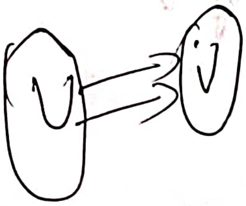
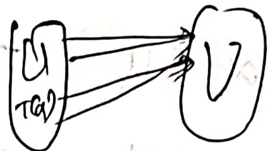
Unit - 2

Linear Transformation





onto
Isomorphism.

One - one
Homomorphism T

$T(\alpha) : U \to V$

$T(\alpha)$

* $\frac{d}{dx}(f(x))$

* $T(\alpha) = 0 \longrightarrow$ Nullity space.

* $\int_0^x f(x)\,dx$

null space of vector

Total dimension of space $=$ rank(V) + $N(V)$

Let U & V be a 2 vector space over the same field F. A mapping $T(\alpha) : U \to V$ is said to be a linear Transformation from U into V which associates to each element $\alpha(U)$ to a unique element $T(\alpha)$ of V such that

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad \forall\, \alpha, \beta \in U$$

$$\&\quad \text{all scalars } \forall\, a, b \in F$$

Properties of linear Transformation :

U & V such that $T(\alpha): U \to V$

zero vectors of V

i] $T(0) = 0$, $\alpha \in U$

ii] $T(-\alpha) = -T(\alpha)$ $\forall \alpha \in U$

iii] $T(\alpha - \beta) = T(\alpha) - T(\beta)$, $\forall \alpha, \beta \in U$

iv] $T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(\alpha_n)$.

Theorem 1:-

Let U & V be the finite 2D vector space over the field F & let $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ be an ordered basis for U & let set of $\{\beta_1, \beta_2, \dots, \beta_n\}$ then there is a precisely one LT $T(\alpha): U \to V$ such that $T(\alpha_j) = \beta_j$

Proof.

$$\{\alpha_1, \alpha_2, \dots, \alpha_n\} \in U$$
$$\downarrow T(\alpha_1) \quad \downarrow T(\alpha_2) \quad \dots \quad \downarrow T(\alpha_n)$$
$$\{\beta_1, \beta_2, \dots, \beta_n\} \in V$$

$T(\alpha_j) = \beta_j$; $j = 1, 2, \dots, n$.

one-to-one transformation.

Now

$\{\alpha_1, \alpha_2, \dots, \alpha_n\} \in U$

$\Rightarrow \alpha \in U$

$a_1, a_2, \dots, a_n \in \cancel{F}$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

$\alpha = \sum_{i=1}^{n} a_i \alpha_i$

$T(\alpha) : U \to V$

$T(\alpha) : \alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n$.

$T(\alpha) \in V$

$T(\alpha_i) = \beta_i \longrightarrow$ linear Transformation.

$T(a\alpha + b\beta) = \left[a \sum_{i=1}^{n} a_i\alpha_i + b \sum_{i=1}^{n} b_i\alpha_i\right]$

$= T\left[\sum_{i=1}^{n} a\, a_i + \sum_{i=1}^{n} b\, b_i\right] \alpha_i$

Q) Let P_1 & P_2 be the linear operators on R^2 defined as follows

$T_1(x_1, x_2) = (x_2, x_1)$ ST $T_1 \cdot T_2 \neq T_2 \cdot T_1$

$T_2(x_1, x_2) = (x_1, 0)$

$\Rightarrow$ Let $\alpha(x_1, x_2) \in R^2$

$(T_1 T_2)\alpha = T_1[T_2(\alpha)]$

$= T_1[T_2(x_1, x_2)] = T_1[x_1, 0]$

$(T_1 T_2)\alpha = (0, x_1)$ — ①

$$(T_2 T_1)\alpha = T_2(T_1(\alpha))$$
$$= T_2(x_2, x_1)$$
$$(T_2 T_1)\alpha = (x_2, 0) \quad \text{---} \ (2)$$

we can note that from (1) & (2)

$$(T_1 \cdot T_2)\alpha \neq (T_2 \cdot T_1)\alpha.$$

Hence the proof.

Q) Let V be a vector space of all polynomial functions in 'x' with the coefficients in field R of real no. Let ~~t & t~~ D & T be the two LO on V defined by

$$D[f(x)] = \frac{d}{dx}[f(x)]$$
$$T[f(x)] = \int_0^x f(x)\,dx.$$

for every $f(x) \in D$

ST $DT = I$ & $TD \neq I$.

A: Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$ where $a_0, a_1, a_2, \dots a_n \in R$.

Now

$$DT(f(x)) = D[T(f(x))]$$
$$= D\left[\int_0^x (a_0 + a_1x + a_2x^2 + \dots + a_nx^n)\,dx\right]$$
$$= D\left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \dots + a_n\frac{x^{n+1}}{n+1}\right]$$
$$DT(f(x)) = \frac{d}{dx}\left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \dots + a_n\frac{x^{n+1}}{n+1}\right]$$
$$DT(f(x)) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$
$$DT(f(x)) = f(x)$$
$$\boxed{DT = I}$$

- $$TD(f(x)) = T\left[\frac{d}{dx}[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n]\right]$$
$$= T\left[a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}\right].$$
$$= \int_0^x (a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1})\,dx$$
$$TD(f(x)) = a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$
$$TD(f(x)) \neq f(x).$$
$$\cancel{TD} \quad \boxed{TD \neq I}$$

Algebra of LT :-

Vector spaces U & V & T_1 & T_2 be two LT

- $(T_1 + T_2)\alpha = [T_1(\alpha) + T_2(\alpha)] \longrightarrow \in V$

$(T_1 + T_2)U = \in V$

- $c(T\alpha) = c[T(\alpha)]$

- **Closure property** :-

$$T_1 + T_2 = T_2 + T_1$$

$$T_1, T_2 \in L(U, V)$$

- **Associative property** :-

$$[(T_1 + T_2) + T_3]\alpha = [T_1 + (T_2 + T_3)]\alpha$$

- **Commutative** :-

$$T_1 + T_2 = T_2 + T_1$$

- **Existence of identity** :

$$T + O = O + T$$

- **Existence of inverse** :-

$$(T) + (-T) = O$$

- **Distributive property** :-

$$a[(T_1 + T_2)\alpha] = a T_1(\alpha) + a T_2(\alpha)$$

Linear operator

$\alpha \in V$, $T(\alpha) \in V$

$L(T) \to$ Transform.

