6 LINEAR TRANSFORMATIONS

6.1 INTRODUCTION

The concept of homomorphism is easily carried to vector spaces. To begin with linear transformation (vector space homomorphism) is a function from one vector space to another. Like a ring homomorphism, it is supposed to preserve both the vector space operations. The process of taking functional values and performing the vector space operation should be commutative. This requires that the scalar field in case of either space should be the same.

Definition. *Let U and V be two vector spaces over the same field F. A mapping $T:U \to V$ is said to be a linear transformation from U into V which associates to each element α of U to a unique element $T(\alpha)$ of V such that*

$$T\,(a\alpha+b\beta) = aT\,(\alpha)\,+bT(\beta)$$

for all α and β in U and all scalars a, b in F.

REMARKS

- Linear transformation is also known as vector space homomorphism.
- If a linear transformation is onto, then it is known as isomorphism.

ILLUSTRATIONS

(1) If V is any vector space over F, then the identity transformation I, defined by $I(\alpha)=\alpha$, $\forall\ \alpha \in V$ is a linear transformation from V into V. Also the zero transformation 0 denoted by $0(\alpha) = 0$, is a linear transformation.

(2) Let F be a field of real numbers and let V be the vector space of all polynomials, then a mapping

$$D : V \to V$$

given by $D[f(x)] = \dfrac{d}{dx}[f(x)]$, $\forall\ f(x) \in V$ is a linear transformation.

Since for any $f(x)$ and $g(x) \in V$ and a, $b \in F$

$$D[af(x)+bg(x)] = \frac{d}{dx}[af(x)+bg(x)] = \frac{d}{dx}[af(x)]+\frac{d}{dx}[bg(x)]$$

$$= a\frac{d}{dx}[f(x)]+b\frac{d}{dx}[g(x)] = aD[f(x)]+bD[g(x)]$$

(3) Let R be the field of real numbers and let V be the vector space of all functions from R into R which are continuous.

Then a mapping $T : V \to V$ given by

$T[f(x)] = \int_0^x f(t)\,dt$ is a linear transformation.

For any $f(x), g(x) \in V$ and $a, b \in R$

$$\therefore \quad T[af(x)+bg(x)] = \int_0^x [af(t)+\int_0^x bg(t)]dt$$

$$= \int_0^x af(t)dt + \int_0^x bg(t)dt$$

$$= a\int_0^x f(t)dt + b\int_0^x g(t)dt = aT[f(x)] + bT[g(x)]$$

(4) Let V be the vector space of all $m \times n$ matrices over a field F and let P be a fixed $m \times n$ matrix and Q be a fixed matrix of order $n \times n$.
Then a mapping $T : V \to V$ given by $T(A) = PAQ, \forall A \in V$ is a linear transformation.
For any two matrices, $A, B \in V$ and $a, b \in F$

$$T(aA + bB) = P(aA + bB)Q = (aPA + bPB)Q$$

$$= aPAQ + bPBQ = aT(A) + bT(B).$$

6.2 SOME DEFINITIONS

(i) Linear operator : Let $V(F)$ be a vector space. Then a linear transformation from V into V is called a linear operator.

(ii) Zero transformation : Let U and V be two vector spaces over the same field F. Then the zero transformation of U into V is a mapping T defined by

$$T(\alpha) = 0 \ \forall \ \alpha \in U$$

where 0 is the zero vector of V.

(iii) Identity transformation : Let $V(F)$ be a vector space then a linear transformation $I : V \to V$ is said to be identity transformation defined by

$$I(\alpha) = \alpha \ \forall \ \alpha \in V$$

(iv) Negative of a linear transformation : Let U and V be two vector spaces over the same field F. Let T be a linear transformation of U into V. Then a linear transformation $-T$ of U into V defined by $(-T)(\alpha) = -[T(\alpha)] \ \forall \ \alpha \in V$
is called the negative of a linear transformation T.

6.3 PROPERTIES OF LINEAR TRANSFORMATIONS

THEOREM 1. *Let $U(F)$ and $V(F)$ be two vector spaces and T be a linear transformation of U into V. Then*

(i) $T(0) = 0$, where 0 on LHS is the zero vector of U and 0 on RHS is the zero vector of V.

(ii) $T(-\alpha) = -T(\alpha) \ \forall \ \alpha \in U$

(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta) \ \forall \ \alpha, \beta \in U$

(iv) $T(\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots + a_n T(\alpha_n)$
$\forall \ \alpha_1, \alpha_2, \ldots, \alpha_n \in U$ and $a_1, a_2, \ldots, a_n \in F$

Proof. (i) If $\alpha \in U$, then $T(\alpha) \in V$. Since V is a vector space, then we have

$$T(\alpha) + 0 = T(\alpha)$$

$$\Rightarrow \quad T(\alpha) + 0 = T(\alpha + 0) \quad [\because \alpha + 0 = \alpha \text{ in}$$

$$\Rightarrow \quad T(\alpha) + 0 = T(\alpha) + T(0) \quad [\because T \text{ is linear}$$

$$\Rightarrow \quad 0 = T(0)$$

[By left cancellation for addition in

(ii) We have, $\forall\, \alpha \in U$

$$T[\alpha + (-\alpha)] = T(\alpha) + T(-\alpha) \qquad [\because T \text{ is linear.}]$$
$$\Rightarrow \quad T(0) = T(\alpha) + T(-\alpha)$$
$$\Rightarrow \quad 0 = T(\alpha) + T(-\alpha) \qquad [\because T(0) = 0]$$
$$\Rightarrow \quad T(-\alpha) = -T(\alpha)$$

(iii) We have, $\forall\, \alpha, \beta \in U$

$$T[\alpha + (-\beta)] = T(\alpha) + T(-\beta) \qquad [\because T \text{ is linear.}]$$
$$\Rightarrow \quad T(\alpha - \beta) = T(\alpha) - T(\beta) \qquad [\text{Using part (ii)}]$$

(iv) Since $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n$ is a linear combination of vectors of U. Now we shall prove the result by induction on n.

For $n = 1$, $\quad T(\alpha_1 a_1) = \alpha_1 T(a_1) \qquad [\because T \text{ is linear.}]$

For $n = 2$, $\quad T(\alpha_1 a_1 + \alpha_2 a_2) = a_1 T(\alpha_1) + a_2 T(\alpha_2) \qquad [\because T \text{ is linear.}]$

Suppose the result is true for $n - 1$ values, *i.e.*

$$T(\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_{n-1} a_{n-1}) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots + a_{n-1} T(\alpha_{n-1}) \qquad \ldots(1)$$

Now, $T(\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_{n-1} a_{n-1} + \alpha_n a_n)$

$$= T[(\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_{n-1} a_{n-1})] + a_n T(\alpha_n) \qquad [\because T \text{ is linear.}]$$
$$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots + a_{n-1} T(\alpha_{n-1}) + a_n T(\alpha_n) \qquad [\text{Using (1)}]$$

Hence, the result is proved by induction.

THEOREM 2 *Let U and V be two finite-dimensional vector spaces over the same field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for U and let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be an ordered set in V. Then there is precisely one linear transformation T from U into V such that $T(\alpha_j) = \beta_j$, $j = 1, 2, 3, \ldots n$.*

Proof Since the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of $U(F)$, then for each $\alpha \in U$, there are some scalars $a_1, a_2, \ldots, a_n$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = \sum_{i=1}^{n} a_i\alpha_i$$

For this vector α we define $T : U \to V$ given by

$$T(\alpha) = a_1\beta_1 + a_2\beta_2 + \ldots + a_n\beta_n = \sum_{i=1}^{n} a_i\beta_i$$

Then T is well defined for each vector α in U and a vector $T(\alpha) \in V$. From the definition it is clear that $T(\alpha_j) = \beta_j$ for each j.

Now we shall show that T is linear. For this if $\alpha = \sum_{i=1}^{n} a_i\alpha_i$ and $\beta = \sum_{i=1}^{n} b_i\alpha_i$ are any two vectors in U, then for all $a, b \in F$, we have

$$T(a\alpha + b\beta) = \left[a\sum_{i=1}^{n} a_i\alpha_i + b\sum_{i=1}^{n} b_i\alpha_i\right] = T\left(\sum_{i=1}^{n} aa_i\alpha_i + \sum_{i=1}^{n} bb_i\alpha_i\right)$$
$$= T\left(\sum_{i=1}^{n} aa_i + \sum_{i=1}^{n} bb_i\right)\alpha_i = \sum_{i=1}^{n}(a_i a_i + b_i b_i)\beta_i$$

$$= a\sum_{i=1}^{n} a_i\beta_i + b\sum_{i=1}^{n} b_i\beta_i$$

$$= aT\left(\sum_{i=1}^{n} a_i\alpha_i\right) + bT\left(\sum_{i=1}^{n} b_i\alpha_i\right) = aT(\alpha) + bT(\beta)$$

Now, we shall show the uniqueness of T.

Let if possible, T_1 be another linear transformation from U into V such that $T_1(\alpha_j) = \beta_j$, $j = 1, 2, ..., n$.

Then for any vector $\alpha = \sum_{i=1}^{n} a_i\alpha_i$, we have

$$T_1(\alpha) = T_1\left(\sum_{i=1}^{n} a_i\alpha_i\right) = \sum_{i=1}^{n} a_iT_1(\alpha_i) \qquad [\because T_1 \text{ is linear}]$$

$$= \sum_{i=1}^{n} a_i\beta_i \qquad [\because T_1(\alpha_i) = \beta_i]$$

$$= T\left(\sum_{i=1}^{n} a_i\alpha_i\right) = T(\alpha) \qquad [\because \alpha \text{ is an arbitrary vector}]$$

$$\Rightarrow \quad T_1 = T$$

Hence T is unique.

6.4 ALGEBRA OF LINEAR TRANSFORMATIONS

THEOREM 1. *Let U and V be two vector spaces over the field F. Let T_1 and T_2 be two linear transformations from U into V then the function (T_1+T_2) defined by*

$$(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha),\ \forall \alpha\in U$$

is a linear transformation from U into V. If c is any element of F, then the function (cT) defined by

$$(cT)(\alpha) = cT(\alpha)$$

is a linear transformation from U into V.

The set of all transformations $L(U, V)$ from U into V, together with the addition and scalar multiplication defined above, is a vector space over the field F.

Proof. For $\alpha, \beta\in U$ and $a, b\in F$, we have

$$(T_1+T_2)(a\alpha+b\beta) = T_1(a\alpha+b\beta) + T_2(a\alpha+b\beta) \qquad [\text{By definition}]$$

$$= [aT_1(\alpha) + bT_1(\beta)] + [aT_2(\alpha) + bT_2(\beta)]$$

$$[\because T_1 \text{ and } T_2 \text{ are linear transformations}]$$

$$= [aT_1(\alpha) + aT_2(\alpha)] + [bT_1(\beta) + bT_2(\beta)]$$

$$= a(T_1+T_2)(\alpha) + b(T_1+T_2)(\beta)$$

$\therefore$ T_1+T_2 is a linear transformation.

Again, T is linear transformation and c is any scalar, then for $\alpha, \beta\in U$ and $a, b\in F$.

We have

$$(cT)(a\alpha+b\beta) = c[T(a\alpha + b\beta)] \qquad [\text{By definition}]$$

$= c[aT(\alpha) + bT(\beta)]$ [$\because$ T is linear transformation.]

$= c[aT(\alpha)] + c[bT(\beta)] = (ca)T(\alpha) + (cb)T(\beta)$

$= (ac)T(\alpha) + (bc)T(\beta)]$

[$\because$ Multiplication is commutative in F.]

$= a(cT)(\alpha) + b(cT)(\beta)$

$\therefore$ cT is a linear transformation.

Now we shall show that the set of all linear transformations $L(U, V)$ from U into V forms a vector space with respect to the above defined compositions.

First we show that $\{L(U, V), +\}$ is an abelian group :

(i) Closure Property.

If $T_1, T_2 \in L(U, V)$, then we have already proved that $T_1 + T_2$ is linear transformation, so that $T_1, T_2 \in L(U,V)$.

(ii) Associative Property.

For all $T_1, T_2, T_3 \in L(U,V)$ and for all $\alpha \in U$, we have

$[(T_1 + T_2) + T_3](\alpha) = (T_1 + T_2)(\alpha) + T_3(\alpha)$

$= [T_1(\alpha) + T_2(\alpha)] + T_3(\alpha)$

$= T_1(\alpha) + [T_2(\alpha) + T_3(\alpha)]$

[$\because$ Addition is associative in V.]

$= T_1(\alpha) + (T_2 + T_3)(\alpha) = [T_1 + (T_2 + T_3)](\alpha)$

$\therefore$ $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$

(iii) Commutative Property.

For all $T_1, T_2, \in L(U,V)$ and $\alpha \in U$, we have

$(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)$

$= T_2(\alpha) + T_1(\alpha)$ [$\because$ Addition is commutative in V.]

$= (T_2 + T_1)(\alpha)$

$\therefore$ $T_1 + T_2 = T_2 + T_1$

(iv) Existence of Identity .

The zero transformation, denoted by 0 and defined by $0(\alpha) = 0, \forall \alpha \in U$ is a linear transformation.

Also if $T \in L(U, V)$, then

$(T + 0) = 0 + T = T$, for all T.

$\therefore$ $0 \in L(U, V)$ and is identity transformation.

(v) Existence of Inverse.

For each $T \in L(U, V)$, there exists $(-T) \in L(U,V)$, defined by

$(-T)(\alpha) = -T(\alpha), \forall \alpha \in U$, $(-T)$ is linear and $T + (-T) = (-T) + T = 0$.

$\therefore$ $(-T)$ is the additive inverse of T.

(vi) Distributive Property.

For all $T_1, T_2 \in L(U, V), \alpha \in U$ and $a \in F$,

$a[(T_1 + T_2)](\alpha) = a(T_1 + T_2)(\alpha) = a[T_1(\alpha) + T_2(\alpha)]$

$= aT_1(\alpha) + aT_2(\alpha) = (aT_1 + aT_2)(\alpha)$

$$\therefore \quad a(T_1 + T_2) = aT_1 + aT_2$$

Also, for all $T \in L(U, V)$ and $\alpha \in U$, $a, b \in F$,

$$[(a+b)T](\alpha) = (a+b)T(\alpha) = aT(\alpha) + bT(\alpha) = (aT + bT)(\alpha)$$

$$\Rightarrow \quad (a+b)T = aT + bT$$

(vii) For all $T \in L(U,V)$, $\alpha \in U$ and $a, b \in F$,

$$[(ab)T](\alpha) = (ab)T(\alpha) = a[bT(\alpha)] = a(bT)(\alpha)$$

$$(ab).T = a.(b\,T)$$

(viii) If 1 is the unity in F, then for all $T \in L(U, V)$ and $\alpha \in U$

$$(1.T)(\alpha) = 1.T(\alpha) = T(\alpha)$$

$\therefore$ Hence $L(U, V)$ is a vector space.

