Vibrations of Single Degree of Freedom Systems

CEE 541. Structural Dynamics
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Fall, 2018

This document describes free and forced dynamic responses of single degree of freedom (SDOF) systems. The prototype single degree of freedom system is a spring-mass-damper system in which the spring has no damping or mass, the mass has no stiffness or damping, the damper has no stiffness or mass. Furthermore, the mass is allowed to move in only one direction. The horizontal vibrations of a single-story building can be conveniently modeled as a single degree of freedom system. Part 1 of this document describes some useful trigonometric identities. Part 2 shows how damped SDOF systems vibrate freely after being released from an initial displacement with some initial velocity. Part 3 covers the response of damped SDOF systems to persistent sinusoidal forcing.

Consider the structural system shown in Figure 1, where:

$f(t)$ = external excitation force
$x(t)$ = displacement of the center of mass of the moving object
m = mass of the moving object, $f_I = \frac{d}{dt}(m\dot{x}(t)) = m\ddot{x}(t)$
c = linear viscous damping coefficient, $f_D = c\dot{x}(t)$
k = linear elastic stiffness coefficient, $f_S = kx(t)$

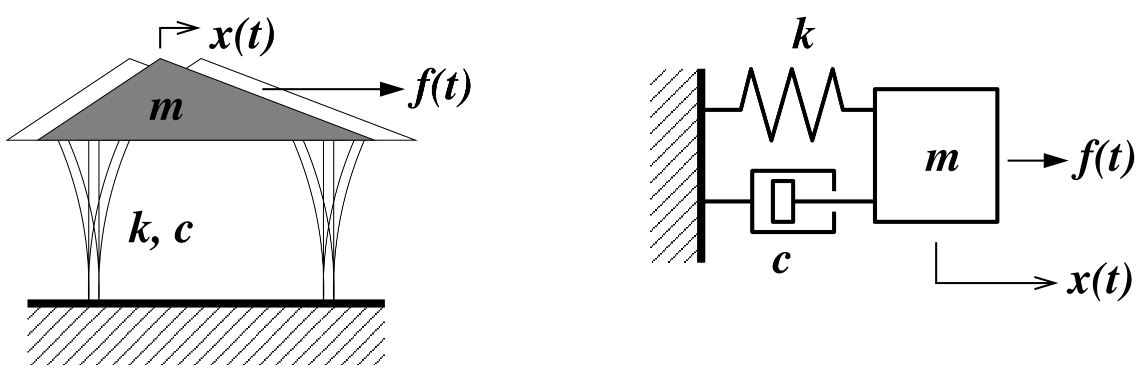


Figure 1. The proto-typical single degree of freedom oscillator.

The kinetic energy $T(x, \dot{x})$, the potential energy, $V(x)$, and the external forcing and dissipative forces, $p(x, \dot{x})$, are

$$T(x, \dot{x}) = \frac{1}{2} m (\dot{x}(t))^2 \tag{1}$$

$$V(x) = \frac{1}{2} k (x(t))^2 \tag{2}$$

$$p(x, \dot{x}) = -c\dot{x}(t) + f(t) \tag{3}$$

The general form of the differential equation describing a SDOF oscillator follows directly from Lagrange's equation,

$$\frac{d}{dt}\frac{\partial T(x, \dot{x})}{\partial \dot{x}} - \frac{\partial T(x, \dot{x})}{\partial x} + \frac{\partial V(x)}{\partial x} - p(x, \dot{x}) = 0 \ , \tag{4}$$

or from simply balancing the forces on the mass,

$$\sum F = 0 : f_I + f_D + f_S = f(t) \ . \tag{5}$$

Either way, the equation of motion is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t), \quad x(0) = d_o, \ \ \dot{x}(0) = v_o \tag{6}$$

where the initial displacement is d_o, and the initial velocity is v_o.

The solution to equation (6) is the sum of a homogeneous part (free response) and a particular part (forced response). This document describes free responses of all types and forced responses to simple-harmonic forcing.

1 Trigonometric and Complex Exponential Expressions for Oscillations

1.1 Constant Amplitude

An oscillation, $x(t)$, with amplitude $\bar{X}$ and frequency ω can be described by sinusoidal functions. These sinusoidal functions may be equivalently written in terms of complex exponentials $e^{\pm i\omega t}$ with complex coefficients, $X = A + iB$ and $X^* = A - iB$. (The complex constant X^* is called the *complex conjugate* of X.)

$$x(t) = \bar{X} \cos(\omega t + \theta) \tag{7}$$

$$= a \cos(\omega t) + b \sin(\omega t) \tag{8}$$

$$= X \ e^{+i\omega t} + X^* \ e^{-i\omega t} \tag{9}$$

To relate equations (7) and (8), recall the cosine of a sum of angles,

$$\bar{X}\cos(\omega t+\theta)=\bar{X}\cos(\theta)\cos(\omega t)-\bar{X}\sin(\theta)\sin(\omega t) \tag{10}$$

Comparing equations (10) and (8), we see that

$$a=\bar{X}\cos(\theta)\ , \qquad b=-\bar{X}\sin(\theta)\ , \qquad \text{and} \qquad a^2+b^2=\bar{X}^2\ . \tag{11}$$

Also, the ratio b/a provides an equation for the phase shift, θ,

$$\tan(\theta)=-\frac{b}{a} \tag{12}$$

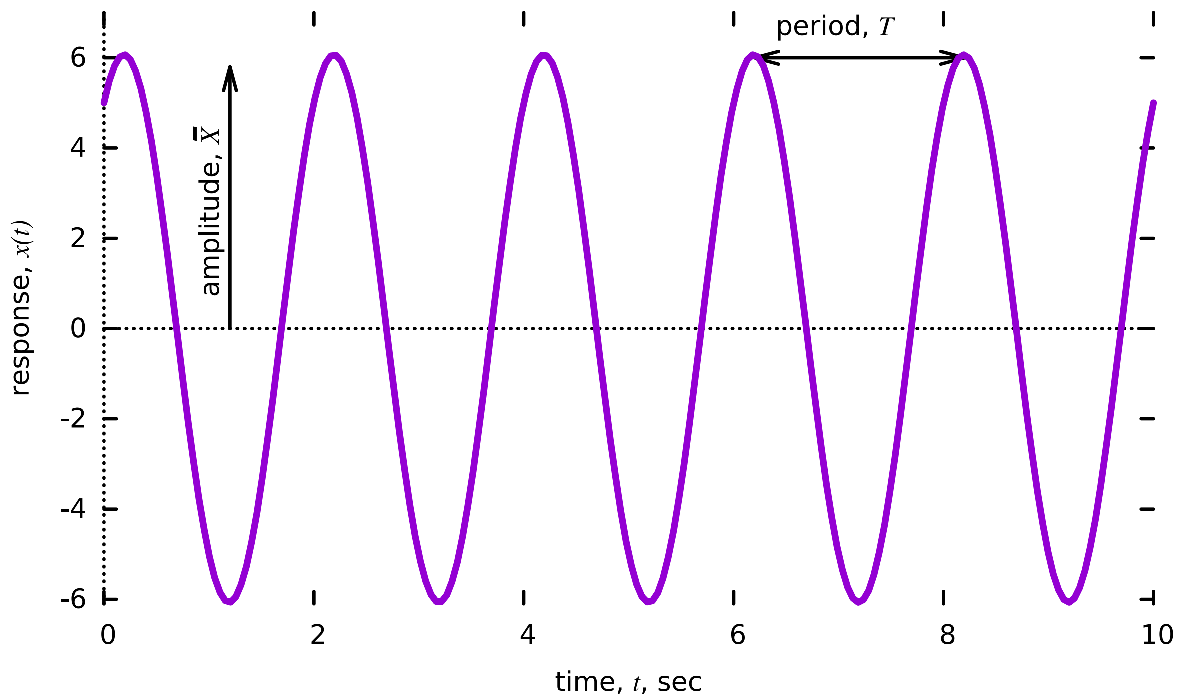


Figure 2. A constant-amplitude oscillation.

To relate equations (8) and (9), recall the expression for a complex exponent in terms of sines and cosines,

$$\begin{aligned} X\ e^{+i\omega t}+X^*\ e^{-i\omega t} &= (A+iB)\ (\cos(\omega t)+i\sin(\omega t))+ \\ &\quad\ (A-iB)\ (\cos(\omega t)-i\sin(\omega t)) && (13) \\ &= A\cos(\omega t)-B\sin(\omega t)+iA\sin(\omega t)+iB\cos(\omega t)+ \\ &\quad\ A\cos(\omega t)-B\sin(\omega t)-iA\sin(\omega t)-iB\cos(\omega t) \\ &= 2A\cos(\omega t)-2B\sin(\omega t) && (14) \end{aligned}$$

Comparing equations (14) and (8), we see that

$$a = 2A \; , \qquad b = -2B \; , \quad \text{and} \quad \tan(\theta) = \frac{B}{A} \; . \tag{15}$$

Any sinusoidal oscillation $x(t)$ can be expressed equivalently in terms of equations (7), (8), or (9); the choice depends on the application, and the problem to be solved. Equations (7) and (8) are easier to interpret as describing a sinusoidal oscillation, but equation (9) can be much easier to work with mathematically. These notes make use of all three forms.

One way to interpret the complex exponential notation is as the sum of complex conjugates,

$$Xe^{+i\omega t} = [A\cos(\omega t) - B\sin(\omega t)] + i[A\sin(\omega t) + B\cos(\omega t)]$$

and

$$X^* e^{-i\omega t} = [A\cos(\omega t) - B\sin(\omega t)] - i[A\sin(\omega t) + B\cos(\omega t)]$$

as shown in Figure 3. The sum of complex conjugate pairs is real, since the imaginary parts cancel out.

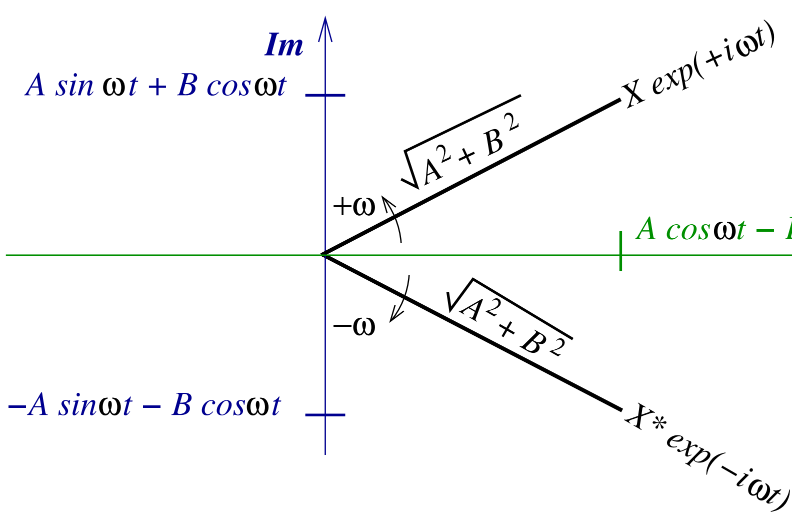


Figure 3. Complex conjugate oscillations.

The amplitude, $\bar{X}$, of the oscillation $x(t)$ can be found by finding the the sum of the complex amplitudes $|X|$ and $|X^*|$.

$$\bar{X} = |X| + |X^*| \;\; = \;\; 2\sqrt{A^2 + B^2} = \sqrt{a^2 + b^2} \tag{16}$$

Note, again, that equations (7), (8), and (9) are all equivalent using the relations among (a, b), (A, B), $\bar{X}$, and θ given in equations (11), (12), (15), and (16).

1.2 Decaying Amplitude

The dynamic response of damped systems decays over time. Note that damping may be introduced into a structural system through diverse mechanisms, including linear viscous damping, nonlinear viscous damping, visco-elastic damping, friction damping, and plastic deformation. ll but linear viscous damping are somewhat complicated to analyze, so we will restrict our attention to linear viscous damping, in which the damping force f_D is proportional to the velocity, $f_D = c\dot{x}$.

To describe an oscillation which decays with time, we can multiply the expression for a constant amplitude oscillation by a positive-valued function which decays with time. Here we will use a real exponential, $e^{\sigma t}$, where $\sigma < 0$.

