■ 0.1 Hit-and-Run

The ball walk does not mix rapidly from all starting points. While this hurdle can be overcome by starting with a deep point and carefully maintaining a warm start, it is natural to ask if there is a simple process that does truly mix rapidly from any starting point. Hit-and-Run satisfies this requirement.

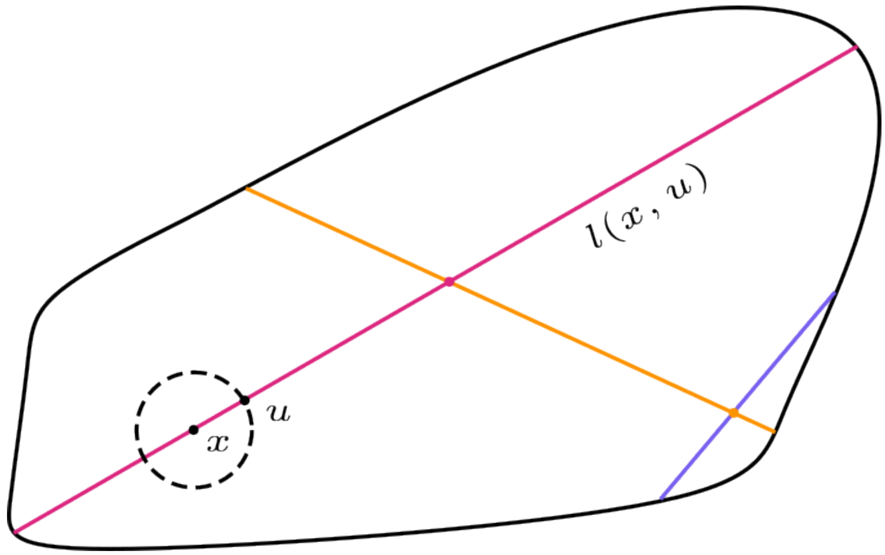


Figure 1: Hit-and-Run Algorithm

Algorithm 1: `Hit − and − Run`

Input: starting point x_0 in a convex body K.
Repeat T times: at current point x,

1. Pick a uniform random direction ℓ through x.
2. Go to uniform random point y on the chord of K induced by ℓ.

return x.

Since hit-and-run is a symmetric Markov chain, the uniform distribution on K is stationary for it. To sample from a general density proportional to $f(x)$, in Step 2, we sample y according to the density proportional to f restricted to the random line ℓ.

Next we give a formula for the next step distribution from a point u.

Lemma 0.1. *The next step distribution of Hit-and-Run from a point u is given by*

$$P_u(A) = \frac{2}{(S^{n-1})} \int_A \frac{dx}{x - u^{n-1}\ell(u,x)}$$

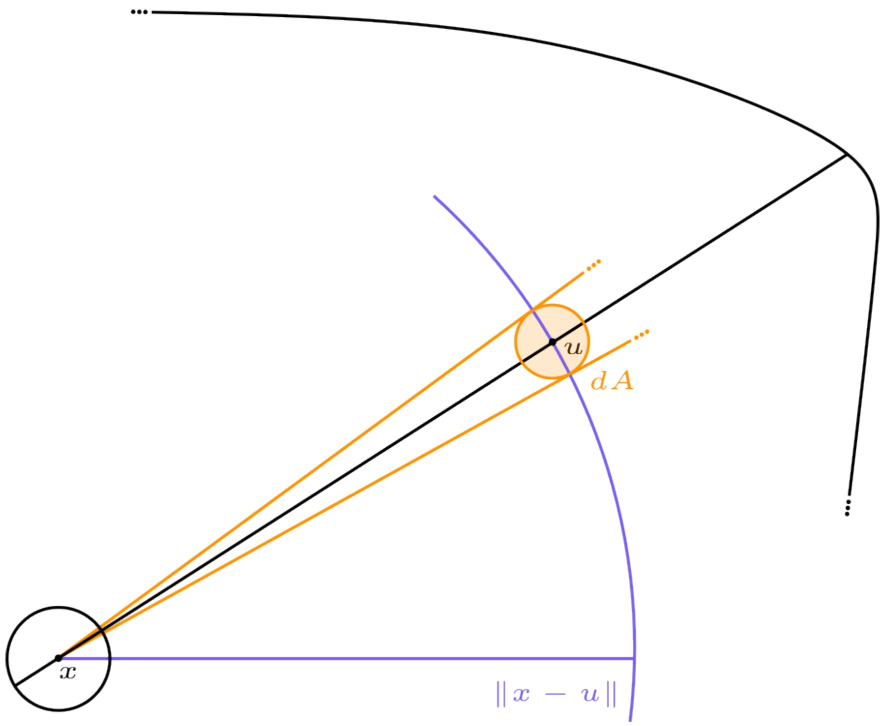
where A is any measurable subset of K and $\ell(u,x)$ is the length of the chord in K through u and x.

