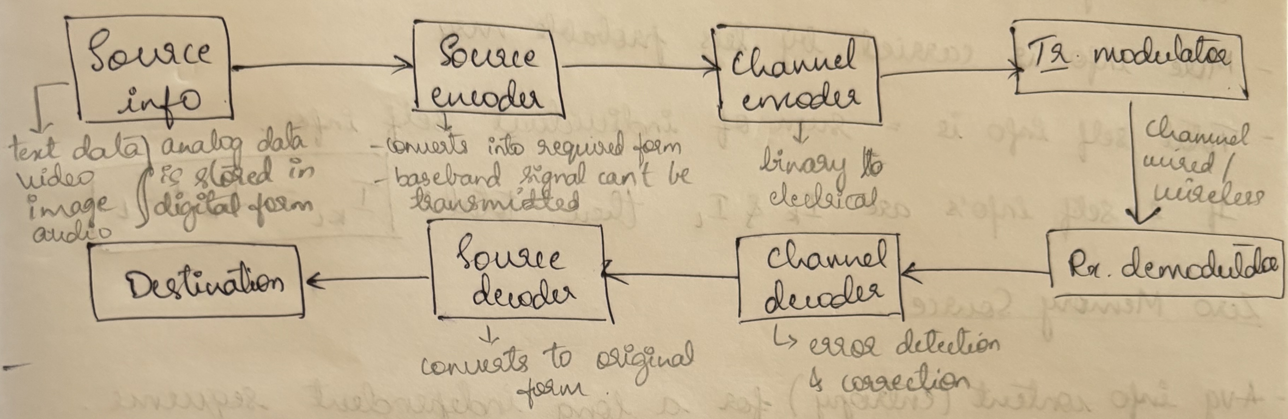
Information Theory

Digital comm. block diagram.



Measure of information / self information.

$S_k = \{S_1, S_2, --- S_q\}$ - source that produces info.

$P_k = \{P_1, P_2 --- P_q\}$ - probabilities.

Amount of info / self info, $\boxed{I_k = \log_2 \frac{1}{P_k}}$

- Info: sun rises in east $\rightarrow P_k = 1 \quad I_k = \log_2 \frac{1}{1} = 0$

  $\therefore$ not an info.

- Less probable (rare) $\rightarrow$ contains more info.

Problem - The binary symbols '0' & '1' are transmitted with probabilites $\frac{1}{4}$ & $\frac{3}{4}$ resp. Find corresponding self info.

$\rightarrow \quad I_k = \log_2 \frac{1}{P_k}$.

$0 \rightarrow I_k = \log_2 \frac{1}{1/4} = \log_2 4 = 2$ bits.

$1 \rightarrow I_k = \log_2 \frac{1}{3/4} = \log \frac{4}{3} = 0.415$ bits.

Observations on self info.

- It cannot be -ve. Each msg must contain certain info.
- Lowest probable self info is 0.
- More info is carried by less probable msg.
- Total self info is = sum of individual self info.

  If 2 self info's are I_k & I_l then total :- $\boxed{I_{kl} = I_k + I_l}$

Zero Memory Source.

- Avg info content (entropy) for a long independent sequence.

  $S = \{S_1, S_2 \, --- \, S_q\}$ $\quad P = \{P_1, P_2 \, --- \, P_q\}$.

  Length of sequence = L symbols.

  $P_1 L$ no. of messages of type S_1

  $P_2 L$ " " " S_2

  ⋮

  $P_q L$ " " " S_q

  $$I_k = \log \frac{1}{P_k}$$

  $P_1 L$ no. of message of type S_1 contain $P_1 L \log \frac{1}{P_1}$ bits of info.

  $P_2 L$ " " " " " S_2 " $P_2 L \log \frac{1}{P_2}$ " "

  ⋮

  $P_q L$ " " " " S_q " $P_q L \log \frac{1}{P_q}$ " "

  $$I_{total} = P_1 L \log \frac{1}{P_1} + P_2 L \log \frac{1}{P_2} + \ldots + P_q L \log \frac{1}{P_q}$$

  $$\boxed{I_{total} = L \sum_{i=1}^{q} P_i \log \frac{1}{P_i}}$$

Average info = $\frac{I_{total}}{L}$

Entropy = $\boxed{H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}}$ bits / msg symbol.

3) A discrete source emits one of 6^{th} symbols once every ms. The symbol prob. are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ & $\frac{1}{32}$ resp. Find source entropy & info rate.

$\rightarrow H(S) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

$= \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) + \frac{1}{16}\log_2(16) + \frac{1}{32}\log_2(32) + \frac{1}{32}\log_2(32)$

$= \frac{31}{16} = \underline{1.9375}$ bits / msg signal.

$R_s = r_s H(S)$ $\qquad$ r_s = msg symbol rate.

$= 1000 \times 1.9375$ $\qquad$ $r_s = 1$ msg/ms. $= 1000$ sec.

$\underline{\underline{R_s = 1937.5 \text{ bps}}}$

Find relation b/w Hartleys, nats & bits.