$\alpha(U) \xrightarrow{T} T(\alpha)(V)$

Algebra of LO :-

$TI = IT$ identity operator.

Associative

$$T_1(T_2 T_3) = (T_1 T_2) T_3$$

- **distributive** :-

$$T_1(T_2 + T_3) = T_1 T_2 + T_1 T_3$$

- $c(T_1 T_2) = (cT_1) T_2 = (cT_2) T_1$

- $OT = TO = O$, O being linear operator.

Range & Nullspace of Linear Transform

Range space :-

Dimension of T

$T(\alpha) \longrightarrow \rho(\alpha)$ = Dimensionality of Range space T

if $T(\alpha) = 0 \to$ Nullspace of LT

$[\underbrace{v_1, v_2, \ldots}_{\text{linearly Independ}}] \Rightarrow$ basis $\Rightarrow$ Rangespace of LT

Statement :-

Let 'U' & 'V' be the vectorspace over the field F & let 'T' be the LT from U into V. Suppose 'U' is finite dimensional

rank(T) + nullity (T) = dimension(U)

$$\rho(T) + N(T) = \dim(T)$$

Proof :- Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the basis of $N(T)$

Let $\dim(U)$ be 'n'

then basis of Vectorspace are $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of 'U'

Let $a_i \in F$

$$\sum_{i=1}^{k} a_i T(\alpha_i) = 0 \Rightarrow T\left[\sum_{i=1}^{k} a_i \alpha_i\right] = 0.$$

$\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \rightarrow$ basis of N_T

$$\Rightarrow \sum_{i=1}^{k} a_i \alpha_i \in N_T$$

~~Scalars $\{b_1, b_2, \ldots, b_k\}$~~

$T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)$ These transformation spans in range of T.

i.e., $T(\alpha) \in R_T$

$\Rightarrow \alpha = a_1 \alpha_{k+1} + a_2 \alpha_{k+2} + \cdots + a_n \alpha_n$

$T(\alpha) \rightarrow$ linear $\Rightarrow$ Basis for - Range of Vector.

$\dim(R_T) = n-k.$

$= \dim(U) - \dim(N_T)$

$$\Rightarrow \boxed{\dim(R_T) + \dim(N_T) = \dim(U)}$$

Product of LT :-

$T(\alpha) \, U \rightarrow V$

$S[T(\alpha)] \rightarrow V$

$ST(\alpha) \rightarrow V$

Invertible

$T(\alpha) \, U \rightarrow V$

$T(\alpha) : U \rightarrow V$

$T^{-1}(\alpha) : V \rightarrow U.$

Theorem :-

Let U & V be the vector space over the same field F & let T be a linear transformation from $U \rightarrow V$. If T is invertible then T^{-1} is a LT from V into U.

Proof :- $T(\alpha) = \beta$

$T^{-1}(\beta) = \alpha.$

$\alpha_1, \alpha_2 \in U$, $a, b \in F$

$T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$

$T(\alpha_1) = \beta_1 \iff T^{-1}(\beta_1) = \alpha_1$

$T(\alpha_2) = \beta_2 \iff T^{-1}(\beta_2) = \alpha_2$

$aT^{-1}(\beta_1) + bT^{-1}(\beta_2) = T^{-1}(a\beta_1) + T^{-1}(b\beta_2)$

$= T^{-1}(a\beta_1 + b\beta_2)$

Non singular LT :-

$\cdot$ U & V. $\longrightarrow$ non singular.

$T(\alpha) = 0$

$\Rightarrow \alpha = 0$

Theorem:-

Let T be a LT from U into V then T is non singular if and only if T carries each linearly Independent subset of U onto a LID subset of V $|A| \neq 0$

$S = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$

$S = \{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_p)\}$

$T(\alpha_1) = 0 \Rightarrow \alpha_1 = 0$

Theorem :

Let U & V finite dimensional vectorspace over the field F such that $\dim(U) = \dim(V)$. If T is a LT from $U \to V$ then the following are equivalent.

i) T is invertible

ii) T is non singular

iii) T is onto ie. Range of T is V

iv) If $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is basis of U' then $\{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\}$ is a basis of V'

i) T is ...

T - onto one 

ii) T is non singular. 

Let $\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_n)}_{LID}$ standard basis

$T(\alpha_1) ; T(\alpha_2), \ldots, T(\alpha_n) \longrightarrow V$

$\beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots + a_n T(\alpha_n)$

$\longrightarrow$ T linear.

$\therefore$ Range of T is in V

Q] Describe explicitly LT from V^3 into V^3 which has its range the subspace spanned by $(1,0,-1)$ & $(1,2,2)$

A standard basis $= \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (1,0,-1)$

$T(0,1,0) = (1,2,2)$

$T(0,0,1) = (0,0,0)$

(x,y,z) element is V_3 space

$T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$

$= x(1,0,-1) + y(1,2,2) + z(0,0,0)$

$T(x,y,z) = (x+y, 2y, -x+2y)$

Co-ordinate vectors :-

$V \in F$

$B = \{ \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \}$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

$$[\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Matrix Representation of LT :-

$U \quad \& \quad V$

$B = \{ \alpha_1, \alpha_2, \dots, \alpha_n \}$ } Std basis of Vector.

