UNIT - 5

Canonical Forms

Similarity of Linear transformation

$\alpha \to V$

$T(\alpha) \Rightarrow \in V$

$T(\alpha) : V \to U$

T^{-1} is present $T^{-1}(\alpha) : U \to V$

Speciality of LT

$T(\alpha) : V \to V$

Invarient subspace

$T : V \to V$

W is subspace of V

$\alpha \in W$

$\forall \alpha \in W \Rightarrow T(\alpha)\ TW \to V$

Theorem:- If W is a subspace under $T \in A(V)$. then T induces a linear transformation T_q on the co-efficient space V divided by W defined by $T_q(\alpha + W) = T(\alpha) + W$. Further if T satisfies the polynomial $q(x) \in F(x)$ then sof this T_q thus the minimal polynomial of T_q divided by minimal polynomial of T

$V/W \Rightarrow$ Quotient space

$W \Rightarrow$ subspace.

$\Rightarrow$

$$T_q(\alpha + W) = T(\alpha) + W$$

$$q(x) \in F(x)$$

(i) T_q

$\alpha + W$ & $\beta + W \in V/W$

$\alpha+w = \beta+w$

$\Rightarrow \alpha-\beta \in w$

$T(\alpha-\beta) = T(\alpha)-T(\beta) \in w$

$T(\alpha)+w = T(\beta)+w$

$T_q(\alpha+w) = T_q(\beta+w)$

(ii) T_q is linear transformation

$\alpha+w,\ \beta+w \in w$

$T_q\{(\alpha+w)+(\beta+w)\} = T_q(\alpha+\beta+w)$

$= T(\alpha+\beta)+w$

$= T(\alpha)+T(\beta)+w$

$= T_q(\alpha+w)+T_q(\beta+w)$

$T_q\{c(\alpha+w)\} = T_q[c(\alpha+w)]$

$= T(c\alpha)+w = cT(\alpha)+w$

$= cT_q(\alpha+w)$

T_q is linear.

$T_q^n = (T_q)^n \quad \forall n \geq 0$

$\alpha+w \in V/w$

$T_q^2(\alpha+w) = T^2(\alpha)+w$

$= T_q(T_q(\alpha+w))$

$= T_q^2(\alpha+w)$

$(T_q^n) = (T_q)^n \quad \forall n \geq 0$

$q(x) \in F(x)$

$q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$q\,T_q(\alpha + \omega) = q\,T(\alpha) + \omega$

$= a_n T^n(\alpha) + a_{n-1} T^{n-1}(\alpha) + \dots a_0\, T(\alpha) + \omega$

$= \Sigma \alpha_i\, T_q^i(\alpha) + \omega.$

$= \Sigma \alpha_i\, T^i(\alpha) + \omega$

$= a_i\, \Sigma\, (T_q)^i(\alpha + \omega) = q\,(T_q)(\alpha + \omega)$

T_q is root of $q(x) = 0$

Invarient Direct sum decomposition

$T: V \rightarrow V$

$\omega_1, \omega_2 \dots \omega_n$ is subspace in V.

$V = \omega_1 \oplus \omega_2 \oplus \dots \oplus \omega_n$

T

If $V = \omega_1 + \omega_2 \dots + \omega_n$ where n_i is dimension of each subspace ω_i and every subspace is invarient $T \in A(V)$ then the basis of V can be found so that the matrix of T is basis of the form

$$\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & & \\ 0 & 0 & 0 & A_n \end{bmatrix}$$ where each A_i is $n \times n$ matrix of linear transformation induced by T on ω_i

Proof :- Let $\{\alpha_1^{(1)}, \alpha_2^{(1)} \dots \alpha_n^{(1)}\} \rightarrow \omega_1$

$\{\alpha_1^{(2)}, \alpha_2^{(2)} \dots \alpha_n^{(2)}\} \rightarrow \omega_2 \quad \omega_1, \omega_2 \dots \omega_n$

