T_q $\alpha + W$ & $\beta + W$ · $V|W$.

$\alpha + W = \beta + W \Rightarrow \alpha - \beta \in W$

$T(\beta) \in W$

10 CANONICAL FORMS

10.1 INTRODUCTION

The relation of similarity (in matrices) arise when we study the various matrix representation of linear transformation of vector space into itself. Under similarity, the rank of matrix A is invariant since two similar matrices are certainly equivalent and rank is even invariant under equivalence.

Now, to check the similarity of two linear transformation, we have to compute a particular canonical form for each and check if these are same.

10.2 SIMILARITY OF LINEAR TRANSFORMATIONS

Definition 1. *Let V(F) be the n-dimensional vector space over the field F and A(V) be the set of all linear transformation from V to V. Then two linear transformation S, T∈A(V) are said to be similar if ∃ an invertible linear transformation C∈ A(V) such that*

$$T = CSC^{-1}$$

The similarity of linear transformation is also an equivalence relation. Thus, we can decompose A(V) into equivalence classes, each such classes is called similarity class.

Definition 2. *The special form of matrix representation (in some basis of V) of linear transformation in each similarity class is called canonical forms.*

Therefore, as we earlier said, in order to check the similarity of two linear transformations, we have to form a particular canonical form for each matrix representation of linear transformation and then we have to verify if these are the same.

10.3 INVARIANT SUBSPACE

Let $T:V\to V$ be a linear transformation. Then a subspace W of V is said to be invariant under T if $T(W) \subset W$ i.e., $\forall \alpha \in W$, $T(\alpha) \in W$.

THEOREM 1. *If W is a subspace invariant under T∈ A(V), then T induces a linear transformation T_q on quotient space V/W defined by $T_q(\alpha + W) = T(\alpha) + W$.*
Further, if T satisfies the polynomial $q(x) \in F[x]$, then so is T_q. Thus the minimal polynomial of T_q divide the minimal polynomial of T.

Proof. We have to show that T_q is well defined and also T_q is linear.

(i) T_q is well defined.

Take two elements $\alpha + W$ and $\beta + W$ of $V|W$ such that

$\alpha + W = \beta + W$, this implies $\alpha - \beta \in W$

Now, $T(\alpha - \beta) = T(\alpha) - T(\beta) \in W$ [W is T-invariant.]

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Similarity of LT

so, $\quad T(\alpha)+W = T(\beta) + W$

$\Rightarrow \quad T_q(\alpha + W) = T_q(\beta + W)$

Thus, T is well-defined.

(ii) $\mathbf{T_q}$ is linear transformation. For $(\alpha + W), (\beta + W) \in V/W$

We have

$$
\begin{aligned}
T_q\{(\alpha + W) + (\beta+W)\} &= T_q(\alpha+\beta+W)\\
&= T(\alpha+\beta)+W\\
&= T(\alpha)+T(\beta)+W\\
&= T(\alpha)+W+T(\beta)+W && [\because T \text{ is linear.}]\\
&= T_q(\alpha+W)+T(\beta+W)
\end{aligned}
$$

Also,
$$
\begin{aligned}
T_q\{C(\alpha+W)\} &= T_q(C\alpha+W)\\
&= T(C\alpha)+W = CT(\alpha)+W && [\because T \text{ is linear.}]\\
&= C\{T(\alpha)+W\}\\
&= CT_q(\alpha+W)
\end{aligned}
$$

So, T_q is linear.

Again $\alpha + W \in V/W$, then

$$
\begin{aligned}
T_q^{\,2}(\alpha+W) &= T^2(\alpha)+W = T(T(\alpha))+W\\
&= T_q(T(\alpha)) + W = T_q(T_q(\alpha+W))\\
&= T_q^2(\alpha+W)
\end{aligned}
$$

Similarly, we can prove $(T_q^{\,n}) = (T_q)^n;\ \forall\, n \geq 0$

Now, for a polynomial $q(x) \in F[x]$ where

$$q(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_0$$

$$
\begin{aligned}
q(T_q)(\alpha+W) &= q(T)(\alpha)+W\\
&= a_n T^n(\alpha) + a_{n-1}T^{n-1}(\alpha) + \ldots\ldots + a_0 I(\alpha) + W\\
&= \Sigma a_i T^i(\alpha) + W = \Sigma a_i (T^i(\alpha)+W)\\
&= \Sigma a_i T_q^{\,i}(\alpha+W)\\
&= a_i (T_q)^i(\alpha+W)\\
&= q(T_q)(\alpha+W)
\end{aligned}
$$

Hence T_q satisfy the polynomial $q(x)=0$ *i.e.*, $q(T_q)=0$. Thus, T_q is root of $q(x)=0$.

10.4 INVARIANT DIRECT-SUM DECOMPOSITIONS

Let $T:V\to V$ be a linear transformation such that V is the direct sum of T-invariant subspaces $W_1, W_2, \ldots\ldots, W_r$, *i.e.*, $V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_r$, *where* $T(W_i) \subset W_i$ for $i \in \mathrm{N}$ of T_i in the linear transformation restricted to W_i. Then T is said to be decomposable into operator T, so T can be written as $T = T_1 \oplus T_2 \ldots\ldots \oplus T_r$ and subspace $W_1, W_2, \ldots\ldots W_r$ are said to be T-invariant direct sum decomposition of V.

THEOREM 1. *If $V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_r$ where n_i is dimension of each subspace W_i and every subspace is invariant under $T \in A(V)$, then a basis of V can be found so that the matrix of T in this basis is of the form*

$$
\begin{bmatrix}
A_1 & 0 & \cdots & 0\\
0 & A_1 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots\\
0 & 0 & \cdots & A_r
\end{bmatrix}
$$

where each A_i is an $n_i \times n_i$ matrix of linear transformation induced by T on W_i.

Proof. Let $\left\{\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_n^{(1)}\right\}, \left\{\alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(2)}\right\} \ldots \left\{\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \alpha_n^{(r)}\right\}$ be the basis of $W_1, W_2, \ldots\ldots, W_r$ respectively.

Since $V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_r$, therefore

$$\left\{\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_n^{(1)}, \alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(2)}, \ldots\ldots, \alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \alpha_n^{(r)}\right\}$$

form a basis of V. Also, each W_i is T-invariant, so that $T(\alpha_j^{\,i}) \in W_i$ and it is linear combination of $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$ *i.e.*,

$$T\left(\alpha_j^{(i)}\right) = a_1^{(i)}\alpha_1^{(i)} + a_2^{(i)}\alpha_2^{(i)} + \ldots + a_n^{(i)}\alpha_n^{(i)} \qquad \ldots\ldots(1)$$

for every $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$.

