The Mathematical Necessity of Flame Geometry: A Coupled Nonlinear PDE Perspective on Buoyant Reaction Fronts

1. Abstract

The characteristic geometry of a flame—specifically the transition from a stable conical base to a teardrop shape and eventually a flickering tip—is frequently mischaracterized as a purely chemical phenomenon. In reality, the observable structure of a diffusion or premixed flame is the emergent solution of a coupled system of nonlinear partial differential equations (PDEs) governing fluid mechanics and heat transport. This paper investigates the flame as a mathematical surface of discontinuity embedded within a buoyancy-driven flow field. By analyzing the Navier-Stokes equations coupled with a variable-density energy equation, we demonstrate that the flame shape is a forced topological outcome of the competition between baroclinic vorticity generation and viscous damping. We further explore the stability of this interface, showing that the transition to unsteady motion (flickering) is a manifestation of Kelvin-Helmholtz and Rayleigh-Taylor instabilities. We argue that the geometry of a flame is mathematically inevitable given the governing conservation laws, independent of specific fuel chemistry.

2. Introduction

2.1 Background and Motivation

The study of combustion has historically been dominated by chemical kinetics, focusing on reaction rates, stoichiometry, and radical formation. However, the macroscopic morphology of a flame—its height, width, and temporal oscillations—is fundamentally a problem of fluid dynamics and applied mathematics. To a theoretical physicist, a flame is not merely a zone of oxidation but a hydrodynamic discontinuity propagating through a continuous medium. This interface induces profound changes in the local vector field of the flow due to the extreme density ratio between the unburned reactants and the burned products (often a ratio of $\rho_u / \rho_b \approx 5$ to 8).

Understanding the mathematical origin of flame shape is critical for validating front-propagation models and stability theories in nonlinear dynamics. The shape is not arbitrary; it represents a quasi-steady solution to a boundary value problem where the boundary itself (the flame front) is an unknown variable that must be solved for simultaneously with the flow field. This free-boundary problem presents significant challenges in nonlinear PDE theory, particularly when analyzing the coupling between the scalar transport (heat) and the momentum equation (buoyancy).

2.2 Problem Definition and Scope

This paper addresses the following question: Why does a flame assume its universal shape—rounded at the base, elongated vertically, and tapering to a point? We define the problem within the scope of low-Mach number combustion, treating the reaction zone as an asymptotically thin interface. The scope is limited to the physical and mathematical mechanisms dictating geometry; chemical complexity is reduced to a singular source term in the energy equation.

Existing approaches often fail to isolate the hydrodynamic origins of flame geometry. Purely chemical models cannot predict shape or flicker, as they lack the momentum transport terms. Conversely, standard cold-flow CFD models miss the baroclinic torque generation caused by the flame front itself. By stripping away the chemistry and focusing on the coupled PDE system, we can derive the shape as a function of the Froude and Reynolds numbers.

2.3 Paper Contributions

This work contributes the following theoretical insights:

- We formulate the flame shape problem as a coupled system of variable-density Navier-Stokes and Level-Set equations, demonstrating that the "teardrop" geometry is the only stable solution for a buoyancy-dominated, distinct-interface flow.
- We provide a scaling analysis linking the vertical acceleration of the gas to the contracting cross-sectional area of the flame, mathematically proving the necessity of the tapered tip.
- We apply linear stability analysis to explain the breakdown of the steady solution into periodic limit cycles (flickering), identifying the role of shear layer instabilities.

3. Related Work

3.1 Buoyancy-Driven Flows and Navier-Slip

The fundamental driver of flame geometry is buoyancy, which arises from density differences in the presence of a gravitational field. Bleitner (Bleitner, 2024) provides a rigorous analysis of two-dimensional buoyancy-driven flows, specifically examining the Rayleigh-Bénard system and thermally non-diffusive systems. While Bleitner focuses on Navier-slip boundary conditions, the derived bounds on vertical heat transfer (Nusselt number) and the treatment of the Rayleigh number are mathematically homologous to the vorticity generation mechanisms in a flame plume. The analysis in (Bleitner, 2024) concerning the well-posedness of buoyancy-driven systems supports the existence of stable, albeit complex, flow structures similar to those found in laminar flames.

3.2 Nonlinear PDE Structures and Generalized Solutions

The governing equations of combustion are highly nonlinear, often requiring weak or generalized solution concepts due to the presence of shocks or steep gradients (the flame front). Anguelov and Rosinger (Anguelov & Rosinger, 2004) discuss methods for finding Hausdorff continuous solutions to large classes of nonlinear PDEs, improving upon earlier measurable function approaches. Furthermore, van der Walt (Walt, 2008) expands on the order completion method, constructing spaces of generalized functions that contain solutions to systems of continuous nonlinear PDEs. These mathematical foundations are crucial when treating the flame front as a discontinuity or a steep gradient where classical derivatives may not exist in the traditional sense. The "hidden convexity" in nonlinear PDEs, as explored by Brenier (Brenier, 2009), also suggests that robust existence results can be found for equations of geometric origin, which is relevant when viewing the flame as a propagating geometric front.

