Unit-2

Linear Transformations.

Let U & V be two vector spaces over the same field F. A mapping
Df. $T: U \to V$ is said to be a linear transformation from UxV which associates to each element α of U to a unique element $T(\alpha)$ of V such that $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ for all α and β in U & all scalars a, b in F.

Some illustrations:

1. If V is any vectorspace over F then identity transformation from VxV defined by $I(\alpha) = \alpha, \forall \alpha \in V$ is a linear transformation. Also zero transformation denoted by $O(\alpha) = 0$ is a linear transformation.

2. Let F be a field of real numbers and let V be the vector space of all polynomials. Then a mapping $D: V \to V$ given by $D[f(x)] = \frac{d}{dx} f(x), \forall f(x) \in V$ is a linear transformation.

Since for any $f(x)$ & $g(x) \in V$ and $a, b \in F$. $D[af(x) + bg(x)] = \frac{d}{dx}\left[af(x) + bg(x)\right]$

$$= \frac{d}{dx}[a f(x)] + \frac{d}{dx}[b g(x)]$$

$$= a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$

$$= a[D f(x)] + b[D g(x)]$$

3. Let R be the field of real numbers and let V be the vector space of all functions from RxR which are continuous. Then a mapping $T: V \to V$ is given by $T[f(x)] = \int_0^x f(t).dt$

For any vector $f(x), g(x) \in V$ & $a, b \in R$

$$\therefore T[af(x) + bg(x)] = \int_0^x [af(t) + bg(t)].dt$$

$$= \int_0^x a.f(t).dt + \int_0^x bg(t).dt$$

$$= aT[f(x)] + b[T g(x)]$$

4. Let V be the vectorspace of all MxN matrices over a field F & let P be a fixed mxn matrix and Q be a fixed matrix of order mxn

Then a mapping $T: V \to V$ given by $T(A) = PAQ$ $\forall$ $A \in V$ is a linear transformation.

For any two matrices $A, B \in V$ & $a, b \in F$. $T(aA + bB) = P(aA+bB)Q$

$$= (aPA + bPB)Q$$
$$= aPAQ + bPBQ$$
$$= a\,T(A) + b\,T(B)$$

Some other definitions:

1. <u>Linear operator</u> :- Let $V(F)$ be a vectorspace. Then a linear transformation from $V \times V$ is called a linear operator.

2. <u>Zero transformation</u> :- Let U and V be two vectorspaces over the same field F then the zero transformation of $U \times V$ is a mapping defined by $T(\alpha) = 0$ $\forall$ $\alpha \in U$ where 0 is the 0 vector of V.

3. <u>Identity transformation</u> :- Let $V(F)$ be a vectorspace then a linear transformation $I: V \to V$ is said to be identity transformation defined by $I(\alpha) \in \alpha$ $\forall$ $\alpha \in V$.

4. <u>Negative of a linear transformation</u> :- Let U and V be two vector spaces over the same field F. Let T be a linear transformation of $U \to V$ ~~from~~. Then a linear transformation $-T$ of $U \to V$ defined by $(-T)(\alpha) = -[T(\alpha)]$ $\forall$ $\alpha \in V$ is called Negative of a linear transformation.

Properties of linear transformations

Let $U(F)$ and $V(F)$ be two vectorspaces and T be a linear transformation of $U \times V$. Then

(i) $T(0) = 0$ where 0 on LHS is the vector of U and zero on RHS is the zero vector V.

(ii) $T(-\alpha) = -T(\alpha)$ $\forall$ $\alpha \in U$

(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)$ $\forall$ $\alpha, \beta \in U$.

(iv) $T[\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n] = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n)$
$\forall$ $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n \in U$ & $a_1, a_2, a_3 - a_n \in F$

(i) Proof: If $\alpha \in U$ then $T(\alpha) \in V$. Since V is the vectorspace, we have, $T(\alpha) + 0 = T(\alpha)$

$$\cancel{T(\alpha+0)} = \quad T(\alpha) + 0 = T(\alpha+0)$$

$$T(\alpha) + 0 = T(\alpha) + T(0)$$

$$0 = T(0)$$

(ii) we have $\forall \alpha \in U$

$$T[\alpha + (-\alpha)] = T(\alpha) + T(-\alpha)$$

$$T(0) = T(\alpha) + T(-\alpha)$$

$$0 = T(\alpha) + T(-\alpha)$$

$$-T(\alpha) = T(-\alpha)$$

(iii) For all $\alpha, \beta \in U$

$$T(\alpha + -\beta) = T(\alpha) + T(-\beta)$$

$$T(\alpha - \beta) = T(\alpha) - T(\beta)$$

(iv) Since $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$ is a linear combination of vectors of U. Now we shall prove ~~that~~ the result by induction of n.

For $n=1$, $T(\alpha_1 a_1) = a_1 T(\alpha_1)$

For $n=2$, $T(\alpha_1 a_1 + \alpha_2 a_2) = a_1 T(\alpha_1) + a_2 T(\alpha_2)$

Suppose the result is true of $(n-1)$ values i.e.,

$$T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_{n-1}\alpha_{n-1}) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_{n-1} T(\alpha_{n-1})$$

For n values,

$$T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(\alpha_n)$$

Hence the result is proved by induction.

Theorem 81 Let U, V & W be vectorspace over F. Let T_1 be a linear transformation from ~~U to V~~ U into V and T_2 be a l. tr. from V into W. Then the composed fn T_2T_1 is defined by

$$T_2T_1(\alpha) = T_2(T_1(\alpha)) \quad \forall \alpha \in U$$

Proof:- For $\alpha, \beta \in U$ and $a, b \in F$ we have $T_2T_1(a\alpha + b\beta) =$

$$T_2[T_1(a\alpha + b\beta)]$$
$$= T_2[aT_1(\alpha) + bT_1(\beta)]$$
$$= a(T_2T_1)(\alpha) + b(T_2T_1)(\beta)$$

$\therefore$ T_2T_1 is a lin. tr. from U into W.

Linear operator:-

If V is a vectorspace over the field F, then a lin. tr. from $V \to V$ is called a linear operator.

Algebra of linear operators

Let V be a vectorspace over the field F and let T, T_1, T_2 & T_3 be linear operators on V. and let c be an element in F. Then

(i) $IT = TI = T$, I being an identity operator

(ii) $T_1(T_2 + T_3) = T_1T_2 + T_1T_3$

$(T_2 + T_3)T_1 = T_2T_1 + T_3T_1$

(iii) $T_1(T_2T_3) = (T_1T_2)T_3$

(iv) $c(T_1T_2) = T_1(cT_2)$

(v) $TO = OT = O$

Proof

(i) For $\alpha \in V$

$$IT(\alpha) = I(T(\alpha)) = T(\alpha)$$
$$IT = T$$
$$TI(\alpha) = T(I(\alpha)) = T(\alpha)$$
$$TI = T$$
$$IT = TI = T$$

(ii) ~~For any~~ $\forall$ $\alpha \in V$

$$[T_1(T_2+T_3)](\alpha) = T_1[(T_2+T_3)(\alpha)]$$
$$= T_1[T_2(\alpha)+T_3(\alpha)]$$
$$= T_1T_2(\alpha)+T_1T_3(\alpha)$$
$$= [T_1T_2+T_1T_3](\alpha)$$

$$\therefore [T_1(T_2+T_3)] = T_1T_2+T_1T_3$$

$$\text{111}^{\text{ly}} [(T_2+T_3)T_1] = T_2T_1+T_3T_1$$

(iii) $\forall \alpha \in V$

$$[T_1(T_2T_3)](\alpha) = [T_1(T_2T_3)(\alpha)]$$
$$= T_1[T_2(T_3(\alpha))]$$
$$= T_1T_2[T_3(\alpha)]$$
$$= [(T_1T_2)T_3](\alpha)$$

$$T_1(T_2T_3) = (T_1T_2)T_3$$

(iv) $\forall \alpha \in V$, $c \in F$

$$[c(T_1T_2)](\alpha) = c[(T_1T_2)(\alpha)]$$
$$= c[T_1(T_2(\alpha))]$$
$$= cT_1[T_2(\alpha)]$$
$$= [(cT_1)T_2](\alpha)$$

$$c(T_1T_2) = (cT_1)T_2$$

$$\text{111}^{\text{ly}}\ c(T_1T_2) = T_1(cT_2)$$

(v)

Problems

wnth

1. Let T_1 & T_2 be linear operators on R^2 defined as follows.

$T_1(x_1, x_2) = (x_2, x_1)$

$T_2(x_1, x_2) = (x_1, 0)$

$T_1 T_2 \neq T_2 T_1$

$\Rightarrow$ Let $\alpha = (x_1, x_2) \in R^2$ then $(T_1 T_2)(\alpha) = T_1(T_2(\alpha))$

$= T_1(T_2(x_1, x_2))$

$= T_1(x_1, 0)$

$[T_1 T_2(\alpha)] = (0, x_1)$ and $(T_2 T_1)(\alpha) = T_2[T_1(\alpha)]$

$= T_2[T_1(x_1, x_2)]$

$= T_2[x_2, x_1]$

$= (x_2, 0)$

Therefore clearly we can see that $(T_1 T_2)(\alpha) \neq T_2 T_1(\alpha)\ \forall\ \alpha \in R$

$\therefore T_1 T_2 \neq T_2 T_1$

2. Let $V(R)$ be the vectorspace of all polynomial functions in x with coefficients in the field R of real numbers. Let D & T be two linear operators on V defined by $D(f(x)) = \frac{d}{dx} f(x)$ and $T(f(x)) = \int_0^x f(x)\,dx$ for every $f(x) \in V$. Then ST $D.T = I$ and $TD \neq I$