$I = \log_{10}\frac{1}{P}$ Hartleys, $\quad I = \log_e \frac{1}{P}$ nats, $\quad I = \log_2 \frac{1}{P}$ bits.

$1 \text{ Hartley} = \frac{I}{\log_{10}\frac{1}{P}} = \frac{\log_e \frac{1}{P}}{\log_{10}\frac{1}{P}} = \frac{-\log_e P}{-\log_{10} P}$ $\qquad$ $\left[\log_a b = \frac{1}{\log_b a}\right]$

$1 \text{ Hartley} = \frac{\log_P 10}{\log_P e}$

1 Hartleys = $\log_e 10$ nats

$\boxed{1 \text{ Hartley} = 2.303 \text{ nats}}$

1 nat = $\log_2 e$ bits & $\log_2 10$ bits $\approx$ $\qquad$ 1 Hartley = $\log_2 10$ bits

$= 1.443$ bits $\qquad$ $= 3.322$ bits.

4) A code is composed of dots & dashes. Assuming that a dash is 3 times as long as dot & has $\frac{1}{3}$ prob of occurrence. Calculate
a) Info in dot & dash $\quad$ b) entropy of dot dash code
c) Avg rate of info if a dot lasts for 10ms & this time is allowed b/w symbols.

$\rightarrow P_{dot} + P_{dash} = 1$ $\qquad P_{dash} = \frac{1}{3} P_{dot}$

$P_{dash} = 1 - \frac{3}{4}$ $\qquad 1 = P_{dot} + \frac{1}{3} P_{dot}$

$\boxed{P_{dash} = \frac{1}{4}}$ $\qquad \boxed{P_{dot} = \frac{3}{4}}$

a) $I_{dot} = \log \frac{1}{P_{dot}} = \underline{0.415 \text{ bit}}$ $\quad I_{dash} = \log \frac{1}{P_{dash}} = \underline{2 \text{ bits}}$

b) $H(S) = \sum_{i=1}^{2} P_i \log_2 1/P_i$ $\qquad$ duration of $\cdot$ = 10ms, " $-$ " dash = 30ms

$= \frac{3}{4} \log_2 \left(\frac{4}{3}\right) + \frac{1}{4} \log_2 (4) = \underline{0.8113 \text{ bits/m. sym}}.$

c) $R_s = r_s H(S)$ $\qquad r_s = 4$ sym/100m sec.

$\boxed{R_s = 32.45 \text{bps}}$. $\qquad r_s = 40$ sym/sec.

5) Find H(S) ~~of~~ source in nats/sym of a source that emits 1 out of 4 sym a, b, c & d resp. in a statistical independent sequence.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ & $\frac{1}{8}$

$\rightarrow H(S) = \sum P_i \log_2 \frac{1}{P_i} = \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) \times 2$

$\boxed{H(S) = 1.75 \text{ bps}}$

1 nats = 1.443 bits $\qquad$ 1 Hartley = ~~$\log_2$~~ 3.32 bits

? = 1.75 bits $\qquad$? = 1.75 bits

$H(S) = \frac{1.75}{1.443} = \underline{1.213 \text{ nats/sym}}.$ $\qquad \underline{H(S) = 0.52 \text{ Hartley/sym.}}$

Properties of Entropy.

$H(S) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i} = \sum_{i=1}^{2} P_i I_i$

i) Prob. should vary b/w 0 & 1.

ii) Entropy fn is symmetric fn about its argument of entropy.

$H[P_k, (1-P_k)] = H[(1-P_k), P_k] \quad k = 1 \ldots q$

value of H(S) remains same,

$P_A = \{P_1, P_2, P_3\}$ $\quad P_B = \{P_2, P_3, P_1\}$ $\quad P_C = \{P_3, P_1, P_2\}$
$S_A \quad S_B \quad S_C$

$H(S_A) = H(S_B) = H(S_C)$

iii) Extremal property - finding out upper bound / max entropy.
lower bound will always be 0. $\qquad \log q - H(S)$ → no. of symbols.

$H(S)_{max} = \log_2 q$ bits/m. s.

Properties of Additivity

$H'(S)$ - subgrouping → shouldn't decrease the entropy → $[H'(S) \geq H(S)]$.

Source efficiency, $\boxed{\eta_s = \frac{H(S)}{H(S)_{max}} \times 100}$

Source redundancy, $\boxed{R_{ns} = 1 - \eta_s} \times 100$.

Problem

1) A discrete msg source S, emits 2 indep. sym x & y with prob. 0.55 & 0.45 resp. Calculate η & R_{ns}.

$\rightarrow H(S) = 0.55 \log_2 \frac{1}{0.55} + 0.45 \log_2 \frac{1}{0.45} = \underline{0.9927 \text{ bits/m. s.}}$

$H(S)_{max} = \log_2(2) = \cancel{0}.9927\,1$ bits/m-s.