Exercise 0.2. Prove Lemma 0.1.

The main theorem of this section is the following [**?**].

Theorem 0.3. *[**?**]The conductance of Hit-and-Run in a convex body K containing the unit ball and of diameter D is $\Omega(1/nD)$.*

This implies a mixing time of $O\left(n^2 D^2 \log(M/\varepsilon)\right)$ to get to within distance ε of the target density starting from an M-warm initial density. By taking one step from the initial point, we can bound M by $(D/d)^n$ where d is the minimum distance of the starting point from the boundary. Hence this gives a bound of $\tilde{O}(n^3 D^2)$ from *any* interior starting point.



The proof of the theorem follows the same high-level outline as that of the ball walk, needing two major ingredients, namely, one-step coupling and isoperimetry. Notably, the isoperimetry is for a non-Euclidean notion of distance. We begin with some suitable definitions.

Define the "median" step-size function F as the $F(x)$ such that

$$\Pr(x - y \leq F(x)) = \frac{1}{8}$$

where y is a random step from x.

We also need a non-Euclidean notion of distance, namely the classical cross-ratio distance. For points u, v in K, inducing a chord $[p, q]$ with these points in the order p, u, v, q, the cross-ratio distance is

$$d_K(u, v) = \frac{u - vp - q}{p - uv - q}.$$

It is related to the Hilbert distance (which is a true distance) as follows:

$$d_H(u, v) = \ln\left(1 + d_K(u, v)\right).$$

The first ingredient shows that if two points are close geometrically, then their next-step distributions have significant overlap.

Lemma 0.4. *For two points $u, v \in K$ with*

$$d_K(u, v) < \frac{1}{8} \text{ and } u - v \leq \frac{2}{\sqrt{n}} \max\left\{F(u), F(v)\right\}$$

we have $d_{TV}(P_u, P_v) < 1 - \frac{1}{500}$.

The second ingredient is an isoperimetric inequality (independent of any algorithm). The cross-ratio distance has a nice isoperimetry inequality.

Theorem 0.5. *For any partition S_1, S_2, S_3 of a convex body K,*

$$(S_3) \geq d_K(S_1, S_2) \frac{(S_1)(S_2)}{(K)}.$$

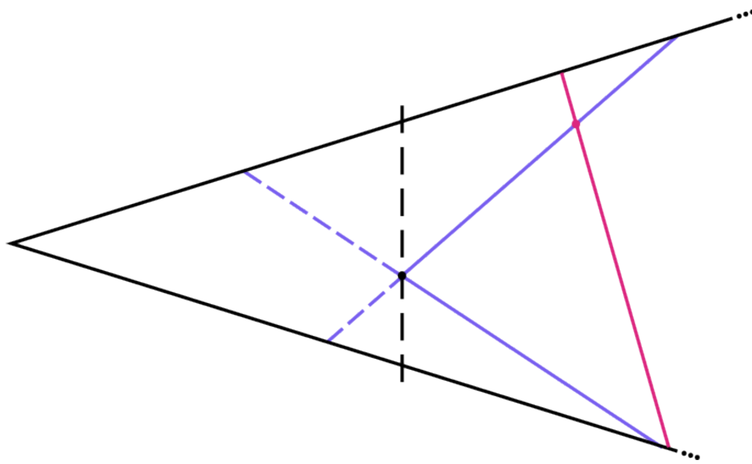
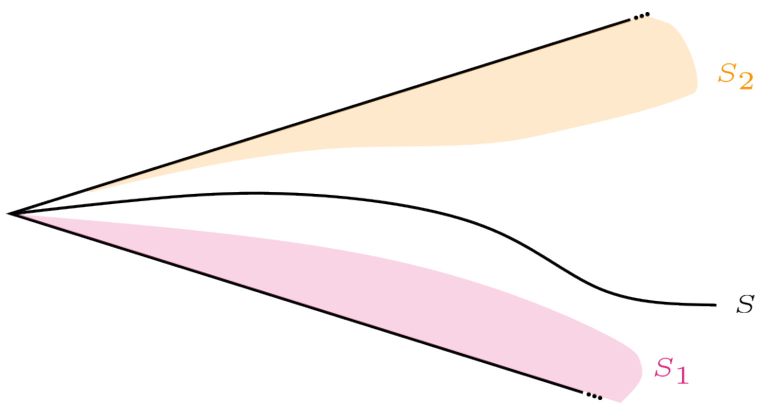


Figure 2: Hit-and-Run from a corner



However, this will not suffice to prove a bound on the conductance of all subsets. The reason is that we cannot guarantee a good lower bound on the *minimum* distance between subsets S_1, S_2. Instead, we will need a weighted isoperimetric inequality, which uses an average distance.

