3) $2x - 3y + 10z = 3$; $-x + 4y + 2z = 20$; $5x + 2y + z = -12$

$\Rightarrow$ $AX = B$

$$\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

$R_2 \leftarrow R_2 + R_1/2$

$R_3 \leftarrow R_3 + \frac{5}{2} R_1$

$x = -4$
$y = 3$
$z = 2$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 19/2 & -24 \end{bmatrix}$$

$R_3 \leftarrow R_3 - \frac{19}{5} R_2$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/5 \end{bmatrix} = U \quad ; \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -5/2 & 19/5 & 1 \end{bmatrix}$$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -5/2 & 19/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

- $y_1 = 3$
- $-\frac{1}{2} y_1 + y_2 = 20 \Rightarrow y_2 = \frac{43}{2}$
- $-\frac{5}{2} y_1 + \frac{19}{5} y_2 + y_3 = -12$

$y_3 = -\frac{431}{5}$

$UX = Y$

$$\begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 43/2 \\ -431/5 \end{bmatrix}$$

$x = -4$ $\quad$ $y = 3$ $\quad$ $z = 2$

* Vector spaces.

In order to discuss vector space we use the set of vector & scalars. To define a vector space we need a field 'F' & elements of F is scalar. In addition to that we need two operators $\Rightarrow$ vector add & scalar mul. This is defined using internal composition

& external composition.

- Internal composition - Let R be any set if $a*b \in R\ \forall a,b$ & $a*b$ is unique & this $*$ is known internal compo
- External composition - Let B be set of vectors & F b field. Then binary operation defined b/w vectors & scalar is called external composition.

If $a \circ \alpha \in V\ \forall\ \alpha \in V$ & $a \in F$, $a \circ \alpha$ is unique.

★ Intro to vector spaces:

Let G be an non-empty set & $*$ be binary operation defined on it. Then the structure (is said to be a group if following axioms are satisfied.

1) Closure ppt: $a*b \in G\ \forall\ a,b \in G$
2) Associative: $a*(b*c) = (a*b)*c\ \forall\ a,b,c \in G$
3) Existence of identity - There exists an element $e \in G$ $a*e = e*a = a\ \forall\ a \in G$
4) Existence of inverse: For each element $a \in G\ \forall\ b \in G$ such that $a*b = b*a = e$, where b is inverse of a & $b = a^{-1}$

5)★ Commutative/ Abelian ~~field~~ Group:

A grp $(G,*)$ is said to be abelian if $a*b = b*a$ $\forall\ a,b \in G$

The grp which are not abelian called non commutative grp.

★ Finite & Infinite grp:

★ Order of a grp: No of elements in a finite grp is called order of a grp.

- infinite - infinite order.

* Definition of Field:

Let F be a non empty set equipped with 2 binary operations - add, mul i.e $\forall a,b \in F$, $a+b \in F$ & $a \cdot b \in F$.

The algebraic structure $(F,+,\cdot)$ is said to be field if it satisfies following.

1) add is associative : $(a+b)+c = a+(b+c) \; \forall a,b,c \in F$
2) add is commutative : $a+b = b+a \; \forall a,b \in F$
3) There exists an identity element zero in F such that $a+0=a=0+a \; \forall a \in F$
4) To each element $a \in F$, there exists $a+(-a)=0$
5) mul is commutative : $a \cdot b = b \cdot a \; \forall a,b \in F$
6) mul is associative : $(a \cdot b) \cdot c = a \cdot (b \cdot c) \; \forall a,b,c \in F$
7) There exists a non-zero element in F such that $(a \cdot 1) = (1 \cdot a) = a \cdot \in F$
8) To every non zero element $a \in F$ there exists an element a^{-1} in F such that $a \cdot a^{-1} = 1$
9) Distributive ppt : $a \cdot (b+c) = a \cdot b + a \cdot c \; \forall a,b,c \in F$

* Sub field:

Let F be a field. A non empty subset 'K' of F is said to be sub field of F, if K is closed wrt add & mul in F & K.

* Vector space:

Let V be a non empty set of vectors & F be a field, then an algebraic structure $(V,+,\cdot)$ together with 2 binary operation - vector add, scalar mul is said to vector space over F. if satisfy following

1) $(V,+)$ is an abelian grp.
2) $a(\alpha+\beta) = a\alpha + a\beta \; \forall \alpha,\beta \in V$ & $a \in F$

3) $(a+b)\alpha = a\alpha + b\alpha \ \forall \ \alpha \in V;\ a, b \in F$
4) $(a \cdot b)\alpha = a(b \cdot \alpha) \ \forall \ \alpha \in V;\ a, b \in F$
5) $1 \cdot \alpha = \alpha \ \forall \ \alpha \in V$

vector sp V over F is denoted by $V(F)$

★ Show that a field K can be regarded as a vector spa.. over any sub field F of K.

→ WKT $F \subset K$, K contains set of vectors & F contains set of scalar values.
We need to verify different ppts to prove F is subset of K using add & mul.
So we will consider one scalar & one vector.
$a\alpha \in K \ \forall \ a \in F \ \& \ \alpha \in K$
If 1 is unity element of K, then 1 is also unity element of sub field F.

1) $a(\alpha+\beta) = a\alpha + a\beta \ \forall \ \alpha \in F \ \& \ \alpha, \beta \in K$
2) $(a+b)\alpha = a\alpha + b\alpha \ \forall \ a, b \in F \ \& \ \alpha \in K$
3) $(ab)\alpha = a(b\alpha) \ \forall \ a, b \in F \ \ \alpha \in K$
4) $1\alpha = \alpha \ \forall \ \alpha \in K,\ 1 \in F$

by above observation K is a vector space over F which is denoted as $K(F)$

★ ST the set of all ordered n tuples forms a vector space over a field F.

eg: $R^n = \{(a_1, a_2 \dots a_n)\} : a_i \in R$

→ If a_1, a_2, a_3 upto a_n are 'n' elements of field 'F' then an ordered set $\alpha = (a_1, a_2 \dots a_n)$ is called an 'n' tuple over F.
Now we shall show that V is a vector sp.. wrt add composition & scalar mul.

1) Closure ppt: $\forall \alpha = (a_1, a_2, \dots a_n) \in V$
& $\beta = (b_1, b_2, \dots b_n) \in V$

if $\alpha + \beta = (a_1+b_1, a_2+b_2 \dots a_n+b_n) \in V$
Since $a_1+b_1, a_2+b_2 \dots, a_n+b_n$ are all elements of F so that $\alpha+\beta \in V \ \forall \alpha, \beta \in V$
Hence V is closed for addition of n tuples.

2) Associativity of addition in V
$\forall \alpha = (a_1, a_2 \dots a_n)$
$\beta = (b_1, b_2 \dots b_n)$
$\gamma = (c_1, c_2 \dots c_n)$ of V
$\alpha + (\beta+\gamma) = (\alpha+\beta)+\gamma$
$\Rightarrow (a_1, a_2 \dots a_n) + [(b_1, b_2 \dots b_n) + (c_1, c_2 \dots c_n)]$
$= (a_1, a_2 \dots a_n) + [(b_1+c_1), (b_2+c_2) \dots (b_n+c_n)]$
$= [a_1+(b_1+c_1), a_2+(b_2+c_2) \dots a_n+(b_n+c_n)]$
$= [a_1+b_1, a_2+b_2, \dots a_n+b_n] (c_1, c_2 \dots c_n)$
$= (\alpha+\beta)+\gamma$

3) Existence of additive identity in V
consider $\alpha = (a_1, a_2 \dots a_n) \in V$
$\alpha + 0 = (a_1, a_2 \dots a_n) + (0, 0, \dots 0)$
$= (a_1+0, a_2+0, \dots a_n+0)$
$\alpha + 0 = \alpha$

4) Existence of additive inverse in V
$\alpha = (a_1, a_2 \dots a_n) \in V$
$-\alpha = (-a_1, -a_2 \dots -a_n) \in V$
$\alpha + (-\alpha) = 0 \quad \forall \ 0 \in V$
$-\alpha + \alpha = 0 \quad \forall \ 0 \in V$

5) Commutativity of add in V

$\alpha = (a_1\ a_2\ a_3 \dots a_n) \in V$

$\beta = (b_1\ b_2\ b_3 \dots b_n) \in V$

$\alpha + \beta = (a_1+b_1,\ a_2+b_2 \dots a_n+b_n)$

$= (b_1+a_1,\ b_2+a_2 \dots b_n+a_n)$

$\alpha + \beta = \beta + \alpha$

* Now we observe that $a(\alpha+\beta) = a\alpha + a\beta$, $\alpha, \beta \in V$, $a \in F$

$a[a_1, a_2 \dots a_n + b_1 \dots b_n]$

$a[a_1+b_1,\ a_2+b_2 \dots a_n+b_n]$

$a(a_1+b_1),\ a(a_2+b_2) \dots a(a_n+b_n)$

$aa_1,\ aa_2 \dots a a_n + ab_1\ ab_2 \dots ab_n$

$a(\alpha) + a(\beta)$

- $\alpha(a+b) = \alpha a + \alpha b \quad a, b \in F \quad \alpha \in V$

$\alpha(a+b) = (a_1\ a_2 \dots a_n)(a+b)$

$= (a+b)a_1\ (a+b)a_2 \dots (a+b)a_n$

$= aa_1 + a_1 b\quad aa_2 + ba_2 \dots aa_n + ba_n$

$= a(a_1\ a_2 \dots a_n) + b(a_1 \dots a_n)$

$= a\alpha + b\alpha.$

- $\forall (a, b) \in F$ & $\alpha \in V$, $(ab)\alpha = a(b\alpha)$

$(ab)(a_1\ a_2 \dots a_n) = aba_1\ aba_2 \dots aba_n$

$= a(ba_1\ ba_2 \dots ba_n)$

$= a(b\alpha)$

- If 1 is unity element of F & $\alpha \in V$ then PT $1\alpha = \alpha$

$1(a_1\ a_2 \dots a_n) = 1a_1\ 1a_2 \dots 1a_n$

$= a_1\ a_2 \dots a_n$

$= \alpha$

$\therefore$ can be denoted as $V_n(F)$

* ST the set of all $m \times n$ matrices with their elements has real nos is a vector space over field F of real nos wrt add of matrices as add of vectors & multiplication of matrix by scalar as scalar mul.