$B' = \{ \beta_1, \beta_2, \dots, \beta_n \}$

$T(\alpha_1), T(\alpha_2)$ $\quad T(\alpha_i)$ — LT from U to V

$T(\alpha_1) = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n$

$T(\alpha_2) = a_{21}\beta_1 + a_{22}\beta_2 + \dots + a_{2n}\beta_n$

$\vdots$

$T(\alpha_n) = a_{n1}\beta_1 + a_{n2}\beta_2 + \dots + a_{nn}\beta_n.$

$$\Rightarrow \begin{bmatrix} T(\alpha_1) \\ T(\alpha_2) \\ \vdots \\ T(\alpha_n) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\Rightarrow [T]_B = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

Q) Let V be a vector space of polynomials in P over the field F containing only Real no of b/w

$D: V \to V$

$D[P(t)] = \frac{d}{dt} P(t)$

$B = [1, t, t^2, t^3]$

Find matrix of LT :

Sol $B = [1, t, t^2, t^3] \to$ Basis $\in V$

$D(1) = 0 = 0 + 0t + 0t^2 + 0t^3$

$D(t) = 1 = 1 + 0t + 0t^2 + 0t^3$

$D(t^2) = 2t = 0 + 2t + 0t^2 + 0t^3$

$D(t^3) = 3t^2 = 0 + 0 + 3t^2 + 0t^3$

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Change of Basis :-

$\{\alpha_1, \alpha_2 \cdots \alpha_n\} \to V$

$\{\beta_1, \beta_2 \cdots \beta_n\}$

$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n$

$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n$

Transition matrix.

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}_{n \times n}.$$

Q] Let $\{(1,0), (0,1)\}$ & $\{(1,1), (-1,0)\}$ be the basis in R^2 domain. Find the change of basis.

A.
$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2$

$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2$

$\Rightarrow (1,1) = a_{11}(1,0) + a_{12}(0,1)$

$(-1,0) = a_{21}(1,0) + a_{22}(0,1)$

$\Rightarrow \boxed{1 = a_{11}}$; $\boxed{1 = a_{12}}$ $\therefore P = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$\boxed{-1 = a_{21}}$; $\boxed{0 = a_{22}}$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Theorem:-

Let P be a Transition Matrix from basis $B \to B'$ in a vector space V then for any vector $\alpha \in V$

$P[\alpha]_{B'} = [\alpha]_B$

$[\alpha]_{B'} = P^{-1}[\alpha]_B$

$B - (\alpha_1, \alpha_2 \cdots \alpha_n)$

$B' - (\beta_1, \beta_2 \cdots \beta_n)$

A. Let V n-dimensional vector space consisting

$B = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$; $B' = \{\beta_1, \beta_2 \cdots \beta_n\}$

$\beta_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n$

$\beta_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n$

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}, a_{ij} \in F$$

$\alpha \in V$

$\alpha = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$

$\alpha = b_1\{a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n\} + b_2\{a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n\} + \cdots$

$\cdots + b_n\{a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n\}$

$$[\alpha]_{B'} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad [\alpha]_B = \begin{bmatrix} b_1 a_{11} + b_2 a_{12} + \cdots + b_n a_{1n} \\ b_1 a_{21} + b_2 a_{22} + \cdots + b_n a_{2n} \\ \vdots \\ b_1 a_{n1} + b_2 a_{n2} + \cdots + b_n a_{nn} \end{bmatrix}$$

$P[\alpha]_{B'} = [\alpha_B] \to \times P^{-1}$

$[\alpha]_{B'} = P^{-1}[\alpha_B]$ //

Similarity of Linear Transformation

Let S & T be two LT.

S if there is invertible LT

$T = PSP^{-1}$

Trace of LT :-

1] $tr(\lambda A) = \lambda \, tr(A)$

2] $tr(A+B) = tr(A) + tr(B)$

3] $tr(AB) = tr(BA)$

Q] Let T be a LT In 2D space. where $T(x,y) = (2x, \frac{1}{2}y)$

Find the matrix associated with T with an ordered basis

A] ~~To~~ std basis = $\{(0,1), (1,0)\}$

$T(1,0) = (2,0)$

$T(0,1) = (0, \frac{1}{2})$

$T(1,0) = (2,0) = 2(1,0) + 0(0,1) = (2,0)$

$T(0,1) = (0, \frac{1}{2}) = 0(1,0) + \frac{1}{2}(0,1) = (0, \frac{1}{2})$

$$[T]_B = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Q] Find [illegible] space defined as

$T(a,b,c) = (2b+c, a-4b, 3a)$ w.r.t the ordered basis B & also with B'

i] $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

ii] $B' = \{(1,1,1), (1,1,0), (1,0,0)\}$

i] $T(1,0,0) = (0, 1, 3)$

$T(0,1,0) = (2, -4, 0)$

$T(0,0,1) = (1, 0, 0)$

$T(1,0,0) = 0(1,0,0) + 1(0,1,0) + 3(0,0,1)$

$T(0,1,0) = 2(1,0,0) + (-4)(0,1,0) + 0(0,0,1)$

$T(0,0,1) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$$[T]_B^T = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -4 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T$$

$$\therefore [T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

ii] $T(1,1,1) = (3, -3, 3)$

$T(1,1,0) = (2, -3, 3)$

$T(1,0,0) = (0, 1, 3)$.

$\therefore$ $\cancel{T(1,1,1) = 3(1,0,0) + (-3)(0,1,0) + 3(0,0,1)}$

$\cancel{T(1,1,0) = 2(1,0,0) + (-3)(0,1,0) + 3(0,0,1)}$

$T(\cancel{1,0},0) =$

$(a,b,c) = x(1,1,1) + y(1,1,0) + z(1,0,0)$

$a = x + y + z$

$b = x + y$

$\boxed{c = x}$

$z = a - c - b + c$.

$\therefore$ $\boxed{x = c}$

$\boxed{y = b - c}$

$\boxed{z = a - b}$

$\cancel{\boxed{z = a - b - 2c}}$

$(a,b,c) = c(1,1,1) + (b-c)(1,1,0) + (a-b)(1,0,0)$ $\rightarrow$ ①

$T(1,1,1) = (3,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 6(1,0,0)$

$T(1,1,0) = (2,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 5(1,0,0)$

$T(1,0,0) = (0,1,3) = 3(1,1,1) + (-2)(1,1,0) + (-1)(1,0,0)$

$$\therefore [T]_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Q] Let T be a linear operator in 2D space defined by $T(x,y) = (2y, 3x - y)$. Find the matrix representation of T relative to the basis $\{(1,3), (2,5)\}$.