Thus, matrix representation of T with respect to basis V is obtained by (1) which is

$$
\begin{bmatrix}
A_1 & 0 & \cdots & 0\\
0 & A_1 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots\\
0 & 0 & \cdots & A_r
\end{bmatrix}
$$
, where A_i is the matrix of T_i induced on W_i by T.

Now, we are in position to discuss some important canonical form for checking the similarity of two linear transformation. They are of following types:

(i) Normal form

(ii) Triangular form

(iii) Jordan form

(iv) Rational form

10.5 NORMAL FORM

A matrix A is said to be in *normal form* if it can be written as $A = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$, where I_r is square identity matrix of order r.

THEOREM 1. *Let $T: U \to V$ be a linear transformation and $\mathrm{rank}(T) = r$. Then there exists bases of U and V such that matrix representation of T has the form $A = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$, $I_r \to$ represents the identity matrix of order r.*

Proof. Let the dim. $U = m$ and dim. $V = n$. Let W be kernel of T. Now rank of T is r so the dimension of kernel space of T is $m-r$. Consider $\{\alpha_1, \alpha_2, \ldots, \alpha_{m-r}\}$ be the basis of W. By extension theorem, it can be extended to form the basis of U. Let this extension be $\{v_1, v_2, \ldots, v_n, \alpha_1, \alpha_2, \ldots, \alpha_{m-r}\}$. By setting a transformation $T(v_i) = u_i$, the set $\{u_1, u_2, \ldots, u_r\}$ form a basis of image (T) and thus base can be extended to form the basis of V. Let this base be

$$\{u_1, u_2 \ldots u_r, u_{r+1} \ldots u_n\}$$

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Similarity of LT

Here, we can observe that every v_i under T can be written as the linear combination of u_i as

$$\begin{aligned}
T(v_1)&=u_1=1u_1+0u_2+\ldots+0u_r+0u_n\\
T(v_2)&=u_2=0u_1+1u_2+\ldots+0u_r+0u_n\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
T(v_r)&=u_r=0u_1+0u_2+\ldots+1u_r+0u_n\\
T(\alpha_1)&=0=0u_1+0u_2+\ldots+0u_r+0u_{r+1}+\ldots+0u_n\\
T(\alpha_2)&=0=0u_1+0u_2+\ldots+0u_r+0u_{r+1}+\ldots+0u_n\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
T(\alpha_{m-r})&=0=0u_1+0u_2+\ldots+0u_r+0u_{r+1}+\ldots+0u_n
\end{aligned}$$

so the matrix representation of T is given by

$$A=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

Hence, $A=\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$.

10.6 TRIANGULAR FORM

Let $T:V\to V$ be a linear transformation on V over F, then the matrix of T in the basis $\{\alpha_1,\alpha_2\ldots,\alpha_n\}$ of V is triangular if

$$\begin{aligned}
T(\alpha_1)&=a_{11}\alpha_1\\
T(\alpha_2)&=a_{21}\alpha_1+a_{22}\alpha_2\\
T(\alpha_3)&=a_{31}\alpha_1+a_{32}\alpha_2+a_{33}\alpha_3\\
&\ldots \quad \ldots \quad \ldots \quad \ldots\\
T(\alpha_n)&=a_{n1}\alpha_1+a_{n2}\alpha_2+a_{nn}\alpha_n
\end{aligned}$$

THEOREM 1. *If $T\in A(V)$ has all its characteristic root in F, then there is a basis of V in which matrix representation of T is triangular.*

Proof. This result can be proved by induction on the dimension of V

(i) If dim. $V=1$, then every matrix representation is a matrix of order 1×1 which is trivially triangular.

(ii) Let this result hold good for all vector space over F of dimension $n-1$. Let dim $V=n>1$. If $\lambda_1\in F$ be a characteristic root of T, then $\exists$ a non-zero eigen-vector α_1 corresponding to λ_1 such that $T(\alpha_1)=a_{11}\alpha_1$. It is due to the fact that T has all its characteristic roots in F. Let W be the one dimensional subspace of V spanned by α_1 and T invariant, then quotient space $V_q=V/W$, and

$$\text{dim. } V_q = \text{dim. } V - \text{dim. } W = n-1.$$

Now, T induces a linear transformation T_q on V_q, whose minimal polynomial divides the minimal polynomial or T so all roots of minimal polynomial of T_q are also the roots of minimal polynomial of T and hence all roots lie in F.

Thus V and T satisfy the hypothesis of the theorem.

Now, dim. of $V_q = n-1$. Then by the hypothesis of induction, there is a basis for V_q as $\{\bar{\alpha}_2,\bar{\alpha}_3,\ldots,\bar{\alpha}_n\}$ of V_q such that

$$\begin{aligned}
T_q(\bar{\alpha}_2)&=a_{22}\bar{\alpha}_2\\
T_q(\bar{\alpha}_3)&=a_{32}\bar{\alpha}_2+a_{33}\bar{\alpha}_3\\
&\ldots \quad \ldots \quad \ldots\\
T_q(\bar{\alpha}_n)&=a_{n2}\bar{\alpha}_2+a_{n3}\bar{\alpha}_3+\ldots+a_{nn}\bar{\alpha}_n
\end{aligned}$$

Now, elements $\{\alpha_2,\alpha_3,\ldots\ldots,\alpha_n\}$ being the elements of V, also belong to cosets $\bar{\alpha}_2,\bar{\alpha}_3,\ldots,\bar{\alpha}_n$ respectively i.e., $\bar{\alpha}_i=\alpha_i+W$.

Now,
$$T_q(\bar{\alpha}_2)=a_{22}\bar{\alpha}_2$$
$$\Rightarrow \quad T_q(\alpha_2+W)=a_{22}(\alpha_2+W) \Rightarrow T(\alpha_2)+W=a_{22}\alpha_2+W$$
$$\Rightarrow \quad T(\alpha_2)-a_{22}\alpha_2\in W$$

But W is spanned by α_1 so

$$T(\alpha_2)-a_{22}\alpha_2=a_{21}\alpha_1$$
$$\Rightarrow \quad T(\alpha_2)=a_{21}\alpha_1+a_{22}\alpha_2$$

Similarly, for $\bar{\alpha}_3,\bar{\alpha}_4,\ldots,\bar{\alpha}_n$ we have $T(\alpha_i)=a_{i1}\alpha_1+a_{i2}\alpha_2+\ldots\ldots+a_{in}\alpha_n$.

In this way, we get

$$\begin{aligned}
T(\alpha_1)&=a_{11}\alpha_1\\
T(\alpha_2)&=a_{21}\alpha_1+a_{22}\alpha_2\\
&\ldots \quad \ldots \quad \ldots\\
T(\alpha_n)&=a_{n1}\alpha_1+a_{n2}\alpha_2+a_{nn}\alpha_n
\end{aligned}$$

Hence the matrix of T in the basis $\{\alpha_1,\alpha_2\ldots\ldots\alpha_n\}$ is triangular.