$\eta = \frac{0.9927}{1} = \underline{99.27\,\%}.$

$R_{ns} = 1 - 0.9927 = 0.0073 = \underline{0.73\,\%}$

2) A pair of dice are tossed simult. The outcome of 1st dice is recorded as x_1 & that of 2nd dice is as x_2. Two events are defined as follows:

$A = \{(x, x_2)$ such that $x_1 + x_2 \leq 7\}$

$B = \{(x, x_2)$ such " $x_1 > x_2\}$

which event conveys more info. Support your ans.

(11 12 13 14 15 16) (21 22 23 24 25)

$\rightarrow P(A) = \frac{21}{36} = \frac{7}{12} = 0.58 \quad P(B) = \frac{15}{36} = \frac{5}{12} = 0.416.$

$I_A = \log_2 \frac{1}{P_A} \qquad I_B = \log_2 \frac{1}{P_B} \qquad \boxed{I_B > I_A}$

$I_A = 0.777$ bits $\qquad I_B = 1.263$ bits

3) Shortly before a horse race, a bookmaker believes that several horses entered in race have following prob of winning:

horse A = 0.04
B = 0.42
C = 0.31
D = 0.12
E = 0.11

} Probabilities.

He then receives a msg that owing to minor injury one of the horses is not participating in race. Explain how would you access from info theory point of view, the info value of this msg:

a) If the horse in question is known
b) " " is not known

$\rightarrow$ a) $I_A = \log_2 \frac{1}{0.04}$
$= 4.643$ bits

$I_B = \log_2 \frac{1}{0.42}$
$= 1.251$ bits

$I_C = \log_2 \frac{1}{0.31}$
$= 1.68$ bits

$I_D = \log_2 \frac{1}{0.12}$
$= 3.058$ bits

$I_E = \log_2 \frac{1}{0.11}$
$= 3.184$

b) $H(S) = 0.04 \log_2 \frac{1}{0.04} + 0.42 \log_2 \frac{1}{0.42} + 0.31 \log_2 \frac{1}{0.31} +$
$0.12 \log_2 \frac{1}{0.12} + 0.11 \log_2 \frac{1}{0.11} = \underline{\underline{1.9173}}$

$H(S)_{max} = \log_2 q = \underline{\underline{2.322}}$

4) B & W TV picture consists of 525 lines of picture info. Assume that each line consists of 525 picture ele (pixel) & that each ele can have 256 brightness levels. Picture are repeated at the rate of 30 frame/sec. Calculate average rate of info conveyed by TV set to viewer.

$\rightarrow$ $525 \times 525 \rightarrow$ (lines × pixels).

$q = (256)^{525 \times 525} = 256 \times 525 \times 525 = 70560000$

$H(S)_{max} = \log_2 q = 26.07.$

$R_s = r_s H(S) = 30 \times 26.07 = 782.1$

Extension of zero-memory source.

$S = \{S_1, S_2\} \quad P = \{P_1, P_2\} \quad P_1 + P_2 = 1.$

$[$no. of basic source symbol$]^{\text{extension}} \rightarrow [2]^2 = 4$ (2nd extension)

S_1S_1 occurs with probabilities $P_1P_1 = P_1^2$
S_1S_2 " " $P_1P_2 = P_1P_2$
S_2S_1 " " $P_1P_2 \quad P_1P_2$
S_2S_2 " " $P_2P_2 = P_2^2$

$P_1^2 + 2P_1P_2 + P_2^2 = 1.$

$H(S) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i} \rightarrow \boxed{H(S) = P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}}$

$\left[\log \frac{1}{a^b} = b \log \frac{1}{a}\right]$

$H(S^2) = \sum_{i=1}^{4} P_i \log \frac{1}{P_i}$

$= P_1^2 \log \frac{1}{P_1^2} + P_1P_2 \log \frac{1}{P_1P_2} + P_1P_2 \log \frac{1}{P_1P_2} + P_2^2 \log \frac{1}{P_2^2}.$

$= 2P_1^2 \log \frac{1}{P_1} + 2P_1P_2 \log \frac{1}{P_1P_2} + 2P_2^2 \log \frac{1}{P_2}.$

$= 2P_1^2 \log \frac{1}{P_1} + 2P_1P_2 \log \frac{1}{P_1} + 2P_1P_2 \log \frac{1}{P_2} + 2P_2^2 \log \frac{1}{P_2}.$

$= 2P_1(P_1+P_2) \log \frac{1}{P_1} + 2P_2(P_1+P_2) \log \frac{1}{P_2}$

$\therefore \boxed{P_1 + P_2 = 1}. \rightarrow = 2P_1 \log \frac{1}{P_1} + 2P_2 \log \frac{1}{P_2}$
$= 2\left(P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}\right) = \underline{2H(S)}.$

3^{rd} extension $\Rightarrow [2]^3 = 8$.

$S_1S_1S_1$ occurs with probabilities $P_1P_1P_1 = P_1^3$
$S_1S_1S_2$ " " " $= P_1^2P_2$
$S_1S_2S_2$ " " " $= P_1P_2^2$
$S_1S_2S_1$ " " " $= P_1^2P_2$
$S_2S_2S_2$ " " " $= P_2^3$
$S_2S_2S_1$ " " " $= P_2^2P_1$
$S_2S_1S_1$ " " " $= P_1^2P_2$
$S_2S_1S_2$ " " " $= P_1P_2^2$

$H(S^3) = \sum_{i=1}^{8} P_i \log \frac{1}{P_i}$ Finally, $H(S^3) = 3H(S)$.

