Pole placement Technique - Back ground

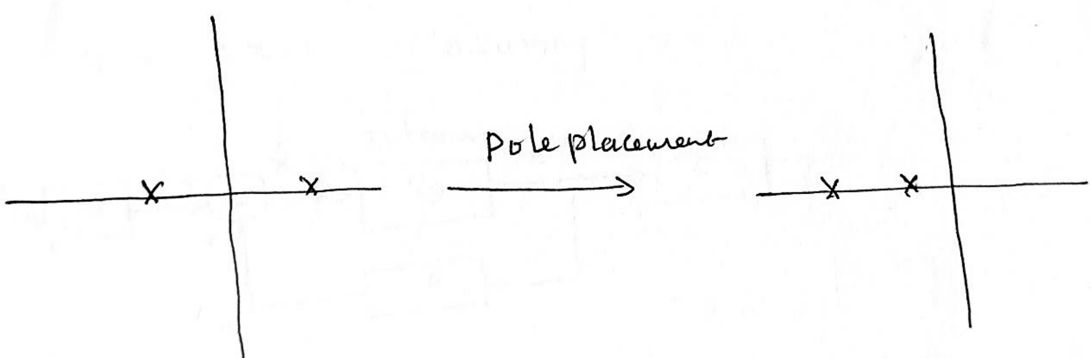
$$\dot{x} = \underbrace{Ax}_{\substack{\text{Dynamics} \\ \text{Captured} \\ \text{here}}} + \underbrace{Bu}_{\substack{\text{How the system} \\ \text{responds to inputs.}}}$$

A Controller has to modify the A matrix to change dynamics.

$$\text{Eigen values }(A) = \underbrace{\text{poles of the system}}_{\text{Locations of poles dictates stability.}}$$

Moving poles $\Rightarrow$ choose system stability.

If eigen values are at undesirable locations use pole placement to move them to the desired location.



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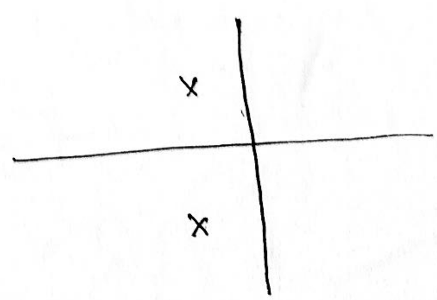


Introduction to full-state Feed back Control - pole placement technique

→ In the transfer function based technique, Compensators are designed to predominantly control the response of second order systems. By adjusting the control gain, poles and zeroes of the Compensator, the adverse effect of the system is compensated.

→ The effect of higher-order poles are either neglected or Compensated separately using notch filters.

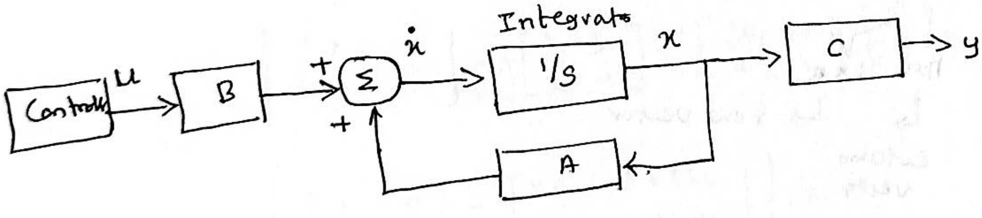
→ In case of full-state feed-back control, controllers could be designed to regulate the behaviour of all the poles of the system.

→ In reality, only some of the states are measured while the rest are estimated using numerical solution, as practically it is not possible to sense all the states of the system

Block diagram

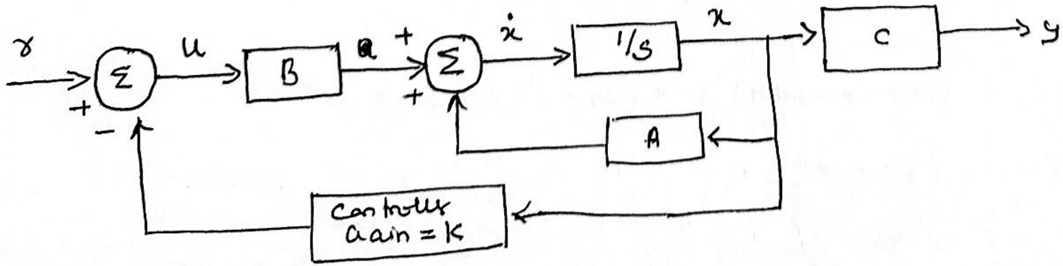
$$\dot{x} = Ax + Bu$$

$$y = cx \qquad \text{Assuming } D = 0.$$



open loop

Let us include the Controller in the feed back path.