REMARK

- The vector space $L(U, V)$ is also denoted by Hom. (U, V), i.e. (the set of all homomorphism from U into V).

THEOREM 2 *Let U be an m-dimensional vector space over the field F, and let V be an n-dimensional vector space over F. Then the vector space $L(U, V)$ is finite dimensional and has dimension mn.*

Proof Since U and V both are finite dimensional vector spaces of dimensions m and n respectively, therefore, let

$$\beta = \{\alpha_1, \alpha_2, \dots \alpha_m\} \text{ and } \beta' = \{\beta_1, \beta_2, \dots \beta_n\}$$

be the ordered basis of U and V respectively.

For each pair of integers (i, j) with $1 \le i \le m$ and $1 \le j \le n$, we define a linear transformation T_{ij} from U into V by

$$T_{ij}(\alpha_k) = \begin{cases} 0; & \text{if } k \ne j \\ \beta_i; & \text{if } k = j \end{cases}$$

The existence and uniqueness of above linear transformations follows from preceding theorem. It is obvious that there are mn linear transformations of the type T_{ij}, so we claim that these mn transformations form a basis of $L(U, V)$.

(i) For mn scalars a_{ij}, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij} = \mathbf{0} \quad \text{[Zero transformation]}$$

$$\Rightarrow \quad \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij}(\alpha_k) = \mathbf{0}(\alpha_k), \forall \alpha_k \in U, 1 \le k \le n$$

$$\Rightarrow \quad \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij}(\alpha_k) = \mathbf{0}$$

$$\Rightarrow \quad \sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}T_{ij}(\alpha_k)\right) = \mathbf{0}$$

$$\Rightarrow \sum_{i=1}^{n}[a_{i1}T_{i1}(\alpha_k) + a_{i2}T_{i2}(\alpha_k) + \dots + a_{im}T_{im}(\alpha_k)] = \mathbf{0}$$

$$\Rightarrow \sum_{i=1}^{n} a_{i1}T_{i1}(\alpha_k)+\sum_{i=1}^{n} a_{i2}T_{i2}(\alpha_k)+....+\sum_{i=1}^{n} a_{im}T_{im}(\alpha_k)=0$$

$$\Rightarrow \begin{aligned} & a_{11}T_{11}(\alpha_k)+a_{21}T_{21}(\alpha_k)+...+a_{n1}T_{n1}(\alpha_k) \\ & +a_{12}T_{12}(\alpha_k)+a_{22}T_{22}(\alpha_k)+...+a_{n2}T_{n2}(\alpha_k) \\ & + ... \\ & +a_{1m}T_{1m}(\alpha_k)+a_{2m}T_{2m}(\alpha_k)+...+a_{nm}T_{nm}(\alpha_k)=0 \end{aligned}$$

$$= a_{11}\beta_1+a_{21}\beta_2+....+a_{n1}\beta_n+a_{12}\beta_1+a_{22}\beta_2+....+a_{n2}\beta_n$$

$$..$$

$$+a_{1m}\beta_1+a_{2m}\beta_2+....+a_{nm}\beta_n=0$$

$$\left(\begin{aligned} \because T_{ij}(\alpha_k)=0, (j\neq k) \\ T_{ij}(\alpha_k)=\beta_i, (j=k) \end{aligned}\right)$$

Since $\beta'=\{\beta_1,\beta_2,...,\beta_n\}$ is a basis of V, therefore it is linearly independent so that

$$a_{11}=0=a_{21}=...=a_{n1}$$
$$a_{12}=0=a_{22}=...=a_{n2}$$
$$..............................$$
$$..............................$$
$$a_{1m}=0=a_{2m}=...=a_{nm}$$

Thus, $\{T_{ij}: 1\le i\le m, i\le j\le n\}$ is linearly independent.

(ii) Now we show that $\{T_{ij}: 1\le i\le m, i\le j\le n\}$ spans $L(U, V)$. For this, let T be an arbitrary linear transformation from U into V, i.e. $T\in L(U,V)$.

For $\alpha_j\in U$, $T(\alpha_j)\in V$ and $\beta'=\{\beta_1,\beta_2,...,\beta_n\}$ is a basis of V so that

$$T(\alpha_j)=a_{1j}\beta_1+a_{2j}\beta_2+...+a_{nj}\beta_n;\ 1<j<n$$

where $a_{1j}, a_{2j},...,a_{nj}$ are the coordinates of vector $T(\alpha_j)$ in β'.

$$T(\alpha_j)=\sum_{i=1}^{n} a_{ij}\beta_i=\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij}(\alpha_j)$$

$$\Rightarrow \qquad T=\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij}$$

$\Rightarrow$ $\{T_{ij}: 1\le i\le m, 1\le j\le n\}$ generates $L(U, V)$.

$\Rightarrow$ $\{T_{ij}: 1\le i\le m, 1\le j\le n\}$ is a basis of $L(U, V)$.

Hence $L(U, V)$ is finite dimensional and dim. $L(U,V)=mn$.

Let U, V and W be vector spaces aver the field F. Let T_1 be a linear transformation from U into V and T_2 be a linear transformations from V into W, then the composed function T_2T_1 is defined by

$$(T_2T_1)(\alpha)=T_2[T_1(\alpha)], \text{ for all } \alpha\in U$$

is a linear transformation from U into W.

For $\alpha,\beta\in U$ and $a,b\in F$, we have

$$(T_2T_1(a\alpha+b\beta))=T_2[T_1(a\alpha+b\beta)]=T_2[aT_1(\alpha)+bT_1(\beta)) [\because T_1 \text{ is linear.}]$$
$$=a(T_2T_1)(\alpha)+b(T_2T_1)(\beta) \qquad [\because T_2 \text{ is linear.}]$$

$\therefore$ T_2T_1 is a linear transformation from U into W.

6.5 LINEAR OPERATOR

Definition. *If V is a vector space over the field F, then a linear transformation from V into V is called a linear operator.*

In case of above theorem if U, V and W are replaced by V, then T_1 and T_2 are linear operators on the space V and $T_2 T_1$ is also a linear operator on V. Thus the vector space $L(V, V)$ has a 'multiplication' defined on it by composition. In this case the operator $T_1 T_2$ is also defined but in general $T_2 T_1 \neq T_1 T_2$. Therefore, if T is a linear operator on V, then we can compose T with T as follows :

$$T^2 = T\,T$$
$$T^3 = T\,T\,T$$

in general, $\quad T^n = T\,T...T \text{ (n times) for } n = 1, 2, 3, ...$

REMARK

- If $T \neq 0$, then we define $T^0 = 1$ (Identity transformation).

6.6 ALGEBRA OF LINEAR OPERATORS

THEOREM 1. *Let V be a vector space over the field F and let T, T_1, , T_2 and T_3 be linear operators on V and let c be an element in F, then*

(i) $IT = TI = T$, *I being an identity operator.*

(ii) $T_1 (T_2 + T_3) = T_1T_2 + T_1T_3$; $(T_2+T_3)\, T_1 = T_2\, T_1 + T_3T_1$

(iii) $T_1 (T_2T_3) = (T_1T_2)T_3$

(iv) $c(T_1T_2) = (cT_1)T_2 = T_1(cT_2)$

(v) $TO = OT = 0$, *0 being zero linear operator.*

Proof. (i) For $\alpha \in V$

$$(IT)(\alpha) = I[T(\alpha)] = T(\alpha) \qquad [\because I(\alpha) = \alpha]$$
$$IT = T$$

Also, $\quad (TI)(\alpha) = T[I(\alpha)] = T(\alpha)$

$\Rightarrow \quad TI = T$

Thus, $\quad IT = TI = T$

(ii) For any $\alpha \in V$

$$[T_1(T_2 + T_3)](\alpha) = T_1[(T_2+T_3)(\alpha)] = T_1\,[T_2\,(\alpha) + T_3\,(\alpha)]$$
$$= (T_1T_2)(\alpha) + (T_1T_3)(\alpha) = (T_1T_2+T_1T_3)(\alpha)$$

$\therefore \quad T_1(T_2+T_3) = T_1T_2+T_1T_3$

Similarly,

$$(T_2+T_3)T_1 = T_2T_1+T_3T_1$$

(iii) For any $\alpha \in V$

$$[T_1(T_2T_3)](\alpha) = T_1[(T_2T_3)(\alpha)] = T_1[T_2(T_3(\alpha))]$$
$$= (T_1T_2)[T_3(\alpha)] = [(T_1T_2)T_3](\alpha)$$

$\therefore \quad T_1(T_2T_3) = (T_1T_2)T_3$

(iv) For any $\alpha \in V$, $c \in F$

$$[c(T_1T_2)](\alpha) = c[(T_1T_2)(\alpha)] = c\,[T_1(T_2(\alpha))]$$
$$= (cT_1)[T_2(\alpha)] = [(cT_1)T_2](\alpha)$$

$\therefore \quad c(T_1T_2) = (cT_1)T_2$

Also, $[c(T_1T_2)](\alpha) = (cT_1)[T_2(\alpha)] = T_1(cT_2(\alpha)] = T_1[(cT_2)](\alpha)$

$\therefore \quad c(T_1T_2) = T_1(cT_2)$

Thus, $\quad c(T_1T_2) = (cT_1)T_2 = T_1(cT_2)$

(v) For any $\alpha \in V$,

$$(TO)(\alpha) = T[0(\alpha)] = T(0) \qquad [\because 0(\alpha) = 0]$$
$$= 0$$

Similarly, $\quad OT = 0.$

6.7 RANGE AND NULL SPACE OF A LINEAR TRANSFORMATION

(I) **Range space of a linear transformation.** If T is a linear transformation from U into V, then the range of T is a subspace of V. Let R_T be the range of T, that is, the set of all vectors β in V such that $T(\alpha) = \beta$ for some $\alpha \in U$,

i.e. $R_T = \{\beta \in V : T(\alpha) = \beta, \text{ for some } \alpha \in U\}$.

If U is finite dimensional, then the dimension of range of T is called rank of T and is denoted by $\rho(T)$.

(II) **Null space of a linear transformation.** If T is a linear transformation from a vector space U into a vector space V, then the null space of T denoted by $N(T)$ is the set of all vectors α in U such that $T(\alpha) = 0$, where 0 is the zero vector in V, *i.e.*,

$N(T) = \{\alpha \in U : T(\alpha) = 0\}$.

If U is finite-dimensional, then the dimension of null space $N(T)$ is called nullity of T and is denoted by $n(T)$.

REMARK

- Kernel of T is also known as null space of T.

THEOREM 1. *Let U and V be vector spaces over the field F and let T be a linear transformation from U into V. Suppose U is finite-dimensional. Then*

$$rank\ (T) + nullity\ (T) = dim.\ U$$

i.e., $$\rho(T) + n(T) = dim.\ U$$

Proof. Let $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ be the basis of N_T, the null space of T. Let the dimension of U be n, so that $\alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n \in U$ such that $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ forms a basis of U. Therefore, dim. N_T, $= k$ and dim. $U = n$.

We claim that $[T(\alpha_{k+1}), T(\alpha_{k+2}),..., T(\alpha_n)]$ is a basis of range of T.

For scalars $a_i \in F$ we have

$$a_{k+1}T(\alpha_{k+1}) + a_{k+2}\,T(\alpha_{k+2}) + ... + a_n\,T(\alpha_n) = 0$$

$$\Rightarrow \quad \sum_{i=k+1}^{n} a_i T(\alpha_i) = 0 \quad \Rightarrow \quad T\left(\sum_{i=k+1}^{n} a_i\alpha_i\right) = 0$$

$$\Rightarrow \quad \sum_{i=k+1}^{n} a_i\alpha_i \in N_T$$

Since $\{\alpha_1, \alpha_1, ..., \alpha_k\}$ is the basis of N_T, so that for some scalars $b_1, b_2, ..., b_k$, we have

$$\sum_{i=k+1}^{n} a_i\alpha_i = b_1\alpha_1 + b_2\alpha_2 + ... + b_k\alpha_k$$

$$\Rightarrow \quad b_1\alpha_1 + b_2\alpha_2 + ... + b_k\alpha_k - \sum_{i=k+1}^{n} a_i\alpha_i = 0$$

$$\Rightarrow \quad b_1\alpha_1 + b_2\alpha_2 + ... + b_k\alpha_k + (-a_{k+1})\alpha_{k+1} + ... + (-a_n)\alpha_n = 0$$

Since $\alpha_1, \alpha_2, ..., \alpha_n$ are linearly independent, we must have

$$b_1 = 0 = b_2 = ... = b_k = a_{k+1} = a_n.$$

$\therefore$ $[T(\alpha_{k+1}), T(\alpha_{k+2}), ..., T(\alpha_n)]$ is linearly independent.

Now, we shall show that $T(\alpha_{k+1}), T(\alpha_{k+2}), ..., T(\alpha_n)]$ spans range of T.

For this, let $T(\alpha) \in R_T$ (range of T) for some $\alpha \in U$.

Since $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ spans U so that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$$

For a_i' $s \in F$, we have

$$\begin{aligned} T(\alpha) &= T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) \\ &= T(a_1\alpha_1) + T(a_2\alpha_2) + ... + T(a_n\alpha_n) \quad [\because T \text{ is linear.}] \\ &= a_1T(\alpha_1) + a_2T(\alpha_2) + ... + a_kT(\alpha_k) + a_{k+1}T(\alpha_{k+1}) + ... + a_nT(\alpha_n) \\ &= a_{k+1}T(\alpha_{k+1}) + a_{k+2}T(\alpha_{k+2}) + ... + a_nT(\alpha_n) \\ & \qquad [\because T(\alpha_i) = 0, 1 \le i \le k] \end{aligned}$$

Thus, $T(\alpha_{k+1})...T(\alpha_n)$ spans R_T.

Hence $[T(\alpha_{k+1}), ..., T(\alpha_n)]$ is a basis of R_T.

Accordingly, $\dim . R_T = n - k = \dim . U - \dim . N_T$

$\therefore$ $\dim . R_T + \dim . N_T = \dim . U$

Hence $\text{rank}(T) + \text{nullity}(T) = \dim . U$

6.8 PRODUCT OF LINEAR TRANSFORMATIONS

THEOREM 1 *Let U(F), V(F) and W(F) be the vector spaces. Let T be a linear transformation from U into V and S a linear transformation from V into W. Then the composite function ST, called the product of linear transformations, defined by*

$$(ST)(\alpha) = S[T(\alpha)] \ \forall \ \alpha \in U$$

is a linear transformation from U into W.

Proof Since $T : U \to V$ is a linear transformation, hence $T(\alpha) \in V$ for $\alpha \in U$.

Also, $S : V \to W$ is a linear transformation, hence $S(\beta) \in W$ for $\beta \in V$.

