~~X(4)~~

Unit - 2.

Fast Fourier Transform.

FFT refers to algorithms that compute DFT in a numerically efficient manner. Many such algorithms are available. However we present 2 such algorithms.

i) Decimation in time ~~(DITFFT)~~. (DITFFT)

ii) Decimation in frequency FFT (DIFFFT)

DITFFT:

- The decimation in time algorithm uses 'divide and conquer approach.
- In the following presentation the number of points is assumed as a power of 2 i.e., $N = 2^P$.
- The DIT approach is one of breaking the N point transform into $\frac{N}{2}$ point transform, then breaking each $\frac{N}{2}$ point transform into 2 $\frac{N}{4}$ point transforms. and ~~contdtiving~~. continuing this process until 2 point DFTs are obtained.
- In other words the N point DFT is performed as several 2 point DFTs.
- Let $x(n)$ represent a sequence of length N where N is a power of 2.
- Decimate this sequence into sequences of length $\frac{N}{2}$ one composed of even indexed values of ~~x(n)~~ $x(n)$ and the other is odd indexed values of $x(n)$.

Given sequence ; $x(0), x(1), \ldots x(N-2), x(N-1)$.

Even indexed: $x(0), x(2) \ldots x(N-2)$.

Odd indexed: $x(1), x(3) \ldots x(N-1)$.

$$G(k) = \sum_{r=0}^{\frac{N}{2}-1} g(r)\, W_{\frac{N}{2}}^{kr} \longrightarrow ⑦$$

$$G(k) = \sum_{r=0}^{\frac{N}{2}-2} g(r)\, W_{\frac{N}{2}}^{kr} + \sum_{r=1}^{\frac{N}{2}-1} g(r)\, W_{\frac{N}{2}}^{kr}$$

$$g(r) = \{g(0), g(1), \ldots\ g(\tfrac{N}{2}-2), g(\tfrac{N}{2}-1)\}.$$

Put $r=2l$ in 1st term, $r=2l+1$ in 2nd term.

$$G(k) = \sum_{l=0}^{\frac{N}{4}-1} g(2l)\, W_{\frac{N}{2}}^{2kl} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, W_{\frac{N}{2}}^{k(2l+1)}$$

$$G(k) = \sum_{l=0}^{\frac{N}{4}-1} g(2l)\, W_{\frac{N}{2}}^{2kl} + W_{\frac{N}{2}}^{k} \sum_{l=0}^{\frac{N}{4}-1} b(l)\, W_{\frac{N}{2}}^{2kl}$$

$$G(k) = A(k) + W_{\frac{N}{2}}^{k} B(k) \qquad 0 \le k \le \frac{N}{4}-1 \longrightarrow ⑧$$

$$H(k) = C(k) + W_{\frac{N}{2}}^{k} D(k) \qquad 0 \le k \le \frac{N}{4}-1 \longrightarrow ⑨$$

$A(k), B(k), C(k), D(k) \rightarrow$ periodic $\frac{N}{4}$.

$$⑧ \Rightarrow G(k) = A\left(k-\frac{N}{4}\right) + W_{\frac{N}{2}}^{k} B\left(k-\frac{N}{4}\right); \quad \frac{N}{4} \le k \le \frac{N}{2}-1 \longrightarrow ⑩$$

$$⑨ \Rightarrow H(k) = C\left(k-\frac{N}{4}\right) + W_{\frac{N}{2}}^{k} D\left(k-\frac{N}{4}\right); \quad \frac{N}{4} \le k \le \frac{N}{2}-1 \longrightarrow ⑪$$

$k=0$ & 1 eqn ⑧ & ⑨

$k=2$ & 3 eqn ⑩ & ⑪.

$k=0$: $G(0) = A(0) + W_4^0 B(0)$

$H(0) = C(0) + W_4^0 D(0)$.

$k=1$: $G(1) = A(1) + W_4^1 B(1)$

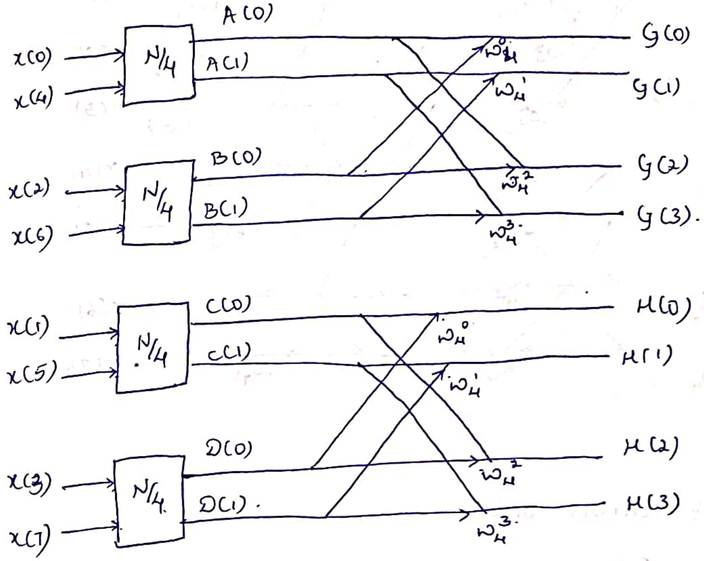
$H(0) = C(1) + W_4^1 D(1)$.

$k=2$: $G(2) = A(0) + W_4^2 B(0)$

$H(2) = C(0) + W_4^2 D(0)$.

$k=3$: $G(3) = A(1) + W_4^3 B(1)$.

$H(3) = C(1) + W_4^3 D(1)$.



To calculate 2 point DFT, each $\frac{N}{4}$ DFT has two $\frac{N}{8}$ point DFTs.

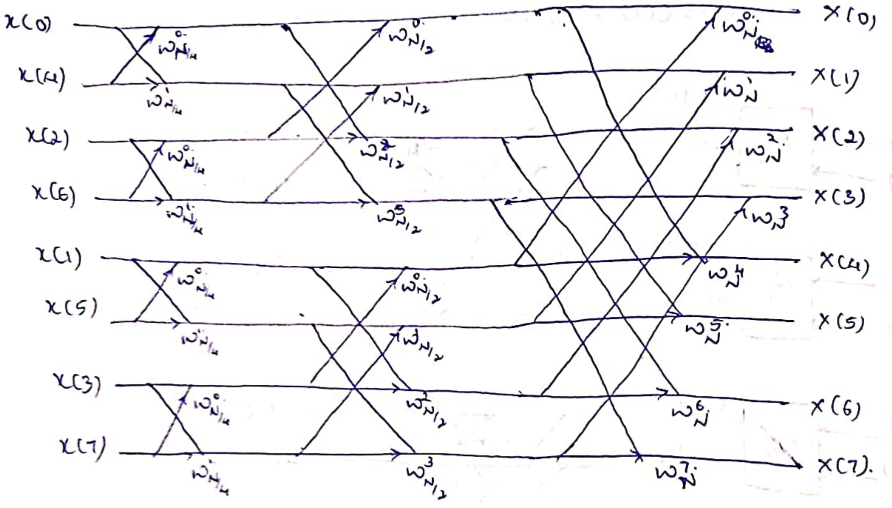
Two point DFT of $x(0)$ & $x(4)$.

$$A(k) = \sum_{n=0}^{\frac{N}{4}-1} x(n)\, W_{\frac{N}{4}}^{kn} \qquad 0 \le k \le \frac{N}{4}-1.$$

$$N=8 \Rightarrow A(k) = \sum_{n=0}^{1} x(n)\, W_2^{kn} \qquad 0 \le k \le 1$$

$k=0$: $A(0) = x(0) + W_2^0 x(4)$.

$k=1$: $A(1) = x(0) + W_2^1 x(4)$



Observation:

1. i/p data is bit reversed order.

i/p	o/p
x(0) 000	x(0) 000.
x(4) 100.	x(1) 001.
x(2) 010.	x(2) 010.
x(6) 110	x(3) 011.
x(1) 001	x(4) 100.
x(5) 101	x(5) 101.
x(3) 011	x(6) 110.
x(7) 111	x(7) 111.

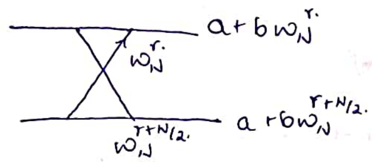
2. o/p are in proper order.

3. Basic computational block is called butterfly diagram.

Further reduction:

Cooky Tukey algorithm.

Butterfly configuration.



can be further simplified to reduce no of complex multiplications.

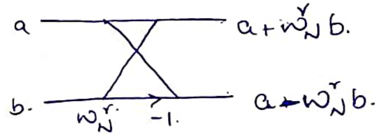
$W_N^{r+N/2} = W_N^r \cdot W_N^{N/2}$

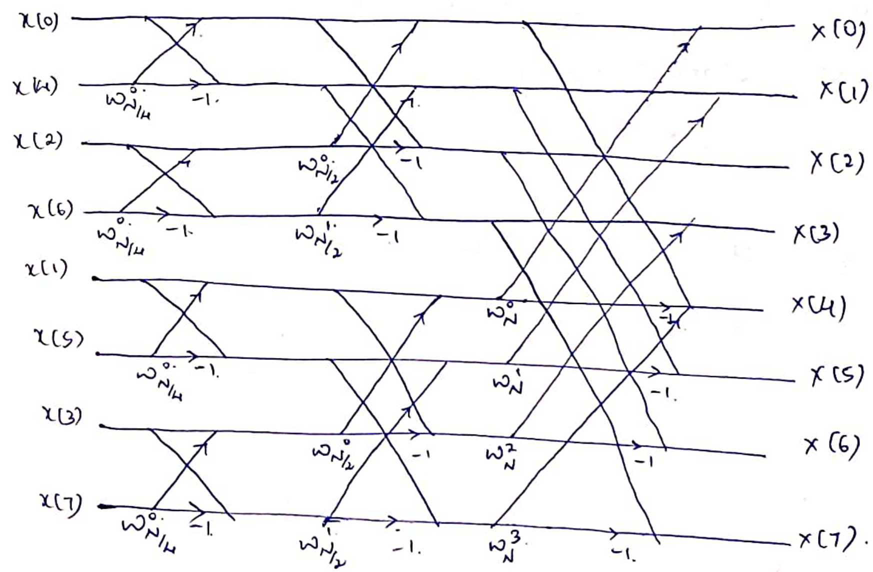
$= W_N^r \cdot (-1)$.

$= -W_N^r$

$W_N^{N/2} = e^{-j\frac{2\pi}{N}\times\frac{N}{2}}$

$= -1$.





Computation time:

Direct computation:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad 0 \le k \le N-1.$$

No of complex multiplications $= N^2$

No of complex additions $= N(N-1)$

Ex: $N = 8$

complex multiplications $= 64$.

complex additions $= 56$

Radix-2 DITFFT algorithm:

No of stages $= \log_2 N$

In each stage no of butterflies $= \frac{N}{2}$

In each butterfly no of multiplications $= 01$.

In each butterfly no of additions $= 02$.

Total no of multiplications $= \frac{N}{2} \log_2 N$

Total no of additions $= N \log_2 N$

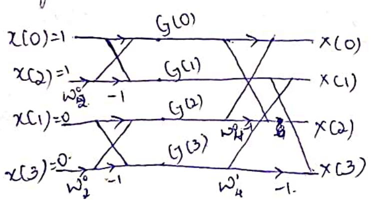
Ex: $N = 8$.

stages $= 3$, ~~no of butterflies~~

total no of multiplications $= 12$.

additions $= 24$.

ompute 4 point DFT using radix-2 DITFFT algorithm. Verify he result by direct computation. $x(n) = \{1, 0, 1, 0\}$.



$G(0) = x(0) + W_2^0 x(2) = 1 + 1 = 2$

$G(1) = x(0) - W_2^0 x(2) = 1 - 1 = 0$

$G(2) = x(1) + W_2^0 x(3) = 0 + 0 = 0.$

$G(3) = x(2) - W_2^0 x(3) = 0 - 0 = 0.$

$X(0) = G(0) + W_4^0 G(2) = 2 + 0 = 2.$

$X(1) = G(1) + W_4^1 G(3) = 0 + (-j)(0) = 0.$

$X(2) = G(0) - W_4^0 G(2) = 2 - 0 = 2$

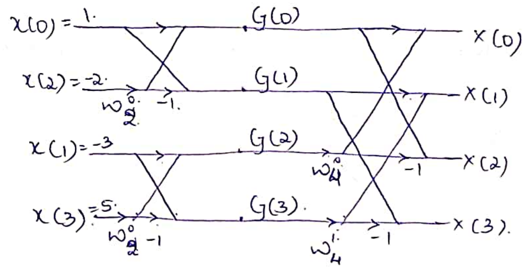
$X(3) = G(1) - W_4^1 G(3) = 0 - (-j)(0) = 0.$

$X(k) = \{2, 0, 2, 0\}.$

Direct computation:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

2. Compute 4 point DFT of the sequence $x(n) = \{1, -3, -2, 5\}$ using radix-2 DITFFT algorithm. Verify the result by direct computation.



$G(0) = x(0) + W_2^0 x(2) = 1 - 2 = -1.$

$G(1) = x(0) - W_2^0 x(2) = 1 - (-2) = 3.$

$G(2) = x(1) + W_2^0 x(3) = -3 + 5 = 2.$

$G(3) = x(1) - W_2^0 x(3) = -3 - 5 = -8.$

$X(0) = G(0) + W_4^0 G(2) = -1 + (1)(2) = 1.$

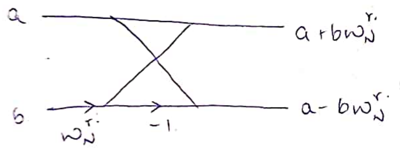
$X(1) = G(1) + W_4^1 G(3) = 3 + (-j)(-8) = 3+8j.$

$X(2) = G(0) - W_4^0 G(2) = -1 - 2 = -3.$

$X(3) = G(1) - W_4^1 G(3) = 3-(-j)(-8) = 3-8j$

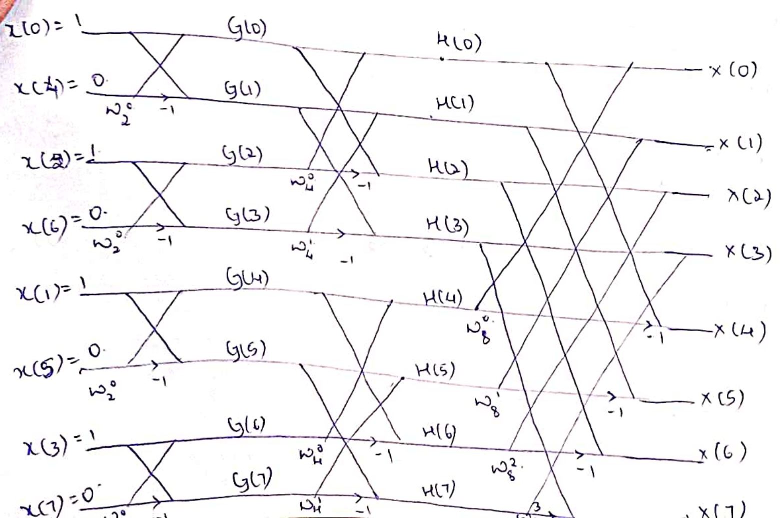
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1-3-2+5 \\ 1+3j+2+5j \\ 1+3-2-5 \\ 1-3j+2-5j \end{bmatrix} = \begin{bmatrix} 1 \\ 3+8j \\ -3 \\ 3-8j \end{bmatrix}$$

3. Find the 8 point DFT of the sequence $x(n) = \{1,1,1,1,0,0,0,0\}$ using DITFFT radix-2 algorithm. use the butterfly diagram given below.