$\dot{x} = Ax + Bu$

$u = r - Kx$

Set $r = 0$ $\Rightarrow$ $\boxed{u = -Kx}$ Control law

$\dot{x} = Ax - BKx$ $\Rightarrow$ $\dot{x} = (A - BK)x$.

- We set $r = 0$ only to study how system dynamics
- By changing matrix A, we can change the Eigen values of the system.

For $\dot{x} = Ax + Bu$

the Eigen values / Poles are given by

$$|SI - A| = 0$$

For $\dot{x} = (A - BK)x = A_{CL}\, x$ where $A_{CL} = (A - BK)$

the Eigen values are given by

$$|SI - (A - BK)| = 0$$

Thus by controlling the gain K, it is possible to place the Eigen values / poles at the desired locations.

This method is called <u>pole placement technique.</u>

<u>Note.</u> SI is $n \times n$ matrix.

$A - BK$ is also $n \times n$ matrix.

$A \to n \times n$, $B \to n \times 1$ ↳ column vector, $K \to 1 \times n$ ↳ row vector

$$K = [k_1 \;\; k_2 \;\; k_3 \; \cdots \; k_n]$$

Example -1

Given that $A = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Design the Controller using pole placement technique so that the closed loop poles are placed at $-5 \pm j2$.

Step1 check whether the given system is Controllable.

$$Q = [B \quad AB] = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$$

$|Q| = -3 \neq 0 \Rightarrow$ Hence system is fully Controllable

Step2 Given $\lambda_1 = -5+j2$ $\lambda_2 = -5-j2$

CE: $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$

$$\lambda^2 + 10\lambda + 29 = 0 \quad --- \quad (1)$$

check Eigenvalues of A using $|\lambda I - A| = 0$

$\Rightarrow \lambda = -1.1623, 5.1623$

$\Rightarrow$ given System is unstable.

To make it stable we have to move RH pole to Left hand side.

CE with Controller is

$$|\lambda I - (A - BK)| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \left\{ \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} - \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{2\times 1} \underbrace{[k_1 \quad k_2]}_{1\times 2} \right\} \right| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \left\{ \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -2k_1 & -2k_2 \\ k_1 & k_2 \end{bmatrix} \right\} \right| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2k_1 & 3+2k_2 \\ 2-k_1 & 4-k_2 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 2k_1 & -(3+2k_2) \\ k_1 - 2 & \lambda - 4 + k_2 \end{vmatrix} = 0$$

$$\lambda^2 + (-4 - 2k_1 + k_2)\lambda + (11k_1 - 6 - 4k_2) = 0 \quad --- \quad (2)$$

Comparing (1) & (2)

$10 = -4 - 2k_1 + k_2$ $k_1 = 30.33$

$29 = 11k_1 - 6 - 4k_2$ $k_2 = 74.67$

Example 2

Given that $\dot{x} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 3 \end{bmatrix} u.$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(a) Is the system Controllable

(b) Is the system stable

(c) Design the matrix K to place the poles at $-2 \pm 2i$

Ans : (a) The system is Controllable

(b) $\lambda = 2, -1 \Rightarrow$ System is unstable

(c) $K = [6.8750 \quad 6.25]$

Example 3

Given that $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

(a) Is the system Controllable

(b) Is the system stable

(c) Design the matrix K to place the poles at $-2, -1 \pm j$

Ans:

(a) Controllable

(b) $\lambda = 0, -1, -2$ stable

(c) $K = [0.4 \quad 0.4 \quad 0.1]$

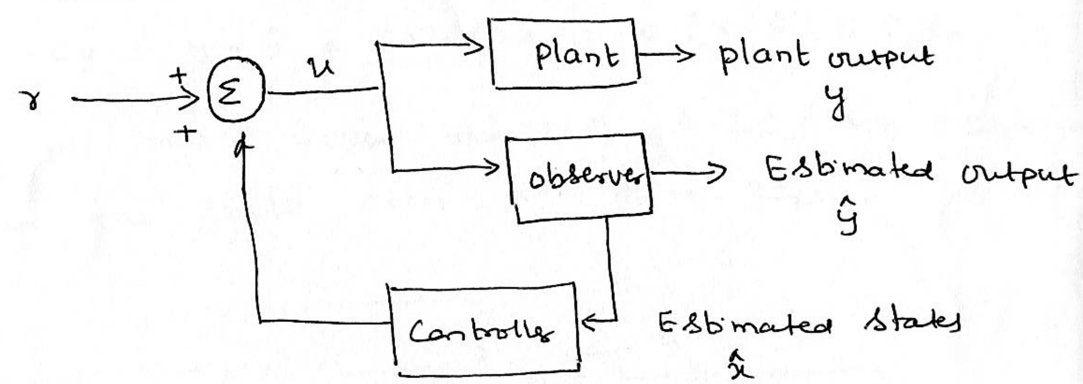
Observer design

→ State feedback control assumes that, we can measure all the states.

