Since, the time shift is circular, we can write the above equation as

$$x((n-m))_N = \frac{1}{N}\sum_{k=0}^{N-1}\left[X(k)W_N^{km}\right]W_N^{-kn}$$

$$\Rightarrow \quad x((n-m))_N = \text{IDFT}\left[X(k)W_N^{km}\right]$$

or

$$\text{DFT}\{x((n-m))_N\} = W_N^{km}X(k)$$

In terms of the transform pair, we can write the above equation as

$$x((n-m))_N \overset{\text{DFT}}{\longleftrightarrow} W_N^{km}X(k)$$

Example 3.15 Find the 4-point DFT of the sequence, $x(n) = (1, -1, 1, -1)$. Also, using time shift property, find the DFT of the sequence, $y(n) = x((n-2))_4$.

☐ Solution

Given, $N = 4$.

We know that

$$W_4^0 = 1, \quad W_4^1 = -j$$

$$W_4^2 = -1, \quad W_4^3 = j$$

Hence,

$$\begin{aligned}
X(k) &= \text{DFT}\{x(n)\} \\
&= \sum_{n=0}^{3} x(n)W_4^{kn}, \quad 0 \leqslant k \leqslant 3 \\
&= 1 \times W_4^{0k} - 1 \times W_4^{k} + 1 \times W_4^{2k} - 1 \times W_4^{3k} \\
&= 1 - W_4^{k} + W_4^{2k} - W_4^{3k}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad X(0) &= 1 - 1 + 1 - 1 = 0 \\
X(1) &= 1 - W_4^1 + W_4^2 - W_4^3 = 0 \\
X(2) &= 1 - W_4^2 + W_4^4 - W_4^6 \\
&= 1 - W_4^2 + W_4^0 - W_4^2 = 4 \\
X(3) &= 1 - W_4^3 + W_4^6 - W_4^9 \\
&= 1 - W_4^3 + W_4^2 - W_4^1 = 0
\end{aligned}$$

Given,

$$y(n) = x((n-2))_4$$

Applying circular time shift property, we get

$$\begin{aligned}
Y(k) &= W_4^{2k}\,X(k), \quad k = 0, 1, 2, 3 \\
\Rightarrow \quad Y(0) &= W_4^0\,X(0) = 0 \\
Y(1) &= W_4^2\,X(1) = 0
\end{aligned}$$

$$Y(2) = W_4^4 X(2) = W_4^0 X(2) = 4$$
$$Y(3) = W_4^6 X(3) = W_4^2 X(3) = 0$$

Hence,
$$Y(k) = (\underset{\substack{\uparrow \\ k=0}}{0}, 0, 4, 0)$$

Example 3.16 Suppose $x(n)$ is a sequence defined on $0-7$ only as $(0, 1, 2, 3, 4, 5, 6, 7)$.

a. Illustrate $x((n-2))_8$.

b. If DFT $\{x(n)\} = X(k)$, what is the DFT$\{x((n-2))_8\}$?

☐ Solution

a. Given
$$x(n) = (0, 1, 2, 3, 4, 5, 6, 7)$$

To generate $x((n-2))_8$, move the last 2 samples of $x(n)$ to the beginning.

That is,
$$x((n-2))_8 = (6, 7, 0, 1, 2, 3, 4, 5)$$

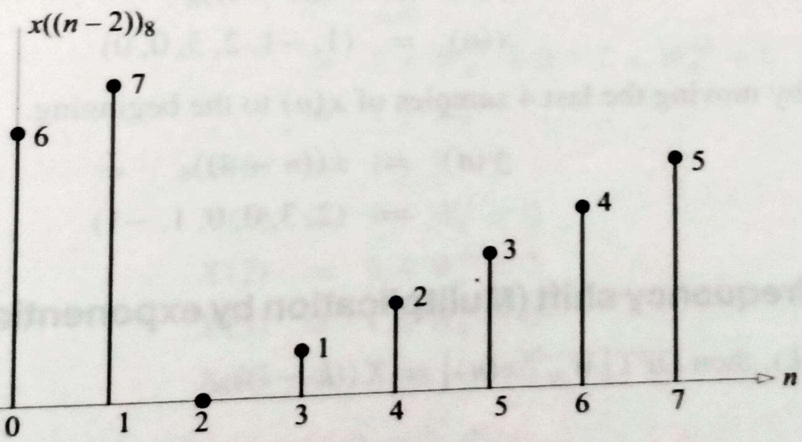


Fig. Ex.3.16 Sequence $x((n-2))_8$.

It should be noted that $x((n-2))_8$ is implicity periodic with a period $= N = 8$.

b. Let
$$y(n) = x((n-2))_8$$

Applying circular time shift property, we get
$$Y(k) = W_8^{2k} X(k)$$

Example 3.17 Let $X(k)$ donote a 6-point DFT of a length-6 real sequence, $x(n)$. The sequence is shown in Fig. Ex.3.17. Without computing the IDFT, determine the length-6 sequence, $y(n)$ whose 6-point DFT is given by, $Y(k) = W_3^{2k} X(k)$.

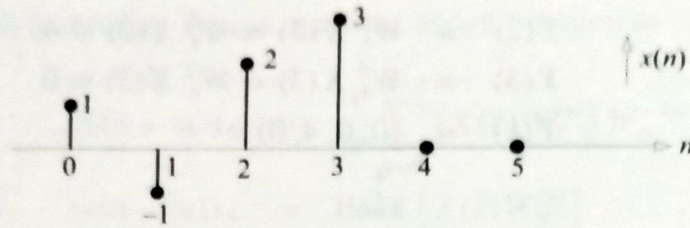


Fig. Ex.3.17 Sequence $x(n)$ for Example.3.17.

