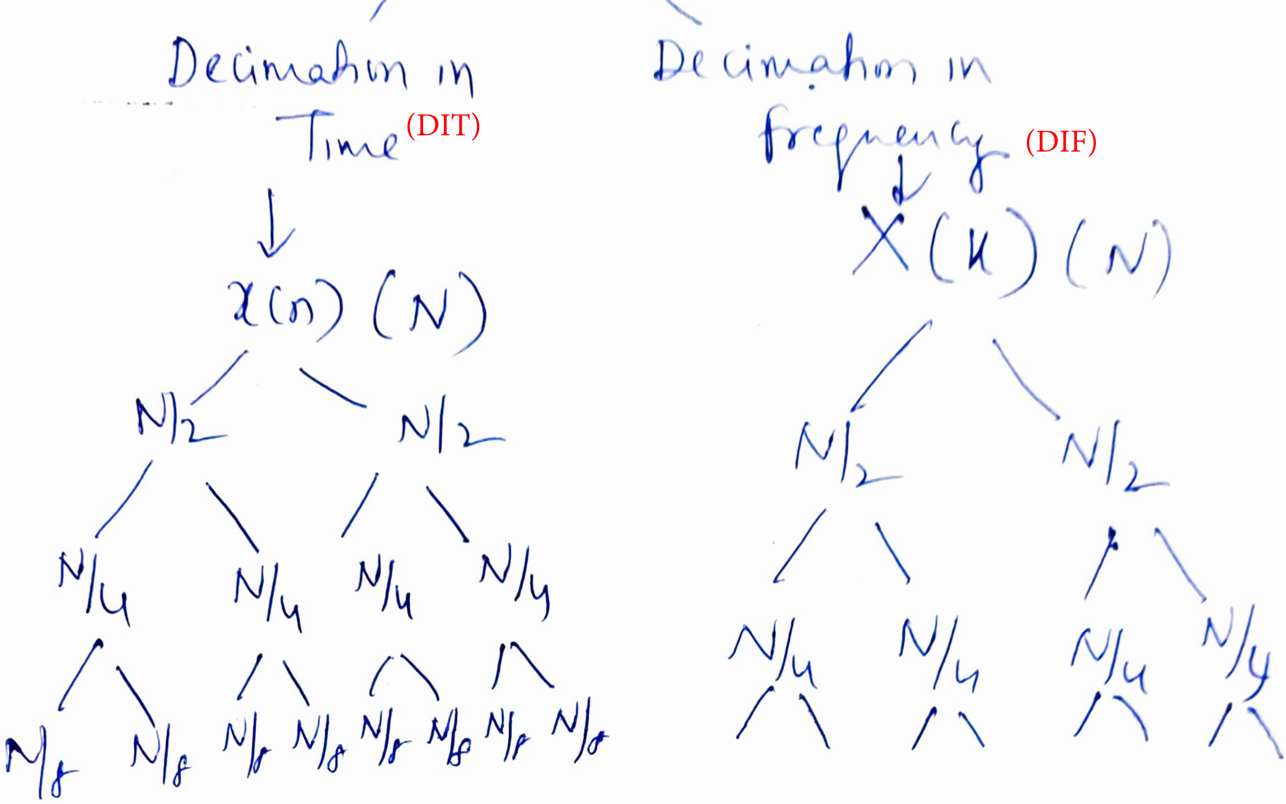
1. FFT(2)
2. Radix-2 DIT FFT.   2

Fast Fourier Transform

①

->It is the name given to the set algorithms used to solve DFT. This is also the name of function used to find DFT in Matlab.
-> It is used to solve quickly.



Radix-2 FFT algorithm.

- Decimate the sequence until we get 2 pt. sequence.
- $N = 2^x$

Previously, we required N^2 multiplications and N*(N-1) additions. This is reduced by FFT which used divide and conquer algorithm.
Most popularly used id Radix-2.
Firstly, we make the no. of points in question to closest power of 2(in Radix-2 algo).
For Radix-3, we make a sequence with length equal to power of 3.
This is done by adding trailing zeros.
All smaller sub-sequence computations can be done simultaneously by DSP preocessor.

Radix-2 DIT FFT Algorithm

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$\Rightarrow X(k) = \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn}$$

$n = 2r$ $\qquad$ $n = 2r+1$

$$= \sum_{r=0}^{N/2-1} x(2r) W_N^{2rk} + \sum_{r=0}^{N/2-1} x(2r+1) W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{N/2-1} x(2r) W_{N/2}^{rk} + W_N^{k} \sum_{r=0}^{N/2-1} x(2r+1) W_{N/2}^{rk}$$

$$W_N^{2rk} = e^{-j\frac{2\pi}{N}2rk} = e^{-j\frac{2\pi}{N/2}rk} = W_{N/2}^{rk}$$

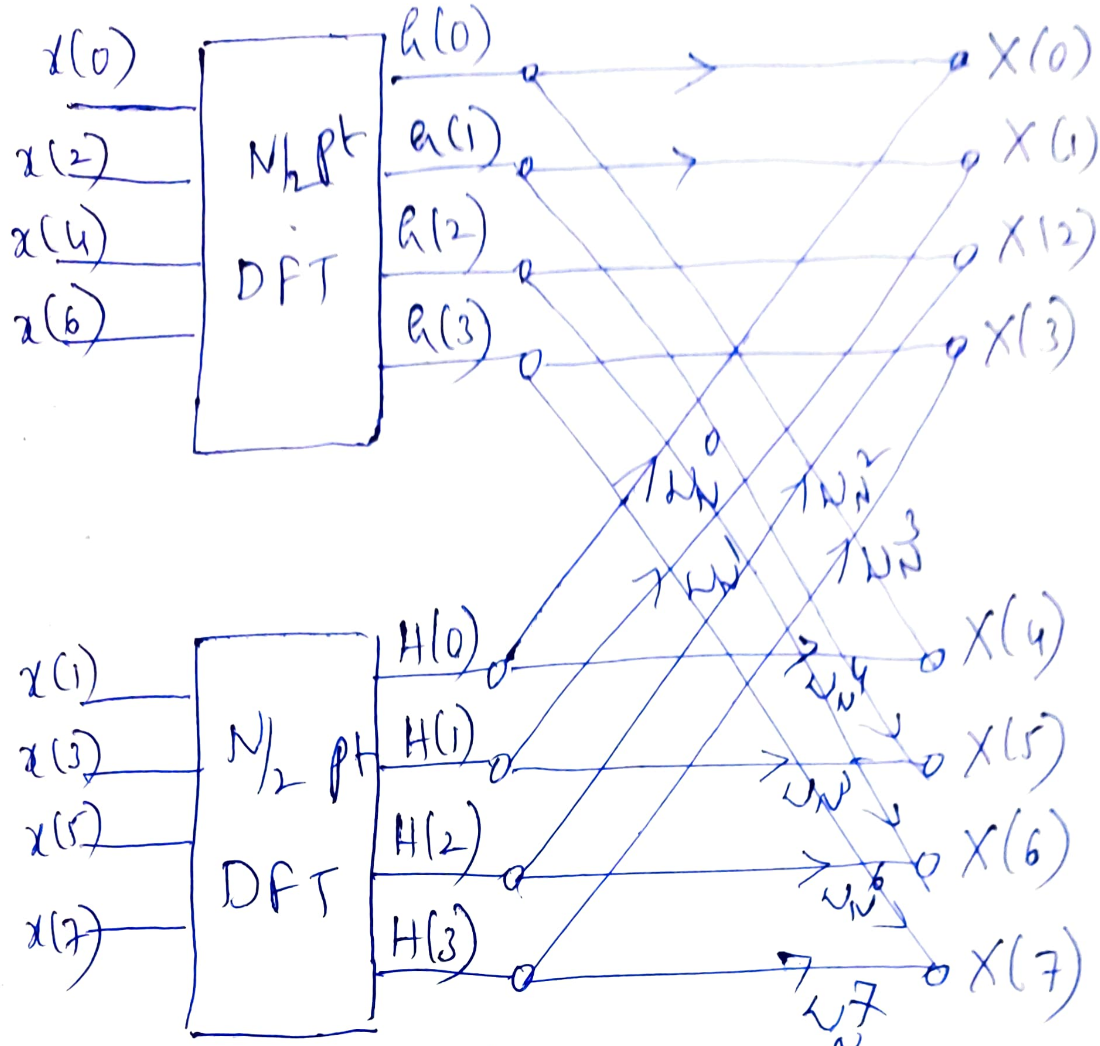
$$= G(k) + W_N^{k} H(k)$$

But W is persiodic with period 8(N).

G(k) and H(k) are periodic with period 4(8/2). Hence G(4) = G(0) and so on until G(7) = G(3).

$G(k)$ and $H(k)$ are $N/2$ Point DFT's.

