c. $y(t) = C x(t) + \overset{0}{\cancel{D}} u(t)$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} + 2e^{-2t} \\ -1 - 2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$\rightarrow y(t) = \underline{-1 - 2e^{-t} + 2e^{-2t}}$

HW

2 Determine the response for a system described by $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_1$ $+u$
$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i/p is unit step. Also find $y(t)$ $C = [-1 \ \ 1]$

$$\rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

12/12/23 Cannonical forms:-

Minimal components for the realization of transfer function

Controllable cannonical form | Observable cannonical form

Controllability }
Observability } control system

- If a system is fully controllable it can be realized in controllable cannonical form.
- If a system is fully observable it can be realized in observable cannonical form

* Controllable cannonical form:

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{1 \cdot s^3 + a_2 s^2 + a_1 s + a_0} \quad \text{; N}^r \text{ degree} = \text{D}^r \text{ degree}$$

$$\frac{Y(s)}{X_1(s)} \frac{X_1(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{1} \cdot \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\frac{Y(s)}{X_1(s)} = b_3 s^3 + b_2 s^2 + b_1 s + b_0 \rightarrow (1)$$

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \rightarrow (2)$$

DATE

$Y(s) = X_1(s)\left[b_3 s^3 + b_2 s^2 + b_1 s + b_0\right]$

$Y(s) = b_3 s^3 X_1(s) + b_2 s^2 X_1(s) + b_0 X_1(s)$

$y = b_3 \dddot{x}_1 + b_2 \ddot{x}_1 + b_1 \dot{x}_1 + b_0 x_1$ $\quad x_1 \rightarrow$ state variable

Order of the system = 3

$\Rightarrow$ Three state variables

Time	Freq
$x(t)$	$X(s)$
$y(t)$	$Y(s)$
$\frac{d}{dt}$ $\rightarrow$	s
$\frac{d^2}{dt^2}$ $\rightarrow$	s^2
$\frac{dx}{dt} = \dot{x}$;	$\frac{d^2x}{dt^2} = \ddot{x}$

$\dot{x}_2 = x_1 - x_2 + u(t)$

x_1

$x_2 = \dot{x}_1$

$x_3 = \dot{x}_2 = \ddot{x}_1$

$\dot{x}_3 = \dddot{x}_1$

Substituting these in previous eqn, we have

$y = b_3 \dot{x}_3 + b_2 x_3 + b_1 x_2 + b_0 x_1 \rightarrow (3) \rightarrow$ o/p eqn

Consider eqn (2). Crossmultiply $U(s) = X_1(s)\left[s^3 + a_2 s^2 + a_1 s + a_0\right]$

$U(s) = s^3 X_1(s) + a_2 s^2 X_1(s) + a_1 s X_1(s) + a_0 X_1(s)$

$u = \dot{x}_3 + a_2 x_3 + a_1 x_2 + a_0 x_1$

$\dot{x}_3 = u - a_2 x_3 - a_1 x_2 - a_0 x_1 \rightarrow (4)$

Substituting (4) in eqn (3)

