ as

$$(2I)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq 2I.$$

Hence W is not a subspace of V.

EXERCISE 5.3

1. Show that the complex field C is a vector space over the field R of reals.
2. Let V be the set of all pairs of real numbers and let F be the field of real numbers and define
   $(x, y) + (x_1, y_1) = (3y + 3y_1, -x - x_1)$ and $c(x, y) = (3cy, -cx)$.
   Show that V is a vector space over F.
3. Show that the set $W = \{(a_1, a_2, 0) : a_1, a_2 \in F\}$ is a subspace of $V_3(F)$.
4. Show that the set W of the elements of the vector space $V_3(\text{R})$ of the form $(x + 2y, y, -x+3y)$ where $x, y \in R$ is a subspace of $V_3(\text{R})$.
5. Prove that the set of all solutions (a, b, c) of the equation $a+b+2c = 0$ is a subspace of vector space $V_3(\text{R})$.
6. Prove that the arbitrary intersection of subspaces of a vector space is a subspace.
7. Let $V = R^3$. Show that the set $W = \{(a,b,c) : a,b,c \in Q]$ is not a subspace of R^3.

Hint to Selected Problems

1. Let C be the set of vectors and R the set of scalars, since C is a field so that $a\alpha \in \text{C}$, this show that C is closed under scalar multiplication. $1 \in \text{R}$ so $1 \in \text{C}$.
   Also for $\alpha, b \in \text{C}$ and $a, b \in \text{R}$,
   $a, b \in \text{R} \Rightarrow a, b \in \text{C}$ $\quad (\because \text{R} \subseteq \text{C})$
   $\therefore$ $a(\alpha + \beta) = a\alpha + a\beta$ and $(a + b)\alpha = a\alpha + b\alpha$.
   also, $(ab)\alpha = a(b\alpha)$, $1.\alpha = \alpha$, $\forall\, \alpha \in \text{C}$. Hence C is a vector space over R.
4. $W = \{(x+2y, y, -x+3y) : x, y \in \text{R}\}$, then show that $a\alpha + b\beta \in W\ \forall\ \alpha, \beta \in W$ and $a, b \in \text{R}$.
   $a\alpha + b\beta = a(x_1 + 2y_1, y_1, -x_1 + 3y_1] + b(x_2 + 2y_2, y_2, -x_2 + 3y_2]$

$= (ax_1 + 2ay_1, ay_1, -ax_1 + 3ay_1) + (bx_2 + 2by_2, by_2, -bx_2 + 3by_2)$
$= (ax_1 + bx_2 + 2(ay_1 + by_2), ay_1 +, by_2 - (ax_1 + bx_2) + 3(ay_1 + by_2)) \in W$

$\therefore$ $a\alpha + b\beta \in W \Rightarrow W$ is a subspace.

7. $(a, b, c) \in W$, $(a, b, -c) \in W$ as $-c \in Q$.
   Now $(a, b, c) + (a, b, -c) = (2a, 2b, 0) \notin W$. Hence W is not a subspace.

5.10 LINEAR COMBINATION OF VECTORS

Definition. *Let V be a vector space over a field F and $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$, then any vector $\alpha \in V$ can be expressed as below:*

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$$

where $a_1, a_2, \ldots, a_n \in F$, is said to be the linear combination of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition. *Let $V(F)$ be a vector space aver F and let S be any non-empty subset of V, then the set of all linear combination of finite elements of S, is called the linear span of S. It is denoted by $L(S)$. Therefore, we have*

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n : a_1, a_2, \ldots, a_n \in F$$

and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are a finite elements of $S\}$

THEOREM 1. *The linear span $L(S)$ of a non-empty subset S of a vector space $V(F)$ is the smallest subspace of V containing S.*

Proof. By definition of $L(S)$, we have

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n : a_i \in V\}$$

Let $\alpha \in S$, then $\alpha = 1.\alpha$, $1 \in F$, so $\alpha \in L(S)$

$\therefore$ $\quad S \subseteq L(S)$

Now, we shall show that $L(S)$ is a subspace.

Let α, β be any two arbitrary elements of $L(S)$, then

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$; for $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ and for $a_1, a_2, \ldots, a_n \in F$

Also, $\beta = b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m$ for all $a, b \in F$, we have

$a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n) + b(b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m)$
$= (aa_1)\alpha_1 + (aa_2)\alpha_2 + \ldots + (aa_n)\alpha_n + (bb_1)\beta_1 + (bb_2)\beta_2 + \ldots + (bb_m)\beta_m$

This implies that $a\alpha + b\beta$ is a linear combination of finite number of elements of S, so $a\alpha + b\beta \in L(S)$. Hence $L(S)$ is a subspace of V.

Next, we shall show that $L(S)$ is the smallest subspace containing S.

For this there is a subspace W of V containing S. Let $\alpha_1, \alpha_2, \ldots, \alpha_t \in S \subset W$ and W being a subspace, then

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_t\alpha_t \in W; \text{ for all } a_i \in F$$

This implies that W contains all linear combinations of finite elements of S, therefore $L(S) \in W$.

Hence $L(S)$ is the smallest subspace of V containing S.

THEOREM 2. *If S, T are two subsets of a vector space V, then*

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$.

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L[L(S)] = L(S)$

Proof Let α be an arbitrary element of $L(S)$, then

$$\alpha \in L(S) \Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

where, $\alpha_1, \alpha_2, \dots, \alpha_n \in S$ and $a_1, a_2, \dots, a_n \in F$.

(i) Since $S \subseteq T$, so that $\alpha_1, \alpha_2, \dots, \alpha_n \in T$, therefore α is also the linear combination of finite elements of T. This implies $\alpha \in L(T)$.

Thus $\qquad \alpha \in L(S) \Rightarrow \alpha \in L(T)$

Hence $\qquad L(S) \subseteq L(T)$ if $S \subseteq T$.

(ii) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, then from (i), we have

$$L(S) \subseteq L(S \cup T)$$

and $\qquad L(T) \subseteq L(S \cup T)$

$\Rightarrow \qquad L(S) + L(T) \subseteq L(S \cup T)$ ...(1)

Next, let α be an arbitrary element of $L(S \cup T)$, then α is a linear combination of finite elements of $S \cup T$. This implies that some of $\alpha_i \in S$ or some of $\alpha_i \in T$. This shows that α is a linear combination of finite elements of S and finite elements of T, therefore $\alpha \in L(S) + L(T)$. Thus,

$$L(S \cup T) \subseteq L(S) + L(T) \qquad \dots(2)$$

From (1) and (2), we get

$$L(S \cup T) = L(S) + L(T) \qquad \dots(3)$$

(iii) Since $S \subseteq T(S)$, then from (i), we have

$$L(S) \subseteq L[L(S)] \qquad \dots(4)$$

Next let α be any arbitrary element of $L[L(S)]$, then α is a linear combination of finite elements of $L(S)$. Suppose, we have

$$\alpha = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n = \sum_{i=1}^{n} b_i\beta_i$$

where, each $\beta_i \in L(S)$ for all $b_1, b_2, \dots, b_n \in F$. Also each β_i is a linear combination of finite elements of S, so that

$$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1m}\alpha_m$$

$$\beta_1 = a_{21}\alpha_1 + a_{22}\alpha_2 + \dots + a_{2t}\alpha_t$$

....

On putting the values of $\beta_1, \beta_2, \dots$, etc. in (2), we see that α is a linear combination of finite elements of S. Thus $\alpha \in L(S)$

$\therefore \qquad L[L(S)] \subseteq L(S)$ ...(5)

From (4) and (5), we get $L(S) = L[L(S)]$.

THEOREM 3. *The linear sum of two subspaces W_1 and W_2 of a vector space $V(F)$ is generated by their union. That is, $W_1 + W_2 = L(W_1 \cup W_2)$.*

Proof We have already proved that the linear sum of two subspaces is aiso a subspace and linear span of a subset of a vector space is also a subspace.

Therefore, $W_1 + W_2$ and $L(W_1 \cup W_2)$ are subspaces of $V(F)$.

Let α be any arbitrary element of $W_1 + W_2$, then

$$\alpha \in W_1 + W_2$$

$\Rightarrow \qquad \alpha = \alpha_1 + \alpha_2$ for some $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$

Since, $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$, so, $\alpha_1 + \alpha_2 \in W_1 + W_2$

Also, we may write $\alpha = \alpha_1 + \alpha_2 = 1.\alpha_1 + 1.\alpha_2$. This implies that α is a linear combination of finite elements namely α_1 and α_2 of $W_1 \cup W_2$, so that $\alpha \in L(W_1 + W_2)$

$\therefore \qquad \alpha \in W_1 + W_2 \Rightarrow \alpha \in L(W_1 \cup W_2)$

Thus, $\qquad W_1 + W_2 \subseteq L(W_1 \cup W_2)$. ... (1)

But $L(W_1 \cup W_2)$ being the smallest subspace containing $W_1 \cup W_2$, and since $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$, therefore

$$L(W_1 \cup W_2) \subseteq W_1 \cup W_2 \qquad \dots(2)$$

From (1) and (2), we get

$$W_1 + W_2 = L(W_1 \cup W_2)$$

5.11 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

In this section, we shall discuss the concept of linear dependence or linear independence which lays the foundations for the key notions (viz., dimensions) of the theory of vector spaces.

Definition 1. *Let $V(F)$ be a vector space over a field F. Then a finite set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of vectors of V is said to be linearly dependent if there exists scalars $a_1, a_2, \dots, a_n$ not all of them equal to zero such that*

$$a_1\alpha_1 + a_2\alpha_2, \dots, + a_n\alpha_n = 0$$

Definition 2. *Let V(F) be a vector space over F. Then a finite set of vectors $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of V is said to linearly independent if for every expressions of the type.*

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$$

where $a_1, a_2, \dots, a_n \in F$ implies $a_1 = 0, a_2 = \dots = a_n$.

REMARKS

- S is linearly independent
  - $\Leftrightarrow$ S is not linearly dependent $\Leftrightarrow$ no finite subset of S is linearly dependent.
  - $\Leftrightarrow$ each finite subset of S is linearly independent.
  - $\Leftrightarrow$ whenever $\sum_{i=1}^{n} a_i\alpha_i = 0, a_i \in F, \alpha_i \in S, i = 1, 2, \dots, n$ and $n \in N$, then each $a_i = 0$.
- Any infinite set is linearly independent if it's every finite subset is linearly independent otherwise it is linearly dependent.

THEOREM 1. *If $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ are linearly independent, then every element in their linear span has a unique representation in the form*

$$a_1\alpha_1 + a_2\alpha_2, \dots, + a_n\alpha_n \text{ with } a_i \in F.$$

Proof By definition of linear span, we know that every element in the linear span is of the form $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$. Therefore, we only show the uniqueness of the representation.

Let if possible,

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ and $b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$ be two forms of an element in linear span of $\alpha_1, \alpha_2, \dots, \alpha_n$, then

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$$

$(a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + \ldots + (a_n - b_n)\alpha_n = 0$

Since, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent so, we have

$$\Rightarrow \quad a_1 - b_1 = 0,\ a_2 - b_2 = 0, \ldots, a_n - b_n = 0$$

$$\Rightarrow \quad a_1 = b_1,\quad a_2 = b_2, \ldots,\quad a_n = b_n.$$

Hence, every element in the linear span of linearly independent vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ has a unique form of $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$.

THEOREM 2. *If $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$, then either they are linearly independent or some α_k is a linear combination of preceding ones $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$.*

Proof. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ are linearly independent, then nothing is to prove. So assume that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not linearly independent. Then, there are some $a_i \in F$ which are non-zero such that

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$$

Let k be the largest integer for which $a_k \neq 0$. Since $a_i = 0$ for $i > k$, and

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k = 0$$

$$\Rightarrow \quad \alpha_k = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 - \ldots - a_{k-1}\alpha_{k-1}) \qquad [\because a_k \neq 0]$$

$$\Rightarrow \quad \alpha_k = (-a_k^{-1}a_1)\alpha_1 + (-a_k^{-1}a_2)\alpha_2 + \ldots + (-a_k^{-1}a_{k-1})\alpha_{k-1}$$

$\Rightarrow$ α_k is a linear combination of preceding ones $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$.

Solved Examples

Example 1. *Is the vector (2, –5, 3) in the subspace of R^3 spanned by the vectors (1, –3, 2), (2, –4, –1), (1, –5, 7)?*

Solution. Let $\alpha = (2, -5, 3)$ and $\alpha_1 = (1, -3, 2)$, $\alpha_2 = (2, -4, -1)$ and $\alpha_3 = (1, -5, 7)$

Again let $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$ where $a_1, a_2, a_3 \in R$

Then $(2,-5,3) = a_1(1,-3,2) + a_2(2,-4,-1) + a_3(1,-5,7)$

$$\Rightarrow \quad (2,-5,3) = (a_1 + 2a_2 + a_3, -3a_1 - 4a_2 - 5a_3, 2a_1 - a_2 + 7a_3)$$

$$\therefore \quad a_1 + 2a_2 + a_3 = 2 \quad \ldots (1)$$

$$-3a_1 - 4a_2 - 5a_3 = -5 \quad \ldots (2)$$

$$2a_1 - a_2 + 7a_3 = 3 \quad \ldots (3)$$

Eliminating a_2 between (1) and (2), we get

$$-a_1 - 3a_3 = -1$$

or $$a_1 = 1 - 3a_3 \quad \ldots (4)$$

Again, eliminating a_2 between (2) and (3), we get

$$11a_1 + 33a_3 = 17 \quad \ldots (5)$$

Here no values of a_3 and a_1 will satisfy both equations (4) and (5). Thus, equations (1), (2) and (3) have no solution.

Hence α cannot be expressed as a linear combination of α_1, α_2 and α_3.

Hence, the vector (2, –5, 3) is not spanned by the vectors (1, –3, 2), (2, –4, –1) and (1, –5, 7).

Example 2. *In the vector space R^3 express the vector (1, –2, 5) as a linear combination of the vectors (1, 1, 1), (1, 2, 3) and (2, –1, 1).*

Solution. Let $\alpha = (1,-2,5), \alpha_1 = (1,1,1), \alpha_2 = (1,2,3)$ and $\alpha_3 = (2,-1,1)$.

