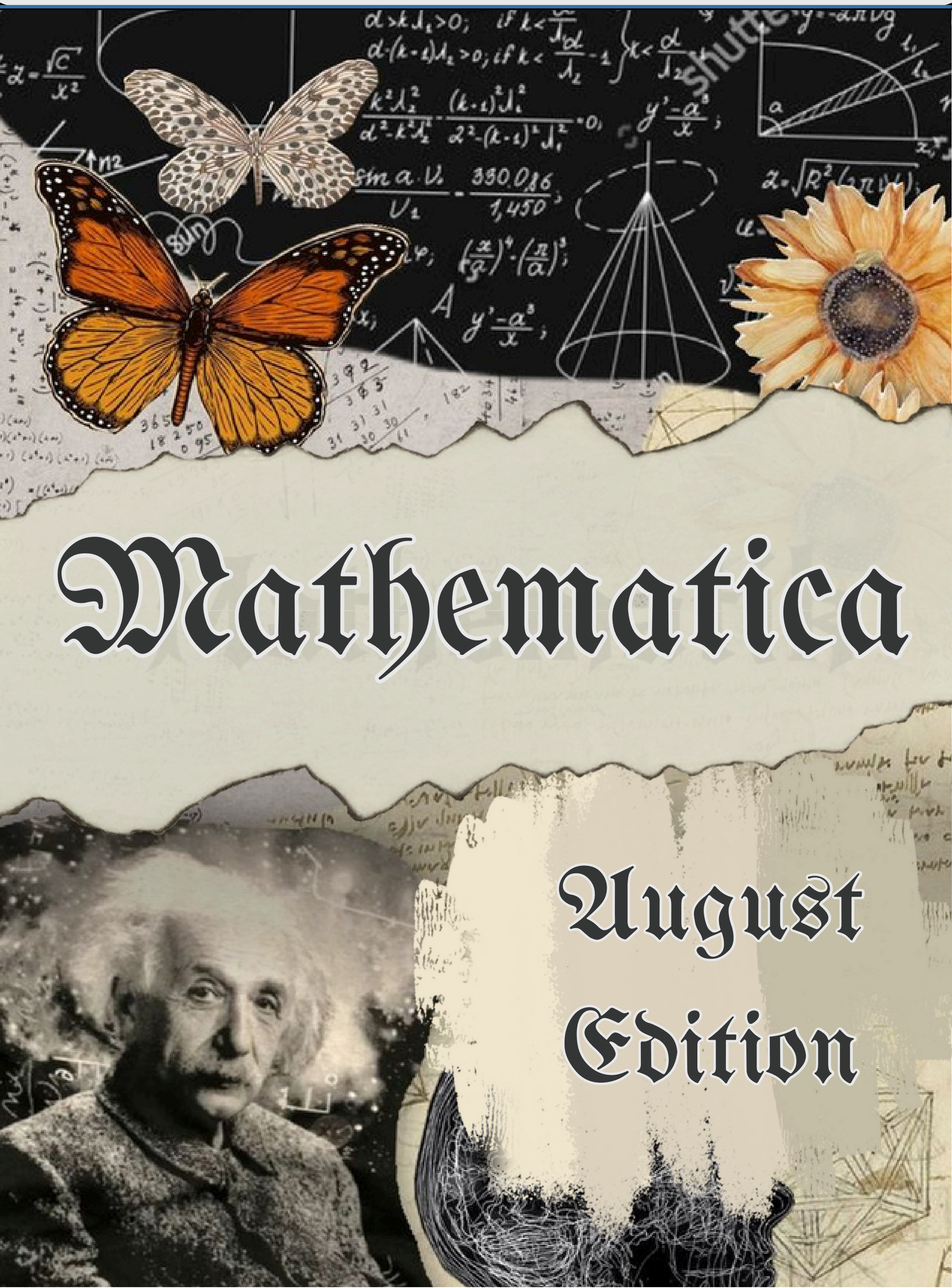

Mathematica

August Edition

Cathedral Math Society

Editor-in-Chief

Aaditya Agarwal

Letter From The Editor:

The Cathedral Math Society was founded to create a space where students could challenge themselves and excel in prestigious mathematical competitions. With this spirit, Mathematica aims to nurture curiosity and provide math problems that encourage critical thinking and exploration.

This second edition brings together a selection of problems from competitions such as Euclid, Fryer, the American Mathematics Competitions (AMC), and the Canadian Intermediate and Senior Mathematics Competitions (CISMC). These problems are intended not only to test ability, but also to spark new ways of thinking and problem-solving. This edition also dives into the mathematics behind basketball.