$P_1^3 + 3P_1^2P_2 + 3P_1P_2^2 + P_2^3 = 1$

$\therefore \boxed{H(S^n) = nH(S)}$

Problem.

1) A ZMS has a source alphabet $S = S_1S_2S_3$ with prob. $P = \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$. Find $H(S)$ also determine $H(S^2)$ & verify that $H(S^2) = 2H(S)$ $[3]^2 = 9$

$\rightarrow H(S) = P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2} + P_3 \log \frac{1}{P_3}$

$H(S) = \frac{3}{2}$

$S_1S_1 = P_1^2$
$S_1S_2 = P_1P_2$
$S_1S_3 = P_1P_3$
$S_2S_1 = P_2P_1$
$S_2S_2 = P_2^2$
$S_2S_3 = P_2P_3$
$S_3S_1 = P_3P_1$
$S_3S_2 = P_3P_2$
$S_3S_3 = P_3^2$

$H(S^2) = P_1^2 + 2P_1P_2 + 2P_1P_3 + 2P_2P_3 + P_2^2 + P_3^2$

$= \sum_{i=1}^{9} P_i \log \frac{1}{P_i}$

$= P_1^2 \log \frac{1}{P_1^2} + 2P_1P_2 \log \frac{1}{P_1P_2} + 2P_1P_3 \log \frac{1}{P_1P_3} + P_2^2 \log \frac{1}{P_2^2} + P_3^2 \log \frac{1}{P_3^2}$

$+ 2P_2P_3 \log \frac{1}{P_2P_3}$

$H(S^2) = 3 = 2H(S) = 2\left(\frac{3}{2}\right) = 3$.

2) A source emits one of the 4 probable msgs. m_1, m_2, m_3 & m_4, with prob $\frac{7}{16}, \frac{5}{16}, \frac{1}{8}, \frac{1}{8}$ resp. Find $H(S)$. List all elements for 2^{nd} extension of source. Hence S.t $H(S^2) = 2H(S)$.

$\rightarrow H(S) = \sum_{i=1}^{4} P_i \log_2 \frac{1}{P_i} = \underline{1.796}$. $[4]^2 = 16$.

$m_1m_1 = P_1^2$	$m_2m_1 = P_1P_2$	$m_3m_1 = P_1P_3$	$m_4m_1 = P_1P_4$
$m_1m_2 = P_1P_2$	$m_2m_2 = P_2^2$	$m_3m_2 = P_2P_3$	$m_4m_2 = P_2P_4$
$m_1m_3 = P_1P_3$	$m_2m_3 = P_2P_3$	$m_3m_3 = P_3^2$	$m_4m_3 = P_3P_4$
$m_1m_4 = P_1P_4$	$m_2m_4 = P_2P_4$	$m_3m_4 = P_4P_3$	$m_4m_4 = P_4^2$

$H(S^2) = \sum_{i=1}^{16} P_i \log \frac{1}{P_i}$.

$= P_1^2 \log \frac{1}{P_1^2} + P_1P_2 \log \frac{1}{P_1P_2} + P_1P_3 \log \frac{1}{P_1P_3} + P_1P_4 \log \frac{1}{P_1P_4}$

$+ P_1P_2 \log \frac{1}{P_1P_2} + P_2^2 \log \frac{1}{P_2^2} + P_2P_3 \log \frac{1}{P_2P_3} + P_2P_4 \log \frac{1}{P_2P_4}$

$+ P_1P_3 \log \frac{1}{P_1P_3} + P_2P_3 \log \frac{1}{P_2P_3} + P_3^2 \log \frac{1}{P_3^2} + P_4P_3 \log \frac{1}{P_4P_3}$

$+ P_1P_4 \log \frac{1}{P_1P_4} + P_2P_4 \log \frac{1}{P_2P_4} + P_3P_4 \log \frac{1}{P_3P_4} + P_4^2 \log \frac{1}{P_4^2}$

$= 3.59$ bits/m.s.

$H(S^2) = 2 \times H(S) = 2 \times 1.796 = \underline{3.59}$

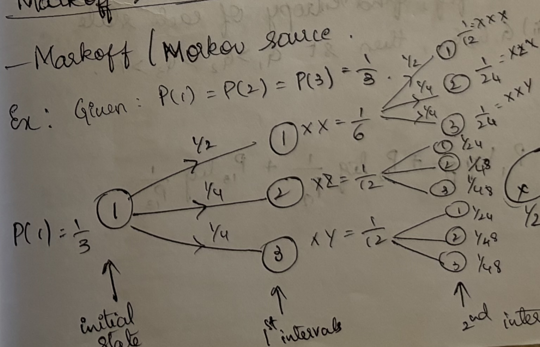
Average info content of symbol in a long dependent sequences.