$y = b_3(u - a_2 x_3 - a_1 x_2 - a_0 x_1) + b_2 x_3 + b_1 x_2 + b_0 x_1$

$y = (b_2 - a_2 b_3) x_3 + (b_1 - a_1 b_3) x_2 + (b_0 - a_0 b_3) x_1 + b_3 u$

$y = \bar{b}_0 x_1 + \bar{b}_1 x_2 + \bar{b}_2 x_3 + b_3 u \rightarrow (5)$

$\bar{b}_0 = b_0 - a_0 b_3$

$\bar{b}_1 = b_1 - a_1 b_3$

$\bar{b}_2 = b_2 - a_2 b_3$

$\dot{x}_1 = x_2 \rightarrow (6)$

$\dot{x}_2 = x_3 \rightarrow (7)$

$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u \rightarrow (8)$

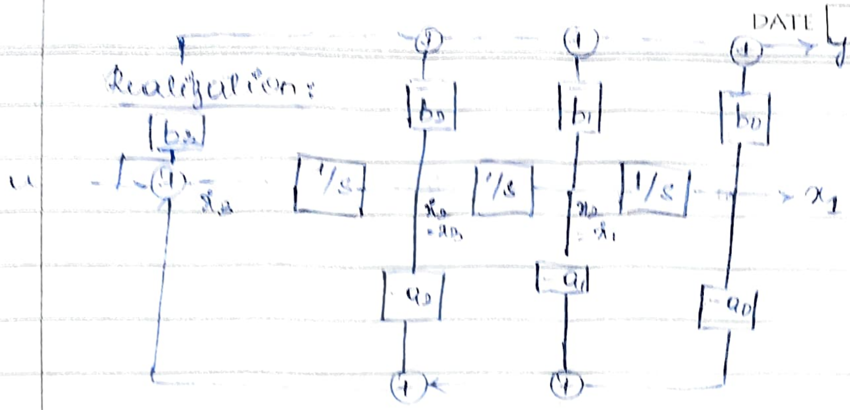
Let us represent state eqns in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{b}_0 & \bar{b}_1 & \bar{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$

Realization

Realization:



Example: $\dfrac{Y(s)}{U(s)} = \dfrac{s^3+12s^2+44s+48}{s^3+9s^2+23s+15} = \dfrac{b_3s^3+b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$

$\dfrac{Y(s)}{U_1(s)} = s^3 + 12s^2 + 44s + 48$

$a_3 = 1;\ a_2 = 9;\ a_1 = 23;\ a_0 = 15$

$b_3 = 1;\ b_2 = 12;\ b_1 = 44;\ b_0 = 48$

$Y(s) = s^3 X_1(s) + 12 s^2 X_1(s) + 44 s X_1(s) + 48 X_1(s)$

$y = \cancel{x}$

$\bar{b}_0 = b_0 - a_0 b_3 = 33$

$\bar{b}_1 = b_1 - a_1 b_3 = 21$

$\bar{b}_2 = b_2 - a_2 b_3 = 3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 33 & 21 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u$$

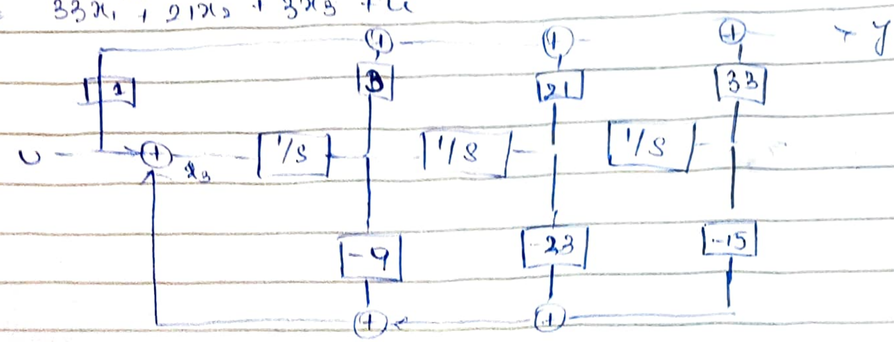
State eq^ns.

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = \cancel{(a_0 x)} - 15x_1 - 23x_2 - 9x_3 + u$

$y = 33x_1 + 21x_2 + 3x_3 + u$

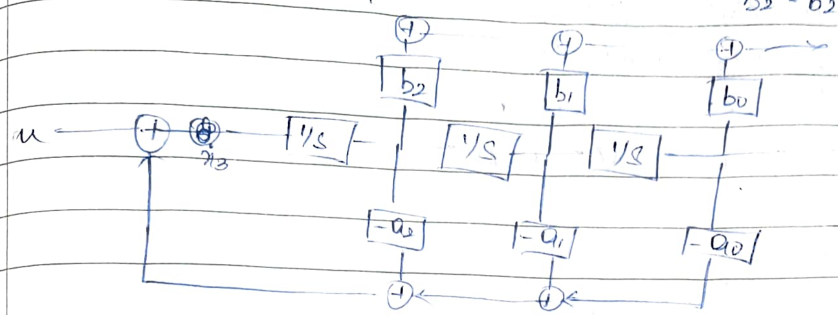


$$Y(s) = \frac{12s^2+44s+48}{s^3+9s^2+23s+15}$$

$b_3 = 0 \Rightarrow \bar{b}_0 = b_0$

$\bar{b}_1 = b_1$

$\bar{b}_2 = b_2$



13/12/23

Observable cannonical form:-

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{1.s^3 + a_2 s^2 + a_1 s + a_0}$$

