$$H(k) = \sum_{m=0}^{N-1} h(m) W_N^{mk}, \quad k = 0, 1, \cdots N-1$$

Substituting, the expressions for $X(k)$ and $H(k)$ in equation (3.15), we get

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{N-1} x(i) W_N^{ik} \sum_{m=0}^{N-1} h(m)\ W_N^{mk}\ W_N^{-nk}$$

Interchanging the order of summations, we get

$$y(n) = \frac{1}{N} \sum_{i=0}^{N-1} x(i) \sum_{m=0}^{N-1} h(m) \sum_{k=0}^{N-1} W_N^{(i+m-n)k}$$

The summation over k equals N when $i = n - m$ and zero for all other i.

Hence,

$$y(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(n-m)\ h(m) \times N$$

Since, the shift $(n - m)$ is circular, we may write the above equation as

$$\begin{aligned} y(n) &= \sum_{m=0}^{N-1} x((n-m))_N h(m) \\ &= h(n) \circledast_N x(n) = x(n) \circledast_N h(n) \quad (\because \text{ of commutative property, refer Example 3.35}) \end{aligned}$$

3.7.8.1 Circular convolution in time-domain is equivalent to multiplication in frequency-domain

$$\text{DFT}\{h(n) \circledast_N x(n)\} = H(k)X(k), \quad k = 0, 1, \cdots N-1$$

Proof:

$$\begin{aligned} \text{DFT}\{h(n) \circledast_N x(n)\} &= \text{DFT}\left\{ \sum_{l=0}^{N-1} h(l)\ x((n-l))_N \right\} \\ &= \sum_{l=0}^{N-1} h(l)\ \underbrace{W_N^{kl} X(k)}_{\text{DFT}\{x((n-l))_N\}} \\ &= H(k)X(k) \end{aligned}$$

Example 3.30 For $x_1(n)$ and $x_2(n)$ given below, compute $x_1(n) \circledast_N x_2(n)$. Take $N = 3$.

$$\begin{aligned} x_1(n) &= (1, 1, 1) \\ x_2(n) &= (1, -2, 2) \end{aligned}$$

☐ Solution

Let
$$y(n) = x_1(n) \circledast_N x_2(n), \quad N = 3$$
$$\triangleq \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

Here, $N = 3$.
The table below demonstrates the computation of $y(n)$ using the above defining equation.

n	$x_1(m)$	$x_2((n-m))_N$	$y(n)$
0	(1, 1, 1)	(1, 2, −2)	$1 \times 1 + 1 \times 2 + 1 \times -2 = 1$
1	(1, 1, 1)	(−2, 1, 2)	$1 \times -2 + 1 \times 1 + 1 \times 2 = 1$
2	(1, 1, 1)	(2, −2, 1)	$1 \times 2 + 1 \times -2 + 1 \times 1 = 1$

Hence, $y(n) = (1, 1, 1)$.

Alternate method

In the Stockham's method of computing circular convolution, we make use of the following block diagram:

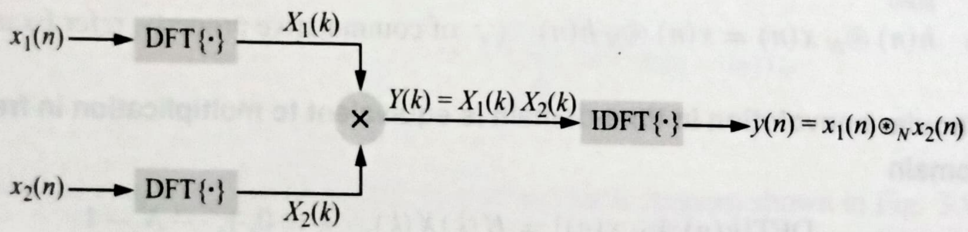

Fig. Ex.3.30 Stockham's method of performing circular convolution of two sequences of length N.

Let us first find $X_1(k)$ and $X_2(k)$.

$$X_1(k) = \sum_{n=0}^{2} x_1(n) W_3^{kn}$$
$$= 1 + 1 \times W_3^k + 1 \times W_3^{2k}, \quad 0 \leqslant k \leqslant 2$$

and
$$X_2(k) = \sum_{n=0}^{2} x_2(n) W_3^{kn}$$
$$= 1 - 2W_3^k + 2W_3^{2k}, \quad 0 \leqslant k \leqslant 2$$

Then,
$$Y(k) = X_1(k) X_2(k)$$
$$= (1 + W_3^k + W_3^{2k}) \times (1 - 2W_3^k + 2W_3^{2k})$$
$$= 1 - 2W_3^k + 2W_3^{2k} + W_3^k - 2W_3^{2k} + 2W_3^{3k}$$
$$+ W_3^{2k} - 2W_3^{3k} + 2W_3^{4k}$$

$$\begin{aligned}
\text{Since,} \quad W_3^{3k} &= W_3^{0k} = 1 \\
\text{and} \quad W_3^{4k} &= W_4^{k} \\
\text{we get,} \quad Y(k) &= 1 + W_3^k + W_3^{2k} \\
\text{Hence,} \quad y(n) &= (1, 1, 1)
\end{aligned}$$

Example 3.31 For the sequences

$$x_1(n) = \cos\left(\frac{2\pi n}{N}\right), \quad x_2(n) = \sin\left(\frac{2\pi n}{N}\right), \quad 0 \leqslant n \leqslant N-1$$

find the N-point circular convolution $x_1(n) \circledast_N x_2(n)$.

□ Solution

Given

$$\begin{aligned}
x_1(n) &= \cos\left(\frac{2\pi n}{N}\right) \\
&= \frac{1}{2}e^{j\frac{2\pi n}{N}} + \frac{1}{2}e^{-j\frac{2\pi n}{N}} \\
&= \frac{1}{2}W_N^{-n} + \frac{1}{2}W_N^{n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{DFT}\{x_1(n)\} &= X_1(k) \\
&= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{-n}W_N^{kn} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{n}W_N^{kn} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k-1)n} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k+1)n} \\
&= \frac{1}{2}N\delta(k-1) + \frac{N}{2}\delta(k+1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
x_2(n) &= \sin\left(\frac{2\pi n}{N}\right) \\
&= \frac{1}{2j}e^{j\frac{2\pi n}{N}} - \frac{1}{2j}e^{-j\frac{2\pi n}{N}} \\
&= \frac{1}{2j}W_N^{-n} - \frac{1}{2j}W_N^{n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{DFT}\{x_2(n)\} &= X_2(k) \\
&= \frac{N}{2j}\delta(k-1) - \frac{N}{2j}\delta(k+1)
\end{aligned}$$

Let

$$y(n) = x_1(n) \circledast_N x_2(n)$$

Then,

$$\begin{aligned}
Y(k) &= X_1(k)X_2(k) \\
\Rightarrow \quad Y(k) &= \frac{N^2}{4j}[\delta(k-1) - \delta(k+1)]
\end{aligned}$$

Please note that, $\delta(k-1)\delta(k+1) = 0$.