Let's draw SFG for $N = 8$

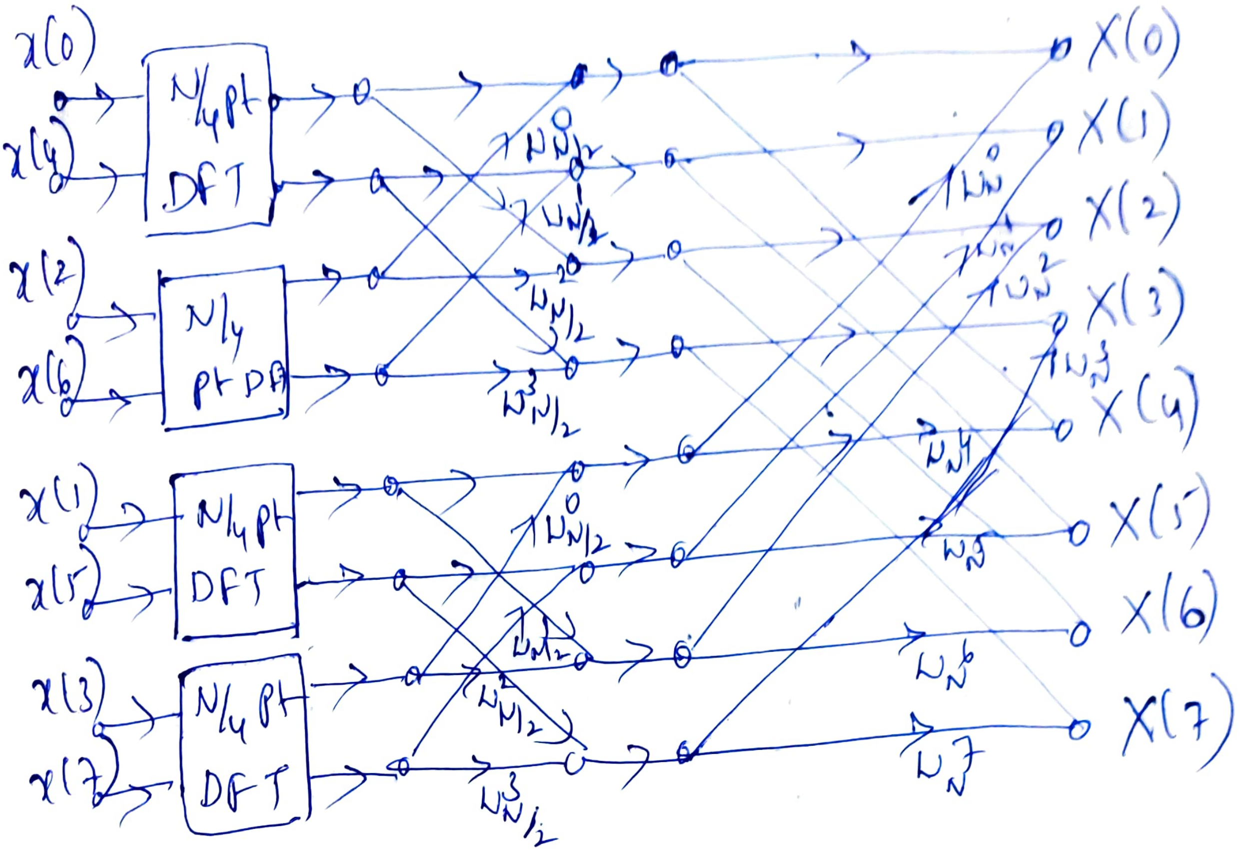


$$G(k) = \sum_{r=0}^{N/2-1} g(r)\, W_{N/2}^{rk}$$

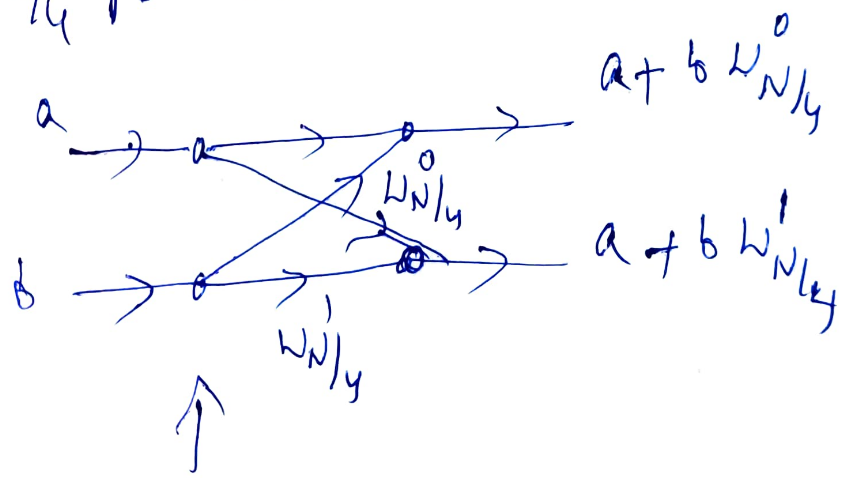
$$= \sum_{r\ \text{even}} + \sum_{r\ \text{odd}}$$

$$= P(k) + W_{N/2}^{k} Q(k)$$

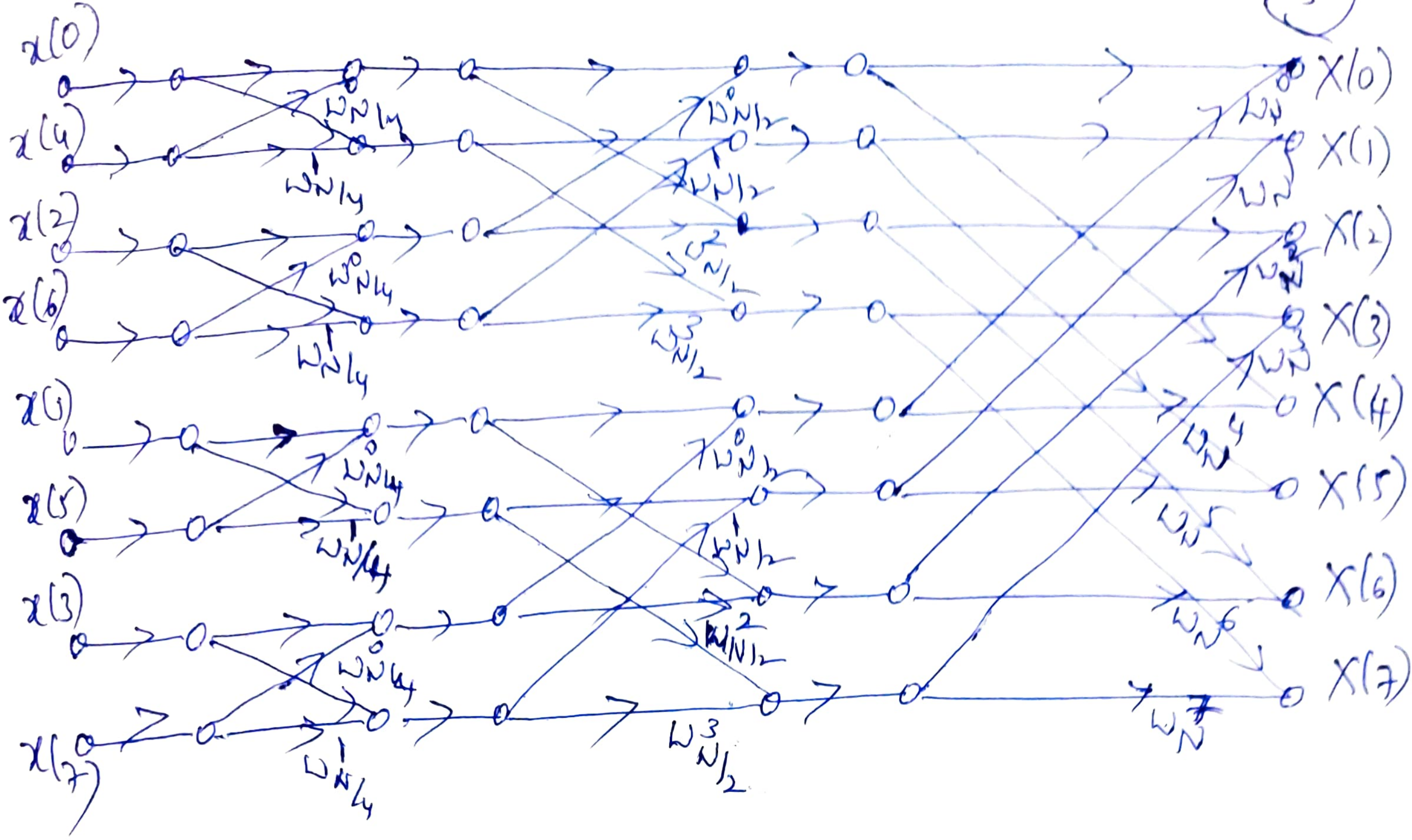
lly $H(k) = R(k) + W_{N/2}^{k} S(k)$



$N/4$ pt DFT $\longrightarrow$ 2 pt. DFT



Butterfly



Observations

1) I/p date is in bit reversed order

x(0)	000	X(0)	000
x(4)	100	X(1)	001
x(2)	010	X(2)	010
x(6)	110	x(3)	011
⋮		⋮	

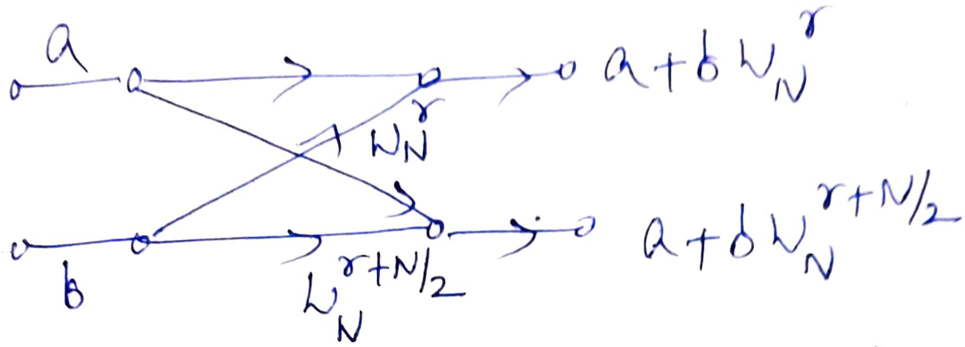
2) O/p date is in proper order (normal order)

3) Basic computational block is called butterfly.

Further reduction:

→ Cooley-Tukey Algorithm

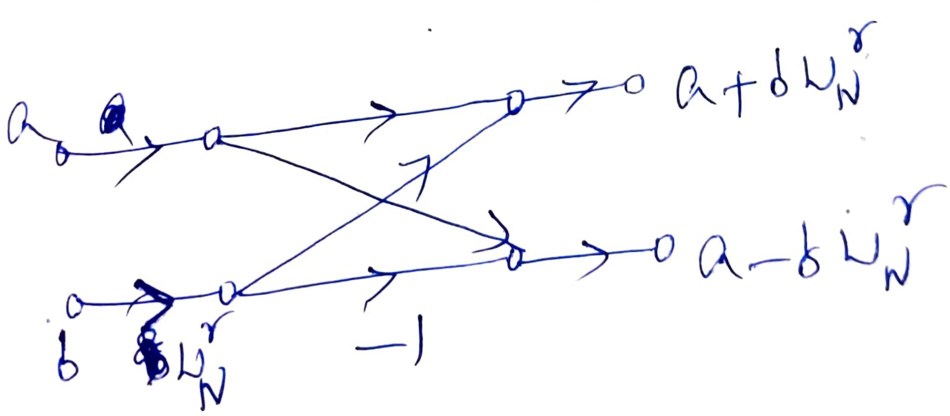
Butterfly Configuration

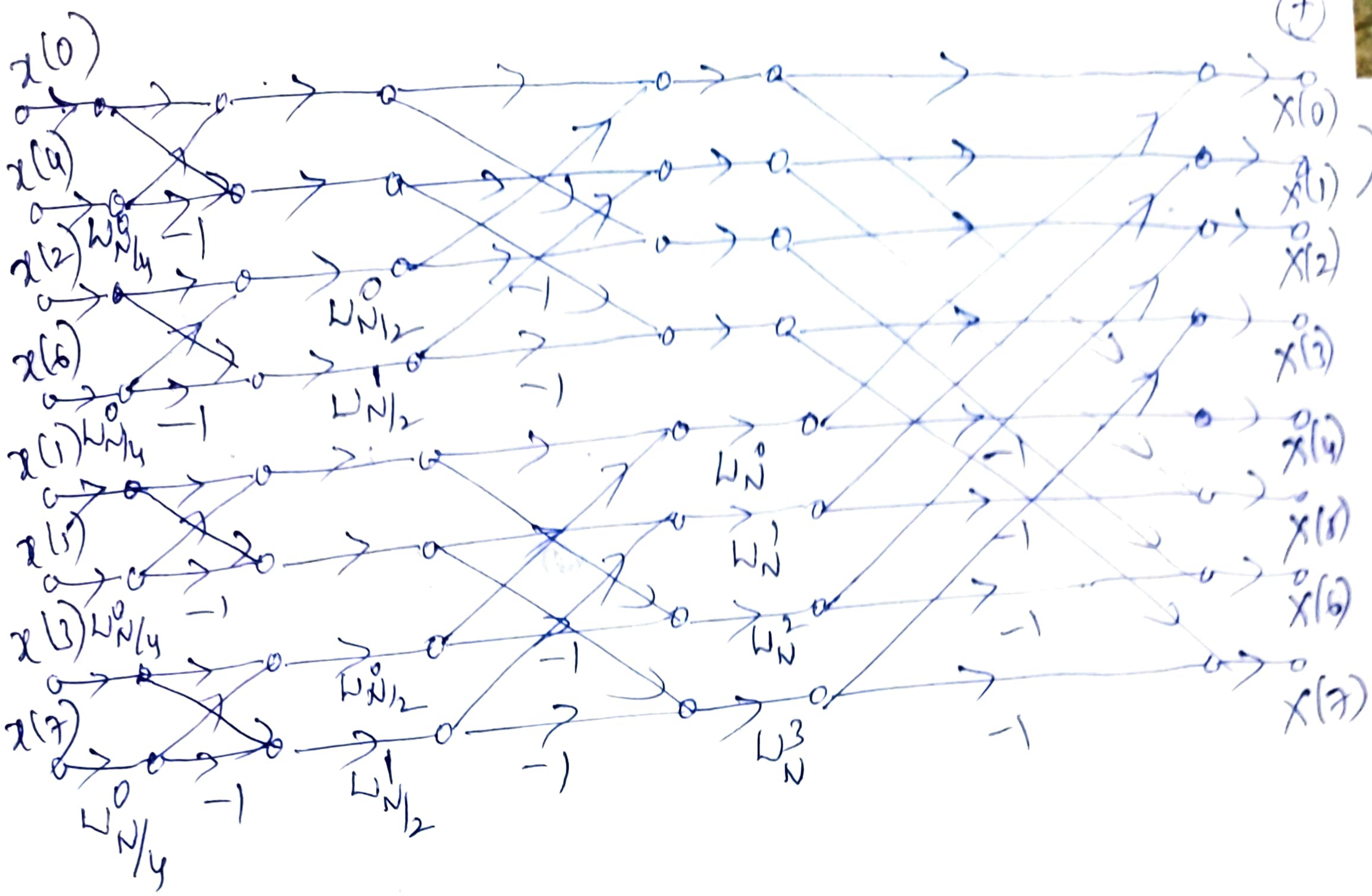


→ Can be further simplified to reduce number of Complex multiplications

$$W_N^{r+N/2} = W_N^r \, W_N^{N/2}$$
$$= -W_N^r$$

$$W_N^{N/2} = e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}}$$
$$= e^{-j\pi}$$
$$= -1$$