Solutions can be accessed through the QR code provided. We encourage you to attempt each question as it often teaches far more than simply reading its solution.

As Paul Halmos said, “The only way to learn mathematics is to do mathematics.” We hope this edition inspires you to do exactly that.

Until September,

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Aaditya Agarwal

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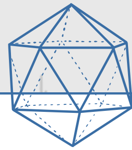
EQUATION INDEX

MATH PROBLEMS



1.0.1 Vertex Paths of Icosahedron

A regular icosahedron is a 20-faced solid where each face is an equilateral triangle and five triangles meet at every vertex. The regular icosahedron shown below has one vertex at the top, one vertex at the bottom, an upper pentagon of five vertices all adjacent to the top vertex and all in the same horizontal plane, and a lower pentagon of five vertices all adjacent to the bottom vertex and all in another horizontal plane. Find the number of paths from the top vertex to the bottom vertex such that each part of a path goes downward or horizontally along an edge of the icosahedron and no vertex is repeated.

1.0.4 Sequences and Series

In the following expression, Melanie changed some of the plus signs to minus signs:

1+3+5+7+...+97+99

When the new expression was evaluated, it was negative. What is the least number of plus signs that Melanie could have changed to minus signs?

A. 14
B. 15
C. 16
D. 17

1.0.2 Permutations and Combinations

How many ways are there to split the integers 1 through 14 into 7 pairs so that in each pair the greater number is at least 2 times the lesser number?

1.0.5 Vertexes of a Rhombic Dodecahedron

A rhombic dodecahedron is a solid with congruent rhombus faces. At every vertex, or edges meet, depending on the vertex. How many vertices have exactly edges that meet?

A. 5
B. 6
C. 7
D. 8
E. 9

MATH PROBLEMS

1.0.6 Properties of Triangles

In the diagram, rectangle PQRS is placed inside rectangle ABCD in two different ways: first, Q at B and R at C; second, with P on AB, Q on BC, R on CD, and S on DA. If AB = 718 and PQ = 250, determine the length of BC.

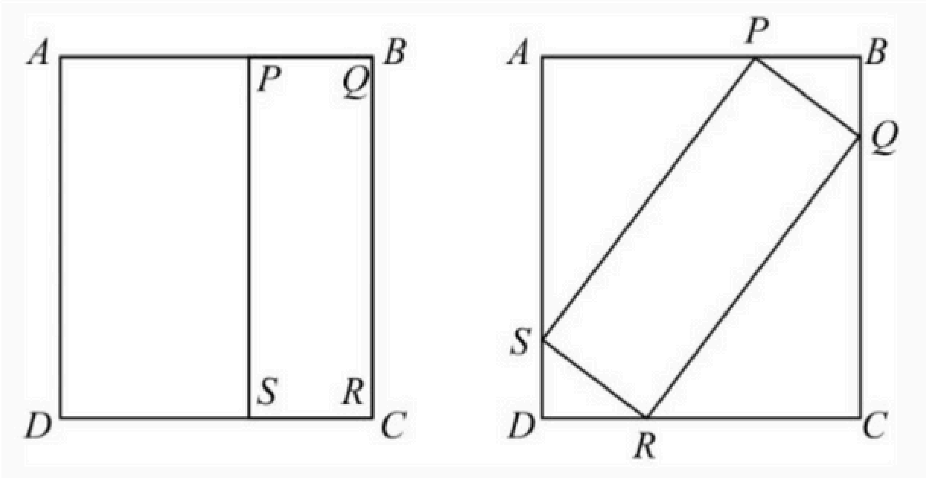
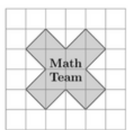


Figure 2: Properties of Triangles

1.0.3 Area of a Figure

The Math Team designed a logo shaped like a multiplication symbol, shown below on a grid of 1-squares. What is the area of the logo in square inches?

A. 10
B. 12
C. 13
D. 14
E. 15

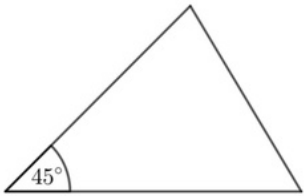


POTW

JUNIOR

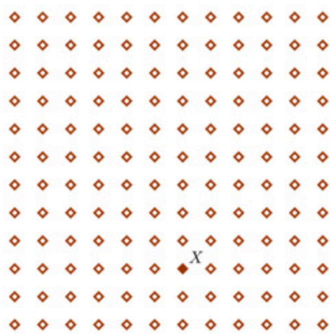
Two Out of Three Angles

While measuring the angles in a triangle, Patricia found the measure of one of the angles is 45°. Once she had measured the other two angles, she noticed that the measures of these two angles are in the ratio 4 : 5. What is the measure of each of the other two angles?