☐ Solution

We may write

$$W_3^{2k} = e^{-j\frac{2\pi}{3}\times 2k}$$
$$= e^{-j\frac{2\pi}{6}\times 4k}$$

Hence,
$$W_3^{2k} = W_6^{4k}$$

It is given in the problem that

$$Y(k) = W_3^{2k}X(k)$$
$$\Rightarrow \quad Y(k) = W_6^{4k}X(k)$$

We know that
$$\text{DFT}\{x((n-m))_N\} = W_N^{mk}X(k)$$
$$\Rightarrow \quad \text{IDFT}\{W_N^{mk}X(k)\} = x((n-m))_N$$

Hence,
$$y(n) = x((n-4))_6$$

Since,
$$x(n) = (1, -1, 2, 3, 0, 0)$$

we get $x((n-4))_6$ by moving the last 4 samples of $x(n)$ to the beginning.

Hence,
$$y(n) = x((n-4))_6$$
$$= (2, 3, 0, 0, 1, -1)$$

3.7.3 Circular frequency shift (Mulitplication by exponential in time-domain)

If $\text{DFT}\{x(n)\} = X(k)$, then $\text{DFT}\left\{W_N^{-ln}x(n)\right\} = X((k-l))_N$.

Proof:

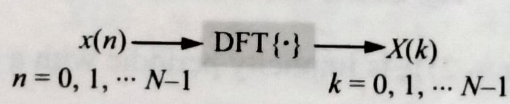


Fig. 3.6 DFT viewed as an operator.

In Fig. 3.6, DFT is viewed as an operator, that is, $\text{DFT}\{x(n)\} = X(k)$. Using the defining equation, we have

$$X(k) = \text{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(k-l) = \sum_{n=0}^{N-1} x(n)W_N^{(k-l)n}$$

Since, the shift in frequency is circular, we may write the above equation as

$$X((k-l))_N = \sum_{n=0}^{N-1} \left[x(n) W_N^{-ln}\right] W_N^{kn}$$

Hence, $$\text{DFT}\left\{x(n) W_N^{-ln}\right\} = X((k-l))_N$$

Example 3.18 Compute the 4-point DFT of the sequence $x(n) = (1, 0, 1, 0)$. Also, find $y(n)$ if $Y(k) = X((k-2))_4$.

☐ Solution

Given $N = 4$.

Also, $W_4^0 = 1, \quad W_4^1 = -j, \quad W_4^2 = -1, \quad W_4^3 = j$.

The DFT of the sequence, $x(n)$ is

$$\begin{aligned}
X(k) &= \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3 \\
&= 1 \times W_4^{0k} + 0 + 1 \times W_4^{2k} + 0 \\
&= 1 + W_4^{2k} \\
\Rightarrow \quad X(0) &= 1 + 1 = 2 \\
X(1) &= 1 + W_4^2 = 0 \\
X(2) &= 1 + W_4^0 = 2 \\
X(3) &= 1 + W_4^2 = 0
\end{aligned}$$

Given $$Y(k) = X((k-2))_4$$

We know that, $$\text{DFT}\{W_N^{-ln} x(n)\} = X((k-l))_N$$

That is, $$y(n) = W_N^{-ln} x(n) \overset{\text{DFT}}{\longleftrightarrow} Y(k) = X((k-l))_N$$

Hence,

$$\begin{aligned}
y(n) &= W_4^{-2n}\, x(n) \\
\Rightarrow \quad y(0) &= W_4^{-0}\, x(0) = 1 \\
y(1) &= W_4^{-2}\, x(1) = 0 \\
y(2) &= W_4^{-4}\, x(2) \\
&= W_4^{-0}\, x(2) = 1 \times 1 = 1 \\
y(3) &= W_4^{-6}\, x(3) = W_4^{-2}\, x(3) = 0
\end{aligned}$$

That is, $$y(n) = (\underset{\substack{\uparrow \\ n=0}}{1}, 0, 1, 0)$$

Example 3.19 In many signal processing applications, we often multiply an infinite length sequence by a window of length N. The time-domain expression for this window is

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right]$$

What is the DFT of the windowed sequence, $y(n) = x(n)w(n)$? Keep the answer interms of $X(k)$.

Solution

Given,

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right], \quad 0 \leqslant n \leqslant N-1$$

$$\Rightarrow \quad w(n) = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}e^{j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)} + \frac{1}{2}e^{-j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)}\right]$$

$$= \frac{1}{2} + \frac{1}{4}e^{j\frac{2\pi n}{N}}e^{-j\pi} + \frac{1}{4}e^{-j\frac{2\pi n}{N}}e^{j\pi}$$

$$= \frac{1}{2} + \frac{1}{4}W_N^{-n} \times (-1) + \frac{1}{4}W_N^{n} \times (-1)$$

$$= \frac{1}{2} - \frac{1}{4}W_N^{-n} - \frac{1}{4}W_N^{n}$$

Given

$$y(n) = x(n)w(n)$$

$$\Rightarrow \quad y(n) = \frac{1}{2}x(n) - \frac{1}{4}x(n)W_N^{-n} - \frac{1}{4}x(n)W_N^{n}$$

We know that, $\text{DFT}\left\{x(n)W_N^{-ln}\right\} = X((k-l))_N$

Hence,

$$Y(k) = \frac{1}{2}X(k) - \frac{1}{4}X((k-1))_N - \frac{1}{4}X((k+1))_N$$

Example 3.20 Let $x(n)$ be a length-N sequence with N-point DFT $X(k)$. Determine the N-point DFTs of the following length-N sequences interms of $X(k)$.

a. $y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$

b. $y_2(n) = \begin{cases} x(n), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$

Example 3.19 In many signal processing applications, we often multiply an infinite length sequence by a window of length N. The time-domain expression for this window is

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right]$$

What is the DFT of the windowed sequence, $y(n) = x(n)w(n)$? Keep the answer interms of $X(k)$.