Hence,
$$y(n) = \frac{N}{2}\sin\left(\frac{2\pi n}{N}\right), \quad 0 \leqslant n \leqslant N-1$$

Example 3.32 Find the 4-point circular convolution of the sequences,

$$x_1(n) = (\underset{\uparrow}{1}, 2, 3, 1)$$

and

$$x_2(n) = (\underset{\uparrow}{4}, 3, 2, 2)$$

using the time-domain approach and verify the result using frequency-domain approach.

☐ Solution

Time-domain approach

Let

$$\begin{aligned} y(n) &= x_1(n) \circledast_N x_2(n), \qquad N = 4 \\ &\triangleq \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N, \quad 0 \leqslant n \leqslant N-1 \end{aligned}$$

n	$x_1(m)$	$x_2((n-m))_N$	$y(n)$
0	(1, 2, 3, 1)	(4, 2, 2, 3)	$1 \times 4 + 2 \times 2 + 3 \times 2 + 1 \times 3 = 17$
1	(1, 2, 3, 1)	(3, 4, 2, 2)	$1 \times 3 + 2 \times 4 + 3 \times 2 + 1 \times 2 = 19$
2	(1, 2, 3, 1)	(2, 3, 4, 2)	$1 \times 2 + 2 \times 3 + 3 \times 4 + 1 \times 2 = 22$
3	(1, 2, 3, 1)	(2, 2, 3, 4)	$1 \times 2 + 2 \times 2 + 3 \times 3 + 1 \times 4 = 19$

Frequency-domain approach

The circular convolution is done using frequency-domain approach by referring the block diagram shown below:

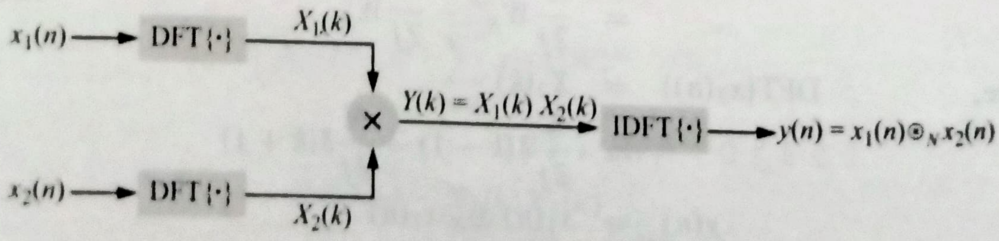

Fig. Ex.3.32 Block diagram for computing IDFT.

$$\text{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{3} x_1(n)W_4^{kn}$$
$$= 1 + 2W_4^k + 3W_4^{2k} + W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

$$\text{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{3} x_2(n)W_4^{kn}$$
$$= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

Hence,
$$\begin{aligned} Y(k) &= X_1(k)X_2(k) \\ &= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k} + 8W_4^k + 6W_4^{2k} + 4W_4^{3k} \\ &\quad +4W_4^{4k} + 12W_4^{2k} + 9W_4^{3k} + 6W_4^{4k} + 6W_4^{5k} \\ &\quad +4W_4^{3k} + 3W_4^{4k} + 2W_4^{5k} + 2W_4^{6k} \\ &= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k} + 4 + 8W_4^k + 6W_4^{2k} + 4W_4^{3k} \\ &\quad +6 + 6W_4^k + 12W_4^{2k} + 9W_4^{3k} + 3 + 2W_4^k + 2W_4^{2k} + 4W_4^{3k} \\ &\quad (\because\ W_4^{4k} = W_4^{0k} = 1,\ W_4^{5k} = W_4^k,\ W_4^{6k} = W_4^{2k}) \end{aligned}$$

Hence,
$$Y(k) = 17 + 19W_4^k + 22W_4^{2k} + 19W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

Taking IDFT, we get

$$y(n) = 17 + 19\delta(n-1) + 22\delta(n-2) + 19\delta(n-3)$$
$$\Rightarrow \quad y(n) = (\underset{\uparrow}{17}, 19, 22, 19)$$

Example 3.33 Let $g(n)$ and $h(n)$ be the two finite-length sequences of length-5 each. If $y_l(n)$ and $y_c(n)$ denote the linear and 5-point circular convolution of $g(n)$ and $h(n)$ respectively, express $y_c(n)$ interms of $y_l(n)$.

☐ Solution

Let
$$g(n) = (g_0, g_1, g_2, g_3, g_4)$$
and
$$h(n) = (h_0, h_1, h_2, h_3, h_4)$$

To find $y_c(n)$:

$$y_c(n) = g(n) \circledast_N h(n), \quad N = 5$$
$$\triangleq \sum_{n=0}^{4} g(n)h((n-m))_5, \quad 0 \leqslant n \leqslant 4$$

n	$g(m)$	$h((n-m))_5$	$y_c(n)$
0	$(g_0, g_1, g_2, g_3, g_4)$	$(h_0, h_4, h_3, h_2, h_1)$	$y_c(0) = g_0h_0 + g_1h_4 + g_2h_3 + g_3h_2 + g_4h_1$
1	$(g_0, g_1, g_2, g_3, g_4)$	$(h_1, h_0, h_4, h_3, h_2)$	$y_c(1) = g_0h_1 + g_1h_0 + g_2h_4 + g_3h_3 + g_4h_2$
2	$(g_0, g_1, g_2, g_3, g_4)$	$(h_2, h_1, h_0, h_4, h_3)$	$y_c(2) = g_0h_2 + g_1h_1 + g_2h_0 + g_3h_4 + g_4h_3$
3	$(g_0, g_1, g_2, g_3, g_4)$	$(h_3, h_2, h_1, h_0, h_4)$	$y_c(3) = g_0h_3 + g_1h_2 + g_2h_1 + g_3h_0 + g_4h_4$
4	$(g_0, g_1, g_2, g_3, g_4)$	$(h_4, h_3, h_2, h_1, h_0)$	$y_c(4) = g_0h_4 + g_1h_3 + g_2h_2 + g_3h_1 + g_4h_0$