Theorem 0.6. *Let S_1, S_2, S_3be a partition of a convex body K. Let $h : K \to_+$ be a function s.t. for any $u \in S_1$,$v \in S_2$, and any x on the chord through u and v, we have*

$$h(x) \leq \frac{1}{3} \min \{1, d_K(u,v)\} .$$

Then,

$$(S_3) \geq_K (h(x)) \min \{(S_1), (S_2)\} .$$

For bounding the conductance, we will use a specific function h. To introduce it, we first define a step-size function $s(x)$:

$$s(x) = \sup \left\{ t : \frac{(x + tB^n \cap K}{(tB^n)} \geq \gamma \right\}$$

for some fixed $\gamma \in (0, 1]$.

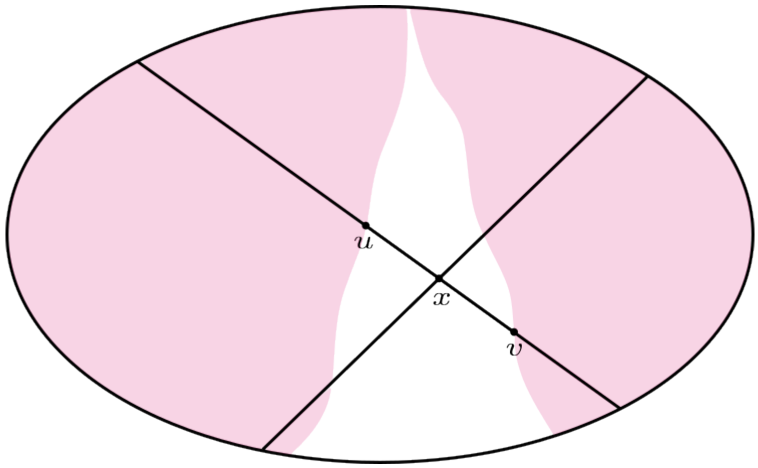
Exercise 0.7. Show that the step-size function is concave over any convex body.

We will need the following relationship between the step-size function and the median step function.

Lemma 0.8. *We have*

$$_K(s(x)) \geq \frac{1-\gamma}{\sqrt{n}}.$$

Moreover, for $\gamma \geq 63/64$, we have $F(x) \geq s(x)/32$.



In the proof of Theorem 0.3, we will set $h(x) = \frac{s(x)}{48\sqrt{n}D}$. We are now ready for that proof.

Proof of Thm. 0.3. Let $K = S_1 \cup S_2$ be a partition of K into measurable sets. We will prove that

$$\int_{S_1} P_x(S_2)\, dx \quad \geq \quad \frac{c}{nD} \min\{(S_1), (S_2)\} \tag{1}$$

Note that since the uniform distribution is stationary,

$$\int_{S_1} P_x(S_2)\, dx = \int_{S_2} P_x(S_1)\, dx.$$

Consider the points that are "deep" inside these sets, i.e., unlikely to jump out of the set:

$$S_1' = \left\{x \in S_1 : P_x(S_2) < \frac{1}{1000}\right\} \text{ and } S_2' = \left\{x \in S_2 : P_x(S_1) < \frac{1}{1000}\right\}.$$

Let S_3' be the rest i.e., $S_3' = K \setminus S_1' \setminus S_2'$.

Suppose $(S_1') < (S_1)/2$. Then

$$\int_{S_1} P_x(S_2)\, dx \geq \frac{1}{1000}(S_1 \setminus S_1') \geq \frac{1}{2000}(S_1)$$

which proves (1).

