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Unit-4

Refer: David Lay textbook

Symmetric matrices and Quadratic forms

Diagonalization of symmetric matrix

1. If possible diagonalize the mat $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

$\rightarrow$ $\Delta(t) = t^3 - 17t^2 + (\cancel{34}+\cancel{29}+\cancel{30})t - 144.$

$= t^3 - 17t^2 + \cancel{102t}\ 90 - 144.$

$t = 3, 6, 8.$

$M = (A-\lambda I) \rightarrow 3$

$= \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$V_1 = (1,1,1)$

$N = (A-\lambda I) \rightarrow 6$

$= \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \rightarrow V_2 \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$

$M = A - \lambda I \rightarrow 8$

$= \begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix}$ $V_3 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$

$\hat{V}_1 = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ $\hat{V}_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$ $\hat{V}_3 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0.57 & 0.57 & 0.57 \\ 0.408 & 0.408 & -0.816 \\ 0.707 & -0.707 & 0 \end{bmatrix}$

P^{-1} will be equal to P^T in case of sym matrix

$10\log \frac{1}{AP^2}$ classmate

$-20\log AP$ Date ____ Page ____

$D = P^{-1}AP$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

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Find a least square solution of $Ax = b$ by (a) constructing the normal eqns for $\hat{x}$. (b) Solving for $\hat{x}$. (c) Compute the least square error.

1. $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

→

$$A^TA = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb.$$

$$\hat{x} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}^{-1}\begin{bmatrix} -4 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$A\hat{x} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\| b - A\hat{x} \| = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \sqrt{9+1+1} = \sqrt{11}$$

2. $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$= \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}^{-1}\begin{bmatrix} -24 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$A\hat{x} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

$$\| b - A\hat{x} \| = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{0}}$$

3. $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

$\rightarrow$ $$A^T A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \quad \hat{x} = \begin{matrix} 6 \\ -6 \end{matrix}$$

$6x_0 + 6x_1 = 6$

$6x_0 + 42 \ldots$

$$A^T b = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{216} \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{16/7} \\ \cancel{-4/7} \end{bmatrix} \quad \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

$$A\hat{x} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\|b - A\hat{x}\| = \begin{bmatrix} 1 \\ +3 \\ -3 \\ 1 \end{bmatrix} = \sqrt{1+9+9+1} = \sqrt{20}$$

4. $$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & +3 \\ +3 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \hat{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\hat{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\|b - A\hat{x}\| = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \underline{\underline{\sqrt{6}}}$$

2 Find the eqn $y = B_0 + B_1 x$ of the least square line that best fits the given data points.

(i) (0,1), (1,1), (2,2), (3,2)

(ii) (1,0), (2,1), (4,2), (5,3)

(iii) (-1,0), (0,1), (1,2), (2,4)

(iv) (2,3), (3,2), (5,1), (6,0)

→ $XB = Y$.

$X^T X B = X^T Y$

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad Y = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$X^T X\, \beta = X^T y$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \beta = \begin{bmatrix} 6 \\ 11 \end{bmatrix} \Rightarrow \beta = \begin{array}{l} \beta_0 = \\ \beta_1 = \end{array} \begin{bmatrix} 9/10 \\ 2/5 \end{bmatrix}$$

$$y = 9/10 + 2/5\, x$$

(ii)

$$x = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \end{bmatrix}$$

$$X^T X\, B = X^T y$$

$$\begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \end{bmatrix} \qquad \begin{array}{l} B_0 = \\ B_1 = \end{array} \begin{bmatrix} -3/5 \\ 7/10 \end{bmatrix}$$

$$y = \frac{-3}{5} + \frac{7}{10} x$$

(iii)

$$x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$X^T X \beta = X^T y \Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} \rightarrow \beta = \begin{bmatrix} 11/10 \\ 13/10 \end{bmatrix}$$

$$y = \frac{11}{10} + \frac{13}{10} x$$

(iv)

$$x = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \quad y = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

$$X^T X B = X^T y$$

$$\begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix} = \begin{bmatrix} 43/10 \\ -7/10 \end{bmatrix}$$

$$y = \frac{43}{10} - \frac{7}{10} x$$

3. Find the least square line $y = \beta_0 + \beta_1 x$ that best fits the data $(-2,0)$ $(-1,0)$, $(0,2)$, $(1,4)$, $(2,4)$. Assuming the first & last data points are less reliable weigh them half as much as the three interior points