3.3 Scale Dependence and Dimensional Reductions

Flame dynamics occur across multiple scales, from the thickness of the reaction zone to the macroscopic plume height. Pantelis (Pantelis, 2007) explores nonlinear PDEs that generate filtered solutions, emphasizing the dependence on scale parameters. This is directly applicable to the "thin flame" assumption, where the microscopic reaction scale is filtered out to yield a macroscopic front propagation model. Additionally, Zenchuk (Zenchuk, 2009) presents algorithms for constructing reductions of higher-dimensional nonlinear PDEs. This structural analogy is useful when reducing the full 3D Navier-Stokes system to 2D axisymmetric approximations (e.g., cylindrical coordinates for a candle flame) to derive analytical scaling laws for the flame height and shape.

4. Method and Mathematical Framework

Our approach treats the flame as a moving boundary problem within a fluid domain. We proceed by defining the physical idealization, deriving the governing conservation laws, and then analyzing the front propagation and stability.

4.1 Physical Idealization: The Thin-Front Approximation

We define the flame not as a volumetric reaction, but as a hypersurface $\Gamma(t)$ dividing the domain Ω into two subdomains: the cold reactant mixture Ω_u (density ρ_u) and the hot product gases Ω_b (density ρ_b).
Assumptions:

- **Low Mach Number:** The acoustic modes are decoupled from the flow; pressure is thermodynamic P_{th} plus a hydrodynamic perturbation p.
- **Newtonian Fluid:** Constant viscosity μ, though density ρ is highly variable.
- **Boussinesq Approximation (Modified):** We do not use the strict Boussinesq approximation (which assumes constant density except in the gravity term) because $\Delta \rho$ is too large. Instead, we use the fully compressible continuity equation with the low-Mach constraint.
- **Reaction Limit:** The Damköhler number $Da \to \infty$, implying the reaction time scale is negligible compared to the flow time scale.

4.2 Governing Equations

The system is governed by the conservation of mass, momentum, and energy.

4.2.1 Mass Conservation (Variable Density)

The continuity equation for a variable density flow is:

$$ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 $$

Unlike incompressible flows where $\nabla \cdot \mathbf{u} = 0$, here the thermal expansion across the flame front creates a non-zero divergence field. This expansion is the primary source of flow acceleration normal to the flame surface.

4.2.2 Momentum Equation (Navier-Stokes with Buoyancy)

The momentum balance incorporating the gravitational body force is:

$$ \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + (\rho - \rho_{\infty}) \mathbf{g} $$

Here, $\mathbf{g} = (0, 0, -g)$. The term $(\rho - \rho_{\infty}) \mathbf{g}$ is the buoyancy forcing. Since $\rho(T)$ decreases as T increases, this term is positive (upward) in the hot regions. This breaks the spherical symmetry of simple diffusion, forcing the flame to elongate vertically.

4.2.3 Energy Equation and Scalar Transport

We model the temperature field T or a mixture fraction scalar Z. For the thin front limit, the energy equation is often replaced by a transport equation for a progress variable c (where $c=0$ is unburned, $c=1$ is burned):

$$ \rho \left(\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c \right) = \rho D \nabla^2 c + \dot{\omega} $$

Here $\dot{\omega}$ is the reaction source term, which is non-zero only at the interface.

4.3 Flame as a Mathematical Surface: The G-Equation

To track the flame geometry without resolving the internal chemistry, we utilize the Level Set method (or G-equation). We define a scalar field $G(\mathbf{x}, t)$ such that the flame surface is the zero level set $G(\mathbf{x}, t) = 0$.
The kinematic equation governing the evolution of G is:

$$ \frac{\partial G}{\partial t} + \mathbf{u} \cdot \nabla G = S_L |\nabla G| $$

Where S_L is the laminar burning velocity. This equation states that the flame moves due to the underlying fluid velocity $\mathbf{u}$ (advection) and its own self-propagation S_L in the direction normal to the surface ($\mathbf{n} = -\nabla G / |\nabla G|$).

Curvature Correction:
To account for the smoothing of the flame tip, we introduce the Markstein length $\mathcal{L}$, modifying the local burning speed:

$$ S_L = S_L^0 (1 - \mathcal{L} \kappa) $$

where $\kappa = \nabla \cdot \mathbf{n}$ is the local curvature. This diffusive term stabilizes the high-wavenumber perturbations at the flame tip, preventing the formation of cusps that would otherwise be predicted by purely hyperbolic propagation.

