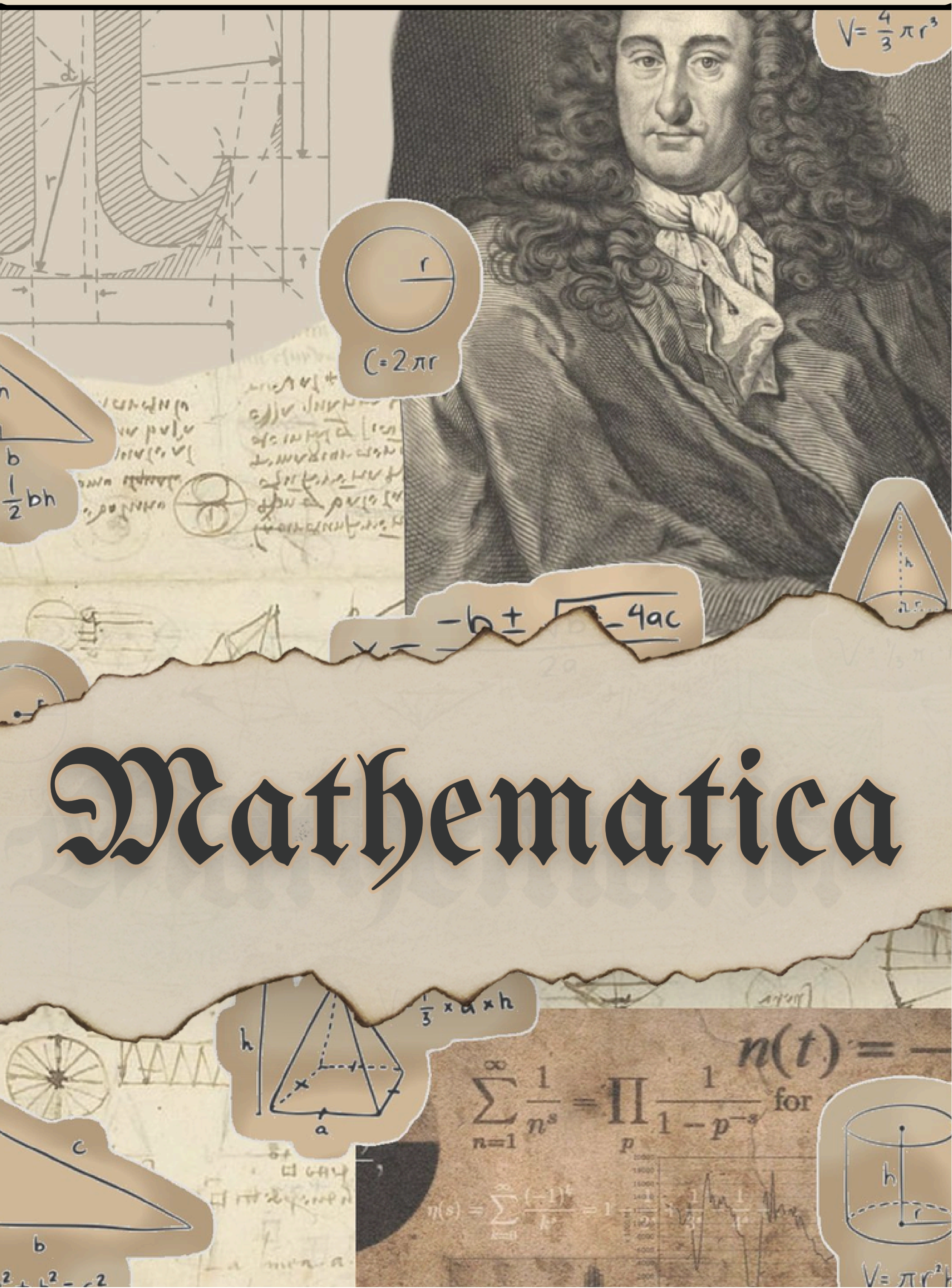

Mathematica

Letter From The Editor:

The Cathedral Math Society was established with the goal of providing opportunities to learn from teammates and compete and excel in prestigious tournaments. With this goal in mind, the Math Magazine aims to foster critical thinking and provide higher level problems for those with a sense of mathematical inquiry.

Featuring problems for all grades, from prestigious competitions such as Euclid, Fryer, American Math Competition (AMC) and Canadian Intermediate and Senior Mathematics Competition (CISMC), students can attempt problems to test their abilities, and of course, learn in the process.

All solutions have been attached in the QR Code, to refer to after problems have been solved or attempted. The most important thing to keep in mind is to ensure that you at least attempt questions, and not merely view the solution. By design, there are multiple alternate paths, steps and reasoning for each problem. “The essence of mathematics lies in its freedom.” — Georg Cantor

Until August,

Editor-in-Chief

~ Aaditya agarwal

Editor-in-chief

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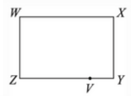
EQUATION INDEX

MATH PROBLEMS

Geometric Cylinder

Q. Rectangle $WXYZ$ has $WX = 4$, $WZ = 3$, and $ZV = 3$. The rectangle is curled without overlapping into a cylinder so that sides WZ and XY touch each other. In other words, W touches X and Z touches Y. The shortest distance from W to V through the inside of the cylinder can be written in the form:

$$\sqrt{\frac{(a+b\pi)^2}{c\pi^2}}$$



where a, b and c are positive integers. The smallest possible value of $a+b+c$ is:

A) 12 B) 26 C) 18 D) 19 E) 36

Permutations & Combinations

Q. A new language uses only the letters A, B, C, D, and E. The letters A and E are called vowels, while the letters B, C, and D are called consonants. A sequence of letters is called a word if it does not include the same letter twice in a row, and it does not include two vowels in a row.

How many words are there in this language that are 10 letters long and that begin with a vowel?