$\rightarrow$ Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots \in V$ where $a_0 a_1, a_2 \in R$ then

$DT(f(x)) = D[T(f(x))]$

$= D\left[\int_0^x (a_0 + a_1 x + a_2 x^2 + \dots).dx\right.$

$= D\left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots) dx\right]$

$= \frac{d}{dx}\left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots)\right]$

$= a_0 + a_1 x + a_2 x^2 + \dots$

$= I[\cdot f(x)]$ $\qquad DT f(x) = I[f(x)]\ \forall f(x) \in V.$

$TD(f(x)) = T\left[\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \cdots)\right]$

$= T\left[a_1 + 2a_2x + 3a_3x^2 + \cdots\right)$

$= \int_0^x a_1 + 2a_2x + 3a_3x^2 + \cdots) \, dx$

$= a_1x + a_2x^2 + a_3x^3 + \cdots \neq f(x) \neq I[f(x)]$

$\therefore TD[f(x)] \neq I[f(x)] \; \forall f(x) \in V.$

$TD \neq I.$

Hence $DT \neq T.D.$

3. Let $V(R)$ be the vectorspace of all polynomials in x with co-efficients in the field R. Let D and T be two linear transformations on V defined by $D(f(x)) = \frac{d}{dx} f(x) \; \forall f(x) \in V$ $T(f(x)) = x(f(x)). \; \forall f(x) \in U$

Then ST $DT \neq TD$ also $DT - T.D = I$

→ Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots \in V$ where $a_0, a_1, a_2 \in R$ then

$DT(f(x)) = D[T f(x)]$

$= D[x f(x)]$

$= \frac{d}{dx}[x f(x)]$

$= x f'(x) + f(x).$

$TD[f(x)] = T[D f(x)]$

$= T[f'(x)]$

$= x f'(x).$

$\therefore$ From the above eq^{n}s we can say that $DT[f(x)] \neq TD[f(x)]$

$DT \neq TD \; \forall f(x) \in U$

$DT - TD = x f'(x) + f(x) - x f'(x)$

$= f(x)$

$= I[f(x)]$

4. Range and Null space of linear transformation.

1. Range space: If T is a lin. tr from $U \to V$ then the range of T is a subspace of V. Let R_T be the range of T, i.e., the set of all vectors β in V such that $T(\alpha) = \beta$ for some $\alpha \in U$

i.e, $R_T = \{\beta \in V : T(\alpha) = \beta$ for some $\alpha \in U\}$. If U is finite dimensional, then $\dim R_T$ is called rank of T and is denoted by $\rho(T)$

2 Null space:

If T is a linear transformation from $U \to V$, then the null space of T denoted by $N(T)$ (or) N_T is the set of all vectors α in U such that $T(\alpha) = 0$ where 0 is the 0 vector in V.

$$N_T = \{\alpha \in U : T(\alpha) = 0,\ 0 \in U\}$$

If U is finite dimensional then the finite dimension is denoted by nullity of T. ~~N(T)~~ $n(T)$

Theorem: Let U & V be vectorspaces over the field F and let T be a linear transformation from U onto V. Suppose U is finite dimensional then rank (T) + nullity (T) = dim U i.e., $\rho(T) + N(T) = \dim U$

Product of linear transformations:-

Let $U(F)$, $V(F)$ & $W(F)$ be the vectorspace. Let T be a lin tr from $U \to V$ and S be a lin. tr from $V \to W$. Then the composite fn ST called the product of lin. tr. defined by $ST(\alpha) = S[T(\alpha)]\ \forall \alpha \in U$. is a lin tr from $U \to W$. ~~Since~~

Since $T: U \to V$ is a linear tr hence $T(\alpha) = \beta$, $\beta \in V$ and $\alpha \in U$. Also transformation S maps from $V \to W$ & S is a lin tr Hence $S(\beta) \in W$ for $\beta \in V$.

$\therefore$ For $\beta = T(\alpha) \in V$, $S[T(\alpha)] \in W$

Let $\alpha_1, \alpha_2 \in U$ & $a, b \in F$ then $(ST)(a\alpha_1 + b\alpha_2) = S[T(a\alpha_1 + b\alpha_2)]$

$$= S[aT(\alpha_1) + bT(\alpha_2)]$$
$$= aST(\alpha_1) + bST(\alpha_2)$$
$$= a(ST)(\alpha_1) + b(ST)(\alpha_2)$$

$\therefore$ ST is a lin. tr.

Polynomials in a lin operator:

Let T be a linear operator on a vectorspace $V(F)$, then $T.T$ is a lin operator on V since the product of lin. operators is an associative operation $\therefore$ if n is a +ve integer, then we define $TT = T^2$, $TT.T = T^3$, $TT \cdots T = T^n$

In general, T^n is a lin. operator on V. Also $T^0 = I$.
If m & n are non -ve integers then we see that $T^m T^n = T^{m+n}$
$(T^m)^n = T^{mn}$

Invertible lin. tr :-

A linear tr T from a vectorspace $U(F) \to V(F)$ is called invertible or regular if there exists a unique lin. tr T^{-1} from $V(F) \to U(F)$ such that T^{-1} is the identity lin. tr on U & TT^{-1} is the identity tr on V. Furthermore T is invertible if :

(i) T is one to one
(ii) T is onto
i.e. $R(T) = V$.

Theorem 1: Let U & V be vectorspaces over the same field F & let T be lin. tr from $U \to V$. If T is invertible then T^{-1} is a lin. tr from $V \to U$.

Proof :- Since T is invertible hence for each $\beta \in V$, there is a unique $\alpha \in U$ such that $T(\alpha) = \beta$ iff $T^{-1}(\beta) = \alpha$ $\quad T(\alpha) = \beta \Leftrightarrow T^{-1}(\beta) = \alpha$

Now we shall ST T^{-1} is lin.

Let $\alpha_1, \alpha_2 \in U$ & $a, b \in F$

$$T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$$

$\therefore$ T is lin.

But for β_1, β_2 in V, there are unique α_1 & α_2 in U respectively such that $T(\alpha_1) = \beta_1 \Leftrightarrow T^{-1}(\beta_1) = \alpha_1$; $T(\alpha_2) = \beta_2 \Leftrightarrow T^{-1}(\beta_2) = \alpha_2$

Thus we have

$$T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2$$
$$a\alpha_1 + b\alpha_2 = T^{-1}(a\beta_1 + b\beta_2)$$
$$aT^{-1}(\beta_1) + bT^{-1}(\beta_2) = T^{-1}(a\beta_1 + b\beta_2)$$

Hence T^{-1} is a lin. tr

Theorem 2:
Let T_1 be an invertible lin tr from $U(F) \to V(F)$ & T_2 be an invertible lin tr from $V(F) \to U(F)$. Then $T_1 . T_2$ is invertible and $(T_2 T_1)^{-1} = T_1^{-1} T_2^{-1}$

Proof ~~Pt~~ To show T_2T_1 is invertible we shall st mapping is one on one and onto

If $\alpha_1 \alpha_2 \in U$ such $(T_2T_1)(\alpha_1) = (T_2T_1)(\alpha_2)$ then

$$T_2[T_1(\alpha_1)] = T_2[T_1(\alpha_2)]$$

$$T_1(\alpha_1) = T_2(\alpha_2)$$

$$\alpha_1 = \alpha_2$$

Thus T_2T_1 is one to one mapping. Also T_1 and T_2 being onto mapping then for each $\beta \in V$, there exists a unique $\alpha \in U$ such that $T_1(\alpha) = \beta$. Then for each $\gamma \in W$ there exists a unique $\beta \in V$ such that $T_2(\beta) = \gamma$.

$\therefore$ There exists $\beta \in V$ for γ obtained by $T_2(\beta)$ there exists $\alpha \in U$

$\gamma = T_2(T_1(\alpha))$ $\therefore$ $\gamma = (T_2T_1)\alpha$

$\therefore$ T_2T_1 is onto mapping. Hence T_2T_1 is invertible

Also $(T_2T_1)(T_1^{-1}T_2^{-1}) = T_2(T_1T_1^{-1})T_2^{-1}$

$$= (T_2 I)T_2^{-1}$$

$$= T_2T_2^{-1}$$

$$= I$$

$$T_1^{-1}T_1 = I$$

Hence $(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}$

<u>Non-singular lin. tr</u>

Let U & V be vectorspaces over the field F. Then a lin. tr T from $U \to V$ is called non singular if the null space of T is zero

Thus if T is non-singular then $T(\alpha) = 0$ i.e, $\alpha = 0$

Also when T is non-singular & $\alpha, \beta \in U$, $T(\alpha) = T(\beta) \Rightarrow T(\alpha) - T(\beta) = 0$

$$T(\alpha - \beta) = 0$$

$$\alpha - \beta = 0$$

$$\alpha = \beta$$

Hence T is non singular and it implies one to one mapping

Theorem 1: Let T be a lin. tr from $U(F) \to V(F)$. Then T is non singular iff T ~~carries~~ carries each lin. independent subset of U into a lin. independent subset of V.

Proof: Let us first prove that T is non-singular.