$\therefore$ For $\beta = T(\alpha) \in V$, $S(T(\alpha)) \in W$

Thus, $(ST)(\alpha) \in W$, therefore, ST is a function from U into W. Now we shall show that ST is a linear transformation from U into W. Let $\alpha_1, \alpha_2 \in U$ and $a, b \in F$, then

$$\begin{aligned} (ST)(a\alpha_1 + b\alpha_2) &= S[T(a\alpha_1 + b\alpha_2)] \\ &= S[aT(\alpha_1) + bT(\alpha_2)] \quad [\because T \text{ is linear}] \\ &= aS(T(\alpha_1)) + bS(T(\alpha_2)) \quad [\because S \text{ is linear}] \\ &= a(ST)(\alpha_1) + b(ST)(\alpha_2) \end{aligned}$$

Hence ST is a linear transformation.

REMARK

- In above theorem, if U, V and W are replaced by V, then T and S are linear operators on the vector space V and ST is also a linear operator on V. Also TS exists and is a linear operator on V. However, in general $TS \ne ST$.

Solved Examples

Example 1. *Let T_1 and T_2 be linear operators on R^2 defined as follows:*

$$T_1(x_1, x_2) = (x_2, x_1)$$

and

$$T_2(x_1, x_2) = (x_1, 0)$$

show that

$$T_1T_2 \ne T_2T_1.$$

Solution. Let $\alpha = (x_1, x_2) \in R^2$. Then

$$\begin{aligned} (T_1T_2)(\alpha) &= T_1(T_2(\alpha)) \\ &= T_1[T_2(x_1, x_2)] \\ &= T_1(x_1, 0) \\ &= (0, x_1) \end{aligned}$$

and

$$\begin{aligned} (T_2T_1)(\alpha) &= T_2(T_1(\alpha)) \\ &= T_2[T_1(x_1, x_2)] \\ &= T_2(x_2, x_1) \\ &= (x_2, 0) \end{aligned}$$

Clearly, $(T_1T_2)(\alpha) \ne (T_2T_1)(\alpha) \ \forall \ \alpha \in R^2$

Hence, $T_1T_2 \ne T_2T_1$.

Example 2. *Let V(R) be the vector space of all polynomial functions in x with coefficients in the field R of real numbers. Let D and T be two linear operators on V defined by*

$$D(f(x)) = \frac{d}{dx}f(x)$$

and

$$T(f(x)) = \int_0^x f(x)dx$$

for every $f(x) \in V$. Then show that $DT = I$ (Identity opearator) and $TD \ne I$.

Solution. Let $f(x) = a_0 + a_1x + a_2x^2 + ... \in V$, where $a_1, a_2, ..., \in R$. Then

$$\begin{aligned} (DT)(x) &= D[T(f(x))] \\ &= \left[\int_0^x f(x)dx\right] = D\left[\int_0^x (a_0 + a_1x + a_2x^2 + ...)dx\right] \\ &= D\left[a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + ...\right] \\ &= \frac{d}{dx}\left[a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + ...\right] \\ &= a_0 + a_1x + a_2x^2 + ... = f(x) = I(f(x)). \end{aligned}$$

$\therefore$ $(DT)(f(x)) = I(f(x)) \ \forall \ f(x) \in V.$

Thus, $DT = I$

Now, $(TD)(f(x)) = T[D(f(x))]$

$$= T\left[\frac{d}{dx}f(x)\right] = T\left[\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \ldots)\right]$$

$$= T[a_1 + 2a_2x + \ldots] = \int_0^x (a_1 + 2a_2x + \ldots)dx$$

$$= a_1x + a_2x^2 + \ldots \neq f(x) = I(f(x))$$

$\therefore$ $(TD)(f(x)) \neq I(f(x))\ \forall\ f(x) \in V.$

Thus, $(TD) \neq I$

Hence in general $DT \neq TD$.

Example 3 *Let V(R) be the vector space of all polynomials in x with coefficients in the field R. Let D and let T be two linear transformations on V defined by*

$$D(f(x)) = \frac{d}{dx}f(x)\ \forall\ f(x) \in V$$

and $$T(f(x)) = xf(x)\ \forall\ f(x) \in V$$

then show that $DT \neq TD$. Also, show that $DT - TD = I$.

Solution Let $f(x) \in V$. Then

$$(DT)(f(x)) = D[T(f(x))] = D[xf(x)]$$

$$= \frac{d}{dx}[xf(x)] = f(x) + x\frac{d}{dx}f(x) \quad \ldots(1)$$

Also, $$(TD)(f(x)) = T(D(f(x)))$$

$$= T\left(\frac{d}{dx}f(x)\right) = x.\frac{d}{dx}f(x) \quad \ldots(2)$$

Therefore, from (1) and (2), we can say that there exists $f(x) \in V$ such that

$$(DT)(f(x)) \neq (TD)(f(x))$$

Hence, $DT \neq TD$.

Also, $(DT)(f(x)) - (TD)(f(x)) = f(x) = I(f(x))$

$\therefore$ $DT - TD = I$

6.9 POLYNOMIALS IN ...

The set $L(V, V)$ of linear operators on V is a vector space over the field. If $a_0, a_1, a_2, \ldots, a_n \in F$, then

$$p(T) = a_0I + a_1T + a_2T_2 + \ldots + a_nT^n \in L(V, V).$$

Thus, $p(T)$ is also a linear operator on V, we call it as a polynomial in linear operator T.

6.10 INVERTIBLE LINEAR TRANSFORMATION

A linear transformation T from a vector space $U(F)$ into $V(F)$ is called invertible or regular if there exists a unique linear transformation T^{-1} (called the inverse of T) from $V(F)$ into $U(F)$ such that (T^{-1}) is the identity linear transformation on U and (TT^{-1}) is the identity transformation on V.

Furthermore, T is invertible iff

(i) T is one-to-one.

(ii) T is onto, *i.e.*, $R(T) = V$

THEOREM 1. *Let U and V be vector spaces over the same field F and let T be a linear transformation from U into V. If T is invertible, then T^{-1} is a linear transformation from V into U.*

Proof. Since T is invertible, hence for each $\beta \in V$, there is a unique $\alpha \in U$ such that

$$T(\alpha) = \beta \Leftrightarrow T^{-1}(\beta) = \alpha$$

Now, we shall show that T^{-1} is linear.

For $\alpha_1, \alpha_2 \in U$ and $a, b \in F$

$$T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2) \quad [\because T \text{ is linear.}]$$

But for β_1 and β_2 in V, there are unique $\alpha_1, \alpha_2 \in U$ respectively, such that

$$T(\alpha_1) = \beta_1 \Leftrightarrow T^{-1}(\beta_1) = \alpha_1$$

and $$T(\alpha_2) = \beta_2 \Leftrightarrow T^{-1}(\beta_2) = \alpha_2$$

Thus, we have

$$T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2$$

$$\Rightarrow \quad a\alpha_1 + b\alpha_2 = T^{-1}(a\beta_1 + b\beta_2)$$

$$\Rightarrow \quad aT^{-1}(\beta_1) + b\,T^{-1}(\beta_2) = T^{-1}(a\beta_1 + b\beta_2) \quad [\because a\alpha_1 + b\alpha_2 \text{ is unique in } V.]$$

Hence T^{-1} is a linear ...

Now, $(TD)(f(x)) = T[D(f(x))]$

$$= T\left[\frac{d}{dx}f(x)\right] = T\left[\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \ldots)\right]$$

$$= T[a_1 + 2a_2x + \ldots] = \int_0^x (a_1 + 2a_2x + \ldots)dx$$

$$= a_1x + a_2x^2 + \ldots \neq f(x) = I(f(x))$$

$$(TD)(f(x)) \neq I(f(x)) \ \forall \ f(x) \in V.$$

$\therefore$ $(TD) \neq I$

Thus,

Hence in general $DT \neq TD$.

Example 3. *Let V(R) be the vector space of all polynomials in x with coefficients in the field R. Let D and let T be two linear transformations on V defined by*

$$D(f(x)) = \frac{d}{dx}f(x) \ \forall \ f(x) \in V$$

and $$T(f(x)) = xf(x) \ \forall \ f(x) \in V$$

then show that $DT \neq TD$. Also, show that $DT - TD = I$.

Solution. Let $f(x) \in V$. Then

$$(DT)(f(x)) = D[T(f(x))] = D[xf(x)]$$

$$= \frac{d}{dx}[xf(x)] = f(x) + x\frac{d}{dx}f(x) \quad \ldots(1)$$

Also, $$(TD)(f(x)) = T(D(f(x)))$$

$$= T\left(\frac{d}{dx}f(x)\right) = x.\frac{d}{dx}f(x) \quad \ldots(2)$$

Therefore, from (1) and (2), we can say that there exists $f(x) \in V$ such that

$$(DT)(f(x)) \neq (TD)(f(x))$$

Hence, $DT \neq TD$.

Also, $(DT)(f(x)) - (TD)(f(x)) = f(x) = I(f(x))$

$\therefore$ $DT - TD = I$

6.9 POLYNOMIALS IN A LINEAR OPERATOR

Let T be a linear operator on a vector space V(F). Then TT is a linear operator on V. Since the product of linear operators is an associative operation, therefore, if n is a positive integer, then we define

$$T^2 = TT$$

$$T^3 = TTT$$

in general, $T^n = TT\ldots T$ (n times)

Clearly, T^n is also a linear operators on V. Also, $T^0 = I$, which is defined as identity operator.

If m and n are non-negative integers, then we see that

$$T^mT^n = T^{m+n}$$

and $$(T^m)^n = T^{mn}$$

The set $L(V, V)$ of linear operators on V is a vector space over the field. If $a_0, a_1, a_2, \ldots, a_n \in F$, then

$$p(T) = a_0I + a_1T + a_2T_2 + \ldots + a_nT^n \in L(V, V).$$

Thus, $p(T)$ is also a linear operator on V, we call it as a polynomial in linear operator T.

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A linear transformation T from a vector space $U(F)$ into $V(F)$ is called invertible or regular if there exists a unique linear transformation T^{-1} (called the inverse of T) from $V(F)$ into $U(F)$ such that (T^{-1}) is the identity linear transformation on U and (TT^{-1}) is the identity transformation on V.

Furthermore, T is invertible iff

(i) T is one-to-one.

(ii) T is onto, i.e., $R(T) = V$

THEOREM 1. *Let U and V be vector spaces over the same field F and let T be a linear transformation from U into V. If T is invertible, then T^{-1} is a linear transformation from V into U.*

Proof. Since T is invertible, hence for each $\beta \in V$, there is a unique $\alpha \in U$ such that

$$T(\alpha) = \beta \Leftrightarrow T^{-1}(\beta) = \alpha$$

Now, we shall show that T^{-1} is linear.

For $\alpha_1, \alpha_2 \in U$ and $a, b \in F$

$$T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2) \quad [\because T \text{ is linear.}]$$

But for β_1 and β_2 in V, there are unique $\alpha_1, \alpha_2 \in U$ respectively, such that

$$T(\alpha_1) = \beta_1 \Leftrightarrow T^{-1}(\beta_1) = \alpha_1$$

and $$T(\alpha_2) = \beta_2 \Leftrightarrow T^{-1}(\beta_2) = \alpha_2$$

Thus, we have

$$T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2$$

$$\Rightarrow a\alpha_1 + b\alpha_2 = T^{-1}(a\beta_1 + b\beta_2) \quad [\because a\alpha_1 + b\alpha_2 \text{ is unique in } V.]$$

$$\Rightarrow aT^{-1}(\beta_1) + b\,T^{-1}(\beta_2) = T^{-1}(a\beta_1 + b\beta_2)$$

Hence T^{-1} is a linear transformation.

THEOREM 2. *Let T_1 be an invertible linear transformation from U(F) into V(F) and T_2 an invertible linear transformation from V(F) into W(F). Then T_1T_2 is invertible and $(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}$.*

Proof. To show T_2T_1 is invertible, we shall show that it is one-one and onto.

If $\alpha_1, \alpha_2 \in U$ such that $(T_2T_1)(\alpha_1) = (T_2T_1)(\alpha_2)$, then

$$(T_2T_1)(\alpha_1) = (T_2T_1)(\alpha_2) \Rightarrow T_2[T_1(\alpha_1)] = T_2[T_1(\alpha_2)]$$

$$\Rightarrow T_1(\alpha_1) = T_1(\alpha_2) \quad [\because T_2 \text{ is one-one.}]$$

$$\Rightarrow \alpha_1 = \alpha_2 \quad [\because T_1 \text{ is one-one.}]$$

Thus, T_2T_1 is one-one.

Also, T_1 and T_2 being onto, then for each $\beta \in V$, there exists a unique $\alpha \in U$ such that

$$T_1(\alpha) = \beta$$

and for each $\gamma \in W$, there exists a unique $\beta \in V$ such that $T_2(\beta) = \gamma$.

Thus, $\gamma \in W \Rightarrow$ there exists $\beta \in V : \gamma = T_2(\beta)$.

$\Rightarrow$ there exists $\alpha \in U : \gamma = T_2(T_1(\alpha))$ $[\because T_1(\alpha) = \beta]$

$\Rightarrow$ there exists $\alpha \in U : \gamma = (T_2 T_1)(\alpha)$.

Therefore $(T_2 T_1)$ is onto. Hence $(T_2 T_1)$ is invertible.

Also, $(T_2 T_1)(T_1^{-1} T_2^{-1}) = T_2(T_1 T_1^{-1})T_2^{-1} = (T_2 I)T_2^{-1} = T_2 T_2^{-1} = I$

Similarly $(T_1^{-1} T_2^{-1})(T_2 T_1) = T_1^{-1}(T_2^{-1} T_2)T_1 = T_2^{-1}(IT_1) = T_1^{-1} T_1 = I$

Hence $(T_2 T_1^{-1}) = T_1^{-1} T_2^{-1}$

6.11 NON-SINGULAR LINEAR TRANSFORMATIONS

Let U and V be vector spaces over the field F. Then a linear transformation T from U into V is called non-singular if the null space of T is (0).

Thus, if T is non-singular, then

$$T(\alpha) = 0 \Rightarrow \alpha = 0$$

Also, when T is non-singular and $\alpha, \beta \in U$

$$T(\alpha) = T(\beta) \Rightarrow T(\alpha) - T(\beta) = 0$$

$\Rightarrow$ $T(\alpha - \beta) = 0$ $[\because T \text{ is linear.}]$

$\Rightarrow$ $\alpha - \beta = 0$ $[\because T \text{ is non-singular.}]$

$\Rightarrow$ $\alpha = \beta$

Hence T is non-singular, implies that T is one-one.

... linear transformation from $U(F)$ into $V(F)$. Then T is non-singular if ...

THEOREM 2 *Let U and V be finite dimensional vector spaces over the field F such that dim. U = dim. V. If T is a linear transformation from U into V, then the following are equivalent:*

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, that is, the range of T is V.

(iv) If $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a basis of U, then $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is a basis of V.