- 199680
- 199968
- 199584
- 199872
- 199776

Repeating Decimal Fractions

There are F fractions $\frac{m}{n}$ with the properties:

- m and n are positive integers with $m < n$,
- $\frac{m}{n}$ is in lowest terms,
- n is not divisible by the square of any integer larger than 1, and
- the shortest sequence of consecutive digits that repeats consecutively and indefinitely in the decimal equivalent of $\frac{m}{n}$ has length 6.

Note: The length of the shortest sequence of consecutive digits that repeats consecutively and indefinitely in $0.\overline{12745} = 0.127457457457\ldots$ is 3, and the length of the shortest sequence of consecutive digits that repeats consecutively and indefinitely in $0.\overline{5}$ is 1.

We define $G = F + p$, where the integer F has p digits. What is the sum of the squares of the digits of G?

A) 170 B) 168 C) 217 D) 195 E) 181

MATH PROBLEMS

Probability Q1

Q. Andreas, Boyu, Callista, and Diane each randomly choose an integer from 1 to 9, inclusive. Each of their choices is independent of the others and the same integer can be chosen more than once. The probability that the sum of their four integers is even is equal to $\frac{N}{6561}$ for some positive integer N. What is the sum of the squares of the digits of N?

Probability Q2

Q. Leilei, Jerome and Farzad write a test independently. The probability that Leilei passes the test and Jerome fails the test is $\frac{3}{20}$. The probability that Jerome passes and Farzad fails is $\frac{1}{4}$. The probability that Leilei and Farzad both pass is $\frac{2}{5}$.

Determine the probability that at least one of Leilei, Jerome and Farzad fails the test.

Combinatorics – Lattice Counting

Q. Makayla finds all the possible ways to draw a path in a 5×5 diamond-shaped grid. Each path starts at the bottom of the grid and ends at the top, always moving one unit northeast or northwest. She computes the area of the region between each path and the right side of the grid. Two examples are shown in the figures below. What is the sum of the areas determined by all possible paths?

Integration with Parameters

Q. A migratory bird begins flying at dawn and stops at dusk. Its instantaneous speed over the course of the daylight hours is modeled by the function:

$$v(t) = v_{\max} \cdot \sin^2\left(\frac{\pi t}{D}\right)$$

where:

- $v_{\max}$ is the bird's maximum speed,
- D is the total duration of daylight (in hours),
- t is the time since dawn (in hours).

(a) Find a closed-form expression for the total distance flown by the bird during the day.

(b) Determine the exact or approximate time $T^* \in [0, D]$ when the bird has covered half of the total distance.

Problem of the Week

1 Grade 8 and below

1.1 Tankmates

Niall has three large fish tanks and would like to put some tetras and guppies in each tank. He currently has 19 tetras and 18 guppies and doesn't want any of his tanks to contain more guppies than tetras. Also, he would like each tank to have at least 5 tetras and 3 guppies.

Determine the largest number of fish that can be in one of his fish tanks.

1.2 They Take the Cake

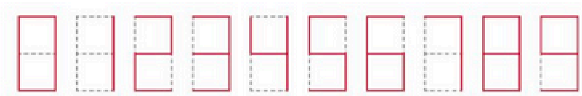
Jessica, Callista, Peter, and Monica went to the bakery to buy seven cakes. Each cake costs \$ 9.00. Jessica paid \$ 27.00, Callista paid \$ 9.00, and Peter paid \$ 22.50. Monica paid the remaining amount. They divide the cakes so that the fraction of the total that each person paid is equal to the fraction of the total amount of cake that each person receives.

What amount of cake should each person receive?

2 Grade 9 and 10

2.1 The Clock Works

Halina's clock uses a digital LED display where each digit is represented by seven LED segments that are either on or off, as shown.



Sometimes some of the LED segments stop working. When the top-most horizontal LED segment stopped working, both the digit 1 and the digit 7 appeared as shown. This was a problem because Halina couldn't distinguish between them.

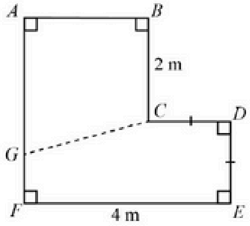
Halina replaced the broken LED segment, but then a week later found that a different LED segment had stopped working. However, this time, she was still able to distinguish between all ten digits.

What is the largest number of LED segments that can be broken at the same time, while still allowing Halina to distinguish between all ten digits?

2.2 Dividing Line

The Bobsie twins share an L-shaped room. The area of the entire room is $11.2\,\text{m}^2$. The twins are not getting along, so their parents decide to partition the room with tape so that each child has exactly the same area.

The layout of their room is represented by ABCDEF in the diagram. The partitioning tape, indicated by a dashed line, will travel from C to a point G on AF.

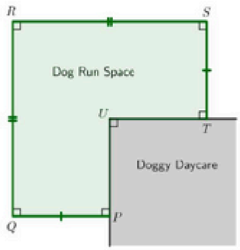


Where should G be located on AF in order to split the room into two smaller rooms of equal area?

3 Grade 11 and 12

3.1 Run Dog Run

At POTW Doggy Care, there is a need for a new outdoor dog run space. The layout of the dog run space is represented by P QRST U in the diagram below.



The lengths of the two longer sides, QR and RS, are to be the same, and the lengths of the two shorter sides, PQ and ST, are to be the same. There will be right angles at each corner.

The dog run space is to be built using a fence along PQ, QR, RS, and ST, and using the walls of the daycare along PU and TU. The total fencing to be used is 30 m. Determine the dimensions of the dog run space that will give the maximum area for the dog run.