A. $T(x,y) = (2y, 3x - y)$.

- $T(1,3) = (6, 0)$
- $T(2,5) = (10, 1)$

$(a,b) = p(1,3) + q(2,5)$

$a = p + 2q$ $\Rightarrow$ $3a = 3p + 6q$

$b = 3p + 5q$. $\quad$ $-\ b = 3p + 5q$

$\boxed{3a - b = q}$

$\therefore$ $a = p + 6a - 2b$.

$p = -5a + 2b$.

$\therefore$ $(a,b) = (-5a+2b)(1,3) + (3a-b)(2,5)$

• $T(1,3) = (1,0) = -30(1,3) + 18(2,5)$

• $T(2,5) = (0,1) = -48(1,3) + 29(2,5)$

$$\therefore [T]_B = \begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$$

Q] Let T be a Linear operator on $\mathbb{R}^3$ defined by

$$T(x,y,z) = (3x+z,\ -2x+y,\ -x+2y+4z)$$

PT T is invertible & find the formula for T^{-1}

$\Rightarrow$ $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ be std basis in $\mathbb{R}^3$ space

$T(1,0,0) = (3, -2, -1) = 3(1,0,0) + (-2)(0,1,0) + (-1)(0,0,1)$

$T(0,1,0) = (0, 1, 2) = 0(1,0,0) + 1(0,1,0) + 2(0,0,1)$

$T(0,0,1) = (1, 0, 4) = 1(1,0,0) + 0(0,1,0) + 4(0,0,1)$

$$\therefore A = [T]_B^T = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}^T$$

$|A| \neq 0 \Rightarrow T$ is invertible

$|A| = 3(4) + 1(-4+1)$

$= 12 - 3$

$\boxed{|A| = 9}$ $\therefore T$ is invertible

$\cancel{adj(A)}$

$$(A:I) = \begin{bmatrix} 3 & -2 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & 0 & 1 & 0 \\ 1 & 0 & 4 & : & 0 & 0 & 1 \end{bmatrix}$$

$$(A:I) = \begin{bmatrix} 1 & -2 & -9 & : & 1 & 0 & -2 \\ 0 & 1 & 2 & : & 0 & 1 & 0 \\ 1 & 0 & 4 & : & 0 & 0 & 1 \end{bmatrix}$$

$$(A:I) = \begin{bmatrix} 1 & -2 & -9 & : & 1 & 0 & -2 \\ 0 & 1 & 2 & : & 0 & 1 & 0 \\ 0 & 2 & 13 & : & -1 & 0 & 3 \end{bmatrix}$$

$$(A:I) = \begin{bmatrix} 1 & 0 & -5 & : & 1 & 2 & -2 \\ 0 & 1 & 2 & : & 0 & 1 & 0 \\ 0 & 0 & 9 & : & -1 & -2 & 3 \end{bmatrix}$$

$$\cancel{[T]_B^T = A^{-1} = \frac{1}{9}}$$

$$\cancel{(A:I) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 \end{bmatrix}}$$

$$[T]_B^T = A^{-1} = \frac{1}{9}\begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & 2 \\ -3 & -6 & 3 \end{bmatrix}$$

$\alpha = (p, q, r) \longrightarrow \mathbb{R}^3$, B – std basis in $\mathbb{R}^3$

$$[\alpha]_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\left[T^{-1}(\alpha)\right]_B = A^{-1}[\alpha]_B = \frac{1}{9}\begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & 2 \\ -3 & -6 & 3 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$T^{-1}(p,q,r) = \left[\frac{4p+2q-r}{9}, \frac{8p+13q+2r}{9}, \frac{-3p-6q+3r}{9}\right]$$

Q] Describe explicitly the LT $T: R^2 \longrightarrow R^3$ such that $T(2,3) = (4,5)$

$T(1,0) = (0,0)$

Let $\alpha = (2,3)$; $\beta = (1,0)$, $a, b \in R$.

$a\alpha + b\beta = 0(0,0)$

$a(2,3) + b(1,0) = (0,0) \Rightarrow (2a+b, 3a) = (0,0)$

$\Rightarrow 2a+b = 0.$

$3a = 0.$

$\Rightarrow \boxed{a=0}$ & $\boxed{b=0}$

$\therefore$ The vectors are linearly independent & forms basis in R^2 dimensional space.

$(x,y) = p\alpha + q\beta$

$(x,y) = (2p+q, 3p)$

$x = 2p+q$, $y = 3p. \Rightarrow \boxed{p = \frac{y}{3}}$

$\boxed{q = \frac{3x-2y}{3}}$

$(x,y) = \frac{y}{3}(2,3) + \frac{3x-2y}{3}(1,0)$

$T(x,y) = T\left(\frac{y}{3}(2,3) + \frac{3x-2y}{3}(1,0)\right)$

$= \frac{y}{3} T(2,3) + \frac{3x-2y}{3} T(1,0)$

$= \frac{y}{3}(4,5) + \frac{3x-2y}{3}(0,0)$

$T(x,y) = \left(\frac{4y}{3}, \frac{5y}{3}\right).$

Unit-3 Eigen Values & Eigen Vectors.

Algebraic Multiplicity & Geometric multiplicity of Eigen value

$|A - \lambda I| = (\lambda-\lambda_1)^2(\lambda-\lambda_2)$ AM.

$|A - \lambda I| = (\lambda-\lambda_1)(\lambda-\lambda_2)^2$

$\boxed{GM \geq AM \quad m \geq n}$

Eigen values & Eigen vectors of Linear Transformation :-

V - finite dimension Vector space F

T - linear operator $\quad \alpha \neq 0 \in V$

$\lambda \rightarrow$ Scalar in field F

$T(\alpha) = \lambda\alpha.$

PTO

__Diagonalisation of Matrix :-__

$A = \left[\;\; \right]_{2\times 2}$ → linearly independent

$P = (v_1 \;\; v_2)$

Eigen values - $\lambda_1 ; \lambda_2$

Eigen vector - $v_1 \quad v_2$

If

$P^{-1}AP = D$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Q) For the given matrix, find the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix also find the diagonal matrix.