PROOF **(i) $\Rightarrow$ (ii) :** Since T is invertible, so it is one-one and onto, therefore, T is non-singular.

(ii) $\Rightarrow$ (iii): Let T be non-singular and let $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the basis of U, then the set $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is linearly independent subset of V, but dim. U = dim. V, therefore $\{T(\alpha_1), ..., T(\alpha_n)\}$ is a basis for V.

For any $\beta \in V$, $a_1, a_2, ..., a_n \in F$, we have

$$\beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) + ... + a_n T(\alpha_n)$$

$$\beta = T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) \quad [\because T \text{ is linear.}]$$

$\Rightarrow$ $\beta \in R_T$

Thus, $V \subseteq R_T$, but $R_T \subseteq V$

$\therefore$ $R_T = V$

i.e. the range of $T = V$

(iii) $\Rightarrow$ (iv) : Suppose range of $T = V$. Let the set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of ...

$\Rightarrow$ there exists $\alpha \in U : \gamma = T_2(T_1(\alpha))$ $[\because T_1(\alpha) = \beta]$

$\Rightarrow$ there exists $\alpha \in U : \gamma = (T_2 T_1)(\alpha)$.

Therefore $(T_2 T_1)$ is onto. Hence $(T_2 T_1)$ is invertible.

Also, $(T_2T_1)(T_1^{-1}T_2^{-1}) = T_2(T_1T_1^{-1})T_2^{-1} = (T_2 I)T_2^{-1} = T_2T_2^{-1} = I$

Similarly $(T_1^{-1}T_2^{-1})(T_2T_1) = T_1^{-1}(T_2^{-1}T_2)T_1 = T_2^{-1}(IT_1) = T_1^{-1}T_1 = I$

Hence $(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}$

6.11 NON-SINGULAR LINEAR TRANSFORMATIONS

Let U and V be vector spaces over the field F. Then a linear transformation T from U into V is called non-singular if the null space of T is (0).

Thus, if T is non-singular, then

$$T(\alpha) = 0 \Rightarrow \alpha = 0$$

Also, when T is non-singular and $\alpha, \beta \in U$

$$T(\alpha) = T(\beta) \Rightarrow T(\alpha) - T(\beta) = 0$$

$\Rightarrow$ $T(\alpha - \beta) = 0$ $[\because T$ is linear.$]$

$\Rightarrow$ $\alpha - \beta = 0$ $[\because T$ is non-singular.$]$

$\Rightarrow$ $\alpha = \beta$

Hence T is non-singular, implies that T is one-one.

THEOREM 1. *Let T be a linear transformation from $U(F)$ into $V(F)$. Then T is non-singular if and only if T carries each linearly independent subset of U onto a linearly independent subset of V.*

Proof. Let us first suppose that T is non-singular. Now, let

$$S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$$

be an arbitrary linearly independent subset of U. Then we have to show that the set

$$S_1 = \{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_k)\}$$

is linearly independent subset of V.

For scalars $a_1, a_2, \ldots a_k \in F$ we have

$$a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots + a_k T(\alpha_k)\} = 0$$

$\Rightarrow$ $T(a_1\alpha_1) + T(a_2\alpha_2) + \ldots + T(a_k\alpha_k)\} = 0$ $[\because T$ is linear.$]$

$\Rightarrow$ $T(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k) = 0$ $[\because T$ is linear.$]$

$\Rightarrow$ $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k = 0$ $[\because T$ is non-singular.$]$

$\Rightarrow$ $a_1 = a_2 = \ldots = a_k = 0$ $[\because S$ is linear independent.$]$

Hence S_1 is linearly independent.

Conversely, suppose that T carries each linearly independent subset of U into a linearly independent subset of V. Let α be a non-zero vector in U, then $\{\alpha\}$ is linearly independent so is $\{T(\alpha)\}$. Consequently, $T(\alpha) \neq 0$ because the set consisting of the zero vector alone is dependent. Therefore, the null space of T is the zero space and hence T is non-singular.

THEOREM 2. *Let U and V be finite dimensional vector spaces over the field F such that dim. U = dim. V. If T is a linear transformation from U into V, then the following are equivalent:*

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, that is, the range of T is V.

(iv) If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of U, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_2)\}$ is a basis of V.

Proof. **(i) $\Rightarrow$ (ii) :** Since T is invertible, so it is one-one and onto, therefore, T is non-singular.

(ii) $\Rightarrow$ (iii): Let T be non-singular and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the basis of U, then the set $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is linearly independent subset of V, but dim. U = dim. V, therefore $\{T(\alpha_1), \ldots, T(\alpha_n)\}$ is a basis for V.

For any $\beta \in V$, $a_1, a_2, \ldots, a_n \in F$, we have

$$\beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots + a_n T(\alpha_n)$$

$$\beta = T(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n)$$

$\Rightarrow$ $\beta \in R_T$ $[\because T$ is linear.$]$

Thus, $V \subseteq R_T$, but $R_T \subseteq V$

$\therefore$ $R_T = V$

i.e. the range of $T = V$

(iii) $\Rightarrow$ (iv) : Suppose range of $T = V$. Let the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of U so that an arbitrary element $\alpha \in U$ is expressible as linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$\therefore$ $\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$ for some scalars, $b_1, b_2, \ldots, b_n \in F$

$\Rightarrow$ $T(\alpha) = T(b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n)$

$= b_1 T(\alpha_1) + b_2 T(\alpha_2) +, \ldots, + b_n T(\alpha_n)$ $[\because T$ is linear.$]$

This shows that each element of range of T is expressible as a linear combination of $\{T(\alpha_1) + T(\alpha_2), \ldots, T(\alpha_n)\}$. Thus, the set

$$\{T(\alpha_1) + T(\alpha_2), \ldots, T(\alpha_n)\}.$$

spans R_T. Since $R_T = V$. Also, dim. U = dim. $V = n$.

Hence $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ forms a basis of V.

(iv) $\Rightarrow$ (i): Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of U such that $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of V.

Let α be an arbitrary element of U, then for $b_1, b_2, \ldots, b_n \in F$, we have

$$\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$$

Now, $T(\alpha) = 0$

$\Rightarrow$ $T(b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n) = 0$

$\Rightarrow$ $b_1T(\alpha_1) + b_2T(\alpha_2) + \ldots + b_nT(\alpha_n) = 0$ $[\because T$ is linear.$]$

$\Rightarrow$ $b_1 = b_2 = \ldots = b_n = 0$

$[\because \{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is linearly independent $]$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n = 0 \Rightarrow \alpha = 0$

Hence, T is non-singular and therefore T is one-one.

Also, $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ spans V and range of T is V. Consequently, T is one-one and hence T is invertible.

THEOREM 3. *A linear transformation T on a finite dimensional vector space is invertible iff T is non-singular.*

Proof. Let $V(F)$ be a vector space and let dim. $V = n$ and T be a linear transformation on V. If T is invertible, then it is one-one and hence T is non-singular.

Conversely, if T is non-singular, then T is one-one. Now in order to prove that T is invertible we will show that T is onto. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V, then we shall show that $S' = \{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is a basis of V.

S' is linearly independent : Let $a_1, a_2, ..., a_n \in F$ and let

$$a_1T(\alpha_1) + a_2T(\alpha_2) + ... + a_nT(\alpha_n) = 0$$

$$\Rightarrow \quad T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) = 0$$

$$\Rightarrow \quad T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) = T(0) \qquad [\because T(0) = 0]$$

$$\Rightarrow \quad a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = 0 \qquad [\because T \text{ is one-one.}]$$

$$\Rightarrow \quad a_1 = 0, a_2 = 0, ..., a_n = 0 \qquad [\because S \text{ is linearly independent.}]$$

$\therefore$ S' is linearly independent.

Since dim. $V = n$ and S' contains n linearly independent vectors. Therefore, S' must be a basis of V. Thus each vector of V can be expressed as a linear combination of vectors of S'.

Let $\alpha \in V$. Then there exists $c_1, c_2, ..., c_n \in F$ such that

$$\alpha = c_1T(\alpha_1) + c_2T(\alpha_2) + ... + c_nT(\alpha_n)$$

$$\Rightarrow \quad \alpha = T(c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n)$$

Now $c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n \in V$ and α is the T-image. Hence T is onto.

THEOREM 4. *A linear transformation T on a finite dimensional vector space is invertible iff T is onto.*

Proof. Let $V(F)$ be a finite dimensional vector space and let T be a linear transformation on V. If T is invertible, then T is onto.

Conversely, Let T be onto. Now, in order to prove that T is invertible we shall prove that T is one-one. Let dim. $V = n$.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V, then we claim that $S' = \{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is also a basis of V.

Let α be any element of V and T is onto V, then there exists $\beta \in V$ such that $T(\beta) = \alpha$.

Also, $\quad \beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ for $a_1, a_2, ..., a_n \in F$

Then, $\quad \alpha = T(\beta) = T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$

$$\Rightarrow \quad \alpha = a_1T(\alpha_1) + a_2T(\alpha_2) + ... + a_nT(\alpha_n)$$

$$\Rightarrow \quad L(S') = V.$$

Since dim. $V = n$ and S' is a subset of V containing n vectors with $L(S') = V$, then S' must be a basis of V. Thus S' is linearly independent.

Let $\gamma = c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n$ and $\delta = d_1\alpha_1 + d_2\alpha_2 + ... + d_n\alpha_n$ be any element of V. We have

$$T(\gamma) = T(\delta)$$

$$\Rightarrow \quad T(c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n) = T(d_1\alpha_1 + d_2\alpha_2 ...$$

$$\Rightarrow \quad c_1T(\alpha_1) + c_2T(\alpha_2) + ... + c_nT(\alpha_n) = d_1T(\alpha_1) + d_2T ...$$

$$\Rightarrow \quad (c_1 - d_1)T(\alpha_1) + (c_2 - d_2)T(\alpha_2) + ... + (c_n - d_n)T ...$$

$$\Rightarrow \quad c_1 - d_1 = 0, c_2 - d_2 = 0, ..., c_n - d_n = 0 \qquad [\because S ...$$

$$\Rightarrow \quad c_1 = d_1, c_2 = d_2, ..., c_n = d_n$$

$$\Rightarrow \quad \gamma = \delta.$$

Hence T is one-one.

Solved Examples

Example 1. *Describe explicitly a linear transformation from $V_3(R)$ into $V_3(R)$ which has its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.*

Solution. We know that the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of $V_3(R)$.

Also $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a subset of $V_3(R)$ which has same number of vectors as the above basis set has. Then there exists a unique linear transformation T from $V_3(R)$ into $V_3(R)$ such that

$$T(1, 0, 0) = (1, 0, -1)$$

$$T(0, 1, 0) = (1, 2, 2)$$

$$T(0, 0, 1) = (0, 0, 0)$$

Now the vectors $T(1, 0, 0)$, $T(0,1, 0)$ and $T(0, 0,1)$ span the range of T *i.e.* the vectors $(1, 0, -1)$, $(1, 2, 2)$ and $(0, 0, 0)$ span the range of T. Thus the range of T is the subspace of $V_3(R)$ spanned by the set $\{(1, 0, -1), (1, 2, 2)\}$ because the vector $(0, 0, 0)$ can be omitted from the spanning set.

Let (x, y, z) be any element of $V_3(R)$, then

$$(x, y, z) = x(1, 0, 0) + y(0,1, 0) + z(0, 0,1)$$

$$\Rightarrow \quad T(x, y, z) = xT(1, 0, 0) + yT(0,1, 0) + zT(0, 0,1)$$

$$\Rightarrow \quad T(x, y, z) = x(1, 0, -1) + y(1, 2, 2) + z(0, 0, 0)$$

$$\therefore \quad T(x, y, z) = (x + y,\ 2y, -x + 2y)$$

which is the required linear transformation.

Example 2. *Let T be a linear operator on a vector space $V(F)$. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator on $V_2(R)$ such that $T^2 = 0$ but $T \neq 0$.*

Solution. Since $T^2 = 0$, then for $\alpha \in V$

$$T^2(\alpha) = 0(\alpha) \Rightarrow T[T(\alpha)] = 0$$

$$T(\alpha) \in N(T) \qquad \text{[By definition of null space]}$$

But $\quad T(\alpha) \in R(T)\ \forall\ \alpha \in V$

$$\therefore \quad R(T) \subset N(T)$$

Hence when $T^2 = 0$, the range of T is contained in null space of T.

Next, let T be a linear map from $V_2(R)$ into $V_2(R)$ such that

$$T(a, b) = (0, a)\ \forall\ (a, b) \in V_2(R)$$

Obviously, $\quad T \neq 0.$

Also, $\quad T^2(a, b) = T[T(a, b)] = T[(0, a)] = (0, 0) = 0(a, b)$

$$\Rightarrow \quad T^2 = 0.$$

Hence, T is non-singular and therefore T is one-one.

Also, $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ spans V and range of T is V. Conversely,
T is one-one and hence T is invertible.

THEOREM 3 *A linear transformation T on a finite dimensional vector space is invertible iff T is non-singular.*

Proof Let $V(F)$ be a vector space and let dim. $V = n$ and T be a linear transformation on V. If T is invertible, then it is one-one and hence T is non-singular.

Conversely, if T is non-singular, then T is one-one. Now in order to prove T is invertible we will show that T is onto. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis, then we shall show that $S' = \{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is a basis of V.

S' is linearly independent : Let $a_1, a_2, ..., a_n \in$ F and let

$$a_1T(\alpha_1) + a_2T(\alpha_2) + ... + a_nT(\alpha_n) = 0$$
$$\Rightarrow \quad T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) = 0$$
$$\Rightarrow \quad T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) = T(0) \qquad [\because T(0) = 0]$$
$$\Rightarrow \quad a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = 0 \qquad [\because T \text{ is one-one}]$$
$$\Rightarrow \quad a_1 = 0, a_2 = 0, ..., a_n = 0 \qquad [\because S \text{ is linearly independent}]$$

$\therefore S'$ is linearly independent.

Since dim. $V = n$ and S' contains n linearly independent vectors. Therefore S' must be a basis of V. Thus each vector of V can be expressed as a linear combination of vectors of S'.

Let $\alpha \in V$. Then there exists $c_1, c_2, ..., c_n \in F$ such that

$$\alpha = c_1T(\alpha_1) + c_2T(\alpha_2) + ... + c_nT(\alpha_n)$$
$$\Rightarrow \quad \alpha = T(c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n)$$

Now $c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n \in V$ and α is the T-image. Hence T is onto.

THEOREM 4 *A linear transformation T on a finite dimensional vector space is invertible iff T is onto.*

Proof Let $V(F)$ be a finite dimensional vector space and let T be a linear transformation on V. If T is invertible, then T is onto.

Conversely, Let T be onto. Now, in order to prove that T is invertible we shall prove that T is one-one. Let dim. $V = n$.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V, then we claim that $S' = \{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is also a basis of V.

Let α be any element of V and T is onto V, then there exists $\beta \in V$ such that $T(\beta) = \alpha$.