3.2 Summing up a Sequence 2

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Shweta writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. How many different possible values of n are there?

$$24, 12, 6, \ldots.$$

****

SOLUTION TO THE POTW

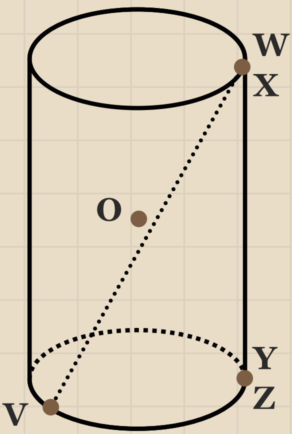
Geometric Cylinder

By Garvit Sood

After the cylinder is created, it is clear that WY is vertical and so is perpendicular to the plane of the base of the cylinder.

Hence, we can see that $\angle Y = 90°$ and that $\triangle VYW$ is right-angled at $\angle Y$. Using the Pythagorean Theorem, we can then say that $WV^2 = WY^2 + VY^2$. We also know that WY will be equal to the breadth of the original rectangle, or WZ : WY = WZ = 3.

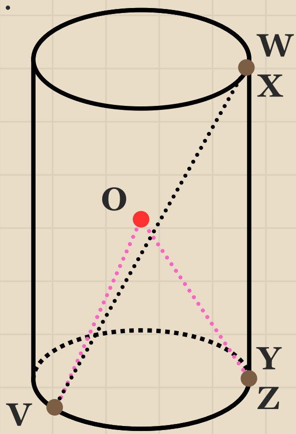
Let O be the centre of the circular base of the cylinder.



In the original rectangle, we know that ZY = WX = 4, and that ZV = 3. Hence, VY = ZY - ZV = 1 = ¼ ZY.

Since ZY is now the circumference of the circular base of the cylinder, this means that V is ¼ of the way around the circumference of the base of the cylinder.

Hence, we can say that $\angle YOV = 90°$, since 90° is one quarter of a full circular angle. Thus, $\triangle YOV$ is right angled at O. Using the Pythagorean Theorem: $VY^2 = YO^2 + VO^2$.



Geometric Cylinder

By Garvit Sood

Since VO and YO are radii of the circular base of the cylinder, they are equal. Hence, $VY^2 = 2YO^2$

To find the radius of the circular base of the cylinder, we can use the formula for the circumference of a circle, which is $2\pi r$. Since we know that the circumference of the base of the cylinder is 4, hence $r = \frac{4}{2\pi} = \frac{2}{\pi}$

Since YO = r, $VY^2 = 2\left(\frac{2}{\pi}\right)^2 = \frac{8}{\pi^2}$

Thus,

$WV^2 = WY^2 + VY^2$

$$WV^2 = 3^2 + \frac{8}{\pi^2} = \frac{9\pi^2 + 8}{\pi^2}$$

$$WV = \sqrt{\frac{8 + 9\pi^2}{\pi^2}}$$

Since the coefficient of π in the denominator is 1, it is not possible to reduce the values of a, b, or c any further. Hence, a = 8, b = 9, and c = 1. Hence, a + b + c = 18

Answer: C) 18

Permutations & Combinations

By Alekha Kilachand

First, let's break down what the question is asking of us. We want to find the number of 10-letter words starting with a vowel we can make with the vowels “A” and “E” and the consonants “B”, “C” and “D”.

The following rules also must be followed:

- No letter is repeated twice in a row
- No two vowels in a row are allowed

Permutations & Combinations

By Alekha Kilachand

The words that are 1 letter long are A, B, C, D, E. The given set comprises of all combinations:

{AB, AC, AD, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EB, EC, ED}.

Let's assume V_N to be the number of words of length N that begin with a vowel and C_N to be the number of words that begin with a consonant.

Thus $V_1 = 2$ and $V_2 = 6$
Thus $C_1 = 3$ and $C_2 = 12$

CASE 1

Let's assume a word starts with a vowel and is of length N. The Number of such words would be V_N.

If this word starts with a vowel and is of length N, if the vowel is removed, the word starts with a consonant (because two vowels can't be in a row) and is of length N - 1. Since the length of the word with the vowel removed is N - 1 and it starts with a consonant, using our assumption, the number of such words is C_{N-1}.

Since there are two vowels to choose from, the two expressions can be related as:

$$V_N = 2 \bullet C_{N-1}$$

Assuming N as 2, we have discovered V_2 to be 6 and we have discovered C_1 as 3. Hence, our relationship works.

Permutations & Combinations

By Alekha Kilachand

CASE 2

Let's assume a word starts with a consonant and is of length N. The number of such words would be C_N. The *rest of the word* without the consonant could either start with a vowel or a consonant. However the length of the rest of the word will be N - 1.

1. Assuming the *rest of the word* starts with a vowel (the initial consonant is followed by a vowel), the number of such words would be V_{N-1}. Since three such consonants could be added $3 \bullet V_{N-1}$ is the number of words that could be formed if the word started with a consonant followed by a vowel.

Eg: CAB. Here N = 3.

1. Assuming the *rest of the word* starts with another consonant (so the initial consonant is followed by another consonant), the number of such words would be C_{N-1}. Since only 2 other consonants could be added (not repeating the same one) $2 \bullet C_{N-1}$ is the number of words that could be formed if the word started with a consonant followed by another consonants

Eg: CBA. Here N = 3.

$$C_N = (3 \bullet V_{N-1}) + (2 \bullet C_{N-1})$$

Starting with our initial values: $V_1 = 2$, $C_1 = 3$. We can find V_2 and C_2 using the the following formulas we have just found:

1. $V_N = 2 \times C_{N-1}$

2. $C_N = (3 \times V_{N-1}) + (2 \times C_{N-1})$

Permutations & Combinations

By Alekha Kilachand

$V_2 = 2 \times C_1 = 2 \times 3 = 6$

$C_2 = (3 \times V_1) + (2 \times C_1)$
$= (3 \times 2) + (2 \times 3) = 12$

Hence, we can find the values of V_3 and C_3 using the values of V_2 and C_2. We find all the values until V_{10} and C_{10} using the previous values. The table below is the table of values constructed using our previous values.

n	v_n	c_n
1	2	3
2	6	12
3	24	42
4	84	156
5	312	564
6	1128	2064
7	4128	7512
8	15 024	27 408
9	54 816	99 888
10	199 776	364 224

We want to find the no. of valid words that start with a vowel and are 10 letters (V_{10}).

