(ii) Let (a, b, c) be any element of $V_3(R)$, then there exists $x, y, z \in R$ such that

$$(a, b, c) = x\,(1, 1, 1) + y\,(1, 1, 0) + z\,(1, 0, 0)$$

$$\Rightarrow \quad (a, b, c) = (x + y + z, x + y, x)$$

$$\Rightarrow \quad x + y + z = a, x + y = b, x = c$$

$$\therefore \quad x = c, y = b - c, z = a - b$$

$$\therefore \quad (a, b, c) = c\,(1, 1, 1) + (b - c)\,(1, 1, 0) + (a - b)\,(1, 0, 0) \quad \text{...(1)}$$

Now $T(1, 1, 1) = (3, -3, 3)$, $T(1, 1, 0) = (2, -3, 3)$ and $T(1, 0, 0) = (0, 1, 3)$. Then from (1), we get

$$T(1, 1, 1) = (3, -3, 3) = 3(1, 1, 1) - 6(1, 1, 0) + 6(1, 0, 0)$$

$$T(1, 1, 0) = (2, -3, 3) = 3(1, 1, 1) - 6(1, 1, 0) + 5(1, 0, 0)$$

$$T(1, 0, 0) = (0, 1, 3) = 3\,(1, 1, 1) - 2\,(1, 1, 0) - 1(1, 0, 0)$$

Therefore, the matrix of T relative to B' is given by

$$[T]_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Example 3. *Let T be a linear operator on R^2 defined by $T(x, y) = (2y, 3x - y)$. Find the matrix representation of T relative to the basis $\{(1, 3), (2, 5)\}$.*

Solution. Let (x, y) be any element of R^2. Then there exist $a, b \in R$ such that

$$(x, y) = a\,(1, 3) + b\,(2, 5)$$

$$\Rightarrow \quad (x, y) = (a + 2b, 3a + 5b)$$

$$\Rightarrow \quad a + 2b = x, 3a + 5b = y$$

Solving these equations, we get

$$a = 2y - 5x, b = 3x - y,$$

$$\therefore \quad (x, y) = (2y - 5x)(1, 3) + (3x - y)(2, 5) \quad \text{...(1)}$$

Since, $\quad T(x, y) = (2y, 3x - y)$

Then, $\quad T(1, 3) = (6, 0), T(2, 5) = (10, 1)$

Now from (1) $\quad T(1, 3) = (6, 0) = -30(1, 3) + 18(2, 5)$

$$T(2, 5) = (10, 1) = -48\,(1, 3) + 29\,(2, 5)$$

Therefore, the matrix of T relative to the given basis is $\begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$.

Example 4. *Show that the vector $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.*

Solution. Let $B' = \{\alpha_1, \alpha_2, \alpha_3\}$. First, we shall show that B' is linearly independent.

Let $a, b, c \in R$ such that $\quad a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$

$$\Rightarrow \quad a(1, 0, -1) + b\,(1, 2, 1) + c\,(0, -3, 2) = (0, 0, 0)$$

$$\Rightarrow \quad (a + b, 2b - 3c, -a + b + 2c) = (0, 0, 0) \quad \text{...(1)}$$

$$\Rightarrow \quad a + b = 0, 2b - 3c = 0, -a + b + 2c = 0$$

The coefficient matrix of these equations is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{vmatrix} = 1(4 + 3) - 1(0 - 3) = 10 \neq 0$$

$\Rightarrow$ rank of $A = 3$, which is the number of variables a, b, c. Hence, the system of equation (1) has only zero solution, *i.e.* $a = 0, b = 0, c = 0$.

Therefore, B' is linearly independent containing 3 elements since dim. $R_3 = 3$, hence B' forms a basis for R^3.