$W_8^0 = 1$ $\quad W_8^1 = 0.707 - 0.707i$ $\quad W_8^2 = -i$ $\quad W_8^3 = -0.707 - 0.707j$

$W_8^4 = -1$ $\quad W_8^5 = -0.707 + 0.707i$ $\quad W_8^6 = i$ $\quad W_8^7 = 0.707 + 0.707i$



$G(0) = x(0) + W_2^0 x(4) = 1$

$G(1) = x(0) - W_2^0 x(4) = 1$

$G(2) = x(2) + W_2^0 x(6) = 1$

$G(3) = x(2) - W_2^0 x(6) = 1$

$G(4) = x(1) + W_2^0 x(5) = 1$

$G(5) = x(1) - W_2^0 x(5) = 1$

$G(6) = x(3) + W_2^0 x(7) = 1$

$G(7) = x(3) - W_2^0 x(7) = 1.$

$H(0) = G(0) + W_4^0 G(2) = 2.$

$H(1) = G(1) + W_4^1 G(3) = 1-j$

$H(2) = G(0) - W_4^0 G(2) = 0.$

$H(3) = G(1) - W_4^1 G(3) = 1+j$

$H(4) = G(4) + W_4^0 G(6) = 2$

$H(5) = G(5) + W_4^1 G(7) = 1-j$

$H(6) = G(4) - W_4^0 G(6) = 0$

$H(7) = G(5) - W_4^1 G(7) = 1+j$

$X(0) = H(0) + W_8^0 H(4) = 2+2 = 4.$

$X(1) = H(1) + W_8^1 H(5) = 1-j + (0.707-0.707i)(1-j) = 1-2.414j$

$X(2) = H(2) + W_8^2 H(6) = 0 - i(0) = 0.$

$X(3) = H(3) + W_8^3 H(7) = 1+j + (-0.707-0.707i)(1+j) = 1-0.414j$

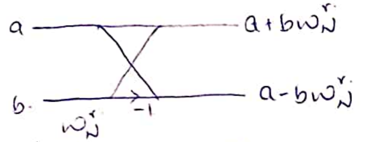
$X(4) = H(0) - W_8^0 H(4) = 2-2 = 0.$

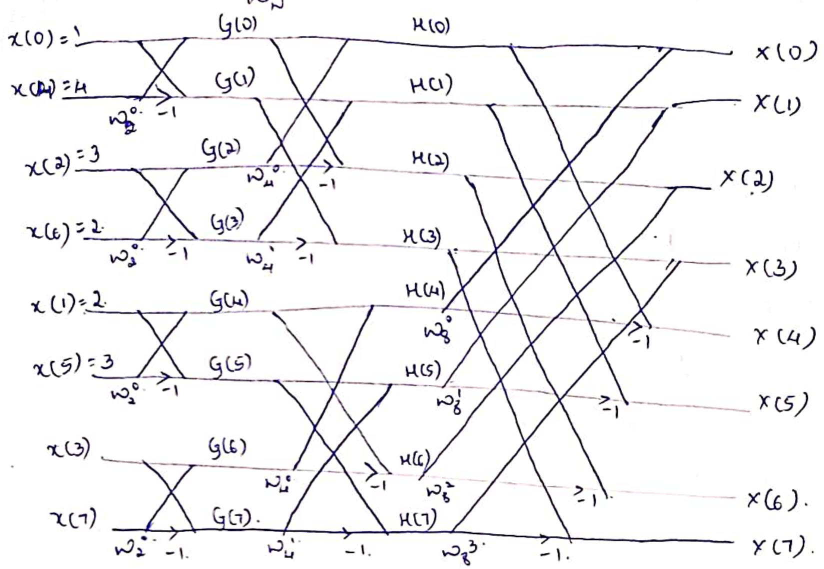
$X(5) = H(1) - W_8^1 H(5) = 1+0.414j$

$X(6) = H(2) - W_8^2 H(6) = 0.$

$X(7) = H(3) - W_8^3 H(7) = 1+2.414j.$

4. Find the 8 point DFT of the sequence {1,2,3,4,4,3,2,1} using DIT FFT radix-2 algorithm. The basic computational block known as a butterfly should be shown in figure.





$G(0) = x(0) + W_2^0 x(4) =$

$G(1) = x(0) - W_2^0 x(4) =$

$G(2) = x(2) + W_2^0 x(6) =$

$G(3) = x(2) - W_2^0 x(6) =$

$G(4) = x(1) + W_2^0 x(5) =$

$G(5) = x(1) - W_2^0 x(5) =$

$G(6) = x(3) + W_2^0 x(7) =$

$G(7) = x(3) - W_2^0 x(7) =$

• Computing IDFT using radix-2 algorithm.

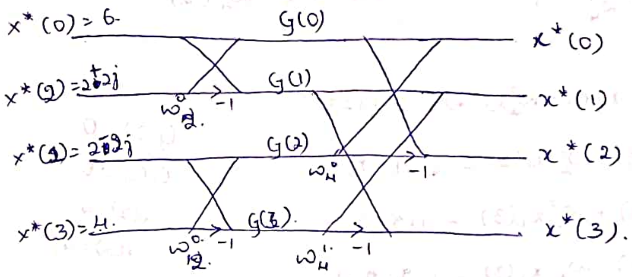
1. Find 4 point sequence $x(n)$ if the DFT sequence are

$X(0) = 6,\ X(1) = 2-2j \quad X(2) = 2+2j \quad X(3) = 4.$

a) use radix-2 DIT FFT algorithm.

b) verify the result by direct computation.

$$x(n) = \frac{1}{N}\left\{ \text{DFT}\,[X^*(k)] \right\}^*$$



$G(0) = X^*(0) + W_2^0 X^*(2) = 6 + 2+2j = 8+2j$

$G(1) = X^*(0) - W_2^0 X^*(2) = 6 - 2-2j = 4-2j$

$G(2) = X^*(1) + W_2^0 X^*(3) = 2-2j+4 = 6-2j$

$G(3) = X^*(1) - W_2^0 X^*(3) = 2-2j-4 = -2-2j$

$x^*(0) = G(0) + W_4^0 G(2) = 8+2j+6-2j = 14.$

$x^*(1) = G(1) + W_4^1 G(3) = 4-2j + (-j)(-2-2j) = 2$

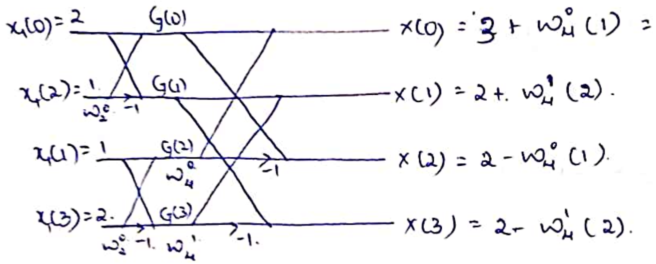
$x^*(2) = G(0) - W_4^0 G(2) = 8+2j - (6-2j) = 2+4j.$

$x^*(3) = G(1) - W_4^1 G(3) = 4-2j - (-j)(-2-2j) = 6-4j.$

$$\begin{bmatrix} x^*(0) \\ x^*(1) \\ x^*(2) \\ x^*(3) \end{bmatrix} = \frac{1}{4}$$

Compute circular convolution of the following sequence using radix-2 DITFFT algorithm. $x_1(n) = \{2,1,1,2\}$, $x_2(n) = \{1,-1,-1,1\}$.

$x_1(n) = \{2,1,1,2\}$ $\quad x_2(n) = \{1,-1,-1,1\}$



$G(0) = x_1(0) + W_2^0 x_1(2) = 2+1=3.$

$G(1) = x_1(0) - W_2^0 x_1(2) = 2-1 = 1$

$G(2) = x_1(1) + W_2^0 x_1(3) = 1+2=3.$

$G(3) = x_1(1) - W_2^0 x_1(3) = 1-2 = -1.$

$X(0) = G(0) + W_4^0 G(2) = 6$

$X(1) = G(1) + W_4^1 G(3) = 1+j$

$X(2) = G(0) - W_4^0 G(2) = 0$

$X(3) = G(1) - W_4^1 G(3) = 1-j$

$G(0) = 0.$

$G(1) = 0.$

$G(2) = 4$

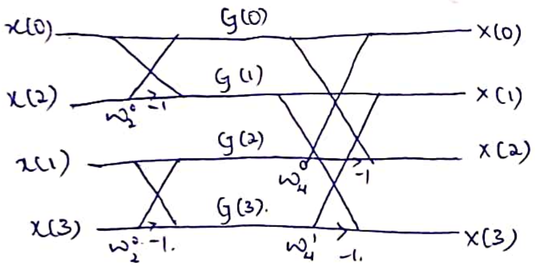
$G(3) = 4j.$

$x^*(0) = 4$

$x^*(1) = 2j-2.$

$x^*(2) = -4$

$x^*(3) = -2-2j$



$G(0) = \quad = 0$

$G(1) = \quad = -2$

$G(2) = \quad = 0.$

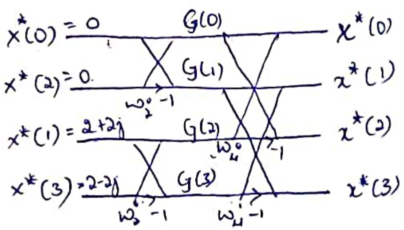
$G(3) = \quad = -2.$

$X(0) = G(0) + W_4^0 G(2) = 0.$

$X(1) = G(1) + W_4^1 G(3) = 2-2j$

$X(2) = G(0) - W_4^0 G(2) = 0.$

$X(3) = G(1) - W_4^1 G(3) = 2+2j$



Radix-2 DIF FFT:

DIF $\rightarrow$ Decimation in frequency.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \; ; \; 0 \le k \le N-1 \longrightarrow ①$$

$$① \Rightarrow X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\tfrac{N}{2}\right) W_N^{k(n+\frac{N}{2})}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{\frac{kN}{2}} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\tfrac{N}{2}\right) W_N^{kn}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\tfrac{N}{2}\right) W_N^{kn}.$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n+\tfrac{N}{2}\right)\right] W_N^{kn} \longrightarrow ②.$$

Decompose $X(k)$ as even and odd index sequence.

Put $k = 2r$ $\quad k = 2r+1.$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r} x\left(n+\tfrac{N}{2}\right)\right] W_N^{2rn}.$$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\left[x(n) + x\left(n+\tfrac{N}{2}\right)\right]}_{g(n)} W_N^{2rn}$$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{\frac{N}{2}}^{rn} \; ; \; 0 \le r \le \tfrac{N}{2}-1 \longrightarrow ③.$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r+1} x\left(n+\tfrac{N}{2}\right)\right] W_N^{(2r+1)n}.$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\tfrac{N}{2}\right)\right] W_N^{(2r+1)n}.$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\tfrac{N}{2}\right)\right] W_N^{n} \cdot W_N^{2rn}.$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\tfrac{N}{2}\right)\right] W_N^{n} W_{\frac{N}{2}}^{rn}$$

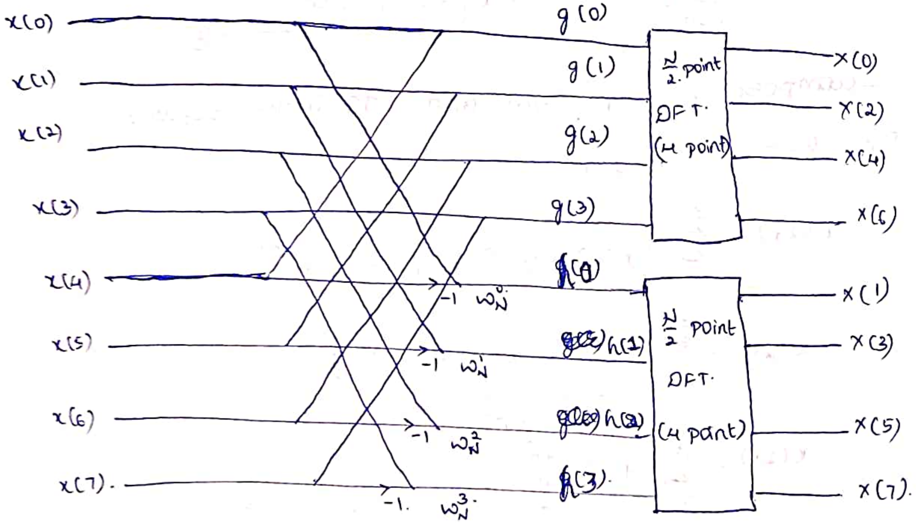
$$x(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} h(n) W_N^n W_{\frac{N}{2}}^{rn} \longrightarrow (4)$$

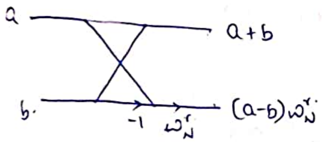
$$g(n) = x(n) + x\left(n+\frac{N}{2}\right) \longrightarrow (5)$$

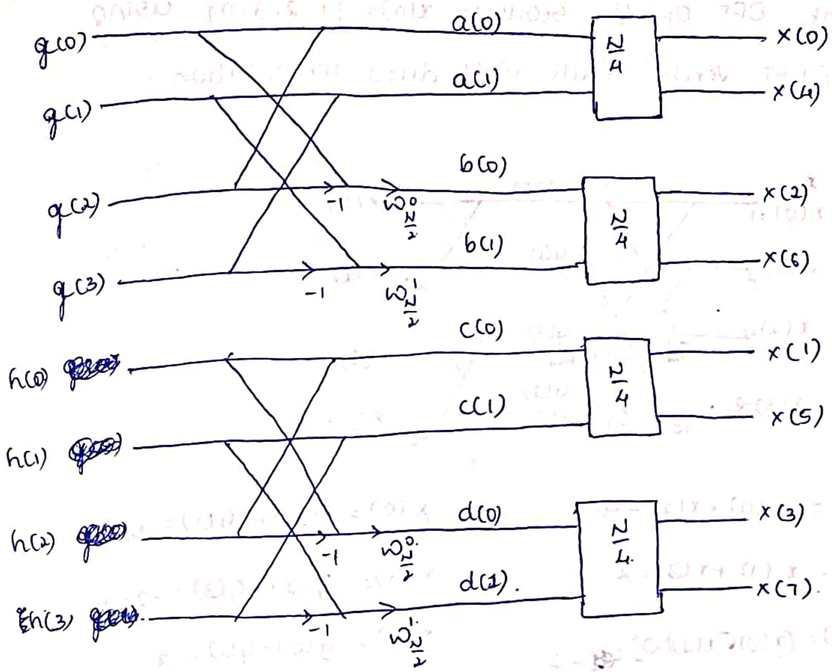
$$h(n) = x(n) - x\left(n+\frac{N}{2}\right) \longrightarrow (6)$$

Put $n \longrightarrow 0$ to 3 in (5) & (6).