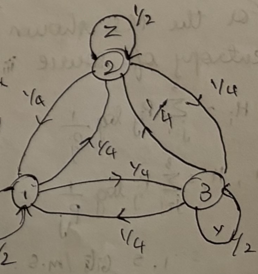
- Depends on previous.

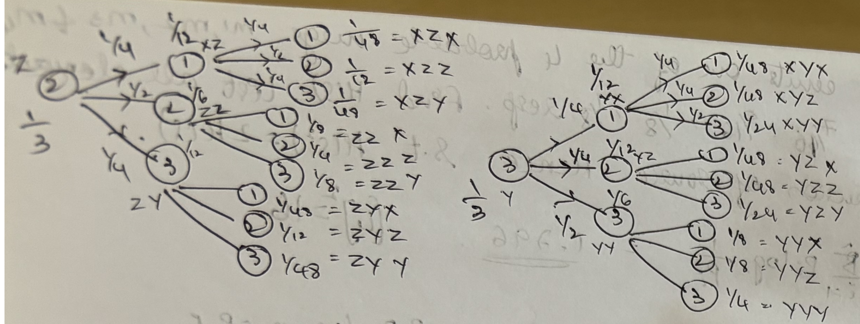
Markoff statical model for info source.

— Markoff / Markov source.

Ex: Given: $P(1) = P(2) = P(3) = \frac{1}{3}$.







Entropy & Info source using Markov source

Entropy of each state, $H_i = \sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}}$

Average entropy, $H = \sum_{i=1}^{n} P_i H_i$

$$H = \sum_{i=1}^{n} P_i \left[\sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}} \right] \text{ bits / m.s}$$

Info. rate : $R_s = r_s H$ bits / sec.

Theorem :- $G_N = \frac{1}{N} \sum_i P(m_i) \log \frac{1}{P(m_i)}$

$G_N = \frac{1}{N} H(S^N)$

Info rate : $\lim_{N \to \infty} G_N = H$ bps

Problem.

i) For the M.S. shown in fig, find entropy of each state
ii) entropy of source iii) $G_1 G_2 G_3$, then st $G_1 > G_2 > G_3 > H$.

$\rightarrow H_i = \sum_{j=1}^{3} P_{ij} \log \frac{1}{P_{ij}}$

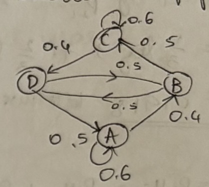
$i = 1$, $H_1 = \sum_{j=1}^{3} P_{ij} \log \frac{1}{P_{ij}} = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}} + P_{13} \log \frac{1}{P_{13}}$

$= 1.5$ bits / m.s.

$H(S) = P_i H_i = P_1 H_1 + P_2 H_2 + P_3 H_3 = \frac{1}{3}(1.5 \times 3) = 1.5$.

2) Consider a state diagram of MS as shown in fig.

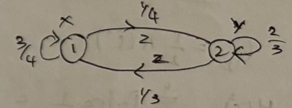
a) compute state probabilities.
b) Find $H(S)$ of each state
c) " " of source.



$\rightarrow P(A) = 0.6 P(A) + 0.5 P(D)$
$P(B) = 0.4 P(B) + 0.5 P(D)$
$P(C) = 0.6 P(C) + 0.5 P(B)$
$P(D) = 0.4 P(C) + 0.5 P(B)$.

3) The given state diagram ~~of a~~ MS shown in fig,

i) Find $H(S)$ of the source.
ii) " G_1, G_2. Hence st $G_1 > G_2 > H$.



$\rightarrow \left[H_i = \sum P_{ij} \log \frac{1}{P_{ij}} \right]$

$P(1) = \frac{3}{4} P(1) + \frac{1}{3} P(2)$ $\qquad$ $P(2) = \frac{2}{3} P(2) + \frac{1}{4} P(1)$

$\frac{1}{3} P(2) = P(1) - \frac{3}{4} P(1)$

$P(2) = \frac{3}{4} P(1)$ $\qquad$ $P(2) = \frac{3}{4} \times \frac{4}{7}$

$P(1) + P(2) = 1$ $\qquad$ $\boxed{P(2) = \frac{12}{28} = \frac{3}{7}}$

$P(1) + \frac{3}{4} P(1) = 1$

$\boxed{P(1) = \frac{4}{7}}$

$$H_1 = \sum_{j=1}^{2} P_{1j} \log \frac{1}{P_{1j}} = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}}$$

$$= \frac{3}{4} \log \frac{4}{3} + \frac{1}{4} \log 4$$

$$H_1 = \underline{0.8113} \text{ bits/m. sym.}$$

$$H_2 = P_{21} \log \frac{1}{P_{21}} + P_{22} \log \frac{1}{P_{22}}$$

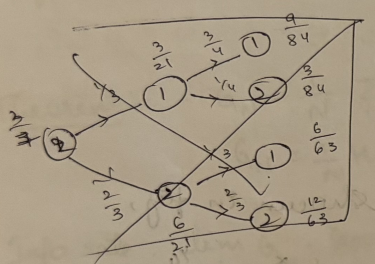
$$= \frac{1}{3} \log 3 + \frac{2}{3} \log \frac{3}{2}$$