To find $y_l(n)$

$$
\begin{aligned}
y_l(n) &= g(n)_\infty * h(n) \\
&\triangleq \sum_{m=-\infty}^{\infty} g(m)h(n-m)
\end{aligned}
$$

i. $n = 0$

$$
\begin{array}{lccccccccc}
 & & & & & \downarrow & & & & \\
g(m): & & & & & g_0 & g_1 & g_2 & g_3 & g_4 \\
h(-m): & h_4 & h_3 & h_2 & h_1 & h_0 & & & & \\
\hline
\end{array}
$$

$$y_l(0) = g_0h_0$$

ii. $n = 1$

$$
\begin{array}{lccccccccc}
 & & & & & \downarrow & & & & \\
g(m): & & & & & g_0 & g_1 & g_2 & g_3 & g_4 \\
h(1-m): & & h_4 & h_3 & h_2 & h_1 & h_0 & & & \\
\hline
\end{array}
$$

$$y_l(1) = g_0h_1 + g_1h_0$$

iii. $n = 2$

$$
\begin{array}{lccccccccc}
 & & & & & \downarrow & & & & \\
g(m): & & & & & g_0 & g_1 & g_2 & g_3 & g_4 \\
h(2-m): & & & h_4 & h_3 & h_2 & h_1 & h_0 & & \\
\hline
\end{array}
$$

$$y_l(2) = g_0h_2 + g_1h_1 + g_2\,h_0$$

iv. $n = 3$

$$
\begin{array}{lccccccccc}
 & & & & & \downarrow & & & & \\
g(m): & & & & & g_0 & g_1 & g_2 & g_3 & g_4 \\
h(3-m)): & & & & h_4 & h_3 & h_2 & h_1 & h_0 & \\
\hline
\end{array}
$$

$$y_l(3) = g_0h_3 + g_1h_2 + g_2h_1 + g_3h_0$$

v. $n = 4$

$$
\begin{array}{lccccc}
g(m): & g_0 & g_1 & g_2 & g_3 & g_4 \\
h(4-m): & h_4 & h_3 & h_2 & h_1 & h_0 \\
\hline
\end{array}
$$

$$y_l(4) = g_0h_4 + g_1h_3 + g_2h_2 + g_3h_1 + g_4h_0$$

vi. $n = 5$

$$\begin{array}{lccccccccc} & \downarrow & & & & & & & & \\ g(m): & g_0 & g_1 & g_2 & g_3 & g_4 & & & & \\ h(5-m): & & h_4 & h_3 & h_2 & h_1 & h_0 & & & \\ \hline \end{array}$$

$$y_l(5) = g_1h_4 + g_2h_3 + g_3h_2 + g_4h_1$$

vii. $n = 6$

$$\begin{array}{lccccccccc} & \downarrow & & & & & & & & \\ g(m): & g_0 & g_1 & g_2 & g_3 & g_4 & & & & \\ h(6-m): & & & h_4 & h_3 & h_2 & h_1 & h_0 & & \\ \hline \end{array}$$

$$y_l(6) = g_2h_4 + g_3h_3 + g_4h_2$$

viii. $n = 7$

$$\begin{array}{lccccccccc} & \downarrow & & & & & & & & \\ g(m): & g_0 & g_1 & g_2 & g_3 & g_4 & & & & \\ h(7-m): & & & & h_4 & h_3 & h_2 & h_1 & h_0 & \\ \hline \end{array}$$

$$y_l(7) = g_3h_4 + g_4h_3$$

ix. $n = 8$

$$\begin{array}{lccccccccc} g(m): & g_0 & g_1 & g_2 & g_3 & g_4 & & & & \\ h(8-m): & & & & & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline \end{array}$$

$$y_l(8) = g_4h_4$$

For $n > 8$, $y_l(n) = 0$

Also, for $n < 0$, $y_l(n) = 0$

Thus, we find that

$$\begin{aligned} y_c(0) &= y_l(0) + y_l(5) \\ y_c(1) &= y_l(1) + y_l(6) \\ y_c(2) &= y_l(2) + y_l(7) \\ y_c(3) &= y_l(3) + y_l(8) \\ y_c(4) &= y_l(4) \end{aligned}$$

Hence, circular convolution equals linear convolution plus aliasing.

Example 3.34 Fig. Ex.3.34 shows two finite-length sequences. Sketch their 6-point circular convolution.

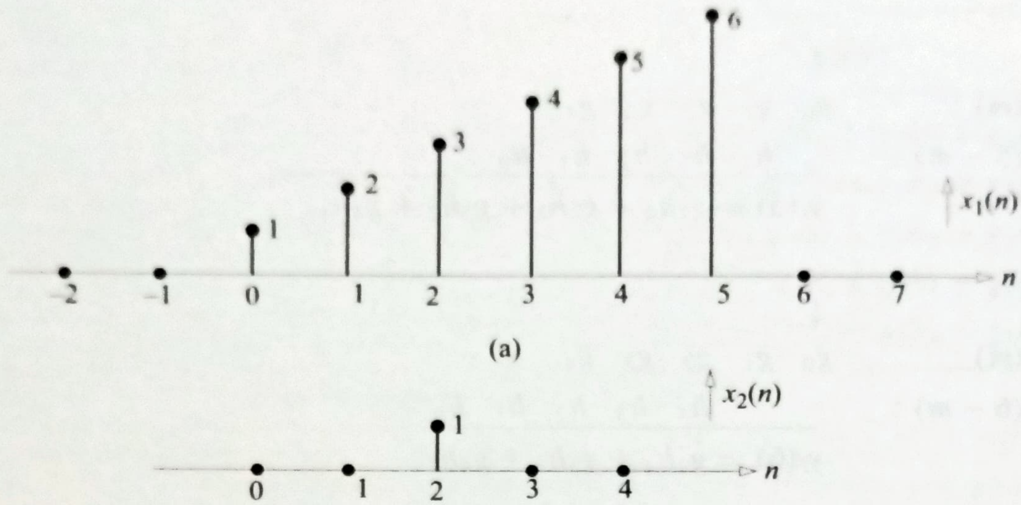

Fig. Ex.3.34 Sequences $x_1(n)$ and $x_2(n)$ for Example.3.34.