$n=0$	$g(0) = x(0) + x(4)$	$h(0) = x(0) - x(4)$
$n=1$	$g(1) = x(1) + x(5)$	$h(1) = x(1) - x(5)$
$n=2$	$g(2) = x(2) + x(6)$	$h(2) = x(2) - x(6)$
$n=3$	$g(3) = x(3) + x(7)$	$h(3) = x(3) - x(7)$



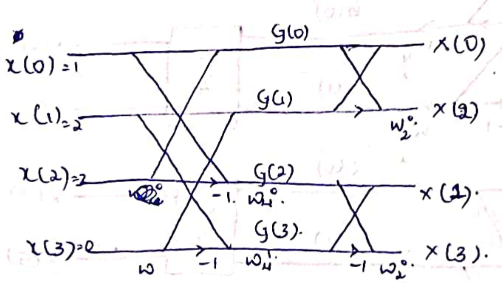




Observation;

1. I/p in proper order.
2. O/p are bit reversible.

Find 4 point DFT of the sequence $x(n) = \{1, 2, 3, 0\}$ using radix-2 DIF FFT. Verify result with direct computation.



$G(0) = x(0) + x(2) = 4$

$G(1) = x(1) + x(3) = 2$

$G(2) = (x(0) - x(2)) W_4^0 = -2$

$G(3) = -2j$

$X(0) = G(0) + G(1) = 6$

$X(1) = G(2) + G(3) = -2 - 2j$

$X(2) = G(0) - G(1) = 2$

$X(3) = G(2) - G(3) = -2 + 2j$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2+3 \\ 1-2j-3 \\ 1-2+3 \\ 1+2j-3 \end{bmatrix}$$

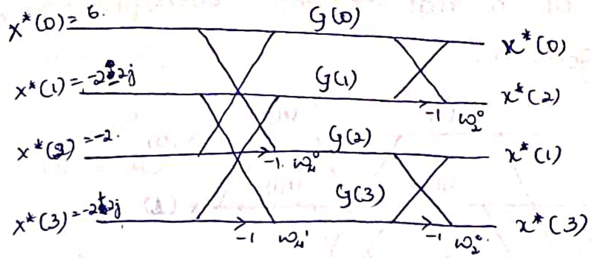
$$= \begin{bmatrix} 6 \\ -2-2j \\ 2 \\ -2+2j \end{bmatrix}$$

Find the 4 point real sequence $x(n)$ if its 4 point DFT samples are $X(0) = 6$, $X(1) = -2+2j$, $X(2) = -2$. ~~others~~ Use DIF FFT algorithm.

$X(k) = X^*(4-k)$

$X(3) = X^*(1)$

$= -2-2j$



$G(0) = X^*(0) + X^*(2) = 4$

$G(1) = X^*(1) + X^*(3) = -4$

$G(2) = (X^*(0) - X^*(2)) W_4^0 = 8$

$G(3) = (X^*(1) - X^*(3)) W_4^1 = -4$

$x^*(0) = G(0) + G(1) = 0$

$x^*(2) = (G(0) - G(1)) W_2^0 = 8$

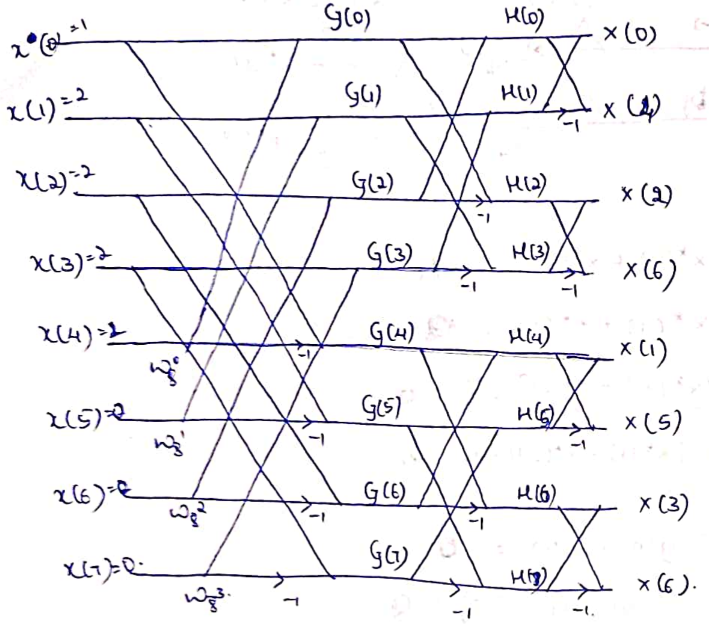
$x^*(1) = G(2) + G(3) = 4$

$x^*(3) = (G(2) - G(3)) W_2^0 = 12$

$x(n) = \frac{1}{4} \{0, 4, 8, 12\}$

$= \{0, 1, 2, 3\}$

Find 8 point DFT of a real sequence $x(n) = \{1,2,2,2,1,0,0,0\}$ using DIF FFT algorithm.



$G(0) = x(0) + x(4) = 2$

$G(1) = x(1) + x(5) = 2$

$G(2) = x(2) + x(6) = 2$

$G(3) = x(3) + x(7) = 2$

$G(4) = (x(0) - x(4)) W_8^0 = 0.$

$G(5) = (x(1) - x(5)) W_8^1 = 1.414 - 1.414j$

$G(6) = (x(2) - x(6)) W_8^2 = -2j$

$G(7) = (x(3) - x(7)) W_8^3 = -1.414 - 1.414j$

$H(0) = G(0) + G(2) = 4$

$H(1) = G(1) + G(3) = 4$

$H(2) = (G(0) - G(2)) W_4^0 = 0.$

$H(3) = (G(1) - G(3)) W_4^1 = 0.$

$G(4) + G(6) = -2j$

$H(6) = (G(4) - G(6)) W_4^0 = 2j$

$H(7) = (G(5) - G(7)) W_4^1 = -2.828j$

$X(0) = H(0) + H(1) = 8$

$X(4) = H(0) - H(1) = 0.$

$X(2) = H(2) + H(3) = 0$

$X(6) = H(2) - H(3) = 0.$

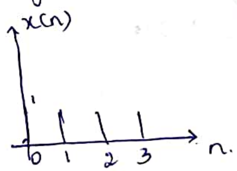
$X(1) = H(4) + H(5) = -4.828j$

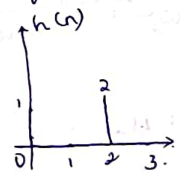
$X(5) = H(4) - H(5) = 0.828j$

$X(3) = H(6) + H(7) = -0.828j$

$X(7) = H(6) - H(7) = 4.828j$

Find 4 point circular convolution of $x(n)$ and $h(n)$ given in figure using radix-2 DIF FFT algorithm.



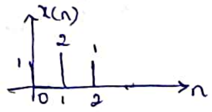


~~Find IDFT using radix-2 DIF FFT algorithm~~

~~X(k) = {~~

Compute IDFT using ~~DIF FFT algorithm. Find x(n)~~

Find the periodic convolution of $x(n)$ and $h(n)$ shown in fig using



a) time domain convolution operation

b) DFT operation. Is this result same as that of part (a)

c) radix-2 FFT & zero padding. Is this result same as that of part (a)

Find 4 point DFT of the sequence $x(n) = \cos(\frac{\pi}{4}n)$ using DIF FFT algorithm.

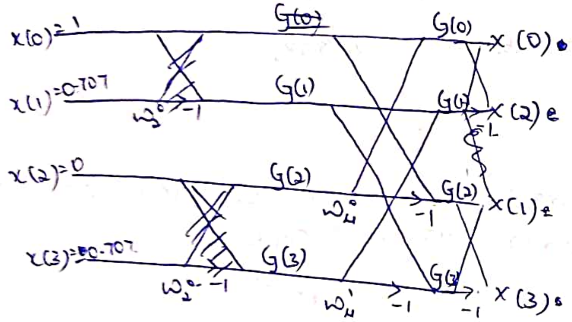
$x(n) = \cos(\frac{\pi}{4}n)$

$x(0) = 1.$

$x(1) = 0.707.$

$x(2) = 0.$

$x(3) = -0.707.$



$G(0) = x(0) + x(2) = 1$

$G(1) = x(1) + x(3) = \cancel{1.414}\ 0.$

$G(2) = (x(0) - x(2))W_4^0 = 1.$

$G(3) = (x(1) - x(3))W_4^1 = -1.414j$

$X(0) = G(0) + G(1) = 1$

$X(1) = G(2) + G(3) = 1 - 1.414j$

$X(2) = G(0) - G(1) = 1.$

$X(3) = G(2) - G(3) = 1 + 1.414j$

Composite radix/Mixed radix FFT: It is used where N is a composite number i.e., a number that has more than 1 prime factor.

Ex: $N = 12$ or 18.

$N = 18 = 2\times3\times3$. $N = 12 = 2\times2\times3$

$N = 9 = 3\times3$

Develop radix-3 DIT FFT algorithm for composite number DFT where $N = 9$.

$N = 9 = 3\times3 \quad m_1\times m_2.$

$$X(k) = \sum_{n=0}^{8} x(n) W_9^{kn}. \qquad 0 \le k \le 8.$$

$$X(k) = \underbrace{\sum_{n=0}^{2} x(3n) W_9^{3nk}}_{X_1(k)} + \sum_{n=0}^{2} x(3n+1) W_9^{k(3n+1)} + \sum_{n=0}^{2} x(3n+2) W_9^{(3n+2)k} \longrightarrow (1)$$

$$X(k) = X_1(k) + W_9^k X_2(k) + W_9^{2k} X_3(k) \longrightarrow (2).$$

$$X_1(k) = \sum_{n=0}^{2} x(3n) W_9^{3nk}.$$

$\cancel{X_2(k) = \sum_{n=0}^{2}}$

$$= x(0)W_9^0 + x(3)W_9^{3k} + x(6)W_9^{6k} \longrightarrow (3)$$

$$X_2(k) = \sum_{n=0}^{2} x(3n+1) W_9^{3nk}$$

$$= x(1)W_9^0 + x(4)W_9^{3k} + x(7)W_9^{6k} \longrightarrow (4)$$

$$X_3(k) = \sum_{n=0}^{2} x(3n+2) W_9^{3nk}.$$

$$= x(2) + x(5)W_9^{3k} + x(8)W_9^{6k} \longrightarrow (5).$$

From ②,

$X(0) = X_1(0) + W_9^0 X_2(0) + W_9^0 X_3(0)$.

$X(1) = X_1(1) + W_9^1 X_2(1) + W_9^2 X_3(1)$

$X(2) = X_1(2) + W_9^2 X_2(2) + W_9^4 X_3(2)$

$X(3) = X_1(0) + W_9^3 X_2(0) + W_9^6 X_3(0)$

$X(4) = X_1(1) + W_9^4 X_2(1) + W_9^8 X_3(1)$

$X(5) = X_1(2) + W_9^5 X_2(2) + W_9^{10} X_3(2)$

$X(6) = X_1(0) + W_9^6 X_2(0) + W_9^{12} X_3(0)$

$X(7) = X_1(1) + W_9^7 X_2(1) + W_9^{14} X_3(1)$

$X(8) = X_1(2) + W_9^8 X_2(2) + W_9^{16} X_3(2)$.

From ③,

$x_1(0) = x(0) + W_9^0 x(3) + W_9^0 x(6)$

$x_1(1) = x(0) + W_9^3 x(3) + W_9^6 x(6)$

$x_1(2) = x(0) + W_9^6 x(3) + W_9^{12} x(6)$.

From ④,

$X_2(0) = x(1) + W_9^0 x(4) + W_9^0 x(7)$

$X_2(1) = x(1) + W_9^3 x(4) + W_9^6 x(7)$

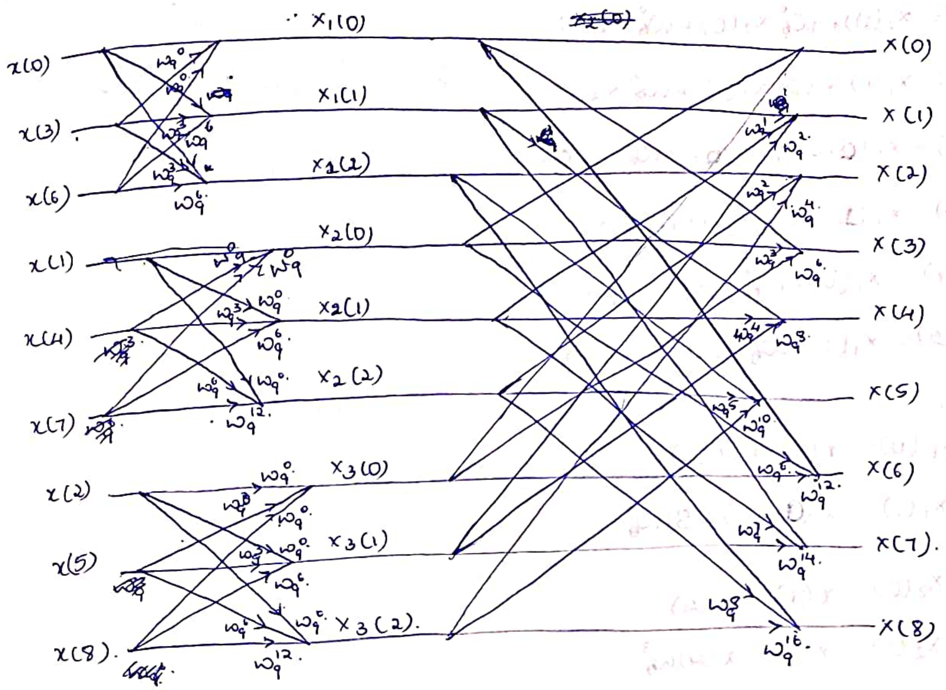
$X_2(2) = x(1) + W_9^6 x(4) + W_9^{12} x(7)$

From ⑤,

$X_3(0) = x(2) + W_9^0 x(5) + W_9^0 x(8)$

$X_3(1) = x(2) + W_9^3 x(5) + W_9^6 x(8)$

$X_3(2) = x(2) + W_9^6 x(5) + W_9^{12} x(8)$.



Develop DITFFT algorithm for N=6 and draw signal flow graph for a) N=6=3×2 and b) N=6=2×3. c) using the flow graph compute 6 point DFT for the sequence $x(n) = \{1, -1, 2, -2, 3, -3\}$ and verify result by direct computation.

$$X(k) = \sum_{n=0}^{5} x(n) W_6^{kn}.$$

$$= \sum_{n=0}^{1} x(3n) W_9^{3nk} + \sum_{n=0}^{1} x(3n+1) W_9^{(3n+1)k} + \sum_{n=0}^{1} x(3n+2) W_9^{(3n+2)k}.$$

$X(k) = X_1(k) + W_9^{k} X_2(k) + W_9^{2k} X_3(k)$.