Cross multiply:-

$$Y(s)\,[s^3 + a_2 s^2 + a_1 s + a_0] = U(s)\,[b_3 s^3 + b_2 s^2 + b_1 s + b_0]$$

$$s^3 Y + a_2 s^2 Y + a_1 s Y + a_0 Y = b_3 s^3 U + b_2 s^2 U + b_1 s U + b_0 U$$

associated with Y

Keep highest power of s, on LHS

$$s^3 Y = b_3 s^3 U + b_2 s^2 U + b_1 s U + b_0 U - a_2 s^2 Y - a_1 s Y - a_0 Y$$

$$s^3 Y = b_3 s^3 U + [b_2 U - a_2 Y] s^2 + [b_1 U - a_1 Y] s + [b_0 U - a_0 Y]$$

Dividing throughout by s^3 we have

$$Y = b_3 U + [b_2 U - a_2 Y] s^{-1} + [b_1 U - a_1 Y] s^{-2} + [b_0 U - a_0 Y] s^{-3}$$

Let $Y = b_3 U + X_1 \rightarrow ①$ $\Rightarrow$ ILT $\rightarrow$ $\boxed{y = b_3 u + x_1}$ ⊛

$X_1 = [b_2 U - a_2 Y] s^{-1} + [b_1 U - a_1 Y] s^{-2} + [b_0 U - a_0 Y] s^{-3}$ × by s.

$sX_1 = [b_2 U - a_2 Y] + s^{-1}[b_1 U - a_1 Y] + s^{-2}[b_0 U - a_0 Y]$

$sX_1 = b_2 U - a_2 Y + X_2 \rightarrow$ ILT $\Rightarrow$ $\boxed{\dot{x}_1 = b_2 u - a_2 y + x_2}$ ⊛

where

$X_2 = s^{-1}[b_1 U - a_1 Y] + s^{-2}[b_0 U - a_0 Y]$ × by s.

$sX_2 = b_1 U - a_1 Y + s^{-1}[b_0 U - a_0 Y]$

$sX_2 = b_1 U - a_1 Y + X_3 \rightarrow$ $\boxed{\dot{x}_2 = b_1 u - a_1 y + x_3}$ ⊛

where

$X_3 = s^{-1}[b_0 U - a_0 Y]$ × by s.

$sX_3 = b_0 U - a_0 Y \Rightarrow$ $\boxed{\dot{x}_3 = b_0 u - a_0 y}$ ⊛

what was the printer, giving letters to HOD?

DATE

Consider

$$\dot{x}_1 = b_2 u - a_2 y + x_2$$

Let us substitute for y: $y = b_3 u + x_1$

$$\dot{x}_1 = b_2 u + x_2 - a_2(b_3 u + x_1)$$

$$= b_2 u + x_2 - a_2 b_3 u - a_2 x_1$$

$$\dot{x}_1 = -a_2 x_1 + x_2 + (b_2 - a_2 b_3) u$$

Now consider,

$$\dot{x}_2 = b_1 u - a_1 y + x_3$$

$$\dot{x}_2 = b_1 u - a_1 (b_3 u + x_1) + x_3$$

$$= b_1 u - a_1 b_3 u - a_1 x_1 + x_3$$

$$= -a_1 x_1 + x_3 + (b_1 - a_1 b_3) u$$

$$\dot{x}_3 = b_0 u - a_0 y$$

$$= b_0 u - a_0 (b_3 u + x_1)$$

$$= b_0 u - a_0 b_3 u - a_0 x_1$$

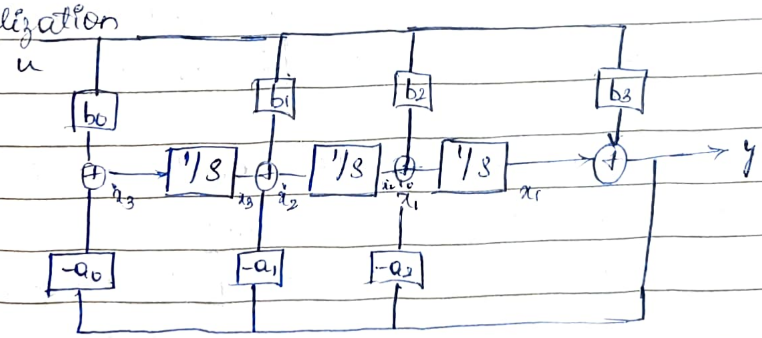
$$= -a_0 x_1 + (b_0 - a_0 b_3) u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 - a_2 b_3 \\ b_1 - a_1 b_3 \\ b_0 - a_0 b_3 \end{bmatrix} u$$