→ Let $M_{mn} = \{A, B, C \ldots\}$ be set of all $m \times n$ matrices.
We shall ST $M_{mn}(F)$ will form abelian grp in add

- Closure ppt: $\forall (A, B) \in M_{mn}$
  we have $A+B \in M_{mn}$
- Associativity: $\forall (A, B, C) \in M_{mn}$
  we have $A+(B+C) = (A+B)+C$
- Existance of identity: If ~~$A \in M_{mn}$~~ O null matrix of order $m \times n$ $O \in M_{mn}$ & also, matrix $A \in M_{mn}$, then we have $A+O = A = O$
  here O is additive identity in given vector space M_{mn}
- Existence of inverse: If $A \in M_{mn}$, $-A \in M_{mn}$. $\forall A \in M_{mn}$ we have $A+(-A) = O = (-A)+A$, here $-A$ is additive inverse of A
- Commutative: $\forall A, B \in M_{mn}$ we have $A+B = B+A$. If $a \in F$ & $A = [a_{ij}]_{m \times n} \in M_{mn}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now we observe that

i) $\forall A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ in M_{mn} & $a \in F$, the $a(A+B) = aA + aB$

$$= a([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n})$$
$$= a([a_{ij} + b_{ij}]_{m \times n})$$
$$= [a a_{ij}]_{m \times n} + [a b_{ij}]_{m \times n}$$

$$a(A+B) = aA + aB$$

ii) $\forall a, b \in F$ & $A = [a_{ij}] \in M_{mn}$

$$(a+b)A = aA + bA$$
$$= (a+b)[a_{ij}]_{m \times n}$$
$$= [(a+b) a_{ij}]_{m \times n}$$
$$= [(a a_{ij} + b a_{ij}]_{m \times n}$$
$$= [a a_{ij}]_{m \times n} + [b a_{ij}]_{m \times n}$$

$= a[a_{ij}]_{m\times n} + b[a_{ij}]_{m\times n}$

$(a+b)A = aA + bA$

iii) For all $a, b \in F$ & $A = [a_{ij}]_{m\times n} \in M_{m\times n}$

$(ab)A = a(bA)$

$= (ab)\,[a_{ij}]_{m\times n}$

$= [(ab)(a_{ij})]_{m\times n}$

$= [a(ba_{ij})]_{m\times n}$

$= a\,[ba_{ij}]_{m\times n}$

$(ab)A = a(bA)$

iv) Since $1 \in F$ & $A = (a_{ij})_{m\times n} \in M_{m\times n}$

$1A = A$

$= 1\cdot[a_{ij}]_{m\times n} = [1a_{ij}]_{m\times n}$

$= [a_{ij}]_{m\times n}$

$1A = A$

* Vector subspaces [vector space within a vector space]

Let W be a non empty subset of V, where V is a vector space over a field 'F'. Then W is said to be a vector subspace $V(F)$ if W is itself a vector space over F wrt the same operations as defined on 'V'.

For eg: the set $W = \{(a,b,0) : a, b \in R\}$ is a subspace of $R^3(R)$

* Elementary ppts of vector subspace.

* Theorem 1:

The necessary & sufficient condition for a non-empty subset 'W' of $V(F)$ to be a subspace is that

i) $\alpha \in W,\ \beta \in W \Rightarrow \alpha - \beta \in W$

ii) $a \in F,\ \alpha \in W \Rightarrow a\alpha \in W$

Proof: Suppose W is a subspace of vector space $V(F)$, then $\beta \in W \Rightarrow -\beta \in W$

$\therefore \alpha \in W,\ -\beta \in W \Rightarrow \alpha + (-\beta) \in W$

$\alpha - \beta \in W$

ii) $a \in F,\ \alpha \in W \Rightarrow a\alpha \in W$

Conversely, suppose W is a subset of V &

i) $\alpha \in W,\ \beta \in W \Rightarrow \alpha - \beta \in W$

ii) $a \in F,\ \alpha \in W \Rightarrow a\alpha \in W$

Now we have to ST W is subspace, for this purpose we proceed as follows

$\alpha \in W,\ -\alpha \in W \Rightarrow \alpha - \alpha \in W$

$\Rightarrow 0 \in W$ with the help of existence of identity

& $0 \in W,\ \alpha \in W \Rightarrow 0 - \alpha \in W$

$= -\alpha \in W$

Now $\alpha \in W,\ -\beta \in W \Rightarrow \alpha - (-\beta) \in W$

$\alpha + \beta \in W$

This proves that W is a vector subspace of $V(F)$ vector add to form abelian grp.

* Theorem 2:

The necessary & sufficient condition for a non empty subset of W of a vector space $V(F)$ to be a subspace of V is $a, b \in F$ & $\alpha, \beta \in W$ then $a\alpha + b\beta \in W$

Proof: Suppose W is a subspace of vector space $V(F)$.

Then W is ~~rattat~~ closed under vector add & mul.

$\therefore$ We have $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

$b \in F, \beta \in W \Rightarrow b\beta \in W$

$a\alpha \in W, b\beta \in W \Rightarrow a\alpha + b\beta \in W$

Conversely suppose W is a subset of $V(F)$ satisfying above condition, then we have to show that V is subset of $V(F)$ by performing vector addition & scalar multiplication.

Now taking $a=1, b=1$, then

$1 \in F, \alpha, \beta \in W \Rightarrow 1\alpha + 1\beta \in W$

$\alpha + \beta \in W$

W is closed under vector addition.

Now taking $a=0, b=-1$, we have

$a\alpha + b\beta \in W$

$0\alpha + (-1)\beta \in W$

$-\beta \in W$

$\therefore$ additive inverse exists in W.

Now taking $a=0, b=0$

$a\alpha + b\beta \in W$

$0\alpha + 0\beta \in W$

$0 \in W$

existence of identity in vector space W

Since $W \subseteq V$, $\therefore$ vector addition is associative & commutative, thus W is an abbelian grp under vector add.

* Now taking $\beta = 0$, we have

$a\alpha + b\beta \in W$

$a\alpha \in W$

$\therefore W$ is closed under scalar multiplication.

Hence W is vector space & consequently W is a subspace of $V(F)$.

* ST the set $W = \{(a,b,c) : a-3b+4c=0\}$ is a subspace of the 3-tuple space $R^3(R)$

$\rightarrow$ Let $\alpha = (a_1, b_1, c_1)$ & $\beta = (a_2, b_2, c_2)$ be any 2 elements of W, such that $a_1 - 3b_1 + 4c_1 = 0$ —(1)

& $a_2 - 3b_2 + 4c_2 = 0$ —(2)

for $(a,b) \in R$, we have $a\alpha + b\beta = a(a_1, b_1, c_1) + b(a_2, b_2, c_2)$

$= (aa_1, ab_1, ac_1) + (ba_2 + bb_2 + bc_2)$

$= (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$

$\Rightarrow (aa_1 + ba_2) - 3(ab_1 + bb_2) + 4(ac_1 + bc_2)$

$\Rightarrow (aa_1 + ba_2), (-3ab_1 - 3bb_2), (4ac_1 + 4bc_2)$

$\Rightarrow (aa_1 - 3ab_1 + 4ac_1) + (ba_2 - 3bb_2 + 4bc_2)$

$\Rightarrow a(a_1 - 3b_1 + 4c_1) + b(a_2 - 3b_2 + 4c_2)$

from (1) & (2)

$a(0) + b(0) = 0$

$\therefore a\alpha + b\beta \in W$, thus $\alpha \in W, \beta \in W \ \forall a, b \in R$

$\therefore W$ is a subspace of $R^3(R)$.

* ST the set $W = \{(a_1, a_2, 0)\} : a_1, a_2 \in F$ is a subspace of $V_3(F)$

$\rightarrow$ Let $\alpha, \beta \in W$ then $\alpha = (a_1, a_2, 0)$ & $\beta = (b_1, b_2, 0)$

$a_1, a_2, b_1, b_2 \in F$.