REMARKS

- The above theorem can be restated as : "If a square matrix A has all its characteristic roots in F, then A is similar to a triangular matrix, i.e., there exists an invertible matrix P such that $P^{-1}AP$ is triangular."
- If any linear transformation T is represented by a triangular matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & a_{nn}\end{bmatrix}$$

the characteristic polynomial of T is a product of linear factors and is given by

$$\Delta(x)=|A-x|=(x-a_{11})(x-a_{22})\ldots\ldots(x-a_{nn}).$$

THEOREM 2. *If dim. $V=n$ and if $T\in A(V)$ has all its characteristic roots in F, then T satisfy a polynomial of degree n over F.*

Proof. Let $\lambda_1,\lambda_2,\ldots\ldots,\lambda_n$ be the characteristic roots of T. Since T has all its characteristic root in F so there exists a basis $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of V such that

$$T(\alpha_1)=\lambda_1\alpha_1$$

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$$T(\alpha_2) = a_{21}\alpha_1 + \lambda_2\alpha_2$$
$$T(\alpha_3) = a_{31}\alpha_1 + a_{32}\alpha_2 + \lambda_3\alpha_3$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$T(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + \lambda_n\alpha_n$$

The all above relations can be rewritten as

$$(T-\lambda_1 I)(\alpha_1) = 0$$
$$(T-\lambda_2 I)(\alpha_2) = a_{21}\alpha_1$$
$$(T-\lambda_3 I)(\alpha_3) = a_{31}\alpha_1 + a_{32}\alpha_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(T-\lambda_n I)(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + \alpha_{n(n-1)}\alpha_{n-1}$$

Now, $(T-\lambda_2 I)(T-\lambda_1 I)(\alpha_2) = (T-\lambda_1 I)(T-\lambda_2 I)(\alpha_2)$

$$= (T-\lambda_1 I)a_{21}\alpha_1$$
$$= a_{21}(T-\lambda_1 I)(\alpha_1) \qquad \text{[From above relations]}$$

and $(T-\lambda_3 I)(T-\lambda_2 I)(T-\lambda_1 I)(\alpha_3)$

$$= (T-\lambda_2 I)(T-\lambda_1 I)(T-\lambda_3 I)(\alpha_3)$$
$$= (T-\lambda_2 I)(T-\lambda_1 I)(a_{31}\alpha_1 + a_{32}\alpha_2) \qquad \text{[From above relations]}$$
$$= (T-\lambda_2 I)(T-\lambda_1 I)(a_{31}\alpha_1) + (T-\lambda_2 I)(T-\lambda_1 I)(a_{32}\alpha_2)$$
$$= a_{31}(T-\lambda_2 I)(T-\lambda_1 I)(\alpha_1) + a_{32}(T-\lambda_2 I)(T-\lambda_1 I)(\alpha_2)$$
$$= 0 + 0 = 0.$$

Proceeding in the same way, we get

$$(T-\lambda_n I)(T-\lambda_{n-1} I)(T-\lambda_{n-2} I)\ldots(T-\lambda_1 I)(\alpha_n) = 0$$

If $(T-\lambda_n I)(T-\lambda_{n-1} I)\ldots(T-\lambda_1 I)$ is represented by S, then we have $S(\alpha_1) = S(\alpha_2) = \ldots\ldots S(\alpha_n) = 0$. Thus S, being the annihilator of base of V, annihilates all elements of V.

i.e., $$S = 0 \Rightarrow (T-\lambda_n I)(T-\lambda_{n-1} I), \ldots (T-\lambda_1 I) = 0$$

Hence T satisfy the polynomial $q(x) = (x-\lambda_n)(x-\lambda_{n-1}), \ldots (x-\lambda_1)$ in $F[x]$ of degree n.

THEOREM 3. *If a subspace W of V is T-invariant, then T has a matrix representation $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is matrix representation of the restricted T_q of T to W.*

Proof. The proof of this result can be showed simply by matrix representation of T_q. Let $\{\beta_1, \beta_2, \ldots\ldots, \beta_r\}$ be the basis of W then by extension theorem, it can be extended to the basis $\{\beta_1, \beta_2, \ldots\ldots, \beta_r, \alpha_1, \alpha_2, \ldots\ldots, \alpha_s\}$ of V.

Since W is T-invariant, so

$$T_q(\beta_i) = T(\beta_i), \text{ for } i = 1, 2, \ldots, r. \text{ Now we have}$$
$$T_q(\beta_1) = T(\beta_1) = a_{11}\beta_1 + \ldots\ldots + a_{1r}\beta_r$$
$$T_q(\beta_2) = T(\beta_2) = a_{21}\beta_1 + \ldots\ldots + a_{2r}\beta_r$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$T_q(\beta_r) = T(\beta_r) = a_{r1}\beta_1 + a_{r2}\beta_2 + \ldots\ldots + a_{rr}\beta_r$$

and $$T(\alpha_1) = b_{11}\beta_1 + \ldots\ldots + b_{1r}\beta_r + c_{11}\alpha_1 + \ldots\ldots + c_{1s}\alpha_s$$

Continuing in this way, we get

$$T(\alpha_1) = b_{s1}\beta_1 + \ldots\ldots + b_{sr}\beta_r + c_{s1}\alpha_1 + \ldots\ldots + b_{ss}\alpha_s$$

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} & b_{11} & b_{21} & \cdots & b_{s1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{r2} & b_{1r} & b_{2r} & \cdots & b_{sr} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & C_{11} & C_{21} & \cdots & C_{s1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{1s} & C_{2s} & \cdots & C_{ss} \end{bmatrix} = \begin{pmatrix} A & B \\ 0 & S \end{pmatrix}$$

where, $$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1r} & a_{2r} & \cdots & a_{rr} \end{bmatrix}_{r \times r} \qquad B = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{s1} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1r} & b_{2r} & \cdots & b_{sr} \end{bmatrix}_{r \times s}$$

$$C = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{r1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1s} & c_{2s} & \cdots & c_{ss} \end{bmatrix} \qquad O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{s \times r}$$

10.7 NILPOTENT TRANSFORMATION

A linear transformation $T : V \to V$ is said to be nilpotent if $T^n = 0$ for some least positive integer n.

or

Any $T \in A(V)$ is nilpotent then for some $k \in Z^+$, $T^k = 0$ but $T^{k-1} \neq 0$, where k is index of nilpotency.

REMARK

- The characteristic root of nilpotent transformation are zero, So they belong to F and hence all characteristic roots of nilpotent transformation belong to F. Then, we can say that nilpotent linear transformation can always be brought to triangular form over F.