$X_1(k) = x(0) W_9^0 + x(3) W_9^{3k}$

$X_2(k) = x(1) W_9^0 + x(4) W_9^{3k}$

$X_3(k) = x(2) W_9^0 + x(5) W_9^{3k}$.

$X(0) = X_1(0) + W_6^0 X_2(0) + W_6^0 X_3(0) =$

$X(1) = X_1(1) + W_6^1 X_2(1) + W_6^2 X_3(1)$

$X(2) = X_1(0) + W_6^2 X_2(0) + W_6^4 X_3(0)$

$X(3) = X_1(1) + W_6^3 X_2(1) + W_6^6 X_3(1)$

$X(4) = X_1(0) + W_6^4 X_2(0) + W_6^8 X_3(0)$

$X(5) = X_1(1) + W_6^5 X_2(1) + W_6^{10} X_3(1).$

$X_1(0) = x(0) + x(3)$

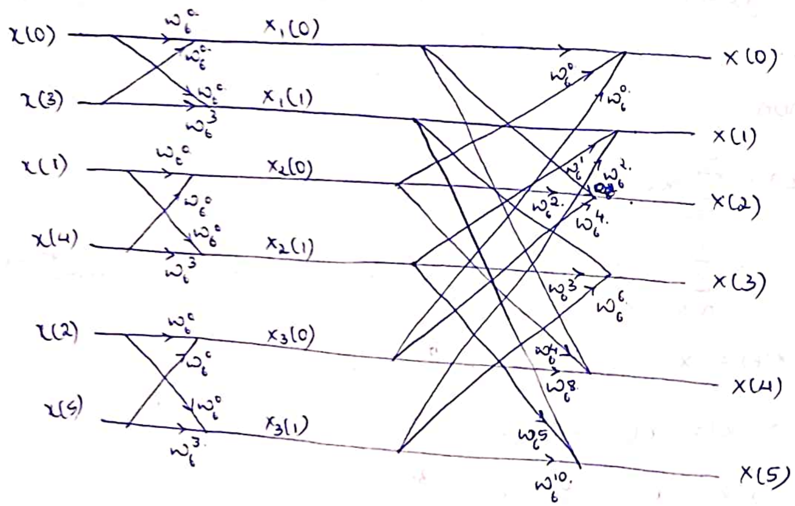
$X_1(1) = x(0) + x(3) W_6^3$

$X_2(0) = x(1) + x(4)$

$X_2(1) = x(1) + x(4) W_6^3$

$X_3(0) = x(2) + x(5)$

$X_3(1) = x(2) + x(5) W_6^3$



b) $X(k) = \sum_{n=0}^{5} x(n) W_6^{kn}$

$X(k) = \sum_{n=0}^{2} x(2n) W_6^{2nk} + \sum_{n=0}^{2} x(2n+1) W_6^{(2n+1)k}.$

$X(k) = X_1(k) + W_6^k X_2(k).$

$X_1(k) = \sum_{n=0}^{2} x(2n) W_6^{2nk}.$

$= x(0) + x(2) W_6^{2k} + x(4) W_6^{4k}.$

$X_2(k) = \sum_{n=0}^{2} x(2n+1) W_6^{2nk}.$

$= x(1) + x(3) W_6^{2k} + x(5) W_6^{4k}.$

$\cancel{X_3(k) = \sum_{n=0}^{2}}$

$X_1(0) = x(0) + x(2) + x(4)$

$X_1(1) = x(0) + x(2) W_6^2 + x(4) W_6^4$

$X_1(2) = x(0) + x(2) W_6^2 + x(4) W_6^4.$

$X_2(0) = x(1) + x(3) + x(5)$

$X_2(1) = x(1) + x(3) W_6^1 + x(5) W_6^2.$

$X_2(2) = x(1) + x(3) W_6^2 + x(5) W_6^4.$

$X(0) = X_1(0) + W_6^0 X_2(0)$

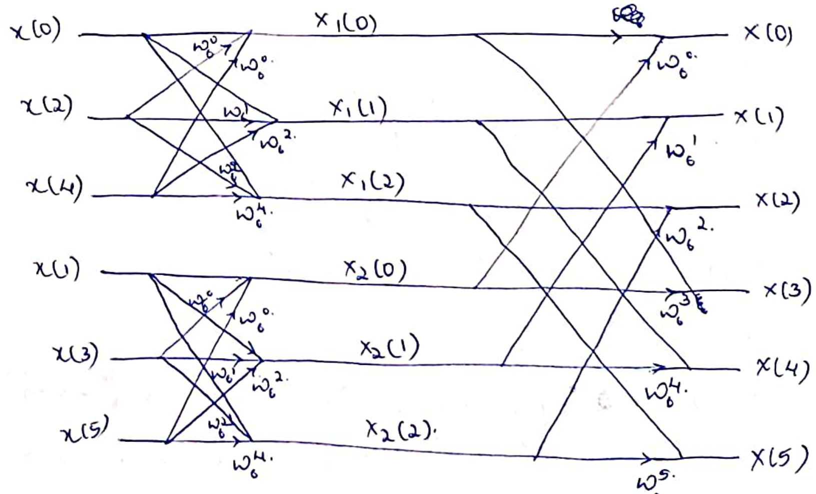
$X(1) = X_1(1) + W_6^1 X_2(1)$

$X(2) = X_1(2) + W_6^2 X_2(2)$

$X(3) = X_1(0) + W_6^3 X_2(0)$

$X(4) = X_1(1) + W_6^4 X_2(1)$

$X(5) = X_1(2) + W_6^5 X_2(2)$



Fast convolution / signal segmentation technique:

a) Overlap add method

b) Overlap save method

Steps:

1. Determine length M which is the length of impulse response data sequence i.e., h(n). Determine M-1.

2. Determine input sequence x(n) size of DFT is 'N'

3. Determine the length of the new data 'L'.

4. Pad M-1 zeros to x(n). Pad ~~N~~-1 Zeroes to h(n).

x(n)	M-1 zeros

h(n)	~~N~~-1 zeros

1. Compute $y(n) = x(n) * h(n)$ using overlap add method and Verify the result by direct computation.

$h(n) = \{1, 1, 1\}$ $\quad x(n) = \{1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 4, 2, 1, -3\}$

$M = 3$

$L = 2^M = 8.$

$~~L~~ = M + N - 1$

$8 = 3 + N - 1$

$N = 6.$

Iteration 1: $h_1(n) = \{1, 1, 1, 0, 0, 0, \overbrace{0, 0}^{N-1 \text{ zeros}}\}$

$x_1(n) = \{1, 2, 0, -3, 4, 2, \overbrace{0, 0}^{M-1 \text{ zeros}}\}$

$X_1(k) = 1 + 2W_8^k - 3W_8^{3k} + 4W_8^{4k} + 2W_8^{5k}.$

$H_1(k) = 1 + W_8^k + W_8^{2k}$

$Y_1(k) = 1 + 2W_8^{k} + W_8^{2k} + 2W_8^{k} + 2W_8^{2k} + 2W_8^{3k} - 3W_8^{3k} - 3W_8^{4k} - 3W_8^{5k}$
$+ 4W_8^{4k} + 4W_8^{5k} + 4W_8^{6k} + 2W_8^{5k} + 2W_8^{6k} + 2W_8^{7k}.$

$= 1 + 3W_8^k + 3W_8^{2k} \ldots$

$y_1(n) = \{1, 3, 3, -1, 1, 3, 6, 2\}.$

Iteration-2: $h_2(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}.$

$x_2(n) = \{-1, 1, -2, 3, 4, 2, 0, 0\}.$

$H_2(k) = 1 + W_8^k + W_8^{2k}.$

$X_2(k) = -1 + W_8^k - 2W_8^{2k} + 3W_8^{3k} + 4W_8^{4k} + 2W_8^{5k}.$

$Y_2(k) = -1 + 0 - 2W_8^{2k} + 2W_8^{3k} + 5W_8^{4k} + 9W_8^{5k} + 6W_8^{6k} + 2W_8^{7k}.$

$y_2(n) = \{-1, 0, -2, 2, 5, 9, 6, 2\}.$

Iteration-3: $h_3(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}.$

$x_3(n) = \{1, -3, 0, 0, 0, 0, 0, 0\}.$

$H_3(k) = 1 + W_8^k + W_8^{2k}.$

$X_3(k) = 1 - 3W_8^k.$

$Y_3(k) = 1 - 2W_8^k - 2W_8^{2k} - 3W_8^{3k}.$

$y_3(n) = \{1, -2, -2, -3\}.$

$y(n) = y_1(n) + y_2(n-N) + y_3(n-2N)$

$= y_1(n) + y_2(n-6) + y_3(n-12).$

1	3	3	-1	1	3	6	2.								
						-1	0	-2	2	5	9	6	2		
												1	-2	-2	-3
1	3	3	-1	1	3	5	2	-2	2	5	9	7	0	-2	-3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \\ -3 & 0 & 2 \\ 4 & -3 & 0 \\ 2 & 4 & -3 \\ -1 & 2 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 1 \\ 4 & 3 & -2 \\ 2 & 4 & 3 \\ 1 & 2 & 4 \\ -3 & 1 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \\ 1 \\ 3 \\ 5 \\ 2 \\ -2 \\ 2 \\ 5 \\ 9 \\ 7 \\ 0 \\ -2 \\ -3 \end{bmatrix}$$

2. Compute $x(n) * h(n) = y(n)$ using overlap add method. Verify the result by direct computation.

$h(n) = \{1, 2, 3\}$ $\quad x(n) = \{1, 4, 3, 2, 5, 4, 3, 2, 1, -5, -4, 4, 5, 2, 3\}$

$M = 3$

$L = 2^3 = 8$

$L = M + N - 1$

$8 = 3 + N - 1$

$N = 6$

Iteration 1: $h_1(n) = \{1, 2, 3, 0, 0, 0, 0, 0\}$

$x_1(n) = \{1, 4, 3, 2, 5, 4, 0, 0\}$

$X_1(k) = 1 + 4W_8^k + 3W_8^{2k} + 2W_8^{3k} + 5W_8^{4k} + 4W_8^{5k}$

$H_1(k) = 1 + 2W_8^k + 3W_8^{2k}$

$Y_1(k) = 1 + 2W_8^k + 3W_8^{2k} + 4W_8^k + 8W_8^{2k} + 12W_8^{3k} + 3W_8^{2k} + 6W_8^{3k}$
$+ 9W_8^{4k} + 2W_8^{3k} + 4W_8^{4k} + 6W_8^{5k} + 5W_8^{4k} + 10W_8^{5k} + 15W_8^{6k} +$
$4W_8^{5k} + 8W_8^{6k} + 12W_8^{7k}.$

$= 1 + 6W_8^k + 14W_8^{2k} + 20W_8^{3k} + 18W_8^{4k} + 20W_8^{5k} + 23W_8^{6k} + 12W_8^{7k}$

$y_1(n) = \{1, 6, 14, 20, 18, 20, 23, 12\}.$

$h_2(n) = \{1, 2, 3, 0, 0, 0, 0, 0\}$

$x_2(n) = \{3, 2, 1, -5, -4, 4, 0, 0\}$

$X_2(k) = 3 + 2W_8^k + W_8^{2k} - 5W_8^{3k} - 4W_8^{4k} + 4W_8^{5k}$

$H_2(k) = 1 + 2W_8^k + 3W_8^{2k}.$

$Y_2(k) = 3 + 6W_8^k + 9W_8^{2k} + 2W_8^k + 4W_8^{2k} + 6W_8^{3k} + W_8^{2k} + 2W_8^{3k} + 3W_8^{4k}$
$- 5W_8^{3k} - 10W_8^{4k} - 15W_8^{5k} - 4W_8^{4k} - 8W_8^{5k} - 12W_8^{6k} + 4W_8^{5k} + 8W_8^{6k}$
$+ 12W_8^{7k}.$

$= 3 + 8W_8^k + 14W_8^{2k} + 3W_8^{3k} - 11W_8^{4k} - 19W_8^{5k} - 4W_8^{6k} + 12W_8^{7k}.$

$y_2(n) = \{3, 8, 14, 3, -11, -19, -4, 12\}$

$h_3(n) = \{1, 2, 3, 0, 0, 0, 0, 0\}.$

$x_3(n) = \{5, 2, 3, 0, 0, 0, 0, 0\}$

$X_3(k) = 5 + 2W_8^k + 3W_8^{2k}$

$H_3(k) = 1 + 2W_8^k + 3W_8^{2k}.$

$Y_3(k) = 5 + 2W_8^k + 3W_8^{2k} + 10W_8^k + 4W_8^{2k} + 6W_8^{3k} + 15W_8^{2k} + 6W_8^{3k} + 9W$

$= 5 + 12W_8^k + 22W_8^{2k} + 12W_8^{3k} + 9W_8^k.$

$y_3(n) = \{5, 12, 22, 12, 9\}.$

1	6	14	20	18	20	23	12									
						3	8	14	3	-11	-19	-4	12			
												5	12	22	12	
1	6	14	20	18	20	26	20	14	3	-11	-19	1	24	22	12	9

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \\ 5 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \\ -5 & 1 & 2 \\ -4 & -5 & 1 \\ 4 & -4 & -5 \\ 5 & 4 & -4 \\ 2 & 5 & 4 \\ 3 & 2 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 14 \\ 20 \\ 18 \\ 20 \\ 26 \\ 20 \\ 14 \\ 3 \\ -11 \\ -19 \\ 1 \\ 24 \\ 22 \\ 12 \\ 9 \end{bmatrix}$$

compute $y(n) = x(n) * h(n)$ using overlap save method. Verify the result by direct computation.

$h(n) = \{1, 1, 1\}$ $\quad x(n) = \{1, 2, 0, 3, 4, 2, -1, 1, -2, 3, 4, 2, 1, -3\}$

$M = 3$.

$L = 2^3 = 8$.

$L = M + N - 1$

$N = 6$.

$h_1(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$x_1(n) = \{0, 0, 1, -2, 0, -3, 4, 2\}$

$X_1(k) = W_8^{2k} - 2W_8^{3k} - 3W_8^{5k} + 4W_8^{6k} + 2W_8^{7k}$.

$H_1(k) = 1 + W_8^{k} + W_8^{2k}$

$Y_1(k) = \cancel{1 + W_8^{2k} + W_8^{3k} + W_8^{4k} - 2W_8^{3k} - 2W_8^{4k} - 3W_8^{5k} - 3W_8^{6k} - 3W_8^{7k}}$

$\cancel{+ 4W_8^{6k} + 4W_8^{7k} + 4W_8^{8k} + 2W_8^{7k} + 2W_8^{8k} + 2W_8^{9k}}$

$= \cancel{1 + W_8^{k} + 2W_8^{2k} - 2W_8^{3k}}$

$= W_8^{2k} + 2W_8^{3k} - 3W_8^{5k} + 4W_8^{6k} + 2W_8^{7k} + W_8^{3k} + 2W_8^{4k} - 3W_8^{6k} + 4W_8^{7k} + 2W_8^{8k}$

$+ W_8^{4k} + 2W_8^{5k} - 3W_8^{7k} + 4W_8^{8k} + 2W_8^{9k}$.