Answer: E) 199776

Recurring Decimals

By Aditya Gupta

There are F fractions, which are in the form *m/n*, with *n>m*. “n” is not divisible by the square of any integer ≥ 1, and the shortest sequence of recurring digits in the decimal equivalent is 6. We are asked to find the sum of the squares of the digits of G, where G is defined as the sum of F and the number of digits in F.

A recurring decimal can be represented in the form $0.g_1g_2 \ldots g_p\overline{r_1r_2 \ldots r_q}$ for some integers p ≥ 0, q > 0 and digits $g_1, g_2, \ldots g_p, r_1, r_2, \ldots, r_q$.

Recurring Decimals

By Aditya Gupta

However, since we know that the length of the recurring sequence is 6, we can represent the number as $0.g_1g_2 \dots g_p\overline{r_1r_2 \dots r_6}$

Recurring decimals, can be written in the form $\frac{c}{10^p(10^q-1)}$, for some positive integer c, where p is the number of decimal places before the recurring starts and q is the number of recurring digits. Since we know that q is 6, we can write this as $\frac{c}{10^p(999999)}$.

By equating m/n with $\frac{c}{10^p(999999)}$, we get $c \cdot n = m \cdot 10^p(999999)$

We also know that m/n is in its lowest term, having no common divisor > 1. Therefore, n must be a divisor of $10^p \cdot 999999$.

999999 can be written as $3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Therefore we can write $10^p \cdot 999999$ as $2^p \cdot 5^p \cdot 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Additionally, since n is not divisible by the square of any integer, it cannot be divisible by the square of any prime number either.

Therefore, n must be a divisor of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37 = 1111110$. m/n can be written as $x/1111110$, for some integer x, which lies in $1 \leq x \leq 1111109$. Since each $s/1111110$ is different, the m/n produced by each will be different and the 1111109 fractions will satisfy the conditions.

To ensure that every fraction has a true repeating period of 6, we must remove those fractions which have true repeating periods less than 6, mainly 1,2,3 as they are factors of 6. There are a total of 1109 fractions with denominator 1110, 329 fractions with denominator 330 and 29 fractions with denominator 30.

Recurring Decimals

By Aditya Gupta

However, some of these can also overlap. The final total number of decimals with repeating periods less than 6, will be (1109 + 329 - 29) = 1409

F = 1111109 - 1409 = 1109700. Since F has 7 digits, p = 7 and G = 1109700 + 7 = 1109707

The sum of the squares of the digits of G is $1^2 + 1^2 + 0^2 + 9^2 + 7^2 + 0^2 + 7^2 = 1 + 1 + 81 + 49 + 49 =$ **181**

Answer: E) 181

Probability Q1

By Adyant Gupta

The values of the integers picked by the four people in this sum are unknown, so let's assign variables to the values:

- Andreas - 'a'
- Boyu - 'b'
- Callista - 'c'
- Diane - 'd'

For each variable, there are 9 possible values it could take (any number from 1 to 9). *S*ince the variables cannot be used interchangeably i.e. (a, b, c, d) = (1, 2, 3, 4) is not the same as (a, b, c, d) = (4, 3, 2, 1)

Probability Q1

By Adyant Gupta

Each variable a, b, c and d can take 9 distinct values and the values of one variable are not affected by the values of another variable

The total number of quadruples (a, b, c, d) = 9^4 = **6561**

TIP:

Notice, the denominator of the fraction given in the sum is also 6561

The scenarios are the following

- 4 even, and 0 odd
- 3 even, and 1 odd
- 2 even, and 2 odd
- 1 even, and 3 odd
- 0 even, and 4 odd

We're categorizing the 4 variables to know whether each number is odd or even and predict whether the sum of that set of numbers will be odd or even.

No of Even Numbers	Sum
4	Even
3	Odd
2	Even
1	Odd
0	Even

∴ We know that for a + b + c + d to be even, there are 3 cases:

- All 4 variables are even *(Case 1)*
- 2 of the four variables are even, and 2 are odd *(Case 2)*
- All 4 variables are odd *(Case 3)*

We have to count the number of quadruples for each of the above cases. First, we must establish

- If the value is even, there are 4 possibilities (2, 4, 6, 8) for the integer value

Probability Q1

By Adyant Gupta

- If the value is odd, there are 5 possibilities (1, 3, 5, 7, 9) for the integer value

Case 1:
4 possibilities for each of the four variables, giving 44 or 256 total quadruples

Case 2:
4 possibilities for the two even numbers, and 5 possibilities for the 2 odd numbers

The even integers could be either (a,b), (a,c), (a,d), (b,c), (b,d), or (c,d)

So, a total of 6 possibilities for the two even values (with the two other variables in each case being odd)

For each case, the number of quadruples are $4^2 \times 5^2 = 400$

Considering all 6 cases, the total number of quadruples = $400 \times 6 = 2400$

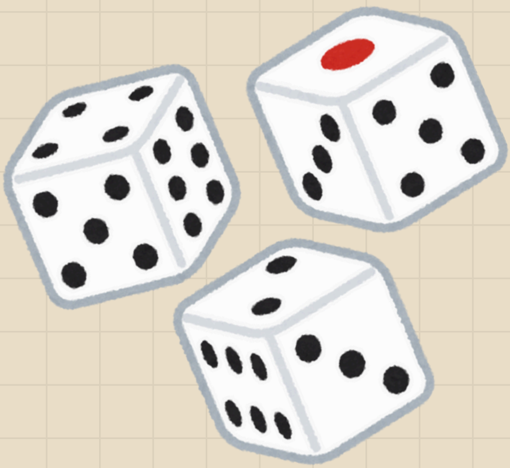
Case 3:
5 possibilities for all four variables, giving 54 or 625 total quadruples