WKT $a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$

$= (aa_1, aa_2) + (bb_1, bb_2)$

$= (aa_1 + bb_1), (aa_2 + bb_2), 0$

Since $(aa_1 + bb_1), (aa_2 + bb_2) \in F$ $\therefore a\alpha + b\beta \in F$

Hence W is a subspace of $V_3(F)$.

* Let W be the collection of all elements from space $M_2(F)$ of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. S.T W is a subspace of $M_2(F)$. where $W = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in F \right\}$

→ Let $\alpha, \beta \in W$, $\alpha = \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix}$ & $\beta = \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$

$\forall\ a_1, b_1, a_2, b_2 \in F$

consider $a\alpha + b\beta = 0$

$$a\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} + b\begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$$

$$\begin{bmatrix} aa_1 & ab_1 \\ -ab_1 & aa_1 \end{bmatrix} + \begin{bmatrix} ba_2 & bb_2 \\ -bb_2 & ba_2 \end{bmatrix}$$

$$\begin{bmatrix} aa_1 + ba_2 & ab_1 + bb_2 \\ -ab_1 - bb_2 & aa_1 + ba_2 \end{bmatrix}$$

$(aa_1 + ba_2), (ab_1 + bb_2) \in F$

$a\alpha + b\beta \in W$

Hence W is subspace of $M_2(F)$.

* If a_1, a_2, a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of elements of field F, such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is subspace of $V_3(F)$.

→ Let α, β $W = \{(x_1, x_2, x_3) : a_1x_1 + a_2x_2 + a_3x_3 = 0\}$ $a_1, a_2, a_3 \in F$ are fixed.

Let $\alpha, \beta \in W$ & $\alpha = (x_1, x_2, x_3)$ & $\beta = (y_1, y_2, y_3)$

$a_1x_1 + a_2x_2 + a_3x_3 = 0$

$a_1y_1 + a_2y_2 + a_3y_3 = 0$

we have $a\alpha + b\beta = 0$

$a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$

$\Rightarrow (aa_1x_1 + aa_2x_2 + aa_3x_3) + (ba_1y_1 + ba_2y_2 + ba_3y_3)$

$(ax_1 + by_1), (ax_2 + by_2), (ax_3 + by_3)$

$a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3) = 0$

$a_1ax_1 + a_1by_1 + a_2ax_2 + a_2by_2 + a_3ax_3 + a_3by_3 = 0$

$a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3) = 0$

$a(0) + b(0) = 0$

$0 \in W$

* Find if following sets of vectors $\alpha = (a_1, a_2, a_3 \dots a_n) \in R^n$ are subspaces of R^n $(n \geq 3)$, (a) all α such that $a_1 \leq 0$ (b) all α such that a_3 is an integer. (c) all α such that $a_2 + 4a_3 = 0$ (d) all α such that $a_1 + a_2 + \dots + a_n = k$ (constant)

→ (a) Let $W = \{(a_1, a_2, a_3 \dots a_n) : a_1 \leq 0\}$

Let $\alpha, \beta \in W$, $\alpha = (a_1, a_2 \dots a_n)$ & $\beta = (b_1, b_2 \dots b_n)$

$a\alpha + b\beta = a(a_1, a_2 \dots a_n) + b(b_1, b_2, \dots b_n)$

$= (aa_1, aa_2 \dots aa_n) + (bb_1, bb_2, \dots bb_n)$

$= (aa_1 + bb_1, aa_2 + bb_2, \dots aa_n + bb_n)$

Since $a_1 \leq 0$ & $b_1 \leq 0$ & if $a \leq 0$ & $b \leq 0$ then

$aa_1 > 0$ & $bb_1 > 0$

so that $(aa_1 + bb_1) > 0$, thus $a\alpha + b\beta \in W$

Hence W is not a subspace of $R^n(R)$

(b) Let $W = \{(a_1, a_2, a_3 \dots a_n) : a_3 \text{ is integer}\}$

Let $\alpha = (a_1, a_2, 2, a_4 \dots a_n) \in W$

consider $a = 1/3 \in R$, now

$a\alpha = \frac{1}{3}(a_1, a_2, 2, \dots a_n)$

$= (\frac{1}{3}a_1, \frac{1}{3}a_2, \frac{2}{3} \dots \frac{a_n}{3})$

$a\alpha \notin W$

hence W is not a subspace of $R^n(R)$

(c) Let $W = \{(a_1, a_2, a_3 \dots a_n) : a_2 + 4a_3 = 0\}$

Let $\alpha = (a_1, a_2 \dots a_n)$ & $\beta = (b_1, b_2 \dots b_n)$

$a_2 + 4a_3 = 0$ & $b_2 + 4b_3 = 0$

$a\alpha + b\beta = 0$

$a(a_1, a_2 \dots a_n) + b(b_1, b_2, \dots b_n)$

$\Rightarrow (aa_1, aa_2 \dots aa_n) + (bb_1, bb_2 \dots bb_n)$

$\Rightarrow (aa_1 + bb_1), (aa_2 + bb_2) \dots (aa_n + bb_n)$

according to given func $(aa_2 + bb_2) + 4(aa_3 + bb_3) = 0$

$aa_2 + 4aa_3, \; bb_2 + 4bb_3 = 0$

$a(a_2 + 4a_3), \; b(b_2 + 4b_3) = 0$

$a(0) + b(0) = 0$

since $(aa_n + bb_n) \in R$

$\therefore a\alpha + b\beta \in W$

Hence W is a subspace of $R^n(R)$.

(d) Let $W = \{(a_1, a_2, \dots a_n) : a_1 + a_2 \dots + a_n = k\}$

→ If $k = 0$, then $W \in R^n(R)$

but if $k \neq 0$, then $W \notin R^n(R)$

* Let V be vector space of all 2×2 matrices over field R ST W is not subspace of V, where W contains all 2×2 matrices with zero determinant.

→ Let $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $a, b \in R$ & $a, b \neq 0$

$A + B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow |A+B| = ab$

$\therefore W$ is not subspace of V as $|A+B| \neq 0$

Algebra of Subspace

* **Theorem 1:**

The intersection of any two subspaces of a vectorspace is a subspace.

→ Let $V(F)$ be a vector space over F & W_1, W_2 be 2 subspaces of $V(F)$. Then we have to show that intersection of W_1 & W_2 is subspace of $V(F)$

Let $\alpha, \beta \in W_1 \cap W_2$, which also indicates $\alpha, \beta \in W_1$ & $\alpha, \beta \in W_2$, since W_1 & W_2 are subspaces of V we have $a, b \in W$, $\alpha, \beta \in W_1$

$\Rightarrow a\alpha + b\beta \in W_1$ — ①

III^{ly} $a, b \in F$ & $\alpha, \beta \in W_2$

$\Rightarrow \alpha a + b\beta \in W_2$ — ②

from ① & ② we get,

$a, b \in F$ & $\alpha, \beta \in W_1 \cap W_2$

$a\alpha + b\beta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of $V(F)$.

* **Theorem 2:**

The intersection of an arbitrary collection of subspaces of a vector space is also a ~~vector~~ sub space.

Let $\{W_\lambda : \lambda \in x\}$ be an arbitrary collection of subspaces of vector space V. Then we have to ST ~~$\alpha \& \beta$~~ $\cap\{W_\lambda : \lambda \in x\}$ is a subspace of V.

Let us consider α & $\beta \in W_\lambda$ & $a, b \in F$. $\therefore \alpha, \beta \in W_\lambda$ for each $\lambda \in x$, then we have

$a\alpha + b\beta \in W_\lambda$ for each $\lambda \in x$

$\therefore a\alpha + b\beta \in \cap\{W_\lambda : \lambda \in x\}$

Hence $\cap\{W_\lambda : \lambda \in x\}$ is a subspace of V.

★ Theorem 3:

The union of 2 subspace of a vector space is not necessarily a subspace.

→ Let W_1, W_2 be subspace of vectorspace V.

where $W_1 = \{(a_1, a_2, 0) : a_1, a_2 \in F\}$

$W_2 = \{(a_1, 0, a_3) : a_1, a_3 \in F\}$

By observing above values we can say that W_1 & W_2 are subspaces of vector space $R^3(R)$

Now if we consider the elements for the given subspaces & assign the numerical value such that $\alpha = (1,2,0)$ & $\beta = (3,0,5) \in (W_1 \cup W_2)$

Then for scalars $a=1$ & $b=2$ & substitute in subspace eq^n, $a\alpha + b\beta = 1(1,2,0) + 2(3,0,5)$

$= (1,2,0) + (6,0,10)$

$= (7,2,10) \notin (W_1 \cup W_2)$ (may not)

Thus if $\alpha \in (W_1 \cup W_2)$ & $\beta \in (W_1 \cup W_2)$, then it is not necessarily => that $a\alpha + \beta b \in (W_1 \cup W_2)$ for some $a, b \in F$

★ Theorem 4:

Union of 2 subspaces of a vectorspace is a subspace iff 1 is contained in another.

→ Let $V(F)$ be a vector space & W_1 & W_2 be 2 subspaces of V.

Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then we can ST $W_1 \cup W_2$ is a subspace of V

Suppose $W_1 \cup W_2 = W_2$ if $W_1 \subseteq W_2$ & W_2 is a subspace of $W_1 \cup W_2$. Also $W_1 \cup W_2 = W_1$ if & W_1 W_1 is a subspace of W_2.