☐ Solution

Given,

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right], \quad 0 \leqslant n \leqslant N-1$$

$$\Rightarrow \quad w(n) = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}e^{j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)} + \frac{1}{2}e^{-j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)}\right]$$

$$= \frac{1}{2} + \frac{1}{4}e^{j\frac{2\pi n}{N}}e^{-j\pi} + \frac{1}{4}e^{-j\frac{2\pi n}{N}}e^{j\pi}$$

$$= \frac{1}{2} + \frac{1}{4}W_N^{-n} \times (-1) + \frac{1}{4}W_N^{n} \times (-1)$$

$$= \frac{1}{2} - \frac{1}{4}W_N^{-n} - \frac{1}{4}W_N^{n}$$

Given

$$y(n) = x(n)w(n)$$

$$\Rightarrow \quad y(n) = \frac{1}{2}x(n) - \frac{1}{4}x(n)W_N^{-n} - \frac{1}{4}x(n)W_N^{n}$$

We know that,

$$\text{DFT}\left\{x(n)W_N^{-ln}\right\} = X((k-l))_N$$

Hence,

$$Y(k) = \frac{1}{2}X(k) - \frac{1}{4}X((k-1))_N - \frac{1}{4}X((k+1))_N$$

Example 3.20 Let $x(n)$ be a length-N sequence with N-point DFT $X(k)$. Determine the N-point DFTs of the following length-N sequences interms of $X(k)$.

a. $y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$

b. $y_2(n) = \begin{cases} x(n), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$

☐ Solution

a. We know that

$$\text{DFT}\{x((n-m))_N\} = W_N^{mk}X(k)$$

Given

$$y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$$

Hence,

$$Y_1(k) = \alpha\, \text{DFT}\{x((n-m_1))_N\} + \beta \text{DFT}\{x((n-m_2))_N\}$$

$$= \alpha W_N^{m_1 k}X(k) + \beta W_N^{m_2 k}X(k)$$

Example 3.19 In many signal processing applications, we often multiply an infinite length sequence by a window of length N. The time-domain expression for this window is

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right]$$

What is the DFT of the windowed sequence, $y(n) = x(n)w(n)$? Keep the answer interms of $X(k)$.

Solution

Given,

$$\begin{aligned}
w(n) &= \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right], \quad 0 \leqslant n \leqslant N-1 \\
\Rightarrow \quad w(n) &= \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}e^{j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)} + \frac{1}{2}e^{-j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)}\right] \\
&= \frac{1}{2} + \frac{1}{4}e^{j\frac{2\pi n}{N}}e^{-j\pi} + \frac{1}{4}e^{-j\frac{2\pi n}{N}}e^{j\pi} \\
&= \frac{1}{2} + \frac{1}{4}W_N^{-n} \times (-1) + \frac{1}{4}W_N^{n} \times (-1) \\
&= \frac{1}{2} - \frac{1}{4}W_N^{-n} - \frac{1}{4}W_N^{n}
\end{aligned}$$

Given

$$\begin{aligned}
y(n) &= x(n)w(n) \\
\Rightarrow \quad y(n) &= \frac{1}{2}x(n) - \frac{1}{4}x(n)W_N^{-n} - \frac{1}{4}x(n)W_N^{n}
\end{aligned}$$

We know that, $\text{DFT}\left\{x(n)W_N^{-ln}\right\} = X((k-l))_N$

Hence,

$$Y(k) = \frac{1}{2}X(k) - \frac{1}{4}X((k-1))_N - \frac{1}{4}X((k+1))_N$$

Example 3.20 Let $x(n)$ be a length-N sequence with N-point DFT $X(k)$. Determine the N-point DFTs of the following length-N sequences interms of $X(k)$.

a. $y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$

b. $y_2(n) = \begin{cases} x(n), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$

Solution

a. We know that

$$\text{DFT}\{x((n-m))_N\} = W_N^{mk}X(k)$$

Given

$$y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$$

Hence,

$$\begin{aligned}
Y_1(k) &= \alpha\, \text{DFT}\{x((n-m_1))_N\} + \beta \text{DFT}\{x((n-m_2))_N\} \\
&= \alpha W_N^{m_1 k}X(k) + \beta W_N^{m_2 k}X(k)
\end{aligned}$$

b. Given

$$y_2(n) = \begin{cases} x(n), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

$$\Rightarrow \quad y_2(n) = \frac{1}{2}[x(n) + (-1)^n x(n)]$$

$$\Rightarrow \quad y_2(n) = \frac{1}{2}[x(n) + e^{-j\pi n} x(n)]$$

$$= \frac{1}{2}\left[x(n) + e^{-j\frac{2\pi}{N}\frac{N}{2}n} x(n)\right]$$

$$\Rightarrow \quad y_2(n) = \frac{1}{2}\left[x(n) + W_N^{\frac{N}{2}n} x(n)\right]$$

We know that $\quad \mathrm{DFT}\{W_N^{-ln} x(n)\} = X((k-l))_N$

Hence, $\quad Y_2(k) = \frac{1}{2}\left[X(k) + X\left(\left(k + \frac{N}{2}\right)\right)_N\right]$

3.7.4 Symmetry: real-valued sequences

If the sequence $x(n), n = 0, 1, \cdots N-1$ is *real*, then its DFT is such that

$$X(k) = X^*(N-k), \quad k = 0, 1, \cdots N-1$$

***Proof*:**

We know that

$$X(k) = \mathrm{DFT}\{x(n)\}$$

$$\triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Taking conjugates on both the sides, we get

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

Since $x(n)$ is real, we have $x^*(n) = x(n)$. As a consequence of this, the above equation reduces to

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}\, W_N^{Nn} \qquad (\text{since } W_N^{Nn} = 1)$$

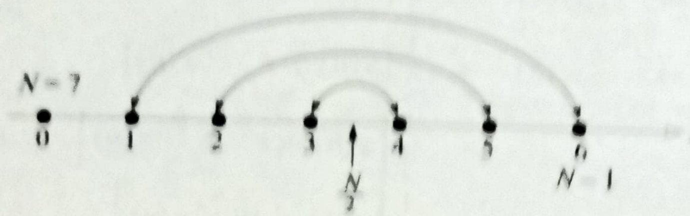
$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

Hence, $\quad X^*(k) = X(N-k)$

The above equation conveys the message that the DFT of a real sequence possesses conjugate symmetry about the midpoint.

If N is odd, the conjugate symmetry is about $\frac{N}{2}$. The index, $k = \frac{N}{2}$ is called the *folding index*. This aspect is illustrated in Fig. 3.7.