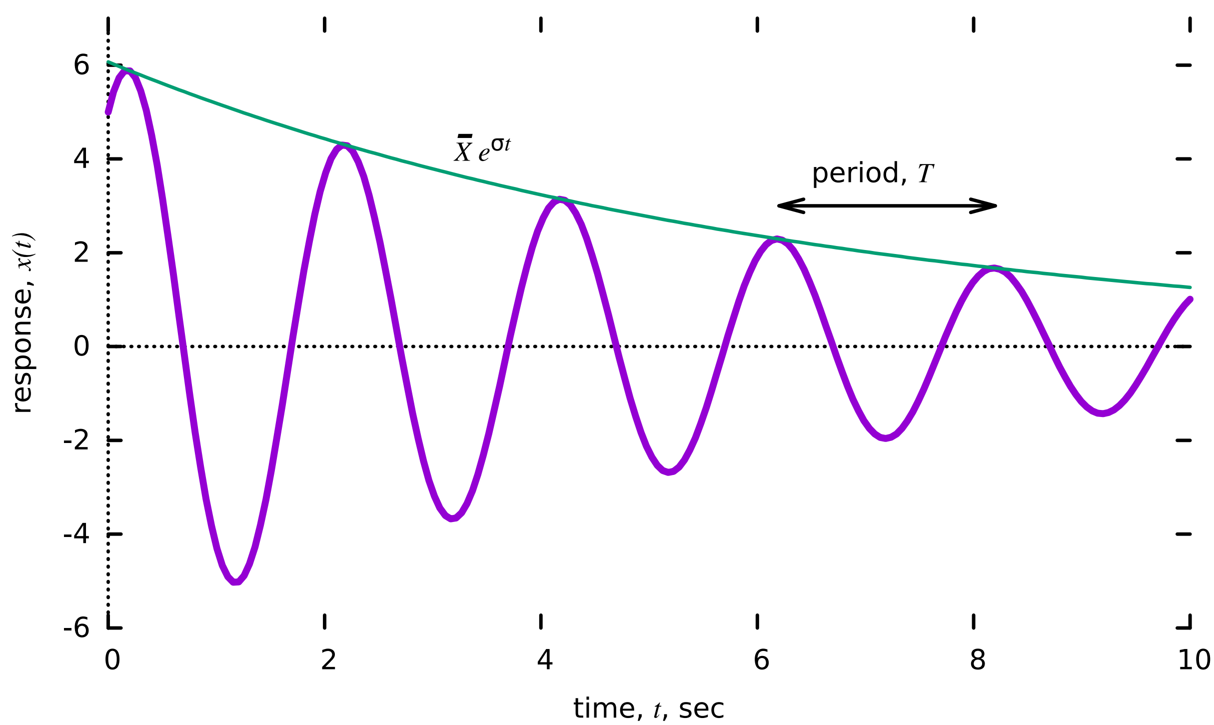


Figure 4. A decaying oscillation.

Multiplying equations (7) through (9) by $e^{\sigma t}$,

$$
\begin{aligned}
x(t) &= e^{\sigma t}\ \bar{X}\ (\cos(\omega t + \theta)) && (17)\\
&= e^{\sigma t}\ (a\cos(\omega t) + b\sin(\omega t)) && (18)\\
&= e^{\sigma t}\ (Xe^{i\omega t} + X^* e^{-i\omega t}) && (19)\\
&= Xe^{(\sigma+i\omega)t} + X^* e^{(\sigma-i\omega)t} && (20)\\
&= Xe^{\lambda t} + X^* e^{\lambda^* t} && (21)
\end{aligned}
$$

Again, note that *all* of the above equations are *exactly* equivalent. The exponent λ is complex, $\lambda = \sigma + i\omega$ and $\lambda^* = \sigma - i\omega$. If σ is negative, then these equations describe an oscillation with exponentially decreasing amplitudes. Note that in equation (18) the unknown constants are σ, ω, a, and b. Angular frequencies, ω, have units of radians per second. Circular frequencies, $f = \omega/(2\pi)$ have units of cycles per second, or Hertz. Periods, $T = 2\pi/\omega$, have units of seconds.

In the next section we will find that for an un-forced vibration, σ and ω are determined from the mass, damping, and stiffness of the system. We will see that the constant a equals the initial displacement d_o, but that the constant b depends on the initial displacement and velocity, as well mass, damping, and stiffness.

2 Free response of simple oscillators

Using equation (21) to describe the free response of a single degree of freedom system, we will set $f(t) = 0$ and will substitute $x(t) = Xe^{\lambda t}$ into equation (6).

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \ , \quad x(0) = d_o \ , \quad \dot{x}(0) = v_o \ , \tag{22}$$

$$m\lambda^2 X e^{\lambda t} + c\lambda X e^{\lambda t} + kXe^{\lambda t} = 0 \ , \tag{23}$$

$$(m\lambda^2 + c\lambda + k)Xe^{\lambda t} = 0 \ , \tag{24}$$

Note that m, c, k, λ and X do *not* depend on time. For equation (24) to be true for all time,

$$(m\lambda^2 + c\lambda + k)X = 0 \ . \tag{25}$$

Equation (25) is trivially satisfied if $X = 0$. The *non-trivial solution* is $m\lambda^2 + c\lambda + k = 0$. This is a quadratic equation in λ which has the roots,

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}. \tag{26}$$

The solution to a homogeneous second order ordinary differential equation requires two independent initial conditions: an initial displacement and an initial velocity. These two independent initial conditions are used to determine the coefficients, X and X^* (or A and B, or a and b) of the two linearly independent solutions corresponding to λ_1 and λ_2.

The amount of damping, c, qualitatively affects the quadratic roots, $\lambda_{1,2}$, and the free response solutions.

- **Case 1** $c = 0$ "undamped"
  If the system has no damping, $c = 0$, and

  $$\lambda_{1,2} = \pm i\sqrt{k/m} = \pm i\omega_{\rm n} \ . \tag{27}$$

  This is called the *natural frequency* of the system. Undamped systems oscillate freely at their natural frequency, $\omega_{\rm n}$. The solution in this case is

  $$x(t) = Xe^{i\omega_{\rm n}t} + X^*e^{-i\omega_{\rm n}t} = a\cos\omega_{\rm n}t + b\sin\omega_{\rm n}t \ , \tag{28}$$

which is a *real-valued* function. The amplitudes depend on the initial displacement, d_o, and the initial velocity, v_o.

- **Case 2** $c = c_{\rm c}$ "critically damped"
  If $(c/(2m))^2 = k/m$, or, equivalently, if $c = 2\sqrt{mk}$, then the discriminant of equation (26) is zero, This special value of damping is called the *critical damping* rate, $c_{\rm c}$,
  $$c_{\rm c} = 2\sqrt{mk} \; . \tag{29}$$
  The ratio of the actual damping rate to the critical damping rate is called the *damping ratio*, ζ.
  $$\zeta = \frac{c}{c_{\rm c}} \; . \tag{30}$$
  The two roots of the quadratic equation are real and are repeated at
  $$\lambda_1 = \lambda_2 = -c/(2m) = -c_{\rm c}/(2m) = -2\sqrt{mk}/(2m) = -\omega_{\rm n} \; , \tag{31}$$
  and the two basic solutions are equal to each other, $e^{\lambda_1 t} = e^{\lambda_2 t}$. In order to admit solutions for arbitrary initial displacements and velocities, the solution in this case is
  $$x(t) = x_1 \; e^{-\omega_{\rm n} t} + x_2 \; t \; e^{-\omega_{\rm n} t} \; . \tag{32}$$
  where the real constants x_1 and x_2 are determined from the initial displacement, d_o, and the initial velocity, v_o. Details regarding this special case are in the next section.

- **Case 3** $c > c_{\rm c}$ "over-damped"
  If the damping is greater than the critical damping, then the roots, λ_1 and λ_2 are distinct and real. If the system is over-damped it will not oscillate freely. The solution is
  $$x(t) = x_1 \; e^{\lambda_1 t} + x_2 \; e^{\lambda_2 t} \; , \tag{33}$$
  which can also be expressed using hyperbolic sine and hyperbolic cosine functions. The real constants x_1 and x_2 are determined from the initial displacement, d_o, and the initial velocity, v_o.

- **Case 4** $0 < c < c_{\rm c}$ "under-damped"
  If the damping rate is positive, but less than the critical damping rate, the system will oscillate freely from some initial displacement and velocity. The roots are complex conjugates, $\lambda_1 = \lambda_2^*$, and the solution is

$$x(t) = X \; e^{\lambda t} + X^* \; e^{\lambda^* t} \;, \tag{34}$$

  where the complex amplitude depends on the initial displacement, d_o, and the initial velocity, v_o.

We can re-write the dynamic equations of motion using the new dynamic variables for natural frequency, $\omega_{\rm n}$, and damping ratio, ζ. Note that

$$\frac{c}{m} = c\frac{\sqrt{k}}{\sqrt{k}}\frac{1}{\sqrt{m}\sqrt{m}} = \frac{c}{\sqrt{k}\sqrt{m}}\frac{\sqrt{k}}{\sqrt{m}} = 2\frac{c}{2\sqrt{km}}\sqrt{\frac{k}{m}} = 2\zeta\omega_{\rm n}. \tag{35}$$

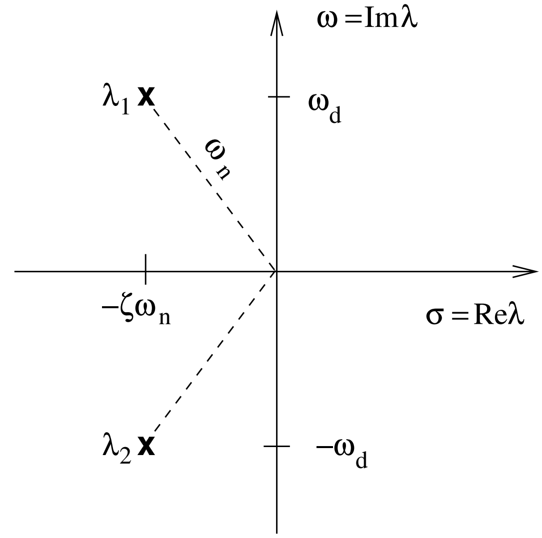
$$\begin{aligned}
m\ddot{x}(t) + c\dot{x}(t) + kx(t) &= f(t), && (36)\\
\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) &= \frac{1}{m}f(t), && (37)\\
\ddot{x}(t) + 2\zeta\omega_{\rm n}\;\dot{x}(t) + {\omega_{\rm n}}^2\;x(t) &= \frac{1}{m}f(t), && (38)
\end{aligned}$$

The expression for the roots $\lambda_{1,2}$, can also be written in terms of $\omega_{\rm n}$ and ζ.

$$\begin{aligned}
\lambda_{1,2} &= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\;, && (39)\\
&= -\zeta\omega_{\rm n} \pm \sqrt{(\zeta\omega_{\rm n})^2 - {\omega_{\rm n}}^2}\;, && (40)\\
&= -\zeta\omega_{\rm n} \pm \omega_{\rm n}\sqrt{\zeta^2 - 1}\;. && (41)
\end{aligned}$$

Some useful facts about the roots λ_1 and λ_2 are:

- $\lambda_1 + \lambda_2 = -2\zeta\omega_{\rm n}$
- $\lambda_1 - \lambda_2 = 2\ \omega_{\rm n}\ \sqrt{\zeta^2 - 1}$
- ${\omega_{\rm n}}^2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 - \frac{1}{4}(\lambda_1 - \lambda_2)^2$
- $\omega_{\rm n} = \sqrt{\lambda_1 \lambda_2}$
- $\zeta = -(\lambda_1 + \lambda_2)/(2\omega_{\rm n})$



2.1 Free response of critically-damped systems

The solution to a homogeneous second order ordinary differential equation requires two independent initial conditions: an initial displacement and an initial velocity. These two initial conditions are used to determine the coefficients of the two linearly independent solutions corresponding to λ_1 and λ_2. If $\lambda_1 = \lambda_2$, then the solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are not independent. In fact, they are identical. In such a case, a new trial solution can be determined as follows. Assume the second solution has the form

$$
\begin{aligned}
x(t) &= u(t) x_2 e^{\lambda_2 t} \ , & (42)\\
\dot{x}(t) &= \dot{u}(t) x_2 e^{\lambda_2 t} + u(t)\lambda_2 x_2 e^{\lambda_2 t} \ , & (43)\\
\ddot{x}(t) &= \ddot{u}(t) x_2 e^{\lambda_2 t} + 2\dot{u}(t)\lambda_2 x_2 e^{\lambda_2 t} + u(t)\lambda_2^2 x_2 e^{\lambda_2 t} & (44)
\end{aligned}
$$

substitute these expressions into

$$\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + {\omega_{\rm n}}^2 x(t) = 0 \ ,$$

collect terms, and divide by $x_2 e^{\lambda_2 t}$, to get

$$\ddot{u}(t) + 2\omega_{\rm n}(\zeta - 1)\dot{u}(t) + 2{\omega_{\rm n}}^2(1 - \zeta)u(t) = 0$$

or $\ddot{u}(t) = 0$ (since $\zeta = 1$). If the acceleration of $u(t)$ is zero then the velocity of $u(t)$ must be constant, $\dot{u}(t) = C$, and $u(t) = Ct$, from which the new trial solution is found.

$$x(t) = u(t) x_2 e^{\lambda_2 t} = x_2\ t\ e^{\lambda_2 t} \ .$$

So, using the complete trial solution $x(t) = x_1 e^{\lambda t} + x_2 t e^{\lambda t}$, and incorporating initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$, the free response of a critically-damped system is:

$$x(t) = d_o \, e^{-\omega_\mathrm{n} t} + (v_o + \omega_\mathrm{n} d_o) \, t \, e^{-\omega_\mathrm{n} t} \; . \tag{45}$$

2.2 Free response of underdamped systems

If the system is under-damped, then $\zeta < 1$, $\sqrt{\zeta^2 - 1}$ is imaginary, and

$$\lambda_{1,2} = -\zeta\omega_\mathrm{n} \pm i\omega_\mathrm{n}\sqrt{|\zeta^2 - 1|} = \sigma \pm i\omega. \tag{46}$$

The frequency $\omega_\mathrm{n}\sqrt{|\zeta^2 - 1|}$ is called the *damped natural frequency*, ω_d,

$$\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{|\zeta^2 - 1|} \; . \tag{47}$$

It is the frequency at which under-damped SDOF systems oscillate freely, With these new dynamic variables (ζ, ω_n, and ω_d) we can re-write the solution to the damped free response,

$$\begin{aligned} x(t) &= e^{-\zeta\omega_\mathrm{n} t}(a \cos \omega_\mathrm{d} t + b \sin \omega_\mathrm{d} t), & (48) \\ &= X e^{\lambda t} + X^* e^{\lambda^* t}. & (49) \end{aligned}$$