$\rightarrow$

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad y = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

$$WX = \begin{bmatrix} 1 & -2 \\ 2 & -2 \\ 2 & 0 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} \qquad Wy = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 4 \end{bmatrix}$$

$$(WX)^T(WX) = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ -2 & -2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & -2 \\ 2 & 0 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(WX)^T(WY) = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ -2 & -2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 28 \\ 24 \end{bmatrix}$$

$$(WX)^T(WX)\,\beta = (WX)^T(Wy)$$

$$\begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 28 \\ 24 \end{bmatrix} \qquad \beta = \begin{bmatrix} 2 \\ 3/2 \end{bmatrix}$$

$$y = 2 + \frac{3}{2}x$$

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Orthognally diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$\Delta(t) = t^3 - 12t^2 + (16 - 7 + 16) + 98$

$= t^3 - 12t^2 + 2\cancel{5}t + 98$

$\lambda_1 = -2, \ \lambda_2 = 7, \ \lambda_3 = 7.$

$M = A - \lambda I$

$= \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

$\left.\begin{array}{l} 5x - 2y + 4z = 0 \\ -2x + 8y + 2z = 0 \\ 4x + 2y - 5z = 0 \end{array}\right\} \quad v_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$

$M = A - \lambda I \to 7.$

$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \Rightarrow \begin{array}{l} -4x - 2y + 4z = 0 \\ -2x - y + 2z = 0 \\ 4x + 2y - 4z = 0 \end{array}$

$\to 2x + y - 2z = 0.$

Assume $y = 0$ $z = 1$

$\to v_1 = [1 \ 0 \ 1]$

Assume $y = 1$ $z = 0$ $x = -\frac{1}{2}$ $\to v_2 = [\frac{-1}{2} \ 1 \ 0]$

The eigen vectors are not orthogonal

Although v_1 and v_2 are linearly independant, they are not orthogonal. To make v_1 & v_2 orthogonal, we need to calculate projection of v_2 on v_1. Then vectors will be orthogonal

$z_2 = v_2 - \frac{\langle v_2 . v_1 \rangle}{\langle v_1 . v_1 \rangle} . v_1$

$= [\frac{-1}{2}, 1, 0] - \frac{\{\frac{-1}{2}\}}{2} . [1 \ 0 \ 1]$

$= \cancel{\frac{-1}{2}} \ (\frac{-1}{2}, 1, 0) + \frac{1}{4}\{1, 0, 1\} = \{\frac{-1}{4}, 1, \frac{1}{4}\}$

$\hat{v}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ $\hat{v}_2 = \|(-1, 4, 1)\| = \left(\frac{-1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, \frac{1}{\sqrt{18}}\right)$

$$v_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \quad \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & 4/\sqrt{18} & \frac{-1}{3} \\ -1/\sqrt{2} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}$$

Since A is symmetric, $P^{-1} = P^T$

$$\Rightarrow D = P^{-1}AP$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \\ \frac{-2}{3} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & 2/3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

2 Orthogonally diagonalize the given matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\Delta(t) = t^3 - 5t^2 + (2-8+2)t + 20 = t^3 - 5t^2 - 4t + 20 = 0$$

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Spectral theorem :-

An $m\times n$ symmetric mat A has the following properties

1. mat A has n real eigen values, counting multiplicities.
2. The dim of eigen space for each eigen value λ equals the multiplicity of λ as a root of CE
3. The eigen spaces are mutually orthogonal, in the sense that eigen vectors corresponding to different eigen values are orthogonal
4. mat A is orthogonally diagonalizable

Spectral decomposition:

$$A = PDP^T$$

$$A = \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 V_1 + \cdots + \lambda_n V_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

$$= \lambda_1 V_1 V_1^T + \lambda_2 V_2 V_2^T + \cdots + \lambda_n V_n V_n^T$$

1. Construct a spectral decomposition of the matrix A that has orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & \frac{-1}{\sqrt{5}} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$A = \lambda_1 V_1 V_1^T + \lambda_2 V_2 V_2^T$$

$$= 8 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} + 3 \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3 & -6/5 \\ -6 & 12/5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A$$