10.8 JORDAN CANONICAL FORM

The matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

is called Jordan block matrix belonging to λ. In this matrix A, λ's are on the diagonal and 1's are on the superdiagonal and other elements are zero.

THEOREM 1. *Let $T : V \to V$ be a linear operator whose characteristic and minimal polynomial are respectively given by*

$$\Delta(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \ldots\ldots (x-\lambda_r)^{n_r}$$

and $$m(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \ldots\ldots (x-\lambda_r)^{m_r}$$

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where, λ_i are different scalars, then T has a block diagonal matrix representation

$$J=\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \cdots & \\ & & & J_r \end{bmatrix}$$

where

$$J_i=\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

For each λ_i, the corresponding block J_i have the following properties :

1. *There is at least one J_i of order m, and all other J_i are of order less than or equal to m_i.*
2. *The sum of the orders of J_i is n_i.*
3. *The number J_i equals the geometric multiplicity of λ_i.*
4. *The number J_i of each possible order is uniquely determined by T.*

Proof. We can write T by primary decomposition theorem as

$$T = T_1 \oplus T_2 \oplus \ldots\ldots \oplus T_r.$$

where $(x-\lambda_i)^{m_i}$ is the minimal polynomial of T_i.

Since the minimal polynomial is satisfied by the operator, therefore we have

$$(T_i-\lambda_i I)^{m_i}=0 \qquad \text{for } i = 1, 2, 3, \ldots, r$$

Now taking $N_i = T_i - \lambda_i I$ for $i = 1, 2, 3, \ldots, r$

$\Rightarrow$ $T_i = N_i + \lambda_i I$ and $N_i^{m_i} = 0$

This implies that N_i is nilpotent of index m_i and T_i is the sum of N_i, and scalar operator $\lambda_i I$.

Now, N_i being the nilpotent of index m_i, we can select a basis in which N_i is represented by a canonical form as

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

In this basis $T_i = N_i + \lambda_i I$ can be reduced to a block diagonal matrix whose block are block Jordan matrix J_i. T is direct sum of $T_1, T_2, \ldots\ldots, T_r$ therefore the direct sum of matrix representation of T_i gives the block diagonal matrix representation of T whose diagonal block are matrices J_i which have the following properties:

1. N_i is the nilpotent of index m_i so there is at least one J_i of order m_i.
2. Since T and the block diagonal matrix representation of T have the same characteristic polynomial so that the sum of orders of J_i is n_i.
3. Since the nullity of N_i is equal to the geometric multiplicity of eigenvalue λ_i because characteristic equation of N_i is $(x-\lambda_i)^{m_i}=0$. Hence the number of J_i is equal to the geometric multiplicity of λ_i.
4. Since T_i and N_i are uniquely determined by T. Hence number of J_i of each possible order is uniquely determined by T.

10.9 RATIONAL CANONICAL FORM

The Jordan canonical form is exerted when the minimal polynomials cannot be factored into linear polynomial while in rational canonical form, the minimal polynomial is taken as the product of linear polynomial.

THEOREM 1. *Let $T : V \to V$ be a linear operator with minimal polynomial $b_1(x) = q_1(x)^{l_1} . q_2(x)^{l_2} \ldots q_k(x)^{l_k}$, where $q_1(x), q_2(x), \ldots, q_k(x)$ are distinct monic irreducible polynomial. Then T has a unique block diagonal matrix representation*

$$\begin{bmatrix} C_1 & & & & \\ & C_2 & & & \\ & & C_3 & & \\ & & & \cdots & \\ & & & & C_k \end{bmatrix}$$

where C_i is the companion matrix of polynomial $q_i(x)^{l_i}$ where

$$l_1 = l_{11} \geq l_{12} \geq \ldots \geq l_{1r} \ldots$$
$$l_k = l_{k_1} \geq l_{k_2} \geq \ldots \geq l_{k_r}$$

Proof. By primary decomposition theorem, V can be decomposed as $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$, where each V_i is T-invariant and the minimal polynomial of T_i, linear transformation induced by T on V_i has minimal polynomial $q_i(x)^{l_i} \ldots$ The matrix representation of T_i in some of V_i is companion matrix C_i. But $V = V_1 \oplus V_2 \oplus \ldots\ldots \oplus V_k$.

Thus the matrix of T is

$$\begin{bmatrix} C_1 & & & & \\ & C_2 & & & \\ & & C_3 & & \\ & & & \cdots & \\ & & & & C_k \end{bmatrix}$$

This matrix representation is called rational canonical form and polynomial

$$q_1(x)^{l_{11}}, q_1(x)^{l_{12}}, \ldots, q_1(x)^{l_{1r}} \ldots q_k(x)^{l_{k_1}} \ldots q_k(x)^{l_{k_2}} \ldots q_k(x)^{l_{k_r}}$$

are called the elementary divisor of T.

10.10 RAW AND COLUMN SPACE OF A MATRIX

Definition 1. Let A be an arbitrary $m \times n$ matrix over a field K and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The rows of A *i.e.*, $R_1 = (a_{11}, a_{12}, \ldots, a_m), \ldots, R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$, viewed as vectors in K^n, span a subspace of K^n called the row space of A. That is,

$$\text{rowsp}(A) = \text{span}(R_1, R_2, \ldots, R_m).$$

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Unit-5 Canonical forms

Definition 2. The columns, $C_1, C_2, \ldots, C_n$ of an $m \times n$ matrix A over a field K, viewed as vectors in K_m, span a subspace of K_m called the column space of A. That is,

$$\text{colsp}(A) = \text{span}(C_1, C_2, \ldots, C_n).$$

THEOREM 1. *Row equivalent matrices have the same row space.*

Proof. Suppose we apply elementary row operations on a matrix A : (i) $R_i \leftrightarrow R_j$, (ii) $R_i \to kR_i$, $k \neq 0$, or (iii) $R_i \to kR_j + R_i$ and obtain a matrix B. Then each row of B is clearly a row of A or a linear combination of rows of A. Hence the row space of B is contained in the row space of A. On the other hand, we can apply the inverse elementary row operation on B and obtain A; hence the row space of A is contained in the row space of B. Therefore, A and B have the same row space. Thus any sequence of elementary row operations produces a matrix with the same row space. Hence, row equivalent matrices have the same row space.

THEOREM 2. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be echelon matrices in row canonical form. Then A and B have the same row space if and only if they have the same non zero rows.*

Proof. Clearly, if A and B have the same non zero rows then they have the same row space. Thus we only have to prove the converse.