Now we can solve for X, (or, equivalently, A and B) in terms of the initial conditions. At the initial point in time, $t = 0$, the position of the mass is $x(0) = d_o$ and the velocity of the mass is $\dot{x}(0) = v_o$.

$$\begin{aligned} x(0) = d_o &= X e^{\lambda \cdot 0} + X^* e^{\lambda^* \cdot 0} & (50) \\ &= X + X^* & (51) \\ &= (A + iB) + (A - iB) = 2A = a. & (52) \\ \dot{x}(0) = v_o &= \lambda X e^{\lambda \cdot 0} + \lambda^* X^* e^{\lambda^* \cdot 0}, & (53) \\ &= \lambda X + \lambda^* X^*, & (54) \\ &= (\sigma + i\omega_\mathrm{d})(A + iB) + (\sigma - i\omega_\mathrm{d})(A - iB), & (55) \\ &= \sigma A + i\omega_\mathrm{d} A + i\sigma B - \omega_\mathrm{d} B + \\ &\quad\ \sigma A - i\omega_\mathrm{d} A - i\sigma B - \omega_\mathrm{d} B, & (56) \\ &= 2\sigma A - 2\omega B & (57) \\ &= -\zeta\omega_\mathrm{n} \, d_o - 2 \, \omega_\mathrm{d} \, B, & (58) \end{aligned}$$

from which we can solve for B and b,

$$B = -\frac{v_o + \zeta\omega_{\rm n} d_o}{2\omega_{\rm d}} \qquad \text{and} \qquad b = \frac{v_o + \zeta\omega_{\rm n} d_o}{\omega_{\rm d}} \; . \tag{59}$$

Putting this all together, the free response of an underdamped system to an arbitrary initial condition, $x(0) = d_o, \dot{x}(0) = v_o$ is

$$x(t) = e^{-\zeta\omega_{\rm n} t}\left(d_o \cos\omega_{\rm d} t + \frac{v_o + \zeta\omega_{\rm n} d_o}{\omega_{\rm d}} \sin\omega_{\rm d} t \right) . \tag{60}$$

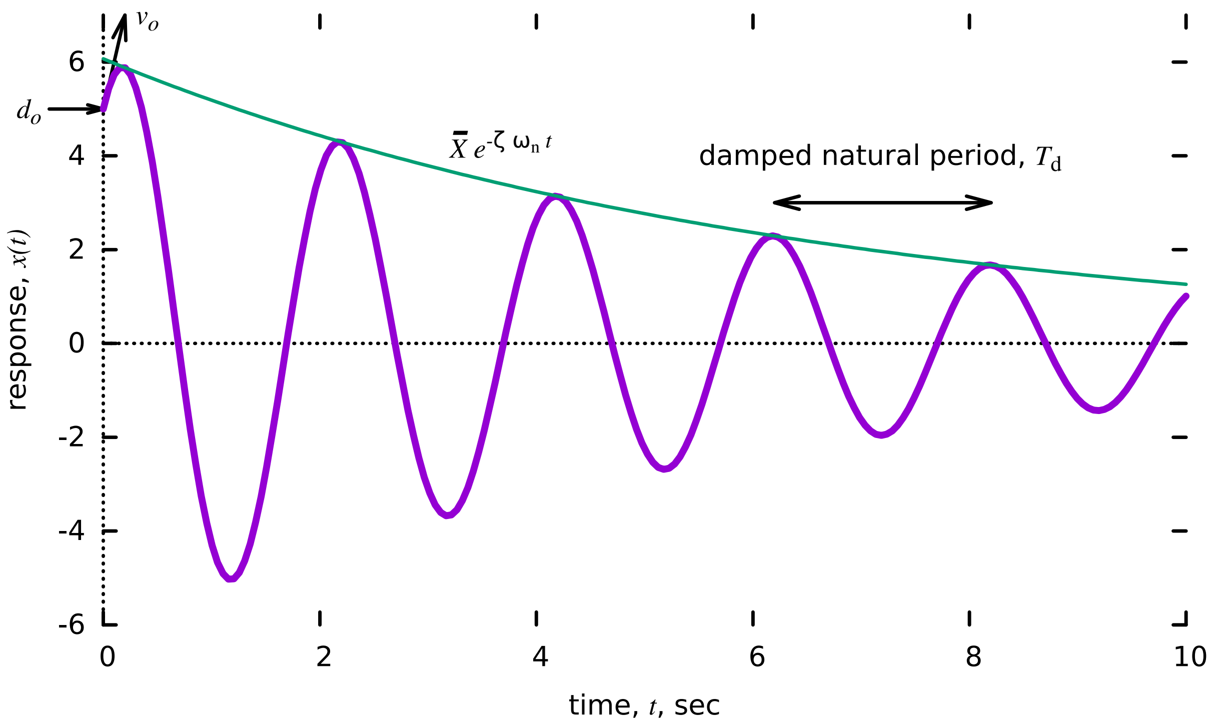


Figure 5. Free response of an under-damped oscillator to an initial displacement and velocity.

2.3 Free response of over-damped systems

If the system is over-damped, then $\zeta > 1$, and $\sqrt{\zeta^2 - 1}$ is real, and the roots are both real and negative

$$\lambda_{1,2} = -\zeta\omega_{\rm n} \pm \omega_{\rm n}\sqrt{\zeta^2 - 1} = -\zeta\omega_{\rm n} \pm \omega_{\rm d}. \tag{61}$$

so, using identities for exponentials in terms of cosh and sinh,

$$\begin{aligned} x(t) &= e^{-\zeta\omega_{\rm n}t}(x_1 e^{+\omega_{\rm d}t} + x_2 e^{-\omega_{\rm d}t}) && (62)\\ x(t) &= e^{-\zeta\omega_{\rm n}t}(x_1(\cosh\omega_{\rm d}t + \sinh\omega_{\rm d}t) + x_2(\cosh\omega_{\rm d}t - \sinh\omega_{\rm d}t)) && (63)\\ x(t) &= e^{-\zeta\omega_{\rm n}t}(a\cosh\omega_{\rm d}t + b\sinh\omega_{\rm d}t)) && (64)\end{aligned}$$

Substituting arbitrary and independent initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$ into equation (64) results in

$$x(t) = e^{-\zeta\omega_{\rm n}t}\left(d_o\ \cosh\omega_{\rm d}t + \frac{v_o + \zeta\omega_{\rm n}d_o}{\omega_{\rm d}}\ \sinh\omega_{\rm d}t\right) . \tag{65}$$

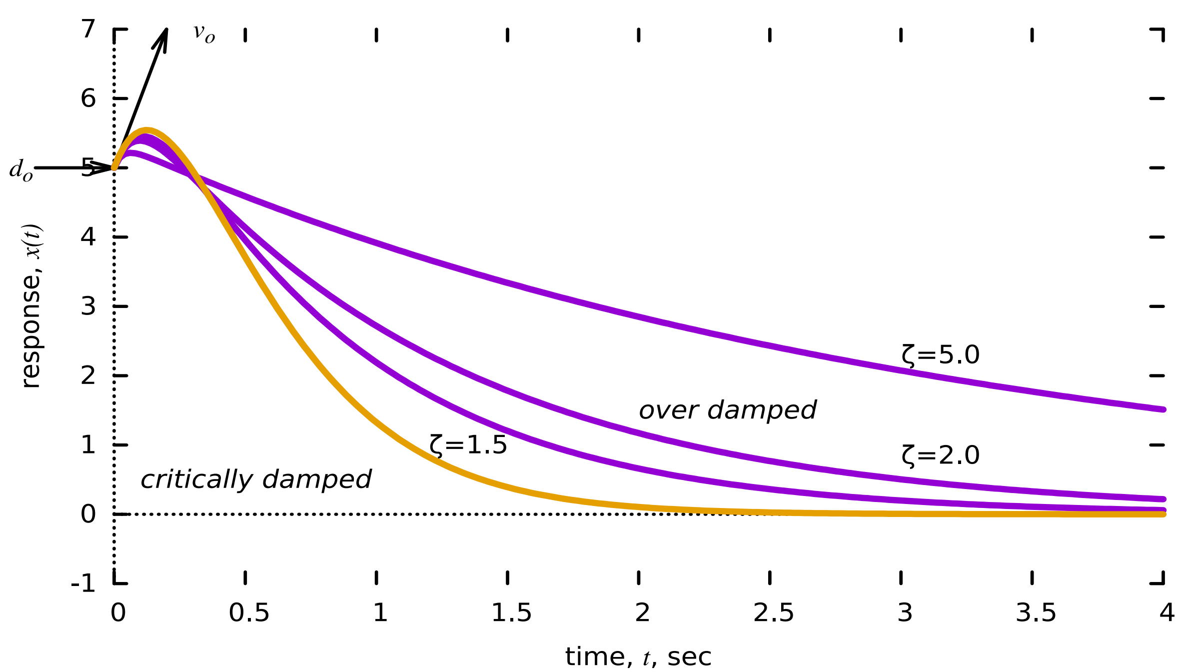


Figure 6. Free response of critically-damped (yellow) and over-damped (violet) oscillators to an initial displacement and velocity.

The undamped free response can be found as a special case of the under-damped free response. While special solutions exist for the critically damped response, this response can also be found as limiting cases of the under-damped or over-damped responses.

2.4 Finding the natural frequency from self-weight displacement

Consider a spring-mass system in which the mass is loaded by gravity, g. The static displacement D_{st} is related to the natural frequency by the constant of gravitational acceleration.

$$D_{\mathrm{st}} = mg/k = g/{\omega_{\mathrm{n}}}^2 \tag{66}$$

2.5 Finding the damping ratio from free response

Consider the value of two peaks of the free response of an under-damped system, separated by n cycles of motion

$$x_1 = x(t_1) \;=\; e^{-\zeta\omega_{\mathrm{n}} t_1}(A) \tag{67}$$

$$x_{1+n} = x(t_{1+n}) \;=\; e^{-\zeta\omega_{\mathrm{n}} t_{1+n}}(A) = e^{-\zeta\omega_{\mathrm{n}}(t_1+2n\pi/\omega_{\mathrm{d}})}(A) \tag{68}$$

The ratio of these amplitudes is

$$\frac{x_1}{x_{1+n}} = \frac{e^{-\zeta\omega_{\mathrm{n}} t_1}}{e^{-\zeta\omega_{\mathrm{n}}(t_1+2n\pi/\omega_{\mathrm{d}})}} = \frac{e^{-\zeta\omega_{\mathrm{n}} t_1}}{e^{-\zeta\omega_{\mathrm{n}} t_1} e^{-2n\pi\zeta\omega_{\mathrm{n}}/\omega_{\mathrm{d}}}} = e^{2n\pi\zeta/\sqrt{1-\zeta^2}}, \tag{69}$$

which is independent of ω_{n} and ω_{d}. Defining the *log decrement* $\delta(\zeta)$ as $\ln(x_1/x_{1+n})/n$,

$$\delta(\zeta) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{70}$$

and, inversely,

$$\zeta(\delta) = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \approx \frac{\delta}{2\pi} \tag{71}$$

where the approximation is accurate to within 3% for $\zeta < 0.2$ and is accurate to within 1.5% for $\zeta < 0.1$.

2.6 Summary

To review, some of the important expressions relating to the free response of a single degree of freedom oscillator are:

$$\bar{X}\cos(\omega t + \theta) = a\cos(\omega t) + b\sin(\omega t) = Xe^{+i\omega t} + X^* e^{-i\omega t}$$

$$\bar{X} = \sqrt{a^2 + b^2}; \quad \tan(\theta) = -b/a; \quad X = A + iB; \quad A = a/2; B = -b/2;$$

$$\left.\begin{aligned} m\ddot{x}(t) + c\dot{x}(t) + kx(t) &= 0 \\ \ddot{x}(t) + 2\zeta\omega_{\mathrm{n}}\dot{x}(t) + {\omega_{\mathrm{n}}}^2 x(t) &= 0 \end{aligned}\right\} \qquad x(0) = d_o,\ \dot{x}(0) = v_o$$

$$\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{c_{\mathrm{c}}} = \frac{c}{2\sqrt{mk}} \qquad \omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{|\zeta^2 - 1|}$$

$$x(t) = e^{-\zeta\omega_{\mathrm{n}} t}\left(d_o \cos\omega_{\mathrm{d}} t + \frac{v_o + \zeta\omega_{\mathrm{n}} d_o}{\omega_{\mathrm{d}}}\sin\omega_{\mathrm{d}} t\right) \qquad (0 \le \zeta < 1)$$

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{1+n}}\right) \qquad \zeta(\delta) = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \approx \frac{\delta}{2\pi}$$

3 Forced sinusoidal response of simple oscillators

When subject to simple harmonic forcing with a forcing frequency ω, dynamic systems initially respond with a combination of a transient response at a frequency ω_{d} and a steady-state response at a frequency ω. The transient response at frequency ω_{d} decays with time, leaving the steady state response at a frequency equal to the forcing frequency, ω. This section examines three ways of applying forcing: forcing applied directly to the mass, inertial forcing applied through motion of the base, and forcing from a rotating eccentric mass.