☐ Solution

Since $x_2(n)$ is a just shifted impulse, the circular convolution coincides with a circular shift of $x_1(n)$ by two points.

$$\begin{aligned} y(n) &= x_1(n) \circledast_6 x_2(n) \\ &= x_1(n) \circledast_6 \delta(n-2) \\ &= x_1((n-2))_6 = (5, 6, 1, 2, 3, 4) \\ & \qquad\qquad\qquad\qquad\quad \uparrow \end{aligned}$$

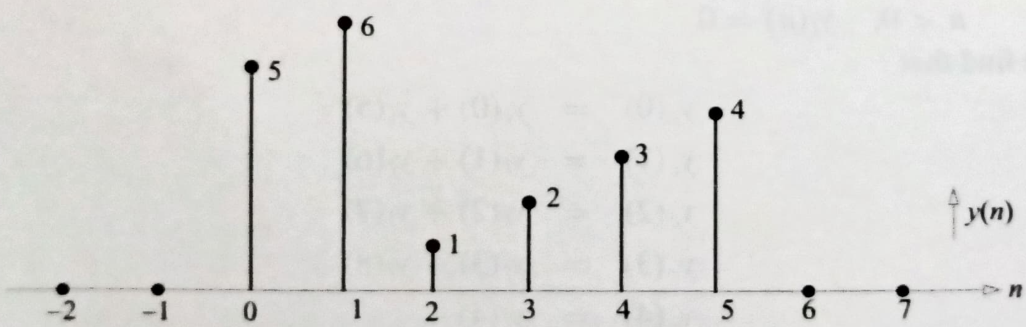

Fig. Ex.3.34(a) 6-point circular convolution of $x_1(n)$ and $x_2(n)$.

Alternate solution

$$\begin{aligned} y(n) &= x_1(n) \circledast_6 x_2(n) \\ &= \sum_{k=0}^{5} x_1(m) \;\; x_2((n-m))_6 \end{aligned}$$

n	$x_1(m)$	$x_2((n-m))_6$	$y(n)$
0	(1, 2, 3, 4, 5, 6)	(0, 0, 0, 0, 1, 0)	5
1	(1, 2, 3, 4, 5, 6)	(0, 0, 0, 0, 0, 1)	6
2	(1, 2, 3, 4, 5, 6)	(1, 0, 0, 0, 0, 0)	1
3	(1, 2, 3, 4, 5, 6)	(0, 1, 0, 0, 0, 0)	2
4	(1, 2, 3, 4, 5, 6)	(0, 0, 1, 0, 0, 0)	3
5	(1, 2, 3, 4, 5, 6)	(0, 0, 0, 1, 0, 0)	4

Example 3.35 Prove the commutative property of circular convolution.

That is, $$x(n) \circledast_N h(n) = h(n) \circledast_N x(n)$$

☐ **Solution**

a. $$x(n) \circledast_N h(n) \triangleq \sum_{m=0}^{N-1} x(m)h((n-m))_N$$

Let $$n - m = p$$

Then, $$x(n) \circledast_N h(n) = \sum_{p=n}^{n-N+1} x((n-p))_N h(p) \quad (\because n-p \text{ is a circular shift})$$

Since both the sequences are implicit periodic, the limits of summations can be changed as follows:

$$x(n) \circledast_N h(n) = \sum_{p=0}^{N-1} h(p)x((n-p))_N$$
$$= h(n) \circledast_N x(n)$$

3.7.9 Multiplication in time

$$\text{DFT}\{x_1(n)x_2(n)\} = \frac{1}{N} X_1(k) \circledast_N X_2(k)$$

Proof:

$$\text{DFT}\{x_1(n)x_2(n)\} \triangleq \sum_{n=0}^{N-1} x_1(n)x_2(n)W_N^{kn} \tag{3.16}$$

From the definition of inverse DFT, we have

$$x_2(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)W_N^{-ln} \tag{3.17}$$

Substituting equation (3.17) in equation (3.16), we get

$$\begin{aligned}
\text{DFT}\{x_1(n)x_2(n)\} &= \sum_{n=0}^{N-1} x_1(n)\frac{1}{N}\sum_{l=0}^{N-1} X_2(l)W_N^{-ln}W_N^{kn} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)\sum_{n=0}^{N-1} x_1(n)W_N^{(k-l)n} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)X_1((k-l))_N \\
&= \frac{1}{N}X_1(k) \circledast_N X_2(k)
\end{aligned}$$

Example 3.36 Find $Y(k)$, if $y(n) = x_1(n)x_2(n)$. Take $x_1(n) = (1, 1, 1, 1, 1, 1, 1, 1)$ and $x_2(n) = \cos(0.25\pi n), 0 \leqslant n \leqslant 7$.

☐ **Solution**

$$\begin{aligned}
X_1(k) \triangleq \text{DFT}\{x_1(n)\} &= \sum_{n=0}^{7} x_1(n)W_8^{kn} \\
&= \sum_{n=0}^{7} 1 \times W_8^{kn}
\end{aligned}$$

We know that
$$\sum_{n=0}^{N-1} a^n = \frac{a^N - 1}{a - 1};\ a \neq 1$$

Hence,
$$\begin{aligned}
X_1(k) &= \frac{W_8^{8k} - 1}{W_8^{k} - 1} \\
&= \begin{cases} 0, & k \neq 0 \\ 8, & k = 0 \end{cases}
\end{aligned}$$

Hence,
$$X_1(k) = (8, 0, 0, 0, 0, 0, 0, 0)$$

Similarly,
$$\begin{aligned}
X_2(k) = \text{DFT}\{x_2(n)\} &= \sum_{n=0}^{7} x_2(n)W_8^{kn} \\
&= \sum_{n=0}^{7} \cos\left(\frac{\pi}{4}n\right) W_8^{kn} \\
&= \sum_{n=0}^{7} \frac{1}{2}\left[e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right] W_8^{kn}
\end{aligned}$$

$$= \sum_{n=0}^{7} \frac{1}{2}\left[W_8^n + W_8^{-n}\right] W_8^{kn}$$

$$= \frac{1}{2}\sum_{n=0}^{7}\left[W_8^{(k+1)n} + W_8^{(k-1)n}\right]$$

We know that, $\displaystyle\sum_{n=0}^{N-1} W_N^{(k-k_0)n} = N\delta(k-k_0)$

Hence, $X_2(k) = \frac{1}{2}[8\delta(k+1) + 8\delta(k-1)]$

$$= 4\delta(k+1) + 4\delta(k-1)$$

$$= \begin{cases} 4, & k = -1 \text{ or } -1+8 = 7 \\ 4, & k = 1 \\ 0, & \text{for all other } k \text{ in } 0 \leqslant k \leqslant 7 \end{cases}$$

Hence, $X_2(k) = (0, 4, 0, 0, 0, 0, 0, 4)$

Recall the property: $\text{DFT}\{x_1(n)\, x_2(n)\} = \frac{1}{N} X_1(k) \circledast_N X_2(k)$