Diamond in the Rough

Using 144 diamonds, the 12 by 12 grid of diamonds below is created. One of the diamonds is coloured and labelled X.



One of the other 143 diamonds in the grid is randomly chosen and is coloured in and labelled Y. What is the probability the line segment connecting X and Y is vertical or horizontal?

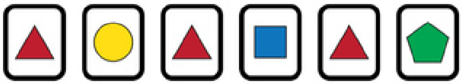
POTW

INTERMEDIATE

Six Cards

Antonia has a set of cards where each card has a shape on one side and a digit from 0 to 9 on the other side. Any two cards with the same shape have the same digit on the other side, and any two cards with different shapes have different digits on the other side.

Antonia lays out the following six cards.



She then flips each card over in place and records the six-digit number they form. For example, if there is a 4 on the other side of the cards with a triangle, a 2 on the other side of the card with a circle, a 7 on the other side of the card with a square, and a 5 on the other side of the card with a pentagon, then the six-digit number they form would be 424 745.

Antonia notices that the six-digit number they form is divisible by 11. Determine the largest and smallest possible six-digit numbers that this could be.

NOTE: You may find the following fact useful:

A number is divisible by 11 exactly when the sum of the digits in the odd digit positions minus the sum of the digits in the even digit positions is divisible by 11. For example, the number 138 248 is divisible by 11 since
$(1 + 8 + 4) - (3 + 2 + 8) = 13 - 13 = 0$ and 0 is divisible by 11.
The number 693 748 is also divisible by 11 since
$(6 + 3 + 4) - (9 + 7 + 8) = 13 - 24 = -11$ and -11 is divisible by 11.

Who Wants Ice Cream?

Xavier made a quilt block to represent an ice cream cone. The quilt block is composed of two isosceles triangles arranged to form a kite. The top triangle represents the ice cream and the bottom triangle represents the cone. The height of the bottom triangle is twice the height of the top triangle. The base of each triangle is $\frac{3}{4}$ of the height of the bottom triangle. If the area of the quilt block of the ice cream cone is 576 units^2, what is its perimeter?

NOTE: You may use the fact that the altitude of an isosceles triangle drawn to the unequal side bisects the unequal side.

POTW

SENIOR

Four Numbers

Norbert has four favourite numbers. Each of these is a three-digit number ABC with the following two properties:

1. The digits A, B, and C are all different.
2. The product $A \times B \times C$ is equal to the two-digit number BC.

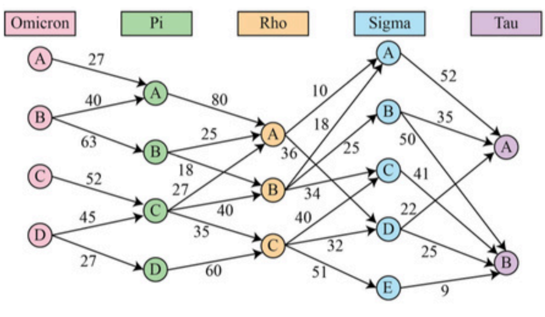
For example, one of Norbert's favourite numbers is 236, since $2 \times 3 \times 6 = 36$.

Find Norbert's other three favourite numbers.

NOTE: It may be helpful to recall that any two-digit number of the form BC can be represented by the sum $10B + C$. For example, $32 = 10(3) + 2$.

The Fantastic Race

As part of The Fantastic Race, teams need to travel on buses from city to city in the order Omicron to Pi to Rho to Sigma to Tau. Each city has several different bus stations to choose from. Nate has created the following map showing all the different bus routes between the five cities, as well as the travel time, in minutes, for each. The different bus stations within each city are labeled A, B, C, etc.