3.1 Direct Forcing

If the SDOF system is dynamically forced with a sinusoidal forcing function, then $f(t) = \bar{F}\cos(\omega t)$, where ω is the frequency of the forcing, in radians per second. If $f(t)$ is persistent, then after several cycles the system will respond only at the frequency of the external forcing, ω. Let's suppose that this *steady-state response* is described by the function

$$x(t) = a\cos\omega t + b\sin\omega t, \tag{72}$$

then

$$\dot{x}(t) = \omega(-a\sin\omega t + b\cos\omega t), \tag{73}$$

and

$$\ddot{x}(t) = \omega^2(-a\cos\omega t - b\sin\omega t). \tag{74}$$

Substituting this trial solution into equation (6), we obtain

$$\begin{aligned} m\omega^2 \quad & (-a\cos\omega t - b\sin\omega t) + \\ c\omega \quad & (-a\sin\omega t + b\cos\omega t) + \\ k \quad & (a\cos\omega t + b\sin\omega t) = \bar{F}\cos\omega t. \end{aligned} \tag{75}$$

Equating the sine terms and the cosine terms

$$(-m\omega^2 a + c\omega b + ka)\cos\omega t \;=\; \bar{F}\cos\omega t \tag{76}$$

$$(-m\omega^2 b - c\omega a + kb)\sin\omega t \;=\; 0, \tag{77}$$

which is a set of two equations for the two unknown constants, a and b,

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{F} \\ 0 \end{bmatrix}, \tag{78}$$

for which the solution is

$$a(\omega) = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \bar{F} \tag{79}$$

$$b(\omega) = \frac{c\omega}{(k - m\omega^2)^2 + (c\omega)^2} \bar{F}. \tag{80}$$

Referring to equations (7) and (12) in section 1.1, the forced vibration solution (equation (72)) may be written

$$x(t) = a(\omega)\cos\omega t + b(\omega)\sin\omega t = \bar{X}(\omega)\cos\left(\omega t + \theta(\omega)\right). \tag{81}$$

The angle θ is the phase between the force $f(t)$ and the response $x(t)$, and

$$\tan(\theta(\omega)) = -\frac{b(\omega)}{a(\omega)} = -\frac{c\omega}{k - m\omega^2} \tag{82}$$

Note that $-\pi < \theta(\omega) < 0$ for all positive values of ω, meaning that the displacement response, $x(t)$, always lags the external forcing, $\bar{F}\cos(\omega t)$. The ratio of the response amplitude $\bar{X}(\omega)$ to the forcing amplitude $\bar{F}$ is

$$\frac{\bar{X}(\omega)}{\bar{F}} = \frac{\sqrt{a^2(\omega) + b^2(\omega)}}{\bar{F}} = \frac{1}{\sqrt{(k - m\omega^2) + (c\omega)^2}}. \tag{83}$$

This equation shows how the response amplitude $\bar{X}$ depends on the amplitude of the forcing $\bar{F}$ and the frequency of the forcing ω, and has units of flexibility.

Let's re-derive this expression using complex exponential notation! The equations of motion are

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \bar{F}\cos\omega t = F(\omega)e^{i\omega t} + F^*(\omega)e^{-i\omega t}. \tag{84}$$

In a solution of the form, $x(t) = X(\omega)e^{i\omega t} + X^*(\omega)e^{-i\omega t}$, the coefficient $X(\omega)$ corresponds to the positive exponents (positive frequencies), and $X^*(\omega)$ corresponds to negative exponents (negative frequencies). Positive exponent coefficients and negative exponent coefficients are independent and may be found separately. Considering the positive exponent solution, the forcing is expressed as $F(\omega)e^{i\omega t}$ and the partial solution $X(\omega)e^{i\omega t}$ is substituted into the forced equations of motion, resulting in

$$(-m\omega^2 + ci\omega + k)\ X(\omega)\ e^{i\omega t} = F(\omega)\ e^{i\omega t}\ , \tag{85}$$

from which

$$\frac{X(\omega)}{F(\omega)} = \frac{1}{(k - m\omega^2) + i(c\omega)}\ , \tag{86}$$

which is complex-valued. This complex function has a magnitude

$$\left|\frac{X(\omega)}{F(\omega)}\right| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\ , \tag{87}$$

the same as equation (83) but derived using $e^{i\omega t}$ in just three simple lines.

Equation (86) may be written in terms of the dynamic variables, $\omega_{\rm n}$ and ζ. Dividing the numerator and the denominator of equation (83) by k, and noting that F/k is a static displacement, $x_{\rm st}$, we obtain

$$\frac{X(\omega)}{F(\omega)} = \frac{1/k}{\left(1 - \frac{m}{k}\omega^2\right) + i\left(\frac{c}{k}\omega\right)}, \tag{88}$$

$$X(\omega) = \frac{F(\omega)/k}{\left(1 - \left(\frac{\omega}{\omega_{\rm n}}\right)^2\right) + i\left(2\zeta\frac{\omega}{\omega_{\rm n}}\right)}, \tag{89}$$

$$\frac{X(\Omega)}{x_{\rm st}} = \frac{1}{(1 - \Omega^2) + i\,(2\zeta\Omega)}, \tag{90}$$

$$\frac{\bar{X}(\Omega)}{x_{\rm st}} = \frac{1}{\sqrt{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}}, \tag{91}$$

where the frequency ratio Ω is the ratio of the forcing frequency to the natural frequency, $\Omega = \omega/\omega_{\rm n}$. This equation is called the *dynamic amplification factor.* It is the factor by which displacement responses are amplified due to the fact that the external forcing is dynamic, not static. See Figure 7.

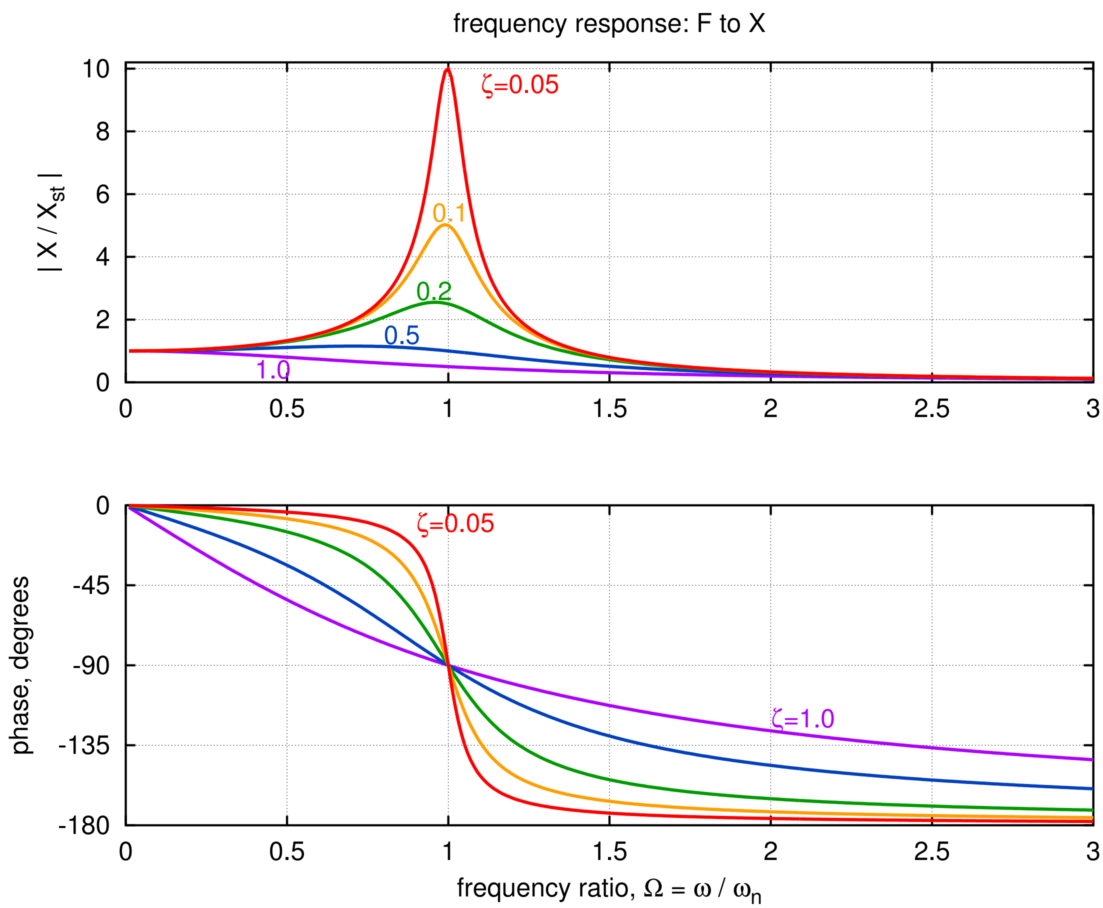


Figure 7. The dynamic amplification factor for external forcing, $\bar{X}/x_{\rm st}$, equation (90).

To summarize, the steady state response of a simple oscillator directly excited by a harmonic force, $f(t) = \bar{F}\cos\omega t$, may be expressed in the form of equation (7)

$$x(t) = \frac{\bar{F}/k}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} \ \cos(\omega t + \theta) \ , \quad \tan\theta = \frac{-2\zeta\Omega}{1-\Omega^2} \qquad (92)$$

or, equivalently, in the form of equation (8)

$$x(t) = \frac{\bar{F}/k}{(1-\Omega^2)^2 + (2\zeta\Omega)^2} \ [\ (1-\Omega^2)\cos\omega t + (2\zeta\Omega)\sin\omega t\ ] \ , \qquad (93)$$

where $\Omega = \omega/\omega_{\rm n}$.

3.2 Inertial Forcing

When the dynamic loads are caused by motion of the supports (or the ground, as in an earthquake) the forcing on the oscillator is the inertial force resisting the ground acceleration, which equals the mass of the oscillator times the ground acceleration, $f(t) = -m\ddot{z}(t)$.

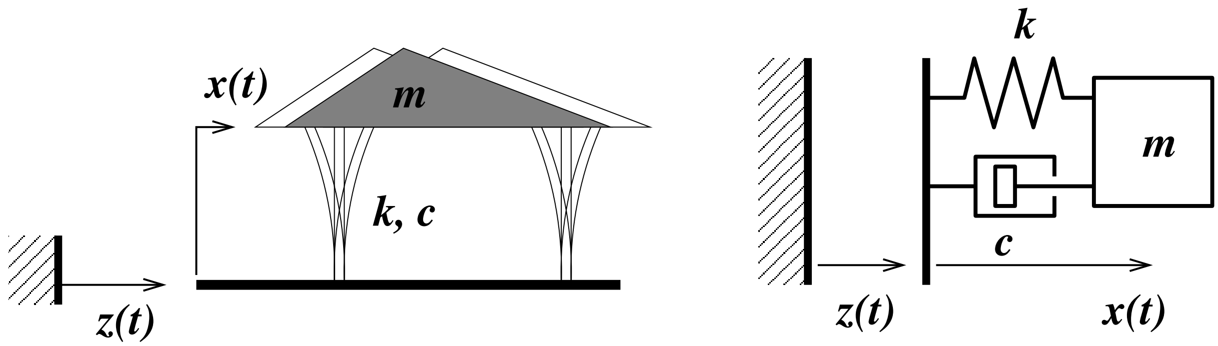


Figure 8. The proto-typical SDOF oscillator subjected to base motions, $z(t)$

$$
\begin{aligned}
m\left(\ \ddot{x}(t) + \ddot{z}(t)\ \right) + c\dot{x}(t) + kx(t) &= 0 && (94)\\
m\ddot{x}(t) + c\dot{x}(t) + kx(t) &= -m\ddot{z}(t) && (95)\\
\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + {\omega_{\rm n}}^2 x(t) &= -\ddot{z}(t) && (96)
\end{aligned}
$$

Note that equation (96) is independent of mass. Systems of different masses but with the same natural frequency and damping ratio have the same behavior and respond in exactly the same way to the same support motion.

If the ground displacements are sinusoidal $z(t) = \bar{Z}\cos\omega t$, then the ground accelerations are $\ddot{z}(t) = -\bar{Z}\omega^2\cos\omega t$, and $f(t) = m\bar{Z}\omega^2\cos\omega t$. Using the complex exponential formulation, we can find the steady state response as a function of the frequency of the ground motion, ω.

$$
m\ddot{x}(t) + c\dot{x}(t) + kx(t) = m\bar{Z}\omega^2\cos\omega t = mZ(\omega)\omega^2 e^{i\omega t} + mZ^*(\omega)\omega^2 e^{-i\omega t} \quad (97)
$$

The steady-state response can be expressed as the sum of independent complex exponentials, $x(t) = X(\omega)e^{i\omega t} + X^*(\omega)e^{-i\omega t}$. The positive exponent parts are independent of the negative exponent parts and can be analyzed separately.

Assuming persistent excitation and ignoring the transient response (the particular part of the solution), the response will be harmonic. Considering the "positive exponent" part of the forcing $mZ(\omega)\omega^2 e^{i\omega t}$, the "positive exponent" part of the steady-state response will have a form $Xe^{i\omega t}$. Substituting these expressions into the differential equation (97), collecting terms, and factoring out the exponential $e^{i\omega t}$, the frequency response function is

$$\begin{aligned}\frac{X(\omega)}{Z(\omega)} &= \frac{m\omega^2}{(k - m\omega^2) + i(c\omega)} \ , \\ &= \frac{\Omega^2}{(1-\Omega^2) + i(2\zeta\Omega)}\end{aligned} \tag{98}$$

where $\Omega = \omega/\omega_{\rm n}$ (the forcing frequency over the natural frequency), and

$$\left|\frac{X(\Omega)}{Z(\Omega)}\right| = \frac{\Omega^2}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} \tag{99}$$

See Figure 9.