Let us consider $S = \{\alpha_1, \alpha_2 \dots \alpha_k\}$ be an arbitrary lin. independent subset of u. Then we have to show that $S_1 = T\{T(\alpha_1), T(\alpha_2) \dots T(\alpha_k)\}$ is linearly independent subset of V. For scalars $a_1, a_2 \dots a_k$ we have $\{a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_k T(\alpha_k)\} = 0$

$$\{T(a_1\alpha_1) + T(a_2\alpha_2) + \dots + T(a_k\alpha_k)\} = 0$$

$$T\{a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k\} = 0$$

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k = 0$$

$$a_1 = 0 = a_2 \dots = a_k$$

Hence S_1 is linearly independent subset of u

Conversely, suppose that T carries each lin. independent subset of u into a lin inde. subset of V. Let α be a non-zero vector in u then α is lin. independent. Also $T(\alpha)$. $\therefore T(\alpha) \neq 0$ $\because$ the set consisting of 0 vector alone is dependent which indicates nullspace of T is zero space Hence T is non-singular.

Theorem 2: Let u & v be finite dimensional v. spaces over the field F such that dim u = dim v. If T is lin. tr from $u \to v$ then the following are equivalent

(i) T is invertible

(ii) T is non singular.

(iii) T is onto mapping i.e., range of T is V.

(iv) If $(\alpha_1, \alpha_2 \dots \alpha_n)$ is a basis of u then $\{T(\alpha_1), T(\alpha_2) \dots T(\alpha_n)\}$ is a basis of V.

Proof: (i) ⇒ (ii)

Since T is invertible, it's one to one & onto mapping. Therefore T is non singular.

(ii) & (iii)

Let T be non-singular & let $\alpha_1, \alpha_2 \cdots \alpha_n$ be the basis of u. Then the set $T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)$ is linearly independant subset of V but $\dim u = \dim V$.

$\therefore$ $T(\alpha_1) \ldots T(\alpha_n)$ is a basis for V.

For any $\beta \in V$, $a_1 a_2 \ldots a_n \in F$ then we have $\beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots$
$\ldots a_n T(\alpha_n)$

$\beta = T\{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n\}$

where $\beta \in R_T$

Thus $V \subseteq R_T$; $V = R_T$; $V = T$

(iii) & (iv)

Suppose range of T equal to V $R_T = V$ let the set $\alpha_1, \alpha_2 \ldots \alpha_n$ be a basis of U so that an arbitrary element $\alpha \in U$ is expressible as lin. combination of $\alpha_1, \alpha_2 \ldots \alpha_n$.

Let $\alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots b_n\alpha_n$ $\forall$ $b_1, b_2 \cdots b_n \in F$

$T(\alpha) = T(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$

$= b_1 T(\alpha_1) + b_2 T(\alpha_2) + \cdots + b_n T(\alpha_n)$

This shows that each element of range of T is expressible as lin. combination of $\{T(\alpha_1) + T(\alpha_2) + \cdots + T(\alpha_n)\}$. Thus the set spans range of T. WKT $R_T = V$ also WKT $\dim u = \dim V = n$. Hence tr fn applied to element of U i.e., α from which we obtain $\{T(\alpha_1) + T(\alpha_2) \ldots T(\alpha_n)\}$ forms the basis of u

(iv) & (i)

Let $\{\alpha_1 \alpha_2 \cdots \alpha_n\}$ be a basis of u such that $\{T(\alpha_1) + T(\alpha_2) \cdots T(\alpha_n)\}$ be the basis of V. Let us consider α which is an arbitrary element of u, then we have $b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$ $\forall$ $b_1, b_2 \cdots b_n \in F$.

$T(\alpha) = 0$
$T(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n) = 0$

$b_1 T(\alpha_1) + b_2 T(\alpha_2) + \cdots + b_n T(\alpha_n) = 0$

$b_1 = 0 = b_2 \cdots = b_n$

$\therefore \alpha = 0$

Hence T is non singular & $\therefore$ T is one to one mapping

Consequently T is also convertible.

Problems:

1. Let T be a linear operator on a vectorspace $V(F)$. If $T^2 = 0$ what can you say about the relation of the range of T to the null space of T. Given an example of a linear operator on $V_2(R)$ such that $T^2 = 0$ But $T \neq 0$

$\Rightarrow$ Since $T^2 = 0$. then for $\alpha \in V$ $T^2(\alpha) = 0(\alpha) \Rightarrow T[T(\alpha)] = 0$

$T(\alpha) \in N(T)$

$T(\alpha) \in R(T)\ \forall \alpha \in V$

$R(T) \subset N(T)$

Hence when $T^2 = 0$ the range of T is contained in $N(T)$.

Ex: Let T be a linear map from $V_2(R) \to V_2(R)$ such that $T(a,b) = (0,a)$ $\forall (a,b) \in V_2(R)$

$T \neq 0$.

Also $T^2(a,b) = T[T(a,b)] = T[(0,a)] = (0,0)$

$\therefore T^2 = 0$

2. If $T: R^3 \to R^3$ is a linear operator defined by $T(x,y,z) = (x+z, x-z, y)$. Show that T is invertible and find $T^{-1}(2,4,6)$

$\Rightarrow$ Let $T(x,y,z) = (0,0,0)$

$(x+z,\ x-z,\ y) = (0,0,0)$

$\Rightarrow x+z = 0,\ x-z = 0\ \ y = 0$

By solving the above eqn's we get

$x = 0\ \ y = 0\ \ z = 0$

$\therefore$ For $\alpha \in R^3$, $T(\alpha) = 0 \Rightarrow \alpha = 0$

Thus T is non singular and hence it is invertible

Now $T(x,y,z) = (p,q,r)$

$x+z = p,\ x-z = q\ \ y = r$

$x = \dfrac{p+q}{2}$ $\quad z = \dfrac{p-q}{2}$

$y = r$

$T^{-1}(p,q,r) = x, y, z$

$= \dfrac{p+q}{2}, r, \dfrac{p-q}{2}$

$T(2,4,6) = (3, 6, -1)$

3. Find two linear operators T & S on $V_2(R)$ such that $T(S)$ but $S.T \neq 0$

→ $T(a,b) = (0,a) \ \forall (a,b) \in V_2(R)$ & $a,b \in R$

$S(a,b) = (a,0) \ \forall (a,b) \in V_2(R)$ & $a,b \in R$

$TS(a,b) = T[S(a,b)]$

$= T[(a,0)]$

$= (0,0) \ \forall (a,b) \in V_2(R)$

$ST(a,b) = S[T(a,b)]$

$= S[$

Co-ordinate vector:

Let V be a finite dimensional v. space over a field F and let $\dim V = n$ then $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a basis of V. Then $\forall \alpha \in V$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ for a_i's $\in F$.

Then the co-ordinate vector of α relative to B which we write as column vector unless otherwise specified is $[\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Matrix representation of linear transformation.

Let U be an m dimensional v. space & V be an n dim v.space Let $B = [\alpha_1, \alpha_2 \cdots \alpha_m]$ & $B' = [\beta_1, \beta_2 \cdots \beta_n]$ be the basis of U and V respectively

Let T be a lin tra from U onto V. Since B' is basis of V so that each $T(\alpha_i)$ is a lin-comb. of elements of B'

For $a_{ij} \in F$ where $1 \leq i \leq m$ $1 \leq j \leq n$

$T(\alpha_i) = a_{ij}\beta_1$

$T(\alpha_1) = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n$

$T(\alpha_2) = a_{21}\beta_1 + a_{22}\beta_2 + \cdots + a_{2n}\beta_n$

$\vdots$

$T(\alpha_m) = a_{m1}\beta_1 + a_{m2}\beta_2 + \cdots + a_{mn}\beta_n$

The transpose of the above matrix of co-efficients denoted by $[T]_B$ is called the matrix representation of T, relative to the ordered basis B

$$[T]_B = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{m \times n}$$

1. Let V be the v. space of polynomials in t over the field of reals R of degree ~~lesser~~ ~~greater~~ than or equal to 3. and let $D: V \to V$ be the differential oper-ator lesser defined by diff of polynomial $D[P(t)] = \frac{d}{dt}[P(t)]$.

→ Let us consider the polynomials for the vectorspace $V(F)$ to compute the matrix of D in basis $B = [1, t, t^2, t^3]$

Now apply the fn.

$D[1] = 0 = 0 + 0t + 0t^2 + 0t^3$
$D(t) = 1 = 1 + 0t + 0t^2 + 0t^3$
$D[t^2] = 2t = 0 + 2t + 0t^2 + 0t^3$
$D[t^3] = 3t^2 = 0 + 0t + 3t^2 + 0t^3$

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*Impor

Theorem:

Let U be an m dim v.space over the field F and V an n dim v.space Let B be an ordered basis for U and B' an ordered basis for V. Let T be any lin. tr from $U \to V$. Then for any vector $\alpha \in U$ $[T]_B [\alpha]_B = [T(\alpha)]_{B'}$

Proof:- Let $B = \{\alpha_1, \alpha_2 \cdots \alpha_m\}$ be an ordered basis for U and $B' = \{\beta_1, \beta_2 \cdots \beta_n\}$ be the ordered basis for V. T is the lin. tr from $U \to V$, Then T is determined by its action on α_i, $1 \le i \le m$.

Each of m vectors $T(\alpha_i)$ is uniquely expressable as a lin. comb of elements of B'

$$T(\alpha_i) = \sum_{j=1}^{n} a_{ij} \beta_j$$ where $a_{i1}, a_{i2} \cdots a_{in}$ are the co-ordinates.