$\{\alpha_1$

$q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$q T_q(\alpha + \omega) = q T(\alpha) + \omega$

$= a_n T^n(\alpha) + a_{n-1} T^{n-1}(\alpha) + \dots a_0 T(\alpha) + \omega$

$= \Sigma \alpha_i T_q^i(\alpha) + \omega.$

$= \Sigma \alpha_i t^i(\alpha) + \omega$

$= a_i \Sigma (T_q)^i(\alpha + \omega) = q(T_q)(\alpha + \omega)$

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Invarient Direct sum decomposition

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proof :- Let $\{\alpha_1^{(1)}, \alpha_2^{(1)} \dots \alpha_n^{(1)}\} \to \omega_1$

$\{\alpha_1^{(2)}, \alpha_2^{(2)} \dots \alpha_n^{(2)}\} \to \omega_2$ $\omega_1, \omega_2 \dots \omega_n$

$\{\alpha_1$

since $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$

W_i is T-invariant

$T(\alpha_j^i) \in W_i$

$T(\alpha_j)^{(i)} \in W_i = a_1^{(i)} \alpha_1^{(i)} + a_2^{(i)} \alpha_2^{(i)}$ 3

T matrix with respect to basis V is

$$\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_r \end{bmatrix}$$

Important Canonical form for checking → similarity of two linear transformation

1) Normal form:-

A → Normal form $A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

I_r → Square identity matrix order r.

r → rank of matrix.

theorem:-

Let T be the LT from U → V and rank of T = r, then there exists a basis of U and such that the matrix respresentation of T has the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

proof :-

Let dim U = m, dim V = n and w be the kernel space of T

Rank of T is r

$m = (r) + (N)$

(r) ↳ rank, (N) ↳ Dimension of null space

$N = m - r$

vectorspace basis $\rightarrow \{v_1, v_2, v_3 \ldots v_n, \alpha_1, \alpha_2 \ldots \alpha_{m-r}\}$

$T(v_i) = u_i$ $(u_1, u_2 \ldots u_m)$ basis of image (T)

$T(\alpha_i) \neq 0$

$\{u_1, u_2 \ldots u_n\} \rightarrow T(v_1) = u_1$ $\quad T(\alpha_1) = u_r$

$T(v_2) = u_2$ $\quad T(\alpha_2) = u_{r+1}$

$T(v_3) = u_3$ $\quad T(\alpha_3) = u_{r+3}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

2) Triangular form :-

$T: V \rightarrow V$ $\qquad T \in A(V)$

Theorem:- If T belongs to $A(V)$ as all x charc root in F then there is a basis of V in which matrix representation V is triangular.

Proof:- dim $V = 1$

dim of basis 1×1

Trivially triangular

functions

vector space $(n-1)$ $\qquad$ dim $V = n > 1$

$\lambda_1 \in F$

$T(\alpha) \rightarrow$ characteristic eq^n

$T(\alpha_1) = a_{11}\alpha_1$

$T(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2$

$\vdots$

$T(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 \ldots + a_{nn}\alpha_n$

W - subspace $\rightarrow V$

Quotient space $T_q = V/W$

$\dim T_q = \dim V - \dim W$
$= n-1$

T induces transformation T_q on V_q
minimal polynomial
char eq^n
$(\lambda-1)(\lambda-2)^2$ 3
$\lambda \to A$
$(A-1)(A-2) \neq 0 \Rightarrow$ Not minimal.
$(A-1)(A-2)^2 = 0 \Rightarrow$ minimal

$\dim V_q = n-1$
B min $\{\bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4 \ldots \bar{\alpha}_n\}$
$T_q(\bar{\alpha}_2) = a_{22}\bar{\alpha}_2$
$T_q(\bar{\alpha}_3) = a_{22}\bar{\alpha}_2 + a_{33}\bar{\alpha}_3$
$\vdots$
$T_q(\bar{\alpha}_n) = a_{n2}\bar{\alpha}_2 + \ldots a_{nn}\bar{\alpha}_n$