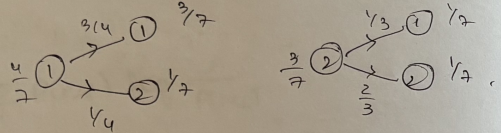
$$\underline{H_2 = 0.918 \text{ bits/msg. sym}}$$

$$H = \sum P_i H_i = P_1 H_1 + P_2 H_2 = \frac{4}{7}(0.8113) + \frac{3}{7}(0.918) = \underline{0.857 \text{ b/ms}}$$

$$G_N = \frac{1}{N} H(\bar{S}^N)$$

$$G_1 = H(\bar{S}) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i} = \frac{4}{7} \log \frac{7}{4} + \frac{3}{7} \log \frac{7}{3} = \underline{0.985}.$$





$$G_2 = \frac{1}{2} H(\bar{S}^2)$$

$$H(\bar{S}^2) = \sum_{i=1}^{4} P_{ij} \log \frac{1}{P_{ij}} = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}} + P_{21} \log \frac{1}{P_{21}} + P_{22} \log \frac{1}{P_{22}}$$

$$\boxed{G_2 = \underline{0.921}}$$

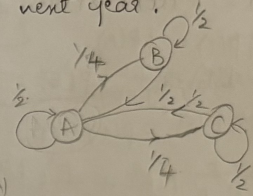
$$G_1 > G_2 > H.$$

Q) Design an info. s/m which gives info every year for about 200 students passing out with BE EC from university, The students can get into one of the 3 fields given below.
The students can get into one of the 3 fields given below.

a) Go abroad for higher studies – A
b) Join civil services / MBA – B
c) " industries in India – C

Based on data given below, construct the model for the source & find source entropy.

- On the avg. 100 students are going abroad.
- Out of 100 going abroad this year, 50 were reported going abroad next year while 25 each went to MBA / C.S or joined industries in India.
- Out of 100 remaining in India this year, 50 continued to do so while 50 went abroad next year.



$P(A) = \frac{1}{2}P(A) + \frac{1}{2}P(B) + \frac{1}{2}P(C)$.

$P(B) = \frac{1}{2}P(B) + \frac{1}{4}P(A) \Rightarrow 1 - \frac{1}{2}P(B) = \frac{1}{4}P(A)$

$2P(B) = P(A)$

$P(B) = \frac{1}{2}P(A)$

$= \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

$P(C) = \frac{1}{2}P(C) + \frac{1}{4}P(A)$

$\frac{1}{4}P(A) = \left[1 - \frac{1}{2}\right]P(C)$

$P(A) = 2P(C)$

$P(C) = \frac{1}{2}P(A) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(A) + P(B) + P(C) = 1$

$\frac{1}{2}P(A) + \frac{1}{2} \times \frac{1}{2}P(A) + \frac{1}{2} \times \frac{1}{2}P(A)$

$+ \frac{1}{2}P(A) + \frac{1}{2}P(A) = 1$

$\frac{1}{2}P(A) + \frac{1}{4}P(A) + \frac{1}{4}P(A) + \frac{1}{2}P(A) \times 2 = 1$

$2P(A) = 1$

$P(A) = 1/2$

P(A)
P(B)
P(C)

H_A
H_B
H_C
H.

$H_A = P_{A1} \log \frac{1}{P_{A}} + P_{A2} \log \frac{1}{P_{A2}} + P_{A3} \log \frac{1}{P_{A3}}$

$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$

$= \frac{3}{2} = \underline{1.5}$

$H_B = P_{B1} \log \frac{1}{P_{B}} + P_{B2} \log \frac{1}{P_{B2}} + P_{B3} \log \frac{1}{P_{B3}}$

$= \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1$

$H_C = P_{C1} \log \frac{1}{P_{C1}} + P_{C2} \log \frac{1}{P_{C2}} + P_{C3} \log \frac{1}{P_{C3}}$

$= \frac{1}{2}\log 2 + 0 + \frac{1}{2}\log 2$

$= \underline{1}$

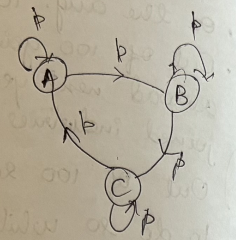
$H = P_A H_A + P_B H_B + P_C H_C$

$= \frac{1}{2} \times \frac{3}{2} + \frac{1}{4} + \frac{1}{4}$

$= \frac{5}{4}$

$\underline{H = 1.25}$

5) For the 1st order MS, the source alphabets $S = \{A, B, C\}$ shown in fig, a) compute probs of states
b) find $H(\bar{S})$ & $H(\bar{S}^2)$