If α be any vector in U, $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$$[\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix}$$

Now $T(\alpha) = T[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n]$

$= a_1[T(\alpha_1)] + a_2[T(\alpha_2)] + \cdots + a_n[T(\alpha_n)]$

$= a_1 a_{11}\beta_1 + a_1 a_{12}\beta_2 + \cdots + a_1 a_{1n}\beta_n + a_2 a_{21}\beta_1 + a_2 a_{22}\beta_2 +$

$\cdots + a_m a_{m1}\beta_1 + a_m a_{m2}\beta_2 + \cdots + a_m a_{mn}\beta_n$

$$[T(\alpha)]_B = \begin{bmatrix} a_1 a_{11} & a_2 a_{21} & \cdots & a_m a_{m1} \\ a_1 a_{12} & a_2 a_{22} & \cdots & a_m a_{m2} \\ \vdots & \vdots & & \vdots \\ a_1 a_{1n} & a_2 a_{2n} & \cdots & a_m a_{mn} \end{bmatrix}_{n\times m} \rightarrow \text{Transpose}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n\times m} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m\times 1}$$

$$[T(\alpha)]_B = [T]_B [\alpha]_B$$

Change of Basis :-

In this section we will see how the representation of a matrix of lin. tr changes if we take another basis.

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V and let $B' = \{\beta_1, \beta_2, \ldots \beta_n\}$ be another basis of V and suppose

$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n$

$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n$

$\vdots$

$\beta_n = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n$

Then the transpose of the co-efficient matrix of the above eq^n is called transition matrix from basis B & B' and it is written as

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

The matrix P is invertible and its P^{-1} is the transition matrix from new basis to old basis.

Ex: Let $\{(1,0),(0,1)\}$ and $\{(1,1),(-1,0)\}$ be two bases of R^2 then
$(1,1) = 1(1,0) + 1(0,1)$ and $(-1,0) = -1(1,0) + 0(0,1)$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Imp

Theorem 1:- Let P be the transition matrix from basis B to basis B' in a V.space V. Then for any vector $\alpha \in V$ $P[\alpha]_{B'} = [\alpha]_B$ and
$[\alpha]_{B'} = P^{-1}[\alpha]_B$.

→ Let V be an n dim V-space and let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \dots \beta_n\}$ be two basis of V and let P be the transition mat from B to B', then we have

$$P = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

Now suppose $\alpha \in V$ such that $\alpha = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n$. Substituting for βs from lin fr eq^n we get $\alpha = b_1a_{11}\alpha_1 + b_1a_{12}\alpha_2 + \dots + b_1a_{1n}\alpha_n + b_2a_{21}\alpha_1 + b_2a_{22}\alpha_2 + \dots + b_2a_{2n}\alpha_n + \dots + b_n a_{n1}\alpha_1 + b_n a_{n2}\alpha_2 + \dots + b_n a_{nn}\alpha_n$

$$[\alpha]_{B'} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ and } [\alpha]_B = \begin{bmatrix} b_1a_{11} + b_1a_{12} + \dots + b_1a_{1n} \\ b_2a_{21} + b_2a_{22} + \dots b_2a_{2n} \\ \vdots \\ b_na_{n1} + b_na_{n2} + \dots + b_na_{nn} \end{bmatrix}$$

$$P[\alpha]_{B'} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Since $P[\alpha]_{B'} = [\alpha]_B \Rightarrow P^{-1}P[\alpha]_{B'} = P^{-1}[\alpha]_B$

$$I[\alpha]_{B'} = P^{-1}[\alpha]_B$$
$$[\alpha]_{B'} = P^{-1}[\alpha]_B$$

Trace of a matrix:

Let there be a square matrix of order n over F then $Tr(A)$ is the sum of the elements of A lying along the principle diagonal of the matrix.

Let $A = [a_{ij}]_{n \times n}$ $tr(A) = a_{11} + a_{22} + \dots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

Theorem:

Let A and B be two square matrices of order n over a field F and $\lambda \in F$. Then

(i) $tr(\lambda A) = \lambda\, tr(A)$

(ii) $tr(A+B) = tr(A) + tr(B)$

(iii) $tr(AB) = tr(BA)$

Proof: Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and $\lambda \in F$ then

$$\lambda A = [\lambda a_{ij}]_{n \times n}$$

(i) $tr(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii}$

$= \lambda \sum_{i=1}^{n} a_{ii}$

$= \lambda\, tr(A)$

(ii) $A+B = [a_{ij} + b_{ij}]_{n \times n}$

$tr(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii})$

$tr(A+B) = tr(A) + tr(B)$

(iii) $AB = [c_{ij}]_{n \times n}$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

and

$BA = [d_{ij}]_{n \times n}$

$d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$

$tr(AB) = \sum_{i=1}^{n} c_{ii}$

$tr(AB) = \sum_{i=1}^{n} \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]$

$= \sum_{k=1}^{n} \left[\sum_{i=1}^{n} a_{ik} b_{ki} \right]$

$\sum_{k=1}^{n} \left[\sum_{i=1}^{n} b_{ki} a_{ik} \right]$

$= \sum_{i=1}^{n} d_{kk}$

$tr(AB) = tr(BA)$

Theorem 2: Similar matrices have the same trace

Proof: Let A and B be two similar matrices, then there exists an invertible matrix C such that $B = C^{-1}AC$

$$\text{Let } D = C^{-1}A$$
$$B = DC$$
$$tr(B) = tr(DC)$$
$$tr(B) = tr(CD)$$
$$tr(B) = tr(CC^{-1}A)$$
$$tr(B) = tr(IA)$$
$$tr(B) = tr(A).$$

Problems:-

1. Let $T: R^2(R) \rightarrow R^2(R)$, where for any $(x, y) \in R^2$, $T(x, y) = (2x, \frac{1}{2}y)$.
Find the matrix associated with T w.r.t the ordered basis $\{(1,0)(0,1)$

→ Let $B = \{(1,0), (0,1)\}$ be an ordered basis of $R^2(R)$ and $T(x, y) = (2x, \frac{1}{2}y)$
$(a,b) = x(1,0) + y(0,1)$
$(a,b) = (x+y)$

Let us apply the tr. fn. $T(1,0) = (2,0) = 2(1,0) + 0(0,1)$
$T(0,1) = (0, 1/2)$ $0(1,0) + \frac{1}{2}(0,1)$

$$\therefore [T]_B = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

2 Find the matrix of the lin. tr T on $V_3(R)$ defined as $(a,b,c) = (2b+c, a-4b, 3a)$
w.r.t to the ordered basis B & also w.r.t ordered basis
B' where (i) $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ (ii) $B' = \{(1,1,1), (1,1,0), (1,0,0)\}$

→ We have (i) $T(a,b,c) = (2b+c, a-4b, 3a)$
then $T(1,0,0) = (0, 1, 3) = 0(1,0,0) + 1(0,1,0) + 3(0,0,1)$
$T(0,1,0) = (2, -4, 0) = 2(1,0,0) + (-4)(0,1,0) + 0(0,0,1)$
$T(0,0,1) = (1, 0, 0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$$[T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Let (a,b,c) be any element of $V_3(\mathbb{R})$, then there exists $\{x,y,z\} \in \mathbb{R}$ such that $(a,b,c) = x(1,1,1) + y(1,1,0) + z(1,0,0)$

$$(a,b,c) = (x+y+z,\ x+y,\ x)$$

$c = x$

$b = x+y$

$a = x+y+z$

$\Rightarrow x = c \quad y = b-x \quad z = a-b \longrightarrow ①$

$T(a,b,c) = (2b+c,\ a-4b,\ 3a)$

$T(1,1,1) = (3,-3,3)$

$T(1,1,0) = (2,-3,3)$

$T(1,0,0) = (0,1,3)$

$(a,b,c) = c(1,1,1) + (b-x)(1,1,0) + (a-b)(1,0,0)$

$T(1,1,1) = (3,-3,3) = 3(1,1,1) - 6(1,1,0) + 6(1,0,0)$

$T(1,1,0) = (2,-3,3) = 3(1,1,1) - 6(1,1,0) + 5(1,0,0)$

$T(1,0,0) = (0,1,3) = 3(1,1,1) - 2(1,1,0) - 1(1,0,0)$

$$[T]_{B'} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

3. Let T be a linear operator on $\mathbb{R}^2$ defined by $T(x,y) = (2y,\ 3x-y)$. Find the matrix representation of T relative to the basis $\{(1,3),(2,5)\}$

$\rightarrow$ Let (x,y) be an element of $\mathbb{R}^2$ then there exists $a,b \in \mathbb{R}$ such that

$(x,y) = a(1,3) + b(2,5)$

$x = a+2b \quad \times 3 \rightarrow \quad 3a+6b = 3x$

$y = 3a+5b \quad\quad 3a+5b = y$

$b = 3x-y$

$a = x-6x+2y$

$a = -5x+2y$

$T(x,y) = (2y, 3x-y)$

$T(1,3) = (6, 0)$

$T(2,5) = (10, 1)$

$T(1,3) = -30(1,3) + 18(2,5)$

$T(2,5) = -48(1,3) + 29(2,5)$

$$\begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$$

31/10/23

4. ST the vector $\alpha_1 = (1,0,-1)$ $\alpha_2 = (1,2,1)$ & $\alpha_3 = (0,-3,2)$ form a basis for R^3. Express each of the std basis vectors as a lin. comb of $\alpha_1, \alpha_2, \alpha_3$.

→ Let $a, b, c \in R$ such that $a v_1 + b\alpha_2 + c\alpha_3 = 0$

$a(1,0,-1) + b(1,2,1) + c(0,-3,2) = 0$.