Hence, $\text{DFT}\{x_1(n)\, x_2(n)\} = \frac{1}{8}\left[\sum_{k=0}^{7} X_1(m) X_2((k-m))_8\right]$

k	$X_1(m)$	$X_2((k-m))_8$	$\frac{1}{8}\left[\sum_{k=0}^{7} X_1(m)X_2((k-m))_8\right]$
0	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 4, 0, 0, 0, 0, 0, 4)	$\frac{1}{8}(0) = 0$
1	(8, 0, 0, 0, 0, 0, 0, 0)	(4, 0, 4, 0, 0, 0, 0, 0)	$\frac{1}{8}(8 \times 4) = 4$
2	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 4, 0, 4, 0, 0, 0, 0)	$\frac{1}{8}(0) = 0$
3	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 4, 0, 4, 0, 0, 0)	$\frac{1}{8}(0) = 0$
4	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 4, 0, 4, 0, 0)	$\frac{1}{8}(0) = 0$
5	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 4, 0, 4, 0)	$\frac{1}{8}(0) = 0$
6	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 4, 0, 4)	$\frac{1}{8}(0) = 0$
7	(8, 0, 0, 0, 0, 0, 0, 0)	(4, 0, 0, 0, 0, 0, 4, 0)	$\frac{1}{8}(8 \times 4) = 4$

Hence, $\text{DFT}\{x_1(n)x_2(n)\} = (0, 4, 0, 0, 0, 0, 0, 4)$

Alternate method

We have,
$$X_2(k) = (0, 4, 0, 0, 0, 0, 0, 4)$$
$$= 4\delta(k-1) + 4\delta(k-7), \quad 0 \leqslant k \leqslant 7$$

Hence,
$$\begin{aligned}\text{DFT}\{x_1(n)x_2(n)\} &= \frac{1}{8}(X_1(k) \circledast_8 [4\delta(k-1) + 4\delta(k-7)]) \\ &= \frac{1}{2}[X_1((k-1))_8 + X_1((k-7))_8] \\ &= \frac{1}{2}[(0, 8, 0, 0, 0, 0, 0, 0) + (0, 0, 0, 0, 0, 0, 0, 8)] \\ &= (0, 4, 0, 0, 0, 0, 0, 4)\end{aligned}$$

3.7.10 Inner product (Parseval)

$$\sum_{n=0}^{N-1} x^*(n)y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)Y(k)$$

***Proof*:**

From the definition of IDFT, we have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}$$

Taking conjugates on both the sides, we get

$$x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{kn}$$

Hence,
$$\begin{aligned}\sum_{n=0}^{N-1} x^*(n)y(n) &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{kn}\right)y(n) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\left(\sum_{n=0}^{N-1} y(n)W_N^{kn}\right) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)Y(k)\end{aligned}$$

Corollary:
If $y(n) = x(n)$, we get

$$\sum_{n=0}^{N-1} x^*(n)x(n) = \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)X(k)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Thus, we have proved that

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Example 3.37 Find the energy of the 4-point sequence,

$$x(n) = \sin\left(\frac{2\pi}{N}n\right), \; 0 \leqslant n \leqslant 3$$

□ **Solution**

Method **1:** Time-domain approach

Given,
$$x(n) = \sin\left(\frac{2\pi}{4}n\right)$$
$$= \sin\left(\frac{\pi}{2}n\right), \; 0 \leqslant n \leqslant 3$$
$$\Rightarrow \quad x(n) = (0, 1, 0, -1)$$

Hence,
$$E = \sum_{n=0}^{N-1} |x(n)|^2$$
$$= \sum_{n=0}^{3} |x(n)|^2$$
$$= 1^2 + 1^2 = 2\text{ J}$$

Method **2:** Frequency-domain approach

From Parseval's theorem, we have

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Hence, let us find $X(k)$.

$$X(k) = \text{DFT}\{x(n)\}$$

$$= \sum_{n=0}^{3} \sin\left(\frac{\pi}{2}n\right) W_4^{kn}$$

$$= \sum_{n=0}^{3} \frac{1}{2j}\left[e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right] W_4^{kn}$$

$$= \frac{1}{2j}\sum_{n=0}^{3}\left[e^{j\frac{2\pi}{4}n} - e^{-j\frac{2\pi}{4}n}\right] W_4^{kn}$$

$$= \frac{1}{2j}\sum_{n=0}^{3}\left[W_4^{-n} - W_4^{n}\right] W_4^{kn}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{3} W_4^{(k-1)n} - \sum_{n=0}^{3} W_4^{(k+1)n}\right]$$

$$= \frac{1}{2j}\left[4\delta(k-1) - 4\delta(k+1)\right]$$

$$= \begin{cases} \frac{4}{2j}, & k = 1 \\ -\frac{4}{2j}, & k = -1 \text{ or } -1+4 = 3 \\ 0, & \text{for all other } k \text{ in the interval}(0,3) \end{cases}$$

Hence,
$$X(k) = \left(0, \frac{4}{2j}, 0, -\frac{4}{2j}\right)$$

Then,
$$E = \frac{1}{4}\sum_{k=0}^{3}|X(k)|^2 = \frac{1}{4}\left[\frac{16}{4} + \frac{16}{4}\right]$$

$$= \frac{1}{4} \times \frac{32}{4} = 2\text{ J}$$

Example 3.38 Let $x(n) = (1, 2, 0, 3, -2, 4, 7, 5)$. Evaluate the following:

(a) $X(0)$, (b) $X(4)$, (c) $\sum_{k=0}^{7} X(k)$, (d) $\sum_{k=0}^{7} |X(k)|^2$

☐ **Solution**

a. By definition,

$$\text{DFT}\{x(n)\} = X(k)$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

Letting $k = 0$, we get

$$\begin{aligned} X(0) &= \sum_{n=0}^{7} x(n) \\ &= 1+2+0+3-2+4+7+5 \\ &= 20 \end{aligned}$$

b. Letting, $k = \frac{N}{2}$ in the expression for $X(k)$, we get

$$\begin{aligned} X\left(\frac{N}{2}\right) &= \sum_{n=0}^{N-1} x(n) W_N^{\frac{N}{2}n} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}\frac{N}{2}n} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j\pi n} \\ &= \sum_{n=0}^{N-1} x(n)(-1)^n \end{aligned}$$

Here, $N = 8$.