Finally, let's consider the motion of the mass with respect to a fixed point. This is called the total motion and is the sum of the base motion plus the motion relative to the base, $z(t) + x(t)$.

$$\begin{aligned}\frac{X+Z}{Z} = \frac{X}{Z} + 1 &= \frac{(1-\Omega^2) + i(2\zeta\Omega) + \Omega^2}{(1-\Omega^2) + i(2\zeta\Omega)} \\ &= \frac{1 + i(2\zeta\Omega)}{(1-\Omega^2) + i(2\zeta\Omega)}\end{aligned} \tag{100}$$

and

$$\left|\frac{X+Z}{Z}\right| = \frac{\sqrt{1 + (2\zeta\Omega)^2}}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} = \mathrm{Tr}(\Omega, \zeta). \tag{101}$$

This function is called the *transmissibility ratio*, $\mathrm{Tr}(\Omega,\zeta)$. It determines the ratio between the total response amplitude $\overline{X+Z}$ and the base motion $\bar{Z}$. See figure 10.

For systems that have a longer natural period (lower natural frequency) than the period (frequency) of the support motion, (i.e., $\Omega > \sqrt{2}$), the transmissibility ratio is less than "1" especially for low values of damping ζ. In such systems the motion of the mass is *less than* the motion of the supports and we say that the mass is *isolated* from motion of the supports.

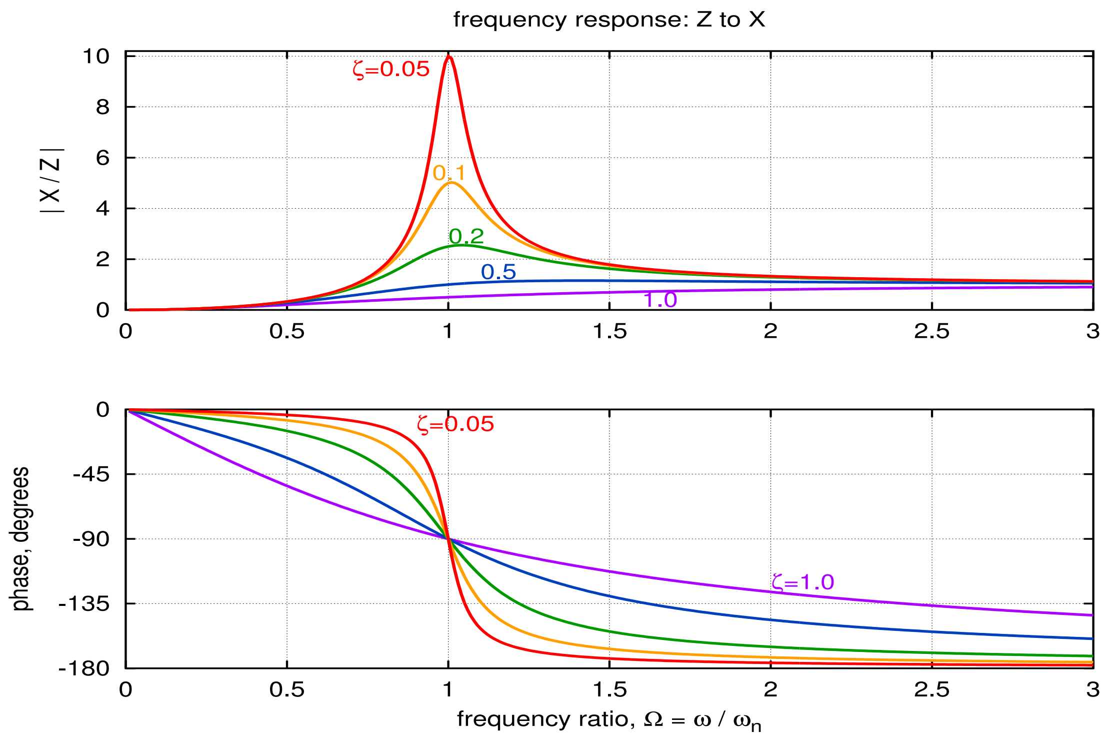


Figure 9. The dynamic amplification factor for inertial loading, $\bar{X}/\bar{Z}$, equation (98).

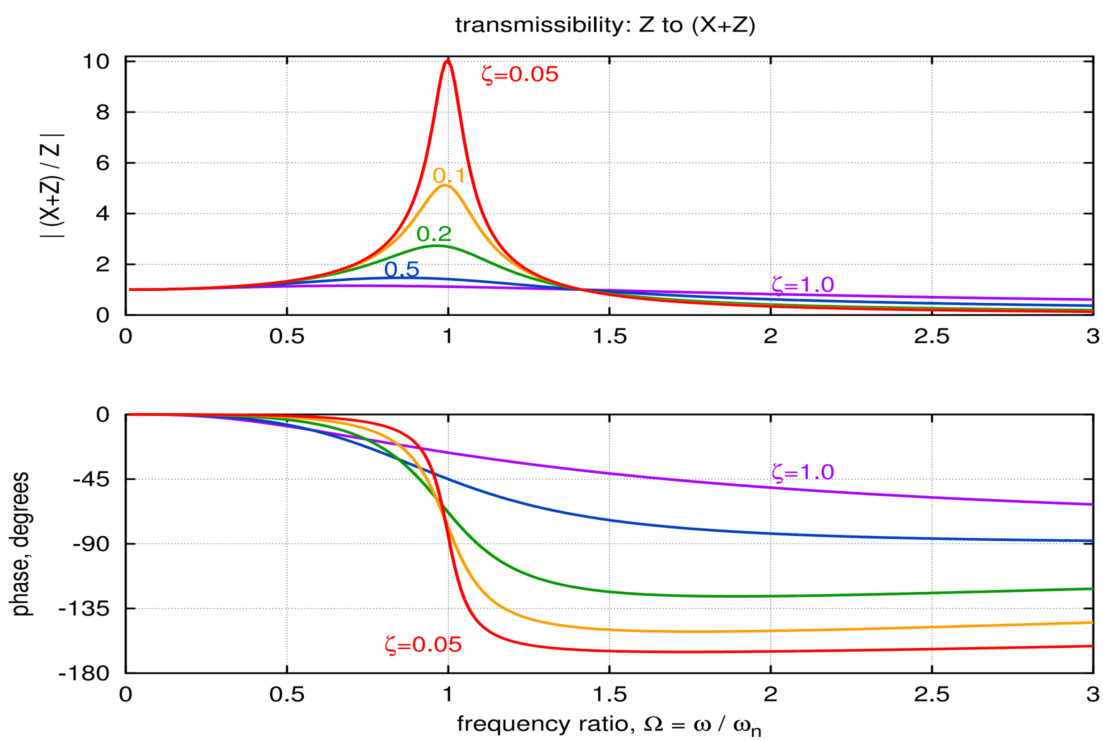


Figure 10. The transmissibility ratio $|(X+Z)/Z| = \text{Tr}(\Omega, \zeta)$, equation (100).

3.3 Eccentric-Mass Forcing

Another type of sinusoidal forcing which is important to machine vibration arises from the rotation of an eccentric mass. Consider the system shown in Figure 11 in which a mass μm is attached to the primary mass m via a rigid link of length r and rotates at an angular velocity ω. In this single degree of freedom analysis, the motion of the primary mass is constrained to lie along the x coordinate and the forcing of interest is the x-component of the reactive centrifugal force. This component is $\mu m r \omega^2 \cos(\omega t)$ where the angle ωt is the counter-clockwise angle from the x coordinate. The equation of motion with

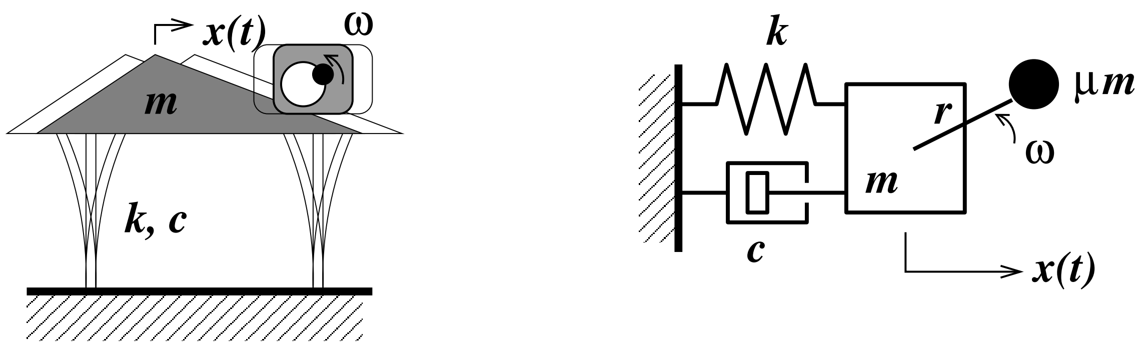


Figure 11. The proto-typical SDOF oscillator subjected to eccentric-mass excitation.

this forcing is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \mu m r \omega^2 \ \cos(\omega t) \tag{102}$$

This expression is simply analogous to equation (84) in which $\bar{F} = \mu \; m \; r \; \omega^2$. With a few substitutions, the frequency response function is found to be

$$\frac{X}{\mu r} = \frac{\Omega^2}{(1-\Omega^2) + i\,(2\zeta\Omega)}, \tag{103}$$

which is completely analogous to equation (98). The plot of the frequency response function of equation (103) is simply proportional to the function plotted in Figure 9. The magnitude of the dynamic force transmitted between a machine supported on dampened springs and the base, $|f_{\mathrm{T}}|$, is the sum of the forces in the springs and the dampers. The ratio of the transmitted force to the force generated by the eccentric mass, $\mu m r \omega^2$, *is* the transmissibility ratio, (101).

$$\frac{|f_{\mathrm{T}}|}{\mu m r \omega^2} = \mathrm{Tr}(\Omega, \zeta) \tag{104}$$

Note, here, though, that the denominator, $\mu m r \omega^2$, increases with ω^2. The transmitted force amplitude increases with $\mu\omega^2 \mathrm{Tr}(\Omega, \zeta)$. Multiplying both sides of equation (104) by Ω^2 we obtain the *transmission ratio*.

$$\frac{|f_{\mathrm{T}}|}{\mu m r {\omega_{\mathrm{n}}}^2} = \Omega^2 \mathrm{Tr}(\Omega, \zeta) \tag{105}$$

Unlike the transmissibility ratio, which asymptotically approaches "0" with increasing Ω, the vibratory force transmitted from an eccentric mass excitation is "0" when $\Omega = 0$ but increases with Ω for $\Omega > \sqrt{2}$. This increasing effect is significant for $\zeta > 0.2$, as shown in Figure 12. For high frequency

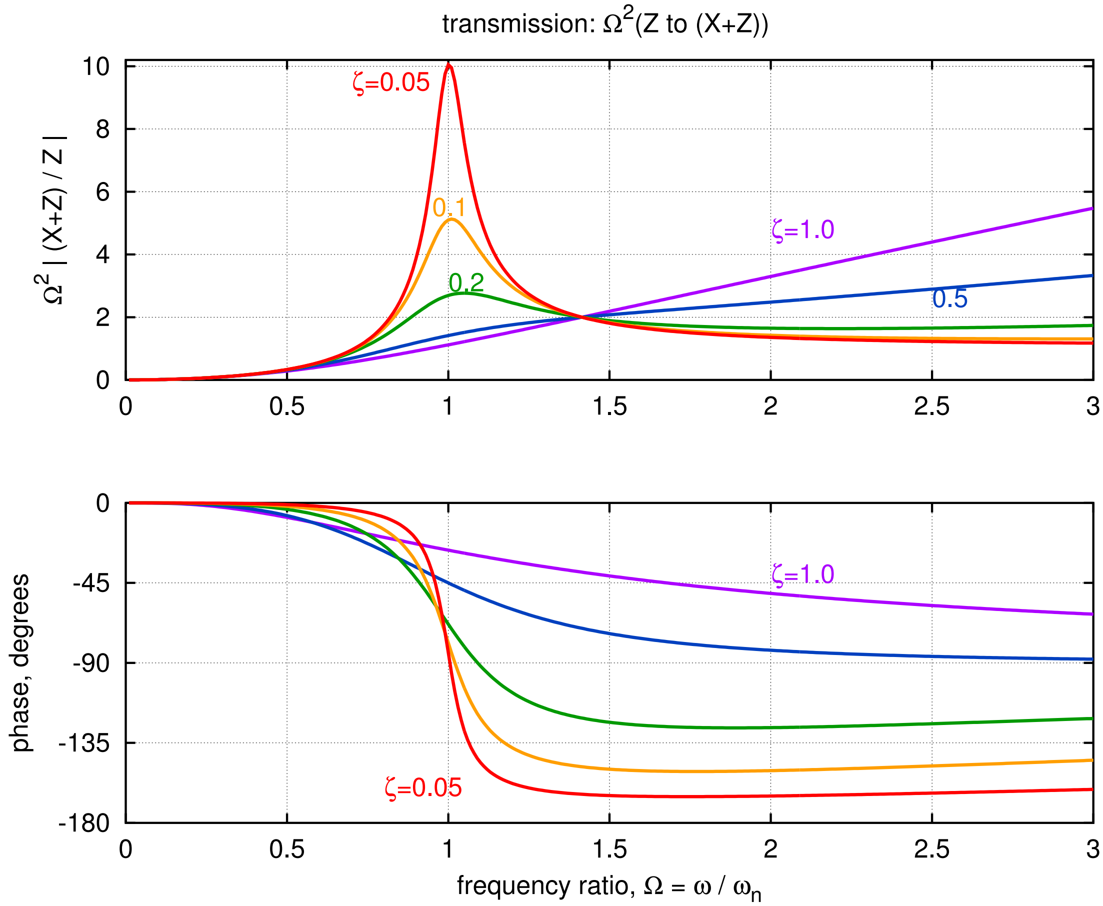


Figure 12. The transmission ratio $\Omega^2 \mathrm{Tr}(\Omega, \zeta)$, equation (105).

ratios ($\omega_{\mathrm{n}} < \omega/2$) and low damping ratios ($\zeta < 0.2$), the force transmitted from the rotating machinery to the floor ($|f_T|$) is less than half of the force generated by the machinery ($\mu m r \omega^2$). Further, for $\zeta \approx 0.2$, the value of the transmitted force is roughly independent of the forcing frequency.