$a+b = 0$

$2b - 3c = 0$

$-a + b + 2c = 0$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} = 1(7) - 1(-3) = 7+3 = 10$$

$\rho(A) = 3$

a, b, c will have zero sol^n

The given values are lin. independent & hence it forms a basis for for given V. space

Let $B = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$ be the std basis for R^3.

$\alpha_1 = (1,0,-1) = 1e_1 + 0e_2 - 1e_3$

$\alpha_2 = (1,2,1) = 1e_1 + 2e_2 + 1e_3$

$\alpha_3 = (0,-3,2) = 0e_1 - 3e_2 + 2e_3$

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}$$

Let P be the tr. matrix from the basis B' to B'

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \Rightarrow |P| = 10$$

$P^{-1} = \frac{1}{|A|} \text{adj}(P)$

Co-factors of elements of 1st row of P are

$A_{11} = 7$ $\quad A_{21} = -2$ $\quad A_{31} = -3$

$A_{12} = 3$ $\quad A_{22} = 2$ $\quad A_{32} = 3$

$A_{13} = 2$ $\quad A_{23} = -2$ $\quad A_{33} = 2$

$\therefore \operatorname{adj}(P) = \begin{bmatrix} 7 & -2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & 2 \end{bmatrix}$

$\Rightarrow P^{-1} = \frac{1}{10}\begin{bmatrix} 7 & -2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & 2 \end{bmatrix}$

Now $e_1 = 1e_1 + 0e_2 + 0e_3$ $\therefore$ Co-ordinate matrix of e_1 related to $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Obtain the co-ordinate matrix of e_1 related to the change matrix B'

$$[e_1]_{B'} = P^{-1}[e_1]_B = \frac{1}{10}\begin{bmatrix} 7 & -2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/10 \\ 3/10 \\ 2/10 \end{bmatrix}$$

$\therefore e_1 = \frac{7}{10}e_1 + \frac{3}{10}\alpha_2 + \frac{2}{10}\alpha_3$

$[e_2]_{B'} = P^{-1}[e_2]_B = \begin{bmatrix} \frac{-2}{10} \\ 2/10 \\ -2/10 \end{bmatrix}$ $\quad e_2 = -\frac{2}{10}\alpha_1 + \frac{2}{10}\alpha_2 - \frac{2}{10}\alpha_3$

$[e_3]_{B'} = P^{-1}[e_3]_B = \begin{bmatrix} \frac{-3}{10} \\ 3/10 \\ 2/10 \end{bmatrix}$ $\quad e_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{2}{10}\alpha_3$

5. Consider the following lin. op G on R^2 & a basis S. $G(x,y) = (2x-7y, 4x+3y)$ $S = \{u_1, u_2\} = \{(1,3),(2,5)\}$

a. Find the mat. representation $G(S)$ of $[G]_S$

b. verify $[G]_S[V]_S = [G(V)]_S$ for the vector $V = (4,-3)$ in R^3.

$\rightarrow$ $G(u_1) = (2(1) - 7(3),\ 4(1) + 3(3))$
$= (-19, 13)$

$G(u_2) = (-31, 23)$

Now find co-ordinates of arbitrary vector $V = (a,b)$ in R^2 related to basis S.

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$x+2y=a \quad \times 3 \Rightarrow \quad 3x+6y=3a \qquad x+6a-2b=a.$

$3x+5y=b \qquad\qquad\quad 3x+5y=b \qquad\qquad \underline{x=-5a+2b.}$

$\overline{y=3a-b}$

$\therefore (a,b) = (-5a+2b)u_1 + (3a-b)u_2$

$V = [-5a+2b,\ 3a-b] \rightarrow$ sub $G(u_1)$ & $G(u_2)$

$[G]_S = \begin{matrix} 121u_1 & -70u_2 \\ 201u_1 & -116u_2 \end{matrix} \Rightarrow \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix} \rightarrow [G]_S.$

use the formula $(a,b) = (-5a+2b)u_1 + (3a-b)u_2$ to get $V=(4,-3) = -26u_1 + 15u_2$

$[V]_S = \begin{bmatrix} -26 \\ 15 \end{bmatrix}$

$[G]_S[V]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix}\begin{bmatrix} -26 \\ 15 \end{bmatrix} = \begin{bmatrix} -131 \\ 80 \end{bmatrix}$

$G(v) = G(4,-3) = \underbrace{(29,7)}_{\text{sub in the fn.}} \longleftrightarrow$ sub in $V = [-5a+2b,\ 3a-b]$

$V = [-131 \quad 80]$

6. Consider the following basis of R^2. $E = \{e_1, e_2\} = \{(1,0),(0,1)\}$ and $S = \{u_1, u_2\} = \{(1,3),(1,4)\}$

a. Find change of basis matrix P from usual basis $[E]_S$

b. Find the change of basis matrix Q from S back to E

c. Find the co-ordinate vector $[v]$ of $v = (5,-3)$ related to S

→ a.

$$A = \begin{matrix} 1 & 1 \\ 3 & 4 \end{matrix} \qquad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

b. $(a,b) = x(1,3) + y(1,4)$

$x + y = a$ $\Rightarrow$ $x = 4a - b$

$3x + 4y = b$ $\quad y = -3a + b.$ $\quad$ or $Q = P^{-1}$

$V = (a,b) = (4a-b)(1,3) + (-3a+b)(1,4)$

$V = (4a-b)u_1 + (-3a+b)u_2$

P_1^{-1}

$$Q = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

Sub $(5,-3)$

c. $[V]_S = \begin{bmatrix} 23 \\ -18 \end{bmatrix}$

Unit-3

Part-A

Diagonalization:

Suppose an N^2 matrix A is given, the mat A is said to be diagonalizable, if there exists a non singular matrix P such that $B = P^{-1}AP$ where B is diagonal

Suppose a lin operator T which maps $T: V \to V$ is given. The lin operator T is said to be diagonalizable if there exists a basis of V such that the mat representation of T relative to the basis S is a diagonal matrix.

we concentrate mainly on conditions under which the lin. operator T is diagonalizable

Polynomials of matrices:-

Consider a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$ over a field K

Consider a sq mat A & we define $f(A)$ as follows.

$f(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$

where I is the identity matrix

In particular we say that mat A is a root of $f(t)$ if $f(A) = 0$ $\therefore$ zero matrix

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$. Let $f(t) = 2t^2 - 3t + 5$ & $g(t) = t^2 - 5t - 2$

Find $f(A)$ & $g(A)$

→ $f(t) = 2t^2 - 3t + 5$

$f(A) = 2A^2 - 3A + 5$

$= 2\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 16 & 14 \\ 21 & 37 \end{bmatrix}$

$g(t) = t^2 - 5t - 2$.

$g(A) = A^2 - 5A - 2$

$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ∴ $g(A)$ is a root of $g(t)$.

2. Let $A = \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}$ find $f(A)$ where a. $f(t) = t^2 - 3t + 7$

b. $f(t) = t^2 - 6t + 13$

→ $A^2 = A \cdot A$

$= \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1-8 & 24 \\ -2-10 & -8+25 \end{bmatrix} = \begin{bmatrix} -7 & 24 \\ -12 & 17 \end{bmatrix}$

a. $f(t) = t^2 - 3t + 7$.

$f(A) = \begin{bmatrix} -7 & 24 \\ -12 & 17 \end{bmatrix} - \begin{bmatrix} 3 & 12 \\ -6 & 15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ \cancel{-18}\ -6 & 9 \end{bmatrix}$

b. $f(t) = t^2 - 6t + 13$

$f(A) = \begin{bmatrix} -7 & 24 \\ -12 & 17 \end{bmatrix} - \begin{bmatrix} 6 & 24 \\ -12 & 30 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$f(A)$ forms a root of $f(t)$

8 **Note:-** Let f and g be polynomials. For any sq. mat A & scalar k -

(i) $(f+g)A = f(A) + g(A)$ (ii) $(fg)A = f(A) \cdot g(A)$. (iii) $(kf)A = kf(A)$

(iv) $f(A) \cdot g(A) = g(A) \cdot f(A)$

Characteristic polynomial and Cayley-Hamilton theorem.

Let $A = [A_{ij}]$ be a sq-matrix. The mat $M = A - tI_n$, where I_n is the n^2 identity matrix and T is an indeterminant. This maybe obtained by substracting t down the diag of A. The -ve of M is the matrix $-M = tI_n - A$ & its determinant is given by

$$\Delta t = \det(tI_n - A)$$
$$= (-1)^n \det(A - tI_n)$$

which is a polynomial in t of degree n is called the characteristic polynomial of t.

$$\Delta(t) = \det(tI_n - A)$$
$$= (t - a_{11})(t - a_{22}) \cdots (t - a_{nn})$$

Examples

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ its characteristic eqn is given by $\Delta(t) = |tI - A|$

$$\Delta(t) = |tI - A|$$

$$= \begin{vmatrix} t-1 & 3 \\ 4 & t-5 \end{vmatrix} = \begin{aligned} &(t-1)(t-5) - 12 \\ &t^2 - 5t - t + 5 - 12 \end{aligned}$$

$$\Delta(t) = \underline{t^2 - 6t - 7}$$

Cayley-Hamilton theorem

Every mat A is a root of its characteristic polynomial

Proof: Suppose $A = [a_{ij}]$ is a triangular matrix. Then $tI - A$ is a triangular mat with diagonal entries $t - a_{ii}$ and hence

$$\Delta(t) = \det(tI_n - A)$$
$$= (t - a_{11})(t - a_{22}) \cdots (t - a_{nn})$$

Observe that the roots of $\Delta(t)$ are diagonal elements of A.

