0.1 Volume Computation by Annealing

For volume computation, we can apply annealing as follows. We consider the following estimator that uses a random sample from the distribution with density proportional to f_i:

$$Y = \frac{f_{i+1}(X)}{f_i(X)}.$$

We see that

$$_{f_i}(Y) = \frac{\int f_{i+1}}{\int f_i}.$$

In the DFK algorithm, this ratio is bounded by a constant (in fact 2) in each phase, giving a total of roughly n phases since the ratio of final to initial integrals is exponential. Instead of uniform densities, we consider

$$f_i(x) = \exp(-a_i x)\chi_K(x) \text{ or } f_i(x) = \exp(-a_i x^2)\chi_K(x).$$

The coefficient a_i (inverse "temperature") will be changed by a factor of $(1+\frac{1}{\sqrt{n}})$ in each phase, which implies that $m = \widetilde{O}(\sqrt{n})$ phases suffice to reach the target distribution. This is perhaps surprising since the ratio of the initial integral to the final is typically $n^{\Omega(n)}$. Yet the algorithm uses only $\widetilde{O}(\sqrt{n})$ phases, and hence estimates a ratio of $n^{\tilde{\Omega}(\sqrt{n})}$ in one or more phases. In the algorithm below, we assume that $B_n \subseteq K \subseteq RB_n$ and $f(x) = \exp(-\|x\|^2/2)\chi_K(x)$.

Algorithm 1: `LV Volume`

1. For $i = 0, \ldots, m = \lceil \sqrt{n} \ln(4R^2 n/\varepsilon) \rceil$, let $a_i = 2n/(1+\frac{1}{\sqrt{n}})^i$ and $f_i = f(x)^{a_i}$.
2. For $i = 0, \ldots, m-1$,
   (a) sample $X^{(1)}, X^{(2)}, \ldots, X^{(N)}$ from π_{f_i}.
   (b) Set $Y_i = \frac{1}{N}\sum_{j=1}^{N} \frac{f_{i+1}(X^{(j)})}{f_i(X^{(j)})}$
3. Let $Y = \prod_{i=1}^{m-1} Y_i$ and return $Y \cdot (\pi/n)^{n/2}$.

The key insight is that even though the expected ratio might be large, its variance is not.

Lemma 0.1. *For $X \sim f_i$ with $f_i(x) = e^{-a_i x}\chi_K(x)$ for a convex body K, or $f_i(x) = f(x)^{a_i}$ for an integrable logconcave function f, we have that the estimator $Y = \frac{f_{i+1}(X)}{f_i(X)}$ satisfies*

$$\frac{(Y^2)}{(Y)^2} \leq \left(\frac{a_{i+1}^2}{(2a_{i+1}-a_i)a_i}\right)^n$$

which is bounded by a constant for $a_i = a_{i+1}\left(1+\frac{1}{\sqrt{n}}\right)$.

Proof. Let $F(a) = \int_K f(x)^a \, dx$. Then

$$(Y) = \frac{F(a_{i+1})}{F(a_i)} \quad \text{and} \quad (Y^2) = \int \frac{f(x)^{2a_{i+1}}}{f(x)^{2a_i}} \cdot \frac{f(x)^{a_i}}{F(a_i)} = \frac{F(2a_{i+1}-a_i)}{F(a_i)}.$$

$$\frac{(Y^2)}{(Y)^2} = \frac{F(2a_{i+1}-a_i)F(a_i)}{F(a_{i+1})^2}.$$

By Lemma **??**, F is logconcave in a and hence

$$\frac{F(2a_{i+1}-a_i)F(a_i)}{F(a_{i+1})^2} \leq \left(\frac{a_{i+1}^2}{(2a_{i+1}-a_i)a_i}\right)^n.$$

For $a_i = a_{i+1}(1 + \frac{1}{\sqrt{n}})$, we have

$$\frac{(Y^2)}{(Y)^2} \le \left(\frac{1/(1+\frac{1}{\sqrt{n}})^2}{(\frac{2}{1+\frac{1}{\sqrt{n}}} - 1)} \right)^n = \left(\frac{1}{(1+\frac{1}{\sqrt{n}})(1-\frac{1}{\sqrt{n}})} \right)^n = \left(1 + \frac{1}{n-1}\right)^n \le 4$$

for $n \ge 2$. ☐

Theorem 0.2. *Let $r = (Y)$ in the LV algorithm. With probability at least $9/10$, we have $|Y - r| \le \frac{\varepsilon}{2} r$.*

Proof. By Lemmas **??** and **??**, we have

$$\begin{aligned}
\frac{(Y)}{(Y)^2} &= \prod_{i=0}^{m-1} \left(1 + \frac{(Y_i)}{(Y_i)^2}\right) - 1 \\
&\le \left(1 + \frac{\varepsilon^2}{32m}\right)^m - 1 \\
&\le \frac{\varepsilon^2}{16}.
\end{aligned}$$

Now the result follows from Chebychev's inequality. ☐

We can now state the main result of this section.