Conjugate symmetry: $X^*(k) = X(N-k)$ or $X(k) = X^*(N-k)$



Conjugate symmetry: $X(k) = X^*(N-k)$

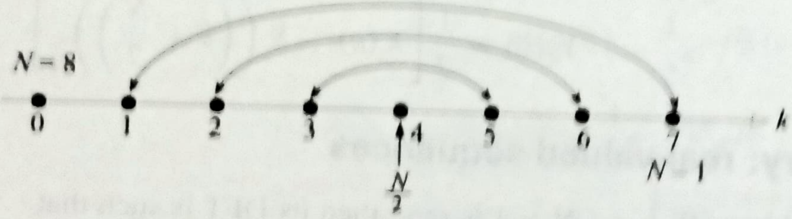


Fig. 3.7 Symmetry of $X(k)$ for $x(n)$ being real.

Conjugate symmetry implies that we need to compute only half of the DFT values to find the entire DFT sequences – a great labor saving help! A similar result holds good for IDFT also.

Example 3.21 Compute the 5-point DFT of the sequence, $x(n) = (1, 0, 1, 0, 1)$ and hence verify the symmetry property.

☐ Solution

We know that $\quad W_N = e^{-j\frac{2\pi}{N}}$

Since, $\quad N = 5, \quad W_5 = e^{-j\frac{2\pi}{5}}$

Hence,

$$W_5^0 = 1$$

$$W_5^1 = e^{-j\frac{2\pi}{5}} = 1\angle{-\frac{2\pi}{5}} = 0.309 - j0.951$$

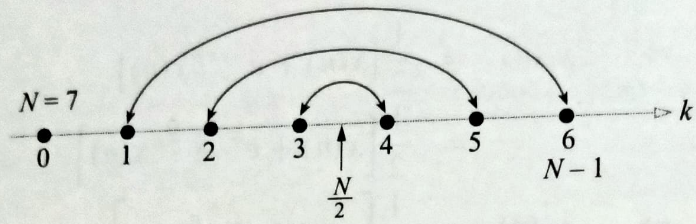
$$W_5^2 = e^{-j\frac{4\pi}{5}} = 1\angle{-\frac{4\pi}{5}} = -0.809 - j0.587$$

$$W_5^3 = e^{-j\frac{6\pi}{5}} = 1\angle{-\frac{6\pi}{5}} = -0.809 + j0.587$$

$$W_5^4 = e^{-j\frac{8\pi}{5}} = 1\angle{-\frac{8\pi}{5}} = 0.309 + j0.951$$

If N is odd, the conjugate symmetry is about $\frac{N}{2}$. The index, $k = \frac{N}{2}$ is called the *folding index*. This aspect is illustrated in Fig. 3.7.

Conjugate symmetry: $X^*(k) = X(N-k)$ or $X(k) = X^*(N-k)$



Conjugate symmetry: $X(k) = X^*(N-k)$

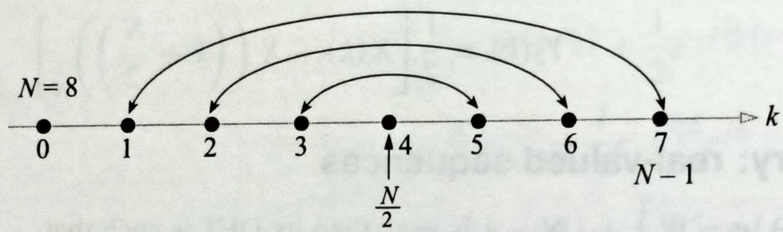


Fig. 3.7 Symmetry of $X(k)$ for $x(n)$ being real.

Conjugate symmetry implies that we need to compute only half of the DFT values to find the entire DFT sequences – a great labor saving help! A similar result holds good for IDFT also.

Example 3.21 Compute the 5-point DFT of the sequence, $x(n) = (1, 0, 1, 0, 1)$ and hence verify the symmetry property.

☐ **Solution**

We know that $\quad W_N = e^{-j\frac{2\pi}{N}}$

Since, $\quad N = 5, \quad W_5 = e^{-j\frac{2\pi}{5}}$

Hence, $\quad W_5^0 = 1$

$$W_5^1 = e^{-j\frac{2\pi}{5}} = 1\angle -\tfrac{2\pi}{5} = 0.309 - j0.951$$

$$W_5^2 = e^{-j\frac{4\pi}{5}} = 1\angle -\tfrac{4\pi}{5} = -0.809 - j0.587$$

$$W_5^3 = e^{-j\frac{6\pi}{5}} = 1\angle -\tfrac{6\pi}{5} = -0.809 + j0.587$$

$$W_5^4 = e^{-j\frac{8\pi}{5}} = 1\angle -\tfrac{8\pi}{5} = 0.309 + j0.951$$

By definition,

$$\begin{aligned}
\text{DFT}[x(n)] &= X(k) \\
&= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1
\end{aligned}$$

Since $N = 5$,

$$X(k) = \sum_{n=0}^{4} x(n) W_5^{kn}, \quad k = 0, 1, 2, 3, 4$$

$$\begin{aligned}
\Rightarrow \quad X(k) &= 1 + 0 + W_5^{2k} + 0 + W_5^{4k} \\
&= 1 + W_5^{2k} + W_5^{4k}
\end{aligned}$$

Hence,

$$\begin{aligned}
X(0) &= 1 + 1 + 1 = 3 \\
X(1) &= 1 + W_5^2 + W_5^4 = 0.5 + j0.364 \\
X(2) &= 1 + W_5^4 + W_5^8 = 1 + W_5^4 + W_5^3 \\
&= 0.5 + j1.538 \\
X(3) &= 1 + W_5^6 + W_5^{12} = 1 + W_5^1 + W_5^2 \\
&= 0.5 - j1.538 \\
X(4) &= 1 + W_5^8 + W_5^{16} = 1 + W_5^3 + W_5^1 \\
&= 0.5 - j0.364
\end{aligned}$$

Verification

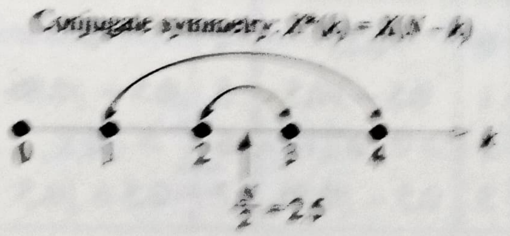


Fig. Ex3.21 Symmetry of $X(k)$ for $x(n)$ being real and $N = 5$.