Let $B = \{e_1, e_2, e_3\}$ be the standard basis for R^3, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

Now, we have

$$\alpha_1 = (1, 0, -1) = 1e_1 + 0e_2 - 1e_3$$

$$\alpha_2 = (1, 2, 1) = 1e_1 + 2e_2 + 1e_3$$

$$\alpha_3 = (0, -3, 2) = 0e_1 - 3e_2 + 2e_3$$

Let P be the transition matrix from the basis B to B', then

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \qquad \therefore |P| = 10$$

Now we shall find P^{-1} :

The cofactors of the elements of the first row of P are

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}, -\begin{vmatrix} 0 & -3 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}, \text{i.e. } 7, 3, 2$$

The cofactors of the elements of the second row of P are

$$-\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}, \text{i.e.} -2, 2, -2$$

The cofactors of the elements of the third row of P are

$$\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}, \text{i.e.} -3, 3, -2$$

$$\text{adj. } P = \text{transpose of the matrix} \begin{bmatrix} 7 & 3 & 2 \\ -2 & 2 & -2 \\ -3 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 & -3 \\ 3 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj.}A}{|P|} = \frac{1}{10}\begin{bmatrix} 7 & -2 & -3 \\ 3 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix}$$

Now $\qquad e_1 = 1e_1 + 0e_2 + 0e_3.$

Therefore, the coordinate matrix of e_1 relative to B is $[e_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

So that the coordinate matrix of e_1 relative to B' is

$$[e_1]_{B'} = P^{-1}[e_1]_B = \frac{1}{10}\begin{bmatrix} 7 & -2 & -3 \\ 3 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/10 \\ 3/10 \\ 1/5 \end{bmatrix}$$

$$\therefore \quad e_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.$$

Similarly, $[e_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [e_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore \quad [e_2]_{B'} = P^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{1}{10}\begin{bmatrix}-2\\2\\-2\end{bmatrix} = \begin{bmatrix}-1/5\\1/5\\-1/5\end{bmatrix}$$

$$[e_3]_{B'} = P^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{10}\begin{bmatrix}-3\\3\\2\end{bmatrix} = \begin{bmatrix}-3/15\\3/10\\1/5\end{bmatrix}$$

$$\therefore \quad e_2 = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3$$

$$e_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.$$

Example 5. *Let T be a linear operator on R^3 defined by*

$$T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$$

Prove that T is invertible and find a formula for T^{-1}.

Solution. Let $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard basis for R^3.

Let A be the matrix of T relative to B, then

$$A = [T]_B$$

Now, $T(1, 0, 0) = (3, -2, 1) = 3\,(1, 0, 0) - 2\,(0, 1, 0) - 1\,(0, 0, 1)$

$T(0, 1, 0) = (0, 1, 2) = 0(1, 0, 0) + 1\,(0, 1, 0) + 2(0, 0, 1)$

and $T(0, 1, 1) = (1, 0, 4) = 1\,(1, 0, 0) + 0\,(0, 1, 0) + 4\,(0, 0, 1)$

$$\therefore \quad A = [T]_B = \begin{bmatrix}3 & 0 & 1\\-2 & 1 & 0\\-1 & 2 & 4\end{bmatrix}$$

Now
$$|A| = \begin{vmatrix}3 & 0 & 1\\-2 & 1 & 0\\-1 & 2 & 4\end{vmatrix} = 3(4-0)+1(-4+1) = 9 \neq 0$$

Since $|A| \neq 0$, therefore A is invertible and hence T is invertible.

Now we shall find A^{-1}. For this we find adj. A.

The cofactors of the first row of A are

$$\begin{vmatrix}1 & 0\\2 & 4\end{vmatrix}, \begin{vmatrix}-2 & 0\\-1 & 4\end{vmatrix}, \begin{vmatrix}-2 & 1\\-1 & 4\end{vmatrix}, \text{i.e. } 4, 8, -3$$

The cofactors of the second row of A are

$$-\begin{vmatrix}0 & 1\\2 & 4\end{vmatrix}, \begin{vmatrix}3 & 1\\-1 & 4\end{vmatrix}, \begin{vmatrix}3 & 0\\-1 & 2\end{vmatrix}, \text{i.e. } 2, 13, -6$$