Theorem 0.3 ([?]). *The LV algorithm estimates the volume of a convex body in ^{n} given by a membership oracle to relative error ε with probability at least $3/4$ using $\widetilde{O}(n^4/\varepsilon^2)$ oracle queries and $\widetilde{O}(n^2)$ arithmetic operations per query.*

Proof. We first note that since f_0 corresponds to a Gaussian of variance $1/2n$, we have

$$\int_{B^n} f_0(x) \ge (1 - 2^{-n}) \int_n f_0(x)\, dx = (1 - 2^{-n}) \left(\frac{\pi}{n}\right)^{n/2}.$$

Next, $a_m \le \varepsilon/(4R^2)$ and hence

$$f_m(x) = e^{-\frac{a_m}{2}x^2} \ge e^{-\varepsilon/4}.$$

Together with the previous theorem, this shows that the algorithm's output is a relative ε approximation of the volume of the input convex body with probability at least $3/4$.

For the complexity, the algorithm has m phases with $O(m/\varepsilon^2)$ samples per phase. The number of membership queries per sample is $\widetilde{O}(n^2R^2)$ from an L_2-warm start using the hit-and-run algorithm for sampling. By Lemma **??**, samples from the i'th phase provide an ℓ_2-warm start for the $(i+1)$'st phase. This gives an overall complexity of $O^*(n^3R^2)$. By Theorem **??**, we can transform the body to near-isotropic position so that $R^2 = O(n)$, establishing the theorem. ☐

The LV algorithm has two parts. In the first it finds a transformation that puts the body in near-isotropic position. The complexity of this part is $\widetilde{O}(n^4)$. In the second part, it runs the annealing schedule, while maintaining that the distribution being sampled is *well-rounded*, a weaker condition than isotropy. Well-roundedness requires that a level set of measure $\frac{1}{8}$ contains a constant-radius ball and the trace of the covariance (expected squared distance of a random point from the mean) to be bounded by $O(n)$, so that R/r is effectively $O(\sqrt{n})$. To achieve the complexity guarantee for the second phase, it suffices to use the KLS bound of $\psi_p \gtrsim n^{-\frac{1}{2}}$. Connecting improvements in the Cheeger constant directly to the complexity of volume computation was an open question for a couple of decades. To apply improvements in the Cheeger constant, one would need to replace well-roundedness with (near-)isotropy and maintain that. However, maintaining isotropy appears to be much harder — possibly requiring a sequence of $\Omega(n)$ distributions and $\Omega(n)$ samples from each, providing no gain over the current complexity of $O^*(n^4)$ even if the KLS conjecture turns out to be true.

A faster algorithm is known for well-rounded convex bodies (any isotropic logconcave density satisfies $\frac{R}{r} = O(\sqrt{n})$ and is well-rounded). This variant of simulated annealing, called Gaussian cooling utilizes the fact that the KLS conjecture holds for a Gaussian density restricted by any convex body, and completely avoids computing an isotropic transformation.

Theorem 0.4 ([?]). *The volume of a well-rounded convex body, i.e., with $R/r = O^*(\sqrt{n})$, can be computed using $O^*(n^3)$ oracle calls.*

0.2 Integration by Annealing

The reader will notice that the LV algorithm of the previous section for volume computation is naturally suited to the more general problem of integrating logconcave functions. In fact, we can apply it directly, with minimal changes. The first step of the algorithm is an affine transformation that we will discuss in the next section.

Algorithm 2: `LV Integration`

1. Put π_f in near-isotropic position.
2. Restrict f to the convex body $K = B(0, 4\sqrt{n}) \cap L_f(M_f e^{-4n})$.
3. Let Y_0 be an estimate of the volume of K.
4. For $i = 0, \ldots, m-1$, let $a_i = \frac{1}{B}(1 + \frac{1}{\sqrt{n}})^i$, $a_m = 1$ and $f_i = f(x)^{a_i}$, with $m-1$ being the largest integer s.t. $a_{m-1} < 1$.
5. For $i = 0, \ldots, m-1$,
   (a) sample $X^{(1)}, X^{(2)}, \ldots, X^{(N)}$ from π_{f_i}.
   (b) Set $Y_i = \frac{1}{N} \sum_{j=1}^{N} f(x)^{a_{i+1} - a_i}$
6. Let $Y = \prod_{i=1}^{m-1} Y_i$ and return $Y_o \cdot Y$.

Theorem 0.5. *The LV algorithm applied to an integrable logconcave function $f :^n \to$, estimates the integral of f to relative error ε with probability at least $3/4$ using $\widetilde{O}(n^4/\varepsilon^2)$ evaluation oracle queries and $\widetilde{O}(n^2)$ arithmetic operations per query.*