Computation time

Direct Computation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad 0 \le k \le N-1$$

No. of Complex multiplications $= N^2$

No. of Complex additions $= N(N-1)$

$N = 8$;

No. of multiplications $= 8^2 = 64$

No. of additions $= 8 \times 7 = 56$

Radix-2 DIT FFT algorithm. (8)

No. of stages = $\log_2 N$

In each stage, No. of butterflies = $N/2$

In each butterfly, No. of multiplications = 01

In each butterfly, No. of additions = 02

$\therefore$ Total No. of $\times^{ns}$ = $N/2 \log_2 N$

Total No. of $+^{ns}$ = $N \log_2 N$

N = 8 :

No. of $\times^{ns}$ = $4 \log_2 8 = 4 \log_2 2^3$

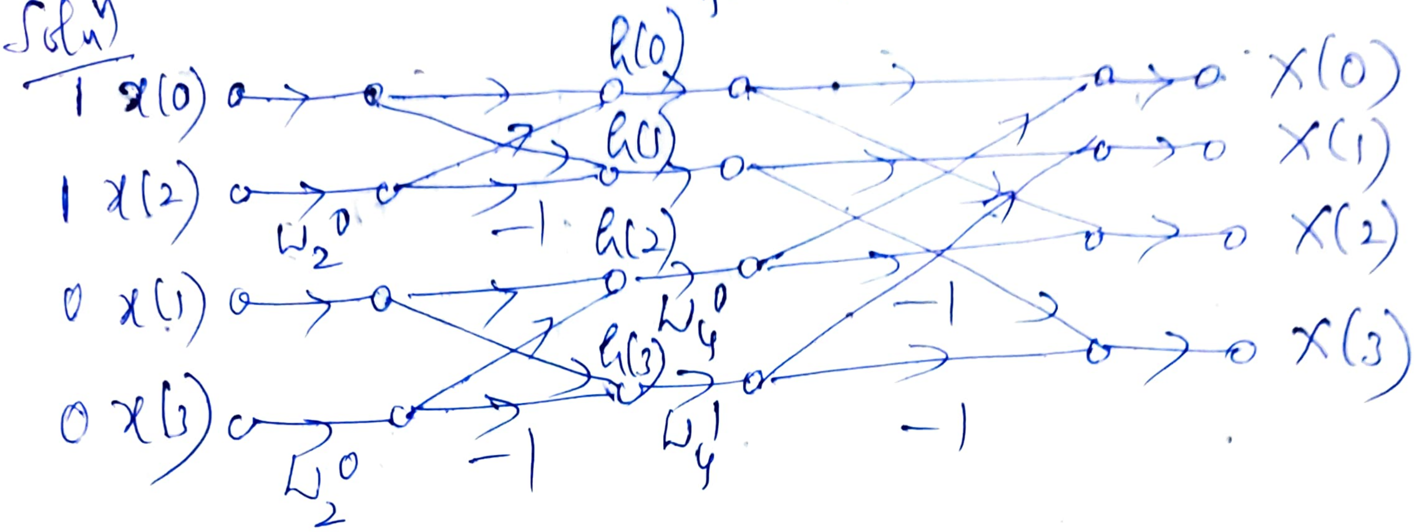
$= 3 \times 4 = \underline{\underline{12}}$

No. of $+^{ns}$ = $8 \log_2 8 = 8 \times 3 = \underline{\underline{24}}$

Pbls: 1) Compute 4 point DFT using Radix-2 DIT FFT algorithm. Verify the result by direct Computation.

$$x(n) = \{\underset{\uparrow}{1}, 0, 1, 0\}$$

Soln



$$G(0) = x(0) + x(2) W_2^0 = 1 + 1 = 2$$

$$G(1) = x(0) - x(2) W_2^0 = 1 - 1 = 0$$

$$G(2) = x(1) + x(3) W_2^0 = 0 + 0 = 0$$

$$G(3) = x(1) - x(3) W_2^0 = 0 - 0 = 0$$

$$X(0) = G(0) + G(2) W_4^0 = 2 + 0 = 2$$

$$X(1) = G(1) + G(3) W_4^1 = 0 + 0 = 0$$

$$X(2) = G(0) - G(2) W_4^0 = 2 - 0 = 2$$

$$X(3) = G(1) - G(3) W_4^1 = 0 - 0 = 0$$

$$X(K) = \{\underset{\uparrow}{2}, 0, 2, 0\}$$

b) Direct Computation

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} \quad 0 \le k \le 3$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$X(k) = \{\underset{\uparrow}{2}, 0, 2, 0\}$$

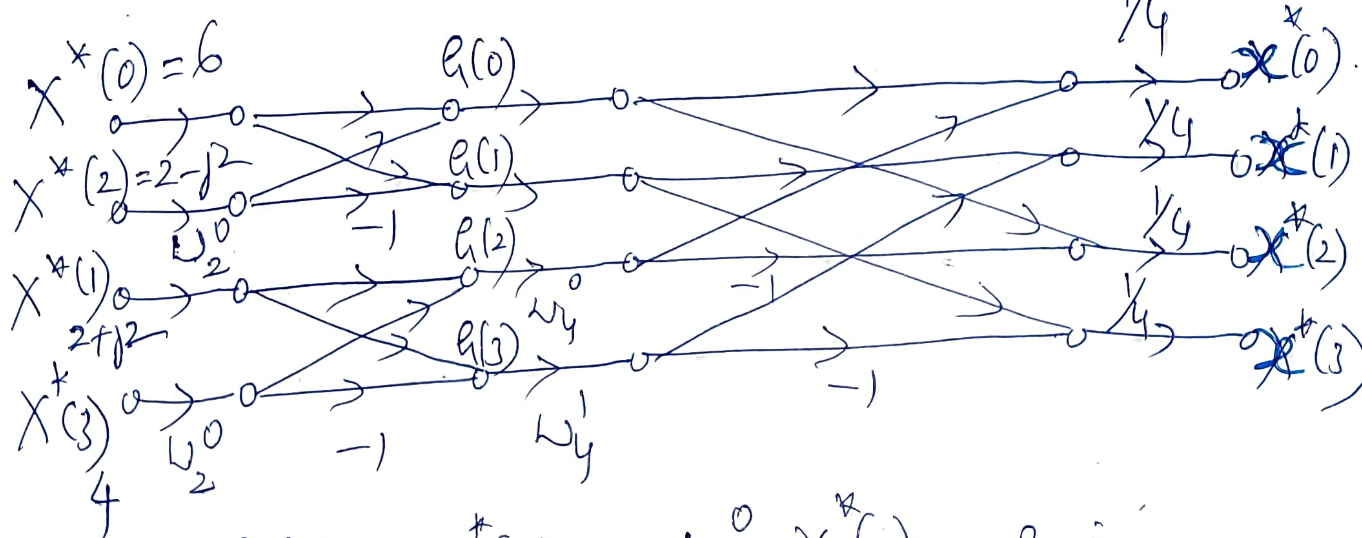
2) Compute 4 point DFT of the sequence $\{\underset{\uparrow}{1}, -3, -2, 5\}$ using Radix-2 DIT FFT algorithm. Verify the result by direct computation.

Computing IDFT using Radix-2 DIT FFT Algorithm

Prob: Find 4 point sequence $x(n)$ if its DFT samples are $X(0) = 6$, $X(1) = 2 - j2$, $X(2) = 2 + j2$, $X(3) = 4$. a) Use Radix-2 DIT FFT algorithm. b) Verify the result by direct Computation

Soln

$$x(n) = \frac{1}{N}\left\{ DFT\left[X^*(k)\right]\right\}^*$$



$$G(0) = X^*(0) + W_2^0\, X^*(2) = 8 - j2$$

$$G(1) = X^*(0) - W_2^0\, X^*(2) = 4 + j2$$

$$G(2) = X^*(1) + W_2^0\, X^*(3) = 6 + j2$$

$$G(3) = X^*(1) - W_2^0\, X^*(3) = -2 + j2$$

$$x^*(0) = \frac{1}{4}\left[G(0) + G(2)\,W_4^0\right]$$

$$= \frac{1}{4}\left[8 - \cancel{j2} + 6 + \cancel{j2}\right] = \frac{14}{4} = \frac{7}{2}$$

$$x^*(1) = \frac{1}{4}\left[G(1) + G(3)\,W_4^1\right]$$

$$= \frac{1}{4}\left[4 + j2 + (-j)(-2 - j2)\right]$$

$$= \frac{1}{4}\left[4 + j2 + 2j + 2\right]$$

$$= \frac{1}{4}\left[6 + j4\right] = \frac{3 + j2}{2} = \frac{3}{2} + j$$

$$x^*(2) = \frac{1}{4}\left[G(0) - W_4^0\,G(2)\right]$$

$$= \frac{1}{4}\left[8 - j2 - 6 - j2\right] = \frac{1}{4}(2 - j4)$$

$$= \frac{1}{2} - j$$

$$x^*(3) = \frac{1}{4}\left[G(1) - G(3)\,W_4^1\right]$$

$$= \frac{1}{4}\left[4 + j2 - (-j)(-2 + j2)\right]$$

$$= \frac{1}{4}\left[4 + \cancel{j2} - j2 - 2\right]$$

$$= \frac{1}{4}\left[2\right] = \frac{1}{2}.$$

$$\therefore x(n) = \left\{ \frac{7}{2},\ \frac{3}{2} - j,\ \frac{1}{2} + j,\ \frac{1}{2} \right\}$$

b) Direct Computation

$$x(n) = \frac{1}{N}\left\{ DFT\left[X^*(k) \right] \right\}^*$$

$$X^*(k) = \{ \underset{\uparrow}{6},\ 2+j2,\ 2-j2,\ 4 \}$$

$$\begin{bmatrix} x^*(0) \\ x^*(1) \\ x^*(2) \\ x^*(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ 2+j2 \\ 2-j2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 14/4 \\ \frac{6+4j}{4} \\ \frac{2-j4}{4} \\ 2/4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3/2+j \\ 1/2-j \\ 1/2 \end{bmatrix}$$

$$\therefore x(n) = \{ \underset{\uparrow}{7/2},\ 3/2-j,\ 1/2+j,\ 1/2 \}$$

Prob 2 Compute circular convolution of the following sequences using Radix-2 DIT FFT algorithm. (14)

$x_1(n) = \{ \underset{\uparrow}{2}, 1, 1, 2 \}$ $x_2(n) = \{ \underset{\uparrow}{1}, -1, -1, 1 \}$

Solution find $X_1(K)$ & $X_2(K)$ using SFG

$X_1(K) = \{ 6, 1+j, 0, 1-j \}$

$X_2(K) = \{ 0, 2+2j, 0, 2-2j \}$

$X_3(K) = X_1(K) \cdot X_2(K)$

$= \{ 0, 4j, 0, -4j \}$

find $x_3(n)$ using SFG.