3.4 Finding the damping from the peak of the frequency response function

For lightly damped systems, the frequency ratio of the resonant peak, the amplification of the resonant peak, and the width of the resonant peak are functions to of the damping ratio only. Consider two frequency ratios Ω_1 and Ω_2 such that $|H(\Omega_1,\zeta)|^2 = |H(\Omega_2,\zeta)|^2 = |H|^2_{\mathrm{peak}}/2$ where $|H(\Omega,\zeta)|$ is one of the frequency response functions described in earlier sections. The frequency ratio corresponding to the peak of these functions Ω_{peak}, and the value of the peak of these functions, $|H|^2_{\mathrm{peak}}$ are given in Table 1. Note that the peak coordinate depends only upon the damping ratio, ζ.

Since $\Omega_2^2 - \Omega_1^2 = (\Omega_2 - \Omega_1)(\Omega_2 + \Omega_1)$ and since $\Omega_2 + \Omega_1 \approx 2$,

$$\zeta \approx \frac{\Omega_2 - \Omega_1}{2} \tag{106}$$

which is called the "half-power bandwidth" formula for damping. For the first, second, and fourth frequency response functions listed in Table 1 the approximation is accurate to within 5% for $\zeta < 0.20$ and is accurate to within 1% for $\zeta < 0.10$.

Table 1. Peak coordinates for various frequency response functions.

$H(\Omega,\zeta)$	Ω_{peak}	$\|H\|^2_{\mathrm{peak}}$	$\Omega_2^2 - \Omega_1^2$
$\frac{1}{(1-\Omega^2)+i(2\zeta\Omega)}$	$\sqrt{1-2\zeta^2}$	$\frac{1}{4\zeta^2(1-\zeta^2)}$	$4\zeta\sqrt{1-\zeta^2}$
$\frac{i\Omega}{(1-\Omega^2)+i(2\zeta\Omega)}$	1	$\frac{1}{4\zeta^2}$	$4\zeta\sqrt{1+\zeta^2}$
$\frac{\Omega^2}{(1-\Omega^2)+i(2\zeta\Omega)}$	$\frac{1}{\sqrt{1-2\zeta^2}}$	$\frac{1}{4\zeta^2(1-\zeta^2)}$	$\frac{4\zeta\sqrt{1-\zeta^2}}{1-8\zeta^2(1-\zeta^2)}$
$\frac{1+i(2\zeta\Omega)}{(1-\Omega^2)+i(2\zeta\Omega)}$	$\frac{\left((1+8\zeta^2)^{1/2}-1\right)^{1/2}}{2\zeta}$	$\frac{8\zeta^4}{8\zeta^4-4\zeta^2-1+\sqrt{1+8\zeta^2}}$	ouch.

4 Real and Imaginary. Even and Odd. Magnitude and Phase.

Using the rules of complex division, it is not hard to show that

$$\mathrm{Re}[H(\Omega,\zeta)] = \mathrm{Re}[H(-\Omega,\zeta)] \tag{107}$$

$$\mathrm{Im}[H(\Omega,\zeta)] = -\mathrm{Im}[H(-\Omega,\zeta)]\ . \tag{108}$$

That is, $\mathrm{Re}[H(\Omega)]$ is an *even function* and $\mathrm{Im}[H(\Omega)]$ is an *odd function*. This fact is true for any dynamical system for which the inputs and outputs are real-valued.

For any frequency response function, the magnitude $|H(\Omega,\zeta)|$ and phase $\theta(\Omega,\zeta)$ may be found from

$$|H(\Omega,\zeta)|^2 = (\mathrm{Re}[H(\Omega,\zeta)])^2 + (\mathrm{Im}[H(\Omega,\zeta)])^2 \tag{109}$$

$$\tan\theta(\Omega,\zeta) = \frac{\mathrm{Im}[H(\Omega,\zeta)]}{\mathrm{Re}[H(\Omega,\zeta)]} \tag{110}$$

Expressions for the magnitude of various frequency response functions are given by equations (90), (99), and (100). The magnitude and phase lag of these functions are plotted in Figures 7, 9, and 10. Real and imaginary parts of $H(\Omega,\zeta)$ are plotted in Figure 13. Note the following:

- The real and imaginary parts are even and odd, respectively.
- The real part of X/x_{st} (force to response displacement) is zero at $\Omega = 1$. The phase at $\Omega = 1$ is 90 degrees.
- The real part of X/Z (support motion to response motion) is zero at $\Omega = 1$. The phase at $\Omega = 1$ is 90 degrees.
- The imaginary part of $i\Omega X/x_{\mathrm{st}}$ (force to response velocity) is zero at $\Omega = 1$. The phase at $\Omega = 1$ is 90 degrees.
- The real part of $i\Omega X/x_{\mathrm{st}}$ (force to response velocity) is maximum at $\Omega = 1$.
- The real part of $i\Omega X/x_{\mathrm{st}}$ (force to response velocity) is positive for all values of Ω.

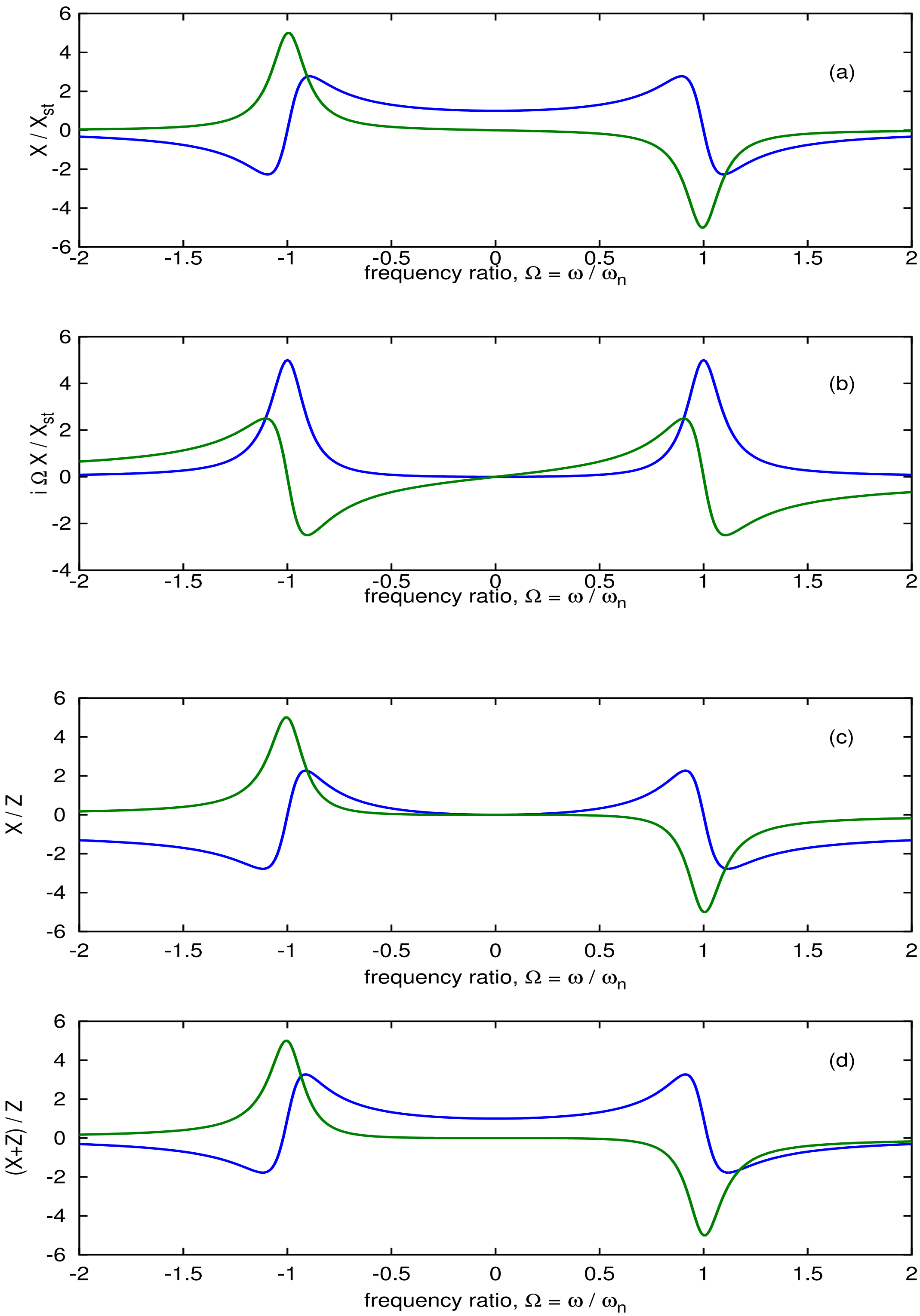


Figure 13. The real (even) and imaginary (odd) parts of frequency response functions, $\zeta = 0.1$.

5 Combined free and sinusoidally-forced response

The analyses of the previous sections describe (a) the transient (homogeneous) response of a simple oscillator to initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$, and (b) the steady-state harmonic (particular) response of a simple oscillator to sinusoidal forcing $f(t) = \bar{F} \cos \omega t$. The harmonic steady state response does not consider the transient from initial conditions.

The combined transient and harmonic response is the sum of the homogeneous solution, (48) or (64) (depending on the value of ζ) and the particular solution, (93). The coefficients a and b are found by matching the *combined response* to the initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$. So doing, for a sinusoidal forcing $f(t) = \bar{F} \cos \omega t$, $\omega_{\rm n} > 0$, $\zeta < 1$, and $\zeta \neq 0$,

$$
\begin{aligned}
x(t) &= e^{-\zeta \omega_{\rm n} t}(a \cos \omega_{\rm d} t + b \sin \omega_{\rm d} t) \\
&+ \frac{\bar{F}/k}{(1-\Omega^2)^2 + (2\zeta\Omega)^2} \left[\, (1-\Omega^2) \cos \omega t + (2\zeta\Omega) \sin \omega t \, \right] , \qquad (111)
\end{aligned}
$$

solving $x(0) = d_o$ for a gives

$$
a = d_o - \frac{\bar{F}/k}{(1-\Omega^2)^2 + (2\zeta\Omega)^2} \, (1-\Omega^2) \qquad (112)
$$

Deriving $\dot{x}(t)$ and solving $\dot{x}(0) = v_o$ for b gives

$$
b = \frac{1}{\omega_{\rm d}} \left(v_o + \zeta \omega_{\rm n} a - \frac{\bar{F}/k}{(1-\Omega^2)^2 + (2\zeta\Omega)^2} \, \omega (2\zeta\Omega) \right) \qquad (113)
$$

Figures 14 and 15 give two examples of combined response. Note that even when initial displacements and velocities are zero, the transient portion of the response can be much larger than the steady-state portion of the response.

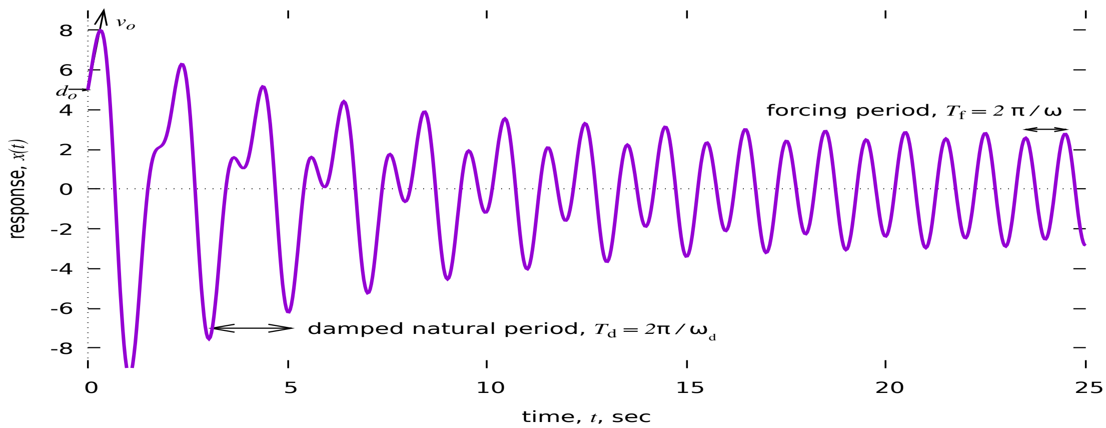


Figure 14. Combined free and forced response of a simple oscillator, $\omega_n = \pi$ rad/s, $\zeta = 0.05$, $d_o = 5$ m, $v_o = 10$ m/s, $\omega = 2\pi$ rad/s, $\bar{F} = 80$ N, $m = 1$ kg