Let $$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

where $a_1, a_2, a_3 \in R$. Then

$$(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$$

$$\Rightarrow \quad (1,-2,5) = (a_1 + a_2 + 2a_3, a_1 + 2a_2 - a_3, a_1 + 3a_2 + a_3)$$

$$\therefore \quad a_1 + a_2 + 2a_3 = 1 \quad \ldots (1)$$

$$a_1 + 2a_2 - a_3 = -2 \quad \ldots (2)$$

$$a_1 + 3a_2 + a_3 = 5 \quad \ldots (3)$$

Eliminating a_1 between (1) and (2), we get

$$-a_2 + 3a_3 = 3 \quad \ldots (4)$$

Eliminating a_1 between (2) and (3), we get

$$-a_2 - 2a_3 = -7$$

or $$a_2 + 2a_3 = 7 \quad \ldots (5)$$

Solving (4) and (5), we get

$$a_3 = 2, a_2 = 3$$

Putting the values of a_2 and a_3 in (1), we get

$$a_1 = -6$$

Hence, (1, –2, 5) = –6(1, 1, 1)+3(1, 2, 3)+2(2, –1, 1)

Example 3. *For what values of m, the vector $(m, 3, 1)$ is a linear combination of the vectors (3, 2, 1) and (2, 1, 0)?*

Solution. Let $\alpha = (m,3,1), \alpha_1 = (3,2,1)$ and $\alpha_2 = (2,1,0)$

Let $\alpha = a_1\alpha_1 + a_2\alpha_2$ where $a_1, a_2 \in R$

Then $(m, 3, 1) = a_1(3, 2, 1) + a_2(2, 1, 0)$

$$\Rightarrow \quad (m, 3, 1) = (3a_1 + 2a_2, 2a_1 + a_2, a_1)$$

$$\therefore \quad 3a_1 + 2a_2 = m \quad \ldots (1)$$

$$2a_1 + a_2 = 3 \quad \ldots (2)$$

$$a_1 = 1 \quad \ldots (3)$$

Solving (1), (2) and (3), we get

$$a_1 = 1, a_2 = 1, m = 5$$

Hence, the reaquired value of m is 5.

Example 4. *In the vector space R^4 determine whether or not the vector (3, 9, –4, –2) is a linear combination of the vectors (1, –2, –0, 3), (2, 3, 0, –1) and (2, –1, 2, 1).*

Solution. Let $\alpha = (3,9,-4,-2), \alpha_1 = (1,-2,0,3), \alpha_2 = (2,3,0,-1)$ and $\alpha_3 = (2,-1,2,1)$.

Let $$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

where $\alpha_1, \alpha_2, \alpha_3 \in R$.

Then, $(3,9,-4,-2) = a_1(1,-2,0,3)+a_2(2,3,0,-1)+a_3(2,-1,2,1)$

$\therefore \quad a_1+2a_2+2a_3=3$ ...(1)

$-2a_1+3a_2-a_3=9$ ... (2)

$2a_3=-4$ ... (3)

$3a_1-a_2+a_3=-2$ ... (4)

Solving (1), (2) and (3), we get

$$a_1=1, a_2=3, a_3=-2$$

These values satisfy the equation (4). Thus the system of equations (1), (2), (3) and (4) is consistent and has a solution.

Hence, α can be written as a linear combination of vectors α_1, α_2 and α_3.

Example 5. *Write the polynomial $f(x) = x^2 + 4x - 3$ over R as a linear combination of the polynomials*

$$f_1(x)=x^2-2x+5, f_2(x)=2x^2-3x \text{ and } f_3(x)=x+3$$

Solution. Let $f(x)=a_1f_1(x)+a_2f_2(x)+a_3f_3(x)$ where $a_1,a_2,a_3 \in R$

Then, $x^2+4x-3=a_1(x^2-2x+5)+a_2(2x^2-3x)+a_3(x+3)$

$\Rightarrow \quad x^2+4x-3=(a_1+2a_2)x^2+(-2a_1-3a_2+a_3)x+5a_1+3a_3$

$\therefore \quad a_1+2a_2=1$ ... (1)

$-2a_1-3a_2+a_3=4$ ... (2)

$5a_1+3a_3=-3$ ... (3)

Eliminating a_2 between (1) and (2), we get

$-a_1+2a_3=11$ ... (4)

Solving (3) and (4), we get

$$a_1=-3, a_3=4$$

Putting these values in (1), we get

$$a_2=2$$

Hence $x^2+4x-3=-3(x^2-2x+5)+2(2x^2-3x)+4(x+3)$

Example 6. *In the vector space R^3, let $\alpha=(1,2,1), \beta=(3,1,5), \gamma=(3,-4,7)$ Show that the subspace spanned by $S=\{\alpha, \beta\}$ and $T=\{\alpha, \beta, \gamma\}$ are the same.*

Solution. We have to show that $L(S)=L(T)$

From given sets S and T we have

$$S \subseteq T \;\Rightarrow\; L(S) \subseteq L(T)$$

Now we show that γ can be expressed as a linear combination of α and β.

Let $\quad \gamma = a\alpha + b\beta$

$\Rightarrow \quad (3,-4,7)=a(1,2,1)+b(3,1,5)$

$\Rightarrow \quad (3,-4,7)=(a+3b, 2a+b, a+5b)$

$\therefore \quad a+3b=3$ ... (1)

$2a+b=-4$ ... (2)

$a+5b=7$ ... (3)

Solving (1) and (2), we get

$$a=-3, b=2$$

These values satisfy the equation (3).

Thus $\quad \gamma = -3\alpha + 2\beta$

Now let $\delta \in L(T)$, then δ can be expressed as a linear combination of α, β and γ but γ can be replaced by $-3\alpha + 2\beta$, therefore, δ can be expressed as a linear combination of α and β.

Thus $\quad \delta \in L(S)$

$\therefore \quad L(T) \subseteq L(S)$

Hence $\quad L(S)=L(T)$

Example 7. *Find a condition on a, b, c such that $\alpha=(a,b,c)$ is a linear combination of vectors (1, –3, 2) and (2, –1,1).*

Solution. Let $\alpha = a_1(1, -3, 2) + a_2(2, -1, 1)$

$\Rightarrow \quad (a,b,c)=a_1(1,-3,2)+a_2(2,-1,1)$

$\Rightarrow \quad (a,b,c)=(a_1+2a_2,-3a_1-a_2,2a_1+a_2)$

$\therefore \quad a_1+2a_2=a$ ... (1)

$-3a_1-a_2=b$ ... (2)

$2a_1+a_2=c$ ... (3)

Solving (1) and (2), we get

$$a_1=-\frac{1}{5}(a+2b), a_2=\frac{1}{5}(3a+b)$$

Putting these values in (3), we get

$$a-3b-5c=0$$

Thus, the system of equations (1), (2) and (3) is consistent iff $a-3b-5c=0$. Hence, α is a linear combination of (1, –3, 2) and (2, –1, 1) if and only if

$$a-3b-5c=0 .$$

Example 8. *Show that (1, 1, 1), (0, 1, 1) and (0, 1, –1) generate R^3.*

Solution. In order to show that (1, 1, 1), (0, 1, 1) and (0, 1, –1) generate R^3, we have to show that any vector of R^3 is a linear combination of (1, 1, 1), (0, 1, 1) and (0, 1, –1).

Let $\quad \alpha=(a,b,c) \in R^3$ and let

$$(a,b,c)=a_1(1,1,1)+a_2(0,1,1)+a_3(0,1,-1)$$

where $a_1,a_2,a_3 \in R$

Then, $\quad (a,b,c)=(a_1, a_1+a_2+a_3, a_1+a_2-a_3)$

$\therefore \quad a_1=a$ ... (1)

$a_1+a_2+a_3=b$ ... (2)

$a_1+a_2-a_3=c$ ... (3)

Solving (1), (2) and (3), we get

$$a_1 = a, a_2 = \frac{b+c-2a}{2}, a_3 = \frac{b-c}{2}.$$

Thus, the system of equations (1), (2) and (3) is consistent and has a solution.
Hence (1, 1, 1), (0, 1, 1) and (0, 1, −1) generate R^3.

Example 9. *Write the matrix* $E = \begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix}$ *as a linear combination of the matri[ces]* $A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}, B = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$ *and* $C = \begin{bmatrix}0 & 2\\ 0 & -1\end{bmatrix}$.

Solution. Let $E = xA + yB + zC$ where $x, y, z \in R$

Then,
$$\begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix} = x\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} + y\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} + z\begin{bmatrix}0 & 2\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}x & x\\ x & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ y & y\end{bmatrix} + \begin{bmatrix}0 & 2z\\ 0 & -z\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}x & x+2z\\ x+y & y-z\end{bmatrix}$$

$$\therefore \quad x = 3, x + 2z = 1, x + y = 1, y - z = -1$$

Solving these equations, we get

$$x = 3, y = -2, z = -1.$$

Since, these values also satisfy the last equation, they form a solution of the system.

Hence $\quad E = 3A - 2B - C$

Example 10. *Show that in the vector space* $V_n(F)$*, the system of n vectors* $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, 0, ..., 1)$ *is linearly independent where 1 denotes the unity of the field F.*

Solution. Let $a_1e_1 + a_2e_2 + ... + a_ne_n = 0$ for $a_i \in F$

Then $a_1(1, 0, 0, ..., 0) + a_2(0, 1, 0, ..., 0) + ... + a_n(0, 0, 0, ..., 1) = (0, 0, 0, ..., 0)$

$\Rightarrow \quad (a_1, a_2, ..., a_n) = (0, 0, 0, ..., 0)$

$\Rightarrow \quad a_1 = 0, a_2 = 0, ..., a_n = 0$

Hence, the set of n vectors $e_1, e_2, ..., e_n$ is linearly independent.

Remark

- In particular {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is a linearly subset of V_3 (F)

Example 11. *Prove that every superset of a linearly dependent set of vectors is linearly dependent.*

Solution. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a linearly dependent set of vectors. Then there exist $a_1, a_2, ... a_n$ not all zero such that

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = 0 \quad ...(1)$$

Let $S' = \{\alpha_1, \alpha_2, ... \alpha_n, \beta_1, \beta_2, ..., \beta_m\}$ be a superset of S.

Then, from (1), we have

$$a_1\alpha_1 + a_1\alpha_2 + ... + a_n\alpha_n + 0.\beta_1 + 0.\beta_2 + ... + 0.\beta_m = 0 \quad ...(2)$$

Therefore, in this relation the scalar coefficients are not all zero. Hence S' is linear[ly] dependent.

Example 12. *Show that the system of three vectors (1, 3, 2), (1, −7, −8), (2, 1, −1) of* V_3 *(R) is linearly dependent.*

Solution. Let $a, b, c \in R$ such that

$$a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = (0,0,0)$$

$$\Rightarrow \quad (a+b+2c, 3a-7b+c, 2a-8b-c) = (0,0,0)$$

$$\therefore \quad \left.\begin{aligned} a+b+2c &= 0\\ 3a-7b+c &= 0\\ 2a-8b-c &= 0\end{aligned}\right\} \quad ...(1)$$

The coefficient matrix is given by

$$A = \begin{bmatrix}1 & 1 & 2\\ 3 & -7 & 1\\ 2 & -8 & -1\end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{vmatrix}1 & 1 & 2\\ 3 & -7 & 1\\ 2 & -8 & -1\end{vmatrix}$$

$$= 1(7+8) - 1(-3-2) + 2(-24+14)$$

$$= 15 + 5 - 20 = 20 - 20 = 0$$

$\Rightarrow$ Rank of A is less than 3 which is less than the number of variables a, b, and c, therefore, the system (1) of homogeneous equations has a non-zero solution. Thus, a, b, c are not all zero. Hence the system of three given vectors (1, 3, 2), (1, −7, −8), (2, 1, −1) is linearly dependent.

Example 13. *Show that S={(1,2,4), (1, 0, 0), (0, 1, 0), (0, 0, 1)} is a linearly dependent subset of the vector space* V_3 *(R) where R is the field of real numbers.*

Solution. We know that the set {(1, 0, 0), (0, 1, 0), (0, 1, 0), (0, 0, 1)} is linearly independent of V_3 (R).

Now, $\quad (1, 2, 4) = 1(1, 0, 0) + 2(0, 1, 0) + 4(0, 0, 1)$

$\Rightarrow$ (1, 2, 4) is a linear combination of vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Hence, {(1, 2, 4), (1, 0, 0), (0, 1, 0), (0, 0, 1)} is linearly dependent.

Example 14. *If* α, β, γ *are linearly independent vectors of a vector space V(F) where F is any field of complex numbers, then so also are* $\alpha+\beta, \beta+\gamma, \gamma+\alpha$.

Solution. Let a, b, c be scalars such that

$$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0$$

$$\Rightarrow \quad (a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0 \quad ...(1)$$

Since α, β, γ are linearly independent, then all the coefficients in the relation (1) must be zero.

$$\therefore \quad \left.\begin{aligned} a+c &= 0\\ a+b &= 0\\ b+c &= 0\end{aligned}\right\} \quad ...(2)$$

Thus, equation (2) represents a system of homogeneous equations. The coefficient matrix of this system is given by

$$A = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$ $|A| = 1(1-0) + 1(1-0) = 2 \neq 0$

$\Rightarrow$ Therefore, the rank of $A = 3$ which is equal to the number of unknowns a, b, c which implies that the system of equations (2) has only zero solution, *i.e.*, $a = 0, b = 0$ and $c = 0$.

Hence $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also linearly independent.