Suppose A and B have the same row space and suppose $R \neq 0$ is the i^{th} row of A. Then there exist scalars $c_1, c_2, \ldots, c_s$ such that

$$R = c_1R_1 + c_2R_2 + \ldots + c_sR_s \qquad \ldots(1)$$

where the R_i are the non-zero rows of B.

The theorem is proved if we show that $R = R_i$, that is, $C_i = 1$ but $c_k = 0$ for $k \neq i$.

Let a_{ij_i} be the leading non zero entry of R. By (1) and we know that

$$a_{ij_i} = c_1b_{1j_i} + c_2b_{2j_i} + \ldots + c_sb_{sj_i} \qquad \ldots(2)$$

Since, b_{ij_i} is a leading non zero entry of B and, since B is row reduced, it is the only nonzero entry in the j_ith column of B. Thus from (2) we obtain $a_{ij_i} = b_{ij_i}$. However, $a_{ij_i} = 1$ and $b_{ij_i} = 1$ since A and B are row reduced; hence $c_i = 1$.

Now suppose $k \neq i$, and b_{kj_k} is the distinguished entry in R_k. By (1) and we know that

$$a_{ij_k} = c_1b_{1j_k} + c_2b_{2j_k} + \ldots + c_sb_{sj_k} \qquad \ldots(3)$$

Since B is row reduced, b_{kk_k} is the only non zero entry in the $j_k{}^{\text{th}}$ column of B; so by (3), $a_{ij_k} = c_kb_{kj_k}$. Since, a_{kj} is a leading non zero entry of A and, since A is row reduced, $a_{ij_k} = 0$. Thus $c_kb_{kj_k} = 0$ and, since $b_{kj} = 0$, $c_k = 0$.

Hence, $R = R_i$.

THEOREM 3. *Let A be any matrix. Then A is row equivalent to a unique matrix in row canonical form.*

Proof. Suppose A is row equivalent to matrices A_1 and A_2 where A_1 and A_2 are in row canonical form. By theorem 1,

$$\text{rowsp}(A) = \text{rowsp}(A_1) \text{ and } \text{rowsp}(A) = \text{rowsp}(A_2); \text{ hence}$$
$$\text{rowsp}(A_1) = \text{rowsp}(A_2)$$

Since A_1 and A_2 are in row canonical form, then by theorem 2

$$A_1 = A_2$$

THEOREM 4. *Matrices A and B have the same row space iff their row canonical forms have the same non zero rows.*

Proof. Let A_1 and B_1 be the row canonical forms of A and B, respectively. Suppose A and B have the same row space. Then

$$\text{rowsp}(A_1) = \text{rowsp}(A) = \text{rowsp}(B) = \text{rowsp}(B_1).$$

By theorem 2, A_1 and B_1 have the same nonzero rows. Then

$$\text{rowsp}(A) = \text{rowsp}(A_1) = \text{rowsp}(B_1) = \text{rowsp}(B).$$

THEOREM 5. *Let A and B be matrices such that the product AB is defined. Then the row space of AB is contained in the row space of B.*

Proof. The rows of AB are R_iB where R_i is the i^{th} row of A. Hence by the above result each row of AB is in the row space of B. Thus the row space of AB is contained in the row space of B.

THEOREM 6. *colsp (AB) $\subseteq$ colsp (A).*

Proof. Using theorem 4, we have

$$\text{colsp}(AB) = \text{rowsp}[(AB)^T]$$
$$= \text{rowsp}(B^TA^T)\ \text{rowsp}(A^T) = \text{colsp}(A).$$

Solved Examples

Example 1. *Suppose W is invariant under $S:V \to V$ and $T: V \to V$. Show that W is also invariant under S+T and ST.*

Solution. Since W is S-invariant and T-invariant, if $\alpha \in W$, then $S(\alpha) \in W$ and $T(\alpha) \in W$.

Now, if $\alpha \in W$, then

$$(S+T)(\alpha) = S(\alpha) + T(\alpha)$$

Since $S(\alpha), T(\alpha) \in W$, so $S(\alpha) + T(\alpha) \in W$

$$(S+T)(\alpha) \in W$$

$\therefore$ W is invariant under S + T.

Also, for $\alpha \in W$.

$$(ST)(\alpha) = S(T(\alpha))$$

Since $T(\alpha) \in W$ and W is invariant under S, so that

$$S(T(\alpha)) \in W$$

$\Rightarrow$ $(ST)(\alpha) \in W$

$\therefore$ W is invariant under ST.

Example 2. *Determine all invariant subspaces of $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ viewed as an operator on R^2.*

Solution. Since R^2 and $\{0\}$ are invariant subspace of A. If A has any other invariant subspaces, then it must be one-dimensional. The characteristic polynomial of A is

$$\Delta(x) = |xI - A|$$
$$= \begin{vmatrix} x-2 & 5 \\ -1 & x+2 \end{vmatrix}$$
$$= (x-2)(x+2) + 5$$
$$= x^2 + 1$$

Clearly, A has no eigenvalue in R so A has no eigenvectors in R^2. Hence, $R^2 + \{0\}$ are the only subspace invariant under A.

Example 3. *Suppose $T: V \to V$ is linear and suppose $T = T_1 \oplus T_2$, with respect to a T-invariant direct sum decomposition $V = V_1 \oplus V_2$. Show that*

(i) *$m(x)$ is the least common multiple of $m_1(x)$ and $m_2(x)$ where $m(x), m_1(x)$ and $m_2(x)$ are the minimal polynomials of T, T_1 and T_2, respectively.*

(ii) *$\Delta(x) = \Delta_1(x)\Delta_2(x)$ where $\Delta(x), \Delta_1(x)$ and $\Delta_2(x)$ are the characteristic polynomials of T, T_1 and T_2 respectively.*

Solution. (i) Since T_1 is induced of T on V_1 and T_2 is induced of T on V_2, therefore the minimal polynomials $m_1(x)$ of T_1 and $m_2(x)$ of T_2 each divides $m(x)$.

Suppose $p(x)$ is a multiple of both $m_1(x)$ and $m_2(x)$, then

$$(P(T_1))(V_1)=0 \quad \text{and} \quad (P(T_2))(V_2)=0$$

Let $\alpha \in V$, then $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in V_1$ and $\alpha_2 \in V_2$

Now,
$$\begin{aligned}(P(T))(\alpha) &= (P(T))(\alpha_1) + (P(T))(\alpha_2)\\ &= 0 + 0\\ &= 0\end{aligned}$$

T is the root of $P(x)$. Hence $m(x) \mid p(x)$. So that $m(x)$ is the least common multiple.