$Y_1(k) = W_8^{2k} + 3W_8^{3k} + 3W_8^{4k} - 1W_8^{5k} + W_8^{6k} + 3W_8^{7k} + 6W_8^{8k} + 2W_8^{9k}$

$= \{\underbrace{6, 2}_{\text{discard}}, 1, 3, 3, -1, 1, 3\}$

$H_2(k) = \{1, 1, 1, 0, 0, 0, 0, 0\}$.

$x_2(n) = \{4, 2, -1, 1, -2, 3, 4, 2\}$.

$H_2(k) = 1 + W_8^{k} + W_8^{2k}$.

$X_2(k) = 4 + 2W_8^{k} - W_8^{2k} + W_8^{3k} - 2W_8^{4k} + 3W_8^{5k} + 4W_8^{6k} + 2W_8^{7k}$.

$Y_2(k) = 4 + 2W_8^{k} - W_8^{2k} + W_8^{3k} - 2W_8^{4k} + 3W_8^{5k} + 4W_8^{6k} + 2W_8^{7k} + 4W_8^{k} + 2W_8^{2k} - W_8^{3k}$

$+ W_8^{4k} - 2W_8^{5k} + 3W_8^{6k} + 4W_8^{7k} + 2W_8^{8k} + 4W_8^{2k} + 2W_8^{3k} - W_8^{4k} + W_8^{5k} - 2W_8^{6k}$

$+ 3W_8^{7k} + 4W_8^{8k} + 2W_8^{9k}$.

$= 4 + 6W_8^{k} + 5W_8^{2k} + 2W_8^{3k} - 2W_8^{4k} + 2W_8^{5k} + 5W_8^{6k} + 9W_8^{7k}$.

$y_2(n) = \{\underbrace{4, 6}_{\text{discard}}, 5, 2, -2, 2, 5, 9\}$.

$H_3(k) = 1 + W_8^{k} + W_8^{2k}$.

$x_3(n) = \{4, 2, 1, -3, 0, 0, 0, 0\}$.

$X_3(k) = 4 + 2W_8^{k} + W_8^{2k} - 3W_8^{3k}$.

$Y_3(k) = 4 + 4W_8^{k} + 4W_8^{2k} + 2W_8^{k} + 2W_8^{2k} + 2W_8^{3k} + W_8^{2k} + W_8^{3k} + W_8^{4k} - 3W_8^{3k}$

$- 3W_8^{4k} - 3W_8^{5k}$

$= 4 + 6W_8^{k} + 7W_8^{2k} + 0W_8^{3k} - 2W_8^{4k} - 3W_8^{5k}$.

$y_3(n) = \{\underbrace{4, 6}_{\text{discard}}, 7, 0, -2, -3\}$

$y(n) = \{1, 3, 3, -1, 1, 3, 5, 2, -2, 2, 5, 9, 7, 0, -2, -3\}$.

3. Compute $y(n) = x(n) * h(n)$ using overlap save method and verify using direct computation.

$h(n) = \{2, -3\}$ $x(n) = \{4, 5, 6, 7, -8, -9, -10, -7, -6, 5, 4, 3, 2, 1\}$

$M = 2$.

$L = 4$.

$L = M + N - 1$.

$4 = 2 + N - 1$.

$N = 3$.

$h_1(n) = \{2, -3, 0, 0\}$.

$x_1(n) = \{0, 4, 5, 6\}$

$X_1(k) = 4W_4^k + 5W_4^{2k} + 6W_4^{3k}$.

$H_1(k) = 2 - 3W_4^k$

$Y_1(k) = 8W_4^k - 12W_4^{2k} + 10W_4^{2k} - 15W_4^{3k} + 12W_4^{3k} - 18W_4^{4k}$.

$y_1(n) = \{18, 8, -2, -3\}$
discard

$x_2(n) = \{6, 7, -8, -9\}$.

$h_2(n) = \{2, -3, 0, 0\}$.

$H_2(k) = 2 - 3W_4^k$.

$X_2(k) = 6 + 7W_4^k - 8W_4^{2k} - 9W_4^{3k}$.

$Y_2(k) = 12 - 18W_4^k + 14W_4^k - 21W_4^{2k} - 16W_4^{2k} + 24W_4^{3k} - 18W_4^{3k} + 27W_4^{4k}$.

$= 39 - 4W_4^k - 37W_4^{2k} + 6W_4^{3k}$.

$y_2(n) = \{39, -4, -37, 6\}$

$x_3(n) = \{-9, -10, -7, -6\}$.

$h_3(n) = \{2, -3, 0, 0\}$.

$H_3(k) = 2 - 3W_4^k$.

$X_3(k) = -9 - 10W_4^k - 7W_4^{2k} - 6W_4^{3k}$.

$Y_3(k) = -18 + 27W_4^k - 20W_4^k + 30W_4^{2k} - 14W_4^{2k} + 21W_4^{3k} - 12W_4^{3k} + 18W_4^{4k}$

$= 0 + 7W_4^k + 16W_4^{2k} + 9W_4^{3k}$.

$y_3(n) = \{0, 7, 16, 9\}$

$x_4(n) = \{-6, 5, 4, 3\}$.

$h_4(n) = \{2, -3, 0, 0\}$.

$X_4(k) = -6 + 5W_4^k + 4W_4^{2k} + 3W_4^{3k}$.

$H_4(k) = 2 - 3W_4^k$

$Y_4(k) = -12 + 18W_4^k + 10W_4^k - 15W_4^{2k} + 8W_4^{2k} - 12W_4^{3k} + 6W_4^{3k} - 9W_4^{4k}$

$= -21 + 28W_4^k - 7W_4^{2k} - 6W_4^{3k}$.

$y_4(n) = \{-21, 28, -7, -6\}$

$x_5(n) = \{3, 2, 1, 0\}$

$h_5(n) = \{2, -3, 0, 0\}$.

$X_5(k) = 3 + 2W_4^k + W_4^{2k}$.

$H_5(k) = 2 - 3W_4^k$.

$Y_5(k) = 6 - 9W_4^k + 4W_4^k - 6W_4^{2k} + 2W_4^{2k} - 3W_4^{3k}$

$= 6 - 5W_4^k - 4W_4^{2k} - 6W_4^{3k}$

$y_5(n) = \{6, -5, -4, -3\}$

$y(n) = \{8, -2, -3, -4, -37, 6, 7, 16, 9, 28, -7, -6, -5, -4, -3\}$

$$\begin{bmatrix} 4 & 0 \\ 5 & 4 \\ 6 & 5 \\ 7 & 6 \\ -8 & 7 \\ -9 & -8 \\ -10 & -9 \\ -7 & -10 \\ -6 & -7 \\ 5 & -6 \\ 4 & 5 \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} +2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -3 \\ -4 \\ -37 \\ 6 \\ 7 \\ 16 \\ 9 \\ 28 \\ -7 \\ -6 \\ -5 \\ -4 \\ -3 \end{bmatrix}$$

Unit-3

Digital filters.

1. Separation of signals that have been combined.
2. Restoration of signals that have been distorted.

Use:

1. Signal separation.
2. Signal restoration.

Classifications:

- IIR — infinite impulse response.
- FIR — finite impulse response.

Difference equation representation of a system with i/p $x(n)$ & o/p $y(n)$:

IIR: $$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k).$$

FIR: $$y(n) = \sum_{k=0}^{N-1} b_k x(n-k).$$

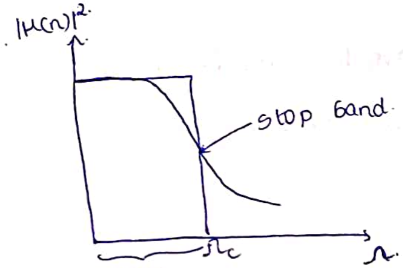
IIR : $$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}}$$

IIR filter design:

Steps:

1. Obtain the specification of equivalent analog filter. Design the analog filter in accordance with specification.
2. Transform the analog filter to an equivalent digital filter.

Analog filter design using Butterworth.

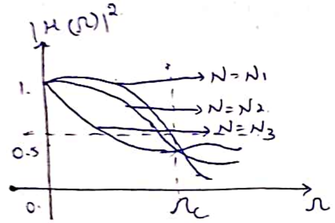


Magnitude squared response of butterworth filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$N \rightarrow$ order of filter.

$\Omega_c \rightarrow$ 3dB cutoff frequency.



Observations:

1. $|H(\Omega)|_{\Omega=0} = 1$ for all N.

2. $|H(\Omega)| = \frac{1}{\sqrt{2}}$ at $\Omega=\Omega_c$

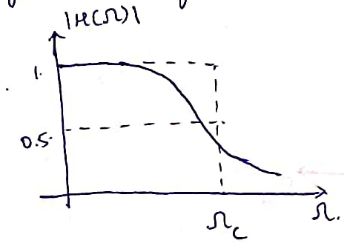
$\Rightarrow 20\log_{10}|H(\Omega)| = -3.01\,dB$ at $\Omega=\Omega_c$.

3. $|H(\Omega)| \rightarrow 0$ as $\Omega \rightarrow \infty$

4. Maximally flat response in the pass band.

$$\left.\frac{d^n|H(\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0 \quad \text{for } n=0,1,\ldots 2N-1$$

5. Monotonically decreasing in the stop band



6. At $\Omega_c=1 \Rightarrow$ normalized LPF.

$$|H(\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$$

$$H(j\Omega)\cdot H(-j\Omega) = \frac{1}{1+\Omega^{2N}}$$

Design of analog lowpass butterworth filter:

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\ldots(s-s_{N-1})}$$

1. N
2. Ω_c
3. Poles $s_0, s_1, \ldots s_{N-1}$

Order of LP butterworth filter

The filter specifications are as follows.

$A_p \le |H(\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_p$

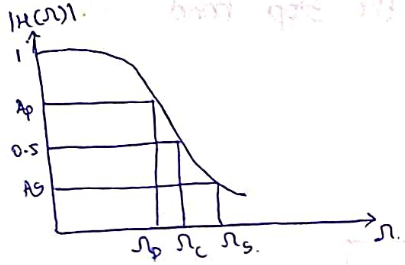
$|H(\Omega)| \le A_s \qquad \Omega \ge \Omega_s$

A_p = pass band gain.

A_s = stop band gain.

Ω_p = pass band edge frequency.

Ω_s = stop band edge frequency.



Pass band gain $\geq A_p$.

Stop band gain $\leq A_s$.

To find minimum value of N.

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

at $\Omega=\Omega_p$

$$A_p^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \longrightarrow ①$$

at $\Omega=\Omega_s$.

$$A_s^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \longrightarrow ②$$

from ①.

$$\frac{1}{A_p^2} - 1 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \longrightarrow ③$$

from ②

$$\frac{1}{A_s^2} - 1 = \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \longrightarrow ④$$

④ ÷ ③.

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{\frac{1}{A_s^2} - 1}{\frac{1}{A_p^2} - 1}.$$

$$N \geq \frac{\log\left[\frac{1}{A_s^2}-1\right] / \left[\frac{1}{A_p^2}-1\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

To find Ω_c.

from ③. $$\frac{\Omega_p}{\Omega_c} = \left[\frac{1}{A_p^2}-1\right]^{1/2N} \longrightarrow ⑤$$

from ④ $$\frac{\Omega_s}{\Omega_c} = \left[\frac{1}{A_s^2}-1\right]^{1/2N} \longrightarrow ⑥$$

Rearranging term in ③.

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}} \qquad \Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}}.$$

Actual cutoff freq is avg of Ω_{cp} & Ω_{cs}.

$$\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2}.$$

To determine poles of $H(s)$

at $s=j\Omega$ $|H(s)| = |H(-s)|$.

$s=j\Omega$ $s^2 = -\Omega^2$.

$$|H(\Omega)|^2 = H(s).H(-s) = \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^N}.$$

Poles of $H(s)$, $H(-s)$ is obtained by equating denominator to 0.

$$1+\left(\frac{-s^2}{\Omega_c^2}\right)^N = 0.$$

$$\left(\frac{-s^2}{\Omega_c^2}\right)^N = -1.$$

$$\frac{-s^2}{\Omega_c^2} = (-1)^{1/N} \longrightarrow ①$$

Now $e^{j(2k+1)\pi} = -1$.

using this equation ①

$$\frac{-s^2}{\Omega_c^2} = e^{j(2k+1)\pi/N}$$

$$s^2 = -\Omega^2 e^{j(2k+1)\pi/N} \Rightarrow s = \pm j\Omega e^{j(2k+1)\pi/2N}$$

Now $e^{j\frac{\pi}{2}} = j$

$$s = \pm\Omega_c e^{j(2k+1)\pi/2N} \cdot e^{j\pi/2} = \pm\Omega_c e^{j(2k+1+N)\pi/2N}$$

$k = 0, 1, \ldots N-1$.

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})}$$

$$N = \frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{2\log\frac{\Omega_s}{\Omega_p}}$$

If A_p and A_s are in dB

A_s in dB $= -20\log_{10} A_s$

$$A_s = 10^{\frac{-A_s \text{ in dB}}{20}}$$

$$\frac{1}{A_s} = 10^{\frac{A_s \text{ in dB}}{10}}$$

$$\frac{1}{A_s^2} = 10^{\frac{A_s(dB)}{10}}$$

$$\frac{1}{A_s^2} = 10^{0.1A_s(dB)}.$$

$|||^{ly}$

$$\frac{1}{A_p^2} = 10^{0.1A_p(\text{in dB})}.$$

$$N = \frac{\log\left[\frac{10^{0.1A_s(dB)}-1}{10^{0.1A_p(dB)}-1}\right]}{2\log\frac{\Omega_s}{\Omega_p}}$$

Frequency transformation:

Analog frequency transformation.

1. Low pass to lowpass:

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} \cdot s.$$

where Ω_{LP} is passband edge frequency of derived filter.

2. Lowpass to highpass:

$$s \rightarrow \frac{\Omega_p \cdot \Omega_{HP}}{s}.$$

where Ω_{HP} is passband edge frequency of highpass filter.

3. Lowpass to bandpass.

$$s \rightarrow \frac{\Omega_p s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$$

where Ω_l = lower band edge frequency

Ω_u = upper band edge frequency.

4. Lowpass to bandstop:

$$s \rightarrow \frac{\Omega_p(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u}.$$

Ω_l = lowerband edge frequency

Ω_u = upper band edge frequency

1. Let $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$ represent tfr function of a lowpass filter with a cutoff frequency of 1 rad/sec. The frequency transformation to find transfer function of the following analog signal.

a) LPF with a cutoff freq of 10 rad/sec.

b) Highpass filter with a cutoff freq of 10 rad/sec.

a) $H(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$

LP - LP transformation.

$\Omega_p = 1$ rad/sec $\quad \Omega_{LP} = 10$ rad/sec

$s \to \dfrac{s}{10}$

$$H(s) = \frac{1}{\frac{s^2}{100}+\frac{\sqrt{2}s}{10}+1}$$

$$= \frac{100}{s^2+10\sqrt{2}s+100}$$

b) LPF $\to$ HPF.