$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 0-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)\left[(3-\lambda)(-\lambda) - 4\right]$

$= (2-\lambda)\left[-3\lambda + \lambda^2 - 4\right]$

$= -6\lambda + 2\lambda^2 - 8 - 3\lambda^2 - \lambda^3 + 4\lambda$

$|A - \lambda I| = -\lambda^3 - 5\lambda^2 - 3\lambda - 2 = 0.$

$\lambda^3 - 5\lambda^2 + 3\lambda + 2 = 0.$

$\Rightarrow \lambda = -1, 4, 2.$

• Now

• $\lambda = -1$

$\therefore$ $(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$(A + I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

$\Rightarrow \left.\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}\right\}$

$x_1 + 3x_3 = 0$

$2x_1 + x_2 = 0.$

$2x_1 + 6x_3 = 0.$

$x_2 - 6x_3 = 0$

$\Rightarrow \boxed{x_2 = 6x_3}$

$\boxed{2x_1 = -x_2}$

$\Rightarrow \boxed{x_1 = -3x_3}$

$\therefore$ $x_3 = c$

$x_2 = 6c$

$x_1 = -3c$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix}$

• $\lambda_2 = 2$.

$(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 + 2x_2 = 0.$

$2x_1 - 2x_2 = 0$

$\boxed{x_1 = 0} \Rightarrow \boxed{x_2 = 0} \Rightarrow \boxed{x_3 = c}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

∴ $\lambda_3 = 4$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} -x_1 + 2x_2 &= 0. \\ 2x_1 - 4x_2 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}\right\} \Rightarrow \boxed{x_1 = 2x_2} \quad \boxed{x_1 = 2x_3}$$

Let $x_3 = c$

$\Rightarrow x_1 = 2c \mid x_2 = c$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} -3 & 0 & 2 \\ 6 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now $|P| = \begin{vmatrix} -3 & 0 & 2 \\ 6 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= -1(-3-12).$

$|P| = 15$

adj$(P) =$ $-1 \quad -5 \quad 6$

$2 \quad -5$

$$P^{-1} = \frac{1}{15}\begin{bmatrix} -1 & 2 & 0 \\ -5 & -5 & 15 \\ 6 & 3 & 0 \end{bmatrix}$$

Now $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Q] ST the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalisable

& also find the diagonalising matrix P.

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$= (-9-\lambda)\left[(3-\lambda)(7-\lambda) - 32\right] - 4\left[(-8)(7-\lambda) + 64\right]$$

$$+ 4\left[-64 + 16(3-\lambda)\right]$$

$$= (-9-\lambda)\left[21 - 10\lambda - \lambda^2 - 32\right] - 4\left[-56 + 8\lambda + 64\right]$$

$$+ 4\left[-64 + 48 - 16\lambda\right]$$

$$= (9+\lambda)\left[\lambda^2 + 10\lambda + 9\right] - 4\left[8\lambda + 8\right] + 4\left[-16 - 16\lambda\right]$$

$$= (9\lambda^2 + 90\lambda + 81 + \lambda^3 + 10\lambda^2 + 9\lambda) - 4\left[24 + 24\lambda\right]$$

$$= \lambda^3 + 19\lambda^2 + 99\lambda + 81 - 96 - 96\lambda.$$

$$= \lambda^3 + 19\lambda^2 + 3\lambda - 15$$

Now ~~$\lambda^3 + 19\lambda^2 + 3\lambda - 18 = 0$~~ $(\lambda+1)^2(3-\lambda) = 0$

$\Rightarrow \lambda = -1, \lambda = -1, 3.$

Now $\lambda = -1$

$$\therefore (A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + x_2 + x_3 = 0.$$

Let $x_2 = c_1$; $x_3 = c_2$

$$\therefore x_1 = \frac{c_1 + c_2}{2}.$$

$$x_1 = \frac{1}{2}c_1 + \frac{1}{2}c_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c_1 + \frac{1}{2}c_2 \\ c_1 \\ c_2 \end{bmatrix}$$

$$X_1 = c_1 \overset{v_1}{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}} + c_2 \overset{v_2}{\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}}$$

• $\lambda = 3.$

$$\begin{bmatrix} -6 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$\cancel{-6 \quad 4 \quad 4}$

$-6x_1 + 4x_2 + 4x_3 = 0$

$\Rightarrow \; -3x_1 + 2x_2 + 2x_3 = 0 \;\Rightarrow\; -3x_1 + 2x_1 + 8x_1 = 0.$

$-8x_1 + 4x_3 = 0$

$\Rightarrow \boxed{x_3 = 2x_1}$

$-16x_1 + 8x_2 + 4x_3 = 0.$

$-16x_1 + 8x_2 + 8x_1 = 0.$

$-8x_1 + 8x_2 = 0$

$\Rightarrow \boxed{x_1 = x_2}$

Let $x_1 = c$, $x_2 = c$, $x_3 = 2c$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -2 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\therefore P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Minimal Polynomial :-

$A = [\;\;]_{2\times 2}$

Characteristic eqn

$\lambda^3 - (\text{trace}(A))\lambda^2 + (\text{adj}A)\lambda - |A| = 0.$

Q) Find the minimal Polynomial for the matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

$|A - \lambda I| = 0.$

Al

$$\lambda^3 - \text{trace}(A)\,\lambda^2 + \overset{\text{Trace}}{\left(\text{adj}(A)\right)}\lambda - |A| = 0$$

$$|A| = \begin{vmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{vmatrix} = 4$$

$\text{trace}(A) = 5$

$$(\text{adj}(A))^T = \left(\begin{vmatrix} -4 & 2 & -6 \\ 12 & -2 & 12 \\ 12 & 2 & 14 \end{vmatrix}\right)^T$$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda-1)(\lambda-2)^2 = 0.$