$\boldsymbol{N = 625 + 2400 + 256 = 3281}$

We must find the sum of the squares of N's digits:
$3^2 = 9$, $2^2 = 4$, $8^2 = 64$, $1^2 = 1$

$9 + 4 + 64 + 1 = \mathbf{78}$

Answer: 78



Probability Q2

By Advik Jha

Q. *Leilei, Jerome and Farzad write a test independently. The probability that Leilei passes the test and Jerome fails the test is 3/20. The probability that Jerome passes and Farzad fails is 1/4. The probability that Leilei and Farzad both pass is 2/5. Determine the probability that at least one of Leilei, Jerome and Farzad fails the test.*

SOLUTION:

Let's define:

- A: L passes
- B: J passes
- C: F passes

Therefore,

- A´ = 1 - A: L fails
- B´ = 1 - B: J fails
- C´ = 1 - C: F fails

Then:

- P $(A \cap B^c) = 3/20$
- P $(B \cap C^c) = 1/4$
- P $(A \cap C) = 2/5$

A (1 - B) = 3/20
A = 3/20 • (1 - B)

Similarly,
B (1 - C)=¼
1 - C = 1/4B
C = 1 - 1/4B
A • C = ⅖

When we substitute to 3rd equation we get:

$$\frac{3}{20}(1-b)*(1-1/4b) = \frac{2}{5}$$

As A(1 - B) = 0, 1 - B ≠ 0
3(1 - 1/4B) = 8(1 - B)

Since B(1 - C) ≠ 0, B ≠ 0
3(4B - 1) = 32B(1 - B)
$12B - 3 = 32B - 32B^2$
$32B^2 - 20B - 3 = 0$
(8B + 1)(4B - 3) = 0

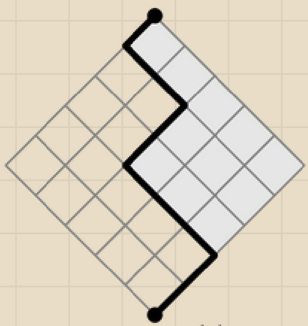
B is ¾ as it is > 0
C = 1 - 1/4B = 1 - ⅓ = ⅔
A = 3/20 • (1 - B) = 3/20 • (¼) = ⅗
Probability that at least 1 fails is 1 - (A • B • C)
= 1 -(¾)(⅔)(⅗)
=1 - (18/60) =1 - (3/10) = 7/10

Answer: 7/10

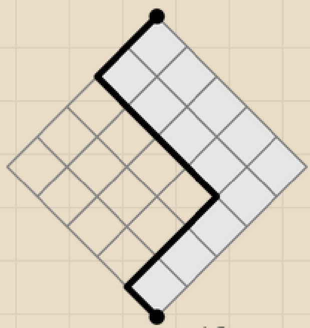
Combinatorics Problem with Lattice-Counting

By Reyansh Aggarwal

Q. *Makayla finds all the possible ways to draw a path in a 5 x 5 diamond-shaped grid. Each path starts at the bottom of the grid and ends at the top, always moving one unit northeast or northwest. She computes the area of the region between each path and the right side of the grid. Two examples are shown in the figures below. What is the sum of the areas determined by all possible paths?*



area = 11



area = 13

SOLUTION:

1. Turn the 5 x 5 diamond sideways to rest it completely on one edge. It becomes a square with 25 smaller unit squares
2. In a 5 x 5 square every path from the bottom corner to the top corner and will have 10 legal steps.
3. 5 steps will be towards **Right** and 5 steps will be **Up**, otherwise you will not end up at the desired point
4. To calculate total number of paths, let us think of this as writing a 10 letter word, made of letters **'R'** and **"U'.**
5. We must choose 5 positions for **R** as the rest will be **U**

Combinatorics Problem with Lattice-Counting

By Reyansh Aggarwal

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

6. This becomes a combination problem, which can be solved using formula where n is the total number of positions and r is the number of positions we must fill.

7. It is equal to $\frac{10!}{5! \cdot 5!} = 252$ *different paths.*

8. Now, since the diamond/square is symmetric, each of the 25 smaller squares are on the right side 50% of the times.

Therefore, the average area on the right side 25 • 50% = 12.5

9. Multiply this by the 252 possible paths we calculated earlier, **252 x 12.5 = 3150**

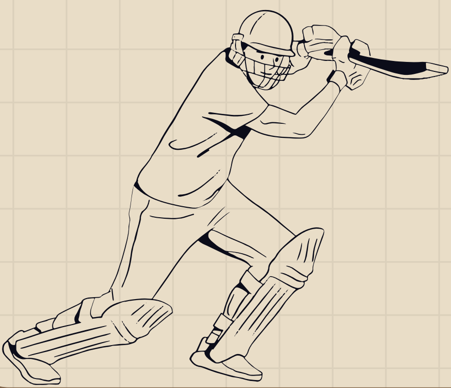
Answer: B) 3150

****

Maths x Cricket DLS

By Reyansh Aggarwal

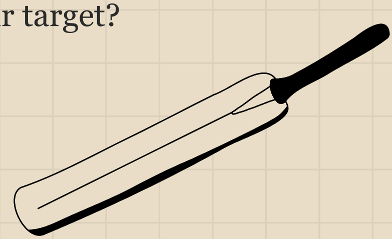
Maths is not only the homework you rush through before a cricket game, it is the unknown 12th player influencing every move silently. Captains sketch angles to slot fielders in the right position, spin bowlers change trajectories or where to pitch based on degree of turn, and analysts crunch probabilities to know whether to bat first or second depending on the conditions. A batter's average, a bowler's economy, even the win-probability graph on our screens, all come from equations, statistics, and computer simulations.