Quadratic forms:

A quadratic form of R^n is a function Q defined by on R^n whose value at a vector x in R^n can be computed by an expression of the form $Q(x) = x^T A x$ where A is an $n \times n$ symmetric matrix. The mat A is called the matrix of quadratic form.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $x^T A x$ for the following matrices

(i) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

$$Q(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix}$$

$$= 4x_1^2 + 3x_2^2$$

(ii) $$Q(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

2. $A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$ $\Delta(t) = t^2 + 4t + -21$

$t = 3, -7$

$M = A - \lambda I$ $\lambda = -7$

$$\begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}$$

$\left.\begin{matrix} 8x - 4y = 0 \\ -4x + 2y = 0 \end{matrix}\right\}$ $2x - y = 0$

$y = 2$

$\Rightarrow x = 1$

$v_1 = \cancel{(2, 1)}$

$(1, 2)$

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$M = A - \lambda I \quad (\lambda = 3)$

$$= \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} =$$

$-2x - 4y = 0$

$-4x - 8y = 0$

$\Rightarrow x + 2y = 0$

$v_2 = (2, -1) \quad y = -1$

$x = 2.$

$\hat{v}_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \quad \hat{v}_2 = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

$Q(x) = x^T A x \qquad x = Py.$

$= (Py)^T A (Py)$

$= y^T P^T A P y$

$= y^T D y$

$$Q(x) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$= 3y_1^2 - 7y_2^2$

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Constrained optimization :-

Read textbook

Singular Value Decomposition

1. Find a SVD of ~~that~~ $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

continue

→ Step 1 For the matrix A find $A^T A$

$$\Rightarrow A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$\Delta(t) = t^3 - 450t^2 + (14400 + 14400 + 3600)t + \cancel{2804400}$

$= \lambda^3 - 450\lambda^2 + 32400\lambda$

$\Rightarrow \lambda = 360, 90, 0$

$\Rightarrow \lambda_1 = 0 \quad \lambda_2 = 90 \quad \lambda_3 = 360$

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$M = A - \lambda I \rightarrow \lambda = 360$

$M = \begin{bmatrix} -280 & 100 & 40 \\ 100 & -190 & 140 \\ 40 & 140 & -160 \end{bmatrix}$ $V_1 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ $\hat{V}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

(ii) $M = \begin{bmatrix} -10 & 100 & 40 \\ 100 & 80 & 140 \\ 40 & 140 & 110 \end{bmatrix}$. $V_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ $\hat{V}_2 =$

(iii) $M = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$. $V_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$V = \underbrace{\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}}_{n \times n \text{ columns}} \rightarrow V^T \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

The singular values of given matrix A are

$\sigma_1 = \sqrt{360} = 6\sqrt{10}$ $\sigma_2 = 3\sqrt{10}$ $\sigma_3 = 0$

$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$ $\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$

Σ → same size as A

$U = [u_1 \ u_2]$

$u_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 54 \\ 18 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} AV_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

$U \Sigma V^T$

$A = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 3/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}_{m \times m \text{ rows of } A} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \frac{1}{3}$

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$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

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$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$$

$$\Delta(t) = t^2 - 50t + 225$$

$$\lambda_1 = 45 \qquad \lambda_2 = 5$$

$$M = A^TA - \lambda_1 I$$

$$= \begin{bmatrix} -20 & 20 \\ 20 & -20 \end{bmatrix} \rightarrow V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \hat{V}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$M = A^TA - \lambda_2 I$$

$$\begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \hat{V}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\sigma_1 = \sqrt{45} = 3\sqrt{5} \qquad \sigma_2 = \sqrt{5} \qquad \cancel{\sigma_2} =$$

$$D = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$U_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{3\sqrt{5}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{5} \\ 3/\sqrt{5} \end{bmatrix}$$

$$U_2 = \frac{1}{\sigma_2} AV_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{5}} & 3/\sqrt{5} \\ 3/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

$A = U \Sigma V^T$

$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

3. Find a SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$\rightarrow$ $A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