Let $(a,b) \in F$, $\alpha, \beta \in (W_1 \cup W_2)$

$\Rightarrow a\alpha + b\beta \in (W_1 \cup W_2)$

Now taking $a=1$ & $b=1$, we have $1\alpha + 1\beta \in W_1 \cup W_2$

$\alpha + \beta \in W_1$ (or) $\alpha + \beta \in W_2$

Suppose $\alpha + \beta \in W_1$ & $\alpha \in W_1$ then $(\alpha + \beta) - \alpha \in W_1$ because W_1 subspace of V.

$\therefore \beta \in W_1$

$|||^{ly}$ if $\beta \in W_2$

then $(\alpha + \beta) - \beta \in W_2$

$\alpha \in W_2$

★ Linear combination of vectors

Let V be a vector space $(\alpha_1, \alpha_2 \ldots \alpha_n) \in V$ then any vector $\alpha \in V$, can be expressed as below.

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$, where $a_1, a_2 \ldots a_n \in F$

is said to be the linear combination of vectors $(\alpha_1 \ldots \alpha_n)$

(or) let $V(F)$ be a vector space over F, let 'S' be any non empty subset of V. Then set of all linear combination of finite elements of 'S' is called linear span of 'S', denoted by $L(S)$

$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \cdots a_n\alpha_n : a_1, a_2 \ldots a_n \in F$

& $\alpha_1, \alpha_2 \ldots \alpha_n \in$ finite element of S$\}$

★ Theorem 1:

The linear span $L(S)$ of a non empty subset 'S' of a vector space $V(F)$ is the smallest subspace of 'V' containing 'S'. We have $L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : a_1 \ldots a_n \in F$

& $\alpha_1, \alpha_2 \cdots \alpha_n \in S\}$

$\alpha \in S$ then $\alpha = 1\alpha$, $1 \in F$, so $\alpha \in L(S)$, $S \subseteq L(S)$

Let α, β be any 2 arbitary elements of $L(S)$

then $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : \alpha_1, \alpha_2 \ldots \alpha_n \in S$

& $\beta = b_1\beta_1 + b_2\beta_2 + \cdots b_m\beta_m : \beta_1, \beta_2 \cdots \beta_n \in S$

Also $(a,b) \in F$

$a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \dots a_n\alpha_n) + b(b_1\beta_1 + b_2\beta_2 + \dots + b_m\beta_m)$

$\Rightarrow (aa_1)\alpha_1 + (aa_2)\alpha_2 + \dots + (aa_n)\alpha_n + (bb_1)\beta_1 + (bb_2)\beta_2 + \dots (bb_m)\beta_m$

Thus $\Rightarrow$ that $a\alpha + b\beta$ is a linear combination of finite no of elements of S.

$\therefore a\alpha + b\beta \in L(S)$

Hence $L(S)$ is subspace of V. & $L(S)$ is the smallest subspace of V containing S.

\+ Theorem 2:

If (S, T) are 2 subsets of vector space V, then

i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

ii) $L(S \cup T) \Rightarrow L(S) + L(T)$

iii) $L(L(S)) \Rightarrow L(S)$

$\rightarrow$ Let α be an arbitrary element of $L(S)$, then $\alpha \in L(S) = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

i) Since $S \subseteq T$, so that $\alpha_1, \alpha_2 \dots \alpha_n \in T$

$\therefore \alpha$ is also the linear combination of finite elements of $T \Rightarrow \alpha \in L(T)$

Hence $S \subseteq T$ & also $L(S) \subseteq L(T)$

ii) Since S is improper subset of $(S \cup T)$ & T is $\subseteq S \cup T$ then we have $L(S) \subseteq L(S \cup T)$

$L(T) \subseteq L(S \cup T)$

$L(S) + L(T) \subseteq L(S \cup T)$ —①

Let α be an arbitrary element of $(S \cup T)$ then α is a linear combination of finite elements of $S \cup T$

$\therefore$ we can say state that $\alpha_i \in S$ & $\alpha_i \in T$

$\therefore \alpha \in L(S) + L(T)$

Thus $L(S \cup T) \subseteq L(T) + L(S)$ —②

$L(S \cup T) = L(T) + L(S)$

iii) Since $S \subseteq T(S)$ then we can write $L(S) \subseteq L(L(S))$

We have $\alpha = b_1\beta_1 + b_2\beta_2 + b_3\beta_3 + \dots + b_n\beta_n$

$= \sum_{i=1}^{n} b_i\beta_i$

where each $\beta_i \in L(S)$. Also, each β_i is a linear combination of finite element of S. so that

$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + a_{13}\alpha_3 + \dots + a_{1m}\alpha_m$

$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3 + \dots + a_{2m}\alpha_m$

On substituting values of β_1 & β_2 in α eqn., we see that α is a linear combination of finite element of S.

* Theorem 3:

The linear sum of 2 subspaces W_1 & W_2 of a ~~vector space~~ $V(S)$ is generated by their union. i.e, $W_1 + W_2 = L(W_1 \cup W_2)$

$\rightarrow$ We have already proved that linear sum of 2 subspaces is also a subspace & linear span of a subset of a vector space is a subspace.

$\therefore W_1 + W_2$ & $L(W_1 \cup W_2)$ are subspaces of $V(F)$

* Linear dependance & independance of vectors:

- LD $\Rightarrow$ Let $V(F)$ be a vector space over a field F then a finite set $\{\alpha_1, \alpha_2 \dots, \alpha_n\}$ of vectors of V is said to be LD if there exists scalar $a_1, a_2 \dots a_n$ not all of them $= 0$ such that $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

- LID $\Rightarrow$ Let $V(F)$ be a vector space over F, then a finite of vectors $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ of V is said to be LID if for every expression of type $(a_1\alpha_1 + a_2\alpha_2 \dots$

$a_n \alpha_n = 0$ where $a_1, a_2 \ldots a_n \in F$ & also $a_1 = a_2 = \ldots = a_n$.

* If the vector $(2,-5,3)$ in the subspace of R^3 spanned by the vector $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$

→ Let $\alpha = (2,-5,3)$ & $\alpha_1 = (1,-3,2)$ $\alpha_2 = (2,-4,-1)$ & $\alpha_3 = (1,-5,7)$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$(2,-5,3) = a_1(1,-3,2) + a_2(2,-4,-1) + a_3(1,-5,7)$

$= (a_1, -3a_1, 2a_1) + (2a_2, -4a_2, -1a_2) + (a_3, -5a_3, 7a_3)$

$(2,-5,3) = (a_1 + 2a_2 + a_3, \; -3a_1 - 4a_2 - 5a_3, \; 2a_1 - a_2 + 7a_3)$

$a_1 + 2a_2 + a_3 = 2$ — ①

$-3a_1 - 4a_2 - 5a_3 = -5$ — ②

$2a_1 - a_2 + 7a_3 = 3$ — ③

eliminate a_2 by considering ① & ②

$-a_1 - 3a_3 = -1$ — ④

eliminate a_2 from ② & ③

$-11a_1 - 33a_3 = -17$ — ⑤

from ④ & ⑤

zero i.e no sol

Since no value of a_3 & a_1 will satisfy ④ & ⑤ eqn ①, ②, ③ doesn't have any sol.

Hence α cannot be expressed linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Hence vector $(2,-5,3)$ is not spanned by vectors given.

* In vector space R^3, express the vectors $(1,-2,5)$ as linear combination of vectors $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$

→ $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$(1,-2,5) = (a_1, a_1, a_1) + (a_2, 2a_2, 3a_2) + (2a_3, -a_3, a_3)$

$\Rightarrow a_1 + a_2 + 2a_3 = 1$ — ①

$a_1 + 2a_2 - a_3 = -2$ — ②

$a_1 + 3a_2 + a_3 = 5$ — ③

from ① & ②

$-a_2 + 3a_3 = 3$ — ④

from ② & ③

$-a_2 - 2a_3 = -7$ — ⑤

sub ④ & ⑤

$5a_3 = 10$

$a_3 = 2$

put in ④

$a_2 = 3$, $a_1 = -6$

$\therefore \alpha = -6\alpha_1 + 3\alpha_2 + 2\alpha_3$

$(1,-2,5) = (-6,-6,-6) + (3,6,9) + (4,-2,2)$

Given vector can be written in form of L.combination of vectors.

* For what value of M the vector $(M,3,1)$ is a L. combination of vector $(3,2,1)$ & $(2,1,0)$

→ $\alpha = a_1\alpha_1 + a_2\alpha_2$

$(m,3,1) = (3a_1, 2a_1, a_1) + (2a_2, a_2)$

$\Rightarrow 3a_1 + 2a_2 = m$

$2a_1 + a_2 = 3$

$a_1 = 1$, $a_2 = 1$, $m = 5$

* In the vector space R^4 determine, whether or not vector $(3,9,-4,-2)$ is a linear combination of vectors $(1,-2,0,3)$, $(2,3,0,-1)$ & $(2,-1,2,1)$

→ $(3,9,-4,-2) = a_1(1,-2,0,3) + a_2(2,3,0,-1) + a_3(2,-1,2,1)$

$3 = a_1 + 2a_2 + 2a_3$

$9 = -2a_1 + 3a_2 - a_3$

$-4 = 2a_3$

$-2 = 3a_1 - a_2 + a_3$

$a_3 = -2$

$a_1 = 1$

$a_2 = 3$

* Write the polynomial $f(x) = x^2+4x-3$ over R as a linear combination of polynomial.