Since $x(n)$ is real,

$$\begin{aligned}
X^*(k) &= X(N-k) \\
&= X(5-k)
\end{aligned}$$

We find that $\quad X^*(1) = X(4)$

and $\quad X^*(2) = X(3)$

Hence, the symmetry property for $x(n)$ being real is verified.

Example 3.22 The first five points of the 8-point DFT of a real-valued sequence are $\{0.25, 0.5 - j0.5, 0, 0.5 - j0.206, 0\}$. Find the remaining three points.

By definition,

$$\begin{aligned}
\text{DFT}\{x(n)\} &= X(k) \\
&= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1
\end{aligned}$$

Since $N = 5$,

$$\begin{aligned}
X(k) &= \sum_{n=0}^{4} x(n) W_5^{kn}, \quad k = 0, 1, 2, 3, 4 \\
\Rightarrow \quad X(k) &= 1 + 0 + W_5^{2k} + 0 + W_5^{4k} \\
&= 1 + W_5^{2k} + W_5^{4k}
\end{aligned}$$

Hence,

$$\begin{aligned}
X(0) &= 1 + 1 + 1 = 3 \\
X(1) &= 1 + W_5^2 + W_5^4 = 0.5 + j0.364 \\
X(2) &= 1 + W_5^4 + W_5^8 = 1 + W_5^4 + W_5^3 \\
&= 0.5 + j1.538 \\
X(3) &= 1 + W_5^6 + W_5^{12} = 1 + W_5^1 + W_5^2 \\
&= 0.5 - j1.538 \\
X(4) &= 1 + W_5^8 + W_5^{16} = 1 + W_5^3 + W_5^1 \\
&= 0.5 - j0.364
\end{aligned}$$

Verification

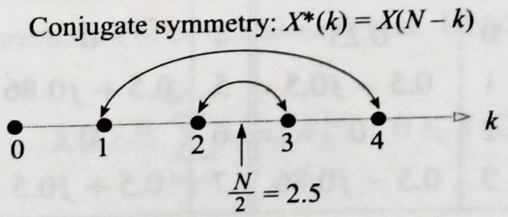


Fig. Ex.3.21 Symmetry of $X(k)$ for $x(n)$ being real and $N = 5$.

Since $x(n)$ is real,

$$\begin{aligned}
X^*(k) &= X(N-k) \\
&= X(5-k)
\end{aligned}$$

We find that

$$X^*(1) = X(4)$$

and

$$X^*(2) = X(3)$$

Hence, the symmetry property for $x(n)$ being real is verified.

Example 3.22 The first five points of the 8-point DFT of a real-valued sequence are $(0.25, 0.5 - j0.5, 0, 0.5 - j0.86, 0)$. Find the remaining three points.

☐ Solution

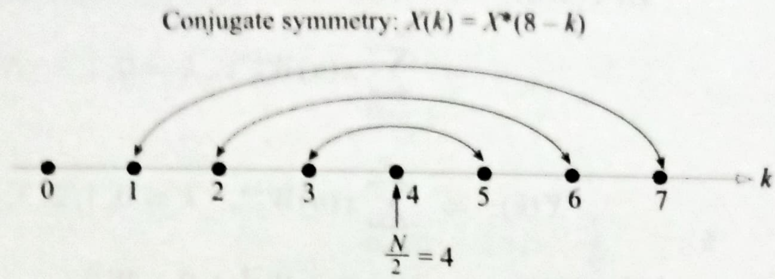


Fig. Ex.3.22 Symmetry of DFT for real signals ($N = 8$).

Since $x(n)$ is real-valued, we have

$$X(k) = X^*(8-k), \quad k = 0, 1, \cdots 7$$

Hence,

$$\begin{aligned} X(5) &= X^*(3) = 0.5 + j0.86 \\ X(6) &= X^*(2) = 0 \\ X(7) &= X^*(1) = 0.5 + j0.5 \end{aligned}$$

Thus, the complete 8-point sequence, $X(k)$ is as tabulated below:

k	$X(k)$	k	$X(k)$
0	0.25	4	0
1	$0.5 - j0.5$	5	$0.5 + j0.86$
2	0	6	0
3	$0.5 - j0.86$	7	$0.5 + j0.5$

Example 3.23 Let $x(n)$ be a real sequence of length-N and its N-point DFT is given by $X(k)$. Show that:

a. $X(N-k) = X^*(k)$,

b. $X(0)$ is real, and

c. if N is even, $X\left(\frac{N}{2}\right)$ is real.

☐ Solution

a. The proof of this part is given in section 3.7.4. However, this being a very important property, we would like to prove this in a slightly different manner.

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(N-k) = \sum_{n=0}^{N-1} x(n)W_N^{(N-k)n}$$

$$= \sum_{n=0}^{N-1} x(n)W_N^{-kn} \quad (\because W_N^{Nn} = 1)$$

Since $x(n)$ is real, we can replace $x(n)$ by $x^*(n)$ in the above expression.

Thus, $$X(N-k) = \sum_{n=0}^{N-1} x^*(n)W_N^{-kn}$$

Hence, $$X(N-k) = X^*(k)$$

b. $$X(k) \triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(0) = \sum_{n=0}^{N-1} x(n)$$

Since $x(n)$ is real, its summation over n is always real. Hence shown.

c. $$X(k) \triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

Letting $k = \frac{N}{2}$ in the above expression, we get

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n)W_N^{\frac{N}{2}n}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}\frac{N}{2}n}$$

$$= \sum_{n=0}^{N-1} x(n)(-1)^n$$

Since $x(n)$ is real, the above summation gives always a real number. Hence shown.