Nowhere is that math-magic clearer than on a rainy day, when rain steals overs and suddenly the target changes without a single ball being bowled. That is the Duckworth–Lewis–Stern (DLS) method at work or what I call Cricket's emergency maths kit. It looks like magic, but it is really a combination of algebra, exponential curves, and percentages.

How DLS Works?

The Problem: A 50 over match suddenly stops at 21.3 overs in the second innings due to rain and 10 overs are lost. Team 2 now has fewer balls to score the target. Giving them the same target is not fair, so what is the fair target?



Maths x Cricket DLS

By Reyansh Aggarwal

Core Concept: Resources, the two most important resources which the batting team has are Overs left (O) and Wickets in hand (W). More O and more W means more resources and more run scoring power.

Resource Table: Statisticians studied thousands of one-day games to build a giant table. This table tells you, if you have X overs and Y wickets, you own Z % of total resources.

They used the Exponential Decay function you meet in Grade 11 Maths along with Regression to build an exponential curve which helped fill this table. The formula for the curve being:

$$R(O,W) = Z_w(1 - e^{-bO})$$

Here, Z_w is the maximum runs that remain if you keep those wickets all fifty overs, and b controls how fast scoring slows down. If it went over your head like mine, it basically means that a team has high run scoring power per over when it has more overs and more wickets.

The magical Duckworth-Lewis method formula:

Team 2's Par Score = Team 1's Score × (Team 2's Resources / Team 1's Resources)

Example:

1. Team 1 bats first and makes 250 runs in 50 overs (they used 100 % of their resources)
2. Rain strikes before Team 2 begins. The innings is cut to 30 overs. They still have all 10 wickets in hand.

Maths x Cricket DLS

By Reyansh Aggarwal

3. The DLS handbook has a 100 × 10 table based on the exponential curve. Umpires will look up Team 2's resources in the table. The entry may look like:

Overs left	Wickets in Hand	Resource %
30	10	75

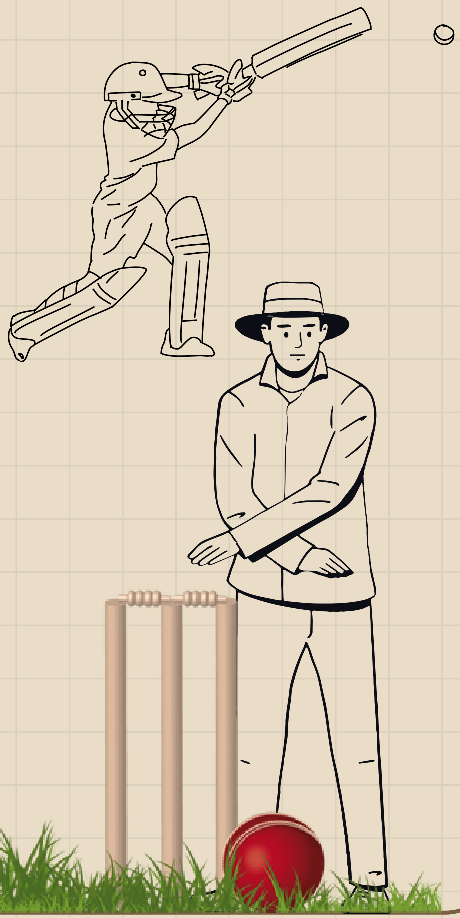
So, with 30 overs and 10 wickets intact, Team 2 has 75 % of a full 50-over innings resources.

4. Par Score = 250 • 75/100 = 187.5, since scores are in whole numbers, **the par score is 188 and the revised target is 189.**

Did you notice the maths in action, percentages, direct proportion, and tables built from the exponential curve. The curve rewards losing less wickets. It factors in how aggressively teams will play based on available resources.

If Team 2 has fewer overs, they will swing for the fences!

Next time when rain stops play, remember the match now belongs to Maths and Maths plays no favourites.



Integration with Parameters

By Vaibhav Sanghai

Q. *A migratory bird begins flying at dawn and stops at dusk. Its* ***instantaneous speed*** *over the course of the daylight hours is modeled by the function:*

$$v(t) = v_{max} \cdot sin^2(\frac{\pi t}{D})$$

where:

- *v_{max} is the bird's maximum speed,*
- *D is the total duration of daylight (in hours),*
- *t is time since dawn (in hours).*

Find a closed-form expression for the ***total distance*** *flown by the bird during the day.*

Determine the exact or approximate time $T^* \in [0, D]$ when the bird has covered ***half of the total distance****.*

The speed function is:

$$v(t) = v_{max} \cdot sin^2(\frac{\pi t}{D})$$

· The speed $v(t)$ starts at 0 at dawn ($t = 0$), peaks at midday ($t = D/2$) where $sin^2(\pi/2) = 1$, and returns to 0 at dusk ($t = D$).

· The graph is a smooth, symmetric hill—ideal for integration—meaning that the acceleration and deceleration profiles mirror each other.

This creates a hill-shaped speed-time graph where the area under the curve represents **total distance**. This shape ensures the bird speeds up, flies fastest mid-day, and slows down symmetrically.