$\rightarrow P(A) = p\,P(A) + p\,P(C)$

$P(B) = p\,P(B) + p\,P(A) \Rightarrow (1-p)P(B) = p\,P(A)$

$P(C) = p\,P(C) + p\,P(B) \qquad P(B) = \left[\frac{p}{1-p}\right]P(A)$

$(1-p)P(C) = p\,P(B)$

$P(C) = \left(\frac{p}{1-p}\right)P(B) = \left(\frac{p}{1-p}\right)\left(\frac{p}{1-p}\right)P(A) = \frac{p^2}{(1-p)^2}P(A)$

$P(A) + P(B) + P(C) = 1$

$P(A) + \left(\frac{p}{1-p}\right)P(A) + \frac{p^2}{(1-p)^2}P(A) = 1$

$P(A)\left[1 + \frac{p}{1-p} + \frac{p^2}{(1-p)^2}\right] = 1$

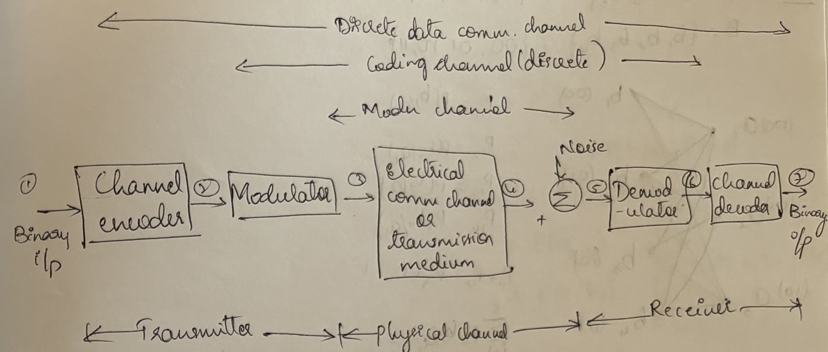
$P(A)\left[(1-p)^2 + p(1-p) + p^2\right] = (1-p)^2$

$P(A)\left[p^2 + p - \cancel{p^2} + 1 + p^2 - 2p\right] = 1 + p^2 - 2p$

$P(A) = \frac{(1-p)^2}{p^2 - p + 1}$

$P(B) = \left(\frac{p}{\cancel{1-p}}\right)\left(\frac{(1-p)^{\cancel{2}}}{p^2-p+1}\right) = \frac{p - p^2}{p^2 - p + 1}$

$P(C) = \left(\frac{p^2}{\cancel{(1-p)^2}}\right)\frac{\cancel{(1-p)^2}}{p^2-p+1} = \frac{p^2}{p^2-p+1}$



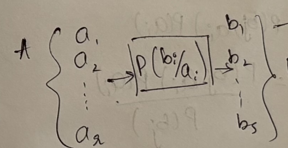
Discrete comm. channel

Representation of channel

i/p 'A' with 'r' symbols. $A = \{a_1, a_2, a_3, \ldots a_r\}$

o/p 'B' " 's' " $B = \{b_1, b_2, b_3, \ldots, b_s\}$

Conditional probabilities $P(b_j/a_i)$



— rep of channel.

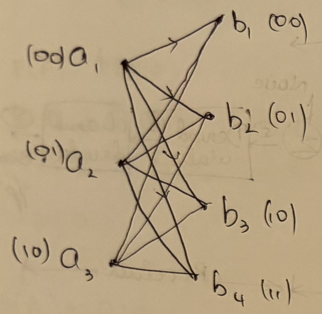
$$P(B/A) = P(b_j/a_i) = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r \end{array}\begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) & \cdots & P(b_s/a_1) \\ P(b_1/a_2) & & & \cdots & \\ P(b_1/a_3) & & & \cdots & \\ \vdots & & & & \\ P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) & \cdots & P(b_s/a_r) \end{bmatrix}_{r \times s}$$

$\downarrow$

Rep of channel matrix / Noise matrix.

Ex : $A = \{a_1 a_2 a_3\} = \{00, 01, 10\}$

$B = \{b_1 b_2 b_3 b_4\} = \{00, 01, 10, 11\}$



$P_{11} = P(b_1/a_1)$

$P_{12} = P(b_2/a_1)$

$P_{13} = P(b_3/a_1)$

$P_{14} = P(b_4/a_1)$

$P_{11} + P_{12} + P_{13} + P_{14} = 1$

$\sum_{j=1}^{s} P(b_j/a_i) = 1$

$P(a_1) + P(a_2) + P(a_3) + \dots P(a_r) = 1$ $\Rightarrow$ Theorem of total probability