Example: Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ & its characteristic eqn is $\Delta(t) = t^2 - 6t - 7$. Now according to CH theorem A is the root of $\Delta(t)$

$\Delta(A) = A^2 - 6A - 7I$.

$$A^2 = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+12 & 3+15 \\ 4+20 & 12+25 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 24 & 37 \end{bmatrix}$$

$$\Delta(A) = \begin{bmatrix} 13 & 18 \\ 24 & 37 \end{bmatrix} - \begin{bmatrix} 6 & 18 \\ 24 & 30 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, suppose A & B are similar matrices $B = P^{-1}AP$ where P is invertible. Then both matrix A & B have same characteristic polynomial

Proof: Let $tI = P^{-1}tIP$ we have $\Delta_B(t) = \det(tI - B)$

$$= \det(P^{-1}tIP - B)$$
$$= \det(P^{-1}tIP - P^{-1}AP)$$
$$= \det[P^{-1}(tI - A)P]$$
$$= \det(P^{-1}) \cdot \det(tI - A) \cdot \det(P)$$

Since $\det(P^{-1}) \cdot \det(P) = I$.

$\Rightarrow \Delta_B(t) = \det(tI - A)$

$\Delta_B(t) = \Delta_A(t)$

Characteristic polynomial of degree 2 & 3:

a. Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then characteristic polynomial $\Delta(t) = t^2 - (a_{11} + a_{22})t + \det(A)$.

$\Delta(t) = t^2 - tr(A)t + \det(A)$

b. Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then characteristic polynomial

$\Delta(t) = t^3 - tr(A)t^2 + (A_{11} + A_{22} + A_{33})t - \det(A)$

$A_{11} + A_{22} + A_{33}$ are co-factors of the given matrix

1. Find the characteristic polynomial for each of the polynomial

a. $A = \begin{bmatrix} 5 & -3 \\ 2 & 10 \end{bmatrix} \rightarrow \Delta(t) = t^2 - 15t + 44$

b. $B = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}$ $\Delta(t) = t^2 - 9t + 20$

c $C = \begin{bmatrix} 5 & -2 \\ 4 & -4 \end{bmatrix}$ $\Delta(t) = t^2 - t - 12$

2. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$

→ $A_{11} = 27 - 6 = 21$

$A_{22} = 9 - 2 = 7$

$A_{33} = 3$

$|A| = 1(21) - 1(-2) + 2(-3)$

$= 17$

$\Delta(t) = t^3 - 13t^2 + 31t - 17$

Diagonalization Eigen values and Eigen vectors

Let A be any square matrix, then A can be represented by a diagonal matrix $D = \text{diag}(k_1, k_2 \dots k_n)$ iff there exists a basis S consisting of column vectors such that $U_1, U_2 \dots U_n$ such that

$AU_1 = k_1 U_1$

$AU_2 = k_2 U_2$

$\vdots$

$AU_n = k_n U_n$

In such a case A is said to be diagonalizable. Furthermore $D = P^{-1}AP$ where P is the non singular matrix whose columns are respectively the basis vectors. $U_1, U_2 \dots U_n$

Definitions

Let A be any sq. mat. A scalar λ is called an eigen value of A if there exists a non-zero column vector v such that $AV = \lambda V$. Any vector satisfying this relation is called an eigen vector of A belonging to the eigen value λ.

Example: Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ and let $V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Find diag mat D & obtain mat A from diag mat D

→ Let us find the eigen ~~values~~ vectors.

$$AV_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = V_1 \qquad \lambda_1 = 1$$

$$AV_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = V_2. \qquad \lambda_2 = 4$$

There fore V_1 & V_2 are eigen vectors A belonging respectively to the eigen values $\lambda_1 = 1$ & $\lambda_2 = 4$. Observe that V_1 & V_2 are linearly independent & hence form a basis of R^2. Accordingly A is diagonalizable by using transition matrix D. $D = P^{-1}AP$

$$P = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ \quad V_1 \quad V_2 \end{array} \Rightarrow P^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\frac{1}{|P|} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

As expected the diagonal elements 1 & 4 are eigen values respectively with eigen vectors V_1 & V_2 which are the column of P

To factorize A from D:

$$A = PDP^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A^m = PD^mP^{-1} \Rightarrow D^4 = \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix}$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 171 & 85 \\ 170 & 86 \end{bmatrix}$$

Suppose $f(t) = t^3 - 5t^2 + 3t + 6$

$f(1) = 1 - 5 + 3 + 6 = 5$

$f(4) = 64 - 80 + 12 + 6 = 2$

$$f(A) = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ -10 & 2 \end{bmatrix}\begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$$

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Computing eigen values & eigen vectors diagonalizing matrices:

Step 1: Find the char polynomial $\Delta(t)$ of A

Step 2: Find the roots of $\Delta(t)$ to obtain the eigen values of A

Step 3: Repeat step 1 & step 2 for each eigen value $\lambda(A)$. From the matrix $M = A - \lambda I$ by subtracting λ down the diagonal of A

Step 4: Find a basis for the sol^n space of the homogeneous system $MX=0$ (we obtain eigen vectors from this.

Step 5: Consider the collection $S=\{V_1, V_2 \dots V_n\}$ of all eigen vectors obtained in step 3. If $M \neq n$, then A is not diagonalizable. If $M=n$ then A is, diagonalizable using $D=P^{-1}AP$.

1. Find the diagonal elements for the given matrix $A=\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$

$\rightarrow$ $\Delta(t) = t^2 - (3)t + -10$

$= t^2 - 3t - 10$

$= (t+2)(t-5)$

$t^2 + 5t - 10$
$-2t - 10.$

$\lambda_1 = 5 \quad \lambda_2 = -2$

$M = A - \lambda I.$

$\rightarrow \lambda_1 = 5$

$= \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$\lambda_2 = -2$

$M = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$MX = 0$

$\left.\begin{array}{l} -x+2y=0 \\ 3x-6y=0 \end{array}\right\} \Rightarrow -x+2y=0$

$y \rightarrow$ free variable

$x \rightarrow$ pivot element

$-x+2y=0$

$y=1 \Rightarrow x=2$

$V_1 = (2,1)$

$MX = 0$

$\left.\begin{array}{l} 6x+2y=0 \\ 3x+y=0 \end{array}\right\} 3x+y=0.$

$y=3 \rightarrow x=-1$

$V_2 = (-1,3)$

Let P be the mat whose cols are V_1 & V_2.

$\Rightarrow P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$D = P^{-1}AP$

$= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

2 Find D for given mat $B = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$

→ Char eqⁿ = $\Delta(t) = t^2 - 8t + 16.$ $\qquad \frac{8 \pm \sqrt{64-64}}{2} = 4.$

$\lambda_1 = \lambda_2 = 4. \quad = (t-4)^2$

$M = A - \lambda I$

$= \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$MX = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \left.\begin{matrix} x - y = 0 \\ x - y = 0 \end{matrix}\right\} x - y = 0.$

$y = 1 \Rightarrow x = 1 \quad V_1 = (1, 1)$

Ex1

The system has only one independent solⁿ i.e. $x = 1, y = 1$. Thus $V = (1,1)$ & its multiples are the only eigen vectors of V. Accordingly B is not diagonalizable since there does not exist a bases consisting of eigen vectors of B.

3. Find D for $A = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$ $\qquad -9+10$

→ $\Delta(t) = t^2 - 0 + 1 = t^2 + 1$

we consider 2 cases.

1 A is matrix over real field R. $\Delta(t)$ has no real roots Thus A has no eigen values and no eigen vectors. A is not diagonalizable

2 A is a mat over the complex field C, $\Delta(t) = (t-i)(t+i)$ has two roots i.e $-i$ which forms the eigen values and A has two independent eigen vectors. ∴ A is diagonalizable over field C

4. Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ is a real symmetric mat. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal

$\frac{7 \pm \sqrt{49-24}}{2}$

→ $\Delta(t) = t^2 - 7t + 6.$

$(t-6)(t-1)$

$$M = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$$

$V_1 \neq MX = \left.\begin{matrix} -4x-2y=0 \\ -2x-y=0 \end{matrix}\right\} -2x-y=0 \Rightarrow y=2 \Rightarrow x=-1$

$V_1 = (-1, 2)$

$$M = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \Rightarrow \left.\begin{matrix} x-2y=0 \\ -2x+4y=0 \end{matrix}\right\} x-2y=0 \quad y=1 \quad x=2$$

$V_2 = (2, 1)$.