→ Some times it is expensive or not feasible to add sensors for every state in a system.

→ An observer or estimator is used to calculate state variables that are not accessible from the plant.

→ The design of observer consists of finding the observer gain, L, so that the transient response / dynamics of the observer is faster than the controlled loop to yield a rapidly updated estimate of the state observer



plant

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

observer

$$\dot{\hat{x}} = A\hat{x} + Bu$$

$$\hat{y} = C\hat{x}$$

→ The plant and observer has the same dynamics because, A is same in both cases.

→ The observer dynamics is modified by introducing an additional term as follows.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$y = Cx \quad \text{and} \quad \hat{y} = C\hat{x}$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})$$

State error

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - [A\hat{x} + Bu + LC(x - \hat{x})]$$

$$\dot{e} = (A - LC)x - (A - LC)\hat{x}$$

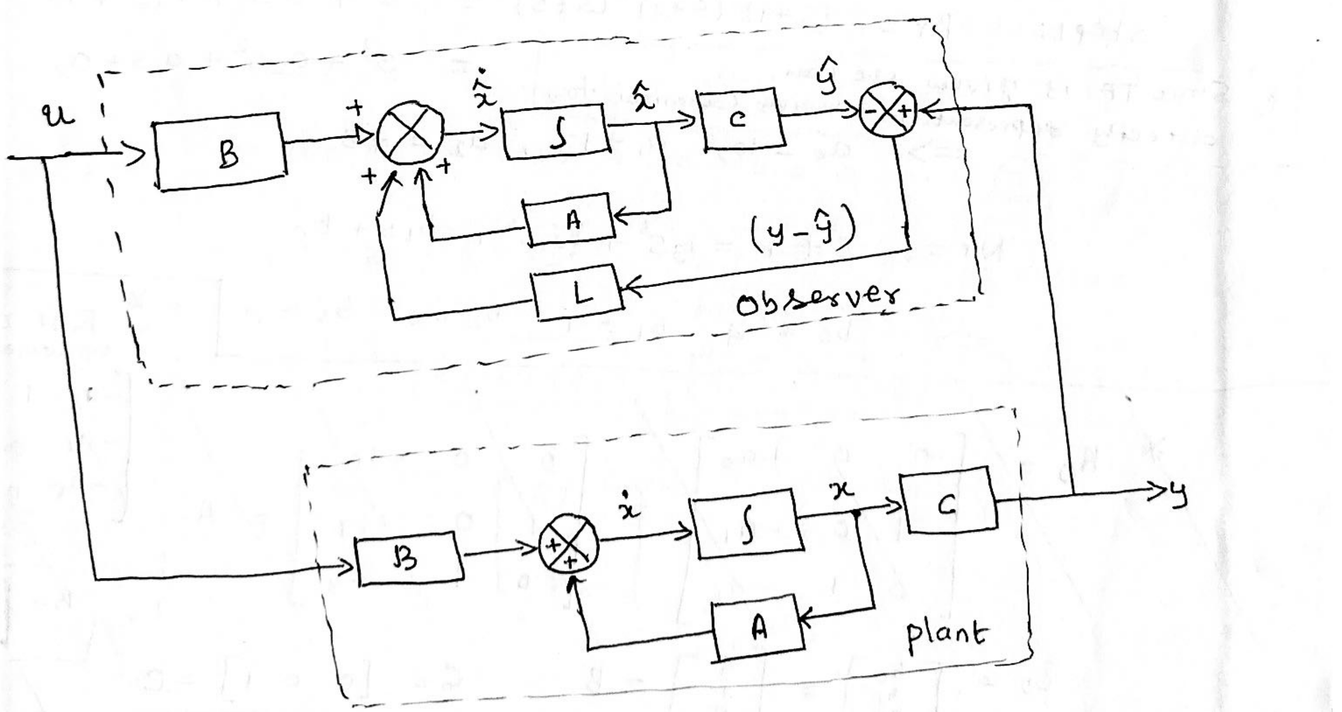
$$\dot{e} = (A - LC)(x - \hat{x})$$

$$\dot{e} = (A - LC)e_x \quad \text{where} \quad e_x = x - \hat{x}$$

Output error

$$e_y = y - \hat{y} = Cx - C\hat{x} = C(x - \hat{x}) = Ce_x$$

If all the Eigen values of A−LC are negative, then the state error will go to zero.



Steps to design the observer

- → Identify the state matrices ; A, B, C, D
- → Calculate the matrix , $sI - (A-LC)$
- → Find the CE of the compensated system

  $$|sI - (A-LC)| = 0$$

- → Determine the desired CE to satisfy the design constraints.
- → Equate the coefficients of to find the observer gain L

Example 1

Given the plant transfer function

$$\frac{s+4}{(s+1)(s+2)(s+5)}$$

Design a suitable observer to place the observer poles at -40, $-10 \pm j20$.