$$y = x_1 + b_3 u$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$

Realization



1. Repeat the procedure by taking $b_3 = 0 \Rightarrow \bar{b}_2 = b_2$; $\bar{b}_1 = b_1$; $\bar{b}_0 = b_0$; $D = 0$

2. Given that $\dfrac{Y(s)}{U(s)} = \dfrac{s^3 + 14s^2 + 50s + 55}{s^3 + 12s^2 + 30s + 15}$

$\rightarrow$ $b_3 = 1 \quad b_2 = 14 \quad b_1 = 50 \quad b_0 = 55$

$a_3 = 1 \quad a_2 = 12 \quad a_1 = 30 \quad a_0 = 15$

$\rightarrow$ $\bar{b}_2 = \cancel{b_2}\ 2$

$\bar{b}_1 = 20$

$\bar{b}_0 = 40$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -12 & 1 & 0 \\ -30 & 0 & 1 \\ -15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 20 \\ 40 \end{bmatrix}$$

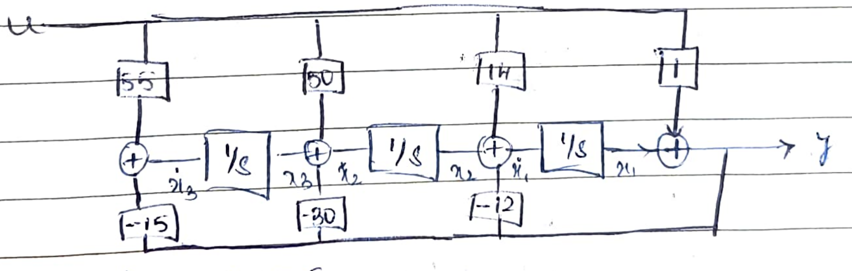
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u.$$

$\dot{x}_1 = -12x_1 + x_2 + 2u$

$\dot{x}_2 = -30x_1 + x_3 + 20u$

$\dot{x}_3 = -15x_1 + 40u \Rightarrow \dot{x}_3 = -15(y-u) + 40u = -15y + 55u$

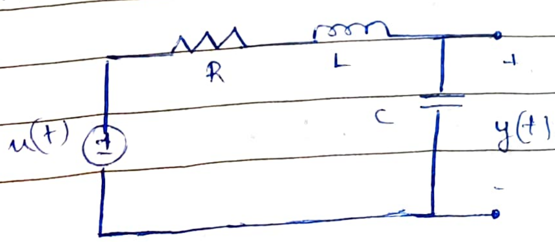
$y = x_1 + u \Rightarrow x_1 = y - u$

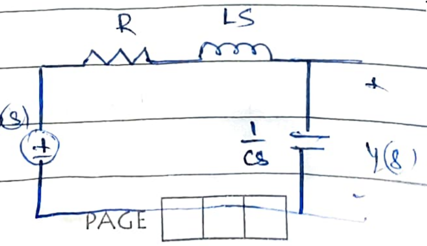


3. $\dfrac{Y(s)}{U(s)} = \dfrac{14s^2 + 50s + 55}{s^3 + 12s^2 + 30s + 15}$

4. $\dfrac{Y(s)}{U(s)} = \dfrac{s^2 + 6}{s^3 + 9s^2 + 23s + 15}$ $\quad b_3 = 0$ $\quad b_1 = 0$

5. For the ckt shown, obtain T.F $Y(s)/U(s)$





classmate

$$Y(s) = \frac{U(s)\, Z_2}{Z_1 + Z_2}$$

$$= \frac{1/Cs}{R + Ls + \frac{1}{Cs}} = \frac{1/C}{Rs + Ls^2 + 1/C} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$b_2 = 0 \quad b_1 = 0 \quad b_0 = 1/LC$

$a_2 = 1 \quad a_1 = R/L \quad a_0 = 1/LC$

Realize in both canonical forms.

15/12/23

State model of Electromechanical system

DC motor

Vtg → [DCM] → speed $\frac{d\theta}{dt} = \omega$

Obtain the state model of DC motor taking armature current, motor speed as state variables.