$\Delta(t) = t^2 - 18t + 0.$

$\lambda_1 = 18 \quad \lambda_2 = 0.$

$M = A^T A - \lambda_1 I$

$= \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \hat{V_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$M = A^T A - 0$

$= \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \hat{V_2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\sigma_1 = 3\sqrt{2} \quad \sigma_2 = 0$

$D = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$U_1 = \frac{1}{\sigma_1} A V_1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

~~$U_2 = \frac{1}{\sigma_2} A V_2 = \frac{1}{\sqrt{2}}$~~

$\hookrightarrow x_1 - 2x_2 + 2x_3$

$x_2 = 1 \quad x_3 = 0$

$\Rightarrow (2, 1, 0) \rightarrow w_1$

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$x_1 - 2x_2 + 2x_3$

$x_2 = 0 \quad x_3 = 1$

$w_2 = (-2, 0, 1)$

$$\hat{w}_1 = w_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\hat{w}_2 = w_2 - \frac{\langle w_2 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1$$

$$= (-2, 0, 1) - \frac{-4+0+0}{5} \frac{(2, 1, 0)}{\sqrt{5}\sqrt{5}}$$

$$= (-2, 0, 1) + \frac{4}{5\sqrt{5}} (2, 1, 0)$$

$$= (-0.4, \frac{4}{5}, 1)$$

$$u_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

$A = U \Sigma V^T$

$$= \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

4 Compute SVD for $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$\Rightarrow$ $A^TA = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ $\quad \lambda_1 = 25 \quad \lambda_2 = 9 \quad \lambda_3 = 0$

$\Delta(t) = t^3 - 34t^2 + (100 - 100 + 25)t - 200$

$M = A - \lambda_1 I$

$= \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$ $\quad V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \hat{V}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$M = A - \lambda_2 I$

$= \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ $\quad \hat{V}_2 = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}$

$M = A - \lambda_3 I$

$= \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ $\quad V_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ $\quad \hat{V}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & 2/3 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}$$

$\sigma_1 = 5 \quad \sigma_2 = 3 \quad \sigma_3 = 0$

$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \quad = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \Sigma$

$u_1 = \frac{1}{\sigma_1} AV_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\qquad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} AV_2 = \begin{bmatrix} 3/\sqrt{18} \\ -3/\sqrt{18} \end{bmatrix}$

$A = U \Sigma V^T$

$= \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

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5. Construct SVD for the given matrix $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$U = 3\times3$

$\Sigma = 3\times2$

$V = 2\times2$

→ $A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

$\Delta(t) = t^2 - 90t - 0$

$\lambda_1 = 90 \quad \lambda_2 = 0$

$M = A - \lambda_1 I$

$= \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} = \begin{matrix} -9x - 27y = 0 \\ -27x - 81y = 0 \end{matrix}$

$x + 3y = 0$

$y = 1 \quad x = -3$

$V_1 = \begin{bmatrix} -3 & 1 \end{bmatrix}$

$M = A - \lambda_2 I$

$= \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

$81x - 27y = 0$
$-27x + 9y = 0$

$3x - y = 0 \Rightarrow y = 3 \quad x = 1$

$V_2 = \begin{bmatrix} 3 & 1 \end{bmatrix}$

$V = \begin{bmatrix} \frac{-3}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$

$\sigma_1 = 3\sqrt{10} \quad \sigma_2 = 0$

$D = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$U_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{\sqrt{10}\times 3\sqrt{10}} \begin{bmatrix} 10 \\ -20 \\ -20 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$

$x - 2x_2 - 2x_3 = 0$

$x_3 = 0$

$x_2 = 1$

$x_1 = 2$

$w_1 = (2,1,0) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$

$x_2 = 0 \quad x_3 = 1$

$w_2 = (2,0,1)$

$U_1 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$

$w_3 =$

$$\hat{w}_2 = w_2 - \frac{\langle w_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (2, 0, 1) - \frac{4}{5}(2, 1, 0)$$

$$= (0.4, -0.8, 1)$$

$$= \sqrt{\frac{4}{25} + \frac{16}{25} + \frac{25}{25}} = \frac{1}{5}\sqrt{45} = \frac{9}{\sqrt{45}}$$