$s \to \dfrac{\Omega_p \Omega_{HP}}{s}$

$s \to \dfrac{10}{s}$

$$H(s) = \frac{1}{\frac{100}{s^2}+\frac{\sqrt{2}(10)}{s}+1}$$

$$= \frac{s^2}{s^2+10\sqrt{2}s+100}$$

2. Let $H(s) = \dfrac{1}{s^2+s+1}$ represent the tfr fn of a LPF. The passband 1 rad/sec. use freq transformation technique to find the

a) HPF with cutoff freq 10 rad/sec.

b) BPF with passband of 10 rad/sec & centre freq 100 rad/sec

a) LPF $\to$ HPF.

$H(s) = \dfrac{1}{s^2+s+1}$

$s \to \dfrac{10}{s}$

$$H(s) = \frac{1}{\frac{100}{s^2}+\frac{10}{s}+1}$$

$$= \frac{s^2}{s^2+10s+100}$$

b). LPF $\to$ BPF.

$$s \to \frac{\Omega_p s^2+\Omega_l \Omega_u}{s(\Omega_u-\Omega_l)}$$

$\Omega_0 = \sqrt{\Omega_l \Omega_u}$

$\Omega_0 = 100$ r/s.

$\Omega_l \Omega_u = 100^2 \to (1)$

$\Omega_p = \Omega_u - \Omega_l = 10 \to (2)$

$\Omega_u - \Omega_l = 10.$

$$\to \frac{(1)s^2+100^2}{s(10)}.$$

$$H(s) = \frac{1}{s^2+s+1}$$

$$= \frac{1}{\left(\frac{s^2+100^2}{10s}\right)^2+\frac{s^2+100^2}{10s}+1}$$

$$= \frac{1}{\frac{(s^2+100^2)^2}{100s^2}+\frac{s^2+100^2}{10s}+1}$$

$$= \frac{100s^2}{(s^2+100^2)^2+100s^2(s^2+100)^2+100s^2}$$

$$= \frac{100s^2}{\cancel{(s^2+100^2)^2+100} \; s^4+100^4+100s^4}$$

Analog Butterworth filter design:

1. $|H(\Omega)|^2 = \frac{1}{1+16\Omega^4}$. Determine the analog filter transfer function $H(S)$.

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$2N = 4$.
$N = 2$.

$$\left(\frac{\Omega}{\Omega_c}\right)^4 = 16$$

$$\left(\frac{\Omega}{\Omega_c}\right)^4 = 2^4$$

$$\Omega_c = \frac{\Omega}{2} \text{ rad/sec}$$

$$P_k = \pm \Omega_c e^{j(2k+1+N)\pi/2N} \qquad k = 0, 1, \dots N-1$$

Let design a normalized LPF $\Omega_c = 1$ rad/sec.

$$P_k = \pm e^{j(2k+1+N)\pi/2N} \qquad N=2,\ k=0,1.$$

Put $k = 0$.

$$P_k = \pm e^{j\left(\frac{3\pi}{4}\right)}$$

$$P_0 = -0.707 + j0.707$$

Put $k = 1$.

$$P_k = \pm e^{j\left(\frac{5\pi}{4}\right)}.$$

$$P_1 = \pm(-0.707 - j0.707).$$

$$H(S) = \frac{1}{(S-P_0)(S-P_1)} = \frac{1}{(S-(-0.707+j0.707))\,[S-(-0.707-j0.707)]}$$

$$= \frac{1}{(S+a-jb)(S+a+jb)}.$$

$$= \frac{1}{(S+a)^2+b^2} = \frac{1}{(S+0.707)^2+(0.707)^2}.$$

$$= \frac{1}{S^2+0.5+1.414S+0.5}$$

$$H(s) = \frac{1}{S^2+1.414S+1}.$$

Apply LP $\rightarrow$ LP $\Rightarrow S \rightarrow \frac{\Omega_p}{\Omega_{Lp}} S.$

$$= \frac{1}{\left(\frac{S}{0.5}\right)^2+1.414\left(\frac{S}{0.5}\right)+1.} = \frac{(0.5)^2}{S^2+0.707S+0.25}$$

2. Determine the normalized butterworth filter with order a) 2 b) 3 c) 4.

$\Omega_c = 1$ rad/sec. $N=2$.

$$P_k = \pm \Omega_c e^{j(2k+1+N)\pi/2N} \qquad k = 0, 1, \dots N-1$$

$$P_k = \pm e^{(2k+1+N)\pi/2N} \qquad N=2, k=0,1.$$

Put $k = 0$.

$$P_0 = \pm e^{j(3\pi/4)}$$

$$\pm(-0.707 + j0.707)$$

Put $k = 1$

$$P_1 = \pm e^{j\left(\frac{5\pi}{4}\right)}$$

$$= \pm(-0.707 - j0.707)$$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{[s-(-0.707+0.707j)][s-(-0.707-0.707j)]}$$

$$= \frac{1}{(s+0.707)^2+(0.707)^2}$$

$$= \frac{1}{s^2+1.414s+1}$$

Apply $LP \to LP$

$$H(s)|_{s\to s} = \frac{1}{s^2+1.414s+1}$$

2) $N=3$

$P_k = \pm e^{j(2k+1+N)\pi/2N}$

Put $k=0$

$P_0 = \pm e^{j(4\pi/6)}$

$= \pm e^{j(2\pi/3)} = -0.5+0.866j$

$k=1$

$P_1 = \pm e^{j\pi}$

$= \pm(-1)$

$k=2$

$P_2 = \pm e^{j(4\pi/3)}$

$= \pm(-0.5-j0.866)$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)}$$

$$= \frac{1}{[(s+0.5)^2+(0.866)^2][s+1]} = \frac{1}{s^3+2s^2+2s+1}$$

$LP \to LP$

$$H(s)|_{s\to s} = \frac{1}{s^3+2s^2+2s+1}$$

3) $N=4$

$P_k = \pm e^{j(2k+1+N)\pi/2N}$

Put $k=0$

$P_0 = \pm e^{j(5\pi/8)}$

$= \pm(-0.382+0.923j)$

$k=1$

$P_1 = \pm e^{j(7\pi/8)}$

$= \pm(-0.923+0.382j)$

$k=2$

$P_2 = \pm e^{j(9\pi/8)}$

$= \pm(-0.923-0.382j)$

$k=3$

$P_3 = \pm e^{j(11\pi/8)}$

$= \pm(-0.382-0.923j)$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)(s-P_3)} = \frac{1}{[(s+0.328)^2+(0.923)^2][(s+0.923)^2+0.328^2]}$$

$$= \frac{1}{s^4+2.5s^3+3.31s^2+2.685s+1}$$

3. $|H(\Omega)|^2 = \frac{1}{1+64\Omega^6}$

$2N=6$

$N=3$

$\left(\frac{\Omega}{\Omega_c}\right)^6 = 64$

$\Omega_c = \frac{1}{2}$ rad/sec.

$P_k = \pm e^{j(2k+1+N)\pi/2N}$

$k=0$

$P_0 = \pm e^{j(2\pi/3)} = \pm(-0.5+0.866j)$

$k=1$

$P_1 = \pm e^{j(\pi)}$

$= \pm(-1)$

$k=2$

$P_2 = e^{\pm j(4\pi/3)}$

$= \pm(-0.5-0.866j)$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)} = \frac{1}{s^3+2s^2+2s+1}$$

$LP \to LP$

$$H(s)|_{s\to\frac{s}{0.5}} = \frac{1}{\left(\frac{s}{0.5}\right)^3+2\left(\frac{s}{0.5}\right)^2+2\left(\frac{s}{0.5}\right)+1} = \frac{0.125}{s^3+s^2+0.5s+0.125}$$

4. Design an analog butterworth to meet the following specifications

1. $0.8 \le |H(\Omega)| \le 1$ $0 \le f \le 100Hz$

$|H(\Omega)| \le 0.2$ $f \ge 5000Hz$

$A_p \le |H(\Omega)| \le 1$ $0 \le \Omega \le \Omega_p$

$|H(\Omega)| \le A_s$ $\Omega \ge \Omega_s$

$A_p = 0.8$

$A_s = 0.2$

$f_p = 1000Hz$ $f_s = 5000Hz$

$\Omega_p = 2\pi f_p = 2000\pi$ rad/s $\Omega_s = 2\pi f_s = 10000\pi$ rad/sec

Step-1: find order of the filter

$$N \geq \frac{\log\left[\frac{1}{A_s^2}-1\right]\Big/\left[\frac{1}{A_p^2}-1\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left[\frac{1}{(0.2)^2}-1\right]\Big/\left[\frac{1}{(0.8)^2}-1\right]}{2\log\left(\frac{10000}{2000}\right)}$$

$N \geq 1.165$

$N \simeq 2$.

Step-2: find Ω_c

$$\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2}$$

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}} = \frac{2000\pi}{\left[\frac{1}{0.8^2}-1\right]^{1/4}} = 7263 \text{ rad/s.}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}} = 14196 \text{ rad/sec.}$$

$$\Omega_c = \frac{7263+14196}{2} = 10729 \text{ rad/s}$$

Step-3: $P_k = \pm\Omega_c\, e^{j(2k+1+N)\pi/2N}$ $\quad \Omega_c = 1$.

$P_k = \pm e^{j(2k+1+N)\pi/2N}$.

$k=0$.

$P_0 = \pm e^{j(3\pi/6)}$

$= \pm(-0.707+j0.707)$

$k=1$

$P_1 = \pm e^{j(5\pi/6)}$

$= \pm(-0.707-j0.707)$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2+1.4145+1.}$$

Step-4: To design LP to LP filter

$s \to \frac{\Omega_c}{\Omega_{cp}} s$

$s \to \frac{s}{10729}$

$$H(s) = \frac{1}{\left(\frac{s}{10729}\right)^2+1.414\left(\frac{s}{10729}\right)+1} = \frac{115111441}{s^2+15170.85+115111441}$$

4. Design an analog butterworth filter for the following specifications

$0.8 \leq |H(\Omega)| \leq 1 \qquad 0 \leq \Omega \leq 0.2\pi$.

$|H(\Omega)| \leq 0.2 \qquad 0.6\pi \leq \Omega \leq \pi$.

$A_p = 0.8 \qquad \Omega_p = 0.2\pi$.

$A_s = 0.2 \qquad \Omega_s = 0.6\pi$.

Step-1:
$$N \geq \frac{\log\left[\frac{1}{A_s^2}-1\right]\Big/\left[\frac{1}{A_p^2}-1\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left[\frac{1}{0.2^2}-1\right]\Big/\left[\frac{1}{0.8^2}-1\right]}{2\log(3)}$$

$N \simeq 2$.

Step-2:
$$\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2}.$$

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}} = \frac{0.2\pi}{\left[\frac{1}{0.8^2}-1\right]^{1/4}} = 0.7256 \text{ rad/sec.}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}} = \frac{0.6\pi}{\left[\frac{1}{0.2^2}-1\right]^{1/4}} = 0.8516 \text{ rad/sec.}$$

$$\Omega_c = \frac{0.7256+0.8516}{2} = 0.7886 \text{ rad/sec.}$$

Step-3:

$P_k = \pm\Omega_c e^{j(2k+1+N)\pi/2N}$ $\quad \Omega_c = 1$.

$P_0 = \pm(-0.707+j0.707)$

$P_1 = \pm(-0.707-j0.707)$.

$$H_1(s) = \frac{1}{s^2+1.4145+1.}$$

Step-4:

$$H(s)\Big|_{s\to\frac{s}{0.7886}} \Rightarrow H(s) = \frac{0.6218}{s^2+1.115s+0.6218}.$$

1. Design a highpass butterworth filter for the given specification -3dB passband attenuation at a frequency of $\Omega_p = 1000$ rad/sec and atleast -15dB attenuation at 500 rad/sec.

$A_p = -3dB$. $\Omega_p = 1000$ rad/sec.

$A_s = -15dB$. $\Omega_s = 500$ rad/sec.

$$N = \frac{\log\left[\frac{10^{0.1A_s (in\ dB)}-1}{10^{0.1A_p (in\ dB)}-1}\right]}{2\log\frac{\Omega_s}{\Omega_p}} = \frac{\log\left[\frac{10^{0.1(-15)}-1}{10^{0.1(-3)}-1}\right]}{2\log\frac{500}{1000}}$$

$$= \frac{-1.2}{-0.602} = 3.$$

Step-2: $\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2}$.

$$\Omega_{cp} = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)} = \frac{1000}{\left(\frac{1}{-3^2}-1\right)^{1/2N}} = 1000.5 \text{ rad/sec.}$$

$$\Omega_{cs} = \frac{500}{10^{0.1A_s(dB)}-1} = 282.6$$

$\frac{1}{A_p^2} = 10^{0.1 A_p (in\ dB)}$.

$$\Omega_c = \frac{1000.5+282.68}{2} = 641.74 \text{ rad/sec.}$$

Step-3: $P_k = \pm\Omega_c e^{j(2k+1+N)\pi/2N}$. $\Omega_c = 1$.

$P_k = \pm e^{j(2k+1+N)\pi/2N}$

k=0.
$P_0 = \pm e^{j(4\pi/6)}$
$= \pm(-0.5+0.866j)$

k=1.
$P_1 = \pm e^{j(\pi)}$
$= \pm(-1)$

k=2.
$P_2 = \pm e^{j(8\pi/6)}$
$= \pm(-0.5-0.866j)$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)} = \frac{1}{(s^2+s+1)(s+1)}$$

$$s \to \frac{\Omega_p\Omega_{HP}}{s} = \frac{641.74}{s}$$

$$H_1(s) = \frac{s^3}{s^3+1283.48s^2+823660.45s+264287930.3}$$

2. Design an analog filter with maximally flat response in the passband & an Acceptable attenuation of 2dB at 20rad/sec. The attenuation in the stop band should be more than 10dB beyond 30 rad/sec.

$A_p = 2dB$.
$\Omega_p = 20$ rad/sec.
$A_s = 10dB$.
$\Omega_s = 30$ rad/sec.

$$N = \frac{\log\frac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)} = 3.37$$

step-2: $\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2}$.

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}}$$

$$= \frac{20}{\left(10^{0.1\times2}-1\right)^{1/8}} = 21.38 \text{ rad/sec.}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}} = \frac{30}{10^{0.1\times10}-1} = 22.795$$

$\Omega_c = 22.09$ rad/sec.

Step-3:
$P_k = \pm\Omega_c e^{j(2k+1+N)\pi/2N}$. $\Omega_c = 1$.

$P_k = \pm e^{j(2k+1+N)\pi/2N}$.

$P_0 = \pm e^{j(5\pi/8)} = \pm(-0.382+0.923j)$

k=1.
$P_1 = \pm e^{j(7\pi/8)} = \pm(-0.923+0.382j)$

k=2.
$P_2 = \pm e^{j(9\pi/8)} = \pm(-0.923-0.382j)$

$k=3$.