$m_1 = (\lambda-1)(\lambda-2)$; $m_2 = (\lambda-1)(\lambda-2)^2$

$m_1 = (A-I)(A-2I)$;

$$m_1 = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 3 & -6 & -6 \\ -1 & 9 & 2 \\ 3 & -6 & -6 \end{pmatrix}$$

$m_1 = 0$

5] $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

$|A - \lambda I| = 0$

$-\lambda^3 + \lambda^2 + 5\lambda + 3 = 0$

$(\lambda+1)^2(\lambda-3) = 0$

$m_1 = (\lambda+1)(\lambda+3)$; $m_2 = (\lambda+1)^2(3-\lambda).$

$m_1 = (A+I)(3I-A)$

$$= \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} 12 & -4 & -4 \\ 8 & \cancel{10} & -4 \\ 16 & -8 & -4 \end{bmatrix}$$

$= 0$

6] Let 'T' be the LT from $R^3 \to R^3$ be defined by.

$T(x,y,z) = (2x+3y-2z,\ 5y+4z,\ x-z)$

Find the characteristic polynomial.

A $T(1,0,0) = (2,0,1)$

$T(0,1,0) = (3,5,0)$

$T(0,0,1) = (-2,4,-1)$

$$[T]_B = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 5 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

Characteristic eqn

$|A - tI| = 0.$

$$\begin{vmatrix} 2-t & 3 & -2 \\ -0 & 5-t & 4 \\ +1 & 0 & -1-t \end{vmatrix} = 0.$$

$(2-t)\left[(5-t)(-1-t)\right] + 1\left[12 + 2(5-t)\right]$

$(2-t)\left[1-(5-t)(1+t)\right] + 1\left[12 + 2(5-t)\right]$

$(2-t)\left[1-\left[5-t^2+4t\right]\right] + 12 + 10 - 2t$

$(2-t)(t^2-4t-4) + 12 + 10 - 2t$

$2t^2 - 8t - 8 - t^3 + 4t^2 + 4t + 12 + 10 - 2t = 0.$

$-t^3 + 6t^2 - 6t + 14 = 0.$

$\boxed{t^3 - 6t^2 + 3t - 12 = 0}$

Q) What is the Algebraic Multiplicity & Geometric multiplicity of $\lambda = -2$ where "$\lambda = -2$" is one of the eigen values of the matrix.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = 0.$$

$(3-\lambda)\left[(5-\lambda)(-2-\lambda)+6\right] - 1\left[-7(-2-\lambda)-6\right] - 1\left[-42+6(5-\lambda)\right] = 0$

$(3-\lambda)\left[6-(5-\lambda)(2+\lambda)\right] - \left[14+7\lambda-6-42+30-6\lambda\right] = 0$

$(3-\lambda)\left[6-\left[10-\lambda^2+3\lambda\right]\right] - \left[\lambda-4\right] = 0.$

$(3-\lambda)\left[6-10+\lambda^2-3\lambda\right] - \lambda + 4 = 0.$

$18 - 30\lambda + 3\lambda^2 - 9\lambda - 6\lambda + 10\lambda - \lambda^3 + 3\lambda^2 - \lambda + 4 = 0.$

$\lambda^3 - 12\lambda - 16 = 0$

$(\lambda+2)^2(\lambda-4) = 0.$

Algebraic multiplicity is 2.

$$|A - \lambda_1 I| = 0.$$

$$(A + 2I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{vmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{vmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} -x_1 + x_2 - x_3 &= 0 \\ -7x_1 + 7x_2 - x_3 &= 0 \end{aligned}\right| \Rightarrow x_3 = 0.$$

$$-6x_1 + 6x_2 = 0$$

$$\Rightarrow \boxed{x_1 = x_2}$$

Let $x_1 = c$

$\therefore x_2 = c$

$x_3 = 0$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Geometric Multiplicity

$\dim E_2 = 1 \rightarrow$ no of sol^n for ~~eigs~~

Q] For the given matrix Find all the eigen values of A. give as a matrix over the

a] Real field R

b] Complex field C

Also find in which case A is diagonalisable

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & 15 & 6 \end{bmatrix} \Rightarrow \text{trace}(\text{adj}(A)) = 21$$

$|A| = 27$

$\text{Trace}(A) = 7,$

$$\lambda^3 - 7\lambda^2 + 21\lambda - 27 = 0.$$

$$(\lambda - 3)(\lambda^2 + 1) = 0.$$

$$\lambda_1 = 3, \quad \lambda_2 = i, \quad \lambda_3 = i$$

a] Real value $-\lambda = 3$, can't be diagonalisable.
If A is a matrix over real field, then A has only one eigen value which is '3' thus A can't be diagonisable.

b] If A is a matrix over complex field, then A has 3 distinct eigen values thus A can be diagonisable.

Orthogonality & Orthogonal Vectors

$U(a_1, b_1, c_1)$ & $V(a_2, b_2, c_2)$

Inner product

$\langle U.V \rangle = a_1 a_2 + b_1 b_2 + c_1 c_2$

$\langle U.V \rangle = 0 \rightarrow$ orthogonal to each other (two vectors)

length of the vector.

$\|U\| = \sqrt{a_1^2 + b_1^2 + c_1^2}$

angle b/w two vector.

$\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$

Orthogonal components :-

S — ~~Subtta~~ Subset of inner product space V

$S^{\perp} = \{ v \in V \ ;\ \langle v, u \rangle = 0$ for every $u \in S \}$

$V^{\perp} = \{ v \in V \ ;\ \langle v, u \rangle = 0 \}$

Q) Verify which of the following vectors are orthogonal to each other.

$u = (1, 1, 1)$; $v = (1, 2, -3)$ & $w = (1, -4, 3)$

$\langle u.v \rangle =$

$\langle u.w \rangle = 1 - 4 + 3 = 0.$ orthogonal

$\langle v.w \rangle = 1 - 8 - 9 = -16.$ — not ⊥

Q) Find a non-zero vector W which is orthogonal to $u_1 = (1, 2, 1)$ & $u_2 = (2, 5, 4)$ in a 3D vector space.