$f_1(x) = x^2-2x+5$, $f_2(x) = 2x^2-3x$, $f_3(x) = x+3$

→ $(1,4,-3) = a_1(1,-2,5) + a_2(2,-3,0) + a_3(0,1,3)$

$1 = a_1 + 2a_2$

$4 = -2a_1 - 3a_2 + a_3$

$-3 = 5a_1 + 3a_3$

$a_1 = -3$, $a_2 = 2$, $a_3 = 4$

* In the vector space R^3, let $\alpha = (1,2,1)$, $\beta = (3,1,5)$, $\gamma = (3,-4,7)$. ST the subspace spanned by $S = \{\alpha, \beta\}$ & $T = \{\alpha, \beta, \gamma\}$ are same.

→ We have to ST $L(S) = L(T)$ from given sets S & T, we have $S \subseteq T$ which can be written as $L(S) \subseteq L(T)$. Now we have to ST γ can be expressed as linear combination of α & β. $\gamma = a\alpha + b\beta$

$(3,-4,7) = a(1,2,1) + b(3,1,5)$

$3 = a + 3b$ — (1)

$-4 = 2a + b$

$7 = a + 5b$ — (2)

(−), $-4 = -2b$

$b = 2$, $a = -3$

$\therefore \gamma = -3\alpha + 2\beta$.

Now let us consider an arbitrary variable $\delta \in L(T)$, then δ can be expressed as L. combination of α, β & γ where γ can be replaced by $-3\alpha + 2\beta$.

Thus $\delta \in L(S)$ $\therefore L(T) \subseteq L(S)$ $\therefore L(T) = L(S)$

* Find the ~~connect~~ condition on (a,b,c) such that $\alpha = (a,b,c)$ is a L. combination of vectors $(1,-3,2)$ & $(2,-1,1)$

→ $\alpha = a_1(1,-3,2) + a_2(2,-1,1)$

$a = a_1 + 2a_2$

$b = -3a_1 - a_2$

$c = 2a_1 + a_2$

$b + c = -a_1$ — (1)

$b = 3b + 3c - a_2$ ⇒ $a_2 = c$

$-2b - 3c = -a_2$

$a_2 = 2b + 3c$ — (2)

$a = -b - c + 4b + 6c$ (substitute in eq^n not used)

⇒ $a = 3b + 5c$

$a - 3b - 5c = 0$

Thus sys of eq^n is consistent iff $a - 3b - 5c$ is ~~con~~ linear, hence α is a linear combination of $(1,-3,2)$ $(2,-1,1)$ iff $a - 3b - 5c = 0$

* ST $(1,1,1)$ $(0,1,1)$ & $(0,1,-1)$ generate R^3

→ ~~In order~~ we have to ST any vector of R^3 is a LC of $(1,1,1)$, $(0,1,1)$ & $(0,1,-1)$

let $\alpha = (a,b,c) \in R^3$ & let $\alpha = a_1 A_1 + a_2 A_2 + a_3 A_3$

consider $\alpha = a_1(1,1,1) + a_2(0,1,1) + a_3(0,+1,-1)$

$\boxed{a = a_1}$

$b = a_1 + a_2 + a_3$ $\quad b + c = 2a_1 + 2a_2$

$c = a_1 + a_2 - a_3$ $\quad -2a + b + c = 2a_2$

$$\boxed{a_2 = \frac{-a+b+c}{2}}$$

$c = a + \left(-a + \frac{b}{2} + \frac{c}{2}\right) - a_3$

$\frac{c}{2} = \frac{b}{2} - a_3$

$$\boxed{a_3 = \frac{b-c}{2}}$$

* ~~Find~~ Write the matrix $E = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as LC of matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$

→ $E = xA + yB + zC$

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x & x \\ x & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ y & y \end{bmatrix} + \begin{bmatrix} 0 & 2z \\ 0 & -z \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x & x+2z \\ x+y & y-z \end{bmatrix}$$

$x = 3$, $x + 2z = 1$, $y = -2$

$z = -1$

* ST the sys of 3 vectors $(1,3,2)$ $(1,-7,-8)$ $(2,1,-1)$ of $V^3(R)$ is LD.

→ Let $(a,b,c) \in R$ such that $a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = (0,0,0)$

$(a + b + 2c ~~=0~~, 3a - 7b + c ~~=0~~, 2a - 8b - c) = (0,0,0)$

$a + b + 2c = 0$, $3a - 7b + c$, $2a - 8b - c = 0$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix} = 15 - (-5) + 2(-24 + 14) = 20 - 20 = 0$$

② & ③ $5a - 15b = 0$

~~(a, b, c)~~ = $\quad 5a = 15b$

$3b + b + 2c = 0$

$4b + 2c = 0$

$4b = 2c$

$c = 2b$.

$a = 3b$

$3b + b + 4b = 0$

$b = 0$, $a = 0$, ~~$c = 0$~~

rank of A is less than 3 i.e less than no of variables

∴ The set of homogeneous eqⁿ has non-zero sol

Hence sys of 3 given vectors are linearly dependent.

* ST $S = \{(1,2,4)\ (1,0,0)\ (0,1,0)\ (0,0,1)\}$ is a LD subset of space $V_3(R)$ where R - field of real nos

→ WKT the set $(1,0,0)$ $(0,1,0)$ $(0,0,1)$ is LID of $V_3(R)$, now we will consider $(1,2,4) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$a = 1$, $b = 2$, $c = 4$, since we obtained non-zero sol for coefficient the given vectors are LD.

* If α, β, γ are LID vectors of a vector space $V(F)$ where F is any field of complex no then so also are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$.

→ Let a, b, c be scalars such that $a(\alpha+\beta)+b(\beta+\gamma)+c(\gamma+\alpha)=0$

$$(a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} a+c=0 \\ a+b=0 \\ b+c=0 \end{matrix}$$

$R_2 \leftarrow R_2 - 1$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

∴ Given vectors has no sol.

$r(A) = 3 = n$ i.e a, b, c

⇒ the sys of eq^n has only zero sol

i.e $a=0$, $b=0$, $c=0$

∴ α, β, γ are LID

$a = b = c = 0$

* If $V_3(R)$, where R is field of real nos examine each of the following sets of vectors for LD.

i) $\{(1,3,2), (1,-7,-8), (2,1,-1)\}$

ii) $\{(0,2,-4), (1,-2,-1), (1,-4,3)\}$

iii) $\{(1,2,0), (0,3,1)\ (-1,0,1)\}$

iv) $\{(-1,2,1)\ (3,0,-1)\ (-5,4,3)\}$

v) $\{(2,3,5)\ (\cancel{4}, 9, 25)\}$

vi) $\{(2,1,2)\ (8,4,8)\}$

→ i) $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -7 & -8 \\ 2 & 1 & -1 \end{bmatrix}$ $\Rightarrow$ $(a, 3a, 2a) + (b, -7b, -8b) + (2c, c, -c)$

Column form.

LD

$a+b+2c=0,\ 3a-7b+c=0,\ 2a-8b-c=0$

ii) $A = \begin{bmatrix} 0 & 2 & -4 \\ 1 & -2 & -1 \\ 1 & -4 & 3 \end{bmatrix}$ $\Rightarrow \cancel{|A|} =$

$(0, 2a, -4a) + (b, -2b, -b) + (c, -4c, 3c)$

$0+b+c=0,\quad 2a-2b\cancel{-4}c=0,\quad -4a-b+3c=0$

$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 2 & -2 & -4 \\ -4 & -1 & 3 \end{vmatrix} = -1(6-16) + 1(-2-8)$

$10 - 10 = 0$

$r(A) < n \quad \therefore$ no sol, thus LD

iii) $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1(3) - 1(2)$

$= 1$, LID

iv) $|A| = \begin{vmatrix} -1 & 3 & -5 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{vmatrix} = -1(4) - 3(6-4) - 5(-2)$

$= -4 - 6 + 10 = 0$ LD

7 - 6

v) $A = \begin{bmatrix} 2 & 4 \\ 3 & 9 \\ 5 & 25 \end{bmatrix}$ $R_2 \leftarrow R_2 - \frac{3}{2}R_1$, $R_3 \leftarrow R_3 - \frac{5}{2}R_1$

$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \\ 0 & 15 \end{bmatrix}$ $R_3 \leftarrow R_3 - 5R_2$

$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$ $r(A) = n$

$\therefore$ unique sol. $\therefore$ LID.

vi) $A = \begin{bmatrix} 2 & 8 \\ 1 & 4 \\ 2 & 8 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1/2$, $R_3 \leftarrow R_3 - R_1$

$A = \begin{bmatrix} 2 & 8 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $r(A) < n$

LD

★ PT in $R(x)$ the vector space of all polynomials in x over R. The system of $p(x) = 1 + x + 2x^2$, $q(x) = 2 - x + x^2$, $r(x) = -4 + 5x + x^2$ is LD.