So we can assume that $(S_1') \geq (S_1)/2$ and similarly $(S_2') \geq (S_2)/2$. Now, for any $u \in S_1'$ and $v \in S_2'$,

$$d_{TV}(P_u, P_v) \geq 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{1}{500}.$$

Applying Lemma 0.4, we get that that one of the following holds:

$$d_K(u, v) \geq \frac{1}{8} \text{ or } u - v \geq \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}$$

We will now prove that Thm. 0.6 using $h(x) = \frac{s(x)}{48\sqrt{n}D}$ where $s(x)$ is defined with $\gamma = 63/64$. To see that this is a valid choice, first note that if $d_K(u, v) \geq \frac{1}{8}$,then we have $h(x) \leq d_K(u, v)/3$, as needed. So we can assume the second condition above holds. Next, noting that x is some point on the chord through u, v, let the endpoints of the chord be p, q. Suppose WLOG that $x \in [u, q]$. Then, by the concavity of $s(x)$, and using the second part of Lemma 0.8, we have,

$$
\begin{aligned}
s(x) &\le \frac{|x-p|}{|u-p|} s(u) \\
&\le 32\frac{|x-p|}{|u-p|} F(u) \\
&\le 16\sqrt{n}\frac{|x-p|}{|u-p|}\,|u-v| \\
&\le 16 d_K(u,v)\sqrt{n} D
\end{aligned}
$$

which again implies the desired condition on h.

Now, applying Theorem 0.6 to the partition S_1', S_2', S_3', and using the first part of Lemma 0.8, we have

$$
\begin{aligned}
(S_3') &\ge {}_K(h(x)) \min\{(S_1'), (S_2')\} \\
&\ge \frac{1}{4000nD} \min\{(S_1), (S_2)\}.
\end{aligned}
$$

We can now prove (1) as follows:

$$
\begin{aligned}
\int_{S_1} P_x(S_2)\,dx &= \frac{1}{2}\int_{S_1} P_x(S_2)\,dx + \frac{1}{2}\int_{S_2} P_x(S_1)\,dx \\
&\ge \frac{1}{2}(S_3')\frac{1}{1000} \\
&\ge \frac{1}{2^{23}nD} \min\{(S_1'), (S_2')\} \\
&\ge \frac{1}{2^{24}nD} \min\{(S_1), (S_2)\}.
\end{aligned}
$$

□

■ 0.2 Dikin walk

Both the ball walk and hit-and-run have a dependence on the "roundness" of the target distribution, e.g., via its diameter or average distance to the center of gravity. Reducing this dependence to logarithmic by rounding is polynomial time but expensive. The current best rounding algorithm (which achieves near-isotropic position) has complexity $O^*(n^3)$. Here we describe a different approach, which is affine-invariant, but requires more knowledge of the convex body. In particular, we will focus on the special case of sampling an explicit polytope $P = \{x : Ax \ge b\}$.

The general Dikin walk is defined as follows. For a convex set K with a positive definite matrix (u) for each point $u \in K$, let

$$
E_u(r) = \left\{x : (x-u)^\top (u)(x-u) \le r\right\}.
$$

Algorithm 2: `DikinWalk`

Input: starting point x_0 in a polytope $P = \{x : \mathbf{A}x \ge b\}$.
Set $r = \frac{1}{8}$.
Repeat T times: at current point x,

1. Pick y from $E_x(r)$.
2. Go to y with probability $\min\left\{1, \frac{(E_x(r))}{(E_y(r))}\right\}$.

return x.

0.2.1 Strong Self-Concordance

We require a family of matrices to have the following properties. Usually but not necessarily, these matrices come from the Hessian of some convex function.

Definition 0.9 (Symmetric self-concordance)**.** For any convex set $K \subset$, we call a matrix function $\mathbf{H} : K \rightarrow^{n\times n}$ is self-concordant if for any $u \in K$, we have

$$-2\|h\|_{\mathbf{H}(u)}\mathbf{H}(u) \preceq \frac{d}{dt}\mathbf{H}(u+th) \preceq 2\|h\|_{\mathbf{H}(u)}\mathbf{H}(u).$$

We call $\mathbf{H}$ is a symmetric $\bar{\nu}$-self-concordant barrier if is self-concordant and for any $u \in K$,

$$E_u(1) \subseteq K \cap (2u - K) \subseteq E_u(\sqrt{\bar{\nu}}).$$

The following lemma shows that self-concordant matrix functions also enjoy a similar regularity as the usual self-concordant functions.