Example 15. *Show that three vectors* $(1, 1, -1), (2, -3, 5)$ *and* $(-2, 1, 4)$ *of* R^3 *are linearly independent.*

Solution. Let $a, b, c \in R$ such that

$$a(1,1,-1) + b(2,-3,5) + c(-2,1,4) = (0,0,0)$$

$\Rightarrow$ $(a+2b-2c, a-3b+c, -a+5b+4c) = (0, 0, 0)$

$$\therefore \left.\begin{aligned} a+2b-2c &= 0 \\ a-3b+c &= 0 \\ -a+5b+4c &= 0 \end{aligned}\right\} \quad \text{...(1)}$$

Here, equation (2) represents a system of homogeneous equations. The coefficient matrix of this system is given by

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -3 & 1 \\ -1 & 5 & 4 \end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{vmatrix} 1 & 2 & -2 \\ 1 & -3 & 1 \\ -1 & 5 & 4 \end{vmatrix}$$

$\Rightarrow$ $|A| = 1(-12-5) - 2(4+1) - 2(5-3)$

$\Rightarrow$ $|A| = -17 - 10 - 4 = -31 \neq 0$

$\Rightarrow$ Rank $(A) = 3$

which is equal to the number of unknowns a, b, c. Therefore, the system (1) has only zero solution, *i.e.*, $a = 0, b = 0, c = 0$.

Hence, the vectors $(1, 1, -1), (2, -3, 5)$ and $(-2, 1, 4)$ of R^3 are linearly independent.

Example 16. *In* $V_3(R)$, *where R is the field of real numbers, examine each of the following sets of vectors for linear dependence:*

(i) $\{(1, 3, 2), (1, -7, -8), (2, 1, -1)\}$

(ii) $\{(0, 2, -4), (1, -2, -1), (1, -4, 3)\}$

(iii) $\{(1, 2, 0), (0, 3, 1), (-1, 0, 1)\}$

(iv) $\{(-1, 2, 1), (3, 0, -1), (-5, 4, 3)\}$

(v) $\{(2, 3, 5), (4, 9, 25)\}$

(vi) $\{2, 1, 2), (8, 4, 8)\}$

Solution. (i) Let $a, b, c \in R$ such that

$$a(1, 3, 2) + b(1, -7, -8) + c(2, 1, -1) = (0, 0, 0)$$

$\Rightarrow$ $(a+b+2c, 3a-7b+c, 2a-8b-c) = (0,0,0)$

$$\therefore \left.\begin{aligned} a+b+2c &= 0 \\ 3a-7b+c &= 0 \\ 2a-8b-c &= 0 \end{aligned}\right\} \quad \text{... (1)}$$

The coefficient matrix of the system of equations is given by

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{vmatrix}$$

$\Rightarrow$ $|A| = 1(7+8) - (-3-2) + 2(-24+14)$

$\Rightarrow$ $|A| = 15 + 5 - 20 = 0$

$\Rightarrow$ Rank $(A) < 3$.

$\Rightarrow$ Therefore, system (1) has non-zero solution.

Hence, the set $\{(1, 3, 2), (1, -7, -8), (2, 1, -1)\}$ is linearly dependent.

(ii) Let $a, b, c \in R$ such that

$$a(0, 2, -4) + b(1, -2, -1) + c(1, -4, 3) = (0, 0, 0)$$

$\Rightarrow$ $(b+c, 2a-2b-4c, -4a-b+3c) = (0, 0, 0)$

$$\therefore \left.\begin{aligned} b+c &= 0 \\ 2a-2b-4c &= 0 \\ -4a-b+3c &= 0 \end{aligned}\right\} \quad \text{...(1)}$$

The coefficient matrix of system (1) is given by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -2 & -4 \\ -4 & -1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{vmatrix} 0 & 1 & 1 \\ 2 & -2 & -4 \\ -4 & -1 & 3 \end{vmatrix}$$

$\Rightarrow$ $|A| = -1(6-18) + 1(-2-8)$

$\Rightarrow$ $|A| = 10 - 10 = 0$

Therefore, system (1) has non-zero solutions.

Hence $\{(0, 2, -4), (1, -2, -1), (1, -4, 3)\}$ is linearly dependent.

(iii) Let $a, b, c \in R$ such that

$$a(1,2,0) + b(0,3,1) + c(-1,0,1) = (0,0,0)$$

$\Rightarrow$ $(a-c, 2a+3b, b+c) = (0,0,0)$

$$\therefore \left.\begin{aligned} a-c &= 0 \\ 2a+3b &= 0 \\ b+c &= 0 \end{aligned}\right\} \quad \text{... (1)}$$

The coefficient matrix of system (1) is given by

$$A=\begin{bmatrix}1&0&-1\\2&3&0\\0&1&1\end{bmatrix}$$

$$\Rightarrow \quad |A|=\begin{vmatrix}1&0&-1\\2&3&0\\0&1&1\end{vmatrix}$$

$$\Rightarrow \quad |A|=1(3-0)-1(2-0)$$

$$\Rightarrow \quad |A|=3-2=1\neq 0$$

$$\Rightarrow \quad \text{Rank }(A)=3$$

which is equal to the number of unknowns a, b, c. Therefore, the system has only zero solution, *i.e.*, $a=0$, $b=0$, $c=0$.

Hence, {(1, 2, 0), (0, 3, 1), (–1, 0, 1)} is linearly independent.

(iv) Let $a, b, c \in$ R such that

$$a(-1,2,1)+b(3,0,-1)+c(-5,4,3)=(0,0,0)$$

$$\Rightarrow (-a+3b-5c, 2a+4c, a-b+3c)=(0,0,0)$$

$$\therefore \quad \left.\begin{aligned}-a+3b-5c&=0\\2a+4c&=0\\a-b+3c&=0\end{aligned}\right\} \quad \text{... (1)}$$

The coefficient matrix of the system (1) is given by

$$A=\begin{bmatrix}-1&3&-5\\2&0&4\\1&-1&3\end{bmatrix}$$

$$\Rightarrow \quad |A|=\begin{vmatrix}-1&3&-5\\2&0&4\\1&-1&3\end{vmatrix}$$

$$\Rightarrow \quad |A|=-1(0+4)-3(6-4)-5(-2-0)$$

$$\Rightarrow \quad |A|=-4-6+10=0$$

$$\Rightarrow \quad \text{Rank }(A)<3.$$

Therefore, the system (1) has non-zero solution.

Hence, the set {(–1, 2, 1), (3, 0, –1), (–5, 4, 3)} is linearly dependent.

(v) Let $a, b \in$R such that

$$a(2, 3, 5)+b(4, 9, 25) = (0, 0, 0)$$

$$\Rightarrow \quad (2a+4b, 3a+9b, 5a+25b) = (0, 0, 0)$$

$$\therefore \quad \left.\begin{aligned}2a+4b&=0\\3a+9b&=0\\5a+25b&=0\end{aligned}\right\} \quad \text{... (1)}$$

The coefficient matrix of the system (1) is given by

$$A=\begin{bmatrix}2&4\\3&9\\5&25\end{bmatrix}$$

Clearly, Rank A=2 which is equal to the number of unknowns a, b. Therefore, the system (1) has only zero solution, *i.e.*, $a=0, b=0$.

Hence, the set {(2, 3, 5), (4, 9, 25)} is linearly independent.

(vi) Let $a, b \in$ R, such that $a(2, 1, 2) + b(8, 4, 8) = (0, 0, 0)$

$$\Rightarrow \quad (2a+8b, a+4b, 2a+8b) = (0, 0, 0)$$

$$\therefore \quad \left.\begin{aligned}2a+8b&=0\\a+4b&=0\\2a+8b&=0\end{aligned}\right\}$$

The coefficient matrix of the system (1) is given by

$$A=\begin{bmatrix}2&8\\1&4\\2&8\end{bmatrix}$$

Clearly, Rank A = 2 which is equal to the number of unknowns a, b.

Therefore, the system has non-zero solution.

Hence, the set {(2, 1, 2), (8, 4, 8)} is linearly dependent.

Example 17. *If α_1, α_2 are vectors of V(F) and a, b∈F, show that the set $\{\alpha_1, \alpha_2, a\alpha_1+b\alpha_2\}$ is linearly dependent.*

Solution. We have

$$(-a)\alpha_1+(-b)\alpha_2+1(a\alpha_1+b\alpha_2)=(-a+a)\alpha_1+(-b+b)\alpha_2$$

$$=0\alpha_1+0\alpha_2$$

$$=0 \text{ (zero vector)}$$

$$\therefore \quad (-a)\alpha_1+(-b)\alpha_2+1(a\alpha_1+b\alpha_2)=0$$

Therefore, in this relation the scalar coefficients are $-a$, $-b$ and 1. As $1\neq 0$, therefore whatever may be the scalars $-a$ and $-b$, the set $\{\alpha_1,\alpha_2,a\alpha_1+b\alpha_2\}$ is linearly dependent.

Example 18. *In the vector space F[x] of all polynomials over the field F the infinite set $S=\{1,x,x^2,x^3,...\}$ is linearly independent.*

Solution. Let $S_n=\{x^{m_1},x^{m_2},x^{m_3},...,x^{m_n}\}$ be any finite subset of S having n vectors, where $m_1, m_2, m_3, ..., m_n$ are some non-negative integers.

Let $a_1, a_2, a_3, ..., a_n$ be scalars such that

$$a_1x^{m_1}+a_2x^{m_2}+a_3x^{m_3}+...+a_nx^{m_n}=0 \quad \text{(Zero polynomial)}$$

$$\Rightarrow \quad a_1x^{m_1}+a_2x^{m_2}+a_3x^{m_3}+...+a_nx^{m_n}=0+0.x+0.x^2+...$$

$$\Rightarrow \quad a_1=0, a_2=0, a_3=0, a_n=0$$

(By the definition of equality of two polynomials)

Thus every finite subset of S is linearly independent.

Hence, S is linearly independent.

Example 19 *Prove that in $R[x]$, the vector space of all polynomials in x over R, the system of $p(x) = 1+x+2x^2$, $q(x) = 2-x+x^2$, $r(x) = -4+5x+x^2$ is linearly dependent.*

Solution Let $a, b, c \in R$ such that

$$a\,p(x)+bq(x)+cr(x) = 0 \qquad \text{(zero polynomial)}$$

$$a(1+x+2x^2)+b(2-x+x^2)+c(-4+5x+x^2) = 0+0.x+0.x^2$$

$$\Rightarrow (a+2b-4c)+(a-b+5c)x+(2a+b+c)x^2 = 0+0.x+0.x^2$$

$$\Rightarrow \therefore \left.\begin{array}{r} a+2b-4c=0 \\ a-b+5c=0 \\ 2a+b+c=0 \end{array}\right\} \qquad \dots(1)$$

The coefficient matrix of the system of equivalent is given by

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow |A| = 1(-1-5)-2(1-10)-4(1+2)$$

$$\Rightarrow |A| = -6+18-12 = 0$$

$\Rightarrow$ Rank $A < 3$ which is the number of unknowns a, b, c.

Therefore, the system (1) has non-zero solution, i.e, not all a, b, c are zero.

Hence $p(x)$, $q(x)$, $r(x)$ are linearly dependent.

Example 20 *Show that the set $\{1, x, 1+x+x^2\}$ is a linearly independent set of vectors in the vector space of all polynomials over the field of real numbers.*

Solution Let $a, b, c \in R$ such that

$$a.1+bx+c(1+x+x^2) = 0 \qquad \text{(zero polynomial)}$$

$$\Rightarrow a+bx+c(1+x+x^2) = 0+0x+0x^2$$

$$\Rightarrow (a+c)+(b+c)x+cx^2 = 0+0x+0x^2$$

$$\left.\begin{array}{r} a+c=0 \\ b+c=0 \\ c=0 \end{array}\right\} \qquad \dots(1)$$

Solving the system (1) of equations, we get

$$a=0, b=0, c=0$$

Hence, $\{1, x, 1+x+x^2\}$ is linearly independent.

Example 21 *Let $V = R^3(R)$. Find a set of linearly independent vector of V which contains vector $(1, 1, 1)$.*

Solution We know that in $R^3(R)$ a set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independent.

Now, we have $(1, 1, 1) = (0, 0, 0) + (0, 1, 0) + (0, 0, 1)$

Thus, we consider a set $S = \{(1,0,0),(0,1,0),(1,1,1)\}$

Now we show that S is linearly independent.

Let $a, b, c \in R$ such that

$$a(1,0,0)+b(0,1,0)+c(1,1,1) = (0,0,0)$$

$$\Rightarrow (a+c, b+c, c) = (0,0,0)$$

$$\therefore \quad a+c=0, \quad b+c=0, \quad c=0$$

Solving these equations, we get

$$a=0, b=0, c=0$$

Hence, the set S is linearly independent which is a required set.

REMARK

- In this question, we may replace any one of the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0,0,1)$ by $(1, 1, 1)$.

Example 22 *Find a maximal linearly independent subsystem of the system of vectors $\alpha_1 = (2, -2, -4)$, $\alpha_2 = (1, 9, 3)$, $\alpha_3 = (-2, -4, 1)$ and $\alpha_4 = (3, 7, -1)$.*

Solution We know that A set S of linearly independent vectors is a maximal linearly independent set if every set of vectors which contains S as a proper subset is linearly dependent.

Let A denote the matrix whose rows are the vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4,

i.e.,
$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

or
$$A = \begin{bmatrix} 2 & -2 & -4 \\ 1 & 9 & 3 \\ -2 & -4 & 1 \\ 3 & 7 & -1 \end{bmatrix}$$

Now we shall reduce this matrix to Echelon form by using the row transformations.

Applying $R_1 \leftrightarrow R_2$, we have

$$A \sim \begin{bmatrix} 1 & 9 & 3 \\ 2 & -2 & -4 \\ -2 & -4 & 1 \\ 3 & 7 & -1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 + 2R_1, R_4 \to R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 9 & 3 \\ 0 & -20 & -10 \\ 0 & 14 & 7 \\ 0 & -20 & -10 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{20}R_2$

$$A \sim \begin{bmatrix} 1 & 9 & 3 \\ 0 & -1 & -1/2 \\ 0 & 14 & 7 \\ 0 & -20 & -10 \end{bmatrix}$$

Applying $R_3 \to R_3 - 14R_2, R_4 \to R_4 + 20R_2$

$$A \sim \begin{bmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in Echelon form having two non-zero rows.

$\Rightarrow$ Rank $A = 2$

Therefore, the maximum number of linearly independent row vectors in the matrix A = Rank $A = 2$.

Clearly, the vectors α_1 and α_2 are linearly independent.

Hence, $\{\alpha_1, \alpha_2\}$ is a maximal linearly independent subsystem of the given system of vectors.