3.7.5 Circular folding

If $\text{DFT}\{x(n)\} = X(k)$, then $\text{DFT}\{x((-n))_N\} = X((-k))_N$.

Proof:

By definition,

$$X(k) = \text{DFT}\{x(n)\}$$
$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Substitute $m = N - n$

Then,
$$X(k) = \sum_{m=N}^{1} x(N-m) W_N^{k(N-m)}$$

Because of the implicit periodicity of $x(n)$, the limits of summation can be changed as shown below.

$$X(k) = \sum_{m=0}^{N-1} x(N-m) W_N^{k(N-m)}$$

Since m is a dummy variable, it can be replaced by n.

Thus,
$$X(k) = \sum_{n=0}^{N-1} x(N-n) W_N^{-kn} \quad (\because W_N^{kN} = 1)$$
$$\Rightarrow \quad X(N-k) = \sum_{n=0}^{N-1} x(N-n) W_N^{-(N-k)n}$$
$$\Rightarrow \quad X(N-k) = \sum_{n=0}^{N-1} x(N-n) W_N^{kn} \quad (\because W_N^{-Nn} = 1)$$

Hence,
$$\text{DFT}\{x(N-n)\} = X(N-k)$$
or
$$\text{DFT}\{x((-n))_N\} = X((-k))_N$$

Example 3.24 Compute the 4-point DFT of the sequence $x(n) = (1, 2, 1, 0)$. Also, find $Y(k)$ if

$$y(n) = x((-n))_N, 0 \leqslant k \leqslant 3$$

□ **Solution**

We know that

$$W_N = e^{-j\frac{2\pi}{N}}$$

Since $N = 4$, we get

$$W_4 = e^{-j\frac{\pi}{2}}$$

Hence,

$$W_4^0 = 1$$
$$W_4^1 = -j$$
$$W_4^2 = -1$$
$$W_4^3 = j$$

By definition,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3$$

$$\Rightarrow \quad X(k) = 1 + 2W_4^k + W_4^{2k}$$

Hence,

$$X(0) = 1 + 2 + 1 = 4$$
$$X(1) = 1 + 2W_4^1 + W_4^2 = -j2$$
$$X(2) = 1 + 2W_4^2 + W_4^4 = 1 + 2W_4^2 + W_4^0 = 0$$
$$X(3) = 1 + 2W_4^3 + W_4^6 = 1 + 2W_4^3 + W_4^2 = j2$$

Thus,

$$X(k) = (4, -j2, 0, j2)$$

Since $x(n)$ is real, it may be noted that the symmetry property: $X(k) = X^*(N-k)$ is observed.

Given

$$y(n) = x((-n))_N$$

Hence,

$$Y(k) = X((-k))_N = X^*(k), \quad 0 \leqslant k \leqslant 3$$

$$\Rightarrow \quad Y(k) = (4, j2, 0, -j2)$$

3.7.6 Symmetry: DFT of real even and real odd sequences

Let $x(n)$ be a length-N real sequence with an N-point DFT given by $X(k)$. If $x(n) = x_e(n) + x_o(n)$, where $x_e(n)$ is the even part and $x_o(n)$ is the odd part of the sequence $x(n)$, then DFT$\{x_e(n)\}$ is purely real and DFT$\{x_o(n)\}$ is purely imaginary.

Proof:

We know that,

$$x_e(n) \triangleq \frac{1}{2}[x(n) + x((-n))_N]$$

Hence,

$$\text{DFT}\{x_e(n)\} = \frac{1}{2}\text{DFT}\{x(n)\} + \frac{1}{2}\text{DFT}\{x((-n))_N\}$$

$$\Rightarrow \quad \text{DFT}\{x_e(n)\} = \frac{1}{2}X(k) + \frac{1}{2}X((-k))_N$$

$$= \frac{1}{2}[X(k) + X((-k))_N]$$

$$= \frac{1}{2}\left[X(k) + X^*(k)\right]$$

Let, $X(k) = A + jB$

Then, $X^*(k) = A - jB$

Hence, $\text{DFT}\{x_e(n)\} = \frac{1}{2}[A + jB + A - jB]$

$$\Rightarrow \text{DFT}\{x_e(n)\} = A$$

Thus, we have proved that the DFT of a real even sequence is purely real.
By definition,

$$x_o(n) = \frac{1}{2}[x(n) - x((-n))_N]$$

$$\Rightarrow X_o(k) = \frac{1}{2}\left[X(k) - X^*(k)\right]$$

$$\Rightarrow X_o(k) = \frac{1}{2}[A + jB - A + jB]$$

Hence, $X_o(k) = jB$

Thus, we find that the DFT of a real odd sequence is purely imaginary.

Example 3.25 Consider the following sequences of length-8 defined for $0 \leqslant n \leqslant 7$.

a. $x_1(n) = (2, 2, 2, 0, 0, 0, 2, 2)$
b. $x_2(n) = (2, 2, 0, 0, 0, 0, -2, -2)$
c. $x_3(n) = (0, 2, 2, 0, 0, 0, -2, -2)$
d. $x_4(n) = (0, 2, 2, 0, 0, 0, 2, 2)$

Which sequences have a real-valued 8-point DFT? Which sequences have an imaginary-valued 8-point DFT?

□ Solution

a. To circularly fold $x_1(n)$, enter the sequence $x_1(n)$ in the clockwise direction along the circumference of a circle with an equal spacing between successive points and read the sequence anticlockwise as shown in Fig. Ex.3.25.

Thus, $x_1((-n))_8 = (2, 2, 2, 0, 0, 0, 2, 2)$

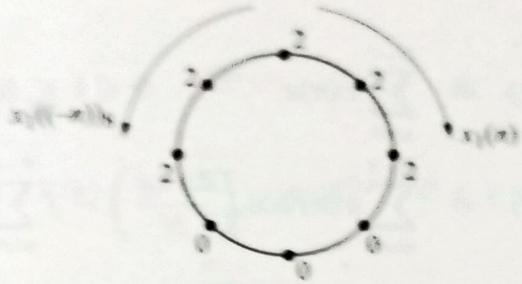


Fig. Ex.3.25 Concept of circular folding.