$x_3(n) = \{ \underset{\uparrow}{0}, -2, 0, 2 \}$

Radix-2 DIF FFT Algorithm

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{k(n+N/2)}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{kn}$$

$$X(k) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k x(n+N/2) \right] W_N^{kn}$$

Separating odd and even terms of $X(k)$

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Even values of $k \Rightarrow k = 2r$)

$$X(2r) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2r} x(n+N/2) \right] W_N^{2rn}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2) \right] W_{N/2}^{rn} \longrightarrow (1)$$

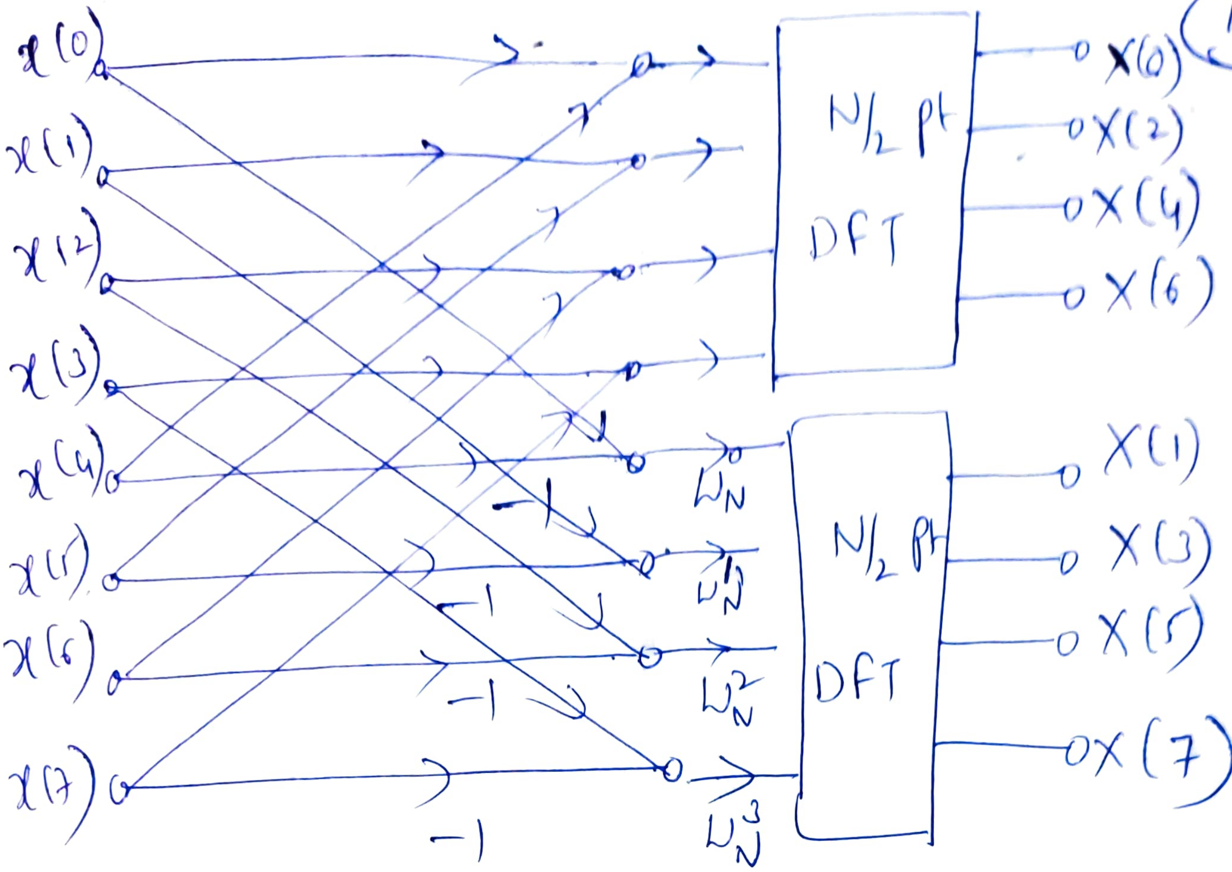
$$W_N^{2rn} = e^{-j\frac{2\pi}{N}2rn} = e^{-j\frac{2\pi}{N/2}rn} = W_{N/2}^{rn}$$

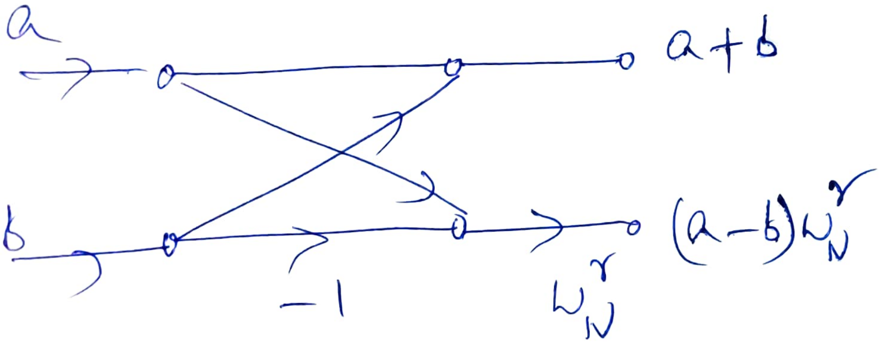
odd values of $k \Rightarrow k = 2r+1$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2r+1} x(n+N/2) \right] W_N^{(2r+1)n}$$

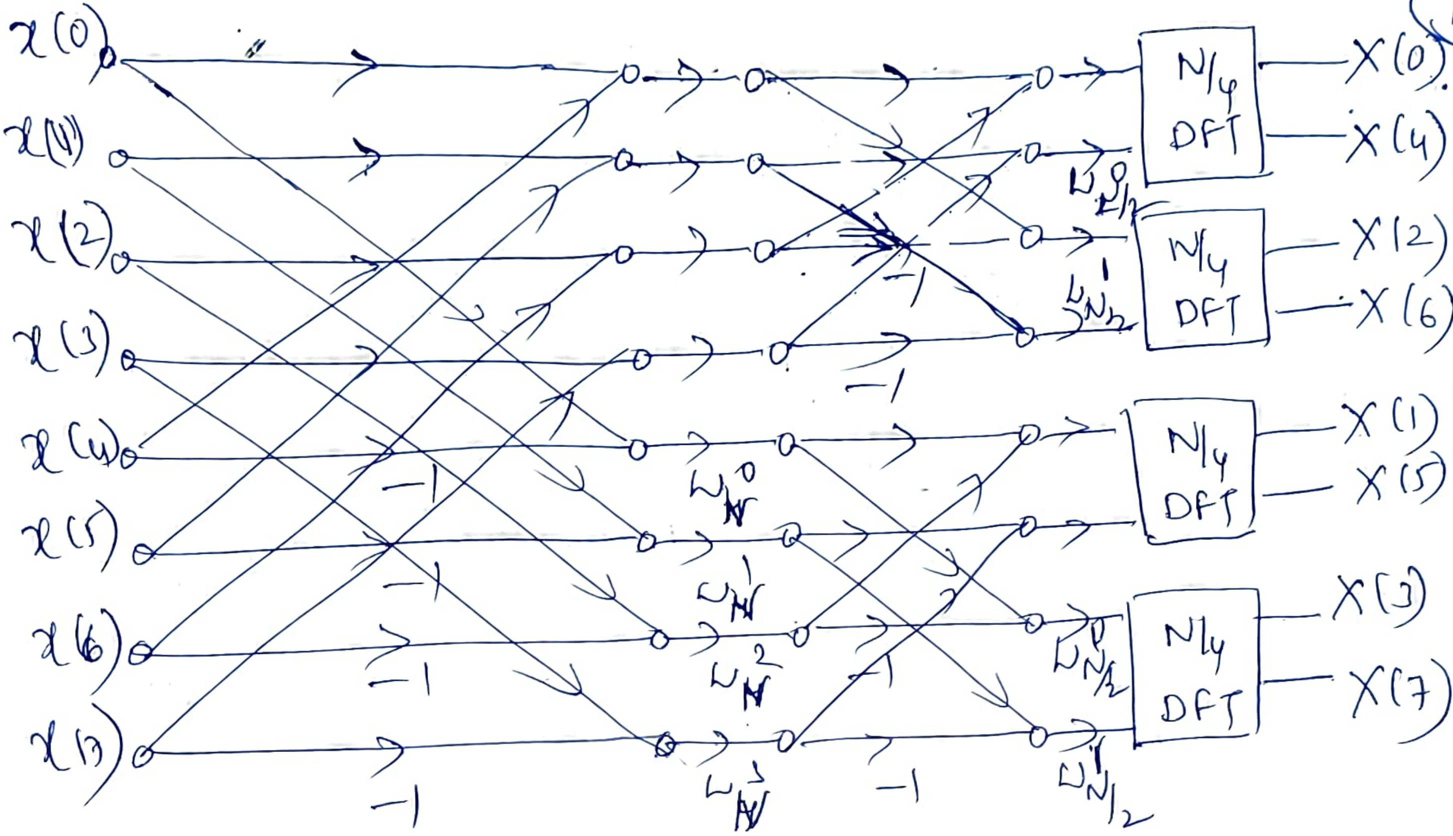
$$= \sum_{n=0}^{N/2-1} \left\{ \left[x(n) - x(n+N/2) \right] W_N^{n} \right\} W_N^{2rn}$$

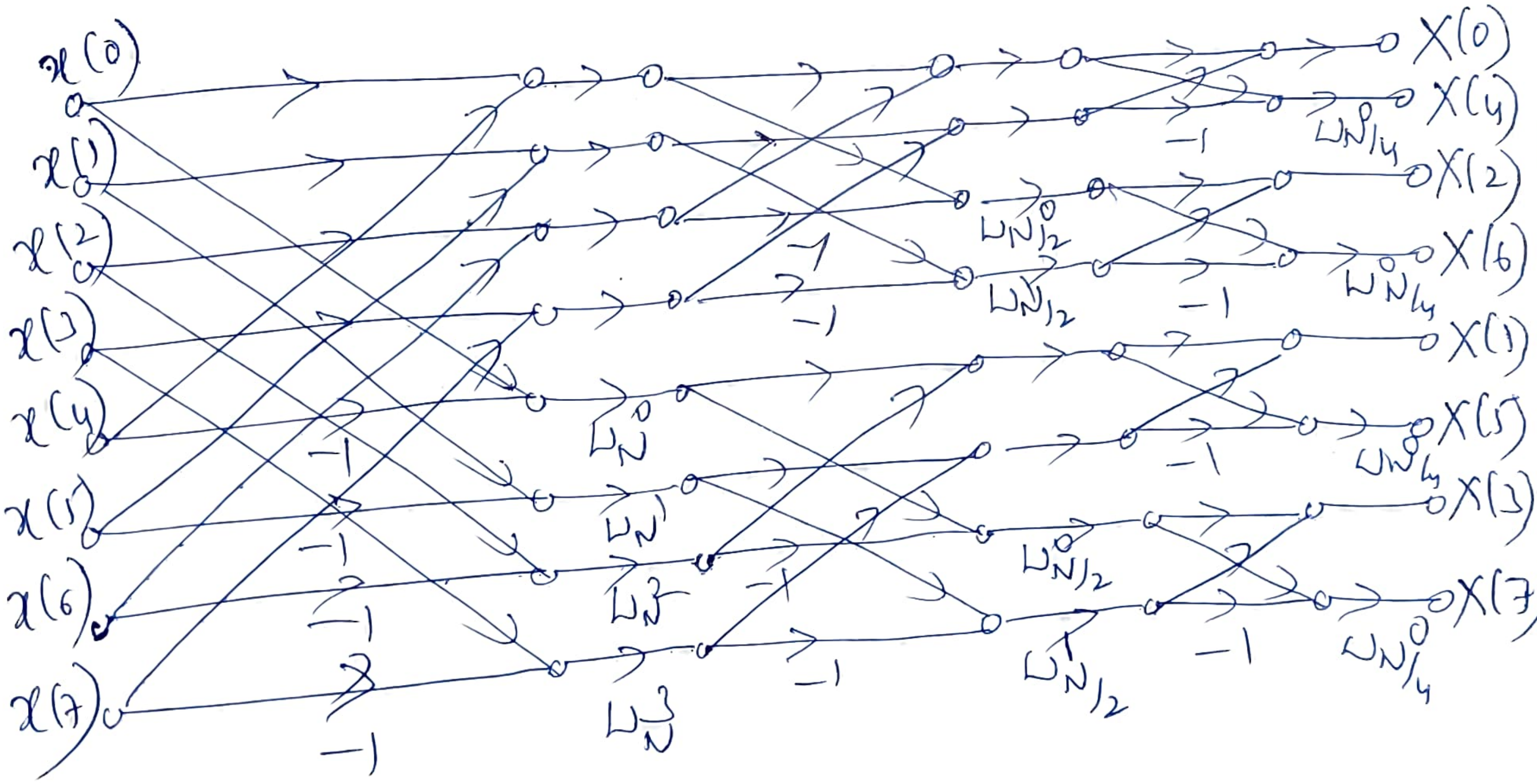
$$= \sum_{n=0}^{N/2-1} \left\{ \left[x(n) - x(n+N/2) \right] W_N^{n} \right\} W_{N/2}^{rn} \longrightarrow (2)$$





N/2 pt DFT is replaced by two N/4 pt DFT's.