Also, $\quad \beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ for $a_1, a_2, ..., a_n \in F$

Then, $\quad \alpha = T(\beta) = T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$

$$\Rightarrow \quad \alpha = a_1T(\alpha_1) + a_2T(\alpha_2) + ... + a_nT(\alpha_n)$$
$$\Rightarrow \quad L(S') = V.$$

Since dim. $V = n$ and S' is a subset of V containing n vectors with $L(S') = V$, then S' must be a basis of V. Thus S' is linearly independent.

Let $\gamma = c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n$ and $\delta = d_1\alpha_1 + d_2\alpha_2 + ... + d_n\alpha_n$ be any element of V. We have

$$T(\gamma) = T(\delta)$$

Since $T^2 = 0$, then for $\alpha \in V$

$$T^2(\alpha) = 0(\alpha) \Rightarrow T[T(\alpha)] = 0$$
$$T(\alpha) \in N(T) \qquad [\text{by definition of null space}]$$

But $\quad T(\alpha) \in R(T) \; \forall \; \alpha \in V$

$\therefore \quad R(T) \subset N(T)$

Hence when $T^2 = 0$, the range of T is contained in null space of T.

Next, let T be a linear map from $V_2(R)$ into $V_2(R)$ such that

$$T(a, b) = (0, a) \; \forall \; (a, b) \in V_2(R)$$

Obviously, $\quad T \neq 0.$

Also, $\quad T^2(a, b) = T[T(a, b)] = T(0, a) = (0, 0) = 0(a, b)$

$\Rightarrow \quad T^2 = 0.$ ...(2)

... + nullity (T^2)

Example 3 If $T: R^3 \to R^3$ is a linear operator defined by $T(x, y, z) = (x + z, x - z, y)$. Show that T is invertible and find $T^{-1}(2, 4, 6)$.

Solution Let $T(x, y, z) = (0, 0, 0)$

$\Rightarrow \quad (x + z, x - z, y) = (0, 0, 0)$

$\Rightarrow \quad x + z = 0, x - z = 0, y = 0$

Solving these equations, we get $x = 0, y = 0, z = 0$. Therefore, for $\alpha \in R^3$, $T(\alpha) = 0 \Rightarrow \alpha = 0$. Thus T is non-singular. Hence T is invertible.

Now $\quad T(x, y, z) = (p, q, r)$

$\Rightarrow \quad (x + z, x - z, y) = (p, q, r)$

$\Rightarrow \quad x + z = p, x - z = q, y = r$

$\Rightarrow \quad x = \dfrac{p+q}{2}, y = r, z = \dfrac{p-q}{2}$

$\Rightarrow \quad T^{-1}(p, q, r) = (x, y, z)$

$\Rightarrow \quad T^{-1}(p,q,r) = \left(\dfrac{p+q}{2}, r, \dfrac{p-q}{2}\right)$

$\therefore \quad T^{-1}(2, 4, 6) = (3, 6, -1)$

Example 4 *A linear transformation T is defined on $V_2(C)$ by $T(a, b) = (\alpha a + \beta b, \gamma a + \delta b)$, where $\alpha, \beta, \gamma, \delta$ are fixed elements of C. Prove that T is invertible if and only if $\alpha\delta - \beta\gamma \neq 0$.*

Solution Since dim. $V_2(C) = 2$. Therefore, T is a linear transformation on a finite dimensional vector space, so that T will be invertible if and only if the null space of T contains only zero vector. The zero vector of $V_2(C)$ is $(0, 0)$.

Thus T is invertible iff $T(x, y) = (0, 0)$

i.e. iff $\quad (\alpha x + \beta y, \gamma x + \delta y) = (0, 0)$

iff $\quad \alpha x + \beta y = 0$ and $\gamma x + \delta y = 0$

Now these equations have only zero solution, i.e. $x = 0, y = 0$ if and only if

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0 \Rightarrow \alpha\delta - \gamma\beta \neq 0$$

Hence T is invertible iff $\alpha\delta - \beta\gamma \neq 0$.

Example 5 *Find two linear operators T and S on $V_2(R)$ such that $TS = 0$ but $ST \neq 0$.*

Solution Consider two linear operators T and S on $V_2(R)$ defined as

$T(a, b) = (a, 0) \ \forall \ (a, b) \in V_2(R)$ and $a, b \in R$

and $\quad S(a, b) = (0, a) \ \forall \ (a, b) \in V_2(R)$ and $a, b \in R$

Now $\quad (TS)(a, b) = T[S(a, b)]$

$= T(0, a)$

$= (0, 0) = 0 \ \forall \ (a, b) \in V_2(R)$

$TS = 0$

and $\quad (ST)(a, b) = S[T(a, b)]$

$= S(a, 0) = (0, a) \ \forall \ (a, b) \in V_2(R)$

$\therefore \quad ST \neq 0$

Example 6 *Let T be a linear transformation on a vector space $V(F)$. Prove that the set of all linear transformation S on V for which $TS = 0$ is a subspace of the vector space of all linear transformations.*

Solution Let $L(V, V)$ denote the set of all linear transformations on vector space $V(F)$.

Let $W = \{S : S$ is a linear transformation on V and $TS = 0\}$

Now we shall prove that W is a subspace of $L(V, V)$.

Let $S_1, S_2 \in W$, then $TS_1 = 0, TS_2 = 0$. If $a, b \in F$ and let $\alpha \in V$, then

$[T(aS_1 + bS_2)](\alpha) = T[(aS_1 + bS_2)(\alpha)]$

[By the product of linear transformations]

$= T[(aS_1)(\alpha) + (bS_2)(\alpha)]$

$= T[aS_1(\alpha) + bS_2(\alpha)]$

$= aT[S_1(\alpha)] + bT[S_2(\alpha)]$ $\quad$ [$\because$ T is linear]

$= a(TS_1)(\alpha) + b(TS_2)(\alpha)$

$= a0(\alpha) + b0(\alpha) = a0 + b0 = 0 = 0(\alpha)$

$\therefore \quad [T(aS_1 + bS_2)](\alpha) = 0(\alpha) \ \forall \ \alpha \in V.$

Thus, $T(aS_1 + bS_2) = 0$, therefore $aS_1 + bS_2 \in W$.

Hence, W is a subspace of $L(V, V)$.

Example 7 *If $T: U \to V$ is a linear transformation and U is finite dimensional, show that U and range of T have the same dimension iff T is non-singular. Determine all non-singular linear transformations*

$$T: V_4(R) \to V_3(R)$$

Solution We know that

dim. U = rank of T + nullity of T

$\Rightarrow \quad$ dim. U = dim. of range of T + dim. of null space

$\therefore$ dim. U = dim. of range of T iff dim. of null space of T is zero, i.e., iff T is non-singular.

Let T be a linear transformation from $V_4(R)$ into $V_3(R)$. Then T will be non-singular iff dim. of $V_4(R)$ = dim. of range of T.

Since dim. $V_4(R) = 4$ and dim. $V_3(R) = 3$, therefore, the dim. of range of $T = 3$ because range of $T \leq V_3(R)$. Thus dim. $V_4(R)$ cannot be equal to the dim. of range of T.

Hence, T cannot be non-singular. Consequently, there can be no non-singular linear transformation from $V_4(R)$ into $V_3(R)$.

Example 8 *Let V be a finite dimensional vector space and T be a linear operator on V. Suppose that rank (T^2) = rank (T). Prove that the range and null space of T are disjoint, i.e. have only the zero vector in common.*

Solution We know that, $\quad$ dim. V = rank (T) + nullity (T) $\quad$ ...(1)

Now, T_2 is also a linear operator on V, then

dim. V = rank (T^2) + nullity (T^2) $\quad$ ...(2)

From (1) and (2), we have

rank (T) + nullity (T) = rank (T^2) + nullity (T^2)

3. *If $T : R^3 \to R^3$ is a linear operator defined by $T(x, y, z) = (x + z, x - z, y)$. Show that T is invertible and find $T^{-1}(2, 4, 6)$.*

Solution. Let $T(x, y, z) = (0, 0, 0)$

$\Rightarrow \quad (x + z, x - z, y) = (0, 0, 0)$

$\Rightarrow \quad x + z = 0, x - z = 0, y = 0$

Solving these equations, we get $x = 0$, $y = 0$, $z = 0$. Therefore, for $\alpha \in R^3$, $T(\alpha) = 0 \Rightarrow \alpha = 0$. Thus T is non-singular. Hence T is invertible.

Now $\quad T(x, y, z) = (p, q, r)$

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$\Rightarrow \quad x + z = p, x - z = q, y = r$

$\Rightarrow \quad x = \dfrac{p+q}{2}, y = r, z = \dfrac{p-q}{2}$

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Solution. Since dim. $V_2(C) = 2$. Therefore, T is a linear transformation on a finite dimensional vector space, so that T will be invertible if and only if the null space of T contains only zero vector. The zero vector of $V_2(C)$ is $(0, 0)$.

Thus, T is invertible iff $T(x, y) = (0, 0)$

i.e. iff $\quad (\alpha x + \beta y, \gamma x + \delta y) = (0, 0)$

iff $\quad \alpha x + \beta y = 0$ and $\gamma x + \delta y = 0$

Now these equations have only zero solution, i.e. $x = 0, y = 0$ if and only if

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Now $\quad (TS)(a, b) = T[S(a, b)]$

$= T(0, a)$

$= (0, 0) = 0 \;\forall (a, b) \in V_2(R)$

$TS = 0$

and $\quad (ST)(a, b) = S[T(a, b)]$

$= S(a, 0) = (0, a) \;\forall\; (a, b) \in V_2(R)$

$\therefore \quad ST \neq 0$

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Solution. Let $L(V, V)$ denote the set of all linear transformations on vector space $V(F)$.

Let $W = \{S : S$ is a linear transformation on V and $TS = 0\}$

Now we shall prove that W is a subspace of $L(V, V)$.

Let $S_1, S_2 \in W$, then $TS_1 = 0$, $TS_2 = 0$. If $a, b \in F$ and let $\alpha \in V$, then

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[By the product of linear transformations]

$= T[(aS_1)(\alpha) + (bS_2)(\alpha)]$

$= T[aS_1(\alpha) + bS_2(\alpha)]$

$= aT[S_1(\alpha)] + bT[S_2(\alpha)]$ $\quad$ [$\because$ T is linear]

$= a(TS_1)(\alpha) + b(TS_2)(\alpha)$

$= a0(\alpha) + b0(\alpha) = a0 + b0 = 0 = 0(\alpha)$

$\therefore \quad [T(aS_1 + bS_2)]\,(\alpha) = 0\,(\alpha) \;\forall\; \alpha \in V.$

Thus, $T(aS_1 + bS_2) = 0$, therefore $aS_1 + bS_2 \in W$.

Hence, W is a subspace of $L(V, V)$.

Example 7. *If $T : U \to V$ is a linear transformation and U is finite dimensional, show that U and range of T have the same dimension iff T is non-singular. Determine all non-singular linear transformations*

$$T : V_4(R) \to V_3(R)$$

Solution. We know that

dim. U = rank of T + nullity of T

$\Rightarrow \quad$ dim. U = dim. of range of T + dim. of null space

$\therefore$ dim. U = dim. of range of T iff dim. of null space of T is zero, i.e., iff T is non-singular.

Let T be a linear transformation from $V_4(R)$ into $V_3(R)$. Then T will be non-singular iff dim. of $V_4(R)$ = dim. of range of T.

Since dim. $V_4(R) = 4$ and dim. $V_3(R) = 3$, therefore, the dim. of range of $T = 3$ because range of $T \leq V_3(R)$. Thus dim. $V_4(R)$ cannot be equal to the dim. of range of T.

Hence, T cannot be non-singular. Consequently, there can be no non-singular linear transformation from $V_4(R)$ into $V_3(R)$.

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Now, T_2 is also a linear operator on V, then

dim. V = rank (T^2) + nullity (T^2) $\quad$ ...(2)

From (1) and (2), we have

rank (T) + nullity (T) = rank (T^2) + nullity (T^2)

$\Rightarrow$ nullity (T) = nullity (T^2) [$\because$ rank (T) = rank (T^2)]

$\Rightarrow$ dim. of null space of T = dim. of null space of T^2.

If $\alpha \in$ null space of T, then

$$T(\alpha) = 0$$

$$T[T(\alpha)] = T(0)$$

$$\Rightarrow \quad T^2(\alpha) = 0 \qquad [\because T(0) = 0]$$

$$\Rightarrow \quad \alpha \in \text{null space of } T^2$$

$\therefore$ null space of $T \subseteq$ null space of T^2.

But null space of T and null space of T^2 are both subspaces of V and have the same dimension.

Then, null space of T = null space of T^2

$\Rightarrow$ null space of $T^2 \subseteq$ null space of T

$$\Rightarrow \quad T^2(\alpha) = 0$$

$$\Rightarrow \quad T(\alpha) = 0$$

Let $\beta \neq 0$ and $\beta \in R(T) \cap N(T)$, then $\beta \in R(T)$ and $\beta \in N(T)$. ...(1)

Now $\beta \in N(T) \Rightarrow T(\beta) = 0$

Also $\beta \in R(T) \Rightarrow \exists\, \alpha \in V$ such that $T(\alpha) = \beta$.

Now $T(\alpha) = \beta$

$\Rightarrow$ $T[T(\alpha)] = T(\beta) = 0$.

Thus there exists $\alpha \in V$ such that $T[T(\alpha)] = 0$ but $T(\alpha) = \beta \neq 0$ which is against to the equation (1). Therefore, there exists no $\beta \in R(T) \cap N(T)$ such that $\beta \neq 0$.

Hence $R(T) \cap N(T) = \{0\}$.

Example 9 *If A and B are linear transformations on vector space V(F), then show that a necessary and sufficient condition that both A and B be invertible is that both AB and BA be invertible.*

Solution **Condition is necessary.** Let A and B be two invertible linear transformations on a vector space V. Then,

$$AA^{-1} = I = A^{-1}A$$

and $$BB^{-1} = I = B^{-1}B$$

Now $$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= (AI)A^{-1}$$
$$= AA^{-1} = I$$

and $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$
$$= B^{-1}(IB) = B^{-1}B = I$$

$\therefore$ $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$

Thus AB is invertible.

Also, we have

$$(BA)(A^{-1}B^{-1}) = I = (A^{-1}B^{-1})(BA)$$

Thus, BA is invertible.

Condition is sufficient. Let AB and BA be both invertible. Then AB and BA are both one-one and onto.

Now we shall show that A and B are invertible. First, we shall show that A is invertible.

A is one-one. Let $\alpha_1, \alpha_2 \in V$. Then

$$A(\alpha_1) = A(\alpha_2)$$

$$\Rightarrow \quad B[A(\alpha_1)] = B[A(\alpha_2)]$$

$$\Rightarrow \quad (BA)(\alpha_1) = (BA)(\alpha_2)$$

$$\Rightarrow \quad \alpha_1 = \alpha_2$$

$\therefore$ A is one-one. [$\because$ BA is one-one.]