Let $w = (x, y, z)$

$\langle u_1 . w \rangle = x + 2y + z = 0.$

$\langle u_2 . w \rangle = 2x + 5y + 4z = 0.$

$\langle u_1 . u_2 \rangle = 2 + 10 + 4 = 16.$

$4x + 8y + 4z = 0$

$2x + 5y + 4z = 0$

$2x + 3y = 0.$

$\boxed{y = -\frac{2}{3}x}$

$2x + 4y + 2z = 0$

$2x + 5y + 4z = 0$

$-y - 2z = 0$

$\boxed{y = -2z}$

$x + 2y + z = 0$

$y + 2z = 0$

Let $z = 1$

$\therefore y = -2$

$\therefore x = 3$

$\therefore \boxed{w = (3, -2, 1)}$

Inner product of $p(t)$

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt.$$

Q] Consider the vector space $p(t)$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt.$$

$f(t) = t + 2$

$g(t) = 6t - 5$

Find the inner product

$\frac{64 - 8}{3} =$

27
-8

$\therefore$ $\langle f, g \rangle = \int_0^1 (t+2)(6t-5)\, dt$

$$= \int_0^1 (6t^2 - 5t + 12t - 10)\, dt$$

$$= \int_0^1 6t^2 + 7t - 10\, dt.$$

$$= 2\, t^3 \Big|_0^1 + \frac{7}{2} t^2 \Big|_0^1 - 10 \cdot t \Big|_0^1$$

20
-11

$$= 2 + \frac{7}{2} - 10$$

$$= \frac{4 + 7 - 20}{2.}$$

$$\boxed{\langle f, g \rangle = \frac{-9}{2.}}$$

Length of funct

$$\|f\|^2 = \int_0^1 f(t)\, f(t)\, dt.$$

$$= \int_0^1 (t+2)^2\, dt = \frac{(t+2)^3}{3} \Big|_0^1 =$$

$$\frac{3^3}{3} - \frac{2^3}{3} = \frac{19}{3}$$

$$\|g\|^2 = \int_0 \ldots$$

$$= \frac{(6t-5)^3}{6 \times 3} \Big|_0^1$$

$$= \frac{1}{18} (1^3 + 5^3).$$

$$= \frac{126}{18} \quad 42.14.$$

43

$$\boxed{\|g\|^2 = \frac{14}{3}}$$

Q] Find I P of the given matrix.

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} ; \quad B \begin{bmatrix} 2 & -6 & 5 \\ 1 & 4 & -3 \end{bmatrix}.$$

$\therefore$ $\langle A, B \rangle = \begin{bmatrix} 8 + (-30) + 30 \\ 7 + 32 + \overset{-27}{\cancel{(\ )}} \end{bmatrix}$

$$\langle A, B \rangle = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = 20.$$

length of A

$$\|A\|^2 = \sqrt{4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2} = 16.46$$

$$\|B\| = \sqrt{2^2 + 6^2 + 5^2 + 1^2 + 4^2 + 3^2} = 9.53.$$

$$\cos\theta = \frac{\langle A, B \rangle}{\|A\|\ \|B\|} = \frac{20}{16.46 \times 9.53}$$

$\Rightarrow \theta =$

Q] Find the angle b/w the vectors

$u = (2, 3, 5)$ & $v = (1, -4, 3)$

A. $\langle u \cdot v \rangle = 2 - 12 + 15 = 5$

$\|u\| = \sqrt{2^2 + 3^2 + 5^2} = 6.16$

$\|v\| = \sqrt{1^2 + 4^2 + 3^2} = 5.0990$

$\cos\theta = \dfrac{5}{6.16 \times 5.09} = 0.159$ - acute

Orthogonal sets

$S = \{ u_1, u_2, \dots, u_n \}$

$\langle u_i, u_j \rangle = 0 \Rightarrow$ for $i \neq j$

↳ orthogonal set

Orthonormal set

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$$

Pythagorean theorem

u, v

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + \cancel{\quad} \; 2\overset{0}{\langle u \cdot v \rangle}$

$\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Find The ON set of vectors for the given vectors & verify pythagorean theorem.

$u = (1, 2, -3, 4)$; $v (3, 3, 1, -2)$; $w (3, -2, 1, 1)$

A. $\langle u \cdot v \rangle = 3 + 8 - 3 - 8 = 0$.

$\langle u \cdot w \rangle = 3 - 4 - 3 + 4 = 0$.

$\langle v \cdot w \rangle = 9 - 8 + 1 - 2 = 0$.

$\|u\| = \sqrt{30}$

$\|v\| = \sqrt{30}$

$\|w\| = \sqrt{15}$

$\hat{u} = \dfrac{u}{\|u\|}$; $\hat{v} = \dfrac{v}{\|v\|}$; $\hat{w} = \dfrac{w}{\|w\|}$

$\|u + v + w\|^2 = \|u\|^2 + \|v\|^2 + \|w\|^2$

$\|(7, 4, -1, 3)\|^2 = 30 + 30 + 15$

$75 = 75$ //

Q] S.T $\{u_1, u_2, u_3\}$ is an orthogonal set where

$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ & $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ & $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

A. $\langle u_1 \cdot u_2 \rangle = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 0$

$\langle u_2 \cdot u_3 \rangle = \begin{bmatrix} +\frac{1}{2} \\ -4 \\ \frac{7}{2} \end{bmatrix} = 0$

$\langle u_1 \cdot u_3 \rangle = \begin{pmatrix} -\frac{3}{2} \\ 2 \\ 2-2 \\ 7\ldots \end{pmatrix} = 0$

Q) The set $S = \{u_1, u_2, u_3\}$ is orthogonal basis in R^3

$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $u_3 = \begin{bmatrix} -1/2 \\ 2 \\ 7/2 \end{bmatrix}$

Express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a LC of the vectors in S.