Remark

- Here we observe that none of α_1, α_2, α_3, and α_4 is a scalar multiple of any of remaining three vectors, so that any two of the given four vectors form a maximal linearly independent subsystem of the given system of vectors.

Exercise 5.4

1. If S is a subspace of V, then prove that $L(S) = S$.
2. Show that $L(S)$ is the intersection of all the subspaces of a vector space V containing S.
3. Find k so that the vector $(1, k, 5)$ is a linear combination of $(1, -3, 2)$ and $(2, -1, 1)$.
4. For which value of k will the vector $(1, -2, k)$ in R^3 be a linear combination of the vectors $(3, 0, -2)$ and $(2, -1, -5)$?
5. Express the vector $(3, 4)$ in R^3 as a linear combination of the vectors $(1, 3)$ and $(-1, 1)$.
6. Is the vector $(3, -1, 0, -1)$ in the subspace of R^4 spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$ and $(1, 1, 9, -5)$?
7. Write $(1, 7, -4)$ as a linear combination of the vectors $(1, -3, 2)$ and $(2, -1, 1)$ in R^3.
8. Write $(2, -5, 4)$ as a linear combination of the vectors $(1, -3, 2)$ and $(2, -1, 1)$ in R^3.
9. Write E as a linear combination of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

where (i) $E = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ (ii) $E = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$.

10. Express $f(x)$ as a linear combination of the polynomials

$$f_1(x) = 2x^2 + 3x - 4, f_2(x) = x^2 - 2x - 3$$

where (i) $f(x) = 3x^2 + 8x - 5$

(ii) $f(x) = 4x^2 - 6x - 1$

11. Find conditions on a, b, $c \in R$ so that $(a, b, c) \in R^3$ belongs to the space spanned by the vectors $(2, 1, 0)$, $(1, -1, 2)$ and $(0, 3, -4)$.
12. Show that the vectors $(1, 2, 3)$, $(0, 1, 2)$ and $(0, 0, 1)$ generate R^3.
13. Show that the vectors $(1, 2, 5)$, $(1, 3, 7)$ and $(1, -1, -1)$ do not generate R^3.
14. Let V be the vector space of all polynomials in x, then show that the polynomials $1, x, x^2, x^3, \ldots$ generate V.

15. In a vector space R^4, let $\alpha_1 = (1,2,-1,3)$, $\alpha_2 = (2,4,1,-2)$, $\alpha_3 = (3,6,3,-7)$, $\beta_1 = (1,2,-4,11)$, $\beta_2 = (2,4,-5,14)$. Show that the subspace spanned by $S = (\alpha_1, \alpha_2, \alpha_3)$ and $T = (\beta_1, \beta_2)$ are the same.
16. In a vector space R^3, let $\alpha_1 = (1,1,-1)$, $\alpha_2 = (2,3,-1)$, $\alpha_3 = (3,1,-5)$, $\beta_1 = (1,-1,-3)$, $\beta_2 = (3,-2,-8)$ and $\beta_3 = (2,1,-3)$. Show that the subspace spanned by $S = \{\alpha_1, \alpha_2, \alpha_3\}$ and $T = \{\beta_1, \beta_2, \beta_3\}$ are the same.
17. Find one vector in R^3 which generates the intersection of W_1 and W_2 where $W_1 = \{(a,b,0) : a, b \in R\}$ and W_2 is the space generated by the vectors $(1, 2, 3)$ and $(1, -1, 1)$.
18. If α, β and γ are vectors such that $\alpha + \beta + \gamma = 0$, then show that α and β span the same subspace as β and γ.
19. Show that the vectors $(1, 0, -2)$, $(0, 2, 1)$, $(-1, 2, 3)$ are linearly dependent in R^3.
20. Show that the vectors $(1, 2, -1)$, $(-1, 1, 0)$, $(1, 3, -1)$ are linearly independent in R^3.
21. Show that the set $\{(1, 2, 1, 0), (3, -4, 5, 6), (2, -1, 3, 3), (-2, 6, -4, -6)\}$ of $V_4(R)$ is linearly dependent.
22. Examine whether the set of vectors $(2, 3, -1)$, $(-1, 4, -2)$ and $(1, 18, -4)$ is linearly dependent or not in $V_3(R)$.
23. Show that the vectors $(1, 1, 0, 0)$, $(0, 1, -1, 0)$ and $(0, 0, 0, 3)$ in R^4 are linearly independent.
24. Determine whether the set $\{(-1, 2, 1), (3, 1, -2)\}$ of vectors in V_3 (Q) is linearly dependent or independent, Q being the field of rational numbers.
25. Show that the vectors $(0, 2, -4)$, $(1, -2, -1)$, $(1, -4, 3)$ in R^3 are linearly dependent. Also express $(0, 2, -4)$ as a linear combination of $(1, -2, -1)$ and $(1, -4, 3)$.
26. Show that the vectors $(1, -2, 1)$, $(2, 1, -1)$, $(7, -4, 1)$ in R^3 are linearly dependent. Also express $(1, -2, 1)$ as a linear combination of $(2, 1, -1)$ and $(7, -4, 1)$.
27. Prove that the four vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ in V_3 (C) form a linearly dependent set but any of three of them are linearly independent.
28. Find a linearly independent subset T of the set $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where $\alpha_1 = (1, 2, -1)$, $\alpha_2 = (-3, -6, 3)$, $\alpha_3 = (2, 1, 3)$, $\alpha_4 = (8, 7, 7)$ in R^3 which spans the same space as S.
29. Let α_1, α_2, α_3, be the vectors of $V(F)$, a, $b \in F$. Show that the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent if the set $\{\alpha_1 + a\alpha_2 + b\alpha_3, \alpha_2, \alpha_3\}$ is linearly dependent.
30. If α, β, γ are linearly independent vectors of $V(F)$ where F is the field of complex numbers, then prove that:

    (i) $\alpha + \beta, \alpha - \beta, \alpha - 2\beta + \gamma$

    (ii) $\alpha + 3\beta - 2\gamma, 2\alpha + \beta - \gamma, 3\alpha + \beta + \gamma$

    are also linearly independent.
31. Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ be a linearly independent set of vectors in R^n:

    (i) Prove that the set $\{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_1 + \alpha_3\}$ is linearly independent.

    (ii) Prove that the set $\{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_1 - \alpha_3\}$ is linearly dependent.
32. Let $\{\alpha_1, \alpha_2, \ldots \alpha_{r-1}, \alpha_r, \alpha_{r+1}, \ldots, \alpha_k\}$ be a linearly independent set of k vectors in R^n and let $\beta_r = \sum_{j=1}^{k} a_j \alpha_j$ with $a_r \neq 0$. Prove that $\{\alpha_1, \alpha_2, \ldots \alpha_{r-1}, \beta_r, \alpha_{r+1}, \ldots, \alpha_k\}$ is linearly independent.
33. Show that there is one vector in the set $\{(1, 1, 0), (0, 1, 1), (1, 0, -1), (1, 0, 1)\}$ which cannot be written as a linear combination of the other vectors in the set.
34. If $V(F)$ is a vector space and $\alpha_1, \alpha_2, \ldots \alpha_n \in V$ and $\lambda_2, \lambda_3, \ldots \lambda_n \in F$ such that $\{\alpha_2 + \lambda_2\alpha_1, \alpha_3 + \lambda_3\alpha_1, \ldots, \alpha_n + \lambda_n\alpha_1\}$ is linearly dependent then show that $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is linearly dependent.
35. Show that the set $\{1, x, x(1-x)\}$ is a linearly independent set of vectors in the space of all polynomials over the real field.
36. Find whether the vectors $2x^3 + x^2 + x + 1$, $x^3 + 3x^2 + x - 2$ and $x^3 + 2x^2 - x + 3$ of $R[x]$, the vector space of all polynomials over the real numer field, and linearly independent or not.
37. Show the set of functions $\{x, |x|\}$ is linearly independent in vector space of continuous functions defined in $(-1, 1)$.
38. Let V be the vector space of functions from R into R. Show that f, g, $h \in V$ are linearly independent where:

    (i) $f(x) = e^{2x}, g(x) = x^2, h(x) = x$

    (ii) $f(x) = \sin x, g(x) = \cos x, h(x) = x$.
39. Prove that a set of vectors which contains zero vector is linearly dependent.
40. Prove that a set consisting of exactly one non-zero vector is linearly independent.

41. Prove that the three non-coplanar vectors are linearly independent.

42. Show that any three non-zero non-coplanar vectors are linearly dependent in R^3.

43. Let V be the vector space of 2×2 matrices over R. Determine whether that matrices

$$A=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}, B=\begin{bmatrix}3 & -1\\ 2 & 2\end{bmatrix}, C=\begin{bmatrix}1 & -5\\ -4 & 0\end{bmatrix}$$

of V are linearly dependent.

44. In each case, determine whether or not the given vector α is linearly dependent on the given set S:

(i) $\alpha=(-2,1,4), S=\{(1,1,1),(0,1,1),(0,0,1)\}$

(ii) $\alpha=(1,2,1), S=\{(1,0,-2),(0,2,1),(1,2,-1),(-1,2,3)\}$

(iii) $\alpha=(-4,4,2), S=\{(2,1,-3),(1,-1,3)\}$

(iv) $\alpha=(1,2,3), S=\{(1,1,1),(0,1,1),(0,0,1)\}$

(v) $\alpha=(2,13,-5), S=\{(1,2,-1),(3,6,-3),(-1,1,0),(0,6,-2),(2,4,-2)\}$

45. Show that every vector in R^3 is dependent vector on the set $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1=(1,0,0)$, $\alpha_2=(1,1,0)$ and $\alpha_3=(1,1,1)$.

46. Find all subsets of maximal linear independent vectors from the following set of vectors $\{(1,0,1,0),(0,1,0,2),(1,1,0,1),(-1,0,2,0)\}$.

47. Test the vectors $(0,1,0,1,1)$, $(1,0,1,0,1)$, $(0,1,0,1,1)$ and $(1,1,1,1,1)$ in V_5 over the field of rational numbers for linear independence.

Answers

3. $k=-8$ **4.** $k=-8$ **5.** $(3,4)=\frac{7}{2}(1,3)+\frac{1}{2}(-1,1)$ **6.** No **7.** $(1,7,-4)=-3(1,-3,2)+2(2,-1,1)$

8. $(2,-5,4)$ is not a linear combination of $(1,-3,2)$ and $(2,-1,1)$ **9. (i)** $E=2A-B+2C$ **(ii)** Not possible **10.** (i) $f(x)=2f_1(x)-f_2(x)$ (ii) Not possible **11.** $2a-4b-3c=0$ **17.** $(-2,5,0)$

22. linearly independent **24.** linearly independent **28.** $\{\alpha_1, \alpha_3\}$ **36.** linearly independent

44. (i) linearly dependent (ii) linearly independent (iii) linearly independent (iv) linearly dependent (v) linearly dependent **46.** all four vectors **47.** linearly dependent

5.12 BASIS OF A VECTOR SPACE

Definition. *Let V be a vector space over a field F and let S be any non-empty subset of V. Then S is said to be a basis of V if*

(i) *S is linearly independent.*

(ii) *$L(S)=V$, i.e., every element of V is a linear combination of finite elements of S.*

For Example: The set $S=\{(1,0,0),(0,1,0),(0,0,1)\}$ forms a basis of $V_3(R)$, and is called usual basis.

Remarks

- The zero space has no basis.
- Every finitely generated vector space has a basis.
- Every non-zero vector space has a basis.
- A vector space many have more than one basis.

5.13 FINITE DIMENSIONAL VECTOR SPACE

Let $V(F)$ be a vector space over a field F and let S be any non-empty subset of V, then $V(F)$ is said to be finite dimensional if S is finite subset of V such that $L(S)=V$. If this set contains n elements, then the dimension of V is n.

THEOREM 1. *If $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is the basis of a vector space $V(F)$, then each element of V is uniquely expressible as a linear combination of elements of S.*

Proof. Since S is the basis of a vector space $V(F)$, then by the definition of basis, each element of V is a linear combination of elements of S. Thus, we only show the uniqueness. Let there be two different sets $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ of scalars corresponding to an element $\alpha \in V$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

and $$\alpha = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n \quad \dots(1)$$

$\Rightarrow \quad a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$

$\Rightarrow \quad a_1\alpha_1 - b_1\alpha_1 + a_2\alpha_2 - b_2\alpha_2 + \dots + a_n\alpha_n - b_n\alpha_n = 0$

$\Rightarrow \quad (a_1-b_1)\alpha_1 + (a_2-b_2)\alpha_2 + \dots + (a_n-b_n)\alpha_n = 0$

Since the set $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is linearly independent so that

$$a_1-b_1=0,\ a_2-b_2=0, \dots, a_n-b_n=0$$

$\Rightarrow \quad a_1=b_1, \quad a_2=b_2, \dots, \quad a_n=b_n.$

Hence, the expression (1) is unique.

THEOREM 2. **(Existence Theorem).** *Every finitely generated vector space has a finite basis.*

Step Outlines : To make the proof easier, use the following steps :

Step 1. *If S is linearly independent, that theorem is obvious.*

Step 2. *Assume S is linearly dependent and obtained the set $S_1=(\alpha_1, \alpha_2, \dots \alpha_{k-1}, \alpha_{k+1}, \dots \alpha_n)$ by eliminating α_k from S.*

Step 3. *Again if S_1 is linearly independent, result is obvious. And if S_1 is linearly dependent, proceed same as in step-2.*

Proof. Let V be a vector space over F which is generated by a finite set (say) $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of vectors of V. Without loss of any generality we may assume that all the elements in S are non-zero, because zero vector in the linear combination of elements of S is zero.

If S is linearly independent, then S forms a finite basis for V and in this case theorem is proved. So, we assume that S is linearly dependent, then there exists some α_k $(2 \le k \le n)$ in S such that α_k is a linear combination of preceding vectors $\alpha_1, \alpha_2, \dots \alpha_{k-1}$. Therefore,

$$\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{k-1}\alpha_{k-1} = \sum_{i=1}^{k-1} a_i\alpha_i \quad \dots(1)$$

for some scalars $a_i's \in F$.