A is onto. Let $\beta \in V$. Since AB is onto, then there exists $\alpha \in V$ such that

$$(AB)(\alpha) = \beta.$$

$$\Rightarrow \quad A[B(\alpha)] = \beta.$$

Thus $\beta \in V$, then there exists $B(\alpha) \in V$ such that $A[B(\alpha)] = \beta$.

$\therefore$ A is onto.

Hence A is invertible.

Similarly, if we interchange the role of AB and BA, we find that B is invertible.

EXERCISE 6.1

1. Describe explicitly the linear transformation T from F^2 to F^2 such that $T(e_1) = (a, b)$, $T(e_2) = (c, d)$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$.
2. If $T : R^2 \to R^2$ is the linear transformation for which $T(1, 1) = 3$ and $T(0, 1) = -2$, find $T(a, b)$.
3. Describe explicitly a linear transformation from $V_3(R)$ into $V_4(R)$ which has its range the subspace spanned by the vectors $(1, 2, 0, 4)$, $(2, 0, -1, -3)$.
4. Find a linear mapping $T : R^3 \to R^4$ whose image is generated by $(1, -1, 2, 3)$ and $(2, 3, -1, 0)$.
5. Let $T : F^2 \to F^2$ be a linear operator defined by $T(x, y) = (x + y, x)\ \forall\ (x, y) \in F^2$, $x, y \in F$. Show that T is invertible and find a rule for a T^{-1} like the one which defines T.
6. If $T : R^3 \to R^3$ is a linear operator defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$, show that T is invertible and find T^{-1} $(2, 4, 6)$.
7. Let $T : R^3 \to R^3$ be a linear operator defined by $T(x, y, z) = (x - 3y - 2z, y - 4z, z) \forall (x, y, z) \in R^3$. Show that T is non-singular and find a formula for T^{-1} and hence find $T^{-1}(1, 2, 3)$.
8. Let T be the (unique) linear operator on C^3 for which $T(1, 0, 0) = (1, 0, i)$, $T(0, 1, 0) = (0, 1, 1)$, $T(0, 0, 1) = (i, 1, 0)$. Show that T is not invertible.
9. Let S and T be the linear operators on R^2 defined by $S(a, b) = (b, a)$ and $T(a, b) = (a, 0)$. Give rules like the one defining S and T for each of the linear transformations $(T + S)$, TS, ST, S^2, T^2.
10. Let V be of finite dimension and T be a linear operator on V for which $TS = I$, for some operator S on V. Show that T is invertible. Also show that $S = T^{-1}$.
11. Show that if two linear transformations of a finite dimensional vector space coincide on a basis of that vector space, then they are identical.
12. If T is a linear transformation on a finite dimensional vector space V such that range of T is a proper subset of V, show that there exists a non-zero element α in V with $T(\alpha) = 0$.
13. Let $T : R^3 \to R^3$ be defined as $T(x, y, z) = (0, x, y)$, show that $T \neq 0$, $T^2 \neq 0$ and $T^3 = 0$.
14. Let T be a linear transformation from a vector space U into a vector space V with kernel of $T \neq 0$. Show that there exists vectors α_1 and α_2 in U such that $\alpha_1 \neq \alpha_2$ and $T\alpha_1 = T\alpha_2$.
15. Let T be a linear transformation from $V_3(R)$ into $V_2(R)$ and let S be a linear transformation from $V_2(R)$ into $V_3(R)$. Prove that the transformation ST is not invertible.
16. Prove that the set of invertible linear operators on a vector space V with the operation of composition forms a group. Check if this group is commutative.
17. If $V(F)$ be the vector space of all polynomials in x and D and T be the two linear operators on V defined by
$$D[f(x)] = \frac{df(x)}{dx}, T[f(x)] = xf(x)$$
for each $f(x) \in V$. Then show that the product of these operators is not commutative, i.e. $DT \neq TD$ and $(TD)^2 = TD + T^2D^2$.

18. If $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\{\beta_1, \beta_2, ..., \beta_n\}$ are linearly independent sets of vectors in a finite dimensional vector space V, then there exists an invertible linear transformation T on V such that

$$T(\alpha_i) = \beta_i, \, i = 1, 2, ..., n$$

19. Let U and V be vector spaces over the same field F and S be an isomorphism of U onto V. Prove that $T \to STS^{-1}$ is an isomorphism of $L(U, U)$ and onto $L(V, V)$.

20. If V is the space of all polynomials of degree $\leq n$ over a field F, prove that the differentiation operator on V is nilpotent.

21. Let $T_1 : R^3 \to R^3$ be a linear operator defined by $T(x, y, z) = (2x, 2x - 5y, 2y + z)$. Find T^{-1}.

22. Let $T : R^3 \to R^3$ be a linear operator defined by $T(x, y, z) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z)$. Is T singular or non-singular ?

23. Let $T : R^3 \to R^2$ be defined as

$$T(x, y) = (2x - 4y, 3x - 6y).$$

Is T non-singular ? If not, find $\alpha \neq 0$ in R^2 such that $T(\alpha) = 0$.

24. Let $T : R^3 \to R^3$ be defined by

$T(x, y, z) = (x + y - 2z, x + 2y + z, 2x + 2y - 3z)$.

Is T non-singular ?

25. Let $T : R^3 \to R^3$ be defined by

$T(x, y, z) = (x + y + z, x + 2y - z, 3x + 5y - z)$.

Is T non-singular? If not, find $\alpha \neq 0$ in R^3 such that $T(\alpha) = 0$.

26. Let V be a vector space of all real polynomials in t. Let $T : V \to V$ be the linear operator defined by $T[f(t)] = tf(t) \; \forall \; f(t) \in V$. Is T singular or non-singular ?

Answers

1. $T(x, y) = (xa + yc, xb + yd)$
2. $T(x, y) = 5x - 2y$
3. $T(x, y, z) = (x + 2y, 2x - y, -4x - 3y)$
4. $T(x, y, z) = (x + 2y, -x + 3y, 2x - y, 3x)$
5. $T^{-1}(p, q) = (q, p - q)$
6. $T^{-1}(2, 4, 6) = (1, 0, -4)$
7. $T^{-1}(p, q, r) = (p + 3q + 14r, q + 4r, r)$; $T^{-1}(1, 2, 3) = (49, 14, 3)$
9. $(T + S)(x, y) = (x + y, x)$; $(TS)(x, y) = (y, 0)$; $(ST)(x, y) = (0, x)$; $(S^2)(x, y) = (x, y)$; $(T^2)(x, y) = (x, 0)$.

16. Non-commutative
21. $T^{-1}(p,q,r) = \left(\dfrac{p}{2}, \dfrac{p-q}{5}, \dfrac{-2p+2q+5r}{5}\right)$
22. Non-singular
23. T is singular and $\alpha = (-2, 1)$ such that $T(\alpha) = 0$
24. T is non-singular.
25. T is singular and $\alpha = (-3, 2, 1)$ such that $T(\alpha) = 0$
26. T is non-singular.

6.12 COORDINATE VECTOR

Let V be a finite dimensional vector space over a field F and let dim. $V = n$, then $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a basis of V and for $\alpha \in V$, suppose that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$$

for a_i's $\in F$. Then the coordinate vector of α relative to β, which we write as a column vector unless otherwise specified or implied, is

$$[\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

6.13 MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

Let U be an m-dimensional vector space over a field F and let V be an n-dimensional vector space over the field F. Let $B = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $B' = \{\beta_1, \beta_2, ..., \beta_n\}$ be the basis of U and

respectively. If T is a linear transformation from U into V, then $T(\alpha_1), T(\alpha_2), ..., T(\alpha_m)$ are vectors in V. Since $B' = \{\beta_1, \beta_2, ..., \beta_n\}$ is a basis of V so that each $T(\alpha_i)$ is a linear combination of the elements of B'. For $a_{ij} \in F, 1 \leq i \leq m, 1 \leq j \leq n$, we have

$$T(\alpha_1) = \{a_{11}\beta_1 + a_{12}\beta_2 + ,..., + a_{1n}\beta_n)$$

$$T(\alpha_2) = \{a_{21}\beta_1 + a_{22}\beta_2 + ,..., + a_{2n}\beta_n)$$

....

$$T(\alpha_m) = \{a_{m1}\beta_1 + a_{m2}\beta_2 + ,..., + a_{mn}\beta_n)$$

Definition. *The transpose of the above matrix of coefficients, denoted by $[T]_B$ is called the matrix representation of T relative to the ordered basis B.*

Thus,

$$[T]_B = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n \times m}$$

For Example : Let V be the vector space of polynomials in 't' over the field of reals R, of degree ≤ 3, and let

$$D : V \to V$$

be the differential operator defined by $D[p(t)] = \dfrac{d}{dt}[p(t)]$

We compute the matrix of D in the basis $B = [1, t, t^2, t^3]$ as follows

$$D(1) = 0 = 0 + 0t + 0t^2 + 0t^3$$

$$D(t) = 1 = 1 + 0t + 0t^2 + 0t^3$$

$$D(t^2) = 2t = 0 + 2t + 0t^2 + 0t^3$$

$$D(t^3) = 3t^2 = 0 + 0t + 3t^2 + 0t^3$$

Thus, the matrix of D relative to B is given by

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

THEOREM 1. *Let U be an m-dimensional vector space aver the field F and V an n-dimensional vector space over the field F. Let B be an ordered basis for U and B' an ordered basis for V. Let T be any linear transformation from U into V. Then for any vector $\alpha \in U$.*

$$[T]_B\,[\alpha]_B = [T(\alpha)]_{B'}$$

Proof. Let $B = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be an ordered basis for U and $B' = \{\beta_1, \beta_2, ..., \beta_n\}$ an ordered basis for V. T is linear transformation from U into V, then T is determined by its action on the vectors α_i, $1 \leq i \leq m$. Each of m vectors $T(\alpha_i)$ is uniquely expressible as a linear combination of elements of B' :

$$T(\alpha_i) = \sum_{j=1}^{n} a_{ij}\beta_j \quad ...(1)$$

where $a_{i1}, a_{i2}, ..., a_{in}$ are the coordinates of $T(\alpha_i)$ in the ordered basis B'.

If α be any vector in U, then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_n$$

$$\therefore \quad [\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

Now, $T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m)$

$$= a_1T(\alpha_1) + a_2T(\alpha_2) + \dots + a_mT(\alpha_m)$$

$$= a_1 \sum_{j=1}^{n} a_{1j}\beta_j + a_2 \sum_{j=1}^{n} a_{2j}\beta_j + \dots + a_m \sum_{j=1}^{n} a_{mj}\beta_j \qquad \text{[using (1)]}$$

$$= aTa\left(\sum_{i=1}^{n} a_i\alpha_i\right) + bT\left(\sum_{i=1}^{n} b_i\alpha_i\right) = aT(\alpha) + bt(\beta)$$

$$= a_1a_{11}\beta_1 + a_1a_{12}\beta_2 + \dots + a_1a_{1n}\beta_n + a_2a_{21}\beta_1 + a_2a_{22}\beta_2 + \dots + a_2a_{2n}\beta_n$$

$$\dots\dots\dots$$

$$+ a_ma_{m1}\beta_1 + a_ma_{m2}\beta_2 + \dots + a_ma_{mn}\beta_n$$

$$\therefore \quad [T(\alpha)]_{B'} = \begin{bmatrix} a_1a_{11} + a_2a_{21} + \dots + a_ma_{m1} \\ a_1a_{12} + a_2a_{22} + \dots + a_ma_{m2} \\ \dots\dots\dots \\ a_1a_{1n} + a_2a_{2n} + \dots + a_ma_{mn} \end{bmatrix}_{n\times m}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n\times m} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m\times 1}$$

$$[T(\alpha)]_{B'} = [T]_B[\alpha]_B.$$

THEOREM 2. *Let U, V and W be vector spaces over the field F of respective dimensions n, m and p. Let T_1 be a linear transformation from U into V and T_2 a linear transformation from V into W. If B, B' and B'' are the ordered bases for the spaces U, V and W respectively, if A is the matrix of T_1, relative to the pair B, B' and B is the matrix of T_2 relative to the pair B' and B'', then the matrix of $(T_2 T_1)$ relative to the pair B, B'' is the product matrix C = BA.*

Proof. Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $B' = \{\beta_1, \beta_2, \dots, \beta_m\}$ and $B'' = \{\gamma_1, \gamma_2, \dots, \gamma_p\}$ be the bases of U, V and W respectively. If α is any vector in U, then

$$[T_1(\alpha)]_{B'} = [T_1]_B[\alpha]_B \qquad \text{[By above theorem]}$$

$$= A[\alpha]_B \qquad [\because A = [T_1]_B]$$

and $\quad [T_2(T_1(\alpha))]_{B''} = [T_2]_{B'}\,[T_1(\alpha)]_{B'} = B[T_1(\alpha)]_{B'} \qquad [\because B = [T_2]_{B'}]$

$$\therefore \quad [(T_2T_1](\alpha)]_{B''} = BA[\alpha]_B.$$

Hence by the definition and uniqueness of the representing matrix, we must have $\quad C = BA$ as the matrix of (T_2T_1) relativeto B, B''.

THEOREM 3. *Let V be an n-dimensional vector space over the field F and B be an ordered basis of V. If T_1 and T_2 are linear operators from V into V, then*

(i) $[T_1+T_2]_B = [T_1]_B + [T_2]_B$

(ii) $[cT_1]_B = c[T_1]_B$, for $c \in F$

(iii) $[T_2\,T_1]_B = [T_1]_B[T_2]_B$.