(ii) Since $T = T_1 \oplus T_2$, then the matrix representation of T is

$$M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A and B are the matrix representations of T_1 and T_2 respectively.

$$\begin{aligned}\Delta(x) &= |xI - M|\\ &= \begin{vmatrix} xI - A & 0 \\ 0 & xI - B \end{vmatrix} = (xI-A)(xI-B)\\ &= \Delta_1(x)\Delta_2(x)\end{aligned}$$

Example 4. *Let $T: V \to V$ be linear and let W be the eigenspace belonging to an eigenvalue λ of T. Show that W is T-invariant.*

Solution. By the definition of eigenspace, we have

$$W = \text{kernel}(T - \lambda I)$$

If $\alpha \in W$, then

$$(T - \lambda I)(\alpha) = 0 \Rightarrow T(\alpha) - \lambda I(\alpha) = 0$$

$$\Rightarrow \quad T(\alpha) = \lambda\alpha \quad \ldots(1)$$

Since W is a subspace of W, so for any scalar $\lambda \in F$ and $\alpha \in W$, we have

$$\lambda\alpha \in W \Rightarrow T(\alpha) \in W \quad \text{[Using (1)]}$$

Hence W is T-invariant.

Example 5. *If $\{W_i\}$ is a collection of T-invariant subspaces of a vector space V. Show that the intersection $W = \cap W_i$ is also T-invariant.*

Solution. Since each W_i is T-invariant, then for $\alpha \in W_i$, we have $T(\alpha) \in W_i$ for every i.

$$\therefore \quad T(\alpha) \in \cap W_i \text{ for every i.}$$

$$\Rightarrow \quad T(\alpha) \in W.$$

Hence W is T-invariant.

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Unit-5: Canonical forms

Similarity of LT

* Canonical forms:

Similarly if LT

1) T_α

$T+W$ α $\beta+W$ · $V|W$.

$$=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}=0$$

Thus, A is a nilpotent matrix of index 2.

Since A is nilpotent of index, thus we can say that M contains the diagonal block matrix of order less than or equal to 2.

Clearly, the rank of $A=2$ and the matrix A is of order 3. So that nullity of $A=3$.

Thus, M will contain 3 diagonal block matrices, in which 2 diagonal block of order 2 each and 1 diagonal block of order 1.

$$M=\begin{bmatrix}M_2&&\\&M_2&\\&&M_1\end{bmatrix}$$

where, $M_2=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $M_1=[0]$

Thus, we have

$$M=\left[\begin{array}{cc|ccc}0&0&0&0&0\\0&0&0&0&0\\\hline 0&0&0&1&0\\0&0&0&0&0\\\hline 0&0&0&0&0\end{array}\right]$$

Example 9. *If T is nilpotent of index k, show that $T^n, n>1$ is nilpotent of index k.*

Solution. Since $T^k=0$ but $T^{k-1}\neq 0$, then

$(T^n)^k=(T^k)^n=0^n=0$

$(T^n)^{k-1}=(T^{k-1})^n\neq 0$ $\quad[\because T^{k-1}\neq 0]$

and thus T^n is nilpotent of index $\leq k$.

Example 10. *Suppose S and T are nilpotent operators which commutes i.e., $ST=TS$, Show that $S+T$ and ST are also nilpotent.*

Solution. Since S and T are nilpotent, so we have $S^m=0$ and $T^n=0$ for some positive integers m and n. Since S and T commutes, then

$$(S+T)^{m+n}=\sum_{r=0}^{m+n} C_r T^{m+n-r}.\,S^r \qquad ...(1)$$

(i) If $r\geq m$, then $S^r=0$. So from (1), we get

$(S+T)^{m+n}=0$

(ii) If $r<m$, so $m+n-r\geq n$, then $T^{m+n-r}=0$

Therefore, from (1), we get

$(S+T)^{m+n}=0$

$\Rightarrow$ $S+T$ is nilpotent.

Now to show that ST is nilpotent.

Suppose $S^m=0$ and $T^n=0$ and $m\leq n$.

Then, $(ST)^n=S^nT^n=S^n=0=0$

If $m\geq n$, then

$(ST)^m=S^mT^m=0.T^m=0$

Thus ST is nilpotent.

Example 11. *Determine all possible Jordan canonical forms for a linear operator $T: V\to V$ whose characteristic polynomial is $\Delta(x)=(x-2)^3(x-5)^2$.*

Solution. Since $x-2$ has exponent 3 in $\Delta(x)$ and $x-5$ has exponent 2 in $\Delta(x)$, therefore Jordan canonical form will be the matrix of order 5×5.

We may write $\Delta(x)$ as

$$\Delta(x)=(x-\lambda_1)^3(x-\lambda_2)^2,\text{ where }\lambda_1=2\text{ and }\lambda_2=5.$$

Now, $\lambda_1=2$ must appear a three times on the main diagonal and $\lambda_2=5$ must appear two times. Hence the possible Jordan canonical forms are

(i) $\left[\begin{array}{ccc|cc}2&1&0&0&0\\0&2&1&0&0\\0&0&2&0&0\\\hline 0&0&0&5&1\\0&0&0&0&5\end{array}\right]$ (ii) $\left[\begin{array}{cc|c|cc}2&1&0&0&0\\0&2&1&0&0\\\hline 0&0&2&0&0\\\hline 0&0&0&5&1\\0&0&0&0&5\end{array}\right]$

(iii) $\left[\begin{array}{ccc|c|c}2&1&0&0&0\\0&2&1&0&0\\0&0&2&0&0\\\hline 0&0&0&5&1\\\hline 0&0&0&0&5\end{array}\right]$ (iv) $\left[\begin{array}{cc|c|c|c}2&1&0&0&0\\0&2&1&0&0\\\hline 0&0&2&0&0\\\hline 0&0&0&5&1\\\hline 0&0&0&0&5\end{array}\right]$

(v) $\left[\begin{array}{c|c|c|cc}2&1&0&0&0\\\hline 0&2&1&0&0\\\hline 0&0&2&0&0\\\hline 0&0&0&5&1\\0&0&0&0&5\end{array}\right]$ (vi) $\left[\begin{array}{c|c|c|c|c}2&1&0&0&0\\\hline 0&2&1&0&0\\\hline 0&0&2&0&0\\\hline 0&0&0&5&1\\\hline 0&0&0&0&5\end{array}\right]$

Example 12. *If A is a complex 5×5 matrix with characteristic polynomial*

$$\Delta(x)=(x-2)^3(x+7)^2$$

and minimal polynomial $m(x)=(x-2)^2(x+7)$. What is the Jordan form for A?

Solution. Here, in $\Delta(x)$, the exponent of $(x-2)$ is 3 and that of $(x+7)$ is 2. Thus, A will be of order 5×5. Also, in $m(x)$, the exponent of $(x-2)$ is 2 and that of $(x+7)$ is 1. Therefore, Jordan form will have one block of order 2 and other two blocks must be of the order 2 or 1. Hence the required Jordan form of A is

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Unit-5: Canonical forms

Similarity of LT

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{bmatrix}$$

Example 13. *Determine all possible Jordan canonical forms for a matrix of order 5 whose minimal polynomial is* $m(x) = (x-2)^2$.