Hence $\quad X(4) = \sum_{n=0}^{7} x(n)(-1)^n = x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$

$$= 1 - 2 + 0 - 3 - 2 - 4 + 7 - 5 = -8$$

c. From the definition of inverse DFT, we have

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1$$

Letting $n = 0$ on both the sides, we get

$$x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)$$

Since $N = 8$, we get

$$\sum_{k=0}^{7} X(k) = 8\,x(0) = 8 \times 1 = 8$$

d. According to Parseval's theorem:

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\Rightarrow \quad \sum_{k=0}^{7} |X(k)|^2 = 8 \sum_{n=0}^{7} |x(n)|^2$$

$$= 8(1 + 4 + 0 + 9 + 4 + 16 + 49 + 25)$$

$$= 864$$

Example 3.39 Let $X(k)$ be a 14-point DFT of length-14 real sequence $x(n)$. The first 8 samples of $X(k)$ are given by

$$X(0) = 12, \quad X(1) = -1 + j3, \quad X(2) = 3 + j4$$

$$X(3) = 1 - j5, \quad X(4) = -2 + j2, \quad X(5) = 6 + j3$$

$$X(6) = -2 - j3, \quad X(7) = 10$$

Find the remaining samples of $X(k)$. Also, evaluate the following:

(a) $x(0)$, (b) $x(7)$, (c) $\sum_{n=0}^{13} x(n)$, (d) $\sum_{n=0}^{13} |x(n)|^2$

☐ Solution

Since $x(n)$ is a real sequence, the following symmetry condition must be satisfied:

$$X(k) = X^*(N - k), \quad 0 \leqslant k \leqslant N - 1$$

Since, $N = 14$,

$$X(k) = X^*(14 - k), \quad 0 \leqslant k \leqslant 13$$

Conjugate symmetry: $X(k) = X^*(14 - k)$

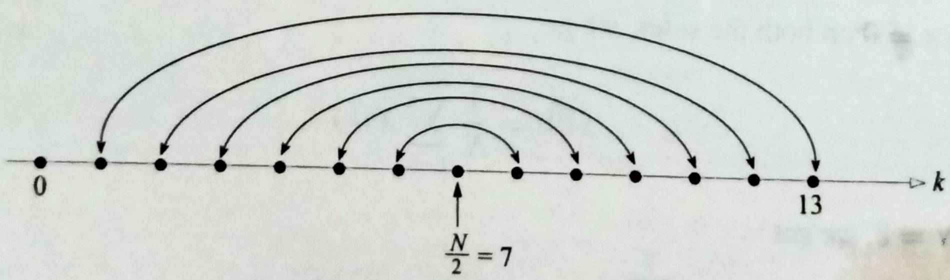

Fig. Ex.3.39 Conjugate symmetry of $X(k)$ for real $x(n)$.

Hence,

$$\begin{aligned} X(8) &= X^*(6) = -2 + j3 \\ X(9) &= X^*(5) = 6 - j3 \\ X(10) &= X^*(4) = -2 - j2 \\ X(11) &= X^*(3) = 1 + j5 \\ X(12) &= X^*(2) = 3 - j4 \\ X(13) &= X^*(1) = -1 - j3 \end{aligned}$$

The result is tabulated below:

k	$X(k)$	k	$X(k)$
0	12	7	10
1	$-1 + j3$	8	$-2 + j3$
2	$3 + j4$	9	$6 - j3$
3	$1 - j5$	10	$-2 - j2$
4	$-2 + j2$	11	$1 + j5$
5	$6 + j3$	12	$3 - j4$
6	$-2 - j3$	13	$-1 - j3$

a. From the definition of N-point inverse DFT, we have

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \leqslant k \leqslant N - 1$$

Since $N = 14$, we get

$$x(n) = \frac{1}{14} \sum_{k=0}^{13} X(k) W_{14}^{-kn}, \quad 0 \leqslant k \leqslant 13$$

Letting $n = 0$ on both the sides of the above equation, we get

$$\begin{aligned} x(0) &= \frac{1}{14} \sum_{k=0}^{13} X(k) \\ &= \frac{1}{14} [X(0) + X(1) + \cdots + X(13)] \\ &= 2.2857 \end{aligned}$$

b. Letting $n = \frac{N}{2}$ in the expression for $x(n)$, we get

$$
\begin{aligned}
x\left(\frac{N}{2}\right) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-k\frac{N}{2}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}k\frac{N}{2}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\pi k} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k)(-1)^k
\end{aligned}
$$

Since $N = 14$, we get

$$
\begin{aligned}
x(7) &= \frac{1}{14}\sum_{k=0}^{13} X(k)(-1)^k \\
&= -0.8571
\end{aligned}
$$

c. By definition:

$$
\begin{aligned}
\mathrm{DFT}\{x(n)\} = X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\
\Rightarrow \qquad X(k) &= \sum_{n=0}^{13} x(n) W_{14}^{kn}
\end{aligned}
$$

Letting $k = 0$ on both the sides of the above equation, we get

$$
X(0) = \sum_{n=0}^{13} x(n) = 12
$$

d. From Parseval's theorem:

$$
\begin{aligned}
\sum_{n=0}^{N-1} |x(n)|^2 &= \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \\
\Rightarrow \quad \sum_{n=0}^{13} |x(n)|^2 &= \frac{1}{14}\sum_{k=0}^{13} |X(k)|^2 \\
&= \frac{1}{14}\left(\begin{array}{c} 144 + 10 + 25 + 26 + 8 + 45 + 13 + 100 \\ +13 + 45 + 8 + 26 + 25 + 10 \end{array}\right) \\
&= 35.5714
\end{aligned}
$$

3.7.11 Circular correlation

Circular correlation of two length-N sequences $x(n)$ and $y(n)$ are defined as

$$r_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Then,

$$\begin{aligned} \mathrm{DFT}\{r_{xy}(l)\} &= R_{xy}(k) \\ &= X(k)Y^*(k) \end{aligned}$$

Proof:

Using the definition of circular convolution, we have

$$r_{xy}(l) = x(l) \circledast_N y^*((-l))_N$$

Hence,

$$\begin{aligned} \mathrm{DFT}\{r_{xy}(l)\} &= R_{xy}(k) \\ &= X(k)Y^*(k) \end{aligned}$$

Example 3.40 For the sequences,

$$x_1(n) = \cos\left(\frac{2\pi}{N}n\right), \quad \text{and} \quad x_2(n) = \sin\left(\frac{2\pi}{N}n\right), \quad 0 \leqslant n \leqslant N-1$$

Compute the following:

a. circular correlation of $x_1(n)$ and $x_2(n)$,

b. circular autocorrelation of $x_1(n)$ and

c. circular autocorrelation of $x_2(n)$.