Lemma 0.10. *Given any self-concordant matrix function on $K \subset$, we define $\|v\|_x^2 = v^\top \mathbf{H}(x)v$. Then, for any $x, y \in K$ with $\|x-y\|_x < 1$, we have*

$$\left(1 - \|x-y\|_x\right)^2 \mathbf{H}(x) \preceq \mathbf{H}(y) \preceq \frac{1}{\left(1-\|x-y\|_x\right)^2}\mathbf{H}(x).$$

Proof. Let $h = y - x$, $x_t = x + th$ and $\phi(t) = h^\top(x_t)h$. Then,

$$|\phi'(t)| = \left|h^\top \frac{d}{dt}(x_t)h\right| \le 2\|h\|_{x_t}^3 = 2\phi(t)^{3/2}.$$

Hence, we have $\left|\frac{d}{dt}\frac{1}{\sqrt{\phi(t)}}\right| \le 1$. Therefore, we have $\frac{1}{\sqrt{\phi(t)}} \ge \frac{1}{\sqrt{\phi(0)}} - t$ and hence,

$$\phi(t) \le \frac{\phi(0)}{(1 - t\sqrt{\phi(0)})^2}. \tag{2}$$

Now we fix any v and define $\psi(t) = v^\top(x_t)v$. Then,

$$|\psi'(t)| = \left|v^\top \frac{d}{dt}(x_t)v\right| \le 2\|h\|_{x_t}\|v\|_{x_t}^2 = 2\phi(t)\psi(t).$$

Using (2) at the end, we have

$$\left|\frac{d}{dt}\ln\psi(t)\right| \le \frac{2\sqrt{\phi(0)}}{(1 - t\sqrt{\phi(0)})}.$$

Integrating both sides from 0 to 1,

$$\left|\ln\frac{\psi(1)}{\psi(0)}\right| \le \int_0^1 \frac{2\sqrt{\phi(0)}}{(1-t\sqrt{\phi(0)})}dt = 2\ln\left(\frac{1}{1-\sqrt{\phi(0)}}\right).$$

The result follows from this, $\psi(1) = v^\top(y)v$, $\psi(0) = v^\top(x)v$, and $\phi(0) = \|x-y\|_x^2$. □

Many natural barriers, including the logarithmic barrier and the LS-barrier, satisfy a much stronger condition than self-concordant. However, this is not always true, as one can construct counterexamples even in one-dimension.

Definition 0.11. For any convex set $K \subset$, we say a matrix function $\mathbf{H} : K \rightarrow^{n\times n}$ is strongly self-concordant if for any $u \in K$, we have

$$\mathbf{H}(x)^{-1/2}D\mathbf{H}(x)[h]\mathbf{H}(x)^{-1/2}{}_F \le 2h_x$$

where $D\mathbf{H}(x)[h]$ is the directional derivative of at x in the direction h.

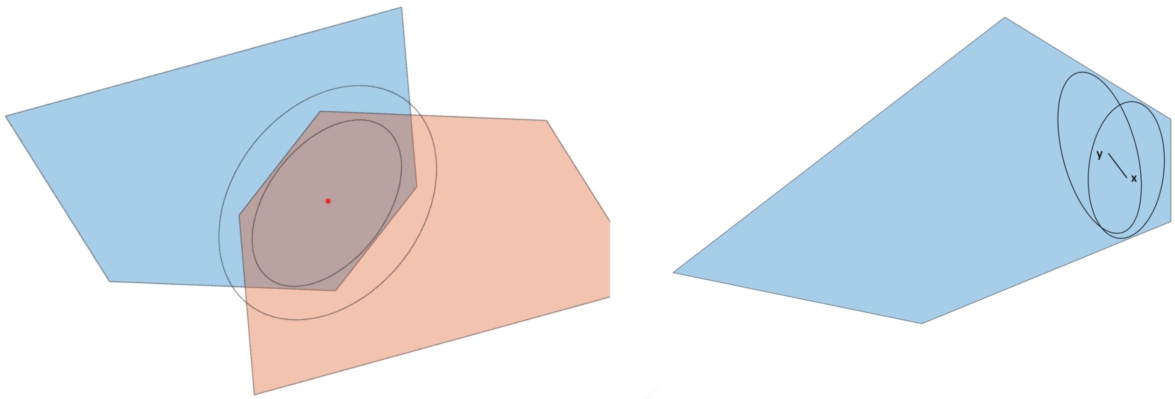


Figure 3: (a) $E_u(1) \subseteq K \cap (2u - K) \subseteq E_u(\sqrt{\bar{\nu}})$. (b) Strong self-concordance measures the rate of change of Hessian of a barrier in the Frobenius norm

Similar to Lemma 0.10, we have a global version of Lemma 0.10.