Armature eqn:

$$e_a = L\frac{di_a}{dt} + i_a R + e_b \rightarrow \text{back emf} \rightarrow (1)$$

$$e_b = K_b \frac{d\theta}{dt} = K_b \omega \rightarrow (2)$$

o/p side:

$$T_a = K_t i_a \rightarrow (3)$$

$$T_a = J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt}$$

$$T_a = J\frac{d\omega}{dt} + B\omega \rightarrow (4)$$

$x_1 = \omega \Rightarrow \dot{x}_1 = \frac{d\omega}{dt}$

$x_2 = i_a \Rightarrow \dot{x}_2 = \frac{di_a}{dt}$

Consider eqn (1)

$$e_a = L\dot{x}_2 + x_2 R + K_b x_1$$

$$\boxed{\dot{x}_2 = \frac{e_a}{L} - \frac{K_b x_1}{L} - \frac{R x_2}{L}}$$

Consider eqn (4) :- $T_a = J\dot{x}_1 + B x_1$

$$K_t x_2 = J\dot{x}_1 + B x_1$$

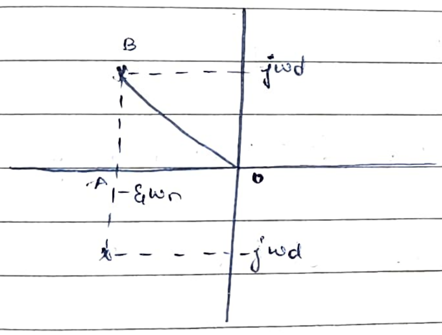
$$\boxed{\dot{x}_1 = \frac{K_t x_2}{J} - \frac{B x_1}{J}}$$

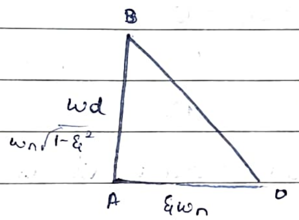
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -B/J & \frac{K_t}{J} \\ \frac{-K_b}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e_a \qquad y = [0 \ 1 \ 0]$$

Design of proportional controller using root-locus method:-

1. $\cdot 1(s) = \dfrac{1}{(s+3)(s+4)(s+5)}$. Design a a P-controller so that

$\xi = 0.5$ $\omega_n = 3.5$. Use Root-locus method





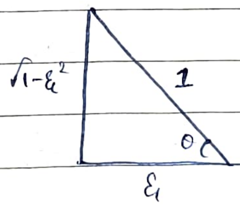
$OB = \sqrt{\omega d^2 + (\xi\omega_n)^2}$

$OB = \sqrt{\omega d^2 + \xi^2\omega_n^2}$

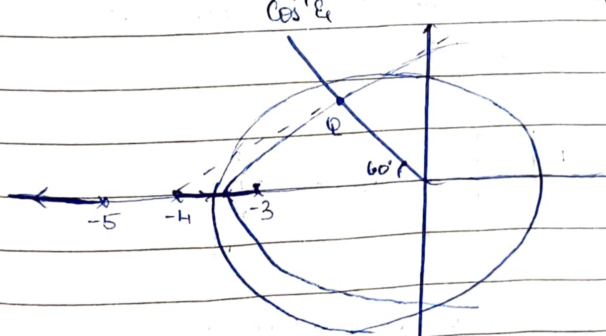
$\omega_n^2(1-\xi^2) + \xi^2\omega_n^2$

$OB = \sqrt{\omega_n^2} = \omega_n$

$\Downarrow$



$\cos\theta = \dfrac{\xi}{1} \rightarrow \theta = \cos^{-1}\xi$



no. asymptotes = 3

angle = 60, 180, 300

$\sigma = \dfrac{-3-4-5-0}{3} = -4$

BAP:

$1 + KL(s) = 0$

$K = -(s+3)(s+4)(s+5) = -(s^2+4s+3s+12)(s+5)$

$= -(s^3+7s^2+12s+5s^2+35s+60)$

$= -(s^3+12s^2+47s+60)$

classmate

$$\frac{dK}{ds} = 0 = 3s^2 + 24s + 47$$
$$= -3.42, \ -4.57$$

Intersection of root locus with img axis

$1 + KL(s) = 0.$

$$1 + \frac{K}{(s+3)(s+4)(s+5)} = 0$$

$(s+3)(s+4)(s+5) + K = 0$

$s^3 + 12s^2 + 47s + 60 + K = 0$

s^3	1	47
s^2	12	$60+K$
s	$\frac{504-K}{12}$	0
s^0	$60+K$	

$\Rightarrow 60+K > 0$

$\boxed{K > -60}$

$504 - K > 0$

$\boxed{K < 504}$

$0 < K < 504$

$12s^2 + 564 = 0$

$$s^2 = \frac{-564}{12} = \sqrt{-47} = 6.85j$$

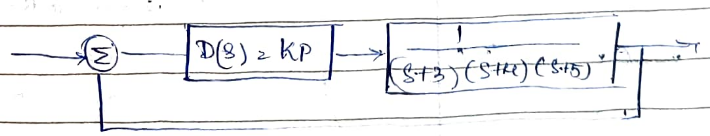
with 3.5 as radius draw a circle with origin as centre