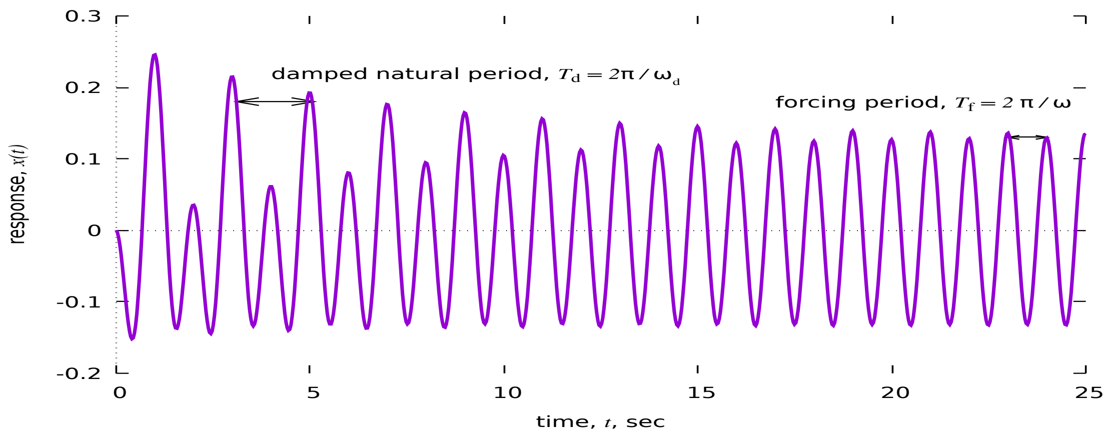


Figure 15. Combined free and forced response of a simple oscillator, $\omega_n = \pi$ rad/s, $\zeta = 0.05$, $d_o = 0$, $v_o = 0$, $\omega = 2\pi$ rad/s, $\bar{F} = -3.9$ N, $m = 1$ kg

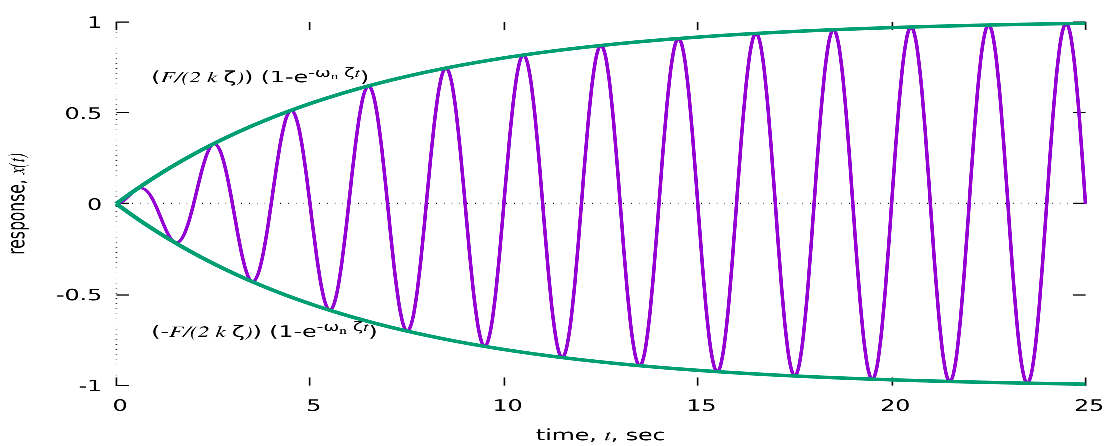


Figure 16. Combined free and forced response of a simple oscillator, in resonance $\omega_n = \pi$ rad/s, $\zeta = 0.05$, $d_o = 0$, $v_o = 0$, $\omega = \omega_n$, $\bar{F} = 1$ N, $m = 1$ kg

6 Undamped Resonance

The combination of free and forced responses is a weighted sum of four linearly independent terms $\cos\omega_{\rm d}t$, $\sin\omega_{\rm d}t$, $\cos\omega t$, and $\sin\omega t$, from which any set of two initial conditions (d_o and v_o), and magnitude, and phase may be specified. If the forcing is applied at the natural frequency and the damping is zero, then $\omega_{\rm d} = \omega_{\rm n} = \omega$, and there are only two linearly independent terms. So solutions of the form of equation (111) can not satisfy the differential equation for undamped resonance

$$\ddot{x}(t) + {\omega_{\rm n}}^2 x(t) = \frac{\bar{F}}{m} \cos\omega_{\rm n}t \tag{114}$$

As seen in the case of damped resonance, Figure 17, the response is enveloped by

$$-\frac{\bar{F}/k}{2\zeta}\left(1 - e^{-\omega_{\rm n}\zeta t}\right) \le x(t) \le +\frac{\bar{F}/k}{2\zeta}\left(1 - e^{-\omega_{\rm n}\zeta t}\right) \tag{115}$$

Without damping, however, we may expect the resonant response to grow without bound. A trial solution in this case could be

$$x(t) = a\ t\cos\omega_{\rm n}t + b\ t\sin\omega_{\rm n}t \tag{116}$$

The solution

$$x(t) = \frac{\bar{F}}{2k}\ t\ \ \sin\omega_{\rm n}t \tag{117}$$

satisfies equation (114) and initial conditions at rest $d_o = 0$, $v_o = 0$.

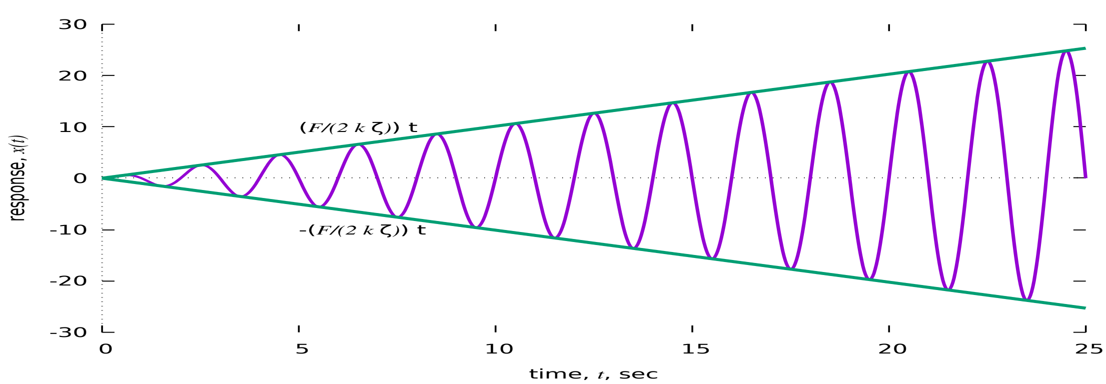


Figure 17. Undamped resonance of a simple oscillator $\omega_{\rm n} = \pi$ rad/s, $\zeta = 0$, $d_o = 0$, $v_o = 0$, $\omega = \omega_{\rm n}$, $\bar{F} = 1$ N, $m = 1$ kg

7 Vibration Sensors

A vibration sensor may be accurately modeled as an inertial mass supported by elements with stiffness and damping (i.e. as a single degree of freedom oscillator). Vibration sensors mounted to a surface ideally measure the velocity or acceleration of the surface with respect to an inertial reference frame. The electrical signals generated by vibration sensors are actually proportional to the velocity of the mass with respect to the sensor's case (for seismometers) or the deformation of the elastic elements of the sensor (for accelerometers). The frequency response functions and sensitivities of seismometers and accelerometers have qualitative differences.

7.1 Seismometers

Seismometers transduce the velocity of a magnetic inertial mass to electrical current in a coil. The transduction element of seismometers is therefore based on Ampere's Law, and seismometers are made from sprung magnetic masses that are guided to move within a coil fixed to the instrument housing. The electrical current generated within the coil is proportional to the velocity of the magnetic mass relative to the coil; so the mechanical input is the velocity of the case $\dot{z}(t) = i\omega Z e^{i\omega t}$ and the electrical output is proportional to the velocity of the inertial mass with respect to the case $v(t) = \kappa\dot{x}(t) = \kappa i\omega X(\omega)e^{i\omega t}$. These variables are related in the frequency domain by equation (98).

$$\frac{V}{i\omega Z} = \frac{\kappa i\omega X}{i\omega Z} = \frac{\kappa X}{Z} = \frac{\kappa\Omega^2}{(1-\Omega^2)+i(2\zeta\Omega)} \tag{118}$$

The sensitivity of seismometers approaches zero as Ω approaches zero. In order for seismometers to be sensitive to very low amplitude motion, the natural frequency of seismometers is designed to be quite small (less than 0.5 Hz, and sometimes less than 0.1 Hz).

In comparison to accelerometers, seismometers are heavy, large, delicate, sensitive, high output sensors which do not require external power or amplification. They require frequency-domain calibration.

7.2 Accelerometers

Accelerometers are themselves simple oscillators, with an input $f(t) = -m\ddot{z}(t)$, and a voltage output proportional to the relative displacement, $v(t) = \kappa x(t)$, where κ is related to the accelerometer sensitivity (in volts/m/s^2). Accelerometers transduce the deformation of inertially-loaded elastic elements within the sensor to an electrical charge, a voltage, or a current. Accelerometers may be designed with many types of transduction elements, including piezo-electric materials, strain-gages, variable capacitance components, and feed-back stabilization.

In all cases, the mechanical input to the accelerometer is the acceleration of the case $\ddot{z}(t) = -\omega^2 Z e^{i\omega t}$, the electrical output is proportional to the deformation of the spring $v(t) = \kappa x(t) = \kappa X e^{i\omega t}$, and these variables are related in the frequency domain by

$$\frac{V}{-\omega^2 Z} = \frac{-\kappa/{\omega_{\rm n}}^2}{(1-\Omega^2) + i(2\zeta\Omega)} \tag{119}$$

So the sensitivity of an accelerometer increases with decreasing natural frequency.

Piezo-electric accelerometers have natural frequencies in the range of 1 kHz to 20 kHz and damping ratios in the range of 0.5% to 1.0%. Because of electrical coupling considerations and the sensitivity of peizo-electric accelerometers to low-frequency temperature transients, the sensitivity and signal-to-noise ratio of piezo-electric accelerometers below 1 Hz can be quite poor.

Micro-machined electro-mechanical silicon (MEMS) accelerometers are monolithic with their signal-conditioning micro-circuitry. They typically have natural frequencies in the 100 Hz to 500 Hz range and are accurate down to frequencies of 0 Hz (constant acceleration, e.g., gravity). These sensors are damped to a level of about 70% of critical damping.

Force-balance accelerometers utilize feedback circuitry to magnetically stabilize an inertial mass. The force required to stabilize the mass is directly proportional to the acceleration of the sensor. Such sensors can be made to measure accelerations in the μg range down to frequencies of 0 Hz, and have natural frequencies on the order of 100 Hz, also with damping of about 70% of critical damping.

Accelerometers are typically light, small, and rugged but require electrical power, amplification, and signal conditioning.

Given the sensor sensitivity, natural frequency and damping ratio, the accelerometer's sensor dynamics can be corrected by numerically differentiating $x(t) = v(t)/\kappa$ once to obtain $\dot{x}(t)$, and again to obtain $\ddot{x}(t)$. Then, the "instrument-corrected" (true) acceleration at each point in time can be obtained from

$$\ddot{z}(t) = -\left(\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + {\omega_{\rm n}}^2 x(t)\right) \tag{120}$$

For sensors with a low natural frequency, this kind of instrument correction is essential. For instruments with natural frequencies greater than about ten times the frequency range of the measured signals, instrument correction is not significantly important. Further, if the signal to noise ratio is less than about 10:1, instrument correction has no discernible effect.

7.3 Design Considerations for Accelerometers

Accelerometers should have a uniform amplitude spectrum and a linear phase spectrum (minimum phase distortion) over the frequency bandwidth of the application. The *phase distortion* is the deviation in the phase lag from a linear phase shift. Figure 18 is a close-up of the frequency-response function of equation (119) over the typical frequency band width of accelerometer applications.

A linear phase spectrum is equivalent to a constant time delay. Consider a phase-lagged dynamic response in which the phase θ changes linearly with frequency, $\theta = \tau\omega$.

$$\begin{aligned} v(t) &= |V| \cos(\omega t + \tau\omega) \\ &= |V| \cos(\omega(t + \tau)) \end{aligned} \tag{121}$$

shifted which is the same as a response that is shifted in time by a constant time increment of τ for all frequencies. As can be seen in Figure 18, a level of damping in the range of 0.67 to 0.71 provides an amplitude distortion of less than 0.5% and a phase distortion of less than 0.5 degrees for a bandwidth up to 30% of the sensor natural frequency. A sensor damping of $\zeta = \sqrt{2}/2$ provides an optimally flat amplitude response and an extremely small phase distortion ($<$ 0.2 degrees?) up to 25% of the sensor natural frequency. For frequency ratios in this range, the phase spectrum is

$$\theta(\Omega) = \tan^{-1}\left[\frac{-2\zeta\Omega}{1-\Omega^2}\right] \approx -2\zeta\Omega \qquad (122)$$

giving a time lag τ of approximately $2\zeta/\omega_{\rm n}$. For $\zeta = \sqrt{2}/2$, the time lag τ evaluates to $\tau = \sqrt{2}/\omega_{\rm n} = \sqrt{2}/(2\pi f_{\rm n}) \approx 0.23T_{\rm n}$.