Lemma 0.12. *Given any strongly self-concordant matrix function on $K \subset$. For any $x, y \in K$ with $\|x - y\|_x <$ 1, we have*

$$\|(x)^{-\frac{1}{2}}((y) - (x))(x)^{-\frac{1}{2}}\|_F \leq \frac{\|x - y\|_x}{(1 - \|x - y\|_x)^2}.$$

Proof. Let $x_t = (1 - t)x + ty$. Then, we have

$$\|(x)^{-\frac{1}{2}}((y) - (x))(x)^{-\frac{1}{2}}\|_F = \int_0^1 \|(x)^{-\frac{1}{2}} \frac{d}{dt}(x_t)(x)^{-\frac{1}{2}}\|_F dt.$$

We note that is self-concordant. Hence, Lemma 0.10 shows that

$$\begin{aligned}
\|(x)^{-\frac{1}{2}} \frac{d}{dt}(x_t)(x)^{-\frac{1}{2}}\|_F^2 &= (x)^{-1} \left(\frac{d}{dt}(x_t)\right) (x)^{-1} \left(\frac{d}{dt}(x_t)\right) \\
&\leq \frac{1}{(1 - \|x - x_t\|_x)^4} (x_t)^{-1} \left(\frac{d}{dt}(x_t)\right) (x_t)^{-1} \left(\frac{d}{dt}(x_t)\right) \\
&\leq \frac{4}{(1 - \|x - x_t\|_x)^4} \|x - x_t\|_{x_t}^2 \\
&\leq \frac{4}{(1 - \|x - x_t\|_x)^6} \|x - x_t\|_x^2
\end{aligned}$$

where we used the assumption in the second inequality and Lemma 0.10 again for the last inequality. Hence,

$$\begin{aligned}
\|(x)^{-\frac{1}{2}}((y) - (x))(x)^{-\frac{1}{2}}\|_F &\leq \int_0^1 \frac{2\|x - x_t\|_x}{(1 - \|x - x_t\|_x)^3} dt \\
&= \int_0^1 \frac{2t\|x - y\|_x}{(1 - t\|x - y\|_x)^3} dt \\
&= \frac{\|x - y\|_x}{(1 - \|x - y\|_x)^2}.
\end{aligned}$$

$\square$

We note that strong self-concordance is stronger than self-concordance since the Frobenius norm is always larger or equal to the spectral norm. As an example, we will verify that the conditions hold for the standard log barrier (Lemma **??**).

The Dikin walk has the following guarantee.

thmdikingeneralThe mixing rate of the Dikin walk for a symmetric, strongly self-concordant matrix function with convex log determinant is $O(n\bar{\nu})$.

Each step of the standard Dikin walk is fast, and does not need matrix multiplication.

Theorem 0.13. *The Dikin walk with the logarithmic barrier for a polytope $\{\mathbf{A}x \geq b\}$ can be implemented in time $O((\mathbf{A}) + n^2)$ per step while maintaining the mixing rate of $O(mn)$.*

The next lemma results from studying strong self-concordance for classical barriers. The KLS constant below is conjectured to be $O(1)$ and known to be $O(\sqrt{\log n})$.

Lemma 0.14. *Let ψ_n be the KLS constant of isotropic logconcave densities in ^{n}, namely, for any isotropic logconcave density p and any set $S \subset$, we have*

$$\int_{\partial S} p(x)dx \geq \frac{1}{\psi_n} \min\left\{\int_S p(x)dx, \int_{^n \setminus S} p(x)dx\right\}.$$

Let (x) be the Hessian of the universal or entropic barriers. Then, we have

$$\mathbf{H}(x)^{-1/2} D\mathbf{H}(x)[h]\mathbf{H}(x)^{-1/2}{}_F = O(\psi_n)h_x.$$

In short, the universal and entropic barriers in ^{n} are strongly self-concordant up to a scaling factor depending on ψ_n.

In fact, the proof shows that up to a logarithmic factor the strong self-concordance of these barriers is *equivalent* to the KLS conjecture.

■ 0.3 Mixing with Strong Self-Concordance

A key ingredient of the proof of Theorem 0.2.1 is the following lemma.

Lemma 0.15. *For two points $x, y \in P$, with $\|x - y\|_x \leq \frac{1}{8\sqrt{n}}$, we have $d_{TV}(P_x, P_y) \leq \frac{3}{4}$.*

Proof. We have to prove two things: first, the rejection probability is small, second the ellipsoids used by the Dikin walk at x, y have large overlap. More precisely, we have

$$\begin{aligned} d_{\mathrm{TV}}(P_x, P_y) &\leq \frac{1}{2}\mathrm{rej}_x + \frac{1}{2}\mathrm{rej}_y + \frac{1}{2}\frac{(P_x \setminus P_y)}{(P_x)} + \frac{1}{2}\frac{(P_y \setminus P_x)}{(P_y)} \\ &= \frac{1}{2}\mathrm{rej}_x + \frac{1}{2}\mathrm{rej}_y + 1 - \frac{1}{2}\frac{(P_x \cap P_y)}{(P_x)} - \frac{1}{2}\frac{(P_x \cap P_y)}{(P_y)} \end{aligned} \tag{3}$$

where rej_x and rej_y are the rejection probabilities at x and y.