Solution. Since the minimal polynomial of a matrix of order 5×5 is $(x-2)^2$, therefore, its characteristic polynomial will be $(x-2)^5$.

Thus, Jordan canonical form must have an Jordan block matrix of order 2 and other must be of order 2 or 1.

Hence the possible Jordan canonical forms are :

(i) $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Example 14. *Suppose* $T : V \to V$ *has characteristic polynomial*

$$\Delta(x) = (x+8)^4(x-2)^3$$

and minimal polynomial

$$M(x) = (x+8)^3(x-1)^2$$

Find the Jordan canonical form of the matrix representation of T.

Solution. Since degree of $\Delta(x)$ is 7 so that Jordan form will be a matrix of order 7×7 , in which –8 will be repeated 4 times on the diagonal and 1 will be three times on the diagonal. Also, $(x+8)$ has the exponent 3 in $m(x)$, therefore Jordan form will have one block of order 3×3 belonging to –8 and $(x-1)$ has the exponent 2 in $m(x)$. So, there must be one block of order 2×2 belonging to 1.

Hence the required Jordan canonical form is

$$\begin{bmatrix} -8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 15. *Find all possible rational canonical forms for 6×6 matrices with minimal polynomial* $m(x) = (x+1)^3$.

Solution. Let $T : V \to V$ be a linear with minimal polynomial $m(x) = (x+1)^3$ and dim. V = 6, then T is one of the following direct sum of companion matrices:

(i) $C((x+1)^3) \oplus C((x+1)^3)$

(ii) $C((x+1)^3) \oplus C((x+1)^2) \oplus C(x+1)$

(iii) $C((x+1)^3) \oplus C(x+1) \oplus C(x+1) \oplus C(x+1)$

Now, $C((1+x)^3) = C(x^3+3x^2+3x+1) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$

$C((1+x)^2) = C(x^2+2x+1) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

$C(1+x) = [-1]$.

Thus, the rational canonical form of T is one of the following

(i) $\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$ (ii)

(iii) $\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

Example 16. Let A be a 4×4 matrix with minim… rational canonical form for A … real field R, (iii) the comple…

Solution. (i) If the field is ration… direct sum of compa…

(a) C(x²…

Now, C…

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Unit-5: Canonical forms

Similarity of LT

Thus the required rational canonical form is

$$\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right]$$

(iii) If the field is field of complex numbers C, then the rational canonical form of A is the direct sum of the companion matrices

$$C(x-i) \oplus C(x+i) \oplus C(x-\sqrt{3}) \oplus C(x+\sqrt{3})$$

Thus the required rational canonical form is

$$\left[\begin{array}{cc|cc} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ \hline 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right]$$

Example 17. *Let V be a vector space of dimension 6 over R and let T be a linear operator whose minimal polynomial is $m(x) = (x^2 - x + 3)(x-2)^2$. Find the rational canonical form of T.*

Solution. Since $\dim.V = 6$, then T is one of the following direct sums of companion matrices:

(i) $C(x^2 - x + 3) \oplus C(x^2 - x + 3) \oplus C((x-2)^2)$

(ii) $C(x^2 - x + 3) \oplus C((x-2)^2) \oplus C((x-2)^2)$

(iii) $C(x^2 - x + 3) \oplus C((x-2)^2) \oplus C(x-2) \oplus C(x-2)$

where $C(q(x))$ is the companion matrix of $q(x)$.

Now $C(x^2 - x + 3) = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$

$$C((t-2)^2) = C(t^2 - 4t + 4) = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix}$$

$$C((x-2)) = [2]$$

Thus the canonical forms of T is one of the following matrices :

(i) $\left[\begin{array}{cc|cc|cc} 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$ (ii) $\left[\begin{array}{cc|cc|cc} 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

(iii) $\left[\begin{array}{cc|cc|c|c} 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right]$

EXERCISE 10.1

1. Prove that the relation of similarity is an equivalence relation in $A(V)$.
2. If A is triangular $n \times n$ matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal, then $(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I) = 0$
3. Suppose $T : V \to V$ is linear. Show that each of the following is invariant under T:
   (i) kernel of T (ii) $\{0\}$
   (iii) image of T (iv) V
4. Determine the invariant subspace of $A = \begin{bmatrix} 2 & -4 \\ 5 & -2 \end{bmatrix}$ viewed as linear operator on:
   (i) R^2 (ii) C^2
5. Suppose A is super triangular matrix (all entries below the main diagonal are 0). Show that A is nilpotent.
6. What is the minimal polynomial of nilpotent matrix A of index k?
7. Show that following matrix are nilpotent :
   $A = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix}; A = \begin{bmatrix} 1 & -3 & 2 \\ 1 & -3 & 2 \\ 1 & -3 & 2 \end{bmatrix}$
   Find also the index of nilpotency in each matrix.
8. Find the canonical nilpotent form of the matrix
   $$A = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$
9. If matrix A and B are similar, then show that A is nilpotent of index k, if and only if B is nilpotent of index k.
10. If F is a field of characteristic zero and if S and T in $A(V)$ are such that $ST - TS$ commutes with S, then $ST - TS$ is nilpotent.

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Unit-5 Canonical forms

Similarity of LT

Chapter Review : A competitive Approach

Selected Terms and Results

TERMS

- **Canonical Forms :** The special form of matrix representation (in some basis of V) of linear transformation in each similarity class is called canonical form.
- **Invariant Subspace :** Let $T : V \to V$ be a linear transformation. Then a subspace W of V is said to be invariant under T if $T(W) \subset W$ i.e., for all $\alpha \in W$, $T(\alpha) \in W$.
- **Normal Form :** A matrix A is said to be in normal form if it can be written as $A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is square identity matrix of order r.
- **Triangular Form :** Let $T: V \to V$ be a linear transformation on V over F, then the matrix of T in the basis $\{\alpha_1, \alpha_2 \ldots\ldots \alpha_n\}$ of V is triangular if

$$T(\alpha_1) = a_{11}\alpha_1$$
$$T(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2$$
$$T(\alpha_3) = a_{31}\alpha_1 + a_{32}\alpha_2 + a_{33}\alpha_3$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$T(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + a_{nn}\alpha_n$$

- **Nilpotent Transformation :** A linear transformation $T : V \to V$ is said to be nilpotent if $T^n = 0$ for some least positive integer n.
- **Jordan Canonical Form :** The matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

is called Jordan block matrix belonging to λ.