But S generates V so that an arbitrary element $\alpha \in V$ is expressible as a linear combination of elements of S.

$\therefore \quad \alpha = b_1\alpha_1 + b_2\alpha_2 + \dots + b_k\alpha_k + \dots + b_n\alpha_n$

$\quad = \sum_{i \ne k} b_i\alpha_i + b_k\alpha_k$, for some $b_i's \in F$

$$= \sum_{i \neq k} b_i \alpha_i + b_k \sum_{i=1}^{k-1} a_i \alpha_i$$

$$= (b_1 + b_k a_1)\alpha_1 + (b_2 + b_k a_2)\alpha_2 + \ldots + (b_{k-1} + b_k a_{k-1})\alpha_{k-1} + (b_{k+1})\,\alpha_{k+1} + \ldots + b_n \alpha_n$$

$\Rightarrow$ α is a linear combination of $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1}$.

Thus the set,

$$S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{k+1} \ldots, \alpha_k\}.$$

Obtained by eliminating some α_k from S also generates V.

If S_1 in linearly independent, then S_1 will form a basis of V and the theorem is proved in this case. If S_1 is linearly dependent, then by above process, we obtain a new set

$$S_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$$

by eliminating some α_i $(i > k)$ from S_1, which generates V. If S_2 is linearly independent, then S_2 will form a basis. If S_2 is linearly dependent, then we continue the above process, till after a finite number of steps we obtain a linearly independent set which generates V. At the most by repeating the above process we may obtain a singleton set which is always linearly independent and it generating V and will form a basis of V. Hence, in every finitely generated vector space there exists a finite basis.

THEOREM 3. *If V is a finite-dimensional vector space and if $a_1, a_2, \ldots, a_m$ span V, then some subset of $\alpha_1, \alpha_2, \ldots, \alpha_m$ forms a basis of V.*

Proof. Since a finite-dimensional vector space has a basis of V containing a finite number of elements. Let these vectors be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus every element in V has a unique representation of the form

$$a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n; \text{ for } a_1, a_2, \ldots, a_n \in F.$$

If $\alpha \in V$, then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$$

Now define a map ϕ from V into $F^{(n)}$ by

$$\phi(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n) = (a_1, a_2, \ldots, a_n)$$

Since $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ is a unique representation so that ϕ is well defined, one-to-one and onto and also preserves the composition.

Thus V is isomorphic to $F^{(n)}$ for some n, where n is the number of elements in some basis of V over F. If some other basis of V has m elements, then V would be isomorphic to $F^{(m)}$. Since both $F^{(n)}$ and $F^{(m)}$ are isomorphic to V, therefore, $F^{(n)}$ and $F^{(m)}$ are isomorphic to each other. This implies $n = m$. Hence the theorem.

THEOREM 4. *If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of $V(F)$ and if $\beta_1, \beta_2, \ldots, \beta_m \in V$, are linearly independent over F, then $m \leq n$.*

Proof. Since the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of $V(F)$, then every element in V is a linear combination of the elements of basis, so in particular, β_m is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Therefore, the set $\{\beta_m, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly dependent. Also, this set spans V because $\alpha_1, \alpha_2, \ldots, \alpha_n$ span V. Thus proper subset of the set $\{\beta_m, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ forms a basis of V. Let this proper set be $\{\beta_m, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $k \leq n - 1$. In forming this new basis at least one α_i is replaced by some one β_j. Repeat this procedure with the set $\{\beta_{m-1}, \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}$ which is obviously linearly dependent, so we can extract a basis of the form $\{\beta_{m-1}, \beta_m, \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_s}\}$ with $s \leq n - 2$. Continuing this procedure with the set $\{\beta_2, \beta_3, \ldots, \beta_{m-1}, \beta_m, \alpha_x, \alpha_y\}$. Since β_1 is not a linear combination of $\{\beta_2, \beta_3, \ldots, \beta_{m-1}, \beta_m, \alpha_x, \alpha_y\}$, therefore, the basis $\{\beta_2, \beta_3, \ldots, \beta_{m-1}, \beta_m, \alpha_x, \alpha_y\}$ must contain some α's. To get this basis. We have introduced $m-1$ β's and each such introduction costs at least one α's and yet there is an 'α' left. Thus $m - 1 \leq n - 1$ implying $m \leq n$.

THEOREM 5. *If V is a finite-dimensional vector space over F, then any two bases of V have the same number of elements.*

Proof. Let S_1 and S_2 be any two bases of $V(F)$ and let

$$S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \text{ and } S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}.$$

Then we shall have to show that $m = n$.

Since S_1 forms a basis of V, so that every element of V is uniquely expressible as a linear combination of the elements of S_1. In particular β_1 is uniquely expressible as a linear combination of S_1. Thus the set $S_3 = \{\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$ is now linearly dependent.

Therefore, there exists an element α_k in S_s which is linear combination of proceeding ones, $\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}$. But every element of V can be expressed as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_m$. Also, α_k is a linear combination of $\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}$. This implies that each element of V is expressible as a linear combination of $\{\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \ldots, \alpha_m\}$. Thus the set

$$S_4 = \{\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_m\}$$

generates V. This set is obtained by adjoining β_1 to S_3 and eliminating α_k and S_3. Since $\beta_2 \in V$ so it is the linear combination of elements of S_4, therefore, the set is obtained by adjoining β_2 to S_4 and eliminating α_1 as before. Thus, the set $S_5 = \{\beta_2, \beta_1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_m\}$ generates V.

Continuing the above manner, we observe that each step consists of an inclusion of one β' s and the exclusion of an α's and the resulting set generates V. Since all the α's cannot be exhausted before the β's. If it is so then a proper subset of S_2 generates V which is a contradiction of linear independence of S_2. Hence $m < n$. Similarly, if we change the role of S_1 and S_2, we obtain $n < m$. Hence $m = n$.

THEOREM 6. **(Extension Theorem).** *If $V(F)$ is a finite dimensional vector space, then every linearly independent subset of V is either a basis of V or can be extended to form a basis of V.*

Proof. Suppose the dimension of $V = n$. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a basis of V and let $S_1 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be a linearly independent subset of V.

Since S is a basis of V so that every element of V is expressible as a linear combination of elements of S, in particular, β_m is a linear combination of elements of S, therefore, the set obtained by adjoining β_m to S is linearly dependent. Since, the superset of a linearly dependent set is linearly dependent, it follows that the set,

$$S_2 = \{\beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n\}$$

is linearly dependent.

Since the set S_1 is linearly independent, then there exists an element α_k in S_2 which can be expressed as a linear combination of preceding ones $\beta_1, \beta_2, \ldots, \beta_m, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}$, therefore each element in V can be expressed as a linear combination of $\beta_1, \beta_2, \ldots, \beta_m, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. Thus the set

$$S_3 = \{\beta_1, \beta_2, \ldots, \beta_m, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n\}$$

is obtained be eliminating α_k from S_2 which generates V. If S_3 is linearly independent, then S_3 forms a basis of V containing S_1. Thus, in this case the theorem is proved.

If S_3 is linearly dependent, then repeat the above process of adjoining and eliminating, till after a finite number of steps we obtain a linearly independent set which generates V and contains S_1. At the most by repeating the above process we may obtain the set S_1 itself which generates V and being linearly independent will form a basis of V.

Hence, either S_1 is a basis of V or can be extended to form a basis of V.

THEOREM 7. *Let V be a finite-dimensional vector space and let dim. $V = n$. Then*

(i) *any subset of V which contains more than n vectors is linearly dependent.*

(ii) *no subset of V which contains less than n vectors can span V.*

Proof. (i) Since $V(F)$ is n-dimensional, every basis of V will contain n vectors. Let S be any subset of V which contains more than n vectors. Let, if possible, S be linearly independent, then by previous theorem either S is a basis of V or can be extended to form a basis of V. But in both cases the basis of V contains more than n elements which is contradictory to the fact that V is n-dimensional. Hence S is linearly dependent.

(ii) Let S be a subset of V which contains less than n elements and which can span V. Then every element of V can be expressed as a linear combination of elements of S. If S is linearly independent, then S forms a basis of V which show that dim. $V < n$ which is contradictory to the fact that dim. $V = n$. On the other hand it S is linearly dependent, then S cannot span V which is again a contradiction because we have assumed that S spans V. Hence in both cases such subset S cannot exist. Consequently no subset of V containing less than n vectors can span V.

5.14 DIMENSION OF SUBSPACE OF A VECTOR SPACE

THEOREM 1. *Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.*

Proof. Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a linearly independent subset of V. Then we shall show that the set

$$S_1 = \{\beta, \alpha_1, \alpha_2, \ldots \alpha_n\}$$

obtained by adjoining β to S is also linearly independent where $\beta \in V$, but not in the subspace of V which is spanned by S.

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct vectors in S such that

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n + b\beta = 0$$

where, all a's. are zero. ...(1)

We actually show that $b = 0$. Let, if possible, $b \neq 0$. Then from (1), we have

$$\beta = \left(-\frac{a_1}{b}\right)\alpha_1 + \left(-\frac{a_2}{b}\right)\alpha_2 + \ldots + \left(-\frac{a_n}{b}\right)\alpha_n$$

$\Rightarrow$ β is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$\Rightarrow$ β is in the subspace of V spanned by $\alpha_1, \alpha_2, \ldots, \alpha_n$.

But it is contradictory to the hypothesis that β is not in the subspace spanned by S. Hence $b = 0$. Consequently, the set S_1 is linearly independent.

THEOREM 2. *If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is a part of a (finite) basis for W.*

Proof. Let S be a linear independent subset of W and let S_1 be a linearly independent subset of W containing S. Then S_1 contains not more than dim. V elements.

If S spans W, then S is a basis for W and in this case, theorem is proved. If S does not span W, then we find a vector β_1 in W such that the set

$$S_2 = S \cup \{\beta_1\}$$

is linearly independent (By theorem 1). If S_2 spans W, it will form a basis for W, If S_2 does not span W, then again we find a vector β_2 in W such that the set

$$S_3 = S \cup \{\beta_1, \beta_2\}$$

is linearly independent.

Continuing above process up to finite number of steps less than dim. V, we get a set,

$$S_m = S \cup \{\beta_1, \beta_2, \ldots, \beta_{m-1}\}$$

which is linearly independent and is a basis for W or a part of basis for W.

THEOREM 3. *If $V(F)$ is a finite-dimensional vector space and W is a subspace of V, then W is finite dimensional and dim. $W \leq$ dim V. In particulars, if W is a proper subspace of V, then dim. $W <$ dim. V. Also $V = W$ if and only if dim. $V =$ dim. W.*

Proof. Let dim. $V = n$. Then every basis of V will contain n vectors of V, therefore, every subset having vectors more than n will be linearly dependent. Thus, a linearly independent set of vectors in W contains at most n-elements. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, with $m \geq n$ being a maximal linearly independent set in W. If α is an arbitrary element in W, then the set

$$S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$$

will be linearly dependent because S being a maximal linearly independent set. Therefore, α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_m$ which shows that S spans V. Hence W is finite-dimensional. Also, dim. $W = m \leq n =$ dim. V.

$\therefore$ dim. $W <$ dim. V.

Next, if $V = W$, then every basis of V is also the basis of W which shows that dim. V = dim W. On the other hand, if dim. V = dim. W, then every basis of W will contain the vectors equal to dim. V so it will also generate V. Thus each one of V and W is generated by some basis. Hence $V = W$.

THEOREM 4 **(Existence of Complementary Subspace).** *Every subspace of a finite dimensional vector space has a complement.*

Proof Let $V(F)$ be a finite dimensional vector space and let W_1 be its subspace. Then our aim is to find out a subspace W_2 of V such that $V = W_1 \oplus W_2$. Since $V(F)$ is a finite dimensional vector space so that W_1 will be finite dimensional. Let $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the basis of W_1, then by extension theorem, we have, S_1 can be extended to form a basis of V. Let this extended set be

$$S_2 = \{\alpha_1, \alpha_2, ...,\alpha_n, \beta_1, \beta_2, ...,\beta_m\} \text{ be the basis of } V.$$

Let us suppose that the set $\{\beta_1, \beta_2, ...,\beta_m\}$ generates a subspace and let this subspace be W_2.

Now we shall show that $V = W_1 \oplus W_2$ or equivalently $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Let γ be an arbitrary element of V and S_2 being the basis for V, then there exists scalars $a_1, a_2, ...,a_n, b_1, b_2, ...,b_m$ such that

$$\gamma = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n + b_1\beta_1 + b_2\beta_2 + ... + b_m\beta_m = \alpha + \beta$$

Where $\quad \alpha = \sum_{i=1}^{n} a_i\alpha_i$ and $\beta = \sum_{j=1}^{m} b_j\beta_j$

Since $\{\alpha_1, \alpha_2, ...,\alpha_n\}$ generates W_1 and $\{\beta_1, \beta_2, ,...,\beta_m\}$ generates W_2, therefore $\alpha \in W_1$ and $\beta \in W_2$.

Thus, every element of V is expressible as the sum of an element of W_1 and an element of W_2.

$\therefore \quad V = W_1 + W_2$

Again, $\quad \alpha = \sum_{i=1}^{n} a_i\alpha_i \in W_1$ and $\beta = \sum_{j=1}^{m} b_j\beta_j \in W_2$

Let if possible $W_1 \cap W_2 \neq \{0\}$. Then there exists a non-zero element which belong to both W_1 and W_2. Let it be x. Then $x \in W_1$ and $x \in W_2$.

$$\therefore \quad x = \sum_{i=1}^{n} a_i'.\alpha_i \text{ and } x = \sum_{j=1}^{m} b_j'.\beta_j$$

$$\Rightarrow \quad \sum_{i=1}^{n} a_i'.\alpha_i = \sum_{j=1}^{m} b_j'.\beta_j$$

$$\Rightarrow \quad \sum_{i=1}^{n} a_i'.\alpha_i + \sum_{j=1}^{m} (-b_j').\beta_j = 0$$

Since the set S_2 is linearly independent, so that

$$a_i' = 0, \text{ for each } i = 1, 2 ,.., n$$

and $\quad b_j' = 0,$, for each $i = 1, 2 ,.., m$

$\Rightarrow \quad x = 0$

which is a contradiction, because we have taken $x \neq 0$.