The point where circle cuts root locus $\rightarrow$ P

$P = -2.5 + j2.25$

$Q = -1.9 + j3.4 \quad \rightarrow \quad \cos^{-1}\xi$

For the system to satisfy the requirements $\xi = 0.5$ & $\omega_n = 3.5$, the root locus of the compensated system shd pass through the points P & Q



The OL tr fn of compensated system is

$$L_1(s) = DG = \frac{K_P}{(s+3)(s+4)(s+5)}$$

For point P to be on root locus, $\left| \frac{K_P}{(s+3)(s+4)(s+5)} \right|_{s=-2.5+j2.25} = 1 \Rightarrow K_P = $

For point Q to be on root locus,

$$\left| \frac{K_P}{(s+3)(s+4)(s+5)} \right|_{s=-1.9+j3.4} = 1 \Rightarrow K_{P2} = 65.7$$

$\therefore$ we can use the controller gain in the range $K_{P1} < K_P < K_{P2}$

$21 < K_P < 65.7$

2/12/23 Controllability of a system:-

A linear time invariant system is said to be completely state controllable at time $t = t_0$ if it is possible to transfer the initial state $x(t_0)$ to any final state $x(t_f)$ by applying a control vector $u(t)$ which has no constraints in its choice in time interval $t_0 \le t \le t_f$.

Condition for the controllability of the system.

$$Q = [B \quad AB \quad A^2B \cdots A^{n-1}B]$$

where

n = No. of state variables

For the system to be completely controllable, controllability mat Q should be a full rank matrix.

NOTE: Consider the following cases

For a square matrix 4x4 full rank is 4.

For a rectangular matrix 4x3 full rank = max(4, 3) = 4

Alternatively, the system is fully controllable if inverse of Q exists.

$$Q^{-1} = \frac{\text{adj } Q}{|Q|} \qquad |Q| \neq 0$$

- The det is zero when the rows/columns are linearly dependant

Ex: $\begin{bmatrix} 2 & 12 \\ 4 & 24 \end{bmatrix}$ Note that elements of 2nd column is obtained by multiplying 1st column by 6

$\hookrightarrow \det = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \rightarrow 1(36-36) - 2(18-18) + 3(12-12) = 0$$

classmate

DATE

3.

- $n=2$

$$Q = [B \quad AB]$$

- $n=3$

$$Q = [B \quad AB \quad A^2B]$$

Example:

1. A control system is described by $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$ 4.

A, B

$n=3$

$$B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 \\ 10 \\ 100 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 10 \\ 90 \\ 990 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & 90 \\ 10 & 100 & 990 \end{bmatrix}$$

$$|Q| = -1000$$

∴ The system is fully controllable

~~$Q_{11} = 900$~~ ~~$Q_{21} = 1000$~~ ~~$Q_{31} = 100$~~

~~$Q_{12} = +900$~~ ~~$Q_{22} = -100$~~

~~$Q_{13} = -100$~~ ~~$Q_{23} = 0$~~

2. Given that $A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & -5 & 4 \\ -5 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -2 \\ 1 & -4 \\ 5 & -3 \end{bmatrix}$ Check whether the system is completely controllable

→

$$AB = \begin{bmatrix} -14 & 5 \\ 15 & 4 \\ 41 & -29 \end{bmatrix} \qquad A^2B = \begin{bmatrix} -108 & 91 \\ 61 & -126 \\ 489 & 219 \end{bmatrix}$$

The $Q = \begin{bmatrix} 0 & -2 & -14 & 5 & -108 & 91 \\ 1 & -4 & 15 & 4 & 61 & -126 \\ 5 & -3 & 41 & -29 & 489 & 219 \end{bmatrix}$

The rows / columns are linearly independent

∴ The sys is completely controllable

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3. $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ $= -1+2 = 1$

Controllable

$A^2B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow |Q| = 0$ Not controllable.

Observability of LTI system:

A LTI system is said to be completely observable if every state, say $x(t_0)$, can be determined by measuring the o/p $y(t)$ in a finite interval of $t_0 \le t \le t_f$. The observability matrix $S = [\,C^T \;\; A^T C^T \; \ldots \; (A^T)^{n-1} C^T\,]$.