$$y = au_1 + bu_2 + cu_3$$

$y \cdot u_1 = \begin{bmatrix} 18 \\ 1 \\ -8 \end{bmatrix} = 11$ | $y \cdot u_2 = \begin{bmatrix} -6 \\ 2 \\ -8 \end{bmatrix} = -12$

$u_1 \cdot u_1 = \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix} = 11$ | $u_2 \cdot u_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = 6$

$y \cdot u_3 = \begin{bmatrix} -3 \\ 2 \\ -28 \end{bmatrix} = -33$

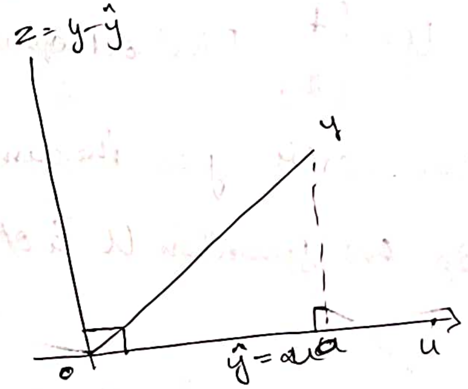
$u_3 \cdot u_3 = \begin{bmatrix} \frac{1}{4} \\ 4 \\ \frac{49}{4} \end{bmatrix}$. $\frac{50}{4} + 4 = \frac{\cancel{66}\ 33}{\cancel{4}} \ 16.5 = \frac{33}{2}$

$$y = \overbrace{\frac{y \cdot u_1}{u_1 \cdot u_1}}^{a} \cdot u_1 + \overbrace{\frac{y \cdot u_2}{u_2 \cdot u_2}}^{b} u_2 + \overbrace{\frac{y \cdot u_3}{u_3 \cdot u_3}}^{c} \cdot u_3$$

projection.

$\therefore \boxed{y = u_1 - 2u_2 - 2u_3}$

Orthogonal Projections



$y = \hat{y} + z$.

$\hat{y} = \alpha u$

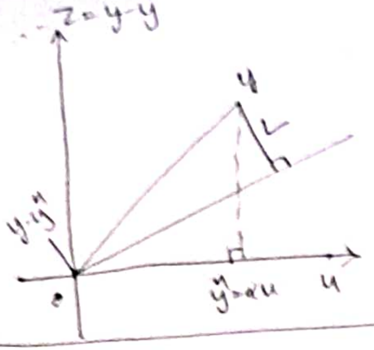
z - vector orthogonal to u

$z = y - \hat{y}$

$\alpha = \frac{y \cdot u}{u \cdot u}$

$\hat{y} = \frac{y \cdot u}{u \cdot u} \cdot u$.

Projection of y on u



Q] Given $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ & $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Find orthogonal projection of y on to u & then write y as the sum of two orthogonal vectors one spanned in u & other orthogonal to u

$$\hat{y} = \frac{y.u}{u.u} . u$$

$$y.u = \begin{bmatrix} 28 \\ 12 \end{bmatrix} = 40$$

$$u.u = \begin{bmatrix} 16 \\ 4 \end{bmatrix} = 20$$

$$\therefore \hat{y} = \frac{40}{20} u$$

$$\boxed{\hat{y} = 2u}$$

$$y = \hat{y} + z$$

$$\hat{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$z = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Now

$$\hat{y} . (y - \hat{y}) = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$$\hat{y} . (y - \hat{y}) = 0$$

$\therefore$

length of vector.

$$l = \| y - \hat{y} \| = \left\| \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Orthonormal set:-

$S = \{u_1, u_2, u_3\}$

$$\langle u_i . u_j \rangle = \begin{cases} 0 , & i \neq j \\ 1 , & i = j \end{cases}$$

Q] S.T $v_1\ v_2\ v_3$ is an orthonormal basis of $\mathbb{R}^3$ where

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix} ; v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} ; v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

$$\langle v_1, v_2 \rangle = \begin{bmatrix} -3/\sqrt{66} \\ 2/\sqrt{66} \\ 1/\sqrt{66} \end{bmatrix} = 0$$

$$\langle v_1, v_3 \rangle = \begin{bmatrix} -3/a \\ -4/a \\ 7/a \end{bmatrix} = 0$$

$$\langle v_2, v_3 \rangle = \begin{bmatrix} 1/b \\ -8/b \\ 7/b \end{bmatrix} = 0$$

$v_1 . v_1 = 1$

$v_2 . v_2 = 1$

$v_3 . v_3 = 1$

Q) Verify the matrix 'U' has an orthonormal columns & ST

$$||U^T x|| = ||x||$$

$$U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}_{3\times2} \; ; \; x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}_{2\times1}$$

$$U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}_{2\times3}$$

$$U^T U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Ux = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 3$$

$$||Ux|| = 3 \; ; \; ||x|| = \sqrt{2+9} = \sqrt{11}$$

Q) Given $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$; $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$; $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Verify that set $[u_1, u_2]$ is an orthogonal basis for $W = \text{Span}(u_1, u_2)$. Write y as sum of the vector in W & vector orthogonal to W.

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \rightarrow \text{projection of } y \text{ onto } w.$$

$$y = \hat{y} + z$$

$$y \cdot u_1 = \begin{bmatrix} 2 \\ 10 \\ -3 \end{bmatrix} = 9. \qquad y \cdot u_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = 3$$

$$u_1 \cdot u_1 = \begin{bmatrix} 4 \\ 25 \\ 1 \end{bmatrix} = 30 \qquad u_2 \cdot u_2 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = 14$$

$$\therefore \hat{y} = \frac{9}{30} u_1 + \frac{3}{14} u_2$$

$$\hat{y} = \frac{3}{10} u_1 + \frac{3}{14} u_2 = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} ;$$

Now

$$z = y - \hat{y}$$

$$z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 6/10 \\ 15/10 \\ -3/10 \end{bmatrix} - \begin{bmatrix} -6/14 \\ 3/14 \\ 3/14 \end{bmatrix}$$

$$z = \begin{bmatrix} 49/35 \\ 0 \\ 98/35 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$