$P_3 = \pm e^{j(11\pi/8)} = \pm(-0.382-0.923j)$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)(s-P_2)(s-P_3)}$$

$$= \frac{1}{s^4+2.6s^3+3.315s^2+2.685s+1.}$$

Step-3:

$$s \rightarrow \frac{22.09}{s}$$

$$H_1(s) = \frac{1}{\left(\frac{22.09}{s}\right)^4 + 2.6\left(\frac{22.09}{s}\right)^3 + 3315\left(\frac{22.09}{s}\right)^2 + 2.685\left(\frac{22.09}{s}\right)+1.}$$

$$= \frac{s^4.}{238112.86+28025.95s+1615.17s^2+59.31s^4.}$$

$$= \frac{s^4}{59.31s^4+1615.17s^2+28025.95s+238112.86.}$$

s-z mapping:

1. Approximation of derivatives.
2. Impulse invariance transformation (IIT).
3. Bilinear transformation.
4. Pole-zero mapping.

IIT:

$$\underset{\text{(analog filter)}}{H(s)} \xrightarrow{\mathcal{L}^{-1}} h(t) \xrightarrow[\text{sampling}]{\text{uniform}} h(nT) \xrightarrow[\text{transform}]{z} \underset{\text{(digital filter)}}{H(z)}$$

Let $H(s)$ be the system function of analog filter. using partial fraction expansion

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k} \longrightarrow ①.$$

Taking inverse laplace transform.

$$h(t) = \sum_{k=1}^{N} C_k e^{P_k t} \quad \text{for } t \geq 0.$$

$$= \sum_{k=1}^{N} C_k e^{P_k t} u(t)$$

Sampling $h(t)$ uniformly gives

$$h(nT) = h(n) = \sum_{k=1}^{N} C_k e^{P_k nT} u(nT) \longrightarrow ②$$

Taking z transform.

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}} \longrightarrow ③$$

from ① & ③ we can write mapping as.

$$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T} z^{-1}}$$

$$H(s) = \frac{1}{s-P_k}$$

$H(s)$ has poles at $s = P_k$ & $H(z)$ has poles at $z = e^{P_k T}$

analog pole at $s = P_k$ is mapped to $z = e^{P_k T}$

$z = e^{sT}$

In pole for $z = re^{j\omega}$ & $s = \sigma + j\Omega$

$re^{j\omega} = e^{(\sigma + j\Omega)T}$

$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$

Equating real & imaginary parts

$$\left.\begin{array}{l} r = e^{\sigma T} \\ \omega = \Omega T \end{array}\right\} \longrightarrow (4)$$

Mapping summary:

A) From (4). if $\sigma < 0$ then $0 < r < 1$.

$\sigma < 0 \Rightarrow$ LHS of s-plane.

$0 < r < 1 \Rightarrow$ area inside unit circle in z-plane.

LHS of s-plane is mapped inside unit circle in z-plane.

B) from (4) if $\sigma > 0$ then $r > 1$.

$\therefore$ RHS of s-plane is mapped outside unit circle in z-plane.

C) from (4) if $\sigma = 0$ the $r = 1$.

$\therefore j\Omega$ axis is mapped to unit circle in z-plane.

Bilinear transformation (BLT): It is obtained by using the trapezoidal formula for numerical integration.

consider analog system

$$y(t) = \frac{dx(t)}{dt} \longrightarrow (1)$$

$$x(t) \longrightarrow \boxed{s} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

$$H(s) = \frac{Y(s)}{X(s)} = s \longrightarrow (2)$$

(1) is now integrated b/w. the limitits.

$$\int_{nT-T}^{nT} y(t)\,dt = \int_{nT-T}^{nT} \frac{dx(t)}{dt}\,dt$$

$$\int_{nT-T}^{nT} y(t)\,dt = x(nT) - x(nT-T) \longrightarrow (3)$$

using trapezoidal rule on LHS of (3)

$$\frac{T\,[y(nT) + y(nT-T)]}{2} = x(nT) - x(nT-T) \longrightarrow (4)$$

using $x(nT) = x(n)$ in (4).

$$\frac{T}{2}[y(n) + y(n-1)] = x(n) - x(n-1) \longrightarrow (5)$$

Applying z transform

$$\frac{T}{2}[Y(z) + z^{-1}Y(z)] = X(z) - z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \longrightarrow (6)$$

Comparing (2) & (6)

$$s \longrightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\Omega = \frac{2}{T}\left[\frac{z-1}{z+1}\right] = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]$$

$$= \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega) + 1}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\left.\begin{aligned}\sigma &= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] \\ \Omega &= \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]\end{aligned}\right\} \rightarrow$$

1. The system function of analog filter is given by $\frac{1}{(s+1)(s+2)}$. Find $H(z)$ using IIT method take sampling frequency as 5 samples/sec.

$$H(s) = \frac{1}{(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

~~$H(s) = A(s+2) + B(s+1)$~~

$A = (s+2)H(s)|_{s=-1}$ $\quad\quad$ $B = (s+1)H(s)|_{s=-2}$

$= 1$ $\quad\quad$ $B = -1$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

using IIT mapping.

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T}z^{-1}}$$

$F = 5$

$$T = \frac{1}{F} = \frac{1}{5} = 0.2\ \text{sec}$$

$$\frac{1}{s+1} \rightarrow \frac{1}{1-e^{-1\times0.2}z^{-1}}$$

$$\frac{1}{s+2} \rightarrow \frac{1}{1-e^{-2\times0.2}z^{-1}}$$

$$H(z) = \frac{1}{1-0.818z^{-1}} - \frac{1}{1-0.67z^{-1}}$$

$$= \frac{z}{z-0.818} - \frac{z}{z-0.67}$$

$$= z\left[\frac{z-0.67-z+0.818}{z^2-0.818z-0.67z+0.548}\right]$$

$$= z\left[\frac{0.148}{z^2-1.488z+0.548}\right]$$

$$= \frac{0.148z^{-1}}{1-1.488z^{-1}+0.548z^{-2}}$$

2. Find $H(z)$ if $H(s) = \frac{b}{(s+a)^2+b^2}$. using this result find $H(z)$ when $H(s) = \frac{1}{s^2+2s+2}$.

$$H(s) = \frac{b}{(s+a)^2+b^2} = \frac{A}{s-(-a+ib)} + \frac{B}{s-(-a-ib)}$$

$$b = A(s+a+ib) + B(s+a-ib)$$

$s = -a-ib$

$b = B(-a-ib+a-ib) \Rightarrow B = b/-2ib = \frac{b}{b}\cdot = -1/2j$

$s = -a+ib \Rightarrow b = A(-a+ib+a+ib)$

$A = b/2ib$

$= \frac{1}{2j}$

~~using IIT method,~~

$$\cancel{\frac{1}{s-p_k} \longrightarrow \frac{1}{1-e^{p_kT}z^{-1}}}$$

$$H(s) = \frac{1/2}{s+a-ib} - \frac{1/2}{s+a+ib} \longrightarrow ①$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_kT}z^{-1}}$$

$$= \sum_{k=1}^{N} \frac{C_k z}{z-e^{P_kT}}$$

using in ①.

$$H(z) = \frac{1}{2j}\left[\frac{z}{z-e^{-(a+jb)T}} - \frac{z}{z-e^{-(a-jb)T}}\right]$$

$$\cancel{= \frac{z}{2j}\left[\frac{z-\cos(a-jb)T + i\sin(a-jb)T - z + \cos(a+jb)T - i\sin(a+jb)}{z^2 - e^{-(a+jb)T}z - e^{-(a-jb)T}z}\right]}$$

$$= \frac{z}{2j}\left[\frac{z - e^{-(a-jb)T} - z + e^{-(a+jb)T}}{z^2 - (e^{-(a+jb)T} + e^{-(a-jb)T})z + e^{-2aT}}\right]$$

$$= \frac{z}{2j}\left[\frac{\cancel{\cos(a+jb)T}\ e^{-aT}\sin bT\, z}{z^2 - 2e^{-aT}\cos bT\, z + e^{-2aT}}\right]$$

$$H(z) = \frac{e^{-aT}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}z^{-2}}$$

$$H(s) = \frac{1}{s^2+2s+2}$$

$$= \frac{1}{(s+1)^2+1^2}$$

3. Find $H(z)$ using IIT. Assume i) $H(s) = \frac{2}{(s+1)(s+2)}$.

ii) $H(s) = \frac{10}{s^2+7s+10}$.

4. convert the analog filter $H(s) = \frac{2}{(s+1)(s+3)}$ into a digital filter using BLT. Take $T = 0.1$ sec.

Mapping of $s \to z$.

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\cancel{H(z)} = \frac{2}{0.1}\left[\frac{z-1}{z+1}\right].$$

$$s = 20\left[\frac{z-1}{z+1}\right].$$

$$H(z) = \frac{2}{20\left[\frac{z-1}{z+1}+1\right] \times 20\left[\frac{z-1}{z+1}+3\right]}$$

$$= \frac{2(z^2+z+1)}{20(z-1+z+1) \times 20(z-1+3z+3)}.$$

$$= \frac{2(z+2)}{40z \times (80z+40)}.$$

$$= \frac{2z+2}{3200z^2+1600z}$$

$$= \frac{2(z^{-1}+z^{-2}+1)}{1600z^{-1}+3200}$$

$$H(z)= \frac{2}{\left[\frac{20z-20}{z+1}+1\right]\times\left[\frac{20z-20}{z+1}+3\right]}$$

$$= \frac{2(z^2+2z+1)}{(20z-20+z+1)\times(20z-20+3z+3)}$$

$$= \frac{2z^2+4z+2}{(21z-19)\times(23z-17)}$$

$$= \frac{2z^2+4z+2}{483z^2-794z+323}$$

$$= \frac{2(1+2z^{-1}+z^{-2})}{323z^{-2}-794z^{-1}+483}$$

2. Convert 2nd order normalized filter to digital filter using BLT. Take T= 1sec.

$$H(s)= \frac{1}{s^2+\sqrt{2}s+1} = \frac{1}{s^2+1.414s+1}$$

$$s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$s \to \frac{2}{1}\left[\frac{z-1}{z+1}\right]$$

$$H(z)= \frac{1}{\left(\frac{2z-2}{z+1}\right)^2+1.414\left(\frac{2z-2}{z+1}\right)+1}$$

$$= \frac{\cancel{(z+1)^2}}{4z^2+4-8z\cancel{+2.828z-2.828+z^2+}}$$

$$= \frac{\cancel{z^2+2z+1}}{\cancel{4z^2-5.172z+2.172}}$$

$$= \frac{\cancel{1+2z^{-1}+z^{-2}}}{\cancel{4-5.172z^{-1}+2.172z^{-2}}}$$

$$= \frac{\cancel{(1+z^{-1})^2}}{\cancel{4-5.172z^{-1}+2.172z^{-2}}}$$

$$= \frac{(z+1)^2}{4z^2+4-8z+(2.828z-2.828)(z+1)+(z^2+2z+1)}$$

$$= \frac{(z+1)^2}{4z^2+4-8z+2.828z^2+2.828z-2.828z-2.828+z^2+2z+1}$$

$$= \frac{(z+1)^2}{7.828z^2-6z+2.172}$$

$$= \frac{(1+z^{-1})^2}{2.172z^{-2}-6z^{-1}+7.828}$$

3. Transform analog filter $H(s)= \frac{s+0.1}{(s+0.1)^2+9}$ into a digital filter using BLT. The digital filter should have resonant frequency $\omega_r = \pi/4$.

$$(s+0.1)^2+9 = (s+0.1-3j)(s+0.1+3j)$$

$$s = -0.1 \pm 3j$$

$$= \sigma \pm j\Omega$$

$$\Omega = 3 \text{ rad/sec.}$$

$$\omega_r = \pi/4.$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$T = \frac{2}{\Omega}\tan\frac{\omega}{2} = 0.276 \text{ sec.}$$

$\frac{2}{T} = 7.242$

$$H(Z) = \frac{7.24\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{\left(7.24\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1\right)^2 + 9}$$

$$= \frac{[7.24(z-1) + 0.1(z+1)](z+1)}{[7.24(z-1) + \frac{0.1}{(z+1)}]^2 + 9(z+1)^2}$$

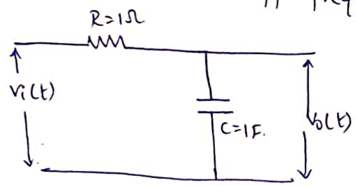
$$= \frac{[7.24z - 7.24 + 0.1z + 0.1](z+1)}{[7.24(z-1) + 0.1(z+1)]^2 + 9(z+1)^2}$$

$$= \frac{7.24z^2 - 7.24z + 0.1z^2 + 0.1z + 7.24z - 7.24 + 0.1z + 0.1}{(7.24z - 7.24 + 0.1z + 0.1)^2 + 9(z+1)^2}$$

$$= \frac{7.34z^2 + 0.2z - 7.14}{\cancel{52.41762^2}\ 53.8756z^2 + 50.9796 + 9z^2 + 9 + 18z}$$

$$= \frac{7.34z^2 + 0.2z - 7.14}{62.8756z^2 + 18z + 59.9796} = \frac{7.34 + 0.2z^{-1} - \cancel{7.14}z^{-2}}{59.9796z^{-2} + 18z^{-1} + 62.8[?]}$$

4. Obtain digital filter equivalent of the analog filter shown in fig using a) IIT b) BLT. Assume sampling frequency $F_s = 8F_c$ where F_c is cutoff frequency of the filter.



$$V_o(t) = \frac{X_C}{R + X_C} V_i(t).$$

$$\frac{V_o(t)}{V_i(t)} = \frac{X_C}{R + X_C}$$

Taking laplace transform.

$$H(s) = \frac{1/s}{1 + 1/s} = \frac{1}{s+1}.$$

cutoff frequency $F_C = \frac{1}{2\pi RC}$.

$$F_S = 8F_C = \frac{4}{\pi RC}.$$

$$F_S = 1.27\ Hz.$$

$$T = 0.785\ sec.$$

a) $\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}.$

$$H(z) = \frac{1}{1 - e^{-1 \times 0.785} z^{-1}}.$$

$$= \frac{1}{1 - 0.455z^{-1}}.$$

b) $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$$= \frac{2}{0.785}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{1}{2.547\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

$$= \frac{(1+z^{-1})}{2.547 - 2.547z^{-1} + 1 + z^{-1}} = \frac{1+z^{-1}}{3.547 - 1.547z^{-1}} = \frac{0.282(1+z^{-1})}{1 - 0.436z^{-1}}$$

Digital Butterworth design:

1. Design a butterworth filter using BLT for the following specifications.

$0.8 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le 0.2\pi$

$|H(\omega)| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$

$A_p = 0.8$ (Take $\frac{2}{T} = 1$ if not given)

$A_s = 0.2$

Step-1: To obtain specifications of corresponding analog filter.

$A_p = 0.8 \qquad A_s = 0.2$

$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} \qquad \Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2}$

$= \tan\left(\frac{0.2\pi}{2}\right) \qquad = \tan \frac{0.6\pi}{2}$

$\Omega_p = 0.325 \; r/s \qquad \Omega_s = 1.376 \; r/s$

Step-2: To determine order N of the filter

$$N = \frac{\log \left[\frac{1}{A_s^2} - 1\right] / \left[\frac{1}{A_p^2} - 1\right]}{2 \log \frac{\Omega_s}{\Omega_p}}$$

$$= \frac{\log \left[{}^{16}/_{9}\right] / \left[{}^{1}/_{24}\right]}{2 \log \left(\frac{1.376}{0.325}\right)}$$

$$= \frac{1.63}{1.25}$$

$= 1.3$

$N \approx 2$.