$\sum_{i=1}^{r} P(a_i) = 1$

$P(b_1) = P(b_1/a_1)P(a_1) + \dots + P(b_1/a_r)P(a_r)$

$P(b_2) = P(b_2/a_1)P(a_2) + \dots + P(b_2/a_r)P(a_r)$

$\vdots$

$P(b_s) = P(b_s/a_1)P(a_1) + \dots + P(b_s/a_r)P(a_r)$

I/p prob $P(a_i)$

O/p " $P(b_j)$

Condition " $P(b_j/a_i)$

I/p " " $P(a_i/b_j)$

$P(a_i/b_j)P(b_j) = P(b_j/a_i)P(a_i)$

$P(a_i/b_j) = \dfrac{P(b_j/a_i)P(a_i)}{P(b_j)}$

Joint probability

$P(a_i, b_j) = P(a_i \cap b_j) = P(b_j/a_i)P(a_i) = P(a_i/b_j)P(b_j)$

$$P(b_j/a_i)P(a_i) = \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array}\begin{bmatrix} P(b_1/a_1)P(a_1) & P(b_2/a_1)P(a_1) & \dots & P(b_s/a_1)P(a_1) \\ P(b_1/a_2)P(a_2) & P(b_2/a_2)P(a_2) & \dots & P(b_s/a_2)P(a_2) \\ \vdots & & & \\ P(b_1/a_r)P(a_r) & \dots & \dots & P(b_s/a_r)P(a_r) \end{bmatrix}$$

$$P(a_i, b_j) = \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array}\begin{bmatrix} P(a_1, b_1) & P(a_1, b_2) & \dots & P(a_1 b_s) \\ P(a_2, b_1) & \dots & \dots & P(a_2 b_s) \\ \vdots & & & \\ P(a_r b_1) & \dots & \dots & P(a_r b_s) \end{bmatrix}$$

$P(A, B)$

Property 1 : $P(b_1) = P(b_1/a_1)P(a_1) + P(b_1/a_2)P(a_2) + \dots + P(b_1/a_r)P(a_r)$

$P(b_1) = P(a_1, b_1) + P(a_2, b_1) + \dots P(a_r, b_1)$

$P(b_2) = P(a_1 b_2) + \dots + P(a_r b_2)$

$\vdots$

$P(b_s) = P(a_1 b_s) + \dots + P(a_r b_s)$

$\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)$

Property 2 - $P(a_1) = P(b_1/a_1)P(a_1) + P(b_2/a_1)P(a_1) + \dots + P(b_s/a_1)P(a_1)$

$P(a_1) = P(a_1, b_1) + P(a_1, b_2) + \dots P(a_1, b_s)$

$P(a_2) = P(a_2, b_1) + \dots P(a_2 b_s)$

$\vdots$

$P(a_r) = P(a_r, b_1) + \dots P(a_r, b_s)$

$\sum_{j=1}^{s} P(a_i, b_j) = P(a_i)$

Property 3 - $P(a_1) + P(a_2) + \dots P(a_r) = P(a_1 b_1) + P(a_1 b_2) + \dots P(a_1 b_s) + P(a_2 b_1) + P(a_2 b_2) + \dots P(a_2 b_s) + P(a_r b_1) + P(a_r b_2) + \dots P(a_r b_s) = 1$

Problem - In a comm. system, a transmitter has 3 i/p symbols i.e., $A = \{a_1 a_2 a_3\}$ & receiver also has 3 symbols i.e, $B = \{b_1 b_2 b_3\}$. The matrix given below shows JPM.

$a_i \backslash b_j$	b_1	b_2	b_3
a_1	$\frac{1}{12}$	*	$\frac{5}{36}$
a_2	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{5}{36}$
a_3	*	$\frac{1}{6}$	×
$P(b_j)$	$\frac{1}{3}$	$\frac{14}{36}$	*

a) Find missing prob in Table.

b) Find $P(b_3/a_1)$ & $P(a_1/b_3)$.

c) Are the events a_1 & b_1 statistically independent? Why?

$P(b_1) + P(b_2) + P(b_3) = 1$

$\frac{1}{3} + \frac{14}{36} + P(b_3) = 1$

$\boxed{P(b_3) = \frac{5}{18}}$

$P(a_1, b_1) + P(a_2 b_1) + P(a_3, b_1) = P(b_1)$

$\frac{1}{12} + \frac{5}{36} + P(a_3, b_1) = \frac{1}{3}$

$P(a_3, b_1) = \frac{1}{3}$

$P(a_1 b_2) + P(a_2, b_2) + P(a_3, b_2) = \frac{14}{36}$

$P(a_1, b_2) = \frac{1}{9}$

$P(a_1 b_3) + P(a_2, b_3) + P(a_3 b_3) = P(b_3)$

$P(a_3, b_3) = 0.$

$P(a_1, b_3) = P(b_3/a_1)\, P(a_1)$

$P(b_3/a_1) = \frac{P(a_1 b_3)}{P(a_1)} = \frac{\frac{5}{36}}{\frac{1}{3}} = 5/12.$

$\frac{1}{3} \rightarrow \left(\frac{1}{12} + \frac{1}{9} + \frac{5}{36}\right)$

~~$P(a_1/b_3) = P($~~

$P(b_3, a_1) = P(a_1/b_3)\, P(b_3)$

$P(a_1/b_3) = \frac{P(b_3 a_1)}{P(b_3)} = \frac{\frac{5}{36}}{\frac{5}{18}} = \frac{1}{2}.$

$P(a_1 \cap b_1) = P(a_1, b_1) = P(a_i)\, P(b_j)$ *

$\frac{1}{12} \neq \frac{1}{3} \times \frac{1}{3}$