Since V_1 & V_2 are orthogonal, normalizing V_1 & V_2 yields orthonormal vectors.

$$\hat{V_1} = \frac{V_1}{\sqrt{V_1^2}} = \left(\frac{-1}{\sqrt5}, \frac{2}{\sqrt5}\right) \qquad \hat{V_2} = \frac{V_2}{\sqrt{V_2^2}} = \left(\frac{2}{\sqrt5}, \frac{1}{\sqrt5}\right)$$

$$P = \begin{bmatrix} \frac{-1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & 1/\sqrt5 \end{bmatrix} \rightarrow P^{-1} = \begin{bmatrix} -1/\sqrt5 & +2/\sqrt5 \\ +\frac{2}{\sqrt5} & +1/\sqrt5 \end{bmatrix} \qquad -\frac{1}{5} - \frac{4}{5} = -\frac{5}{5} = -1$$

$D = P^{-1}AP$

$$= \begin{bmatrix} \frac{-1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

6. Consider q. eq^n $q(x, y) = 2x^2 - 4xy + 5y^2$. Diagonalize the given eq^n

As per earlier problem, we obtain $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

8/4/23

1. Find the char. polynomial of each,

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} +3 & -2 \\ 9 & -3 \end{bmatrix}$$

→ a. $\Delta(t) = t^2 - 3t - 18$

b. $\Delta(t) = t^2 - 5t + 1$

c. $\Delta(t) = t^2 + 9$,

2 Find the char. polynomial $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 6 & 4 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 3 & -4 \end{bmatrix}$

→ a. $t^3 - 6t^2 + (-16-13-6)t - 38$

$t^3 - 6t^2 - 35t - 38$

b. $t^3 - t^2 + (-8-4+20)t + 62$

$t^3 - t^2 + 8t + 62$

$+(-8) - 6(12) - 2(-9)$

$-8 - 72 + 18$

3. Find char. polynomial

$$A = \begin{bmatrix} 2 & 5 & 1 & 1 \\ 1 & 4 & 2 & 2 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$\Delta_1(t) = t^2 - 6t + 3$

$\Delta_2(t) = t^2 - 9t + 28$

$\Rightarrow \Delta(t) = \Delta_1(t).\Delta_2(t)$

$= (t^2-6t+3)(t^2-9t+28)$

$= t^4 - 9t^3 + 28t^2 - 6t^3 + 54t^2 - 168t + 3t^2 - 27t + 84$

$= \underline{t^4 - 15t^3 + 85t^2 - 195t + 84}$

$$B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$\Delta_1(t)$, $\Delta_2(t)$

$\Delta_1(t) = t^2 - 4t + 3$

$\Delta_2(t) = t^2 - 11t + 30$

$\Delta(t) = (t^2-4t+3)(t^2-11t+30)$

$= t^4 - 11t^3 + 30t^2 - 4t^3 + 44t^2 - 120t + 3t^2 - 33t + 90$

$= \underline{t^4 - 15t^3 + 77t^2 - 153t + 90}$

4. Find char poly. of each of the following lin. operators,

a. $F: R^2 \to R^2$ defined by $F(x,y) = (3x+5y, 2x-7y)$

b. $D: V \to V$ defined by $D(f) = df/dt$, $S = \{\sin t, \cos t\}$

→ a) $A = \begin{bmatrix} 3 & 5 \\ 2 & -7 \end{bmatrix}$ → $\Delta(t) = t^2 + 4t - 31$

b. The matrix A representing the differential operator relating to the basis S we get

$D(\sin t) = \cos t$ → $0 \sin t + 1 \cos t$

$D(\cos t) = -\sin t$ → $-1 \sin t + 0 \cos t$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Delta(t) = t^2 + 1$

5. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ a. Find all eigen values and corresponding eigen vectors b. Find a non-singular mat P such that $D = P^{-1}AP$ is diagonal c. Find A^6 and $f(A)$ where $t^4 - 3t^3 - 6t^2 + 7t + 8$ d. Find a real cube root of A i.e., a mat B such that $B^3 = A$ & B has real eigen values.

$\Delta(t) = t^2 - 5t + 4$.

$\frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2} = 4, 1$

$\lambda_1 = 4 \quad \lambda_2 = 1$.

$M = A - \lambda I$

$= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$

$MX = 0$

$\left.\begin{array}{l} -2x + 2y = 0 \\ x - y = 0 \end{array}\right\} x - y = 0$

$y = 1 \quad x = 1$

$V_1 = (1, 1)$

$M = A - \lambda I$

$= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$MX = 0$

$\left.\begin{array}{l} x + 2y = 0 \\ x + 2y = 0 \end{array}\right\} x + 2y = 0.$

$y = 1 \quad x = -2$

$V_2 = (-2, 1)$.

$P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{|P|} \cdot \text{adj } P \Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \Rightarrow P^{-1} \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix}$

$D = P^{-1}AP$

$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow D^6 = \begin{bmatrix} 4096 & 0 \\ 0 & 1 \end{bmatrix}$

c.

$A^6 = PD^6P^{-1}$

$= \begin{bmatrix} 1366 & 2730 \\ 1365 & 2731 \end{bmatrix}$

$f(\lambda_1) = 4^4 - 3(4)^3 - 6(4)^2 + 7(4) + 3 = -1$

$f(\lambda_2) = 1^4 - 3(1)^3 - 6(1)^2 + 7 + 3 = 2$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

d. The real cube root of A can be found by finding cube root of D which is equal to

$$D = \begin{bmatrix} \sqrt[3]{4} & 0 \\ 0 & \sqrt[3]{1} \end{bmatrix}$$

Therefore cube root of A is nothing but mat B is given by

$B = P\sqrt[3]{D}P^{-1}$

$$B = \begin{bmatrix} 1.195 & 0.3916 \\ 0.1958 & 1.3916 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

a. Find all e.vals & e.vecs.

b. is A diagonalizable. if yes then find P such that $D = P^{-1}AP$ is diag.

→ $\Delta(t) = t^3 - 11t^2 + (\underbrace{12+9+18}_{39})t - 45$

$4(12) - 1(6) - 1(-3) = 48 - 6 + 3 = 45$

3	1	-11	39	-45
	1	6	-15	72
	2	-5	24	

$\lambda_1 = 5 \quad \lambda_2 = 3 \quad \lambda_3 = 3.$

$M = A - \lambda I$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}$$

$MX = 0$

$-x + y - z = 0$
$2x - 2z = 0$
$x + y - 3z = 0$

Assume free variable as z

$2y - 4z = 0$
$y = 2z$
$x - z = 0$
$x = z$

$z = 1 \Rightarrow y = 2 \quad x = 1$

$\rightarrow v_1 = (1, 2, 1)$

$M = A - \lambda I$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{matrix} x+y-z=0 \\ 2x+2y-2z=0 \\ x+y-z=0 \end{matrix}$$

$V_2 = (\lambda, \lambda, \lambda)$

y & z are free variable

$\Rightarrow y=-1 \quad z=0.$ $\quad y=0 \quad z=1$

$V_2 = (1,-1,0) \quad V_3 = (1,0,1)$

The basis for $S = V_1, V_2, V_3$ from which we obtain P.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$t^3 - 11t^2 + 39t - 45$

$A^3 - 11A^2 + 39A - 45I$

$\Rightarrow A^3 - 11A^2 + 39AI - 45AA^{-1}$

$A(A^2 - 11A + 39I - 45A^{-1}) = 0.$

$A^{-1} = \frac{1}{45}(A^2 - 11A + 39I)$

$\Rightarrow$ ~~$A_{11} = $~~

~~$A_{12} =$~~

$$P^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 1 & 0 & -1 \\ -0.5 & -0.5 & 1.5 \end{bmatrix}$$

$D = P^{-1}AP$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

7 Let mat $B = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

$3(-10+6) + 1(14-6) + 1(-42+30)$

$-12 + 8 - 12$

$\rightarrow \Delta(t) = t^3 - 0t^2 + \underbrace{(-4+0+-8)}_{-12} t + 16$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -12 & 16 \\ & 0 & 2 & 4 & -16 \\ \hline & 1 & 2 & -8 & 0 \end{array}$$

$\lambda_1 = -4 \quad \lambda_2 = 2 \quad \lambda_3 = \cancel{-8}\ 2$

$\lambda_i = 1, 2, -4$

$M = A - \lambda I$

$$= \begin{bmatrix} \cancel{2} & \cancel{-1} & \cancel{1} \\ \cancel{1} & \cancel{-6} & \cancel{1} \\ \cancel{6} & \cancel{-6} & \cancel{1} \end{bmatrix}$$

~~$2x - y + z = 0$~~
~~$7x - 6y + z = 0$~~
~~$6x - 6y + z = 0$~~

~~$2x - y + z = 0$~~
~~$7x$~~

$$M = \begin{bmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{bmatrix}$$

~~$V_1 = (1, 1, 0)$~~

$7x - y + z = 0$
$7x - y + z = 0$
$6x - 6y + 6z = 0$
$z = 1$

$V_2 = (0, 1, 1)$

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix}$$

$x - y + z = 0$
$-x - 7y + z = 0$
$6x - 6y = 0 \Rightarrow x = y$
$z = 0$
$V_1 = (1, 1, 0)$

~~$M =$~~

$V_1 = (1, 1, 0)$ $V_2 = (0, 1, 1)$ ~~V_3~~

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Three basis

The basis for $S = V_1, V_2$ which has only 2 independent e. vectors. But we require 3 basis vectors to match the dim of D.