Step-3: To determine Ω_c.

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2}.$$

$$\Omega_{cp} = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}} = \frac{0.325}{\left(\frac{1}{0.8^2} - 1\right)^{1/4}} = 0.375 \; r/s$$

$$\Omega_{cs} = \frac{\Omega_s}{\left(\frac{1}{A_s^2} - 1\right)^{1/2N}} = \frac{1.376}{\left(\frac{1}{0.2^2} - 1\right)^{1/4}} = 0.621 \; r/s$$

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2} = 0.5 \; r/s.$$

Step-4: Finding system function of normalized LPF.

$P_k = \pm \Omega_c e^{j(2k+1+N)\pi/2N}$

$N = 2. \qquad \Omega_c = 1 \; r/s.$

$P_0 = \pm(-0.707 + j0.707)$

$P_1 = \pm(-0.707 - j0.707)$

$$H(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2 + 1.414s + 1}$$

Step-5: To find the required H(s) using frequency transformation.

$LP \to LP : s \to \frac{\Omega_p \cdot s}{\Omega_{cp}} \Rightarrow \frac{s}{0.5}$

$$H(s) = H_1(s)\Big|_{s \to \frac{s}{0.5}} = \frac{0.25}{s^2 + 0.707s + 0.25}$$

Step-6: Finding H(z) by applying BLT.

$$s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$s \to \left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = H(s)\Big|_{s \to \left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.25}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.707\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.25}$$

$$= \frac{0.25\,(1+z^{-1})^2}{1+z^{-2}-2z^{-1}+0.707-0.707z^{-2}+0.25+0.25z^{-2}+0.5z^{-1}}$$

$$= \frac{0.25\,(1+z^{-1})^2}{0.543z^{-2}-1.50z^{-1}+1.957}$$

$$H(z) = \frac{0.127\,(1+z^{-1})^2}{1-0.766z^{-1}+0.277z^{-2}}$$

Digital LPF is required to meet the following specifications

a) acceptable pass band attenuation -1.9328 dB.

b) Pass band edge frequency of $\omega_p = 0.2\pi$ rad.

c) Stop band attenuation of -13.9794 dB or higher, beyond $\omega_s = 0.6\pi$ rad.

The filter must have maximally flat frequency response in the pass band. Find $H(z)$ using BLT.

Step-1: To obtain specifications of corresponding analog filter. Take $T = 1$ sec

$$\Omega = \frac{\omega}{T}$$

$$\Omega_p = \frac{\omega_p}{T} = 0.2\pi \qquad \Omega_s = \frac{\omega_s}{T} = 0.6\pi$$

$$A_p = -1.9328 \qquad A_s = -13.9794$$

$$N = \frac{\log\left[\frac{10^{0.1A_s(dB)}-1}{10^{0.1A_p(dB)}-1}\right]}{2\log\frac{\Omega_s}{\Omega_p}} = \frac{1.6315}{0.9542}$$

$$= 1.7$$

$$N = 2$$

Step-3: $\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2}$

$$= \frac{1}{2}\left[\frac{\Omega_p}{(10^{0.1A_p}-1)^{1/2N}} + \frac{\Omega_s}{(10^{0.1A_s}-1)^{1/2N}}\right]$$

$$= \frac{1}{2}\left[\frac{0.2\pi}{0.865} + \frac{0.6\pi}{0.784}\right]$$

$$= 0.726$$

Step-4: To determine

$$P_k = \pm\Omega_c e^{j(2k+1+N)\pi/2N}$$

$N = 2$. $\Omega_c = 1$.

$$P_0 = \pm(-0.707 + j0.707)$$

$$P_1 = \pm(-0.707 - j0.707)$$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2+1.414s+1}$$

Step-5: To find required $H(s)$ using frequency transformation.

LP $\to$ LP

$$s \to \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{0.726}$$

$$H(s) = H_1(s)\Big|_{s \to \frac{s}{0.726}}$$

$= \dfrac{0.526}{s^2+1.02s+0.526}$

$= \dfrac{1.02 \times 0.513}{(s+0.513)^2+(0.513)^2}$

Step-6: Finding H(z) by applying IIT

$$\frac{b}{(s+a)^2+b^2} = \frac{e^{-aT}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$$

$a = 0.513 \qquad b = 0.513$

$$= \frac{0.293z^{-1}}{1-1.043z^{-1}+0.358z^{-2}}$$

Apply IIT to the analog transfer function given by.

~~$H(S) = \dfrac{0.293z^{-1}}{1-1.043z^{-1}+0.358z^{-2}}$~~

$$H(s) = \frac{s^2+4.525}{s^2+0.692s+0.5021}$$

with T=1 sec.

$$H(s) = \frac{s^2+4.525}{s^2+0.692s+0.502} = \frac{s^2+4.525}{(s-0.346)^2+(0.618)^2}$$

$-0.346+0.618j$

$-0.346-0.618j$

$$H(s) = \frac{s^2+4.525}{(s-0.346+0.618j)(s-0.346-0.618j)} = \frac{A}{s-0.346+0.618j} + \frac{B}{s-0.346-0.618j}$$

$s^2+4.525 = A(s-0.346-0.618j) + B(s-0.346+0.618j)$

$s = 0.346+0.618j$

$4.2627+0.4276i = B(1.236j)$

$B = 0.346-3.448j$

Chebyshev filter design:

Two types:

i) Chebyshev -I

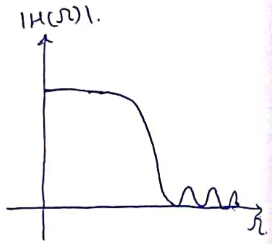
ii) Chebyshev -II

i) Chebyshev-I : *All pole filter

*Ripples in the pass band.

*Stepper roll off than the butterworth filter.

ii) Chebyshev - II :

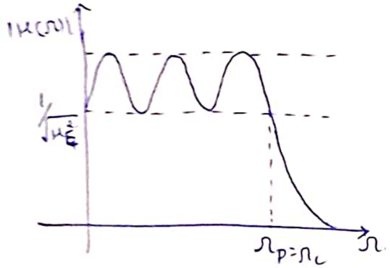


*Pole zero filter.

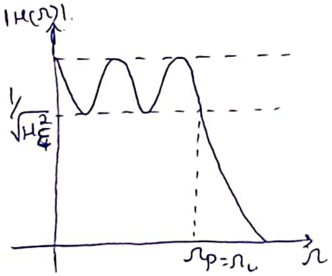
*Doesn't roll-off so faster than chebyshev -I

*Usually called as inverse chebyshev

Chebyshev-I filters: These filters are all pole filters. In the pass band these filters show equiripple behaviour and have montonic characteristics in the stop band.



N = odd.



N = even.

Chebyshev polynomial:

$C_N(\Omega) = \cos(N\cos^{-1}(\Omega))$ for $|\Omega| \le 1$

$C_N(\Omega) = \cosh(N\cosh^{-1}(\Omega))$ for $|\Omega| > 1$.

For $N=0 \Rightarrow C_0(\Omega) = \cos 0 = 1$

$N=1 \Rightarrow C_1(\Omega) = \cos(\cos^{-1}(\Omega)) = \Omega$

Higher order chebyshev polynomials are obtained using the recursive formula.

$C_N(\Omega) = 2\Omega C_{N-1}(\Omega) - C_{N-2}(\Omega)$

1. Find chebyshev polynomial for $N = 2, 3, 4, 5$

$C_N(\Omega) = 2\Omega C_{N-1}(\Omega) - C_{N-2}(\Omega)$.

$N=2 \Rightarrow C_2(\Omega) = 2\Omega C_1(\Omega) - C_0(\Omega)$

$= 2\Omega^2 - 1$.

$N=3 \Rightarrow 2\Omega C_2(\Omega) - C_1(\Omega)$

$C_3(\Omega) = 4\Omega^3 - 2\Omega - \Omega$.

$= 4\Omega^3 - 3\Omega$.

$N=4 \Rightarrow C_4(\Omega) = 8\Omega^4 - 6\Omega^2 - 2\Omega^2 + 1$.

$= 8\Omega^4 - 8\Omega^2 + 1$.

$N=5 \Rightarrow C_5(\Omega) = 16\Omega^5 - 16\Omega^3 + 2\Omega - 4\Omega^3 + 3\Omega$.

$= 16\Omega^5 - 20\Omega^3 + 5\Omega$

Observations: For $\Omega >> 1$ first term dominate

$C_N(\Omega) = 2^{N-1}\Omega^N$

Magnitude function of a chebyshev filter is given by.

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega)} \longrightarrow A$$

where ϵ = Ripple factor

$C_N(\Omega)$ = chebyshev polynomial of order N.

for normalised filter $\Omega_p = \Omega_c = 1$ rad/sec

Observation: a) For $|\Omega| \le 1$, ripple in the pass band $= 1 - \frac{1}{\sqrt{1+\epsilon^2}}$

b) At $\Omega = 1$, $C_N^2(\Omega) = 1$ [$C_N = 1$ always).

at $\Omega = 1$, (A) can be written as

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2} \quad (\text{or})$$

$$|H(\Omega)| = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$\therefore A_p = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1} \longrightarrow (B)$$

for $\Omega >> 1$. $\epsilon^2 C_N^2(\Omega) >> 1$.

$$|H(\Omega)|^2 = \frac{1}{\epsilon^2 C_N^2(\Omega)}$$

$$|H(\Omega)| = \frac{1}{\epsilon C_N(\Omega)}$$

$$|H(\Omega)| \text{ in dB} = 20\log_{10} 1 - 20\log_{10}(\epsilon C_N(\Omega))$$

$$= 0 - 20 \log_{10} [\epsilon\, 2^{N-1} \Omega^N]$$

$$= -20\log \epsilon - 20(N-1)\log_{10} 2 - 20N \log \Omega.$$

$\Omega >> 1 \Rightarrow$ stop band $\therefore \Omega = \Omega_s$

$$|H(\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 20(N-1)\log_{10} 2 - 20N\log_{10} \Omega_s'$$

$$= -20\log_{10}\epsilon - 6(N-1) - 20N\log_{10}\Omega_s'$$

Ω_s' = Normalized stop band edge frequency.

Poles of $H(s)$

$$s_k = \sigma_k + j\Omega_k.$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \cos\left(\frac{2k-1}{2N}\right)\pi.$$

$$k = 1, 2, \ldots N.$$

system function $H(s)$ of chebyshev filter

$$H(s) = \frac{k}{(s-s_1)(s-s_2)\ldots(s-s_N)}$$

$$= \frac{k}{s^N + b_{N-1}s^{N-1} + \ldots b_0}$$

constant k is given by

$$k = \begin{cases} b_0 & \text{for } N \text{ odd.} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & \text{for } N \text{ even.} \end{cases}$$

1) Design analog chebyshev filter to meet following specifications.

passband ripple = -1 dB $\quad 0 \le \Omega \le 10$ r/s.

stop band attenuation = -60 dB $\quad \Omega \ge 50$ r/s.

$\Omega_p = 10$ r/s. $\quad A_p = -1$ dB

$\Omega_s = 50$ r/s. $\quad A_s = -60$ dB

Step-1: Normalized specifications:

$$\Omega_p' = \frac{\Omega_p}{\Omega_p} = 1 \text{ r/s.} \qquad A_p = 1\text{dB.}$$

$$A_s = -60\text{dB}$$

$$\Omega_s' = \frac{\Omega_s}{\Omega_p} = \frac{50}{10} = 5 \text{ r/s.}$$

Step-2: To determine ϵ

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1}$$

$$= \cancel{\sqrt{\frac{1}{1} - 1}} \quad \sqrt{10^{0.1 A_p(\text{in dB})} - 1} = 0.508$$

Step-3: To determine order N

$|H(\Omega)|$ in dB $= -20\log_{10}\epsilon - 6(N-1) - 20\log\Omega_s' \times N$

$-60 = -20\log(0.5) - 6(N-1) - 20\times N\log 5.$

$20N\log 5 + 6(N-1) = 66.02.$

$20N\log 5 + 6N = 72.02.$

$19.97N = 72.02.$

$N = 3.6 \approx 4.$

Step-4: $H_1(s) = \dfrac{K}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$

$s_k = \sigma_k + j\Omega_k.$

$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$

$\cancel{\sigma_0 = -0.368\sin\left(\frac{-1}{8}\pi\right) = 0.14}$

$\sigma_1 = -0.368\sin\left(\frac{\pi}{8}\right) = -0.141.$

$\sigma_2 = -0.368\sin\left(\frac{3\pi}{8}\right) = -0.339$

$\sigma_3 = -0.368\sin\left(\frac{5\pi}{8}\right) = -0.339.$

$\sigma_4 = -0.368\sin\left(\frac{7\pi}{8}\right) = -0.14.$

$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi.$

$\cancel{\Omega_0 = 1.06\cos\left(\frac{-\pi}{8}\right) = 0.98.}$ $\qquad \Omega_4 = -0.98$

$\Omega_1 = 1.06\cos\left(\frac{\pi}{8}\right) = 0.98$

$\Omega_2 = 1.06\cos\left(\frac{3\pi}{8}\right) = 0.40$

$\Omega_3 = 1.06\cos\left(\frac{5\pi}{8}\right) = -0.40.$

$\cancel{s_0 = 0.14 + 0.98j}$

$s_1 = -0.14 + 0.98j$

$s_2 = -0.33 + 0.40j$

$s_3 = -0.34 - 0.40j.$

$s_4 = -0.14 - 0.98j$

$H_1(s) = \dfrac{K.}{(s-(-0.14+0.98j))(s-(-0.34+0.40j))(s-(-0.34-0.40j))(s-(-0.14-0.98j))}.$

$= \dfrac{K.}{[(s+0.14)^2 + 0.98^2][(s+0.34)^2 + (0.40)^2]}.$

$= \dfrac{K.}{(s^2 + 0.28s + 0.98)(s^2 + 0.68s + 0.2756)}$

$b_0 = 0.98 \times 0.2756 = 0.27.$

$N = \text{even} \Rightarrow K = \dfrac{b_0.}{\sqrt{1+\epsilon^2}} = \dfrac{0.27.}{\sqrt{1+0.5^2}.} = 0.241.$

$H_1(s) = \dfrac{0.241}{(s^2+0.28s+0.98)(s^2+0.68s+0.2756)}.$

Step-5: Apply frequency transformation:

LP $\rightarrow$ LP.

$s \rightarrow \dfrac{\Omega_p}{\Omega_{cp}} = \dfrac{s}{10}.$

$H(s) = H_1(s)\Big|_{s=\frac{s}{10}} = \dfrac{0.241}{\left(\left(\frac{s}{10}\right)^2 + \frac{0.28s}{10} + 0.98\right)\left(\left(\frac{s}{10}\right)^2 + \frac{0.68s}{10} + 0.2756\right)}$

$= \dfrac{24.1}{(s^2+2.8s+98)(s^2+6.8s+27.56)}$

5 2. Design an analog chebyshev filter to meet the following specifications

$A_p = 2.5\,dB$ $\quad \Omega_p = 20\ r/s$

$A_s = -30\,dB$ $\quad \Omega_s = 50\ r/s.$