$T_q(\bar{\alpha}_1) = a_{22}\bar{\alpha}_2$
$T_q(\alpha_2 + \omega) = a_{22}(\alpha_2 + \omega)$
$T(\alpha_1) = a_{11}\alpha_1$
$T(\alpha_n) = a_{n1}\alpha_1 + \ldots a_{nn}\alpha_n.$

Theorem

If $\dim V = n$ and $T \in A(V)$ has all its char roots in F then T satisfies a polynomial degree of n over F

proof:- Let $\{\lambda_1, \lambda_2, \ldots \lambda_p\} \in F \to$ char roots of T

$\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ basis $\in V$
$T(\alpha_1) = \lambda_1 \alpha_1$
$T(\alpha_2) = a_{21}\alpha_1 + a_{32}\alpha_2 + \lambda_3\alpha_3$
$\vdots$
$T(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + \lambda_n\alpha_n$

$(T-\lambda_1 I)\alpha_1 = 0$

$(T-\lambda_2 I)\alpha_2 = 0$

$\vdots$

$(T-\lambda_n I)\alpha_n = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots a_{n(n-1)}\alpha_{(n-1)}$

$(T-\lambda_2 I)(T-\lambda I)\alpha_2 = 0$

$(T-\lambda_1 I)(T-\lambda_{n-1} I)(T-\lambda_{n-2} I)\ldots = 0$

Annihilator of basis of V

$S=0 \Rightarrow (T-\lambda_n I)(T-\lambda_{n-1} I)\cdots = 0$

~~Theorem~~:- problems:-

1) W is an invarient subspace under $S: V \to V$ and $T: V \to V$. Show that W is invarient under $S+T$ & ST

$\Rightarrow$ As W is S invarient and T invarient subspace

Let $\alpha \in W$ then $(S+T)\alpha = S(\alpha) + T(\alpha)$

where $S(\alpha), T(\alpha) \in W$

$S(\alpha) + T(\alpha) \in W$

$(S+T)(\alpha) \in W$

Also

$ST(\alpha) = S(T(\alpha))$

$T(\alpha) \in W$ is invarient under S

$S(T(\alpha)) \in W$

$ST(\alpha) \in W$.

Nilpotent transformation

$T: V \to V$

$T(\alpha) \in V.$

:- A linear transformation $T: V \to V$ is said to be nilpotent if $T^n = 0$ for some least +ve integer n.

3

T = (Addition) or (Multiplication)

$T^n \to T(\alpha) = 0$

$n \in R \quad T^n = 0$

$T^{n-1} \neq 0$

n - index of nilponency.

Ex: $T^3 \neq 0$

$\boxed{n \geq 3}$

Jordan canonical form

The matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & & \\ 0 & \lambda & 1 & \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

is called jordan block matrix belonging to λ.

In this matrix the λ's are on diagonal and 1's are on the super diagonal and other elements are equal to zero.

Theorem

Let $T: V \to V$ is a linear operator whose char and minimal polynomial are respectively given by.

$$\Delta(x) = (x-\lambda_1)^{n_1} (x-\lambda_2)^{n_2} \dots (x-\lambda_r)^{n_r}$$

$$m(x) = (x-\lambda_1)^{m_1} (x-\lambda_2)^{m_2} \dots (x-\lambda_r)^{m_r}$$

where λ_i are different scalars the T has a block diagonal matrix representation

$$T = \begin{bmatrix} J_1 & - & - \\ - & J_2 & - \\ - & - & J_r \end{bmatrix}$$

For each λ_i the corresponding J block have the following properties.