☐ **Solution**

Let us first find $X_1(k)$ and $X_2(k)$.

$$\begin{aligned} X_1(k) &= \sum_{n=0}^{N-1} x_1(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\ &= \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right] W_N^{kn} \\ &= \sum_{n=0}^{N-1} \frac{1}{2}\left[W_N^{-n} + W_N^{n}\right] W_N^{kn} \\ &= \frac{1}{2}\left[\sum_{n=0}^{N-1} W_N^{(k-1)n} + W_N^{(k+1)n}\right] \end{aligned}$$

$$= \frac{N}{2}[\delta(k-1)+\delta(k+1)]$$

Similarly, $$X_2(k) = \frac{N}{2j}[\delta(k-1)-\delta(k+1)]$$

a. We know that,

$$\begin{aligned}
\text{DFT}\{r_{xy}(l)\} &= R_{xy}(k) = X(k)Y^*(k) \\
\Rightarrow \quad R_{x_1x_2}(k) &= X_1(k)X_2^*(k) \\
&= \frac{N}{2}[\delta(k-1)+\delta(k+1)] \times \frac{-N}{2j}[\delta(k-1)-\delta(k+1)] \\
&= \frac{-N^2}{4j}[\delta(k-1)-\delta(k+1)] \\
&= \frac{-N}{2} \times \frac{N}{2j}[\delta(k-1)-\delta(k+1)]
\end{aligned}$$

Taking IDFT, we get

$$r_{x_1x_2}(l) = \frac{-N}{2}\sin\left(\frac{2\pi}{N}l\right), \quad 0 \leqslant l \leqslant N-1$$

b.

$$\begin{aligned}
R_{x_1x_1}(k) &= X_1(k)X_1^*(k) \\
\Rightarrow \quad R_{x_1x_1}(k) &= \frac{N}{2}[\delta(k-1)+\delta(k+1)] \times \frac{N}{2}[\delta(k-1)+\delta(k+1)] \\
&= \frac{N}{2} \times \frac{N}{2}[\delta(k-1)+\delta(k+1)]
\end{aligned}$$

Hence, $$r_{x_1x_1}(l) = \frac{N}{2}\cos\left(\frac{2\pi}{N}l\right), \quad 0 \leqslant l \leqslant N-1$$

c.

$$\begin{aligned}
R_{x_2x_2}(k) &= X_2(k)X_2^*(k) \\
&= \frac{N}{2j}[\delta(k-1)-\delta(k+1)] \times -\frac{N}{2j}[\delta(k-1)-\delta(k+1)] \\
&= -\frac{N}{2j} \times \frac{N}{2j}[\delta(k-1)+\delta(k+1)] \\
&= \frac{N}{2} \times \frac{N}{2}[\delta(k-1)+\delta(k+1)]
\end{aligned}$$

Hence, $$r_{x_2x_2}(l) = \frac{N}{2}\cos\left(\frac{2\pi}{N}l\right), \quad 0 \leqslant l \leqslant N-1$$

3.8 Useful DFT Pairs

The DFT of finite sequences defined mathematically quite often results in very unwieldy expressions and explains the absence of many standard DFT pairs. However, the following DFT pairs are useful for many DFT and IDFT manipulations.

$$\delta(n) \xleftrightarrow{\text{DFT}} (1, 1, \cdots 1) \;\text{(constant)}$$

$$(1, 1, \cdots 1)\;\text{(constant)} \xleftrightarrow{\text{DFT}} (N, 0, \cdots 0) = N\delta(k)$$

$$a^n\text{(exponential)} \xleftrightarrow{\text{DFT}} \frac{a^N - 1}{aW_N^k - 1}$$

$$\begin{aligned}\cos\left(\frac{2\pi nk_0}{N}\right)\;\text{(sinusoid)} &\xleftrightarrow{\text{DFT}} \frac{N}{2}[\delta(k-k_0) + \delta(k+k_0)] \\ &= \frac{N}{2}[\delta(k-k_0) + \delta(k-(N-k_0))]\end{aligned}$$

Example 3.41 Find the inverse DFT of the sequence given below:

$$X(k) = \begin{cases} 3, & k = 0 \\ 1, & k = 1, 2, \cdots 9 \end{cases}$$

☐ **Solution**

The given sequence, $X(k)$ may be written as

$$\begin{aligned} X(k) &= 1 + 2\delta(k), \quad k = 0, 1, \cdots 9 \\ \Rightarrow \quad X(k) &= 1 + \frac{2}{10}[10\delta(k)] \end{aligned}$$

We know that, $x_1(n) = \delta(n) \xleftrightarrow{\text{DFT}} X_1(k) = 1$

and $x_2(n) = 1 \xleftrightarrow{\text{DFT}} X_2(k) = N\delta(k)$

Hence, $x(n) = \delta(n) + \frac{1}{5}, \quad 0 \leqslant n \leqslant 9$

3.9 *N*-point DFTs of Two Real Sequences Using a Single *N*-point DFT

Let $g(n)$ and $h(n)$ be two real sequences of length-N each with $G(k)$ and $H(k)$ denoting their respective N-point DFTs. These two N-point DFTs can be computed using a single N-point DFT, $X(k)$ of a length-N complex sequence, $x(n)$ defined as $x(n) = g(n) + jh(n)$.

The DFT operation is linear and hence,

$$X(k) = G(k) + jH(k)$$

$$\text{Also,} \quad g(n) = \frac{x(n)+x^*(n)}{2} \quad \text{and} \quad h(n) = \frac{x(n)-x^*(n)}{2j}$$

$$\Rightarrow \quad G(k) = \frac{1}{2}\left[\text{DFT}\{x(n)\} + \text{DFT}\{x^*(n)\}\right] = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

$$\text{and} \quad H(k) = \frac{1}{2j}\left[\text{DFT}\{x(n)\} - \{x^*(n)\}\right] = \frac{1}{2j}\left[X(k) - X^*(N-k)\right]$$

Example 3.42 Find the 4-point DFTs of two sequences $g(n)$ and $h(n)$ defined below, using a single 4-point DFT.