Proof. Let $\{a_1, a_2, \dots, a_n\}$ be the basis of V, then for $a_{ij} \in F$ and $b_{ij} \in F$, $1 \le i \le n$, $1 \le j \le n$, we have

$$T_1(\alpha_1) = a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1n}\alpha_n$$

$$T_1(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2 + \dots + a_{2n}\alpha_n$$

$$\dots\dots\dots$$

$$T_1(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{nn}\alpha_n$$

$$\therefore \quad [T_1]_B = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Also,

$$T_2(\alpha_2) = b_{11}\alpha_1 + b_{12}\alpha_2 + \dots + b_{1n}\alpha_n$$

$$T_2(\alpha_2) = b_{21}\alpha_1 + b_{22}\alpha_2 + \dots + b_{2n}\alpha_n$$

$$\dots\dots\dots$$

$$T_2(\alpha_n) = b_{n1}\alpha_1 + b_{n2}\alpha_2 + \dots + b_{nn}\alpha_n$$

$$\therefore \quad [T_2]_B = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}$$

(i) $(T_1 + T_2)(\alpha_1) = (T_1)(\alpha_1) + (T_2)(\alpha_1)$

$$= (a_{11} + b_{11})\alpha_1 + (a_{12} + b_{12})\alpha_2 + \dots + (a_{1n} + b_{1n})\alpha_n$$

$$(T_1 + T_2)(\alpha_2) = (T_1)(\alpha_2) + (T_2)(\alpha_2)$$

$$= (a_{21} + b_{21})\alpha_1 + (a_{22} + b_{22})\alpha_2 + \dots + (a_{2n} + b_{2n})\alpha_n$$

$$(T_1 + T_2)(\alpha_n) = (T_1)(\alpha_n) + (T_2)(\alpha_n)$$

$$= (a_{n1} + b_{n1})\alpha_1 + (a_{n2} + b_{n2})\alpha_2 + \dots + (a_{nn} + b_{nn})\alpha_n$$

$$[T_1 + T_2]_B = \begin{bmatrix} (a_{11}+b_{11}) & (a_{21}+b_{21}) & \cdots & (a_{n1}+b_{n1}) \\ (a_{12}+b_{12}) & (a_{22}+b_{22}) & \cdots & (a_{2n}+b_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ (a_{1n}+b_{1n}) & (a_{2n}+b_{2n}) & \cdots & (a_{nn}+b_{nn}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}$$

$$= [T_1]_B + [T_2]_B$$

(ii) $(cT_1)(\alpha_1) = cT_1(\alpha_1) = ca_{11}\alpha_1 + ca_{12}\alpha_2 + \dots + ca_{1n}\alpha_n$

$(cT_1)(\alpha_2) = cT_1(\alpha_2) = ca_{21}\alpha_1 + ca_{22}\alpha_2 + \dots + ca_{2n}\alpha_n$

....

$(cT_1)(\alpha_n) = cT_1(\alpha_n) = ca_{n1}\alpha_1 + ca_{n2}\alpha_2 + \dots + ca_{nn}\alpha_n$

$$\therefore \quad (cT_1)_B = \begin{bmatrix} ca_{11} & ca_{21} & \cdots & ca_{n1} \\ ca_{12} & ca_{22} & \cdots & ca_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ ca_{1n} & ca_{2n} & \cdots & ca_{nn} \end{bmatrix} = c\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} = c[T_1]_B$$

(iii) $(T_2T_1)(\alpha_1) = T_2(T_1(\alpha_1))$

$= T_2(a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1n}\alpha_n)$

$= a_{11}T_2(\alpha_1) + a_{12}T_2(\alpha_2) + \dots + a_{1n}T_2(\alpha_n)$ [$\because T_2$ is linear]

$= a_{11}(b_{11}\alpha_1 + b_{12}\alpha_2 + \dots + b_{1n}\alpha_n)$

$+ a_{12}(b_{21}\alpha_1 + b_{22}\alpha_2 + \dots + b_{2n}\alpha_n)$

....

$+ a_{1n}(b_{n1}\alpha_1 + b_{n2}\alpha_2 + \dots + b_{nn}\alpha_n)$

$= (a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1})\alpha_1$

$+ (a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1n}b_{n2})\alpha_2$

....

$+ (a_{11}b_{1n} + a_{12}b_{2n} + \dots + a_{1n}b_{nn})\alpha_n$

$(T_2T_1)(\alpha_2) = T_2(T_1(\alpha_2)) = T_2(a_{21}\alpha_1 + a_{22}\alpha_2 + \dots + a_{2n}\alpha_n)$

$= a_{21}T_2(\alpha_1) + a_{22}T_2(\alpha_2) + \dots + a_{2n}T_2(\alpha_n)$

$= a_{21}(b_{11}\alpha_1) + b_{12}\alpha_2 + \dots + b_{1n}\alpha_n)$

$+ a_{22}(b_{21}\alpha_1 + b_{22}\alpha_2 + \dots + b_{2n}\alpha_n)$

....

$+ a_{2n}(b_{n1}\alpha_1 + b_{n2}\alpha_2 + \dots + b_{nn}\alpha_n)$

$= a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2n}b_{n1})\alpha_1$

$+ (a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2n}b_{n2})\alpha_2$

....

$+ (a_{21}b_{1n} + a_{22}b_{2n} + \dots + a_{2n}b_{nn})\alpha_n$

Similarly,

$(T_2T_1)(\alpha_n) = T_2(T_1(\alpha_n)) = T_2(a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{nn}\alpha_n)$

$= a_{n1}T_2(\alpha_1) + a_{n2}T_2(\alpha_2) + \dots + a_{nn}T_2(\alpha_n)$

$= a_{n1}(b_{11}\alpha_1 + b_{12}\alpha_2 + \dots + b_{1n}\alpha_n)$

$+ a_{n2}(b_{21}\alpha_1 + b_{22}\alpha_2 + \dots + b_{2n}\alpha_n)$

....

$+ a_{nn}(b_{n1}\alpha_1 + b_{n2}\alpha_2 + \dots + b_{nn}\alpha_n)$

$= (a_{n1}b_{11} + a_{n2}b_{21} + \dots + a_{nn}b_{n1})\alpha_1$

$+ (a_{n1}b_{12} + a_{n2}b_{22} + \dots + a_{nn}b_{n2})\alpha_2$

....

$+ (a_{n1}b_{1n} + a_{n2}b_{2n} + \dots + a_{nn}b_{nn})\alpha_n$

$$\therefore \quad [T_2T_1]_B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1} & a_{21}b_{11} + \dots + a_{2n}b_{n1}, \dots & a_{n1}b_{11} + \dots + a_{nn}b_{n1} \\ a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1n}b_{n2} & a_{21}b_{12} + \dots + a_{2n}b_{n2}, \dots & a_{n1}b_{12} + \dots + a_{nn}b_{n2} \\ \dots & \dots & \dots \\ a_{11}b_{1n} + a_{12}b_{2n} + \dots + a_{1n}b_{nn} & a_{21}b_{1n} + \dots + a_{2n}b_{nn} \dots & a_{n1}b_{1n} + \dots + a_{nn}b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} = [T_2]_B[T_1]_B$$

6.14 CHANGE OF BASIS

It has been shown that we can represent vectors by tuples (column vectors) and linear operators by matrix once we have selected a basis.

In this section we will see how the representation of matrix of linear transformation changes if we take another basis.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of V and let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be another basis of V and suppose

$$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1n}\alpha_n$$
$$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + \dots + a_{2n}\alpha_n$$
....
$$\beta_n = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{nn}\alpha_n$$

Then the transpose of the coefficient matrix of above equation is called the *transition matrix* from the basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ to the basis $\{\beta_1, \beta_2, \dots, \beta_n\}$

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

REMARK

- P is invertible and its P^{-1} is the transition matrix from new basis to old basis.

For Example: Let $\{(1,0),(0,1)\}$ and $\{(1,1),(-1,0)\}$ be two bases of R^2, then $(1,1) = 1.(0,1) + 1(0,1)$ and $(-1,0) = -1(1,0) + 0.(0,1)$

$$\therefore \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

THEOREM 1. *Let P be the transition matrix from a basis B to a basis B' in a vector space V. Then for any vector $a \in V$, $P[\alpha]_{B'} = [\alpha]_B$ and $[\alpha]_{B'} = P^{-1}[\alpha]_B$.*

Proof. Let V be an n-dimensional vector space and let,

$$B=\{\alpha_1,\alpha_2,......,\alpha_n\} \text{ and } B'=\{\beta_1,\beta_2,......,\beta_n\}$$

be two bases of V and let P be the transition matrix from B to B'. Then we have,

$$\beta_1=a_{11}\alpha_1+a_{12}\alpha_2+......+a_{1n}\alpha_n$$
$$\beta_2=a_{21}\alpha_1+a_{22}\alpha_2+......+a_{2n}\alpha_n$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\beta_n=a_{n1}\alpha_1+a_{n2}\alpha_2+......+a_{nn}\alpha_n; \text{ for } a_{ij}\in F$$

$$\therefore\quad P=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{22} & \cdots & a_{n2}\\ \cdots & \cdots & \cdots & \cdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{bmatrix}.$$

Now suppose $\alpha\in V$ such that

$$\alpha=b_1\beta_1+b_2\beta_2+......+b_n\beta_n.$$

Substituting β's from above, we obtain,

$$\alpha=b_1(a_{11}\alpha_1+a_{12}\alpha_2+......+a_{1n}\alpha_n)+b_2(a_{21}\alpha_1+a_{22}\alpha_2+......+a_{2n}\alpha_n)+$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+b_n(a_{n1}\alpha_1+a_{n2}\alpha_2+......+a_{2n}\alpha_n)$$
$$=(b_1a_{11}+b_2a_{12}+......+b_na_{1n})\alpha_1+(b_1a_{12}+b_2a_{22}+......+b_na_{2n})\alpha_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+(b_1a_{1n}+b_2a_{2n}+......+b_na_{2n})\alpha_n$$

$$\text{Thus, } [\alpha]_{B'}=\begin{bmatrix} b_1\\ b_2\\ \cdots\\ b_n\end{bmatrix} \text{ and } [\alpha]_B=\begin{bmatrix} b_1a_{11}+b_2a_{12}+......+b_na_{1n}\\ b_1a_{12}+b_2a_{22}+......+b_na_{2n}\\ \dots\dots\dots\dots\\ b_1a_{1n}+b_2a_{2n}+......+b_na_{nn}\end{bmatrix}$$

Accordingly,

$$P[\alpha]_{B'}=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{22} & \cdots & a_{n2}\\ \cdots & \cdots & \cdots & \cdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{bmatrix}.\begin{bmatrix} b_1\\ b_2\\ \cdots\\ b_n\end{bmatrix}=\begin{bmatrix} b_1a_{11}+b_2a_{12}+......+b_na_{1n}\\ b_1a_{12}+b_2a_{22}+......+b_na_{2n}\\ \dots\dots\dots\dots\\ b_1a_{1n}+b_2a_{2n}+......+b_na_{nn}\end{bmatrix}$$
$$=[\alpha]_B$$

Furthermore, since P is invertible, hence

$$P[\alpha]_{B'}=[\alpha]_B \quad\Rightarrow\quad P^{-1}P[\alpha]_{B'}=P^{-1}[\alpha]_B$$
$$\Rightarrow\quad I[\alpha]_{B'}=P^{-1}[\alpha]_B \quad\Rightarrow\quad [\alpha]_{B'}=P^{-1}[\alpha]_B$$

THEOREM 2 *Let P be the transition matrix from a basis to a basis B' in a vector space V. Then for any linear operator T on V,*

$$[T]_{B'}=P^{-1}[T]_BP.$$

Proof Let α be any vector in V, then we have

$$[T]_B[\alpha]_B=[T(\alpha)]_B \qquad ...(1)$$

and $$P[\alpha]_{B'}=(\alpha)_B \qquad ...(2)$$

$$\Rightarrow\quad [T]_BP[\alpha]_{B'}=[T]_B[\alpha]_B=[T(\alpha)]_B \qquad \text{[Using (1)]}$$
$$\Rightarrow\quad P^{-1}[T]_BP[\alpha]_{B'}=P^{-1}[T(\alpha)]_B=[T(\alpha)]_{B'} \qquad \text{[By theorem (1)]}$$
$$=[T]_{B'}[\alpha]_{B'} \qquad \text{[Using (1)]}$$
$$\Rightarrow\quad P^{-1}[T]_BP=[T]_{B'} \qquad [\because [\alpha]_{B'}\in F \text{ are arbitrary.}]$$

6.15 SIMILARITY OF MATRICES

Let A and B be two square matrices each of order n over the field F. Then B is similar to A if there exists an invertible matrix C of order n over the field F such that

$$B=C^{-1}AC \text{ or } A=CBC^{-1}$$

THEOREM 1 *The relation of similarity is an equivalence relation in the set of all $n\times n$ matrices over the field F.*

Proof Let M_n be the set of all $n\times n$ matrices over the field F.

If $A, B\in M_n$, then B is similar to A if there exists an invertible matrix C in M_n such that

$$B=C^{-1}AC.$$

Now, in order to prove equivalence relation, we shall prove that the relation is reflexive, symmetric and transitive.

Reflexive. Let $A\in M_n$, then there exists an $n\times n$ invertible matrix I_n such that

$$A=I_n^{-1}AI_n$$

where I_n is the unit matrix over F.

Thus, A is similar to A itself.

$\therefore$ The relation of similarity on M_n is reflexive.

Symmetric. Let A, $B\in M_n$ such that A is similar to B, then there exists an invertible matrix $C\in M_n$ such that

$$A=C^{-1}BC$$
$$\Rightarrow\quad CAC^{-1}=C(C^{-1}BC)C^{-1}$$
$$\Rightarrow\quad CAC^{-1}=(CC^{-1})B(CC^{-1})$$
$$\Rightarrow\quad CAC^{-1}=B$$
$$\Rightarrow\quad B=CAC^{-1}$$
$$\Rightarrow\quad B=(C^{-1})^{-1}AC^{-1} \qquad [\because C^{-1} \text{ is invertible.}]$$

$\Rightarrow$ B is similiar to A.

$\therefore$ The relation of similarity on M_n is symmetric.

Transitive. Let $A, B, C\in M_n$ such that A is similar to B and B is similar to C.

Now A is similar to B, then there exists an invertible matrix $P\in M_n$ such that

$$A=P^{-1}BP$$

Also B is similar to C, then there exists an invertible matrix $Q\in M_n$ such that

$$B=Q^{-1}CQ$$

Now
$$A=P^{-1}BP$$
$$\Rightarrow\quad A=P^{-1}(Q^{-1}CQ)P$$
$$\Rightarrow\quad A=(P^{-1}Q^{-1})C(QP)$$
$$\Rightarrow\quad A=(QP)^{-1}C(QP) \qquad [\because (QP)^{-1}=P^{-1}Q^{-1}]$$

$\Rightarrow$ A is similar to C.

$\therefore$ The relation of similarity on M_n is transitive.

Hence, the relation of similarity of matrices is an equivalence relation on the set of $n\times n$ matrices over the field F.

THEOREM 2. *Similar matrices have the same determinant.*

Proof. Let A and B be two square matrices of order $n \times n$ over a field F such that B is similar to A. Then there exists an invertible matrix C of order $n \times n$ over the field F such that

$$B = C^{-1}AC$$

$$\det B = \det(C^{-1}AC)$$

$\Rightarrow$ $\det B = (\det C^{-1})(\det A)(\det C)$

$\Rightarrow$ $\det B = (\det C^{-1})(\det C)(\det A)$

$\Rightarrow$ $\det B = (\det C^{-1}C)(\det A)$

$\Rightarrow$ $\det B = (\det I_n)(\det A)$ $\quad [\because C^{-1}C = I_n]$

$\Rightarrow$ $\det B = \det A$ $\quad [\because \det I_n = 1]$

Hence A and B have the same determinant.