Integration with Parameters

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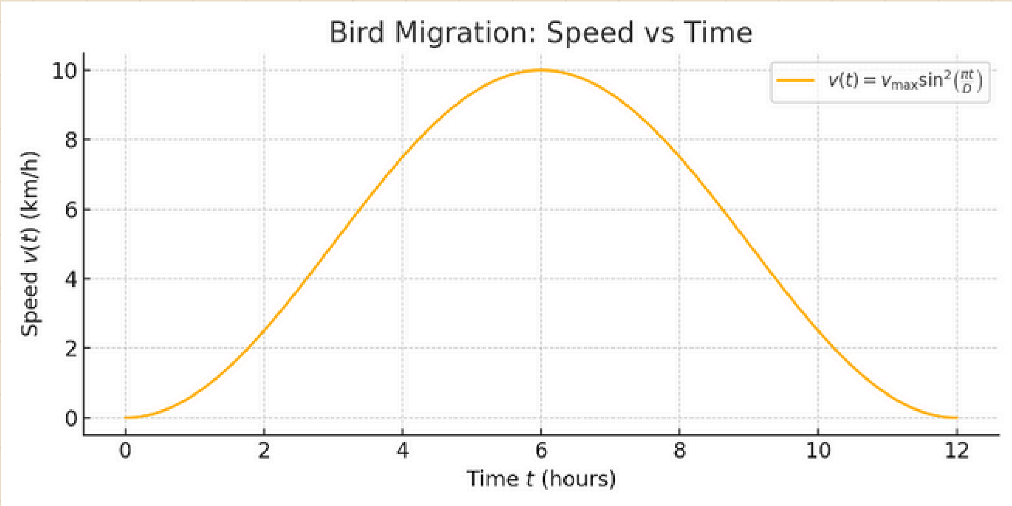

Total distance *S(D)* flown using definite integration:

$$S(D) = \int_0^D v(t)dt = \int_0^D v_{max} \cdot sin^2\left(\frac{\pi t}{D}\right) dt$$

Factor out constants:

$$S(D) = v_{max} \cdot \int_0^D sin^2\left(\frac{\pi t}{D}\right) dt$$

Use the identity:

$$sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Apply to our function and simplify by integrating term-by-term:

$$S(D) = \frac{v_{max}}{2} \cdot \int_0^D \left[1 - cos\left(\frac{2\pi t}{D}\right)\right] dt$$

as $\int_0^D 1dt = D,$

$$\int_0^D cos\left(\frac{2\pi t}{D}\right) dt = \frac{D}{2\pi} \cdot \sin\left(\frac{2\pi t}{D}\right) |_0^D = 0$$

Therefore, the Final Answer for the **total distance**:

$$S(D) = \frac{v_{max} \cdot D}{2}$$

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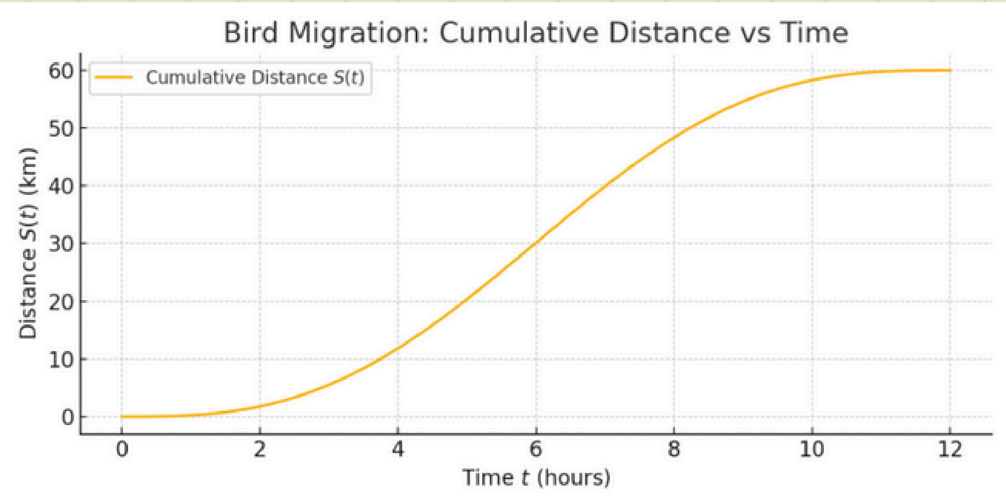


Half distance *S(D)/2* occurs when:

$$S(T^*) = \frac{S(D)}{2} = \frac{v_{max} \cdot D}{4}$$

Recall the general form of *S(t)* for any *t*:

$$S(t) = \frac{v_{max}}{2}\left[t - \frac{D}{2\pi} \cdot \sin\left(\frac{2\pi T^*}{D}\right)\right]$$

Set:

$$\frac{v_{max}}{2}\left[T^* - \frac{D}{2\pi} \cdot \sin\left(\frac{2\pi T^*}{D}\right)\right] = \frac{v_{max} \cdot D}{4}$$

Divide $v_{max}/2$ from both sides:

$$T^* - \frac{D}{2\pi} \cdot \sin\left(\frac{2\pi T^*}{D}\right) = \frac{D}{2}$$

This is a **transcendental equation**, meaning it can't be solved algebraically. But numerically (e.g. via graphing or iteration), we find the solution for **Half Distance**:

$$\mathbf{T^* \approx 0.45D}$$

Srinivasa Ramanujan

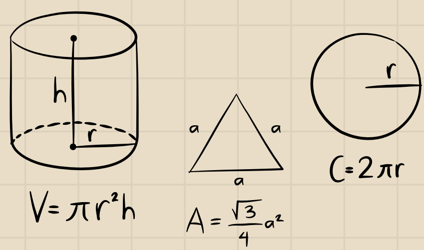
By Vaibhav Sanghai

Ramanujan was born in Kumbakonam, a small town in South India, in 1887. He dedicated his time to learning about numbers. He is well-known as one of the most brilliant mathematicians ever born.