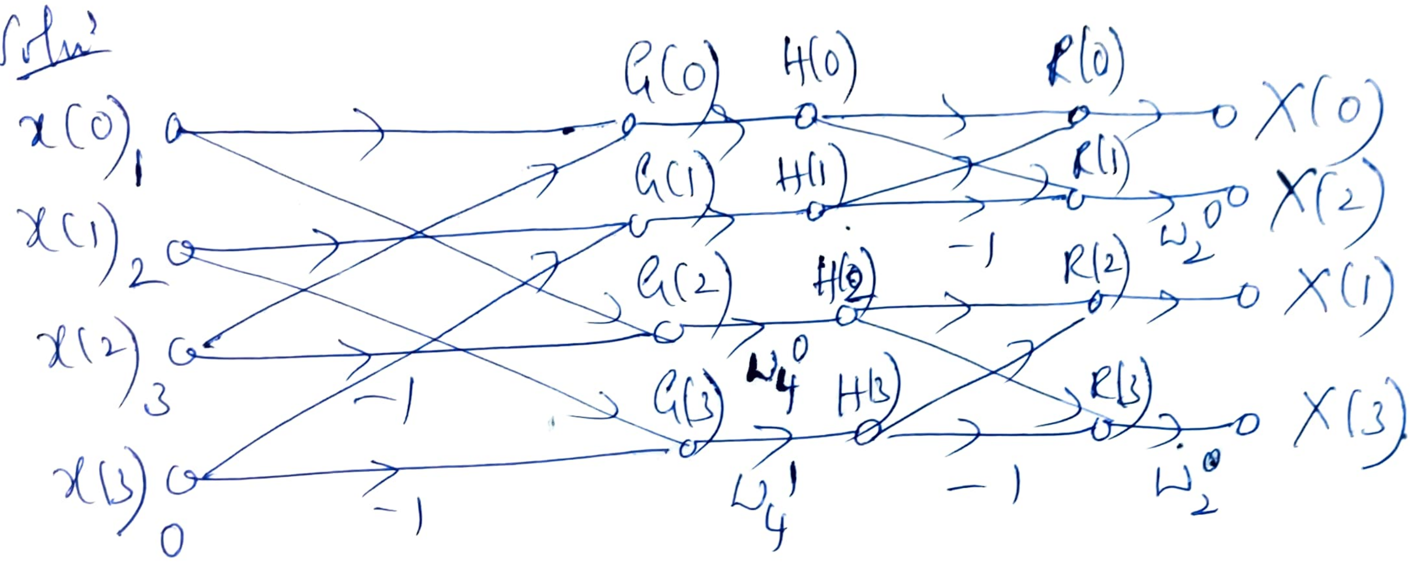


Observations

1) I/p in proper order
2) O/p in bit reversed order

Prob: find 4 point DFT of the sequence $x(n) = \{\underset{\uparrow}{1}, 2, 3, 0\}$ using radix-2 DIF FFT algorithm. verify the result by direct computation. (15)

Soln:



$G(0) = x(0) + x(2) = 1+3 = 4$

$G(1) = x(1) + x(3) = 2+0 = 2$

$G(2) = x(0) - x(2) = 1-3 = -2$

$G(3) = x(1) - x(3) = 2-0 = 2$

$H(0) = G(0) = 4$ $\quad$ $H(1) = G(1) = 2$

$H(2) = G(2)W_4^0 = -2$ $\quad$ $H(3) = G(3)W_4^1 = -2j$

$R(0) = H(0) + H(1) = 4+2 = 6$

$R(1) = H(0) - H(1) = 4-2 = 2$

$R(2) = H(2) + H(3) = -2-2j$

$R(3) = H(2) - H(3) = -2+2j$

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$X(0) = R(0) = 6$

$X(2) = R(1)\, W_2^0 = 2$

$X(1) = R(2) = -2-2j$

$X(3) = R(3)\, W_2^0 = -2+2j$

$$X(k) = \{ \underset{\uparrow}{6}, -2-2j, 2, -2+2j \}$$

b) $$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} \qquad 0 \le k \le 3$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2-2j \\ 2 \\ -2+2j \end{bmatrix}$$

$$X(k) = \{ \underset{\uparrow}{6}, -2-2j, 2, -2+2j \}$$

Prob: find IDFT using Radix-2 DIF FFT algorithm.

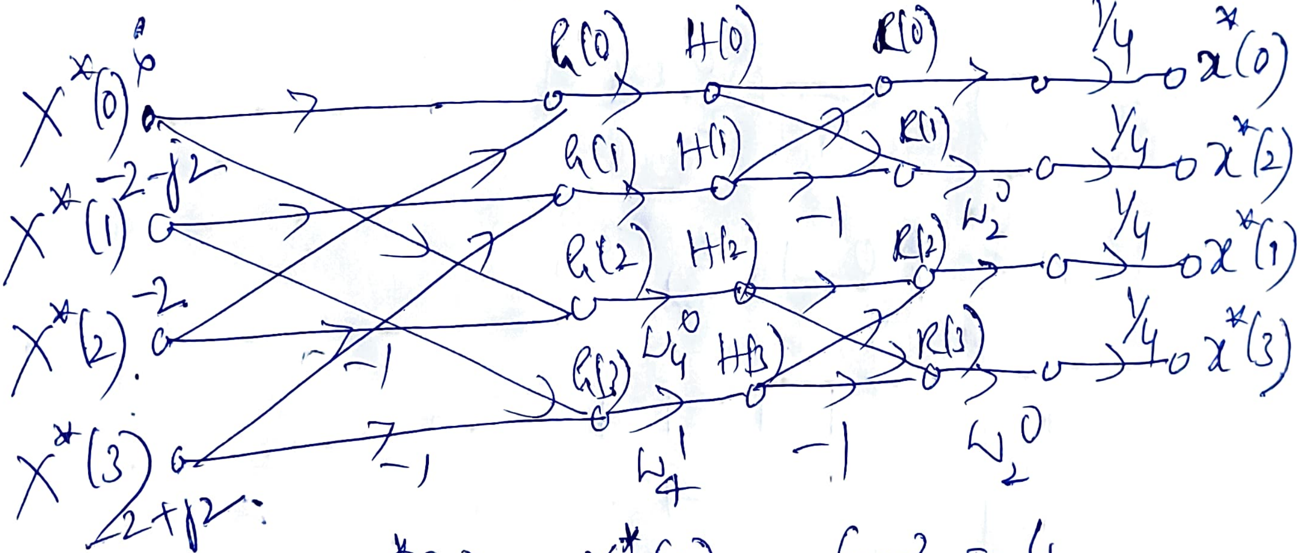
$$X(k) = \{6, -2+j2, -2, -2-j2\}$$

Verify the result by direct computation.

Soln:

$$x(n) = \frac{1}{N}\{DFT[X^*(k)]\}^*$$

$$X^*(k) = \{6, -2-j2, -2, -2+j2\}$$



$$G(0) = X^*(0) + X^*(2) = 6 - 2 = 4$$

$$G(1) = X^*(1) + X^*(3) = -2-j2-2+j2 = -4$$

$$G(2) = X^*(0) - X^*(2) = 6 + 2 = 8$$

$$G(3) = X^*(1) - X^*(3) = -2-j2+2-j2 = -j4$$

$H(0) = G(0)$ $\quad$ $H(1) = G(1)$

$H(2) = G(2)W_4^0 = G(2)$ $\quad$ $H(3) = W_4^1\, G(3) = -j \times -j4$

$= -4$

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$$R(0) = H(0) + H(1) = 4 - 4 = 0$$

$$R(1) = H(0) - H(1) = 4 + 4 = 8$$

$$R(2) = H(2) + H(3) = 8 - 4 = 4$$

$$R(3) = H(2) - H(3) = 8 + 4 = 12$$

$$x^*(0) = \frac{1}{4} R(0) = 0$$

$$x^*(2) = \frac{1}{4} W_2^0 R(1) = 2$$

$$x^*(1) = \frac{1}{4} R(2) = 1$$

$$x^*(3) = \frac{1}{4} W_2^0 \times R(3) = 3$$

$$x(n) = \{\underset{\uparrow}{0}, 1, 2, 3\}$$

b)

$$\begin{bmatrix} x^*(0) \\ x^*(1) \\ x^*(2) \\ x^*(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ -2-j2 \\ -2 \\ -2+j2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x(n) = \{\underset{\uparrow}{0}, 1, 2, 3\}$$

Composite - Radix FFT

Composite or mixed Radix FFT is used when N is a Composite no. ie, a no. which has more than one prime factor.

eg: $N = 12$ or 18.

$N = 18$
$= 2 \times 3 \times 3$

Q:

1) Develop Radix-3 DIT FFT algorithm for Computing DFT for $N=9$.

Soln

$$N = 9 = 3 \times 3 = m_1 \times m_2$$

$$X(k) = \sum_{n=0}^{8} x(n) W_9^{kn} \qquad 0 \le k \le 8$$

$$= \sum_{n=0}^{2} x(3n) W_9^{3nk} + \sum_{n=0}^{2} x(3n+1) W_9^{(3n+1)k} + \sum_{n=0}^{2} x(3n+2) W_9^{(3n+2)k} \quad \text{—(1)}$$