$\therefore$ mat B is not diagonalizable

Let mat $B = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$

$-44 - 16$

$-11 - 64.$

$\Rightarrow \Delta(t) = t^3 - 6t^2 + \underbrace{(0 + -60 + -75)}_{-135} t + -400$

$11(4-4) + 8(32+8) + 4(16+4)$

$320 + 80 \Rightarrow 400$

-5	1	-6	-135	-400
	0	-5	55	$+400$
	1	-11	-80	0

$t^2 - 11t - 80$

$$\frac{11 \pm \sqrt{121 - 120}}{2}$$

$$\frac{11 \pm 1}{2} = 6.5$$

$\lambda_1 = -5$ $\lambda_2 = 16$

$$M = \begin{bmatrix} 16 & -8 & 4 \\ -8 & 4 & -2 \\ 4 & -2 & 1 \end{bmatrix}$$

$16x - 8y + 4z = 0$

$-8x + 4y - 2z = 0$

$4x - 2y + z = 0$

$4x - 2y + z = 0$

$4x - 2y + z = 0.$

$$M = \begin{bmatrix} -5 & -8 & 4 \\ -8 & -17 & +2 \\ 4 & -2 & -20 \end{bmatrix} \rightarrow$$

$-5x - 8y + 4z = 0$

$-8x - 17y - 2z = 0$

$4x - 2y - 20z = 0$

$$D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Q. Find the char. eqn of the mat $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ verify Caley Ham theorem for the given mat & hence find its inverse

$\rightarrow$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)[(2-\lambda)(3-\lambda)] + 2[-4+2\lambda]$$

$(1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2) - 8 + 4\lambda$

$= 6 - 5\lambda + \lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 - 8 + 4\lambda$

$0 = -\lambda^3 + 6\lambda^2 - 7\lambda - 2$

$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0.$

$\Delta(t) = t^3 - 6\lambda^2 + (6 - 1 + 2) t + 2$

$= t^3 - 6t^2 + 7t + 2$

$2 \,|\, 1 \quad -6 \quad 7 \quad 2$

To verify Caley Hamilton theorem we substitute the given matrix to the char eqn

$\rightarrow$ $A^3 - 6A^2 + 7A + 2I \rightarrow ①$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \underline{0}$$

Since determinant for given matrix is non zero, therefore inverse exists

Multiply eqn ① by A^{-1}

$$A^2 - 6A + 7I + 2A^{-1} = 0.$$

$$A^{-1} = \frac{1}{2}\left[A^2 + 6A - 7I\right]$$

$$= \frac{1}{2}\left[-\begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} + 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right]$$

$$= \begin{bmatrix} -3 & 0 & 2 \\ -1 & 0.5 & 0.5 \\ 2 & 0 & -1 \end{bmatrix}$$

10. Consider the matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. Also find D.

→ $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 0-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$-2\lambda + \lambda^2$

$(3-\lambda)(2) - 2(4 - 2\lambda) = 0$

$= -6\lambda + 3\lambda^2 - 8 + 4\lambda = 0.$

$+2\lambda^2 - \lambda^3$

$= -\lambda^3 + 5\lambda^2 - 2\lambda - 8 = 0$

$$\begin{array}{c|cccc} -1 & 1 & -5 & 2 & 8 \\ & 0 & -1 & 6 & -8 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

→ $\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$

$\lambda_1 = -1, 2, 4$

$\frac{6 \pm \sqrt{36-32}}{2}$

$\frac{6 \pm 2}{2}$ $4, 2$

M_2 $A - \lambda I$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

~~$4x + 2y = 0$~~   ~~$2x + y = 0$~~

$2x + y = 0$ ~~$x + 3z = 0$~~

~~$x + 3z = 0$~~

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 2 & -12 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 + 3x_3 = 0$

$x_2 - 6x_3 = 0$

$x_3 = 1$ $\quad V_1 = (-3, 6, 1)$

$x_1 = -3$

$x_2 = 6$

$M = A - \lambda I$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \\ x_1 &= 0 \end{aligned}\right\}$$ NO x_3. $\quad V_2 = (0, 0, 1)$

$\to x_1 = 0 \quad x_2 = 0$ Assume $x_3 = 1$

$M = A - \lambda I$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\left.\begin{aligned} -x_1 + 2x_2 &= 0 \\ 2x_1 - 4x_2 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}\right\} \begin{aligned} x_1 - 2x_2 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$ $\quad V_3 = (2, 1, 1)$

$x_3 = 1 \Rightarrow x_1 = 2$

$x_2 = 1$

$$P = \begin{bmatrix} -3 & 0 & 2 \\ 6 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{-1}{15} & \frac{2}{15} & 0 \\ \frac{-1}{3} & \frac{-1}{3} & 1 \\ \frac{2}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

Do in adj form

or sub P in chal. eqn

$D = P^{-1}AP$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Unit-3

Part-B

Inner product spaces:- and orthogonality

Definition: Let V be a real v. space. Suppose to each pair of vectors $(u, v) \in V$ there is assigned a real no. denoted by u, v. This function is called inner product ~~on~~ of V if it satisfies the following axioms

$[I_1]$ (Linear property): $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$

$[I_2]$ (Symmetric property): $\langle u, v \rangle = \langle v, u \rangle$

$[I_3]$ (Positive definite property): $\langle u_1, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if $u = 0$.

Example: Let V be a real inner product space. Then by linearity property. $\langle 3u_1 - 4u_2, 2v_1 - 5v_2 + 6v_3 \rangle = 6\langle u_1, v_1 \rangle - 15\langle u_1, v_2 \rangle + 18 \langle u_1, v_3 \rangle$
$- 8\langle u_2, v_1 \rangle + 20\langle u_2, v_2 \rangle - 24\langle u_2, v_3 \rangle$

Example 2: Let $u = (1, 3, -4, 2)$, $v = (4, -2, 2, 1)$, $w = (5, -1, -2, 6)$ in R^4

a. ST $\langle 3u - 2v, w \rangle = 3\langle u, w \rangle - 2\langle v, w \rangle$

$\langle u, w \rangle = 5 - 3 + 8 + 12 = 22$

$\langle v, w \rangle = 20 + 2 - 4 + 6 = 24$

$3u - 2v = (3, 9, -12, 6) - (8, -4, 4, 2) = (-5, 13, -16, 4)$.

$\langle 3u - 2v, w \rangle = -25 - 13 + 32 + 24 = 18$

$3\langle u, w \rangle - 2\langle v, w \rangle = 18$

3. Consider vectors $u = (1, 2, 4)$ $v = (2, -3, 5)$, $w = (4, 2, -3)$ in R^3. Find
a. $u.v$ b. $u.w$ c. $v.w$ d. $(u+v).w$ e. $\|u\|$ (f). $\|v\|$

→ a. $u.v = 2 - 6 + 20 = 16$

b. $u.w = 8 - 6 - 15 = -4$

c. $v.w = 8 - 6 - 15 = -13$

d. $(u+v).w = 12 - 2 - 27 = -17$

e. $\|u\| = \sqrt{1+4+16} = \sqrt{21}$

f. $\|v\| = \sqrt{4+9+25} = \sqrt{38}$

4. Consider the following polynomials in $P(t)$ with inner pr. $\langle f, g\rangle = \int_0^1 f(t). g(t). dt$ where $f(t) = t+2$ $g(t) = 3t-2$ $h(t) = t^2-3t-3$

a. find $\langle f, g\rangle$ & $\langle f, h\rangle$

b. find $\|f\|$, $\|g\|$

c normalize f & g

→ a. $\langle f, g\rangle = \int_0^1 (t+2). 3t-2 = \int_0^1 3t^2 - 2t + 6t - 4$

$= t^3 + 2t^2 - 4t \Big|_0^1$

$= (1 + 2 - 4) - 0$

$= \underline{\underline{-1}}$

$\langle f, h\rangle = \int_0^1 (t+2)\, t^2 - 3t - 3$

$t^3 - 3t^2 - 3t + 2t^2 - 6t - 6$

$t^3 - t^2 - 9t - 6$

$= \int_0^1 t^3 - t^2 - 9t - 6 \,.\, dt$

$= \frac{t^4}{4} - \frac{t^3}{3} - \frac{9t^2}{2} - 6t \Big|_0^1$

$= -10.583$

b. $\|f\| = \langle f, f\rangle = \sqrt{\int_0^1 f(t).f(t).dt}$

$= \sqrt{\int_0^1 t^2+4+4t.dt}$

$= \sqrt{6.33}$

$= \underline{\underline{2.51}}$

$\|g\| = \langle g, g\rangle = \sqrt{\int_0^1 9t^2+4-6t.dt}$ $\quad (3t-2)(3$

$= 1$

c. $\hat{f} = \dfrac{f}{\|f\|} = \dfrac{t+2}{2.51} = 0.39t + 0.79$

$\hat{g} = \dfrac{g}{\|g\|} = \dfrac{3t-2}{1} = 3t-2$

Orthogonality:

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal & u is said to be orthogonal to v if $\langle u, v\rangle = 0$. The relation is clearly symmetric i.e. if u is orthogonal to v then v is orthogonal to u.

Examples Find the orthogonality for the vectors $u=(1,1,1)$ $v=(1,2,-3)$ $w=(1,-4,3)$ in R^3. Then find / verify whether $\langle u,v\rangle$, $\langle u,w\rangle$, $\langle v,w\rangle$ is orthogonal to each other or not

$\rightarrow$ $\left.\begin{array}{l}\langle u,v\rangle = 1+2-3 = 0 \\ \langle u,w\rangle = 1-4+3 = 0\end{array}\right\}$ orthogonal

$\langle v,w\rangle = 1-8-9 = -16$ → Not orthogonal

Q. Verify whether the functions Sint & Cost in vector space $C[-\pi,\pi]$ of continuous fns on the closed interval is orthogonal or not

$\rightarrow$ $\langle \sin t, \cos t\rangle = \int_{-\pi}^{\pi} \sin t.\cos t.dt$

$= 0$

3. Find k so that $u = (1, 2, k, 3)$ $v = (3, k, 7, -5)$ in R^4 are orthogonal

$\Rightarrow$ $\langle u, v \rangle = 3 + 2k + 7k - 15 = 0$

$9k = 12$

$k = \frac{12}{9} = \frac{4}{3}$