The cofactors of the third row of A are

$$\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix}, -\begin{vmatrix}3 & 1\\-2 & 0\end{vmatrix}, \begin{vmatrix}3 & 0\\-2 & 1\end{vmatrix}, \text{i.e. } -1, -2, 3$$

adj. A = transpose of the cofactors matrix

$$= \begin{bmatrix}4 & 2 & -1\\8 & 13 & -2\\-3 & -6 & 3\end{bmatrix}$$

$$A^{-1} = \frac{\text{adj.}A}{|A|} = \frac{1}{9}\begin{bmatrix}4 & 2 & -1\\8 & 13 & -2\\-3 & -6 & 3\end{bmatrix}$$

Since we know that
$$[T^{-1}]_B = [T]_B^{-1} = A^{-1}$$

Now we shall find the formula for T^{-1}.

Let $\alpha = (p, q, r)$ be any element of R^3 and B is a standard basis for R^3. Then

$$[\alpha]_B = \begin{bmatrix}p\\q\\r\end{bmatrix}$$

$$[T^{-1}(\alpha)]_B = [T^{-1}]_B[\alpha]_B = A^{-1}[\alpha]_B = \frac{1}{9}\begin{bmatrix}4 & 2 & -1\\8 & 13 & -2\\-3 & -6 & 3\end{bmatrix}\begin{bmatrix}p\\q\\r\end{bmatrix}$$

$$[T^{-1}(\alpha)]_B = \frac{1}{9}\begin{bmatrix}4p+2q-r\\8p+13q-2r\\-3p-6q+3r\end{bmatrix}$$

$$T^{-1}(\alpha) = T^{-1}(p,q,r) = \left(\frac{4p+2q-r}{9}, \frac{8p+13q-2r}{9}, \frac{-3p-6q+3r}{9}\right)$$

Example 6. *Consider the vector space V(R) of all 2×2 matrices over the field R of real numbers. Let T be the linear transformation on V sending each matrix X onto AX, where $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$. Find the matrix of T with respect to the ordered basis $B = \{E_1, E_2, E_3, E_4\}$, for V where*

$$E_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, E_2 = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, E_3 = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, E_4 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Solution. We have
$$T(X) = AX$$

Then
$$T(E_1) = AE_1 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$$

$$= 1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$\therefore \quad T(E_1) = 1E_1 + 0E_2 + 1E_3 + 0E_4$$

$$T(E_2) = AE_2 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

$$= 0\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$\therefore \quad T(E_2) = 0E_1 + 1E_2 + 0E_3 + 1E_4$$

$$T(E_3) = AE_3 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$= 1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$\therefore \quad T(E_3) = 1E_1 + 0E_2 + 1E_3 + 0E_4$$

and
$$T(E_4) = AE_4 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= 0\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$\therefore \quad T(E_4) = 0E_1 + 1E_2 + 0E_3 + 1E_4$

The matrix of T relative to B is $[T]_B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Example 7. *Let a linear map $T : P_3 \to P_2$ be defined by*

$$T(a_0+a_1x+a_2x^2+a_3x^3) = a_3 + (a_2 + a_3)x + (a_0 + a_1)x^2$$

where $P_n[x]$ = set of all polynomials of degree $\le n$. Find the matrix of T with respect to the ordered bases $B = \{1, (x-1), (x-1)^2, (x-1)^3\}$ and $B' = \{1, x, x^2\}$.

Solution. Since B and B' are the bases of P_3 and P_2, respectively, hence we shall express the images of each element of B in terms of an element of B'.