$$X(k) = X_1(k) + W_9^{k} X_2(k) + W_9^{2k} X_3(k) \quad \text{—(2)}$$

$$X_1(k) = \sum_{n=0}^{2} x(3n)\, W_9^{3nk}$$

$$= x(0) + x(3)\, W_9^{3k} + x(6)\, W_9^{6k} \quad \text{— (3)}$$

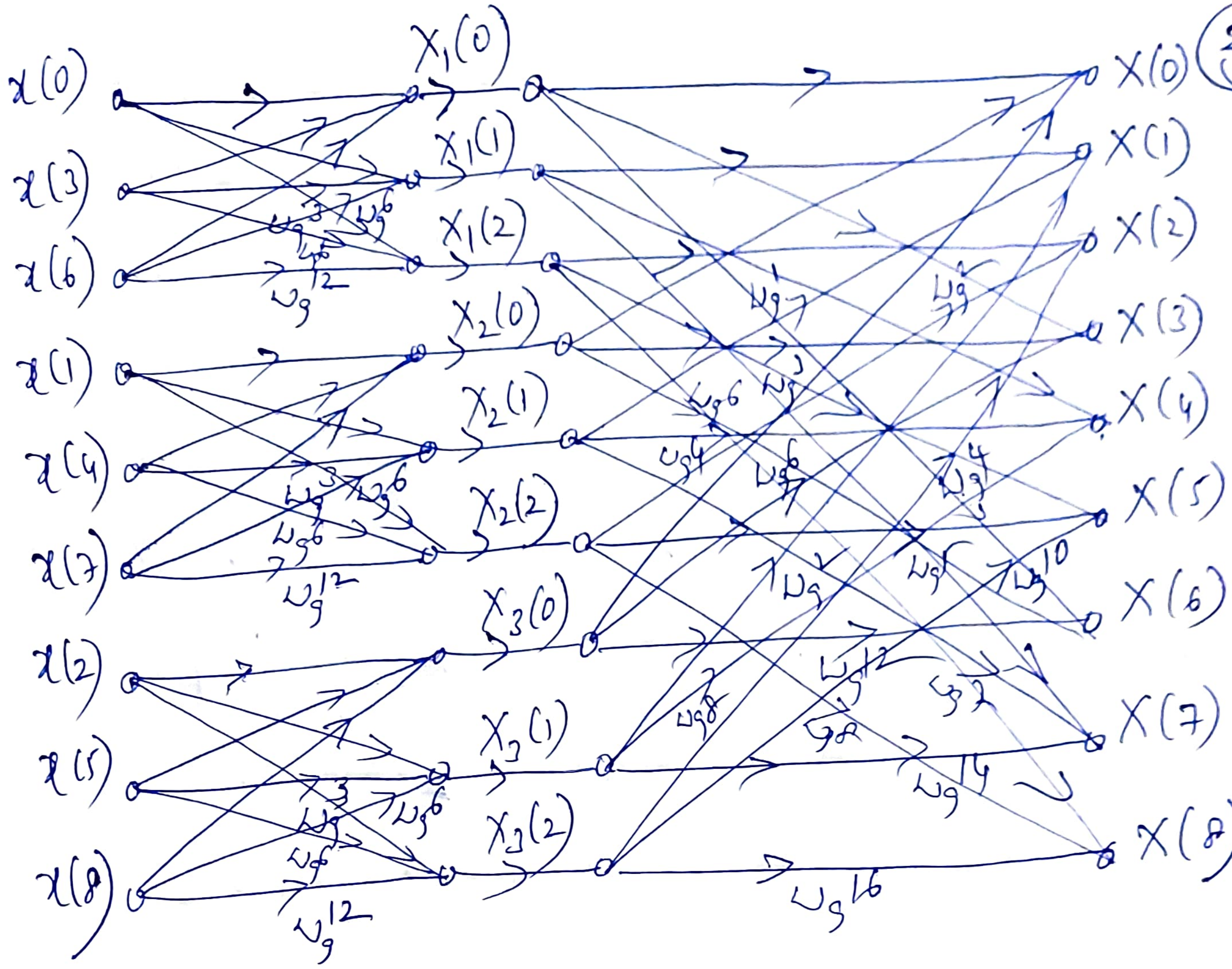
$$X_2(k) = \sum_{n=0}^{2} x(3n+1)\, W_9^{3nk}$$

$$= x(1) + x(4)\, W_9^{3k} + x(7)\, W_9^{6k} \quad \text{— (4)}$$

$$X_3(k) = \sum_{n=0}^{2} x(3n+2)\, W_9^{3nk}$$

$$= x(2) + x(5)\, W_9^{3k} + x(8)\, W_9^{6k} \quad \text{— (5)}$$

$$X(0) = X_1(0) + W_9^{0} X_2(0) + W_9^{0} X_3(0)$$
$$X(1) = X_1(1) + W_9^{1} X_2(1) + W_9^{2} X_3(1)$$
$$X(2) = X_1(2) + W_9^{2} X_2(2) + W_9^{4} X_3(2)$$
$$X(3) = X_1(0) + W_9^{3} X_2(0) + W_9^{6} X_3(0)$$
$$X(4) = X_1(1) + W_9^{4} X_2(1) + W_9^{8} X_3(1)$$
$$X(5) = X_1(2) + W_9^{5} X_2(2) + W_9^{10} X_3(2)$$
$$X(6) = X_1(0) + W_9^{6} X_2(0) + W_9^{12} X_3(0)$$
$$X(7) = X_1(1) + W_9^{7} X_2(1) + W_9^{14} X_3(1)$$
$$X(8) = X_1(2) + W_9^{8} X_2(2) + W_9^{16} X_3(2)$$



from Eq^n (3)

$$X_1(0) = x(0) + x(3)\,W_9^0 + x(6)\,W_9^0$$

$$X_1(1) = x(0) + x(3)\,W_9^3 + x(6)\,W_9^6$$

$$X_1(2) = x(0) + x(3)\,W_9^6 + x(6)\,W_9^{12}$$

2) Develop DIT FFT algorithm (26)
for $N = 6$ and draw SFG for

(a) $N = 6 = 3 \times 2$ (b) $N = 6 = 2 \times 3$

(c) Using the flow graphs compute 6 pt. DFT of the sequence
$x(n) = \{1, -1, 2, -2, 3, -3\}$ and verify the result by direct computation.

soln a) $N = 6 = 3 \times 2 = m_1 \times m_2$

$m_1 = 3 \quad m_2 = 2.$

$$X(k) = \sum_{n=0}^{5} x(n)\, W_6^{nk} \qquad 0 \le k \le 5$$

$$= \sum_{n=0}^{1} x(3n)\, W_6^{3nk} + \sum_{n=0}^{1} x(3n+1)\, W_6^{(3n+1)k}$$

$$+ \sum_{n=0}^{1} x(3n+2)\, W_6^{(3n+2)k}$$

$$= \sum_{n=0}^{1} x(3n)\, W_6^{3nk} + W_6^{k} \sum_{n=0}^{1} x(3n+1)\, W_6^{3nk} + W_6^{2k} \sum_{n=0}^{1} x(3n+2)\, W_6^{3nk}$$

$$X(k) = X_1(k) + W_6^{k} X_2(k) + W_6^{2k} X_3(k) \quad \text{--- (1)}$$

$$X_1(k) = \sum_{n=0}^{1} x(3n)\, W_6^{3nk}$$

$$= x(0) + x(3)\, W_6^{3k} \quad \text{--- (2)}$$

$$X_2(k) = \sum_{n=0}^{1} x(3n+1)\, W_6^{3nk}$$

$$= x(1) + x(4)\, W_6^{3k} \quad \text{--- (3)}$$

$$X_3(k) = \sum_{n=0}^{1} x(3n+2)\, W_6^{3nk}$$

$$= x(2) + x(5)\, W_6^{3k} \quad \text{--- (4)}$$

from (2)

$$X_1(0) = x(0) + x(3)$$

$$X_1(1) = x(0) + x(3)\, W_6^{3}$$

from (3)

$$X_2(0) = x(1) + x(4)$$

$$X_2(1) = x(1) + x(4)\, W_6^{3}$$

from (4)

$$X_3(0) = x(2) + x(5)$$

$$X_3(1) = x(2) + x(5)\, W_6^{3}.$$

from ①

$$X(k) = X_1(k) + W_6^{k} X_2(k) + W_6^{2k} X_3(k)$$

$$X(0) = X_1(0) + X_2(0) + X_3(0)$$

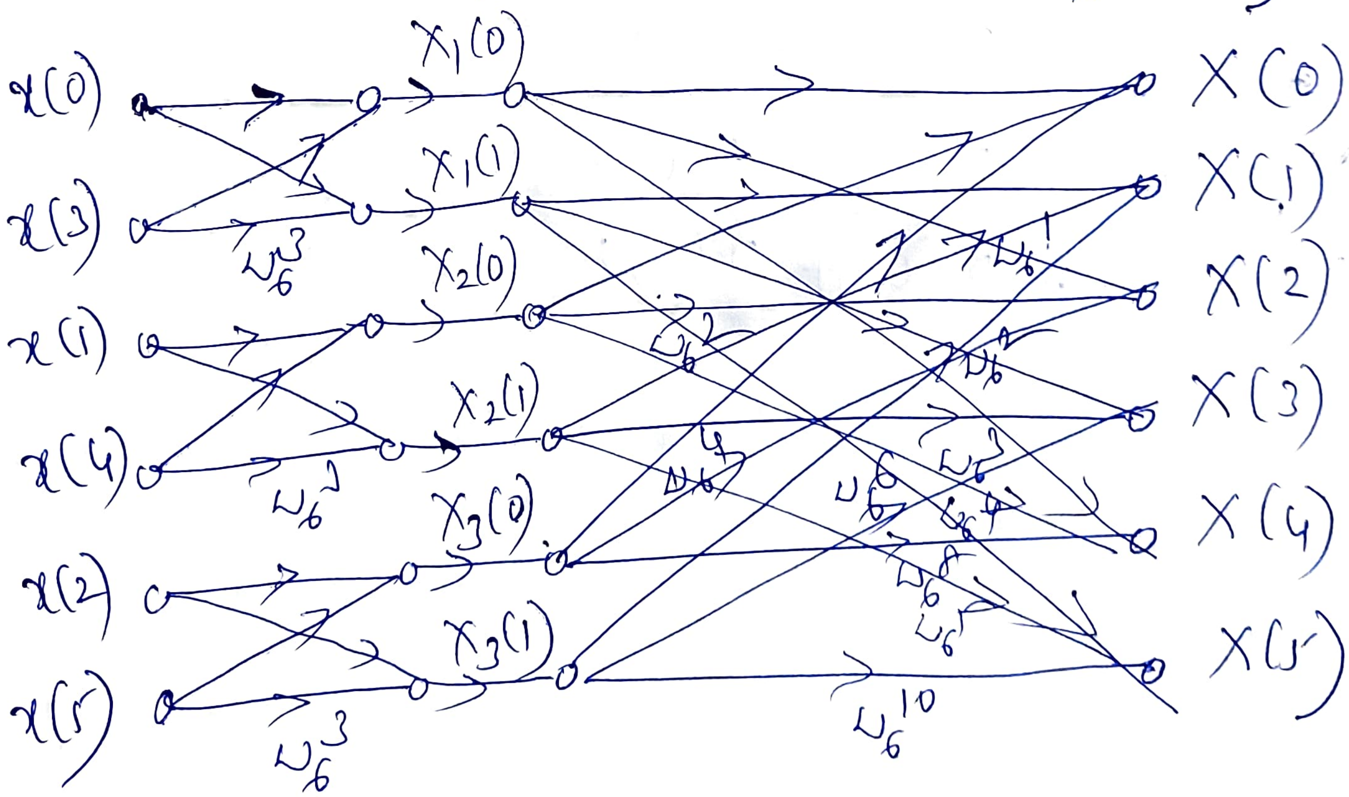
$$X(1) = X_1(1) + X_2(1) W_6^1 + X_3(1) W_6^2$$

$$X(2) = X_1(0) + W_6^2 X_2(0) + W_6^4 X_3(0)$$

$$X(3) = X_1(1) + W_6^3 X_2(1) + W_6^6 X_3(1)$$

$$X(4) = X_1(0) + W_6^4 X_2(0) + W_6^8 X_3(0)$$

$$X(5) = X_1(1) + W_6^5 X_2(1) + W_6^{10} X_3(1)$$



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b) $N = 6 = 2 \times 3 = m_1 \times m_2$

$$X(k) = \sum_{n=0}^{5} x(n)\, W_6^{nk} \qquad 0 \le k \le 5$$

$$= \sum_{n=0}^{2} x(2n)\, W_6^{2nk} + \sum_{n=0}^{2} x(2n+1)\, W_6^{(2n+1)k}$$

$$X(k) = \sum_{n=0}^{2} x(2n)\, W_6^{2nk} + W_6^{k} \sum_{n=0}^{2} x(2n+1)\, W_6^{2nk}$$