In the present context, we find that $x_1(n) = x_1((-n))_8$ and hence $x(n)$ is an even sequence. Also, $x(n)$ is a real sequence and hence $X(k)$ will be purely real.

b. Given

$$x_2(n) = (2, 2, 0, 0, 0, 0, -2, -2)$$

The circularly folded sequence is found to be

$$x_2((-n))_8 = (2, -2, -2, 0, 0, 0, 0, 2)$$

Since $x_2(n)$ is neither odd nor even, $\text{DFT}\{x_2(n)\} = X_2(k)$ is neither purely real nor purely imaginary.

c. Given

$$x_3(n) = (0, 2, 2, 0, 0, 0, -2, -2)$$

The circularly folded sequence is

$$x_3((-n))_8 = (0, -2, -2, 0, 0, 0, 2, 2)$$

Since $x_3(n) = -x_3((-n))_8$, the sequence $x_3(n)$ is an odd sequence. Also, $x_3(n)$ is a real sequence. Hence, $\text{DFT}\{x_3(n)\}$ is purely imaginary.

d. Given

$$x_4(n) = (0, 2, 2, 0, 0, 0, 2, 2)$$

The circularly folded sequence is

$$x_4((-n))_8 = (0, 2, 2, 0, 0, 0, 2, 2)$$

Since $x_4(n) = x_4((-n))_8$, it is an even sequence and being real, its DFT is purely real.

Example 3.26 If $x(n)$ is real and even, then show that its DFT reduces to the following form:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1$$

Solution

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right) - j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Since, $x(n)$ is an even sequence and $\sin\left(\frac{2\pi kn}{N}\right)$ is an odd sequence, their product is an odd sequence. If this odd sequence is summed over one period of $x(n)$, the result is zero.

Hence,
$$X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1$$

Example 3.27 If $x(n)$ is real and odd, then show that:

$$\begin{aligned} \text{DFT}\{x(n)\} &= X(k) \\ &= -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

Solution

$$\begin{aligned} \text{DFT}\{x(n)\} &= X(k) \\ &\triangleq \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

$$\Rightarrow \quad X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right) - j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Since, $x(n)$ is an odd sequence and $\cos\left(\frac{2\pi kn}{N}\right)$ is an even sequence, their product is an odd sequence. If this odd sequence is summed over one period of $x(n)$, the result is zero.

Hence,
$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1$$

3.7.7 DFT of a complex conjugate sequence

Let $x(n)$ be a complex sequence with

$$\text{DFT}\{x(n)\} = X(k), \quad 0 \leqslant k \leqslant N-1$$

Then,
$$\text{DFT}\{x^*(n)\} = X^*(N-k) = X^*((-k))_N$$

Proof

By definition, the DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \tag{3.12}$$

Changing k to $-k$ gives

$$X^*(-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{kn}$$

Since the folding is circular in nature, the above equation may be written as

$$X^*((-k))_N = \text{DFT}\{x^*(n)\} \tag{3.13}$$

Changing k to $N-k$ in equation (3.12) gives

$$\begin{aligned} X^*(N-k) &= \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n} \\ &= \sum_{n=0}^{N-1} x^*(n) W_N^{-Nn} W_N^{kn} \\ &= \sum_{n=0}^{N-1} x^*(n) W_N^{kn} \quad (\because W_N^{-Nn} = 1) \end{aligned}$$

$$\Rightarrow \quad \text{DFT}\{x^*(n)\} = X^*(N-k) \tag{3.14}$$

From equations (3.13) and (3.14), we can write

$$\text{DFT}\{x^*(n)\} = X^*((-k))_N = X^*(N-k)$$

Example 3.28 The 5-point DFT of a complex sequence $x(n)$ is given as

$$X(k) = (j, 1+j, 1+j2, 2+j2, 4+j)$$

Compute $Y(k)$, if $y(n) = x^*(n)$.

Solution

$$\begin{aligned} Y(k) &= \text{DFT}\{y(n)\} \\ &= \text{DFT}\{x^*(n)\} \\ &= X^*((-k))_5 \end{aligned}$$

Proof:

By definition, the DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \tag{3.12}$$

Changing k to $-k$ gives

$$X^*(-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{kn}$$

Since the folding is circular in nature, the above equation may be written as

$$X^*((-k))_N = \text{DFT}\{x^*(n)\} \tag{3.13}$$

Changing k to $N-k$ in equation (3.12) gives

$$\begin{aligned} X^*(N-k) &= \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n} \\ &= \sum_{n=0}^{N-1} x^*(n) W_N^{-Nn} W_N^{kn} \\ &= \sum_{n=0}^{N-1} x^*(n) W_N^{kn} \quad (\because W_N^{-Nn} = 1) \end{aligned}$$

$$\Rightarrow \quad \text{DFT}\{x^*(n)\} = X^*(N-k) \tag{3.14}$$

From equations (3.13) and (3.14), we can write

$$\text{DFT}\{x^*(n)\} = X^*((-k))_N = X^*(N-k)$$

Example 3.28 The 5-point DFT of a complex sequence $x(n)$ is given as

$$X(k) = (j, 1+j, 1+j2, 2+j2, 4+j)$$

Compute $Y(k)$, if $y(n) = x^*(n)$.

Solution

$$\begin{aligned} Y(k) &= \text{DFT}\{y(n)\} \\ &= \text{DFT}\{x^*(n)\} \\ &= X^*((-k))_5 \end{aligned}$$

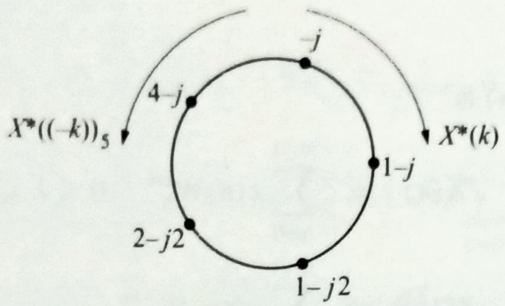


Fig. Ex.3.28 Concept of circular folding.

To find $X^*((-k))_5$, enter the sequence $X^*(k)$ on a circle clockwise and then read the sequence anticlockwise.