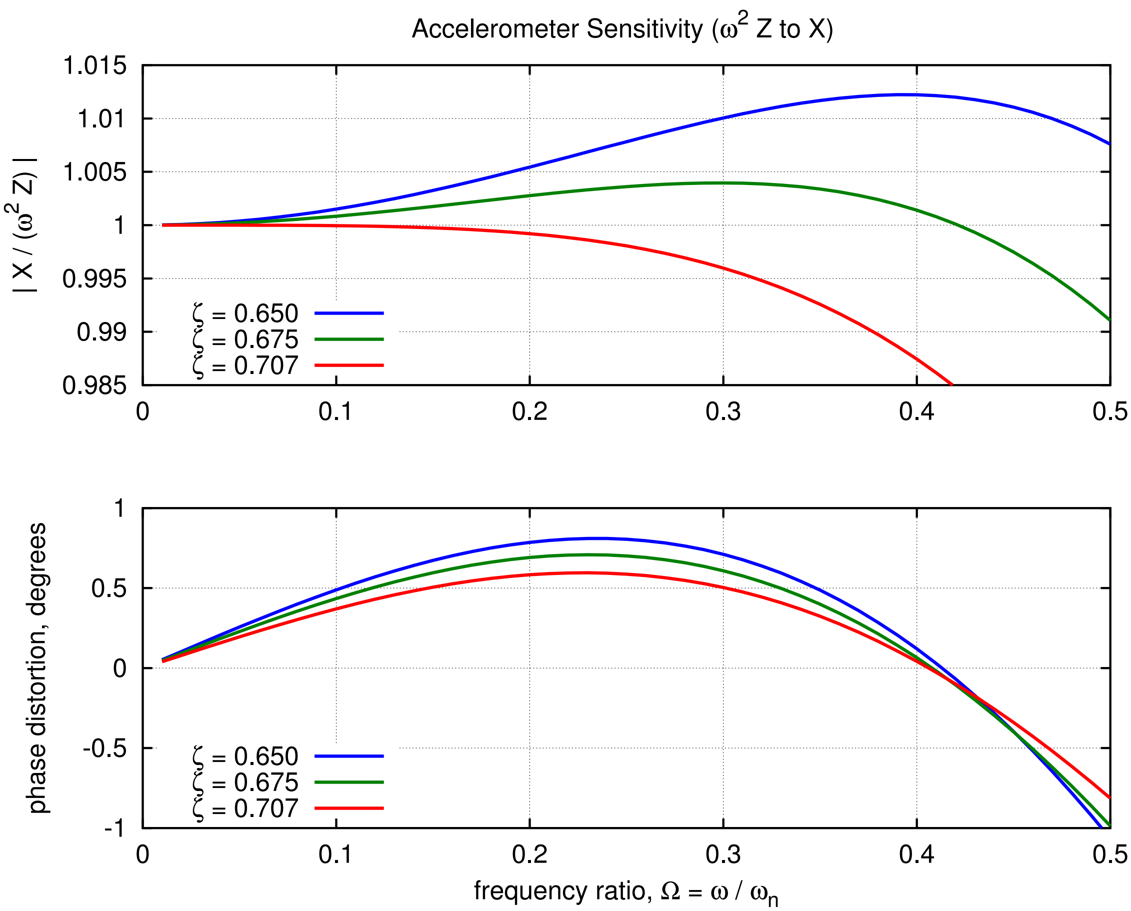


Figure 18. Accelerometer frequency response functions

8 Simulation for exploration

The behavior of simple oscillators may be explored by studying animated simulations of inertially-forced systems with different frequency ratios and damping ratios. The combined free and sinusoidally-forced response in the time domain can be compared to the frequency-domain solution for the steady-state sinusoidal response of oscillators. Animation facilitates the interpretation of magnitude and phase relationships in the context of observations in the time-domain.

Figures 9 and 10 illustrate the behavior in the frequency-domain. Figure 9, labeled frequency response: Z to X, illustrates the dynamic amplification of the oscillator's steady-state displacement response, $x(t) = \bar{X}\cos(\omega t + \theta)$, subjected to sinusoidal ground displacements, $z(t) = \bar{Z}\cos\omega t$. At a frequency ratio ($\Omega = \omega/\omega_{\rm n}$) of one, (the forcing frequency, ω, equals the resonant frequency, $\omega_{\rm n} = \sqrt{k/m}$) the response motion can be many times greater than the support motion ($|X/Z| \gg 1$). As the damping (ζ) increases, the amount of amplification decreases throughout the frequency range. The displacement amplification at resonance ($\Omega = 1$) is equal to $1/(2\zeta)$. For example, if the damping ratio, ζ, equals five percent (0.05), the amplification at resonance is 10. Also note that at resonance, the displacement response lags the base displacement by ninety degrees for all values of damping. If the frequency of the ground displacement is practically zero (the ground simply has a nearly constant static displacement) ($z(t) \approx$ const. and $\ddot{z}(t) \approx 0$) and the relative displacement of the oscillator with respect to its support point is zero ($|X/Z| = 0$). If the frequency of the ground displacement is high ($\Omega > 2$) the oscillator remains relatively stationary while its support point moves. In this case the structural deformation is equal and opposite to the ground displacement. The dynamic amplification factor, $|X/Z|$, is close to 1, and the displacement response lags the ground displacement by 180 degrees.

Figure 10, labeled transmissibility: Z to (X+Z), illustrates the relationship between the support point displacement, Z, and the *total* (or absolute) motion of the oscillator, $X + Z$, the sum of the support-point motion and the relative motion of the oscillator with respect to its support. This relationship is used to analyze the *isolation* of an oscillator from vibrations of its supports The goal in vibration isolation is to minimize or reduce the *total* response $(x + z)$, not the relative response, x; that is, to design for a natural frequency and damping ratio that has very low transmissibility at the frequency (or frequencies) of the support motion. For forcing frequencies near the natural frequency (near resonance), adding damping reduces the dynamic amplification. At a frequency ratio Ω of $\sqrt{2}$, the total response amplitude is equal to the base motion amplitude, regardless of the damping ratio. At higher frequencies ($\Omega > \sqrt{2}$), increasing damping increases the dynamic amplification. The phase relationship for transmissibility is more complicated than the phase relationship for the frequency response function from Z to X. The phase lags at all frequencies, including resonance ($\Omega = 1$), depend on the value of damping.

To further develop your understanding of inertially-forced vibrations, download the simulation programs found in: http://www.duke.edu/~hpgavin/cee541/cee541-mfiles.zip Unzipping `cee541-mfiles.zip` creates a folder called `cee541/m-files`. The program `linear_oscillator_sim.m` simulates the response of a linear oscillator to sinusoidal input, $z(t) = \bar{Z}\cos(2\pi f_g t)$. To use it:

1. Start Matlab and navigate to your new `cee541/m-files` folder.
2. Type `>> help linear_oscillator_sim` for information on how to run the program.
3. Run the program with a command like `>> linear_oscillator_sim(5,1,0.1)` to simulate and animate the response to a 5 Hz forcing frequency of an oscillator with a natural frequency of 1 Hz and a damping ratio of 0.10. Separate the four plots on your computer screen so that you can observe the animation in all four figures as you run simulations.

Using information provided in Figures 9 and 10 predict the amplitudes and phases of X/Z and $(X + Z)/Z$ for a value of the *ground motion frequency* of 1.0 cycles/second, a value of the natural frequency of 1.0 cycles/second, and a damping ratio of 0.10, Then run the simulation using `>> linear_oscillator_sim(1,1,0.1)` Observe the dynamic amplification and phase of the response, and confirm your prediction. Next set the ground motion frequency to 3. Before running the simulation use Figures 9 and 10 to predict the amplitudes and phases of X/Z and $(X+Z)/Z$. Confirm your prediction by running a simulation. `>> linear_oscillator_sim(1,0.3,0.1)` Notice that the oscillator remains relatively stationary while the ground moves beneath it. The total acceleration of the mass is smaller than the acceleration of the base. The relative displacement of the mass with respect to the base is nearly equal and opposite to the displacement of the base. This is considered good isolation performance. The total acceleration response is much less than the ground acceleration, and the relative displacement response is equal and opposite to the ground displacement.

Now, increase the damping ratio from 10 percent to 50 percent, use Figures 9 and 10 to make a new prediction and repeat the simulation. Notice that the total acceleration response is *larger* than when the damping was 10 percent. Note also that the total acceleration response is more 'in phase' with the ground acceleration. Finally increase the damping to 99 percent (0.99). Notice that the amplitude and duration of the initial transient is much shorter. Finally, notice how the total accelerations increase further as the higher damping level couples the mass more tightly to the ground.

Figure 3 in the Matlab simulation displays a Lissajous figure of force vs deformation. This is a plot of the force in the spring and damper $(kx(t)+c\dot{x}(t))$ vs the deformation of the spring, $x(t)$. Note that if the damping ratio is low, the restoring force is basically just $kx(t)$ and the hysteresis plot simply shows the force vs displacement relationship for the spring (a straight line).

As the damping increases, the restoring force hysteresis plot becomes more like an ellipse, and as the damping ratio gets very large the force approaches $c\dot{x}(t)$. The area of the ellipse has units of *force* $\times$ *displacement* and is equal to the energy dissipated per cycle in the damper. The bigger the loops, the more energy is dissipated into heat, and the less energy goes into shaking the oscillator.

Explore the effect of other values of the *ground motion frequency* and the *damping ratio* and notice the relationship between the frequency response (X/Z vs. Ω), transmissibility ((X+Z)/Z vs. Ω) (Figures 9 and 10), the animation, the time history plots, and restoring force hysteresis plots ($kx+c\dot{x}$ vs. x). Note how the amplitude ratio and the phase of the frequency response and the transmissibility are affected by the *ground motion frequency* and the *damping ratio*.

Investigate the significance of transients in the response. The relative importance of the transient is linked to the level of damping. Make a figure of the ratio of the peak displacement response to the ground motion displacement amplitude (`X_pk_displ / Z_displ`) vs the damping ratio, for $\Omega \in [0.5, 1.0, 2.0]$, for values of $\zeta \in [0.02 : 0.02 : 0.5]$. To generate the data for this figure quickly, comment out the plotting function in `linear_oscillator_sim.m` (around line 66), and write a `.m`-file that has a nested loop to compute the data, save that data into a matrix, and plot the data.

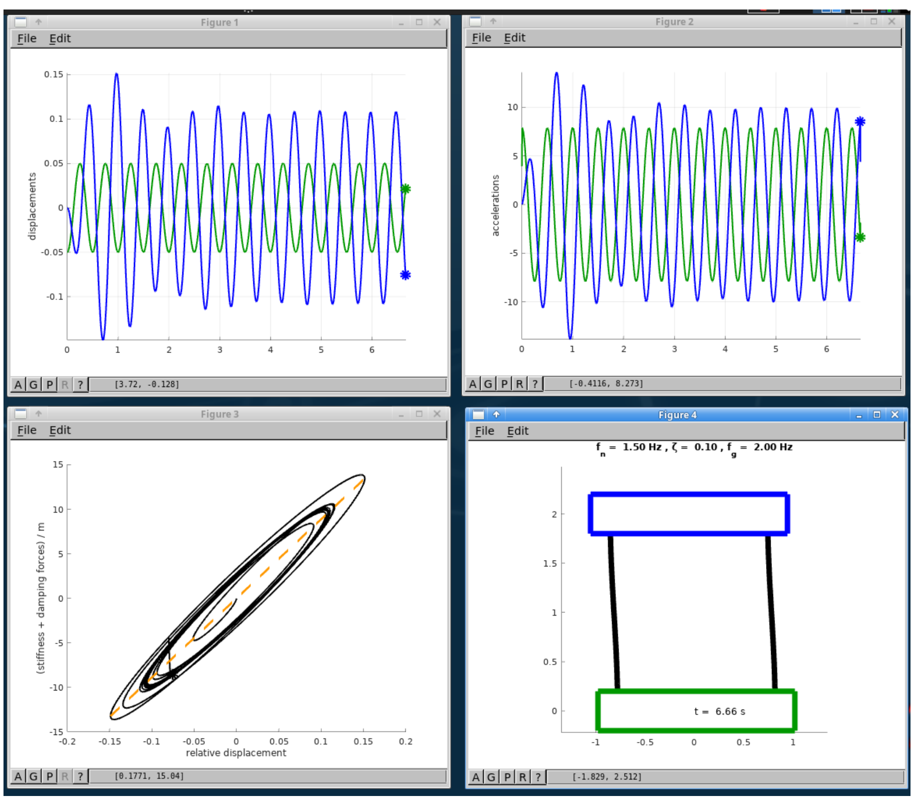


Figure 19. Figures displayed by `linear_oscillator_sim`

8.1 Exercises

1. Referring to Figures 9 and 10, for a structure to isolate its contents from base-excited vibration, should its natural frequency be higher, about the same, or lower than the predominate frequencies in the ground excitation? Why? (Note: $\Omega = \omega/\omega_{\rm n}$ = (forcing frequency)/(natural frequency).)

2. For vibration isolation in linear elastic structural systems, what are the 'pro's' and 'con's' of high damping ($\zeta > 15\%$)?

3. For a sinusoidally-base-excited structure with $\omega_{\rm n} = (1/3)\omega$, determine the value of ζ so that the magnitude of the total response motion $(X+Z)$ is only fifteen percent of the magnitude of the base motion (Z). (You'll need to solve a quadratic equation to do this.)

4. For these values of $\omega/\omega_{\rm n}$ and ζ what is the ratio of the structural deformation amplitude to the ground motion amplitude (X/Z)? Since "$(X+Z)/Z = X/Z+1$" did you expect the answer for X/Z to be -0.85? Why do you think it is not -0.85?

5. Include your plot of `X_pk_displ / Z_displ` vs ζ as described on the previous page.