$$X(k) = X_1(k) + W_6^{k}\, X_2(k) \longrightarrow (1)$$

$$X_1(k) = \sum_{n=0}^{2} x(2n)\, W_6^{2nk}$$

$$\left.\begin{aligned} X_1(0) &= x(0) + x(2) + x(4) \\ X_1(1) &= x(0)\, W_6^{0} + x(2)\, W_6^{2} + x(4)\, W_6^{4} \\ X_1(2) &= x(0)\, W_6^{0} + x(2)\, W_6^{4} + x(4)\, W_6^{8} \end{aligned}\right\} (A)$$

$$W_6^{6} = W_6^{0}; \quad W_6^{8} = W_6^{2}$$

$$X_2(k) = \sum_{n=0}^{2} x(2n+1)\, W_6^{2nk}$$

$$X_2(k) = x(1) + x(3)\, W_6^{2k} + x(5)\, W_6^{4k}$$

$$\left.\begin{aligned}
X_2(0) &= x(1) + x(3) + x(5) \\
X_2(1) &= x(1) + x(3)\, W_6^{2} + x(5)\, W_6^{4} \\
X_2(2) &= x(1) + x(3)\, W_6^{4} + x(5)\, W_6^{8}
\end{aligned}\right\} B$$

$$X(k) = X_1(k) + W_6^{k} X_2(k)$$

$$X(0) = X_1(0) + X_2(0)$$

$$X(1) = X_1(1) + W_6^{1} X_2(1)$$

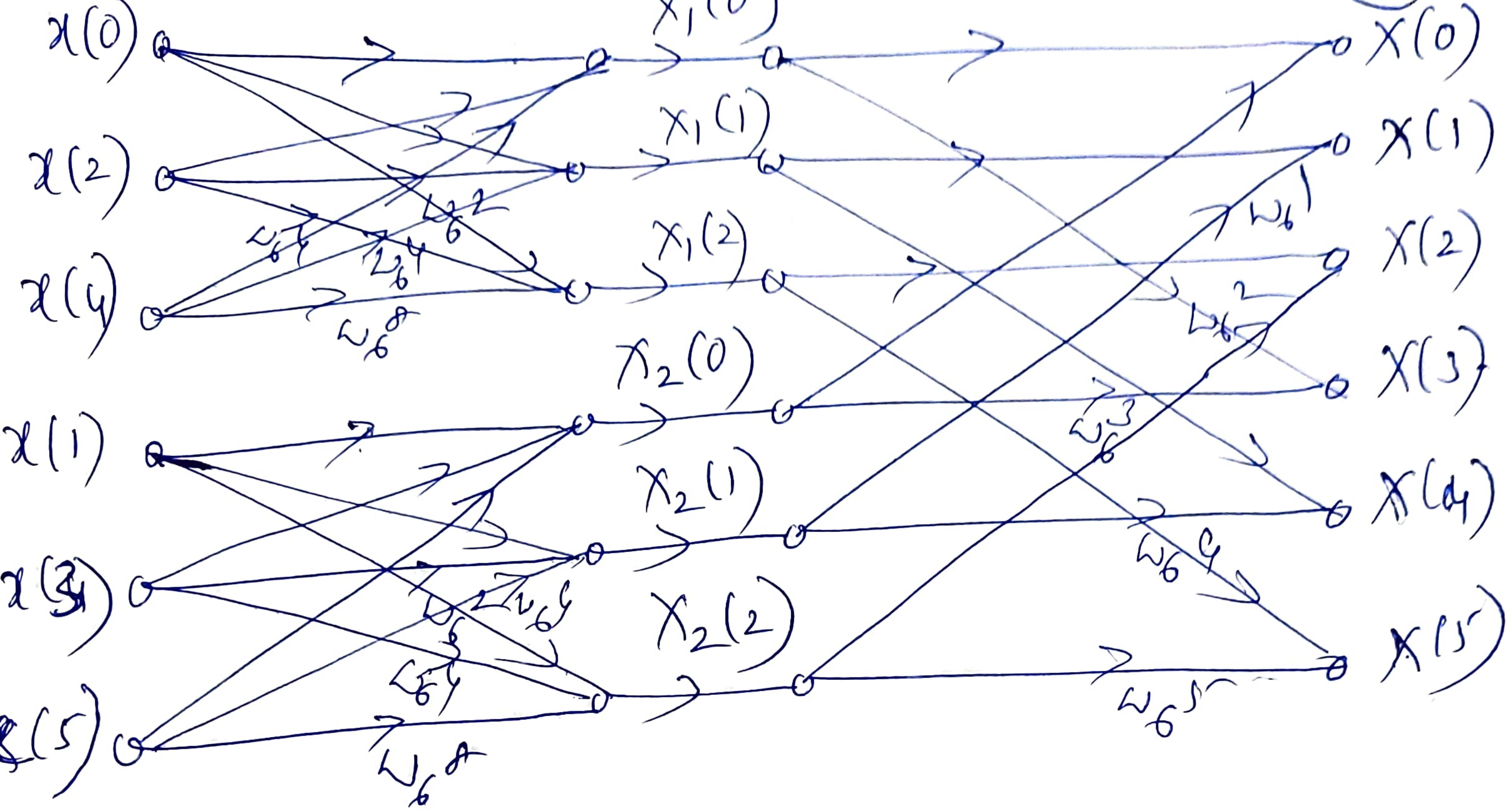
$$X(2) = X_1(2) + W_6^{2} X_2(2)$$

$$X(3) = X_1(0) + W_6^{3} X_2(0)$$

$$X(4) = X_1(1) + W_6^{4} X_2(1)$$

$$X(5) = X_1(2) + W_6^{5} X_2(2)$$

(31)



Fast Convolution techniques / Segmentation techniques.

a) overlap - add method

b) overlap - save method

a) overlap-add method

Prob 1) Compute $y(n) = x(n) * h(n)$ using overlap-add method and verify the result by direct Computation

$$h(n) = \{1, 1, 1\}$$

$$x(n) = \{1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 4, 2, 1, -3\}$$

Solution

$$M = 3$$

$$L = 2^M = 2^3 = 8$$

$$L = M + N - 1 \Rightarrow N = 6$$

Iteration 1

N-1 zeros

$$h_1(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$$

$$x_1(n) = \{1, 2, 0, -3, 4, 2, 0, 0\}$$

M-1 zeros

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$$X_1(k) = 1 + 2W_8^{k} + 0 - 3W_8^{3k} + 4W_8^{4k} + 2W_8^{5k}$$

$$H_1(k) = 1 + W_8^{k} + W_8^{2k}$$

$$Y_1(k) = X_1(k) \cdot H_1(k)$$

$$Y_1(k) = 1 + 3W_8^{k} + 3W_8^{2k} - W_8^{3k} + W_8^{4k}$$
$$+ 3W_8^{5k} + 6W_8^{6k} + 2W_8^{7k}$$

$$y_1(n) = \{1, 3, 3, -1, 1, 3, 6, 2\}$$

<u>Iteration 2</u>

$$h_2(n) = \{1, 1, 1, \underbrace{0, 0, 0, 0, 0}_{N-1 \text{ Zeros}}\}$$

$$x_2(n) = \{-1, 1, -2, 3, 4, 2, \underbrace{0, 0}_{M-1 \text{ Zeros}}\}$$

$$X_2(k) = -1 + W_8^{k} - 2W_8^{2k} + 3W_8^{3k} + 4W_8^{4k} + 2W_8^{5k}$$

$$H_2(k) = 1 + W_8^{k} + W_8^{2k}$$

$$Y_2(k) = X_2(k) \cdot H_2(k)$$
$$= -1 + 0 - 2W_8^{2k} + 2W_8^{3k} + 5W_8^{4k} + 9W_8^{5k}$$
$$+ 6W_8^{6k} + 2W_8^{7k}$$

$$y_2(n) = \{-1, 0, -2, 2, 5, 9, 6, 2\}$$

Iteration 3

$h_3(n) = \{1, 1, 1, \underbrace{0, 0, 0, 0, 0}_{N-1 \text{ zeros}}\}$

$x_3(n) = \{1, -3, 0, 0, 0, 0, 0, 0\}$

$H_3(k) = 1 + W_8^{k} + W_8^{2k}$

$X_3(k) = 1 - 3W_8^{k}$

$Y_3(k) = X_3(k) \cdot H_3(k)$

$= 1 - 2W_8^{k} - 2W_8^{2k} - 3W_8^{3k}$

$y_3(n) = \{1, -2, -2, -3\}$

To find y(n)

$y(n) = y_1(n) + y_2(n-N) + y_3(n-2N)$

$= y_1(n) + y_2(n-6) + y_3(n-12)$

$$
\begin{array}{cccccccccccccccc}
= 1 & 3 & 3 & -1 & 1 & 3 & 6 & 2 & & & & & & & & \\
 & & & & & & -1 & 0 & -2 & 2 & 5 & 9 & 6 & 2 & & \\
 & & & & & & & & & & & & 1 & -2 & -2 & -3 \\
\hline
\{1 & 3 & 3 & -1 & 1 & 3 & 5 & 2 & -2 & 2 & 5 & 9 & 7 & 0 & -2 & -3\}
\end{array}
$$

$L1 = 3 \quad , \quad L2 = 14$

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Length of $y(n) = L_1 + L_2 - 1$

$= 3 + 14 - 1 = 16$

Direct Computation

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \\ -3 & 0 & 2 \\ 4 & -3 & 0 \\ 2 & 4 & -3 \\ -1 & 2 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 1 \\ 4 & 3 & -2 \\ 2 & 4 & 3 \\ 1 & 2 & 4 \\ -3 & 1 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \\ 1 \\ 3 \\ 5 \\ 2 \\ -2 \\ 2 \\ 5 \\ 9 \\ 7 \\ 0 \\ -2 \\ -3 \end{bmatrix}$$

Prob 2

Compute $x(n) * h(n) = y(n)$ using overlap-add method and verify the result by direct computation.

$$h(n) = \{1, 2, 3\}$$

$$x(n) = \{1, 4, 3, 2, 5, 4, 3, 2, 1, -5, -4, 4, 5, 2, 3\}$$

Mixed - Radix DFT

Prob: Develop a DIF FFT algorithm for ~~decomposing~~ Computing DFT for $N=6$ and draw the signal flow graph for (a) $N = 2 \times 3$ (b) $N = 3 \times 2$