(i) There is atleast one J_i of order m and all other J_i of order $\leq m$.

ii) Sum of the orders of J_i is n_i

iii) the number of J_i is equals the Geometric multiplicity of λ_i

(iv) The no of J_i of each possible order is uniquely determined by T

proof:- primary decomposition.

$$T = T_1 \oplus T_2 \oplus T_3$$

$(x-\lambda_i)^{m_i} \rightarrow$ minimal polynomial

$(T_i - \lambda_i I)^{m_i} = 0$ $i = 1, 2, 3 \ldots$

$N_i = T_i - \lambda_i I$

$T_i = N_i + \lambda_i I \Rightarrow N_i^{m_i} = 0$

canonical form as

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 & \\ & & & & \ldots & \end{array}\right] \ldots$$

$T_i = N_i + \lambda_i I \rightarrow$ Reduced to different blocks of different size.

Rational canonical form

minimal polynomial can be represented as product of linear Polynomial.

$T: V \to V$

$b_1(x) = q_1(x)^{l_1} \, q_2(x_2)^{l_2} \dots q_k(x)^{l_k}$ 3

$q_k(x) \Rightarrow$ Distinct monic polynomial

$$\begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & C_3 & \\ & & & C_x \end{bmatrix}$$

$C_1 = C_2 =$ Companion matrix.

$V = V_1 \oplus V_2 \oplus V_3$
$\quad\; \downarrow \quad\; \downarrow \quad\; \downarrow$
$\quad\; C_1 \quad C_2 \quad C_3$

problems

1) Determine all invariant subspaces of A where $A = \begin{bmatrix} 2 & -5 \\ 1 & 2 \end{bmatrix}$ viewed as an operator on R^2

(R^2) & $\{0\} \Rightarrow$ Basic invariant subspaces consider

$\Delta x = |xI - A|$

$= (x-2)(x+2) + 5$

$= x^2 + 1$

Clearly A has no eigen values in R so A has no eigen vectors in R^2. Hence $R^2 + 0$ are the only subspace invariant under A.

2) Let matrix A given by $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Show that it is nilpotent and find its index of nilpotency.

$\Rightarrow$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

3) Show that it is nilpotent and find its index of nilpotency also find nilpotent matrix m in canonical form which is similar to A

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A is nilpotent matrix of index 2.

Diagonal matrix.

Nullity of matrix $= 3$

$$A = \begin{bmatrix} m_2 & & \\ & m_2 & \\ & & m_1 \end{bmatrix}$$

$m_2 = 2 \times 2$

$m_1 = 1 \times 1$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine all possible jordan canonical forms for a linear operator whose char polynomial is

$\Delta x = (x-2)^3 (x-5)^2$

(*) Find all possible rational canonical form for 6×6 matrices with minimal polynomial $m(x) = (x+1)^3$

Sol^n:

$V = 6$

(i) $C(x+\ )^3 \oplus C(x+1)^3$

(ii) $C(x+1)^3 \oplus C(x+1)^2 \oplus C(x+1)$

(iii) $C(x+1)^3 \oplus C(x+1) \oplus C(x+1) \oplus C(x+1)$

(iv) $C(1+x^3) = C[x^3+3x^2+3x+1) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$

$C(1+x^2) = C(x^2+2x+1) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

$C(1+x) = (-1)$

(i) $\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right]$

(ii) $\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

iii)

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

2) Let A be a 4×4 matrix with minimal polynomial $m(x) = (x^2+1)(x^3-3)$. Find the rational canonical form for matrix A if A is a matrix over (i) ~~Rational~~ field F (ii) Real field R (iii) a complex field C

Soln

$m(x) = (x^2+1)(x^3-3)$

(i) Rational ~~rational~~ field F

$C(x^2+1) \oplus C(x^2-3)$

$$\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$C(x^2+1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$C(x^2-3) = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$

ii) Real field R

$C(x^2+1) \oplus C(x+\sqrt{3}) \oplus C(x+\sqrt{3})$

$$\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right]$$

iii) complex field

$C(x-i) \oplus C(x+i) \oplus C(x-\sqrt{3}) \oplus C(x+\sqrt{3})$

$$\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{bmatrix}$$