6.16 SIMILARITY OF LINEAR TRANSFORMATIONS

Let S and T be two linear transformations on a vector space $V(F)$. Then T is similar to S if there exists an invertible linear transformation P on V such that

$$T = PSP^{-1}$$

THEOREM 1. *The relation of similarity is an equivalence relation in the set of all linear transformations on a vector space $V(F)$.*

Proof. Let $L(V, V)$ be the set of all linear transformations on V over a field F. Let S and T be two elements of $L(V, V)$. Then T is similar to S if there exists an invertible linear transformation $P \in L(V, V)$ such that

$$T = PSP^{-1}$$

Now we shall show that the relation of similarity is an equivalence relation.

Reflexive. Let $T \in L(V, V)$, Then there exists an invertible linear transformation $I \in L(V, V)$ such that $T = ITI^{-1}$

where I is an identity transformation.

$\therefore$ T is similar to T.

Symmetric. Let $T, S \in L(V, V)$ such that T is similar to S, then there exists an invertible linear transformation $P \in L(V, V)$ such that

$$T = PSP^{-1}$$

$\Rightarrow$ $P^{-1}TP = P^{-1}(PSP^{-1})P$

$\Rightarrow$ $P^{-1}TP = (P^{-1}P)S(P^{-1}P)$

$\Rightarrow$ $P^{-1}TP = ISI$ $\quad [\because P^{-1}P = I = PP^{-1}]$

$\Rightarrow$ $P^{-1}TP = S$

$\Rightarrow$ $S = P^{-1}TP$

$\Rightarrow$ $S = P^{-1}T(P^{-1})^{-1}$ $\quad [(P^{-1})^{-1} = P]$

$\Rightarrow$ S is similar to T.

$\therefore$ If T is similar to S, then S is similar to T.

Transitive. Let $T_1, T_2, T_3 \in L(V, V)$ such that T_1 is similar to T_2 and T_2 is similar to T_3. Now T_1 is similar to T_2, then there exists an invertible linear transformation $P \in L(V, V)$ such that

$$T = PT_2P^{-1}$$

Also T_2 is similar to T_3, then there exists an invertible linear transformation $Q \in L(V, V)$ such that

$$T_2 = QT_3Q^{-1}$$

Now $\quad T_1 = PT_2P^{-1}$

$\Rightarrow$ $T_1 = P(QT_3Q^{-1})P^{-1}$

$\Rightarrow$ $T_1 = (PQ)T_3(Q^{-1}P^{-1})$

$\Rightarrow$ $T_1 = (PQ)T_3(PQ)^{-1}$ $\quad [\because (PQ)^{-1} = Q^{-1}P^{-1}]$

$\Rightarrow$ T_1 is similar and T_3.

$\Rightarrow$ T_1 is similar to T_2 and T_2 is similar to T_3 then T_1 is similar to T_3.

Hence the relation of similarity of linear transformations on a vector space $V(F)$ is an equivalence relation.

THEOREM 2. *Let T be a linear operator on an n-dimensional vector space $V(F)$ and let B and B' be two ordered bases for V. Then the matrix of T relative to B' is similar to the matrix of T relative to B.*

Proof. Since dim. $V = n$. Let $B = \{\alpha_1, \alpha_2 ..., \alpha_n\}$ and $B' = (\beta_1, \beta_2,..., \beta_n\}$ any two ordered bases of V.

Let $[T]_B = [a_{ij}]_{m \times n}$ be the matrix of T relative to B.

and $[T]_{B'} = [b_{ij}]_{n \times n}$ be the matrix of T relative to B'.

Then we have

$$T(\alpha_j) = \sum_{i=1}^{n} a_{ij}\alpha_i, j = 1, 2, ...n \quad ...(1)$$

and

$$T(\beta_j) = \sum_{i=1}^{n} b_{ij}\beta_i, j = 1, 2, ...n \quad ...(2)$$

Let S be the linear operator on V defined by

$$S(\alpha_j) = \beta_j, j = 1, 2,..., n \quad ...(3)$$

Clearly, S maps a basis B onto a basis B', therefore S is necessary invertible. Let $[S]_B$ be the matrix of S relative to B, then $[S]_B$ is also invertible.

Now if $[S]_B = [p_{ij}]_{n \times n}$, then

$$S(\alpha_j) = \sum_{i=1}^{n} p_{ij}\alpha_i, j = 1, 2, ...n \quad ...(4)$$

We have, $\quad T(\beta_j) = T[S(\alpha_j)]$ $\quad$ [using (3)]

$\Rightarrow$ $T(\beta_j) = T\left[\sum_{i=1}^{n} p_{ij}\alpha_i\right]$ $\quad$ [Using (4)]

$\Rightarrow$ $T(\beta_j) = T\left[\sum_{k=1}^{n} p_{kj}\alpha_k\right]$ $\quad$ [Replacing i by k which is immaterial]

$\Rightarrow$ $T(\beta_j) = \sum_{i=1}^{n} p_{kj}T(\alpha_k)$ $\quad [\because T$ is linear$]$

$\Rightarrow$ $T(\beta_j) = \sum_{k=1}^{n} p_{kj}\sum_{i=1}^{n} a_{ik}\alpha_i$ $\quad$ [Using (1)]

$$T(\beta_j) = \sum_{i=1}^{n}\left(\sum_{i=1}^{n} a_{ik}p_{kj}\right)\alpha_i \quad ...(5)$$

From (2), on replacing i by k, we have

$$T(\beta_j) = \sum_{k=1}^{n} b_{ki}\beta_k$$

$$\Rightarrow \quad T(\beta_j) = \sum_{k=1}^{n} b_{ki} S(\alpha_k) \qquad \text{[Using (3)]}$$

$$\Rightarrow \quad T(\beta_j) = \sum_{k=1}^{n} b_{ki} \sum_{i=1}^{n} p_{ki}\alpha_i \qquad \text{[Using (4) on replacing } j \text{ by } k]$$

$$\Rightarrow \quad T(\beta_j) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} p_{ik}b_{kj}\right)\alpha_i \qquad \text{...(6)}$$

Now from (5) and (6), we have

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}p_{kj}\right)\alpha_i = \sum_{i=1}^{n}\left(\sum_{i=1}^{n} p_{ik}b_{kj}\right)\alpha_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} a_{ik}p_{kj} = \sum_{i=1}^{n} p_{ik}b_{kj}$$

$$\Rightarrow \quad [a_{ik}]_{n\times n}[p_{kj}]_{n\times n} = [p_{ik}]_{n\times n}[b_{kj}]_{n\times n} \qquad \text{[By matrix multiplication]}$$

$$\Rightarrow \quad [T]_B[S]_B = [S]_B[T]_B$$

$$\Rightarrow \quad [S]_B^{-1}[T]_B[S]_B = [T]_B$$

Hence $[T]_B$ is similar to $[T]_B$.

THEOREM 3 *Let V be an n-dimensional vector space over the field F and T_1 and T_2 be two linear operators on V. If there exists two ordered bases B and B' for V such that $[T_1]_B = [T_2]_{B'}$, then T_2 is similar to T_1.*

Proof dim. $V = n$. Let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $B' = \{\beta_1, \beta_2, ... \beta_n\}$ be two ordered bases for V.

Let $\qquad [T_1]_B = [T_2]_{B'} = [a_{ij}]_{n\times n}$

Then we have $\qquad T_1(\alpha_j) = \sum_{i=1}^{n} a_{ij}\alpha_i, j = 1,2,...,n \qquad$ ...(1)

and $\qquad T_2(\beta_j) = \sum_{i=1}^{n} a_{ij}\beta_i, i = 1,2,...,n \qquad$ ...(2)

Let S be the linear operator on V defined by

$$S(\alpha_j) = \beta_j, j = 1, 2, ..., n \qquad \text{...(3)}$$

Clearly, S maps a basis B onto a basis B' of V, therefore, S is invertible.

Now we have

$$T_2(\beta_j) = T_2[S(\alpha_j)] \qquad \text{[using (3)]}$$

$$\Rightarrow \quad T_2(\beta_j) = (T_2S)(\alpha_j). \qquad \text{...(4)}$$

From (2), $\qquad T_2(\beta_j) = \sum_{i=1}^{n} a_{ij}\beta_i$

$$\Rightarrow \quad T_2(\beta_j) = \sum_{i=1}^{n} a_{ij}S(\alpha_i) \qquad \text{[Using (3)]}$$

$$\Rightarrow \quad T_2(\beta_j) = S\left(\sum_{i=1}^{n} a_{ij}\alpha_i\right) \qquad [\because S \text{ is linear.}]$$

$$\Rightarrow \quad T_2(\beta_j) = S[T_1((\alpha_j))]$$

$$\Rightarrow \quad T_2(\beta_j) = (ST_1)(\alpha_j) \qquad \text{...(5)}$$

From (4) and (5), we have

$$(T_2S)(\alpha_j) = (ST_1)(\alpha_j), j = 1, 2, ..., n$$

Since T_2S and ST_1, agree on a basis B on V, then we have

$$T_2S = ST_1$$

$$\Rightarrow \quad T_2SS^{-1} = ST_1S^{-1} \qquad [\because S^{-1} \text{ exists.}]$$

$$\Rightarrow \quad T_2 = ST_1S^{-1}$$

Hence, T_2 is similar to T_1.

6.17 DETERMINANT OF A LINEAR TRANSFORMATION ON A FINITE DIMENSIONAL VECTOR SPACE

Let $V(F)$ be an n-dimensional vector space and T be a linear operator on V. If B and B' be two ordered bases of V, then the matrices $[T]_B$ and $[T]_{B'}$ are similar and similar matrices have the same determinant. Then the determinant of T is the determinant of the matrix of T relative to any ordered basis for V.

6.18 SCALAR TRANSFORMATION

Let $V(F)$ be a vector space and T be a linear transformation on V. Then for a fixed scalar $c \in F$, the linear transformation T on V is said to be a scalar transformation of V if

$$T(\alpha) = c\alpha \ \forall \ \alpha \in V$$

Also, we may write $T = cI$, where I is the identity transformation on V.

6.19 TRACE OF A MATRIX

Let there be a square matrix of order n over a field F. Then the trace of A, denoted by trace A or tr A, is the sum of the elements of A lying along the principal diagonal.

If $A = [a_{ij}]_{n\times n}$, then $\qquad tr\, A = a_{11} + a_{22} + ... + a_{nn} = \sum_{i=1}^{n} a_{ii}$.

THEOREM 1 *Let A and B be two square matrices of order n over a field F and $\lambda \in F$. Then*

(i) $tr\,(\lambda A) = \lambda\, tr\, A$

(ii) $tr\,(A + B) = tr\, A + tr\, B$

(iii) $tr\,(AB) = tr\,(BA)$

Proof Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ and $\lambda \in F$. Then

$$\lambda A = [\lambda a_{ij}]_{n\times n}.$$

(i)
$$tr\,(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii}$$

$$= \lambda \sum_{i=1}^{n} a_{ij} = \lambda\, tr\, A$$

$A + B = [a_{ij} + b_{ij}]_{n\times n}$, then

(ii)
$$\text{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii})$$
$$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{tr}\,A + \text{tr}\,B$$

(iii) By the multiplication of matrices, we have

$$AB = [c_{ij}]_{n\times n}$$

where
$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad \text{...(1)}$$

and
$$BA = [d_{ij}]_{n\times n} \text{ where } d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} \quad \text{...(2)}$$

Now,
$$\text{tr}(AB) = \sum_{i=1}^{n} c_{ii}$$
$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right) \quad \text{[Using (1)]}$$
$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ik}b_{ki}\right)$$

(Interchanging the order of summation)

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ki}a_{ik}\right)$$
$$= \sum_{i=1}^{n} d_{kk} = \text{tr}(BA).$$

THEOREM 2 *Similar matrices have the same trace.*

Proof Let A and B be two similar matrices. Then there exist an invertible matrix C such that

$$B = C^{-1}AC$$

Let $D = C^{-1}A$, then

$$B = DC$$
$$\Rightarrow \quad \text{tr}\,B = \text{tr}\,(DC)$$
$$\Rightarrow \quad \text{tr}\,B = \text{tr}\,(CD) \quad [\because \text{tr}\,(AB) = \text{tr}\,(BA)]$$
$$\Rightarrow \quad \text{tr}\,B = \text{tr}\,(CC^{-1}A)$$
$$\Rightarrow \quad \text{tr}\,B = \text{tr}\,(IA)$$
$$\Rightarrow \quad \text{tr}\,B = \text{tr}\,A$$

6.20 TRACE OF A LINEAR TRANSFORMATION ON A FINITE DIMENSIONAL VECTOR SPACE

Let V be an n-dimensional vector space over a field F and T be a linear operator on V. If B and B' are two ordered bases of V, then the matrices $[T]_B$ and $[T]_{B'}$ are similar and the similar matrices have the same trace. Thus, the *trace of a linear transformation T is the trace of the matrix of T relative to any ordered basis for V.*

Solved Examples

Based on the following Results

- For any matrix A, the trace of A is the sum of the elements of A lying along the principal diagonal
- $tr(\lambda A) = \lambda\, tr(A)$
- $(A^{-1})^{-1} = A$
- $tr(A+B) = tr(A) + tr(B)$
- $tr(AB) = tr(BA)$
- Similar matrices have the same trace.
- The trace of a linear transformation T is the trace of the matrix of T relative to any ordered basis for V.

Example 1. *Let $T: R^2(R) \to R^2(R)$, where for any $(x, y) \in R^2$, $T(x, y) = \left(2x, \frac{1}{2}y\right)$. Find the matrix associated with T with respect to the ordered basis $\{(1, 0), (0, 1)\}$.*

Solution. Let $B = \{(1, 0), (0,1)\}$ be an ordered basis of $R^2(R)$ and $T(x, y) = \left(2x, \frac{1}{2}y\right)$, then

$$T(1, 0) = (2, 0) \text{ and } T(0,1) = \left(0, \frac{1}{2}\right)$$

Now,
$$T(1, 0) = (2, 0) = 2(1, 0) + 0.(0, 1)$$

and
$$T(0, 1) = \left(0, \frac{1}{2}\right) = 0(1, 0) + \frac{1}{2}(0, 1).$$

Thus, the matrix associated with T w.r.t. B is

$$[T]_B = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Example 2. *Find the matrix of the linear transformation T on $V_3(R)$ defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ with respect to the ordered basis B and also with respect to the ordered basis B' where*

(i) $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(ii) $B' = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

Solution. (i) We have

$$T(a, b, c) = (2b + c, a - 4b, 3a)$$

Then
$$T(1, 0, 0) = (0, 1, 3)$$
$$T(0, 1, 0) = (2, -4, 0)$$
$$T(0, 0, 1) = (1, 0, 0)$$

Now
$$T(1, 0, 0) = 0(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1) \quad \text{...(1)}$$
$$T(0, 1, 0) = 2(1, 0, 0) - 4(0, 1, 0) + 0(0, 0, 1) \quad \text{...(2)}$$
$$T(0, 0, 1) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1) \quad \text{...(3)}$$

The matrix of T relative to B is given by

$$[T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$