Thus, the contradiction arises by assuming that $W_1 \cap W_2 \neq \{0\}$

Hence, $W_1 \cap W_2 = \{0\}$.

Consequently

$$V = W_1 \oplus W_2.$$

REMARK

- Here W_2 is the subspace complementary to the subspace W_2 of finite dimensional vector space V.

THEOREM 5. *If W_1 and W_2 are two finite dimensional subspaces of a vector space $V(F)$, then $W_1 + W_2$ is finite dimensional and dim.W_1+dim.W_2=dim.$(W_1 \cap W_2)$+dim.$(W_1 + W_2)$.*

Proof Since W_1 and W_2 are subspaces of V so that $W_1 \cap W_2$ will be a subspace of V and its dimension is finite. Let dim. $W_1 = m$, dim. $W_2 = n$ and dim. $(W_1 \cap W_2) = r$.

Let $\{\alpha_1, \alpha_2, ..., \alpha_r\}$ be a basis of $W_1 \cap W_2$. Therefore, we can extend this basis to a basis of W_1 and also to a basis of W_2.

Let, $\quad S_1 = \{\alpha_1, \alpha_2, ... \alpha_r, \beta_1, \beta_2, ... \beta_{m-r}\}$

and $\quad S_2 = \{\alpha_1, \alpha_2, ... \alpha_r, \gamma_1, \gamma_2, ... \gamma_{n-r}\}$

be the basis of W_1 and W_2 respectively. Consider the set

$$S = \{\alpha_1, \alpha_2, ... \alpha_r, \beta_1, \beta_2, ... \beta_{m-r}, \gamma_1, \gamma_2, ... \gamma_{n-r}\}$$

Now we have to show that S will form a basis for $W_1 + W_2$. For this, we shall show that S is linearly independent and spans $W_1 + W_2$. For this, suppose

$$\sum a_i\alpha_i + \sum b_j\beta_j + \sum c_k\gamma_k = 0 \text{ for } a_i' s, b_j' s, c_k' s \in F.$$

Then $\sum c_k\gamma_k = \sum a_i\alpha_i + \sum b_j\beta_j \Rightarrow \sum c_k\gamma_k \in W_1$

Also, $\sum c_k\gamma_k \in W_2$. It follows that $\sum c_k\gamma_k \in W_1 \cap W_2$ and we have $\sum c_k\gamma_k \in d_i\alpha_i$ for some scalars $d_1, d_2, ..., d_r$. Since the set $\{\alpha_1, \alpha_2, ... \alpha_r, \gamma_1, \gamma_2, ... \gamma_{n-r}\}$ is linearly independent hence all the scalars $c_1 = 0 = c_2 = ... = c_{n-r}$.

Thus $\sum a_i\alpha_i + \sum b_j\beta_j = 0$

and since the set $\{\alpha_1, \alpha_2, ... \alpha_r, \beta_1, \beta_2, ... \beta_{m-r}\}$ is also linearly independent,

then

$$a_1 = 0 = a_2 = ... = a_r$$

and $\quad b_1 = 0 = b_2 = ... = b_{m-r}$

Thus the set $S = \{\alpha_1, \alpha_2, ... \alpha_r, \beta_1, \beta_2, ... \beta_{m-r}, \gamma_1, \gamma_2, ... \gamma_{n-r}\}$ is linearly independent.

Now we shall show that S spans $W_1 + W_2$.

Let α be an arbitrary element of $W_1 + W_2$ then it can be written as $\alpha = \beta + \gamma$ with $\beta \in W_1$ and $\gamma \in W_2$. Now S_1 and S_2 being the basis of W_1 and W_2 respectively, β and γ can be expressed uniquely in the form

$$\beta = \sum_{i=1}^{r} a_i\alpha_i + \sum_{j=1}^{n-r} b_j\beta_j \text{, for some } a_i's \text{ and } b_j's.$$

and
$$\gamma = \sum_{i=1}^{r} e_i\alpha_i + \sum_{j=1}^{n-r} c_j\gamma_j \text{, for some } e_i\text{'s and } c_j\text{'s.}$$

$\therefore$
$$\alpha = \beta + \gamma = \sum_{i=1}^{r} (a_i + e_i)\alpha_i + \sum_{j=1}^{n-r} b_j\beta_j + \sum_{j=1}^{n-r} c_j\gamma_j .$$

$\Rightarrow$ α is a linear combination of elements of S.

$\Rightarrow$ S spans W_1+W_2.

Hence, S is basis of W_1+W_2 so that W_1+W_2 is finite-dimensional with dimensional $(m+n-r)$.

Finally,
$$\dim W_1 + \dim W_2 = m+n = r+(m+n-r)$$
$$= \dim .(W_1 \cap W_2) + \dim .(W_1 + W_2).$$

THEOREM 6. *If a finite dimensional vector space $V(F)$ be the direct sum of its two subspaces W_1 and W_2, then $\dim.V = \dim. W_1 + \dim.W_2$.*

Proof. Since V is finite dimensional, therefore W_1 and W_2 are also finite-dimensional.

Let
$$\dim.W_1 = m,\ \dim.W_2 = n$$

Also
$$V = W_1 \oplus W_2 \text{, implying that}$$

(i) $V=W_1+W_2$

(ii) $W_1 \cap W_2=\{0\}$.

Let $S_1=\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis of W_1 and the set $S_2=\{\beta_1, \beta_2,..., \beta_n\}$ is a basis of W_2.

Now consider a set
$$S_3=\{\alpha_1, \alpha_2,...,\alpha_m, \beta_1,\beta_2,..., \beta_n\}$$

We claim that S_3 forms a basis of V.

For some scalars $a_1,a_2,...,a_m$, $b_1,b_2,..., b_n \in F$, we have
$$a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m+b_1\beta_1+b_2\beta_2+.....+b_n\beta_n=0$$

$\Rightarrow$ $a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m = -(b_1\beta_1+b_2\beta_2+.....+b_n\beta_n)$

$\Rightarrow$ $a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m \in W_1$ as $b_1\beta_1+b_2\beta_2+.....+b_n\beta_n \in W_2$

and $b_1\beta_1+b_2\beta_2+......+b_n\beta_n \in W_2$ as $a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m \in W_1$

$\Rightarrow$ $a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m \in W_1 \cap W_2$

$\Rightarrow$ $b_1\beta_1+b_2\beta_2+......+b_n\beta_n \in W_1 \cap W_2$

But from(i) $W_1 \cap W_2=\{0\}$.

$\Rightarrow$ $a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m = 0$

and $b_1\beta_1+b_2\beta_2+......+b_n\beta_n = 0$

Since S_1 and S_2 both are linearly independent, therefore
$$a_1=0=a_2= ....=a_m,\ b_1=0=b_2= ....=b_n.$$

$\Rightarrow$ S_1 is linearly independent.

Next, let γ be an arbitrary element of V, then
$$\gamma=\alpha+\beta,\ \alpha \in W_1,\ \beta \in W_2 \qquad [\because V=W_1+W_2]$$

Since $\alpha \in W_1 \Rightarrow \alpha \in a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m$ for some a_i's $\in F$.

and $\beta \in W_2 \Rightarrow \beta \in b_1\beta_1+b_2\beta_2+......+b_m\beta_m$ for some b_i's $\in F$.

$\therefore$
$$\gamma = \alpha+\beta = a_1\alpha_1+a_2\alpha_2+......+a_m\alpha_m+b_1\beta_1+b_2\beta_2+.....+b_n\beta_n$$

$\Rightarrow$ S_3 generates V, thus S_3 forms a basis of V.

Accordingly, $\dim.V = m+n = \dim.W_1 + \dim.W_2$.

REMARK

- It should be noted that in case of finite dimensional spaces, since a basis is a maximal linearly independent subset of the vector space, so the dimension of a finite dimensional vector spaces may be regarded as the maximum of numbers of elements in all linearly independent subsets. If this is adopted as definition of the dimension of a vector space, is n iff it has a basis consisting of n elements.

Solved Examples

Example 1. *Let V be the vector space of ordered pairs of complex numbers over the real field R, i.e., let V be the vector space $C^2(R)$. Show that the set $S=\{(1, 0), (i, 0), (0, 1), (0, i)\}$ is a basis for V.*

Solution. First, we shall show that S is linearly independent.

Let $a, b, c, d \in R$, such that
$$a(1,0)+b(i,0)+c(0,1)+d(0,i)=(0,0)$$

$\Rightarrow$
$$(a+ib, c+id)=(0,0)$$

$\therefore$
$$\left.\begin{array}{r} a+ib=0 \\ c+id=0 \end{array}\right\} \qquad ...(1)$$

Solving the system (1), we get
$$a = 0, b = 0, c = 0, d = 0$$

Therefore, S in linearly independent.

Now, we shall show that $L(S) = V$.

Let $(a+ib, c+id)$ be any element of V where $a, b, c, d \in R$.

Then, $(a+ib, c+id) = a(1, 0)+b(i, 0)+c(0, 1) +d(0, i)$.

Therefore, every element of V can be expressible as a linear combination of the elements of S.

Thus, $L(S)= V$

Hence, S is a basis of V.

Example 2. *Show that the set $S=\{(1, 2), (3, 4)\}$ forms a basis for R^2.*

Solution. Since the dimension of R^2 is 2 and S contains 2 elements.

Now we shall show that S is linearly independent.

Let $a, b \in R$ such that
$$a(1,2)+b(3,4)=(0,0)$$

$\Rightarrow$
$$(a+3b, 2a+4b)=(0,0) \qquad ...(1)$$

$\therefore$
$$\left.\begin{array}{r} a+3b=0 \\ 2a+4b=0 \end{array}\right\}$$

The coefficient matrix of the system (1) is
$$A=\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

$\Rightarrow$ $|A| = 4 - 6 = -2 \neq 0$

$\Rightarrow$ Rank $A = 2$ which is equal to number of unknowns.

Therefore, the system (1) has only zero solution, *i.e.*, $a = 0$, $b = 0$ so that S is linearly independent.

Hence S forms a basis of R^2.

Example 3. *Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.*

Solution. Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{and } \alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are the four elements of V.

Now we shall show that S forms a basis of V.

Let $a, b, c, d \in F$ such that

$$a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 = 0$$

$$\Rightarrow a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $a = 0, b = 0, c = 0, d = 0$

Therefore, S is linearly independent.

Next, we shall show that $L(S) = V$.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any element of V.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4$

$\Rightarrow$ $L(S) = V$

Hence S forms a basis of V which has four elements, therefore

$$\dim V = 4$$

Example 4. *Let $\alpha = (1, 2, 1), \beta = (2, 9, 0)$ and $\gamma = (3, 3, 4)$. Show that the set $S = \{\alpha, \beta, \gamma\}$ is a basis of R^3.*

Solution. The dimension of $R^3 = 3$

Now we shall show that S is linearly independent.

Let $a, b, c \in R$ such that

$$a\alpha + b\beta + c\gamma = 0$$

$\Rightarrow$ $a(1,2,1) + b(2,9,0) + c(3,3,4) = (0,0,0)$

$\Rightarrow$ $(a + 2b + 3c, 2a + 9b + 3c, a + 4c) = (0,0,0)$

$$\therefore \left.\begin{aligned} a + 2b + 3c &= 0 \\ 2a + 9b + 3c &= 0 \\ a + 4c &= 0 \end{aligned}\right\} \quad \dots (1)$$

The coefficient matrix of the system (1) is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

$\Rightarrow$ $|A| = 1(36 - 0) - 2(8 - 3) + 3(0 - 9)$

$\Rightarrow$ $|A| = 36 - 10 - 27 = -1 \neq 0$

$\Rightarrow$

$\Rightarrow$ Rank $(A) = 3$ which is equal to the number of unknowns. Therefore the system (1) has only zero solution, *i.e.*, $a = 0, b = 0, c = 0$.

$\therefore$ S is a linearly independent which has 3 elements.

Hence, S forms a basis of R^3.

Example 5. *Consider the basis $S = \{\alpha_1, \alpha_2, \alpha_3\}$ of R^3 where $\alpha_1 = (1, 1, 1)$, $\alpha_2 = (1, 1, 0)$, $\alpha_3 = (1, 0, 0)$. Express $(2, -3, 5)$ in terms of the basis elements $\alpha_1, \alpha_2, \alpha_3$.*

Solution. Since $S = \{\alpha_1, \alpha_2, \alpha_3\}$ forms a basis of R^3.

Then every elements of R^3 can be expressed as the linear combination α_1, α_2 and α_3.

Now for $a, b, c \in R$, we have

$$(2,-3,5) = a\alpha_1 + b\alpha_2 + c\alpha_3 \quad \dots(1)$$

$\Rightarrow$ $(2,-3,5) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$\Rightarrow$ $(2,-3,5) = (a + b + c, a + b, a)$

$$\therefore \quad a + b + c = 2$$
$$a + b = -3$$
$$a = 5$$

Solving these equations, we get

$$a = 5, b = -8, c = 5$$

Putting the values of a, b and c in (1), we have

$$(2,-3,5) = 5\alpha_1 - 8\alpha_2 + 5\alpha_3$$

Example 6. *Show that the vectors $\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$ form a basis of R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.*

Solution. The dimension of $R^3 = 3$.

Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$. Now we shall show that S is linearly independent.

Let $a, b, c \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$\Rightarrow \quad a(1,0,-1)+b(1,2,1)+c(0,-3,2)=(0,0,0)$

$\Rightarrow \quad (a+b, 2b-3c, -a+b+2c)=(0,0,0)$

$\therefore$
$$a+b=0$$
$$2b-3c=0$$
$$-a+b+2c=0$$

The coefficient matrix of the above equations is

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad |A|=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{vmatrix}$$

$\Rightarrow \quad |A|=1(4+3)-1(0-3)$

$\Rightarrow \quad |A|=7+3=10\neq 0$

$\Rightarrow \quad$ Rank $(A)=3$ which is equal to the number of unknowns a, b, c.

Thus, above equation has only zero solution, *i.e.*, $a=0, b=0, c=0$.

Therefore, S is linearly independent.

Also dim $R^3=3$. Hence S forms a basis of R^3.