For the rejection probability at x, we consider the algorithm pick z from $E_x(r)$. Let $f(z) = \ln \det \mathbf{H}(z)$. The acceptance probability of the sample z is

$$\min\left\{1, \frac{(E_x(r))}{(E_z(r))}\right\} = \min\left\{1, \sqrt{\frac{\det(\mathbf{H}(z))}{\det(\mathbf{H}(x))}}\right\}. \tag{4}$$

By the assumption that f is a convex function, we have that

$$\ln \frac{\det(\mathbf{H}(z))}{\det(\mathbf{H}(x))} = f(z) - f(x) \geq \langle \nabla f(x), z - x \rangle. \tag{5}$$

To simplify the notation, we assume $(x) =$. Since z is sampled from unit ball of radius r centered at x, we know that

$$\P(v^\top(z-x) \geq -\epsilon r\|v\|_2) \geq 1 - e^{-n\epsilon^2/2}.$$

In particular, with probability at least 0.99 in z, we have

$$\langle \nabla f(x), z - x\rangle \geq -\frac{4r}{\sqrt{n}}\|\nabla f(x)\|_2. \tag{6}$$

To compute $\|\nabla f(x)\|_2^2$, it is easier to compute directional derivative of ∇f. Note that

$$\begin{aligned}
\|\nabla f(x)\|_2 &= \max_{\|v\|_2=1} \nabla f(x)^\top v \\
&= \max_{\|v\|_2=1} ((x)^{-1} D(x)[v]) \\
&= \max_{\|v\|_2=1} \left((x)^{-\frac{1}{2}} D(x)[v](x)^{-\frac{1}{2}}\right) \\
&\leq \max_{\|v\|_2=1} \sqrt{n}\|(x)^{-\frac{1}{2}} D(x)[v](x)^{-\frac{1}{2}}\|_F \\
&\leq \sqrt{n}
\end{aligned} \tag{7}$$

where the first inequality follows from $|\sum_{i=1}^n \lambda_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^n \lambda_i^2}$ and the second inequality follows from the definition of strong self-concordance.

Combining (4), (5), (6) and (7), we see that with probability at least 0.99 in z, acceptance probability of the sample z is

$$\min\left\{1, \frac{(E_x(r))}{(E_z(r))}\right\} \geq e^{-2r} \geq 0.77 \tag{8}$$

where we used that $r = \frac{1}{8}$. Hence, the overall rejection probability rej_x (and similarly rej_y) satisfies

$$\mathrm{rej}_x \leq 0.24 \qquad \text{and} \qquad \mathrm{rej}_y \leq 0.24. \tag{9}$$

Now, we bound the fraction of volume in the intersection of the ellipsoids at x, y. Again, we can assume that $(x) =$. Then, the strongly self-concordance and Lemma 0.12 shows that

$$\|(y) - \|_F \leq 2\|x - y\|_x \leq \frac{1}{4\sqrt{n}}. \tag{10}$$

In particular, we have that

$$\frac{1}{2} \preceq (y) \preceq \frac{3}{2}. \tag{11}$$

We partition the eigenvalues λ_i of (y) into those of value at least 1 and the rest. Then consider the ellipsoid E whose eigenvalues are $\min\{1, \lambda_i\}$. This is contained in both $E_x(1)$ and $E_y(1)$. We will see that (E) is a constant fraction of the volume of both $E_x(1)$ and $E_y(1)$. First, we compare E and $E_x(1)$

$$\frac{(E)}{(E_x(1))} = \prod_{i:\lambda_i<1} \lambda_i = \prod_{i:\lambda_i<1} (1 - (1 - \lambda_i)) \geq \exp\left(-2\sum_{i:\lambda_i<1}(1 - \lambda_i)\right) \tag{12}$$

where we used that $1 - x \geq \exp(-2x)$ for $0 \leq x \leq \frac{1}{2}$ and $\lambda_i \geq \frac{1}{2}$ (11). From the inequality (10), it follows that

$$\sqrt{\sum_i (\lambda_i - 1)^2} \leq \frac{1}{4\sqrt{n}}.$$

Therefore, $\sum_{i:\lambda_i<1} |\lambda_i - 1| \leq \frac{1}{4}$. Putting it into (12), we have

$$\frac{(P_x \cap P_y)}{(P_x)} = \frac{(E)}{(E_x(1))} \geq e^{-\frac{1}{2}}. \tag{13}$$