$$g(n) = (1, 2, 0, 1)$$

$$\text{and} \quad h(n) = (2, 2, 1, 1)$$

☐ **Solution**

Let $\quad x(n) = g(n) + jh(n), \quad 0 \leqslant n \leqslant 3$

Hence, $\quad x(n) = (1 + j2, 2 + j2, 0 + j1, 1 + j1)$

$$\text{DFT}\{x(n)\} \triangleq X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ 0+j1 \\ 1+j1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

$$\Rightarrow \quad X(k) = (4+j6, 2, -2, j2)$$

$$\Rightarrow \quad X^*(k) = (4-j6, 2, -2, -j2)$$

Hence,

$$G(k) = \frac{1}{2}\left[X(k) + X^*(4-k)\right]$$
$$= \frac{1}{2}[(4+j6, 2, -2, j2) + (4-j6, -j2, -2, 2)]$$
$$= (4, 1-j1, -2, 1+j)$$

and

$$H(k) = \frac{1}{2j}\left[X(k) - X^*(4-k)\right]$$
$$= \frac{1}{2j}[(4+j6, 2, -2, j2) - (4, -j6, -j2, -2, 2)]$$
$$= (6, 1-j, 0, 1+j)$$

Reinforcement Problems

RP-3.1 Two finite sequences

$$x(n) = [x(0), x(1), x(2), x(3)]$$

and

$$h(n) = [h(0), h(1), h(2), h(3)]$$

have DFTs given by

$$X(k) = \text{DFT}\{x(n)\} = (1, j, -1, -j)$$
$$H(k) = \text{DFT}\{h(n)\} = (0, 1+j, 1, 1-j)$$

Use the properties of the DFT and find the following:

a. $X_1(k) = \text{DFT}\{x(3), x(0), x(1), x(2)\}$

b. $X_2(k) = \text{DFT}\{h(0), -h(1), h(2), -h(3)\}$

c. $X_3(k) = \text{DFT}\{y(n)\}$, where $y(n) = x(n) \circledast_4 h(n)$

d. $X_4(k) = \text{DFT}\{x(0), h(0), x(1), h(1), x(2), h(2), x(3), h(3)\}$

☐ Solution

a.

$$x_1(n) = x((n-1))_4$$

$$\Rightarrow \quad X_1(k) = W_4^k X(k), \quad 0 \leqslant k \leqslant 3$$

Hence,

$$X_1(0) = W_4^0 X(0) = (1)(1) = 1$$

$$X_1(1) = W_4^1 X(1) = (-j)(j) = 1$$

$$X_1(2) = W_4^2 X(2) = (-1)(-1) = 1$$

$$X_1(3) = W_4^3 X(3) = (j)(-j) = 1$$

$$\Rightarrow \quad X_1(k) = (1, 1, 1, 1)$$

b.

$$x_2(n) = (h(0), -h(1), h(2), -h(3))$$

$$= (-1)^n h(n) = e^{j\frac{2\pi}{4}2n}$$

$$= W_4^{-2n} h(n)$$

Hence,

$$X_2(k) = H((k-2))_4$$

$$= (1, 1-j, 0, 1+j)$$

c.

$$y(n) = x(n) \circledast_4 h(n)$$

$$\Rightarrow \quad \text{DFT}\{y(n)\} = X(k)H(k)$$

$$\Rightarrow \quad X_3(k) = (1, j, -1, -j) \times (0, 1+j, 1, 1-j)$$

$$= (0, -1+j, -1, -1-j)$$

d. Let

$$x_4(n) = (x(0), h(0), x(1), h(1), x(2), h(2), x(3), h(3)), N = 8$$

$$X_4(k) = \sum_{n=0}^{7} x_4(n) W_8^{kn}$$

$$= \sum_{\substack{n=1 \\ n,\ \text{even}}}^{6} x_4(n) W_8^{kn} + \sum_{\substack{n=1 \\ n,\ \text{odd}}}^{7} x_4(n) W_8^{kn}$$

Letting $n = 2r$ in the first sum and $n = 2r + 1$ in the second, we get

$$X_4(k) = \sum_{r=0}^{3} x_4(2r) W_8^{2rk} + \sum_{r=0}^{3} x_4(2r+1) W_8^{(2r+1)k}$$

$$= \sum_{r=0}^{3} x_4(2r) W_4^{rk} + W_8^k \sum_{r=0}^{3} x_4(2r+1) W_4^{rk}$$

$$= X(k) + W_8^k H(k), \quad k = 0, 1, 2, 3$$

For finding $X_4(k)$ for $k = 4, 5, 6, 7$ we use the above equation with the fact that $X(k)$ and $H(k)$ are periodic with a period equal to 4.

Hence,

$$\begin{aligned}
X_4(0) &= X(0) + W_8^0 H(0) = 1\\
X_4(1) &= X(1) + W_8^1 H(1) = 1.414 + j\\
X_4(2) &= X(2) + W_8^2 H(2) = -1 - j\\
X_4(3) &= X(3) + W_8^3 H(3) = -1.414 - j\\
X_4(4) &= X(4) + W_8^4 H(4)\\
&= X(0) + W_8^4 H(0) = 1\\
X_4(5) &= X(5) + W_8^5 H(5)\\
&= X(1) + W_8^5 H(1) = -1.414 + j\\
X_4(6) &= X(6) + W_8^6 H(6)\\
&= X(2) + W_8^6 H(2) = -1 + j\\
X_4(7) &= X(7) + W_8^7 H(7)\\
&= X(3) + W_8^7 H(3) = 1.414 - j
\end{aligned}$$

Since $x_4(n)$ is a real sequence, the symmetry condition: $X(k) = X^*(8-k)$ is observed.

RP- 3.2 Let $x(n)$ be a finite length sequence with $X(k) = (0, 1+j, 1, 1-j)$. Using the properties of DFT, find DFTs of the following sequences:

a. $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$

b. $x_2(n) = \cos\left(\frac{\pi}{2}n\right) x(n)$

c. $x_3 = x((n-1))_4$

d. $x_4(n) = (0, 0, 1, 0) ⊛_4 x(n)$

☐ Solution

a.

$$\begin{aligned}
x_1(n) &= e^{j\frac{\pi}{2}n} x(n)\\
&= e^{j\frac{2\pi}{4}n} x(n)\\
&= W_4^{-n} x(n)
\end{aligned}$$

Recall the property:

$$\text{DFT}\left\{x(n) W_N^{-ln}\right\} = X((k-l))_N$$

Hence,

$$\begin{aligned}
X_1(k) &= X((k-1))_4\\
&= (1-j, 0, 1+j, 1)
\end{aligned}$$

b.

$$\begin{aligned}
x_2(n) &= \cos\left(\frac{\pi}{2}n\right) x(n)\\
&= \left[\frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n}\right] x(n)
\end{aligned}$$

$$\Rightarrow \quad x_2(n) = \frac{1}{2} W_4^{-n} x(n) + \frac{1}{2} W_4^{n} x(n)$$