RESULTS

- If W is a subspace invariant under $T \in A(V)$, then T induces a linear transformation T_q on quotient space V/W defined by $T_q(\alpha + W) = T(\alpha) + W$.
- If $V = W_1 \oplus W_2 \oplus \ldots \oplus W_r$ where n_i is the dimension of each subspace W_i and every subspace is invariant under $T \in A(V)$, then a basis of V can be found so that the matrix of T in this basis is of the form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_r \end{bmatrix}$$

where each A_i is an $n_i \times n_i$ matrix of linear transformation induced by T on W_r.
- If $T \in A(V)$ has all its characteristic root in F, then there is a basis of V in which matrix representation of T is triangular.
- If dim.V = n and if $T \in A(V)$ has all its characteristic roots in F, then T satisfy a polynomial of degree n over F.
- The characteristic root of nilpotent transformation are zero.
- All characteristic roots of nilpotent transformation belong to F.
- The nilpotent linear transformation can always be brought to triangular form over F.
- If W is invariant under $S : V \to V$ and $T : V \to V$ then W is also invariant under S + T and ST.
- If $T : V \to V$ is linear and W be eigenspace belonging to an eigenvalue λ of T, then W is T-invariant.
- Similar matrices have the same eigenvalues.
- If T is nilpotent of index k, then T^n, n > 1 is also nilpotent of index k.

Review Questions and Project Work

1. If $T : V \to V$ is linear operator on V over a field F of characteristic zero such that tr. $T^k = 0$ for all $k \geq P$, then show that T is nilpotent.
2. Find all possible Jordan canonical forms for those matrix whose characteristic polynomial $\Delta(x)$ and minimal polynomial $m(x)$ are as

follows :

(i) $\Delta(x) = (x-2)^4 . (x-3)^2$,
$m(x) = (x-2)^2 (x-3)^2$

(ii) $\Delta(x) = (x-2)^7$, $m(x) = (x-2)^2$

(iii) $\Delta(x) = (x-3)^4 (x-5)^4$,
$m(x) = (x-3)^2 (x-5)^2$

3. How many possible Jordan forms are there for 6×6 complex matrix with characteristic polynomial $\Delta(x) = (x+2)^4 (x-1)^2$?
4. Find all possible Jordan forms 8×8 matrices having $x^2(x-1)^3$ as minimal polynomial.
5. Show that every complex matrix is similar to its transpose.
6. Prove that the matrix $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ is nilpotent and find its invariant and Jordan form.
7. Determine the Jordan canonical form for matrices

(i) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (ii) [illegible]

8. Show that all complex matr[illegible] which $A^n = I$ are similar.
9. Find all possible rat[illegible] for 6×6 matrix w[illegible] $m(x) = (x^2+2)^2$ (x[illegible]

Objective Type Questions

FILL IN THE BLANKS

1. N is nilpotent if there is some integer r such that N^r = ________.
2. An operator T can be put into ________ if its characteristic and minimal polynomials fields into linear polynomials.
3. T can b[illegible] sum of[illegible]
4. If C[illegible]

TRUE/FALSE

Write 'T' for true and 'F' for false statement.

1. The operator $(T - CI)$ is singular $\Rightarrow$ det(C $-$ λI[illegible] = 0. (T[illegible]
2. det(C $-$ λI) = 0 $\Rightarrow$ C is a characteristic [illegible] of T.

MULTIPLE CHOICE QUESTIONS

Choose the most appropriate o[illegible]

1. Let $T : V \to V$ be linear, the[illegible] following is invariant unde[illegible]
(a) {0} (b)[illegible]
(c) Ker(T)

2. The subspace of A[illegible] riant under T is
(a) {0}
(c) Both [illegible]

3. Let T :[illegible] $T^k(v)$ foll[illegible]

public static void main (String args[]){

9/12/23

Unit-5: Canonical forms

Similarity of LT

Similarly if LT

$\alpha : V$ T^{-1} is present

$T(\alpha) = \gamma \in V$ $T^{-1}(\gamma\alpha) : U \to V$

1) T_q

$T + W Q$ $\beta + W$ · $V|W$.

Answers

FILL IN THE BLANKS

1. 0	2. Jordan canonical form	3. nilpotent	4. 0

TRUE/ FALSE

1. T	2. T	3. T	4. T

MULTIPLE CHOICE QUESTIONS

1. (d)	2. (c)	3. (c)	4. (d)	5. (c)

COMPETITION CORNER
for JRF, NET/SET, GATE Aspirants

SOME FASCINATING FACTS

- If $E_1, E_2, \ldots, E_k$ are the projections associated with the primary decomposition of T, then each E_i is a polynomial in T and accordingly if a linear operator U commute with T then U commutes with each of the E_i i.e., each subspace W_i is invariant under U.
- Let T be a linear operator on the finite dimensional vector space V over the field F. If for the polynomial, there is a diagonalizable operator D on V and a nilpotent operator N on V such that

$T = D + N$ and $DN = ND$

- The characteristic polynomial $f(t)$ and minimal polynomial $m(t)$ of a matrix A have the same irreducible factors.
- The minimal polynomial $m(t)$ of a matrix divides every polynomial that A as a zero. In particular $m(t)$ divides the characteristic polynomial $f(t)$ of A.
- The minimal polynomial of lowest degree over the field F that annihilates A is called minimal polynomial of A.

SOME IMPORTANT ILLUSTRATIONS

- Let T be a linear operator on C^2, the characteristic polynomial for T is either $(x-c_1)(x-c_2)$, where c_1 and c_2 are distinct complex numbers $(x-c)^2$. T is diagonalizable and is represented in the basis of $\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$.
- If $a_0, a_1, \ldots, a_{n-1}$ are complex numbers and let V be the space of all n times differential functions f on the interval of the real time which satisfy the differential equation

$$a_n \frac{d^n f}{dx^n} + a_{n-1}\frac{d^{n-1} f}{dx^{n-1}} + \ldots + a_1 \frac{df}{dx} + a_0 f = 0$$

and if D be the differential operator. Then V is invariant under D.

- Let T be the linear operator on R^2, which represents the standard ordered basis by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then T has no characteristic values.
- If T is a linear operator on the finite dimensional space V then T is diagonalizable if there is a basis for V of each vector, which is the characteristic vector of T.
- The minimal polynomial of $A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$ is given by $m(t) = t^2 - 4t + 3$.
- The minimal polynomial of $A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ is given by $m(t) = (t-1)(t-2)$.
- The minimal polynomial of $A = \begin{bmatrix} 5 & 1 \\ 3 & 7 \end{bmatrix}$ is given by $m(t) = (t-4)(t-8)$.
- The minimal polynomial of the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$ is given by $m(t) = (t-1)^2(t-2)$.