$\Rightarrow$ Let $a, b, c \in R$

$\Rightarrow a p(x) + b q(x) + c r(x) = 0$

$(a + ax + 2ax^2) + (2b - bx + bx^2) + (-4c + 5cx + cx^2) = 0 + 0x + 0x^2$

$\Rightarrow a + 2b - 4c = 0$

$ax - bx + 5cx = 0x$

$2ax^2 + bx^2 + cx^2 = 0x^2$

$A = \begin{bmatrix} 1 & 2 & -4 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$

$|A| = 1(-1-5) - 2(1-10) - 4(1+2)$

$= -6 + 18 - 12 = 0$

$\therefore$ It is LD

★ **Basis of a Vector:**

Let V be a vector space over a field F & S be any non empty subset of V. Then S is said to be a basis of V if

i) S is LID

ii) $L(S) = V$ i.e, every element of V is a LC of finite elements of S.

★ **Finite dimensional Vector Space:**

Let $V(F)$ be a vector space over a field F & let S be any non-empty subset of V, then $V(F)$ is said to be finite dimensional if $\exists S$ is finite set of V such that $L(S) = V$. If this set contains n elements the D. of V is 'n'.

★ **Theorem 1:**

If $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of vector space $V(F)$, then each element of V is uniquely expressible as a LC of elements of S.

$\Rightarrow$ Proof: Since S is a basis of $V(F)$, then by defn of basis each element of V is a LC of elements of S. Thus we need to only show uniqueness.

Let us consider 2 different subsets $\{a_1, a_2, \dots, a_n\}$ & $\{b_1, b_2, \dots, b_n\}$ of scalars corresponding to an element $\alpha \in V$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ & $\alpha = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$

$a_1\alpha_1 + a_2\alpha_2 - b_2\alpha_2 + \dots + a_n\alpha_n - b_n\alpha_n = 0$

$a_1\alpha_1 - b_1\alpha_1 + a_2\alpha_2 - b_2\alpha_2 + \dots + (a_n - b_n).\alpha_n = 0$

$(a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + \dots + (a_n - b_n)\alpha_n = 0$

Since let $S = \{\alpha_1, \dots \alpha_n\}$ is LID

$a_1 - b_1 = 0$, $a_2 - b_2 = 0$ ... $a_n - b_n = 0$

$a_1 = b_1$, $a_2 = b_2$, $a_n = b_n$

∴ Above demonstration shows that given vector forms unique solution for LC of elements of S.

* Dimension of subspace of a vector space.

* Let V be the vector space of ordered pairs of complex no over real field R i.e let V be vector space $C^2(R)$: ST the set $S = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis for V.

→ • First let us prove that set S is linearly independent. Let us consider $a, b, c, d \in R$ such that

$a(1,0) + b(i,0) + c(0,1) + d(0,i) = (0,0)$

$a + ib = 0$

$c + id = 0$

Solving the above eqn we get $a = b = c = d = 0$

∴ The given sys is LID.

• Now we shall ST $L(S) = V$

Let $(a+ib, c+id)$ be any element of V where $a, b, c, d \in R$, then $(a+ib, c+id) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$

∴ Every element of V can be expressed as LC of elements of S, which indicates the given set S is a basis of V.

* ST the sets $S = \{(1,2), (3,4)\}$ forms the basis for R^2

→ • Let us consider $a, b \in R$ such that

$a(1,2) + b(3,4) = (0,0)$

$a + 3b = 0$, $2a + 4b = 0$

$b = 0$ $a = 0$

∴ The given system is LID

• Let $(a+3b, 2a+4b)$ be any element of R^2

=> $(a+3b, 2a+4b) = a(1,2) + b(3,4)$

Thus S forms basis of R^2

* Let V be vector space of all 2x2 matrices over F PT V has dimension 4 by exhibiting a basis for V which has 4 elements.

→ Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where

$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

are the four elements of V.

• Now we shall show that S forms basis of V

let $a, b, c, d \in F$ such that.

$a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 = (0, 0, 0, 0)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

=> $a = b = c = d = 0$

∴ S is LID => (S forms basis of V)

• Next we shall ST $L(S) = V$

let $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

∴ By observing above eqn S forms basis of V which has 4 elements ∴ $\dim V = 4$

* Let $\alpha=(1,2,1)$, $\beta=(2,9,0)$ & $\gamma=(3,3,4)$ ST the set $S=\{\alpha,\beta,\gamma\}$ is a basis of R^3.

→ Let $a,b,c \in R$ such that

$a\alpha + b\beta + c\gamma = (0,0,0)$

$a(1,2,1) + b(2,9,0) + c(3,3,4) = (0,0,0)$

$a+2b+3c=0$

$2a+9b+3c=0$

$a+0b+4c=0$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

$= 36 - 2(5) + 3(-9)$

$= 36 - 10 - 27$

∴ S is LID $|A| \neq -1$

∴ S is a basis of R^3.

* Consider the basis $S=\{\alpha_1,\alpha_2,\alpha_3\}$ of R^3 where $\alpha_1=(1,1,1)$, $\alpha_2=(1,1,0)$, $\alpha_3=(1,0,0)$. Express $(2,-3,5)$ in terms of basis elements $\alpha_1,\alpha_2,\alpha_3$

→ let $a,b,c \in R$, since S forms basis of R^3

LC ⇒ $a\alpha_1 + b\alpha_2 + c\alpha_3 = (2,-3,5)$

$a(1,1,1) + b(1,1,0) + c(1,0,0) = (2,-3,5)$

$a+b+c=2$

$a+b=-3$

$a=5$ $b=-8$ $c=+5$

⇒ $5\alpha_1 + (-8)\alpha_2 + 5\alpha_3 = (2,-3,5)$

* ST the vectors $\alpha_1=(1,0,-1)$, $\alpha_2=(1,2,1)$, $\alpha_3=(0,-3,2)$ form basis of R3. Express each of std basis vectors as a LC of $\alpha_1, \alpha_2, \alpha_3$

→ Let $a,b,c \in R$,

$a(1,0,-1) + b(1,2,1) + c(0,-3,2) = (0,0,0)$

$a+b=0$

$2b-3c=0$

$-a+b+2c=0$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{vmatrix} = 1(+6) - 1(4)$$

$= +10$

(5)×(3)

∴ S is in form of basis of R^3.

• The std basis of R^3 is $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$

Now let $\alpha=(p,q,r)$ be any element of R^3.

Since S forms a basis of R^3, then there exists $\{x,y,z\} \in R$ such that

$(p,q,r) = x(1,0,-1) + y(1,2,1) + z(0,-3,2)$

$x+y \cancel{+0} = p$

$2y - 3z = q$

$-x + y + 2z = r$

→ $2y+2z = p+r$

$2\left(\frac{q+3z}{2}\right) + 2z = p+r$

$q + 3z + 2z = p+r$

• $y = \frac{q+3z}{2} = \frac{+2q+3p+3r}{10}$

$5z = p+r-q$

• $z = \frac{p+r-q}{5}$

• $x = p - \frac{2q+3p+3r}{2}$

Let us consider std basis R^3 $e_1=(1,0,0)=(p,q,r)$

• $e_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3$

• $e_2 = \frac{-1}{5}\alpha_1 + \frac{1}{5}\alpha_2 \; \frac{-1}{5}\alpha_3$

• $e_3 = \frac{-3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3$

* Given that each set S below spans R^3, find basis of R^3 which is contained in S

i) $\{(1,0,2)\ (0,1,1)\ (2,1,5)\ (1,1,3)\ (1,2,1)\}$

ii) $\{(2,6,-3)\ (5,15,-8)\ (3,9,-5)\ (1,3,-2)\ (5,3,-2)\}$

→ i) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_1$, $R_4 \leftarrow R_4 - R_1$, $R_5 \leftarrow R_5 - R_1$

since $\dim R^3$ & S span R^3

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $R_3 \leftarrow R_3 - R_2$, $R_4 \leftarrow R_4 - R_2$, $R_5 \leftarrow R_5 - 2R_2$

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$r(A) = n$, here non-zero rows are corresponding to vectors corresponds to vectors $(1,0,2)$ $(0,1,1)$ $(1,2,1)$ of S

$\therefore$ S contains maximum LID subset $\{(1,0,2), (0,1,1)(1,2,1)\}$

3 elements = $\dim R^3$

Hence the set S forms basis of R^3

ii) $A = \begin{bmatrix} 2 & 6 & -3 \\ 5 & 15 & -8 \\ 3 & 9 & -5 \\ 1 & 3 & -2 \\ 5 & 3 & -2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 5/2 R_1$

$R_3 \leftarrow R_3 - 3/2 R_1$

$R_4 \leftarrow R_4 - 1/2 R_1$

$R_5 \leftarrow R_5 - 5/2 R_1$

$A = \begin{bmatrix} 2 & 6 & -3 \\ 0 & 0 & -0.5 \\ 0 & 0 & -0.5 \\ 0 & 0 & -0.5 \\ 0 & -12 & +3 \end{bmatrix}$ $R_3 \leftarrow R_3 + R_2$

R

$A = \begin{bmatrix} 2 & 6 & -3 \\ 0 & 0 & -0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -12 & +3 \end{bmatrix}$

$\therefore$ basis corresponds to $\{(2,6,-3)\ (5,15,-8)\ (5,3,-2)\}$

* **Linear sum of 2 subspaces:**

Let W_1 & W_2 be 2 subspaces of a vector $V(F)$ then the linear sum of W_1 & W_2 is the set of all those elements each one of which is expressible as the sum of an element of W_1 & W_2 & it can be written as $W_1 + W_2$

$$W_1 + W_2 = \{\alpha + \beta ; \alpha \in W_1, \beta \in W_2\}$$

* **Direct sum of vector subspaces:**

Let W_1 & W_2 be 2 subspaces of a vector space V. Then V is said to be the direct sum of W_1 & W_2 if each element of V can be uniquely expressed as the sum of an element of W_1 & W_2. If V is direct sum of W_1 & W_2, then it can be written as $V = W_1 \oplus W_2$

** The necessary & sufficient condition for V to be direct sum of 2 of its subspaces W_1 & W_2 are

i) $V = W_1 + W_2$

ii) $W_1 \cap W_2 = \{0\}$ **

* **Dimension of Subspace of a Vectorspace:**

* **Theorem:**

Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. ~~Suppose~~ Then set obtained by adjoining β to S is LID.

→ Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be a linearly ID subset of V. Then we shall show the set $S_1 = \{\beta, \alpha_1, \alpha_2 \dots \alpha_n\}$ obtained by adjoining β to S is also LID where $\beta \in V$, but not in subspace of V which is spanned by S

Since $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ are distinct vectors in S we can express those in form of LC i.e $a_1\alpha_1 + a_2\alpha_2 \dots + a_n\alpha_n = 0$

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + b\beta = 0$ where all a's are zero & also b should also be zero to express S_1 as LID.

• If $b \neq 0$, then $\beta = \left(\frac{-a_1}{b}\right)\alpha_1 + \left(\frac{-a_2}{b}\right)\alpha_2 + \dots + \left(\frac{-a_n}{b}\right)\alpha_n$

$\therefore$ This indicates β is LC of $\alpha_1, \alpha_2 \dots \alpha_n$

$\therefore$ set S_1 is LID

★ Theorem 6:

If a finite dimensional vector space $V(F)$ be the direct sum of its 2 subspaces W_1 & W_2 then dimension $V = \dim W_1 + \dim W_2$

→ Since $\dim V$ is finite $\therefore W_1$ & W_2 are also finite dimensional

let $\dim W_1 = m$, $\dim W_2 = n$

$V = W_1 \oplus W_2 \Rightarrow V = W_1 + W_2$

$W_1 \cap W_2 = \{0\}$

Let us consider $S_1 = \{\alpha_1, \alpha_2 \dots \alpha_m\}$ be a basis of W_1

& set $S_2 = \{\beta_1, \beta_2 \dots \beta_n\}$ is a basis of W_2

Now consider a set $S_3 = \{\alpha_1, \alpha_2 \dots \alpha_m, \beta_1, \beta_2 \dots \beta_n\}$

& S_3 forms a basis of V.

For some scalars $a_1, a_2 \dots a_m$ & $b_1, b_2 \dots b_m \in F$

then we can write eqn as $a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m + b_1\beta_1 + \dots + b_n\beta_n = 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m = -(b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n)$

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m \in W_1$

$b_1\beta_1 + b_2\beta_2 + \dots + b_m\beta_m \in W_2$

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m \in W_1 \cap W_2$

$b_1\beta_1 + b_2\beta_2 + \dots + b_m\beta_m \in W_2 \cap W_1$

$\therefore a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m = 0$

$b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n = 0$

Since S_1 & S_2 both are LID $\therefore a_1 = a_2 = \dots = a_m = 0$

also $b_1 = b_2 = \dots = b_n = 0$

Let γ be an arbitary element of V then $\gamma = \alpha + \beta$

$\alpha \in W_1, \beta \in W_2$

$\gamma = a_1\alpha_1 + a_2\alpha_2 \dots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 \dots + b_n\beta_n$

$\Rightarrow$ the elements present in S_3 $\therefore S_3$ forms a basis of V

accordingly $\dim V = m + n$ $\quad \therefore \dim V = \dim W_1 + \dim W_2$

★ Let W be the subspace of $V_4(\mathbb{R})$ generated by vectors $(1,-2,5,-3), (2,3,1,-4), (3,8,-3,-5)$ then

i) Find a basis & $\dim W$

ii) Extend the basis of W to a basis of $V_4(\mathbb{R})$

→ i) Let $S = \{(1,-2,5,-3)\ (2,3,1,-4)\ (3,8,-3,-5)\}$

& $a, b, c, d \in \mathbb{R}$ & $L(S) = W$

$\Rightarrow$ $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 8 \\ 5 & 1 & -3 \\ -3 & -4 & -5 \end{vmatrix}$ Now we shall find maximal LID subset of S. Let A be matrix whose rows are elements of S.

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the echelon matrix 2 non-zero rows representing the coordination vectors $(1,-2,5,-3)$, $(0,7,-9,2)$ that form a basis of rows space ie

$T = \{(1,-2,5,-3), (0,7,-9,2)\}$

$\dim W = 2$

ii) $\dim V_4(\mathbb{R}) = 4$, in order to form basis of $V_4(\mathbb{R})$ we shall extend set T by including 2 vectors $(0,0,1,0)$ $(0,0,0,1)$

$T' = \{(1,-2,5,-3), (0,7,-9,2)\ (0,0,1,0)\ (0,0,0,1)\}$

T' is LID $\because$ of the matrix

$$A' = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix which is in echelon form has 4 non-zero rows. which is LID.

Hence T is a basis of $V_4(R)$ which is obtained by extending a basis of W & dim = 4.

* Let W_1 be the subspace of $V_4(R)$ generated by set of vectors $S = \{(1,1,0,-1)\ (1,2,3,0)\ (2,3,3,-1)\}$ & W_2 the subspace of $V_4(R)$ generated by the set of vectors $T = \{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$

Find i) $\dim(W_1+W_2)$, ii) $\dim(W_1 \cap W_2)$

→ WKT $V = W_1 + W_2 = L(W_1 \cup W_2)$

Then W_1+W_2 is a subspace generated by set of vectors of $S \cup T$, where $S \cup T = \{(1,1,0,-1)\ (1,2,3,0)\ (2,3,3,-1)\ (1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$

Let A be coefficient matrix obtained by $S \cup T$

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & +1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r(A) = 3 = \dim(V)$$

$\therefore \dim(W_1+W_2) = 3$

ii) First we find $\dim(W_1 \& W_2)$

To find $\dim(W_1)$, let us consider matrix A_1 obtained by set S

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \dim(W_1) = 2$$

let A_2 be matrix, whose rows are elements of W_2

$$A_2 = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$\Rightarrow \dim(W_2) = 2$

WKT $\dim(W_1+W_2) = 3 = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$3 = 4 - \dim(W_1 \cap W_2)$

$\dim(W_1 \cap W_2) = 1$

★ Find inverse

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$$

$$A = \left[\begin{array}{cccc|cccc} 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 & 0 \\ 2 & 3 & 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \longleftrightarrow R_2$

$$A = \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 & 0 \\ 2 & 3 & 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 2R_1$

$$A = \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & -2 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2 \quad R_4 \to R_4 - R_2$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & -5 & -1 & -2 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + R_3 \quad R_3 \to -\frac{1}{2} R_3$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -2 & 3/2 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & -3 & -5 & -1 & -2 & 0 & 1 \end{array}\right]$$

$R_4 \to R_4 + 3R_3$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/2 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & -1 & 1 & -3/2 & 1 \end{array}\right]$$

$R_4 \to -2R_4$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/2 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3 & -2 \end{array}\right]$$

$R_1 \to R_1 - R_4 \quad R_2 \to R_2 + R_4$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & 3 & -3 & 2 \\ 0 & 1 & 0 & 0 & 3 & -4 & 4 & -2 \\ 0 & 0 & 1 & 3/2 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3 & 2 \end{array}\right]$$

$R_3 \to R_3 - \frac{3}{2} R_4$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & 3 & -3 & 2 \\ 0 & 1 & 0 & 0 & 3 & -4 & 4 & -2 \\ 0 & 0 & 1 & 0 & -3 & 4 & -5 & 3 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3 & -2 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$$

★ Guassian Elimination Method.