The system is completely observable if S is a full rank matrix or the inverse of S exists $\Rightarrow |S| \neq 0$

1. Determine the observability of a system having $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$n = 3$

$A^T C^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $(A^T)^2 C^T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

* In exam write A^T, $(A^T)^2$ $(A^T)^2 C^T$

$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $|S| = 1$

$\therefore$ This system is completely observable.

HW

2. Check the observability of the system having $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}$

23/12/23

State space model of electromechanical system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 \\ 10 \\ -30 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 10 \\ -30 \\ 70 \end{bmatrix}$$

$$Q = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -30 \\ 10 & -30 & 70 \end{bmatrix} = -1000$$

System is controllable

$$|\lambda I - A| = 0$$

$$\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{bmatrix}$$

$$\lambda(\lambda^2+3\lambda+2) + 1(0)$$

$$\lambda^3 + 3\lambda^2 + 2\lambda = 0$$

$\lambda = -1, -2, 0$ → System is stable

$\lambda_1 = -2, \ \lambda_2 = -1+j \ \lambda_3 = -1-j$

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = 0$$

$$(\lambda+2)(\lambda+1-j)(\lambda+1+j) = 0$$

$$(\lambda+2)((\lambda+1)^2+1) = 0$$

$$(\lambda+2)(\lambda^2+2\lambda+1)+1 = 0$$

$$\lambda^3 + 2\lambda^2 + \lambda + 2\lambda^2 + 4\lambda + 2 + \cancel{\lambda}+2 = 0$$

$$\lambda^3 + 4\lambda^2 + 5\lambda + 4 = 0 \quad \rightarrow (1)$$

$$|\lambda I - (A - BK)| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10K \end{bmatrix} \right|$$

$\lambda \quad -1 \quad 0$

$0 \quad \lambda \quad -1$

$$BK = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10k_1 & 10k_2 & 10k_3 \end{bmatrix}$$

$$A-BK = \begin{bmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ -10k_1 & -2-10k_2 & -3-10k_3 \end{bmatrix}$$

$$|\lambda I - (A-BK)| = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 10k_1 & 2+10k_2 & \lambda+3+10k_3 \end{vmatrix} = 0$$

$$\lambda(\lambda^2+3\lambda+10\lambda k_3-2-10k_2)+1(+10k_1) = 0$$

$$\lambda^3+3\lambda^2+10\lambda^2k_3-2\lambda-10\lambda k_2+10k_1 = 0$$

$$\lambda^3+(3+10k_3)\lambda^2+(+2+10k_2)\lambda+10k_1 = 0 \rightarrow (2)$$

Comparing (1) & (2).

$3+10k_3 = 4$ $\quad\quad$ $+2+10k_2 = 6$ $\quad\quad$ $10k_1 = 4$

$\boxed{k_3 = 0.1}$ $\quad\quad$ $+10k_2 = 4$ $\quad\quad$ $\boxed{k_1 = 0.4}$

$\boxed{k_2 = +0.4}$

$\therefore \boxed{k_1 = 0.4;\ k_2 = 0.4;\ k_3 = 0.1}$

Similarity transformations:

1. Given one state model of system any no. of state models can be derived using similarity transformation

Consider $\dot{x} = Ax + Bu$

$y = Cx + Du$

- Apply linear transformation $x = Pw \rightarrow w = P^{-1}x$ where P is a constant $n\times$ matrix & w is a new state vector.
- It is necessary that P^{-1} shall exist so that w can be determined from x

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

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we have

$$w = P^{-1}x$$

$$\dot{w} = P^{-1}\dot{x} = P^{-1}[Ax + Bu]$$

$$= P^{-1}[APw + Bu]$$

$$\dot{w} = P^{-1}APw + P^{-1}Bu$$

$$y = CPw + Du$$

Taking $A_w = P^{-1}AP$; $B_w = P^{-1}B$

$C_w = CP$; $D_w = D$

we have

$$\boxed{\begin{aligned} \dot{w} &= A_w w + B_w u \\ y &= C_w w + D_w u \end{aligned}}$$

- For each different P for which P^{-1} exists, a different state model of a given system can be formed.