The standard basis of R^3 is $\{e_1, e_2, e_3\}$, where $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$

Now let $\alpha=(p,q,r)$ be any element of R^3. Since S forms a basis of R^3. Then there exists $x, y, z\in R$ such that

$$\alpha = x\alpha_1+y\alpha_2+z\alpha_3 \quad \text{... (1)}$$

$\Rightarrow \quad (p,q,r)=x(1,0,-1)+y(1,2,1)+z(0,-3,2)$

$\Rightarrow \quad (p,q,r)=(x+y, 2y-3z, -x+y+2z)$

$\therefore$
$$p=x+y \quad \text{... (2)}$$
$$q=2y-3z \quad \text{... (3)}$$
$$r=-x+y+2z \quad \text{... (4)}$$

Adding (2) and (4), we get
$$2y+2z=p+r \quad \text{... (5)}$$

Subtracting (3) from (5), we get
$$5z=p+r-q$$

$\therefore$
$$z=\frac{1}{5}(p-q+r)$$

Putting the value of z in (3), we get
$$2y=q+3z=q+\frac{3}{5}(p-q+r)$$

or
$$2y=\frac{3}{5}p+\frac{2}{5}q+\frac{3}{5}r$$

$\therefore$
$$y=\frac{1}{10}(3p+2q+3r)$$

Putting the value of y in (2), we get
$$x=p-y$$

or
$$x=p-\frac{1}{10}(3p+2q+3r)$$

$\therefore$
$$x=\frac{1}{10}(7p-2q-3r)$$

Now, we shall express e_1, e_2, e_3 in terms of $\alpha_1, \alpha_2, \alpha_3$.

So, for $\quad e_1=(1,0,0)=(p,q,r)$

$\Rightarrow \quad p=1, q=0, r=0$

Then $\quad x=\frac{1}{10}(7)=\frac{7}{10}, y=\frac{3}{10}, z=\frac{1}{5}$

Thus from (1), we get
$$e_1=\frac{7}{10}\alpha_1+\frac{3}{10}\alpha_2+\frac{1}{5}\alpha_3$$

For $\quad e_2=(0,1,0)=(p,q,r)$

$\Rightarrow \quad p=0, q=1, r=0$

Then, $\quad x=-\frac{1}{5}, y=\frac{1}{5}, z=-\frac{1}{5}$

$\therefore \quad$ From (1), we get
$$e_2=-\frac{1}{5}\alpha_1+\frac{1}{5}\alpha_2-\frac{1}{5}\alpha_3$$

For $\quad e_3=(0,0,1)=(p,q,r)$

$\Rightarrow \quad p=0, q=0, r=1$

Then, $\quad x=\frac{-3}{10}, y=\frac{3}{10}, z=\frac{1}{5}$

$\therefore \quad$ From (1), we get
$$e_3=\frac{-3}{10}\alpha_1+\frac{3}{10}\alpha_2+\frac{1}{5}\alpha_3.$$

Example 7. *Show that the set $S=\{1, x, x^2, \ldots, x^n\}$ of $n+1$ polynomials in x is a basis of the vectors space $P_n(R)$ of all polynomials in x (of degree at most n) over the field of real numbers.*

Solution. Here $S=\{1, x, x^2, \ldots, x^n\}$ is a subset of $P_n(R)$.

First, we shall show that S is linearly independent.

Let $a_0, a_1, a_2, \ldots, a_n \in R$ such that

$$a_0.1+a_1x+a_2x^2+\ldots+a_nx^n=0(x), \text{i.e., zero polynomial}$$

$\Rightarrow \quad a_0+a_1x+a_2x^2+\ldots+a_nx^n=0+0x+0x^2+\ldots+0x^n$

$$\Rightarrow \quad a_0 = 0, a_1 = 0, a_2 = 0, \ldots, a_n = 0$$

Therefore, S is linearly independent.

Now we shall show that S spans $P_n(R)$.

Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be any polynomial of $P_n(R)$ where $a_0, a_1, a_2, \ldots a_n \in R$.

Therefore, $p(x)$ is a linear combination of the elements of S. Thus S spans $P_n(R)$

Hence S forms a basis of $P_n(R)$.

Example 8. *Given that each set S below spans* R^3, *find a basis of* R^3 *which is contained in S:*

(i) $\{(1, 0, 2), (0, 1, 1), (2, 1, 5), (1, 1, 3), (1, 2, 1)\}$

(ii) $\{(2, 6, -3), (5, 15, -8), (3, 9, -5), (1, 3, -2), (5, 3, -2)\}$

Solution. (i) Let $S = \{(1, 0, 2), (0, 1, 1), (2, 1, 5), (1, 1, 3), (1, 2, 1)\}$

Since dim. $R^3 = 3$ and S spans R^3.

Now we shall find maximal linearly independent set containing S.

Let A be a matrix whose rows are the elements of S. Then

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

We shall reduce this matrix to Echelon form by using row transformation.

Applying $R_3 \to R_3 - 2R_1, R_4 \to R_4 - R_1, R_5 \to R_5 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$ and, $R_5 \to R_5 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

This is an Echelon form which has three non-zero rows. Thus Rank $(A) = 3$. Here non-zero rows are corresponding to the vectors $(1, 0, 2)$, $(0, 1, 1)$ and $(1, 2, 1)$ of S.

Therefore, S contains maximal linearly independent subset $\{(1, 0, 2), (0, 1, 1), (1, 2, 1)\}$ which has 3 elements which is equal to the dimension of R^3. Hence, the set $\{(1, 0, 2), (0, 1, 1), (1, 2, 1)\}$ forms a basis of R^3.

(ii) Let $S = \{(2, 6, -3), (5, 15, -8), (3, 9, -5), (1, 3, -2), (5, 3, -2)\}$.

Since dim. $R^3 = 3$ and S spans R^3.

Now we shall find maximal linearly independent set contained in S.

Let A be a matrix whose rows and the elements of S.

Then $$A = \begin{bmatrix} 2 & 6 & -3 \\ 5 & 15 & -8 \\ 3 & 9 & -5 \\ 1 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix}$$

Now we shall reduce this matrix to echelon form by using row transformations.

Applying $R_4 \leftrightarrow R_1$, we get

$$A \sim \begin{bmatrix} 1 & 3 & -3 \\ 5 & 15 & -8 \\ 3 & 9 & -5 \\ 2 & 6 & -3 \\ 5 & 3 & -2 \end{bmatrix}$$

Applying $R_2 \to R_2 - 5R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 2R_1$ and $R_5 \to R_5 - 5R_1$, we get

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -12 & 8 \end{bmatrix}$$

Applying $R_5 \leftrightarrow R_2$, we get

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -12 & 8 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3, R_5 \to R_5 - 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -12 & 8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is an Echelon form which has three non-zero rows. Thus Rank $(A) = 3$. Here non-zero rows are corresponding to the vectors $(1, 3, -2)$, $(5, 3, -2)$ and $(3, 9, -5)$ of S.

Therefore, S contains maximal linearly independent subset $\{(3, 9, -5), (1, 3, -2), (5, 3, -2)\}$ which has 3 elements which is equal to the dimension of R^3. Hence, the set $\{(3, 9, -5), (1, 3, -2), (5, 3, -2)\}$ forms a basis of R^3.

Example 9. *Given that the set S is a basis of $\mathbb{R}^4$ and that T is linearly independent. Extend T to a basis of $\mathbb{R}^4$, where*

$$S=\{(1,0,0,0),(0,0,1,0),(5,1,11,0),(-4,0,-6,1)\}$$

and $$T=\{(1,0,1,0),(0,2,0,3)\}$$

Solution. Since dim. $\mathbb{R}^4=4$ and T contains two linearly independent elements so in order to extend it to form a basis of $\mathbb{R}^4$ we shall include two elements from S to T till T becomes linearly independent.

First include an element (1, 0, 0, 0) to T then, we have

$$T_1=\{(1,0,1,0),(0,2,0,3),(1,0,0,0)\}$$

Now, we shall check whether T_1 is linearly independent or not.

Let A be a matrix whose rows are the elements of T_1. Then

$$A=\begin{bmatrix}1&0&1&0\\0&2&0&3\\1&0&0&0\end{bmatrix}$$

Applying $R_3\to R_3-R_1$

$$A\sim\begin{bmatrix}1&0&1&0\\0&2&0&3\\0&0&-1&0\end{bmatrix}$$

which is an Echelon form having 3 non-zero rows so that Rank $(A)=3$. Thus, T_1 is linearly independent but T_1 contains 3 elements and dim. $\mathbb{R}^4=4$, so it is not a basis of $\mathbb{R}^4$.

We again include an element (0, 0, 1, 0) to T_1, we have

$$T_2=\{(1,0,1,0),(0,2,0,3),(1,0,0,0)(0,0,1,0)\}.$$

Again we shall test the linear dependence.

Let A be a matrix whose rows are the elements of T_1. Then

$$A=\begin{bmatrix}1&0&1&0\\0&2&0&3\\1&0&0&0\\0&0&1&0\end{bmatrix}$$

Applying $R_3\to R_3\to R_1$, we get

$$A\sim\begin{bmatrix}1&0&1&0\\0&2&0&3\\0&0&-1&0\\0&0&1&0\end{bmatrix}$$

Applying $R_4\to R_4+R_3$

$$A\sim\begin{bmatrix}1&0&1&0\\2&2&0&3\\0&0&-1&0\\0&0&0&0\end{bmatrix}$$

which is an Echelon form having 3 non-zero rows, so Rank $(A)=3$. Thus T_2 is linearly dependent.

Now we include an element (5, 1, 11, 0) in place of (0, 0, 1, 0) to T_2, then we have

$$T_3=\{(1,0,1,0),(0,2,0,3),(1,0,0,0),(5,1,11,0)\}$$

Again we shall test the linear dependence.

Let A be a matrix whose rows are the elements of T_3. Then

$$A=\begin{bmatrix}1&0&1&0\\0&2&0&3\\1&0&0&0\\5&1&11&0\end{bmatrix}$$

Applying $R_3\to R_3-R_1$ and $R_4\to R_4-5R_1$, we get

$$A\sim\begin{bmatrix}1&0&1&0\\0&2&0&3\\0&0&-1&0\\0&1&6&0\end{bmatrix}$$

Applying $R_4\to R_4-\frac{1}{2}R_2$, we get

$$A\sim\begin{bmatrix}1&0&1&0\\0&2&0&3\\0&0&-1&0\\0&0&6&-3/2\end{bmatrix}$$

Applying $R_4\to R_4+6R_3$, we get

$$A\sim\begin{bmatrix}1&0&1&0\\0&2&0&3\\0&0&-1&0\\0&0&0&-3/2\end{bmatrix}$$

which is an Echelon form having 4 non-zero rows so Rank $(A)=4$. Thus T_3 is linearly independent which has 4 elements. Hence the set

$\{(1,0,1,0),(0,2,0,3),(1,0,0,0),(5,1,11,0)\}$ is a basis.

Example 10. *Let W be a subspace of a vector space $V(F)$ of dimension r. Then show that a set of r vectors in W is a basis of W if and only if it is linearly independent.*

Solution. If a set of r vectors in W is a basis, then by definition of a basis, it is linearly independent.

Let $S=\{\alpha_1,\alpha_2,\ldots,\alpha_r\}$ be a set of r linearly independent vectors in W. Then S can be extended to a basis of W. Now S has r elements and dimension of W is r and hence, all bases of W have r elements. In particular, the basis to which S is extended has r elements therefore, S is a basis of W.

Example 11. *Let W be a subspace of a vector space $V(F)$ of dimension r. Then show that a set of r vectors in W is a basis if and only if it spans W.*

If a set of r vectors in W is a basis if and onl if it spans W(By definition).

Solution. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be a set of r vectors which spans W. Then S contains a basis of W.

But any basis of W contains r vectors. Therefore, the basis contained in S is not a proper subset of S, and S is a basis of W.

Example 12. *If n vectors span a vector space V containing r linearly independent vectors, then show that $n \geq r$.*

Solution. Let S be a subset of V containing n vectors and S spans V. Then there exists a linearly independent subset T and S which also spans V. Therefore, T will form a basis of V. If T contains m elements then dim. $V = m$ and $m \geq n$.

Since dim. $V = m$, then any subset of V containing more than m elements will be linealry dependent. Therefore, if there is a linearly independent subset of V containing r vectors, then $r \leq m$ but $m \leq n$, hence $n \leq r$.

Example 13. *Extend the linearly independent subset $\{(1, 0, 1), (0, -1, 1)\}$ of $V_3(\mathrm{R})$ to form a basis of V_3 (R).*

Solution. Since the dim. V_3 (R) = 3 and the subset $\{(1, 0, 1), (0, -1, 1)\}$ has two linearly independent vectors, therefore, in order to extend this subset to form a basis of V_3 (R) we shall include one vector of $V_3(\mathrm{R})$ such that they are linearly independent.

Let $\quad S = \{(1,0,1),(0,-1,1)\}$

Then, $\quad L(S) = \{a(1,0,1)+b(0,-1,1) : a,b \in \mathrm{R}\}$

$= \{a,-b,a+b) : a,b \in \mathrm{R}\}$

Clearly, in this span, the third component is the sum of first and negative of second component. Therefore, we shall include a vector of $V_3(\mathrm{R})$ which does not have this property.

Now there are many such vectors in V_3 (R), one of which is (1, 0, 0). So we include this vector to S. Thus the set $\{(1, 0, 1), (0, -1, 1), (1, 0, 0)\}$ is linearly independent and has 3 elements.

Hence, this set is a required basis of $V_3(R)$ which is the extension of S.

Example 14. *Extend the linearly dependent subset $\{(1, -1, 0, 0), (1, 1, 1, 0)\}$ of V_4 (R) to form a basis of $V_4(\mathrm{R})$.*

Solution. Let $S = \{(1,-1,0,0),(1,1,1,0)\}$. Then

$$L(S) = \{a(1,-1,0,0)+b(1,1,1,0) : a,b \in \mathrm{R}\}$$

$$= \{(a+b,-a+b,b,0) : a,b \in \mathrm{R}\}$$

Clearly, in this span, the fourth coordinate is zero.