Now $\quad T(1) = T(1 + 0\,.\,x + 0.\,x^2 + 0\,.\,x^3)$

$\qquad = 0 + (0 + 0)x + (1 + 0)x^2 = x^2$

$\therefore \quad T(1) = 0.1 + 0\,.\,x + 1.\,x^2$

$\therefore \quad T(x-1) = T(-1 + 1\,.\,x + 0\,.\,x^2 + 0\,.\,x^3)$

$\qquad = 0 + (0 + 0)x + (-1 + 1)x^2 = 0$

$\Rightarrow \quad T(x-1) = 0\,.\,1 + 0\,.\,x + 0\,.\,x^2$

$T[(x-1)^2] = T(1 - 2x + x^2)$

$\qquad = T(1+(-2)x + 1.\,x^2 + 0.\,x^3)$

$\qquad = 0 + (1 + 0)x + (1-2)x^2$

$\qquad = x - x^2.$

$\Rightarrow \quad T[(x-1)^2] = 0.1 + 1.x + (-1)\,x^2$

$\therefore \quad T[(x-1)^3] = T(-1 + 3x - 3x^2 + x^3)$

$\qquad = 1 + (-3 + 1)x + (-1 + 3)\,x^2$

$\qquad = 1 - 2x + 2x^2$

$\Rightarrow \quad T[(x-1)^3] = 1.1\,(-2)x + 2x^2$

Thus, the matrix of T relative to the ordered bases B and B' is

$${}_B[T]_{B'} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

Example 8. *Let A be an $m \times n$ matrix of real entries. Prove that $A = 0$ (null matrix) if and only if trace $(A^TA) = 0$.*

Solution. Let $A = [a_{ij}]_{m \times n}$, then $A^T = [b_{ij}]_{n \times m}$, where $b_{ij} = a_{ij}$

Also, A^TA is a matrix of order $n \times n$.

Let $\quad A^TA = [b_{ij}]_{n \times m}[a_{ij}]_{m \times n} = [c_{ij}]_{n \times n}$

where $\quad c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$

$$\therefore \quad \text{tr}(A^T A) = \sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n}\left(\sum_{k=1}^{m} b_{ik}a_{ki}\right)$$

$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{m} a_{ki}a_{ki}\right) \qquad [\because b_{ik} = a_{ki}]$$

$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{m} a_{ki}^2\right)$$

$$\therefore \quad \text{tr}(A^T A) = \sum_{i=1}^{n}(a_{1i}^2 + a_{2i}^2 + \ldots + a_{mi}^2)$$

If tr $(A^TA) = 0$, then from (1), we have

$$\sum_{i=1}^{n}(a_{1i}^2 + a_{2i}^2 + \ldots + a_{mi}^2) = 0$$

$\Rightarrow$ the sum of the squares of all the elements of $A = 0$

$\Rightarrow$ each element of $A = 0$

$\Rightarrow$ A is a null matrix.

$\Rightarrow$ $A = 0$.

Conversely, If A is null matrix, then A^TA is also a null matrix.

$\qquad \text{tr}(A^TA) = 0.$

Example 9. *Let T and S be linear operators on the finite dimensional vector space $V(F)$, prove that*

(i) $\det(TS) = \det(T)\det(S)$

(ii) T is invertible iff $\det T \ne 0$.

Solution. (i) Let B be any ordered basis of V then we have

$$[TS]_B = [T]_B[S]_B$$

$\Rightarrow \quad \det([TS]_B) = \det([T]_B[S]_B)$

$\Rightarrow \quad \det([TS]_B) = \det([T]_B)\det([S]_B).$

Since the determinant of a linear transformation is equal to the determinant of its matrix with respect to any ordered basis, therefore

$$\det(TS) = \det(T)\det(S)$$

(ii) If T is invertible, then there exists a linear transformation T^{-1} on V such that

$$T^{-1}T = I = TT^{-1}$$

$\Rightarrow \quad \det(T^{-1}T) = \det(I) = \det([I]_B) \qquad$ [For any ordered basis B]

$\Rightarrow \det(T^{-1})\det(T) = 1 \qquad [\because [I]_B$ is a unit matrix.]

Now $\det(T)$ and $\det(T^{-1}) \in F$ and F is a field and in a field the product of elements can be zero iff at least one of them is zero.

$\therefore \quad \det(T^{-1})\det(T) = 1$

$\Rightarrow \qquad \det(T) \ne 0.$

Conversely, Suppose that $\det(T) \ne 0$.

Then for any ordered basis B of V, we have

$$\det([T]_B) \ne 0$$

$\Rightarrow \quad [T]_B$ is invertible.

$\Rightarrow \quad T$ is invertible.