The CE of a matrix A is defined by

$|sI - A| = 0$ or $|\lambda I - A| = 0$

- The eigen values / characteristic values of the matrix are the roots of characteristic equation.

$$|\lambda I - A| = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) = 0$$

- The CE of a system matrix is unchanged through the linear transformation.

$$|\lambda I - A_w| = |\lambda I - A|$$

$$|\lambda I - A_w| = |\lambda I - P^{-1}AP|$$

$$= |\lambda P^{-1}IP - P^{-1}AP|$$

$$= |P^{-1}|\,|\lambda IP - AP|$$

$$= |P^{-1}|\,|\lambda I - A|\,|P|$$

$$|\lambda I - A_w| = |\lambda I - A|$$

Problem:-

1. Given that $\dot{x} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ $\quad y = [1\ 0]x$

→ a. Obtain another state model using similarity transformation

b. ST the eigen values are unchanged after the transformation

- Soln

$$A = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

a.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$A_w = P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1.8 & 0.2 \\ 0.9 & 0.9 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2.7 & 1.1 \\ -0.9 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 1.35 & 0.55 \\ -0.45 & 0.35 \end{bmatrix}$$

$$B_w = P^{-1}B$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$C_w = CP$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$D_w = D = 0$$

$$\dot{w} = \begin{bmatrix} 1.35 & 0.55 \\ -0.45 & 0.35 \end{bmatrix} w + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} w$$

b. $$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix} = \begin{vmatrix} \lambda - 0.8 & -1 \\ 0 & \lambda - 0.9 \end{vmatrix}$$

$$= (\lambda - 0.8)(\lambda - 0.9) = 0$$

$$= \lambda^2 - 0.9\lambda - 0.8\lambda + 0.72 =$$

$$= \lambda^2 - 1.7\lambda + 1.72$$

$$\Rightarrow \lambda_1 = 0.8 \quad \lambda_2 = 0.9$$

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$$|\lambda I - A_w| = \begin{vmatrix} \lambda - 1.35 & -0.55 \\ 0.45 & \lambda - 0.35 \end{vmatrix} = 0$$

$$= (\lambda - 1.35)(\lambda - 0.35) + 0.55 \times 0.45 = 0$$

$$= \lambda^2 - 1.35\lambda - 0.35\lambda + 0.4725 + 0.2475 = 0$$

$$= \lambda^2 - 1.7\lambda + 0.72 = 0$$

$$\lambda_1 = 0.8 \quad \lambda_2 = 0.9$$

∴ Eigen values remain the same.

Q. Given that $\frac{Y(s)}{U(s)} = \frac{5s^2 + 10.5s + 5.8}{s^3 + 1.1s^2 + 0.5s + 0.12}$

a. Obtain another state model using similarity transformation

b. Verify that the eigen values are unchanged after S.T.

→ Note: write the state model in Control canonical form.

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$a_0 = 0.12 \quad a_1 = 0.5 \quad a_2 = 1.1 \quad a_3 = 1$

$b_0 = 5.8 \quad b_1 = 10.5 \quad b_2 = 5 \quad b_3 = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.12 & -0.15 & -1.1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 5.8 & 10.5 & 5 \end{bmatrix}$$

Arbitrarily choose matrix P, such that P^{-1} exist

Let $P = \begin{bmatrix} 3 & 8 & 4 \\ 0 & 7 & 5 \\ 9 & 6 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -0.267 & 0.133 & 0.2 \\ 0.75 & -0.5 & -0.25 \\ -1.05 & 0.9 & 0.35 \end{bmatrix}$

$$A_w = \begin{bmatrix} -0.85 & -2.788 & -1.75 \\ -1.93 & 4.4 & 3.60 \\ 4.50 & -4.9 & -4.65 \end{bmatrix}$$

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$$B_w = P^{-1}B = \begin{bmatrix} 0.2 \\ 0.25 \\ 0.35 \end{bmatrix}$$

$$C_w = ~~PC~~ \quad CP = \begin{bmatrix} 5.8 & 10.5 & 5 \end{bmatrix} \begin{bmatrix} 3 & 8 & 4 \\ 0 & 7 & 5 \\ 9 & 6 & 2 \end{bmatrix} =$$

Example-3

$$\frac{Y(s)}{U(s)} = \frac{10s-5}{s^3 - 2.1s^2 + 1.4s - 0.288}$$

Example-4

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 \end{bmatrix} \quad D = 0$$

a. Obtain $\frac{Y(s)}{U(s)}$

b. Obtain new state model

c. Find $\frac{Y(s)}{U(s)}$ obtained in b.

d. Verify c & a are same