Ramanujan became well-known for his deep fascination with mathematics. His performance in school subjects remained weak as his mind focused entirely on mathematics and numbers. The sixteen-year-old Ramanujan discovered a worn copy of *A Synopsis of Elementary Results in Pure Mathematics* written by G.S. Carr. The book contained thousands of theorems without proofs, but Ramanujan discovered intricate patterns in them. He used them to recreate and expand the theorems without any formal mathematics education.

By his twenties, Ramanujan had researched infinite series, continued fractions and prime numbers. He said that the goddess *Namagiri* sent him divine visions about these mathematical patterns via his dreams.

Ramanujan sent a letter to Cambridge mathematician G.H. Hardy in 1913 that included pages of formulae which he was working on, which left Hardy astounded. Hardy mentioned that the formulae must have been correct because only someone with extraordinary creative abilities could develop them. Hardy invited Ramanujan to join him at Cambridge in England.



Srinivasa Ramanujan

By Vaibhav Sanghai

Ramanujan worked with Hardy and J.E. Littlewood, from 1914 to 1919, to study partition functions, modular forms and mock theta functions. Ramanujan formulated new equations to calculate π, ground-breaking research on Fermat's Last Theorem and his theta function was considered the heart of string theory in Physics.

Hardy once went to meet Ramanujan when he was sick at the hospital. He arrived in a taxi with the number 1729. Ramanujan remarked that 1729 was fascinating because it represented the smallest natural number which could be summed up by two different pairs of cubes ($1^3 + 12^3$ and $9^3 + 10^3$). 1729 is now called the *Ramanujan-Hardy number*.

Ramanujan achieved glorious feats at Cambridge and faced health challenges as well as scarcity caused by war and social alienation. Ramanujan was physically weak and tired when he returned to India in 1919. Sadly, the world lost this prodigy in 1920 at the ripe age of 32. In his brief life, Ramanujan produced around 4,000 theorems, some of which continue to remain unsolved until today.

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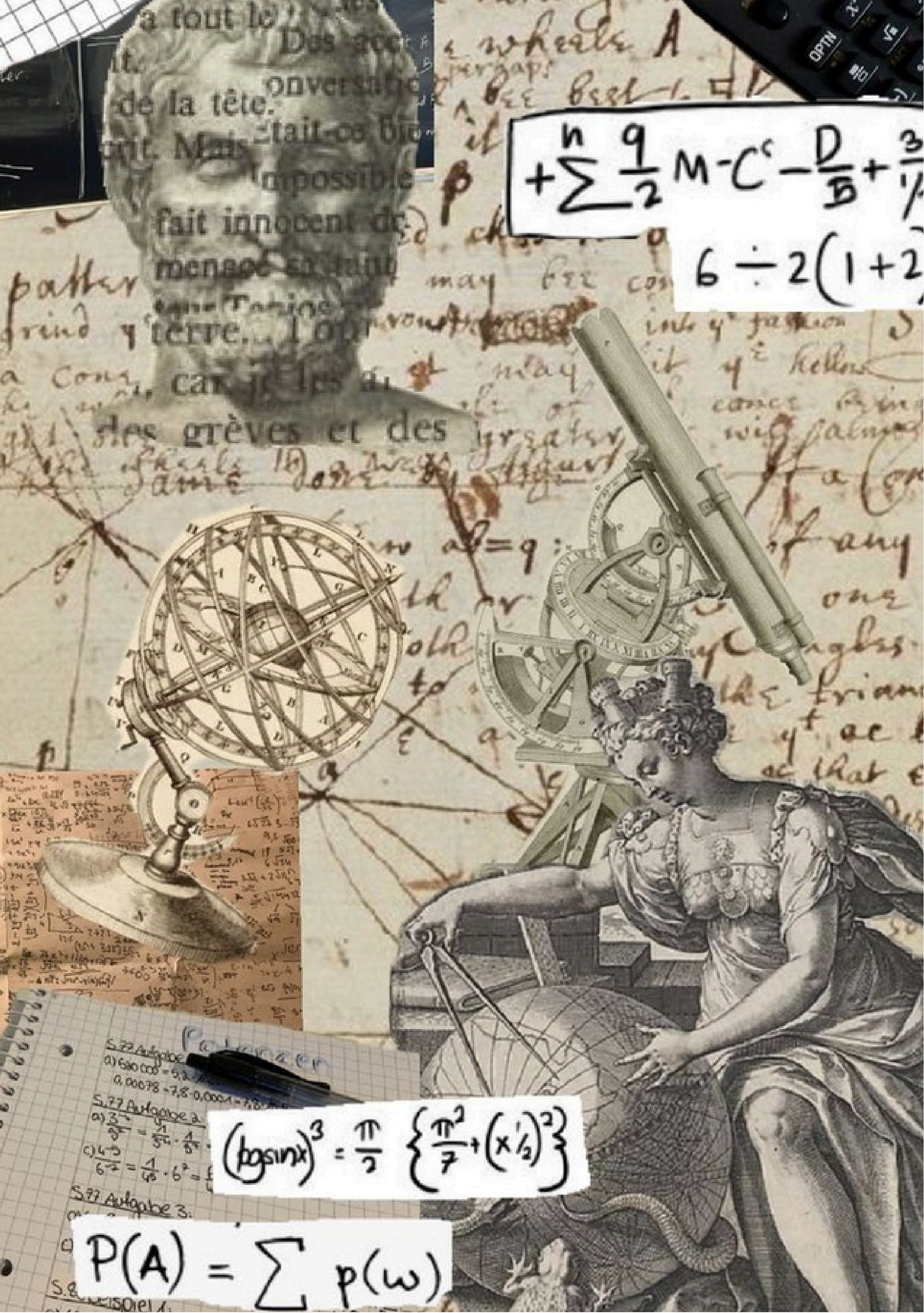
a tout le
de la tête.
impossible
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6 ÷ 2(1+2
P(A) = ∑ p(ω)