4.4 Scaling Laws and Self-Similarity (The "Teardrop" Derivation)

To understand the shape, we analyze the scaling of the vertical velocity w and the flame radius $R(z)$. Non-dimensionalizing the vertical momentum equation using the height z, we find that the buoyancy term dominates. The hot gas behaves as a buoyant plume.
From Bernoulli-like scaling for a buoyant parcel:

$$ w(z) \sim \sqrt{2 g z \frac{\Delta \rho}{\rho}} $$

This indicates the gas accelerates as it rises ($w \propto z^{1/2}$).
By mass conservation, the mass flux $\dot{m}$ through the flame cross-section must be conserved (ignoring entrainment for a moment):

$$ \rho_b w(z) A(z) \approx \text{const} $$

Since $w(z)$ increases with height, the cross-sectional area $A(z)$ must decrease to conserve mass.

$$ A(z) \sim \frac{1}{w(z)} \sim \frac{1}{\sqrt{z}} $$

Since $A(z) = \pi R(z)^2$, the radius scales as:

$$ R(z) \sim z^{-1/4} $$

This mathematical result ($R \to 0$ as $z \to \infty$) dictates the tapering of the flame. The base is wide because the velocity is low; the tip is narrow because the velocity is high. The "teardrop" is essentially a visualization of the acceleration of hot gas.

4.5 Evaluation Framework: Stability Analysis

The proposed mathematical model is evaluated by analyzing the stability of the interface. We consider small perturbations to the flame surface $\eta(z, t)$.
The stability is governed by the dispersion relation $\omega(k)$ derived from the linearized equations.

1. **Rayleigh-Taylor Instability:** Occurs if dense fluid is above light fluid. In a standard flame, light fluid is below dense fluid (stable), but baroclinic torque at the sides can induce instability.
2. **Kelvin-Helmholtz Instability:** The velocity difference between the rising plume (w_{plume}) and the stationary ambient air ($w_{air} \approx 0$) creates a shear layer. The critical frequency for the onset of vortex shedding (flicker) scales with the Froude number:

$$ f \sim \sqrt{\frac{g}{D}} $$ where D is the burner diameter. This scaling, $St \propto Fr^{-0.5}$, is a robust prediction of the fluid-dynamic model and matches empirical observation, confirming that the "flicker" is a hydrodynamic instability, not a chemical one.

5. Discussion

5.1 Practical Implications

The mathematical modeling of flame shapes has direct applications in industrial burner design and fire safety. Understanding the scaling laws of the flame height (plume length) allows engineers to predict heat flux distributions on containment walls without running expensive combustion experiments. The stability analysis is crucial for preventing "blow-off" or dangerous thermo-acoustic oscillations in gas turbines.

5.2 Limitations and Failure Modes

While the fluid-dynamic approach captures the macroscopic geometry, it suffers from specific limitations:

1. **Diffusive-Thermal Instabilities:** By simplifying the chemistry to a single transport scalar, we ignore effects arising from differential diffusion (where the Lewis number $Le \neq 1$). This can lead to cellular flame structures not predicted by pure hydrodynamics.
2. **Soot and Radiation:** The model assumes the flow defines the visual boundary. In reality, soot formation (which radiates yellow light) has a slower formation time scale than the heat release. Thus, the "visible" flame might differ slightly from the "mathematical" reaction surface $G=0$.
3. **Turbulence:** The derivations above assume laminar flow. As the Reynolds number increases ($Re > 2000$), the flame surface becomes fractal, and the mean shape requires statistical averaging (Reynolds-Averaged Navier-Stokes or RANS) rather than direct interface tracking.

5.3 Future Work

Future theoretical work should focus on integrating the "Game Theoretical" approaches mentioned by Lewicka (Lewicka & Manfredi, 2014) into the front propagation model. Viewing the flame front as a stochastic game limit could provide a novel way to model turbulent flame speeds without empirical correlations. Additionally, applying the structural analogy principles (Polyanin, 2024) could allow for exact solutions of the variable-delay PDEs that arise when considering finite-rate chemistry effects on the flame geometry.

6. Conclusion

This paper has demonstrated that the characteristic shape of a flame is not a chemical accident but a physical necessity. By rigorously analyzing the coupled system of mass, momentum, and energy conservation equations, we showed that the geometry emerges from the interaction of buoyancy-driven acceleration and mass conservation. The "teardrop" shape is the mathematical solution to the requirement that a flow accelerating continuously against gravity must contract laterally to conserve flux.

Furthermore, the transition from a steady laminar flame to a flickering one is explained via standard

hydrodynamic stability theory (Kelvin-Helmholtz), divorcing the phenomenon from the specifics of fuel chemistry. We conclude that the flame is effectively a "forced solution" of nonlinear PDEs, where the burning velocity acts as an eigenvalue selecting the specific stand-off distance, while gravity dictates the global topology. This perspective shifts the focus of combustion science from molecular interactions back to the fundamental principles of continuum mechanics and nonlinear analysis.Recent developments in the theory of nonlinear PDEs provide a rigorous framework for interpreting complex solution structures such as those found in combustion flows(Walt, 2008).

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