Thus, $$Y(k) = (-j, 4-j, 2-j2, 1-j2, 1-j)$$

Example 3.29 Consider the sequence

$$x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$$

a. Find the 6-point DFT of the sequence $x(n)$.

b. Find the finite length sequence $y(n)$, which has a DFT equal to the real part of $X(k)$.

☐ Solution

a. We know that, $$W_N = e^{-j\frac{2\pi}{N}}$$

Since $N = 6$, we get $$W_6 = e^{-j\frac{2\pi}{6}}$$

Therefore,

$$W_6^0 = 1$$

$$W_6^1 = \angle -\tfrac{2\pi}{6} = 0.5 - j0.866$$

$$W_6^2 = 1\angle -\tfrac{4\pi}{6} = -0.5 - j0.866$$

$$W_6^3 = 1\angle -\pi = -1$$

$$W_6^4 = 1\angle -\tfrac{8\pi}{6} = -0.5 + j0.866$$

$$W_6^5 = 1\angle -\tfrac{10\pi}{6} = 0.5 + j0.866$$

We know by definition that,

$$\mathrm{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(k) = \sum_{n=0}^{5} x(n) W_6^{kn}, \quad 0 \leqslant k \leqslant 5$$

$$= \sum_{n=0}^{5} [4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)]\, W_6^{kn}$$

Applying sifting property, we get

$$\begin{aligned} X(k) &= 4W_6^{kn}\big|_{n=0} + 3W_6^{kn}\big|_{n=1} + 2W_6^{kn}\big|_{n=2} + W_6^{kn}\big|_{n=3} \\ &= 4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}, \quad k = 0, 1, 2, 3, 4, 5 \end{aligned}$$

Hence,

$$\begin{aligned} X(0) &= 4 + 3 + 2 + 1 = 10 \\ X(1) &= 4 + 3W_6^1 + 2W_6^2 + W_6^3 = 3.5 - j4.33 \\ X(2) &= 4 + 3W_6^2 + 2W_6^4 + W_6^0 = 2.5 - j0.866 \\ X(3) &= 4 + 3W_6^3 + 2W_6^0 + W_6^3 = 2 \\ X(4) &= 4 + 3W_6^4 + 2W_6^2 + W_6^0 = 2.5 + j0.866 \\ X(5) &= 4 + 3W_6^5 + 2W_6^4 + W_6^3 = 3.5 + j4.33 \end{aligned}$$

Since $x(n)$ is a real sequence, we find that $X(k) = X^*(N-k)$ is satisfied.

b. Given

$$Y(k) = \text{Real}\{X(k)\}$$

$$\Rightarrow \quad Y(k) = \frac{1}{2}\left[X(k) + X^*(k)\right]$$

Hence,

$$y(n) = \frac{1}{2}[x(n)] + \frac{1}{2}\,\text{IDFT}\{X^*(k)\}$$

We know that

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \times 1$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \times W_N^{Nk} \quad (\because W_N^{Nk} = 1)$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{(N-n)k}$$

Put

$$N - n = m$$

Then,

$$X^*(k) = \sum_{m=N}^{1} x^*(N-m) W_N^{km}$$

Since m is a dummy variable, it can be replaced by n. Also because of implicit periodicity of $x(n)$, the limit of summation can be changed as follows.

$$
\begin{aligned}
X^*(k) &= \sum_{n=0}^{N-1} x^*(N-n)W_N^{kn} \\
\Rightarrow \quad X^*(k) &= \text{DFT}\{x^*(N-n)\} \\
&= \text{DFT}\{x^*((-n))_N\}
\end{aligned}
$$

Hence,
$$x^*((-n))_N = \text{IDFT}\{X^*(k)\}$$

Thus, we get
$$
\begin{aligned}
y(n) &= \frac{1}{2}[x(n)] + \frac{1}{2}\left[x^*((-n))_N\right] \\
\Rightarrow \quad y(n) &= \frac{1}{2}[(4, 3, 2, 1, 0, 0) + (4, 0, 0, 1, 2, 3)] \\
&= (4, 1.5, 1, 1, 1, 1.5)
\end{aligned}
$$

3.7.8 Circular convolution in time

Let $x(n)$ and $h(n)$ be two sequences of length N.

Then,
$$
\begin{aligned}
y(n) &= x(n) \circledast_N h(n) \\
&= \sum_{m=0}^{N-1} x((n-m))_N \, h(m), \quad 0 \leqslant n \leqslant N-1 \\
&= \sum_{m=0}^{N-1} x(m) \, h((n-m))_N
\end{aligned}
$$

***Proof*:**

The above result can be proved by making use of the block diagram shown in Fig. 3.8.

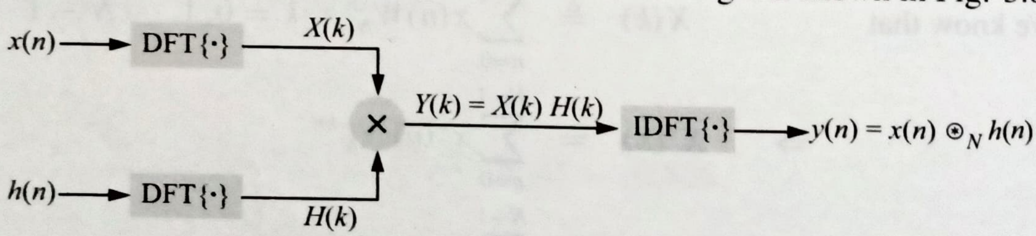


Fig. 3.8 Block diagram used for proving circular convolution.

From Fig. 3.8, we can write

$$Y(k) = X(k)\,H(k)$$

Hence,
$$
\begin{aligned}
y(n) &= \text{IDFT}\{X(k)H(k)\} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k)H(k)W_N^{-kn}, \quad n = 0, 1, \cdots N-1 \qquad (3.15)
\end{aligned}
$$

where
$$X(k) = \sum_{i=0}^{N-1} x(i)W_N^{ik}, \quad k = 0, 1, \cdots N-1$$