Step 1: $Dr = (s+1)(s+2)(s+5) = s^3 + 8s^2 + 17s + 10$

$$= s^3 + a_2 s^2 + a_1 s + a_0$$

* Since TF is given we can directly represent in observer canonical form

$$\Rightarrow a_0 = 10, \quad a_1 = 17, \quad a_2 = +8$$

$$Nr = s + 4 = b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

$$\Rightarrow b_0 = 4 \quad b_1 = 1 \quad b_2 = 0 \quad b_3 = 0$$

* Both are equivalent.

$$A_O = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -10 \\ 1 & 0 & -17 \\ 0 & 1 & -8 \end{bmatrix} = A. \quad \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$B_O = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = B \qquad C_O = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = C$$

Observer Canonical form

Step I: $A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}$; $B = [b_2 \ b_1 \ b_0]$; $C = [1 \ 0 \ 0]$.

$A = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix}$; $B = [0 \ 1 \ 4]$; $C = [1 \ 0 \ 0]$.

Step II $LC = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} [1 \ 0 \ 0] = \begin{bmatrix} L_1 & 0 & 0 \\ L_2 & 0 & 0 \\ L_3 & 0 & 0 \end{bmatrix}$

$A - LC = \begin{bmatrix} -8-L_1 & 1 & 0 \\ -17-L_2 & 0 & 1 \\ -10-L_3 & 0 & 0 \end{bmatrix}$; $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\lambda I - (A - LC) = \begin{bmatrix} (\lambda+8+L_1) & -1 & 0 \\ 17+L_2 & \lambda & -1 \\ 10+L_3 & 0 & \lambda \end{bmatrix}$

$|\lambda I - (A - LC)| = \lambda^2(\lambda+8+L_1) + \lambda(17+L_2) + 10 + L_3 = 0$

$\Rightarrow \lambda^3 + (8+L_1)\lambda^2 + (17+L_2)\lambda + (10+L_3) = 0$ .... (1)

Step III: observer poles : $\lambda_1 = -40$, $\lambda_2 = -10+j20$, $\lambda_3 = -10-j20$

CE: $(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = 0$

$\Rightarrow \lambda^3 + 60x^2 + 1300\lambda + 20{,}000 = 0$ ... (2)

Step IV Comparing (1) & (2) $8 + L_1 = 60$; $17 + L_2 = 1300$

$10 + L_3 = 20{,}000$

$\boxed{L_1 = 52}$ $\boxed{L_2 = 1283}$ $\boxed{L_3 = 19980}$

$[L] = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 1283 \\ 19980 \end{bmatrix}$

Step 2

$$A - LC = \begin{bmatrix} 0 & 1 & -10-L_1 \\ 1 & 0 & -17-L_2 \\ 0 & 1 & -8-L_3 \end{bmatrix}$$

* Instead of sI λI can also be used.

$$|sI - (A-LC)| = s^3 + (8+L_3)s^2 - (17+L_2)s + (10+L_1) \quad \text{---} \; (1)$$

Step 3 observer poles: $-40, \; -10 \pm j20$

$$\Rightarrow \quad CE: \; (s+40)(s+10+j20)(s+10-j20)$$

$$= s^3 + 60s^2 + 1300s + 20{,}000 \quad \text{----} \; (2)$$

Step 4 Equating the coefficients of (1) and (2)

$$8 + L_3 = 60 \quad \Rightarrow \quad L_3 = 52$$

$$10 + L_1 = 20000 \quad \Rightarrow \quad L_1 = 19980$$

$$1300 = -17 - L_2 \quad \Rightarrow \quad L_2 = -1317$$

$$[L] = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} 19980 \\ -1317 \\ 52 \end{bmatrix}$$

Example 2

Design observer for the following system to have 0.8 s settling time (2%) and 16.3 % overshoot.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$t_s = \dfrac{4}{\xi\omega_n} \Rightarrow \xi\omega_n = 5$

$0.163 = e^{-\pi\xi/\sqrt{1-\xi^2}} \Rightarrow \xi =$

$\omega_d = \omega_n\sqrt{1-\xi^2} = 8.66$

Desired observer poles:

$s_d = -\xi\omega_n \pm j\omega_d = -5 \pm j8.66$

$\lambda_1 = -5 + j8.66, \quad \lambda_2 = -5 - j8.66$

check for observability.

$$LC = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} C^T & A^T C^T \end{bmatrix}$$

$A - LC =$

$\lambda I - (A - LC) =$

CE: $|\lambda I - (A - LC)| = 0$ --- (1)

desired CE $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$ -- (2)

Compare coefficients in (1) & (2) and Find L_1 & L_2.