Solution a) $N = 6 = 2 \times 3$

$$X(k) = \sum_{n=0}^{5} x(n) W_6^{nk} \qquad 0 \le k \le 5$$

$$= \sum_{n=0}^{2} x(n) W_6^{nk} + \sum_{n=3}^{5} x(n) W_6^{nk}$$

$$= \sum_{n=0}^{2} x(n) W_6^{nk} + \sum_{n=0}^{2} x(n+3) W_6^{(n+3)k}$$

$$X(k) = \sum_{n=0}^{2} \left[x(n) + x(n+3) W_6^{3k} \right] W_6^{nk} \longrightarrow (1)$$

put $k = 2k$

$$X(2k) = \sum_{n=0}^{2} \left[x(n) + x(n+3) W_6^{6k} \right] W_6^{2nk}$$

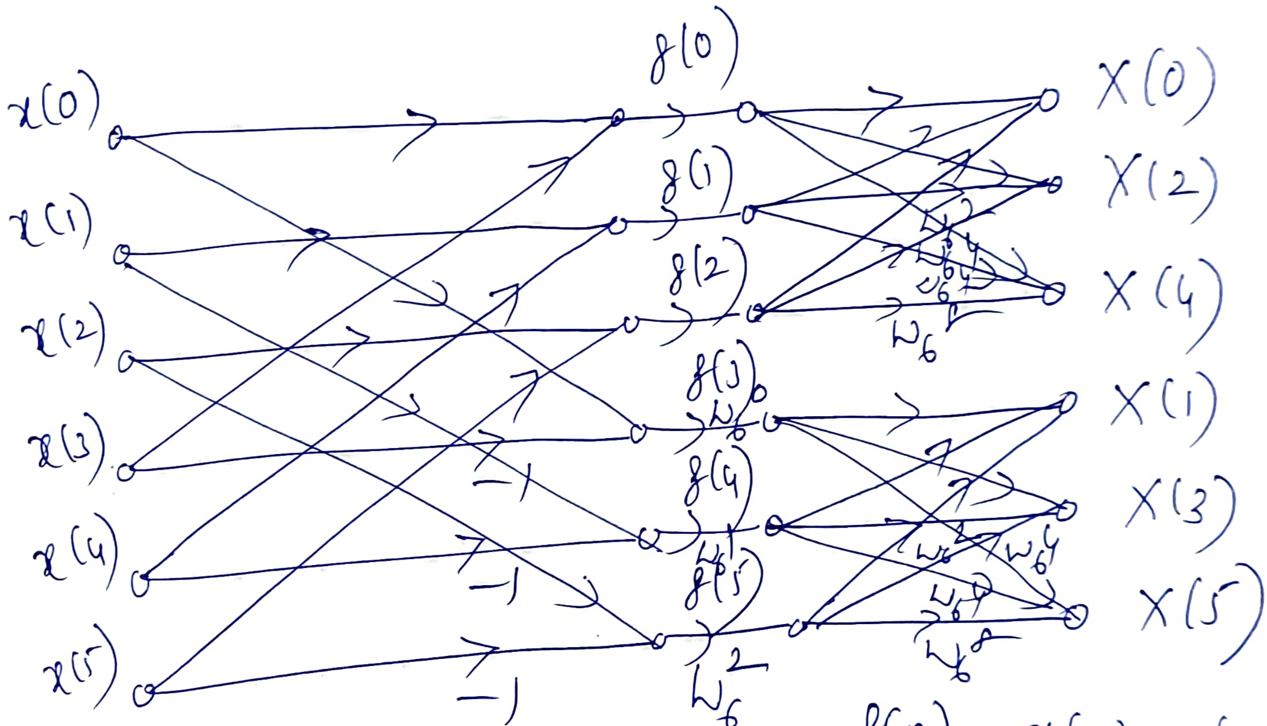
$$= \sum_{n=0}^{2} \left[x(n) + x(n+3) \right] W_3^{nk} \longrightarrow (2)$$

put $k = 2k+1$ (38)

$$X(2k+1) = \sum_{n=0}^{2} \left[x(n) + x(n+3) W_6^{3(2k+1)} \right] W_6^{(2k+1)n}$$

$$= \sum_{n=0}^{2} \left\{ \left[x(n) - x(n+3) \right] W_6^{n} \right\} W_3^{nk}$$

$\longrightarrow$ (3)



from (2)

$X(0) = g(0) + g(1) + g(2)$

$X(2) = g(0) + g(1) W_6^2 + g(2) W_6^4$

$X(4) = g(0) + g(1) W_6^4 + g(2) W_6^8$

$g(n) = x(n) + x(n+3)$

$g(0) = x(0) + x(3)$

$g(1) = x(1) + x(4)$

$g(2) = x(2) + x(5)$

$$g(3) = x(0) - x(3)$$
$$g(4) = x(1) - x(4)$$
$$g(5) = x(2) - x(5)$$

$$X(1) = g(3) + g(4) + g(5)$$
$$X(3) = g(3) + g(4)\, W_6^1 W_6^2 + g(5)\, W_6^2\, W_6^4$$
$$X(5) = g(3) + g(4)\, W_6^1 W_6^4 + g(5)\, W_6^2 W_6^8.$$

b) $N = 6 = 3 \times 2$

$$X(k) = \sum_{n=0}^{5} x(n)\, W_6^{kn} \qquad 0 \le k \le 5$$

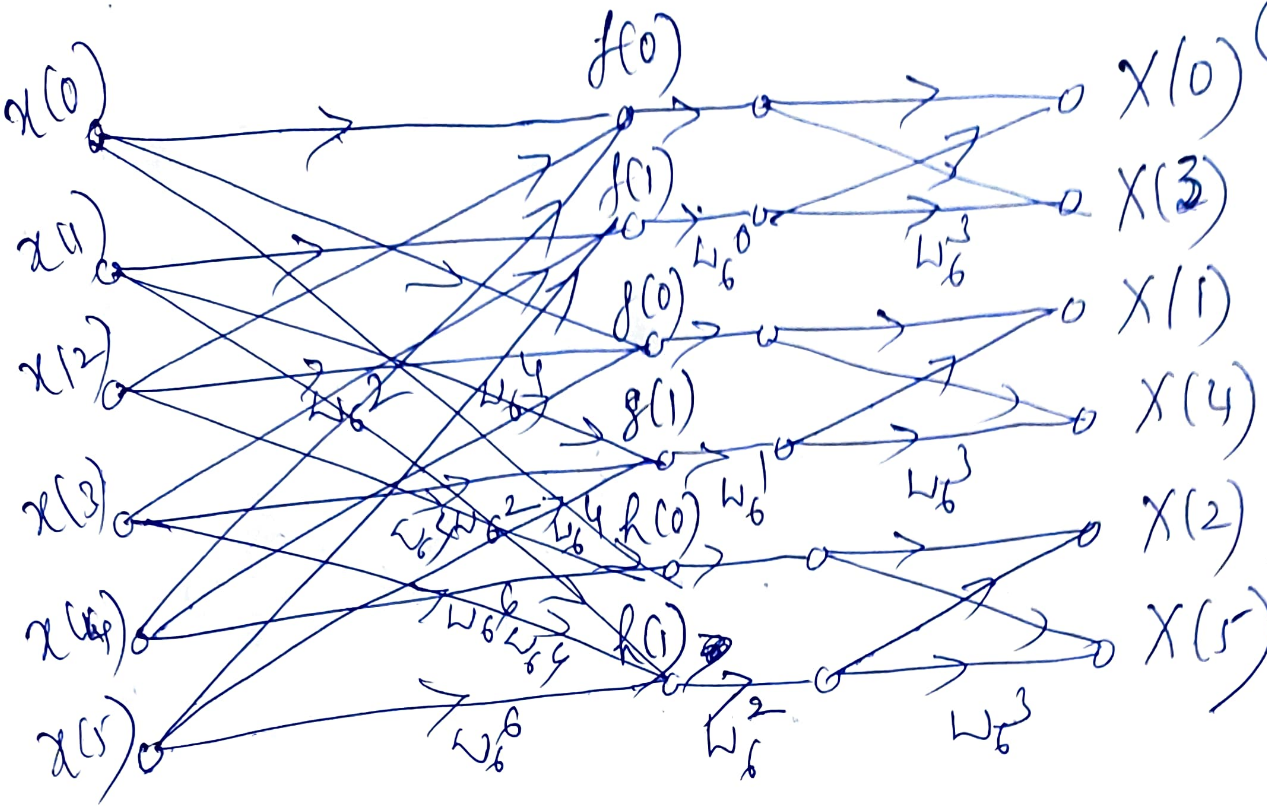
$$= \sum_{n=0}^{1} x(n)\, W_6^{kn} + \sum_{n=2}^{3} x(n)\, W_6^{kn} + \sum_{n=4}^{5} x(n)\, W_6^{kn}$$

$$X(k) = \sum_{n=0}^{1} \left\{ x(n) + x(n+2) W_6^{2k} + x(n+4) W_6^{4k} \right\} W_6^{kn}$$

$$X(3k) = \sum_{n=0}^{1} \left[x(n) + x(n+2) + x(n+4) \right] W_6^{3nk}$$

$$X(3k+1) = \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^2 + x(n+4) W_6^4 \right] W_6^n\, W_6^{3nk}$$

$$X(3k+2) = \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^4 + x(n+4) W_6^6 \right] W_6^{2n} \cdot W_6^{3nk}$$



$$f(n) = x(n) + x(n+2) + x(n+4)$$

$$g(n) = x(n) + x(n+2)\, W_6^2 + x(n+4)\, W_6^4$$

$$h(n) = x(n) + x(n+2)\, W_6^4 + x(n+4)\, W_6^6$$

$X(0) = f(0) + f(1)\, W_6^0$	$X(3) = f(0) + f(1)\, W_6^0\, W_6^3$
$X(1) = g(0) + g(1)\, W_6^1$	$X(4) = g(0) + g(1)\, W_6^1\, W_6^3$
$X(2) = h(0) + h(1)\, W_6^2$	$X(5) = h(0) + h(1)\, W_6^2\, W_6^3$

Overlap- Save method

Prob: Compute $y(n) = x(n) * h(n)$ using overlap- save method. verify the result by direct Computation.

$$h(n) = \{1, 1, 1\}$$

$$x(n) = \{1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 4, 2, 1, -3\}$$

Solu^n

$$M = 3$$

$$L = 2^M = 2^3 = \underline{\underline{8}}$$

$$L = M + N - 1 \Rightarrow \underline{\underline{N = 6}}$$

Iteration 1

N-1 Zeros

$$h_1(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$$

$$x_1(n) = \{0, 0, 1, 2, 0, -3, 4, 2\}$$

M-1 Zeros.

$$H_1(k) = 1 + W_8^k + W_8^{2k}$$

$$X_1(k) = W_8^{2k} + 2W_8^{3k} + 0 - 3W_8^{5k} + 4W_8^{6k} + 2W_8^{7k}$$

$$Y_1(k) = 6 + 2W_8^k + W_8^{2k} + 3W_8^{3k} + 3W_8^{4k} - W_8^{5k} + W_8^{6k} + 3W_8^{7k}$$

$y_1(n) = \{6, 2, 1, 3, 3, -1, 1, 3\}$

discarded

$y_1'(n) = \{1, 3, 3, -1, 1, 3\}$

Iteration 2

N-1 Zeros.

$h_2(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$x_2(n) = \{4, 2, -1, 1, -2, 3, 4, 2\}$

Last two values of $x_1(n)$

$H_2(k) = 1 + W_8^{k} + W_8^{2k}$

$X_2(k) = 4 + 2W_8^{k} - W_8^{2k} + W_8^{3k} - 2W_8^{4k}$
$+ 3W_8^{5k} + 4W_8^{6k} + 2W_8^{7k}$

$Y_2(k) = X_2(k) \cdot H_2(k)$

$= 10 + 8W_8^{k} + 5W_8^{2k} + 2W_8^{3k} - 2W_8^{4k}$
$+ 2W_8^{5k} + 5W_8^{6k} + 9W_8^{7k}$

$y_2(n) = \{10, 8, 5, 2, -2, 2, 5, 9\}$

discarded

$y_2'(n) = \{5, 2, -2, 2, 5, 9\}$

Iteration 3

$h_3(n) = \{1, 1, 1, \underbrace{0, 0, 0, 0, 0}_{N-1 \text{ zeros}}\}$

$x_3(n) = \{\underbrace{4, 2}_{\text{Last two samples of } x_2(n)}, 1, -3, 0, 0, 0, 0\}$

$H_3(k) = 1 + W_8^k + W_8^{2k}$

$X_3(k) = 4 + 2W_8^k + W_8^{2k} - 3W_8^{3k}$

$Y_3(k) = 4 + 6W_8^k + 7W_8^{2k} + 0$
$\quad - 2W_8^{4k} - 3W_8^{5k}$

$y_3(n) = \{\underbrace{4, 6}_{\text{discarded}}, 7, 0, -2, -3\}$

$y_3'(n) = \{7, 0, -2, -3\}$

To find $y(n)$

$y(n) = y_1'(n) + y_2'(n-6) + y_3'(n-12)$

$= \{1, 3, 3, -1, 1, 3, 5, 2, -2, 2, 5, 9,$
$7, 0, -2, -3\}$

Prob: Compute $y(n) = x(n) * h(n)$ (44)

using overlap - Save method and

verify your answer by direct

Computation

$$h(n) = \{2, -3\}$$

$$x(n) = \{4, 5, 6, 7, -8, -9, -10, -7, -6, 5, 4, 3, 2, 1\}$$