Then (0, 0, 0, 1) is clearly not in this span, so that we include this vector to S, we get an enlarged linearly independent set

$$S' = \{(1,-1,0,0),(1,1,1,0),(0,0,0,1)\}$$

Again we extend this set

Now, $\quad L(S') = \{a(1,-1,0,0)+b(1,1,1,0)+c(0,0,0,1) : a,b,c \in \mathrm{R}\}$

$= \{(a+b,-a+b,b,c) : a,b,c \in \mathrm{R}\}$

In this span, we observe that for given a, b and c the second coordinate is always $-a+b$, so we try to find such vector which does not follow this hypothesis.

Clearly, (1, −2, 0, 0) is not in $L(S')$. Now we include this vector to S', we get a set

$$S'' = \{(1,-1,0,0),(1,1,1,0),(0,0,0,1),(1,-2,0,0)\}$$

which is linearly independent and has 4 elements.

Hence, S'' is a required basis of $V_4(\mathrm{R})$ which is the extension of S.

Example 15. *Let W be the subspace of $V_4(\mathrm{R})$ generated by the vectors (1, −2, 5, −3), (2, 3, 1, −4), (3, 8, −3, −5), then*

(i) Find a basis and dimension of W.

(ii) Extend the basis of W to a basis of $V_4(\mathrm{R})$.

Solution. (i) Let $S = \{(1,-2,5,-3),(2,3,1,-4),(3,8,-3,-5)\}$ and $L(S) = W$.

Now we shall find maximal linearly independent subset of S.

Let A be a matrix whose rows are the elements of S, then

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

We shall have to reduce A to an Echelon form by using row transformations.

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

Again applying $R_3 \to R_3 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is an Echelon form and has 2 non-zero rows representing the co-ordinate vectors (1, −2, 5, −3) and (0, 7, −9, 2) that form a basis of rows space *i.e.*, $T = \{(1,-2,5,-3),(0,7,-9,2)\}$ is a basis of W. Thus dim. W = 2.

(ii) Since dim. $V_4(\mathrm{R}) = 4$, so in order to form a basis of $V_4(\mathrm{R})$ we shall extend the set T by including two vectors.

Let us take these two vectors as (0, 0, 1, 0) and (0, 0, 0, 1), so including both in T such that the set

$$T' = \{(1,-2,5,-3),(0,7,-9,2),(0,0,1,0),(0,0,0,1)\}$$

is linearly independent because a matrix.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

whose rows are the elements of T' is in Echelon form with 4 non-zero rows showing that T' is linearly independent.

Hence T' is a basis of $V_4(\mathbb{R})$ which is obtained by extending a basis of W.

Example 16. *Let W be the subspace of $V_4(\mathbb{R})$ generated by the set of vectors*

$$S = \{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$$

and W_2 the subspace of $V_4(\mathbb{R})$ generated by the set of vectors

$$T = \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$$

Find:

(i) $\dim.(W_1 + W_2)$ (ii) $\dim.(W_1 \cap W_2)$

Solution. (i) We know that

$$L(W_1 \cup W_2) = W_1 + W_2$$

Then $W_1 + W_2$ is a subspace generated by the set of vectors of $S \cup T$ where

$S \cup T = \{(1,1,0,-1),(1,2,3,0),(2,3,3,-1),(1,2,2,-2)(2,3,2,-3),(1,3,4,-3)\}$

Let A be a matrix whose rows are the elements of $S \cup T$. Then

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Now we shall reduce A to an Echelon form as follows:

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - R_1, R_5 \to R_5 - 2R_1$ and $R_6 \to R_6 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

Again Applying $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2, R_5 \to R_5 - R_2$ and $R_6 \to R_6 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

Applying $R_4 \leftrightarrow R_3$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

Applying $R_5 \to R_5 - R_3, R_6 \to R_6 - 2R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is an Echleon form and has 3 non-zero rows.

$$\therefore \quad \dim.(W_1 + W_2) = 3$$

(ii) First we find the dim.W_1 and W_2.

Let A_1 be a matrix whose rows are the elements of S, then

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Again applying $R_3 \to R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is an Echelon form and has 2 non-zero rows.

$$\therefore \quad \dim.\ W_1 = 2$$

Let A_2 be a matrix whose rows are the elements of T, then

$$A_2 = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$, we get

$$A_2 \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Again applying $R_3 \to R_3 + R_2$, we get

$$A_2 \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is an Echelon form and has 2 non-zero rows.

$\therefore$ $\quad$ dim. $W_2 = 2$

We know that

$$\dim.(W_1 + W_2) = \dim.W_1 + \dim.W_2 - \dim.(W_1 \cap W_2)$$

$\Rightarrow$ $\quad 3 = 2 + 2 - \dim.(W_1 \cap W_2)$

$\therefore$ $\quad \dim.(W_1 \cap W_2) = 1$

Example 17. *Let V be the vector space of* 2×2 *symmetric matrix over R. Show that dim.*$V = 3$

Solution. An arbitrary 2×2 symmetric matrix is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $a, b, c \in R$.

Now setting (i) $a = 1, b = 0, c = 0$, (ii) $a = 0, b = 1, c = 0$, (iii) $a = 0, b = 0, c = 1$, thus we obtain the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We shall show that the set $S = \{A, B, C\}$ is a basis of V.

First we show that S is linearly independent.

Let $x, y, z \in R$ such that

$$xA + yB + zC = 0$$

$\Rightarrow$ $\quad x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $\quad \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $\quad x = 0, y = 0, z = 0$

$\therefore$ $\quad S = \{A, B, C\}$ is linearly independent.

Now, we show that $L(S) = V$.

Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be any element of V, then we have

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$\Rightarrow$ $\quad \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $\quad \begin{bmatrix} a & b \\ b & c \end{bmatrix} = aA + bB + cC$

$\Rightarrow$ Every element of V is linear combination of elements of S

$\Rightarrow$ $\quad L(S) = V$

Therefore, S forms a basis of V which has 3 elements.

Hence, dim. $V = 3$.

Example 18. *Let W be the subspace of* R^3 *defined by* $W = \{(a, b, c) : a+b+c = 0\}$. *Find a basis and dimension of W.*

Solution. Since $(1, 2, 3) \in R^3$ but $(1, 2, 3) \notin W$, therefore $W \neq R^3$. Thus dim. $W < 3$.

Now $\alpha_1 = (1, 0, -1)$ and $\alpha_2 = (0, 1, -1)$ are two vectors of W. Also α_1 cannot be expressed as a multiple of α_2 so that α_1, α_2 are linearly independent.

Therefore, $\{(1, 0, -1), (0, 1, -1)\}$ is a basis of W and dim. $W = 2$.

Example 19. *Let W be the subspace of* R^3 *defined by* $W = \{(a, b, c) : a = b = c\}$. *Find a basis and dimension of W.*

Solution. Since $(1, 1, 1) \in W$. Let $\alpha = (1, 1, 1)$. Then any vector β of W is of the form $\beta = (a, a, a)$ for all $a \in R$. Thus $\beta = a\alpha$. Therefore α spans W.

Hence $\{(1, 1, 1)\}$ is a basis of W and dim. $W = 1$

Example 20. *Let* W_1 *and* W_2 *be distinct subspaces of V and dim.* $W_1 = 4$, *dim.* $W_2 = 4$ *and dim.* $V = 6$. *Find the possible dimensions of* $W_1 \cap W_2$.

Solution. Since W_1 and W_2 be distinct subspaces of V, then $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

$\therefore$ $\quad$ dim. $(W_1 + W_2) >$ dim. W_1 or dim W_2.

$\Rightarrow$ $\quad$ dim. $(W_1 + W_2) > 4$ $\quad$ ... (1)

Also $\quad$ dim. $V = 6$

$\Rightarrow$ $\quad$ dim. $(W_1 + W_2) \leq 6$ $\quad$ ...(2)

From (1) and (2), we conclude that dim. $(W_1 + W_2)$ is either 5 or 6.

When dim. $(W_1 + W_2) = 5$:

We know that

$\quad$ dim. $(W_1 \cap W_2) =$ dim. $W_1 +$ dim. $W_2 -$ dim. $(W_1 + W_2)$

$\Rightarrow$ $\quad$ dim. $(W_1 \cap W_2) = 4 + 4 - 5$

$\Rightarrow$ $\quad$ dim. $(W_1 \cap W_2) = 3$

When $\quad$ dim. $(W_1 + W_2) = 6$:

$\quad$ dim. $(W_1 \cap W_2) = 4 + 4 - 6$

$\Rightarrow$ $\quad$ dim. $(W_1 \cap W_2) = 2$

Example 21. *Let* W_1 *and* W_2 *be subspaces of* R^3 *for which dim.* $W_1 = 1$, *dim.* $W_2 = 2$ *and* $W_1 \not\subseteq W_2$. *Show that* $R^3 = W_1 \oplus W_2$.

Solution. Since $W_1 \not\subseteq W_2$ then $(W_1 \cap W_2) \subseteq W_1$ and $W_2 \cap W_1 \subseteq W_2$

$\therefore$ $\quad$ dim. $(W_1 \cap W_2) <$ dim. W_1

$\Rightarrow$ $\quad$ dim. $(W_1 \cap W_2) < 1$

$\Rightarrow$ dim. $(W_1 \cap W_2) = 0$

$\Rightarrow$ $W_1 \cap W_2 = \{0\}$

Also, dim. $(W_1 + W_2) >$ dim. W_2

$\Rightarrow$ dim. $(W_1 + W_2) > 2$ ... (1)

But dim. $R^3 = 3$ and $W_1 + W_2 \subseteq R^3$

$\therefore$ dim. $(W_1 + W_2) \leq 3$ ... (2)

From (1) and (2), we conclude that

dim. $(W_1 + W_2) = 3$

$\Rightarrow$ $W_1 + W_2 = R^3$

Thus, $R^3 = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$, hence $R^3 = W_1 \oplus W_2$

Example 22. *Show that if $S = \{\alpha, \beta, \gamma\}$ is a basis of $C^3(C)$, then the set $S' = \{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$ is also a basis of $C^3(C)$.*

Solution. Since dim. $C^3 = 3$, therefore any subset of C^3 having 3 linearly independent vectors will form a basis of C^3.

Further, since $S = \{\alpha, \beta, \gamma\}$ is a basis of C^3, so that α, β, γ are linearly independent. If $S' = \{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$ is linearly independent then it will form a basis of C^3, so we shall test the dependence of S'.

Let $a, b, c \in C$ such that

$$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0$$

$$\Rightarrow \quad (a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0$$

$$\Rightarrow \quad a+c = 0, a+b = 0, b+c = 0 \quad [\because \alpha, \beta, \gamma \text{ are linearly independent.}]$$

$$\Rightarrow \quad a = 0, b = 0, c = 0$$

$\Rightarrow$ S' is linearly independent.

Hence $S' = \{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$ is also a basis of $C^3(C)$.

Example 23. *Give a basis for each of the following vector space over the indicated fields:*

(i) $R(\sqrt{2})$ over R (ii) $Q(2^{1/4})$ over Q

where Q, R are field of rational and real numbers.

Solution. (i) We have $R(\sqrt{2}) = \{a + \sqrt{2}b : a, b \in R\}$

Now zero element of $R(\sqrt{2})$ can be written as $0 = 0 + 0.\sqrt{2}$

Let $S = \{1, \sqrt{2}\}$, then $S \subseteq R(\sqrt{2})$. Now we shall show that S forms a basis of $R(\sqrt{2})$.

Let $a, b \in R$, such that

$$a.1 + \sqrt{2}\,b = 0$$

$$\Rightarrow \quad a + \sqrt{2}b = 0 + 0.\sqrt{2}$$

$$\Rightarrow \quad a = 0, b = 0$$

$\Rightarrow$ S is linearly independent.

Let $x + \sqrt{2}y$ be any element of $R(\sqrt{2})$. Then $x + \sqrt{2}y = x.1 + \sqrt{2}.y$

$\Rightarrow$ every element of $R(\sqrt{2})$ is expressible as the linear combination of elements of S.

$$\Rightarrow \quad L(S) = R(\sqrt{2})$$

Hence, $S = \{1, \sqrt{2}\}$ is a basis of $R(\sqrt{2})$ having two elements so that dim. $R(\sqrt{2}) = 2$

(ii) We have

$$Q(2^{1/4}) = \{a + (2^{1/4})b : a, b \in Q\}$$

The zero element of $Q(2^{1/4})$ is $0 = 0 + (2^{1/4}).0$

Let $S = \{1, 2^{1/4}\}$, then $S \subseteq Q(2^{1/4})$. Now we shall see that S forms a basis of $Q(2^{1/4})$.

Let $a, b \in Q$ such that $a.1 + b.(2^{1/4}) = 0$

$$\Rightarrow \quad a + b(2^{1/4}) = 0 + 0.(2^{1/4})$$

$$\Rightarrow \quad a = 0, b = 0$$

$\Rightarrow$ S is linearly independent.

Let $x + (2^{1/4})y$ be any element of $Q(2^{1/4})$

Then, $x + (2^{1/4})y = x.1 + (2^{1/4}).y$

$\Rightarrow$ every element of $Q(2^{1/4})$ is expressible as a linear combination of elements of S.

$$\Rightarrow \quad L(S) = Q(2^{1/4})$$

Hence, $S = \{1, 2^{1/4}\}$ is a basis of $Q(2^{1/4})$ and dim. $Q(2^{1/4}) = 2$.

Example 24. *Determine dim V/W where V = C(R) and W = R(R).*

Solution. We know that $\{1, i\}$ is a basis of C(R) and $\{1\}$ is a basis of R(R). Therefore, dim $V = 2$ and dim $W = 1$.

Hence, dim V/W = dim V − dim W = 2 − 1 = 1

Example 25. *Let $V = C^2(R)$ and $W = R^2(R)$, find dim (V/W).*

Solution. We know that $\{(1, 0), (0, 1), (1, 0), (i, 0), (0, i)\}$ is a basis of $C^2(R)$ and $\{(1, 0), (0, 1)\}$ is a basis of $R^2(R)$.

So, dim $V = 4$ and dim $W = 2$

Hence, dim V/W = dim V − dim W

= 4 − 2 = 2