Similarly, we have

$$\frac{(P_x \cap P_y)}{(P_y)} = \frac{\prod_{i:\lambda_i<1}\lambda_i}{\prod_{i:\lambda_i}\lambda_i} = \frac{1}{\prod_{i:\lambda_i>1}\lambda_i} \geq \frac{1}{\exp(\sum_{i:\lambda_i>1}(\lambda_i-1))} \geq e^{-\frac{1}{4}}. \tag{14}$$

Putting (9), (13) and (14) into (3), we have

$$d_{\mathrm{TV}}(P_x, P_y) \leq \frac{0.24}{2} + \frac{0.24}{2} + 1 - \frac{e^{-\frac{1}{2}}}{2} + \frac{e^{-\frac{1}{4}}}{2} \leq \frac{3}{4}$$

□

The next lemma establishes isoperimetry. This only needs the symmetric containment assumption. The isoperimetry is for the cross-ratio distance. For a convex body K, and any two points $u, v \in K$,suppose that p, q are the endpoints of the chord through u, v in K, so that these points occur in the order p, u, v, q. Then, the *cross-ratio* distance between u and v is defined as

$$d_K(u,v) = \frac{\|u-v\|_2\|p-q\|_2}{\|p-u\|_2\|v-q\|_2}.$$

Tbis distance enjoyes the following isoperimetric inequality.

Theorem 0.16 ([?]). *For any convex body K, and disjoint subsets S_1, S_2 of it, and $S_3 = K \setminus S_1 \setminus S_2$,we have*

$$(S_3) \geq d_K(S_1, S_2)\frac{(S_1)(S_2)}{(K)}.$$

We now relate the cross-ratio distance to the ellipsoidal norm.

Lemma 0.17. *For $u, v \in K$, $d_K(u,v) \geq \frac{\|u-v\|_u}{\sqrt{\bar{\nu}}}$.*

Proof. Consider the ellipsoid at u. For the chord $[p,q]$ induced by u, v with these points in the order p, u, v, q, suppose that $\|p-u\|_2 \leq \|v-q\|_2$. Then by Lemma 0.2.1, $p \in K \cap (2u - K)$. And hence $\|p-u\|_u \leq \sqrt{\bar{\nu}}$. Therefore,

$$d_K(u,v) = \frac{\|u-v\|_2\|p-q\|_2}{\|p-u\|_2\|v-q\|_2} \geq \frac{\|u-v\|_2}{\|p-u\|_2} = \frac{\|u-v\|_u}{\|p-u\|_u} \geq \frac{\|u-v\|_u}{\sqrt{\bar{\nu}}}.$$

We can now prove the main conductance bound.

*

We follow the standard high-level outline [?]. Consider any measurable subset $S_1 \subseteq K$ and let $S_2 = K \setminus S_1$ be its complement. Define the points with low escape probability for these subsets as

$$S_i' = \left\{x \in S_i : P_x(K \setminus S_i) < \frac{1}{8}\right\}$$

and $S_3' = K \setminus S_1' \setminus S_2'$. Then, for any $u \in S_1'$, $v \in S_2'$, we have $d_{TV}(P_u, P_v) > 1 - \frac{1}{4}$. Hence, by Lemma 0.15, we have $\|u-v\|_u \geq \frac{1}{8\sqrt{n}}$. Therefore, by Lemma 0.17,

$$d_K(u,v) \geq \frac{1}{8\sqrt{n}\cdot\sqrt{\bar{\nu}}}.$$

We can now bound the conductance of S_1. We may assume that $(S_i') \geq (S_i)/2$; otherwise, it immediately follows that the conductance of S_1 is $\Omega(1)$. Assuming this, we have

$$\begin{aligned}\int_{S_1} P_x(S_2)\,dx &\geq \int_{S_3'} \frac{1}{8}dx \geq \frac{1}{8}(S_3') \\ \text{using isoperimetry (Theorem 0.16)} &\geq \frac{1}{8}d_K(S_1', S_2')\frac{(S_1')(S_2')}{(P)} \\ &\geq \frac{1}{512\sqrt{n\bar{\nu}}}\min\{(S_1),(S_2)\}.\end{aligned}$$

□