(a) $2x_1 + 4x_2 + x_3 = 3$

$3x_1 + 2x_2 - 2x_3 = 2$

$x_1 - x_2 + x_3 = 6$

$\to \Rightarrow (1) \times 3/2$

$\cancel{3x_1} + 6x_2 + 3/2\, x_3 = \frac{9}{2}$

$(2) \Rightarrow (-)\ \cancel{3x_1} + 2x_2 - 2x_3 = 2$

$4x_2 + 7/2\, x_3 = 5/2 \quad - (4)$

$(2) \times 1/3 \Rightarrow \cancel{x_1} + 2/3\, x_2 - 4/3\, x_3 = 2/3$

$(3) \Rightarrow (-)\ \cancel{x_1} - x_2 + x_3 = 6$

$5/3\, x_2 - 7/3\, x_3 = -16/3 \quad - (5)$

$x_1 = 2.8, \quad x_2 = -1.16, \quad x_3 = 2.04$

(b) $2x + y + 4z = 12 ; \quad 8x - 3y + 2z = 23 ; \quad 4x + 11y - z = 33$

$(1) \times 3 \Rightarrow 6x + 3y + 12z = 36$

$(2) \Rightarrow (+)\ 8x - 3y + 2z = 23$

$14x + 14z = 59 \quad - (4)$

$(3) \times 2 \Rightarrow \cancel{8x} + 22y - 2z = 66$

$(3) \Rightarrow (-)\ \cancel{8x} - 3y + 2z = 23$

$25y - 4z = 23$

$z = -\left(\frac{23 - 25y}{25}\right)$

* Echeon matrix & row canonical form of matrix.

1) A matrix 'A' is called echeon matrix if following 2 conditions are satisfied.
- All zero rows, if any, are at bottom of matrix.
- Each leading non zero entry in a row is to the right of leading non zero entry in preceeding row.

2) A matrix is said to be in row canonical form if its an echeon matrix ie it satisfies above 2 ppts. & if its satisfy additional 2 ppts.
- Each pivot (leading non-zero element in given row) = 1
- Each pivot element is the only non zero entry in column

* Find the solⁿ for given linear eqⁿ.

1) $x+y-6z=0$; $-3x+y+2z=0$; $x-y+2z=0$

$\rightarrow$ $A = \begin{bmatrix} 1 & 1 & -6 \\ -3 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$AX = B$

$R_2 \rightarrow R_2 + 3R_1$; $R_3 \rightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 4 & -16 \\ 0 & -2 & 8 \end{bmatrix} \qquad 2R_3 + R_2$$

$R_3 \rightarrow 2R_3 + R_2$

$$A = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 4 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow 4y - 16z = 0$

$x + y - 6z = 0$

$z = c$

$y = \frac{16c}{4} = 4c$

$x + 4c - 6c = 0 \Rightarrow x = +2c$

2> $x+2y+z=3$; $2x+5y-z=-4$; $3x-2y-z=5$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 3 & -2 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$ $R_3 \rightarrow R_3 - 3R_1$

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 8R_2$

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{bmatrix}$$

$R_3 \rightarrow \frac{R_3}{-28} \Rightarrow M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & +1 & +3 \end{bmatrix}$

$R_1 \rightarrow R_1 \cancel{+R_3} - 2R_2$ $M = \begin{bmatrix} 1 & 0 & 7 & 23 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$R_1 \rightarrow R_1 - 7R_3$ $M = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$R_2 \rightarrow R_2 + 3R_3$ $M = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$x = 2$, $y = -1$, $z = 3$

$r(A) = 3 \qquad n = 3 \qquad r(M) = 3$

** If $r(A) = n$ then there will be unique sol.

The sol is unique iff $r(A) = n = r(M)$ **

(or) The sol is unique iff $r(A)=n=r(M)$

* Gauss - Jordan: No need to convert into echelon form.

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \qquad R_3 \leftarrow R_3 - 3R_1$

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 8R_2 \qquad R_1 \leftarrow R_1 - 2R_2$

$$M = \begin{bmatrix} 1 & 0 & 7 & 23 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{bmatrix}$$

$R_3 \leftarrow -\frac{1}{28} R_3$

$$M = \begin{bmatrix} 1 & 0 & 7 & 23 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 7R_3 \qquad R_2 \leftarrow R_2 + 3R_3$

$$M = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

3) $x_1 + x_2 - 2x_3 + 4x_4 = 5$; $2x_1 + 2x_2 - 3x_3 + x_4 = 3$; $3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$

$\to$
$$M = \begin{bmatrix} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1 \qquad R_3 \leftarrow R_3 - 3R_1$

$$M = \begin{bmatrix} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{bmatrix}$$

$R_3 \leftarrow R_3/2 \qquad R_3 \leftarrow R_3 - R_2$

$$M = \begin{bmatrix} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = r(M) \neq n$

$\therefore$ Infinite sol

$x_1 + x_2 - 2x_3 + 4x_4 = 5$; $x_3 - 7x_4 = -7$

* x_1 & x_3 are pivot elements $\therefore$ consider arbitrary values to free variables x_2 & x_4

* $x_1 + x_2 - 2x_3 + 3x_4 = 4$; $2x_1 + 3x_2 + 3x_3 - x_4 = 3$; $5x_1 + 7x_2 + 4x_3 + x_4 = 5$

$\to$
$$M = \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_5 - 5R_1$

$$M = \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$M = \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

$r(A) \neq r(M)$

$\therefore$ no sol

degenerate eq^n

★ LDU decomposition / factorization.

1) Suppose $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$ Reduce matrix as LDU factorization

$R_2 \leftarrow R_2 + 3R_1$ $\quad R_3 \leftarrow R_3 - 2R_1$

$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & -3 & 1 \end{bmatrix}$ $\quad R_3 \leftarrow 2R_3 + 3R_2$ (or) $R_3 + \frac{3}{2} R_2$

$A \approx \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} = U$; $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -3/2 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

2) (a) $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 - 2R_1$, $\quad R_3 \leftarrow R_3 + R_1$

$A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix}$ $\quad R_3 \leftarrow R_3 + \frac{5}{2} R_2$

$A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -3/2 \end{bmatrix}$; $U = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -5/2 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3/2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 8 & 1 \\ -5 & -9 & 7 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 - 2R_1$, $\quad R_3 \leftarrow R_3 + 5R_1$

$B = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 7 \\ 0 & 11 & -8 \end{bmatrix}$ $\quad R_2 \leftrightarrow R_3 \Rightarrow$ should not exchange in inverse & decomposition

$B = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 11 & -8 \\ 0 & 0 & 7 \end{bmatrix}$ X → wrong step

Thus above matrix can't be brought to Δ^{le} form without row interchange, ∴ Above matrix isn't LU factorization

★ Solve the following eqn by LU decomposition method & obtain sol for the unknowns.

1) $2x_1 + x_2 + x_3 = 2$; $x_1 + 3x_2 + 2x_3 = 2$; $3x_1 + x_2 + 2x_3 = 2$

→ $AX = B$ $\quad A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - R_1/2$

$R_3 \leftarrow R_3 - 3R_1/2$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & 3/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$ $\quad R_3 \leftarrow R_3 + \frac{1}{5} R_2$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & 3/2 \\ 0 & 0 & 4/5 \end{bmatrix} \neq U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$ check ; $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -1/5 & 1 \end{bmatrix}$

$LY = B$

$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

- $Y_1 = 2$
- $\frac{1}{2} Y_1 + Y_2 = 2 \Rightarrow Y_2 = 1$

$\frac{3}{2} Y_1 - \frac{1}{5} Y_2 + Y_3 = 2$

- $Y_3 = -\frac{4}{5}$

$UX = Y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & 3/2 \\ 0 & 0 & 4/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4/5 \end{bmatrix}$$

- $x_3 = -1$
- $5/2\, x_2 + 3/2\, x_3 = 1 \Rightarrow x_2 = 1$
- $2x_1 + x_2 + x_3 = 2 \Rightarrow x_1 = 1$

2> $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$

→ $AX = B$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

- $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - R_1/2$

$R_3 \leftarrow R_3 - 3/2\, R_1$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & -7/2 & 1/2 \end{bmatrix} \quad R_3 \leftarrow R_3 + 7R_2$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} = U \; ; \; L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}$$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

- $y_1 = 9$
- $1/2\, y_1 + y_2 = 6 \Rightarrow y_2 = 3/2$
- $3/2\, y_1 - 7y_2 + y_3 = 8$

$y_3 = 5$

$UX = Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

- $z = 5/18$
- $1/2\, y + 5/2\, z = 3/2 \Rightarrow y = \frac{29}{18}$
- $2x + 3y + z = 9 \Rightarrow x = \frac{35}{18}$

3> $2x - 3y + 10z = 3$; $-x + 4y + 2z = 20$; $5x + 2y + z = -12$

→ $AX = B$

$$\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

$R_2 \leftarrow R_2 + R_1/2$

$R_3 \leftarrow R_3 + 5/2\, R_1$

$x = -4$
$y = 3$
$z = 2$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 19/2 & -24 \end{bmatrix} \quad R_3 \leftarrow R_3 - 19/5\, R_2$$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/5 \end{bmatrix} = U \; , \; L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -5/2 & 19/5 & 1 \end{bmatrix}$$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -5/2 & 19/5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

- $y_1 = 3$
- $-1/2\, y_1 + y_2 = 20 \Rightarrow y_2 = 43/2$
- $-5/2\, y_1 + 19/5\, y_2 + y_3 = -12$

$y_3 = -\frac{431}{5}$

$UX = Y$

$$\begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 43/2 \\ -431/5 \end{bmatrix}$$

$x = -4 \quad y = 3 \quad z = 2$

* Vector spaces:

In order to discuss vector space we use the set of vector & scalars. To define a vector space we need a field 'F' & elements of F is scalar. In addition to that we need two operators ⇒ vector add & scalar mul. This is defined using internal compositn