Which route from Omicron to Tau gives the shortest total travel time?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

SOLUTION TO THE POTW

Vertex Paths of Icosahedron

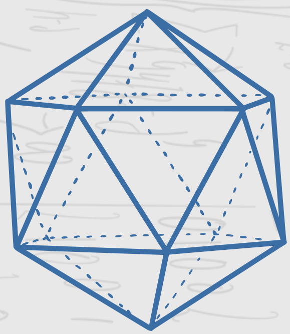
By Ahaan Lakhanpal

A regular icosahedron has top vertex T, bottom vertex B, five upper vertices $U_0,\dots,U_4$ in a horizontal pentagon around T, and five lower vertices $L_0,\dots,L_4$ in a horizontal pentagon around B. Edges connect each U_i to T, to U_{i-1} and U_{i+1}, and to L_i and L_{i+1} (indices mod 5).

Key observations (simple):

1. Because we cannot go up, any path goes:

$T \rightarrow$ some contiguous arc on the upper pentagon $\rightarrow$ one downward edge to a lower vertex $\rightarrow$ some contiguous arc on the lower pentagon $\rightarrow$ B.



(Contiguous arc = a simple path around the 5-cycle; you cannot jump or revisit vertices).

2. For each step there are independent choices:
   - Which upper vertex you go to first (5 choices).
   - Which contiguous upper-arc you follow starting from that vertex.
   - Which of the 2 lower neighbors you descend to from the last upper vertex.
   - Which contiguous lower-arc you follow starting from that lower vertex.

Count the contiguous arcs starting from a fixed vertex on a pentagon.

Fix a start U_0. How many distinct simple paths (contiguous arcs) can start at U_0 and stay on the pentagon?

Vertex Paths of Icosahedron

By Ahaan Lakhanpal

- Length 1 (just U_0): 1 way.
- Length 2, 3, 4, or 5: for each length r (2 – 5) you can go clockwise or counterclockwise, so 2 ways each.

Total = 1 + 2 + 2 + 2 + 2 = 91 arcs starting at a fixed vertex.

(The same count, 9, holds for starting at any fixed lower vertex).

For a fixed initial upper vertex:

- Upper arcs: 9 choices.
- Choice of lower neighbor to descend to: 2 choices.
- Lower arcs from that lower vertex: 9 choices.

So, paths starting via one particular U_i: 9 x 2 × 9 =162

There are 5 possible first upper vertices, so total paths: 5 × 162 = 810

Answer: 810

Sequences & Series

By Jiana Jhaveri

The original sum = 1 + 3 + ... + 99 = 50^2 = 2500

For the new sum to be negative, the total of numbers flipped to negatives must exceed half.

Flip the largest odd numbers to minimize count. The n largest odds are 99, 97, ... 99, 97, ...;

Their sum = 100n – n^2

Test:

- n=14: 100 x 14 – 14^2 = 1204 (too small)
- n=15: 1500 – 225 = 1275 (works)

Answer: B) 15

Permutations & Combinations

By Saisha Damodar

Let our lesser number be n
Let our greater number be a

We need to find out the largest possible number that can be used as the lesser number.

If we take 7, 7 x 2 = 14 which adheres to the criteria, so let's try 8. 8 x 2 = 16, which exceeds 14 and cannot be used.

∴ The largest possible number that can be used as the lesser number is 7.

We require a pair (n, a) where (a > 2n or = 2n, but a < 14 or = 14) and $n \in \{1,2, 3, 4, 5, 6, 7\}$.

The 7 has to be paired with 14. (Only 1 choice)

For 6, there are 2 choices for the greater number - 12 or 13.

For 5, there are 3 choices. This is because the partner of 5 can be anything greater than 10, and two numbers between 10 - 14 have already been used.

If we continue this in a similar manner, we'll see that there are 4 options for a pair with 4, 3 choices for a partner with 3, and 2 choices of a partner with 2 and 1 possible partner of 1, we can get the total number of options as :

1 x 2 x 3 x 4 x 3 x 2 x 1 = 144

Answer: 144

Vertexes of a Rhombic Dodecahedron

By Ananya Kanodia

With 12 rhombi, there are 4 * 12 = 48 total boundaries. Each edge is used as a boundary twice, once for each face on either side. Thus we have 48/2=24 total edges.

Let A be the number of vertices with 3 edges (what the problem asks for) and B be the number of vertices with edges.

∴ 3A + 4B= 48.

Euler's Formula

Vertices (V) + Faces (F) - Edges (E) = 2

1. Euler's formula states that, for all convex polyhedra, V - E + F = 2. In our case, V - 24 + 12 = 2. ∴ V = 14.
2. We know that A + B is the total number of vertices as we are given that all vertices are connected to either 3 or 4 edges. ∴ A + B = 14.
3. We now have a system of two equations. Solving the system yields A = 8.

Answer: 8

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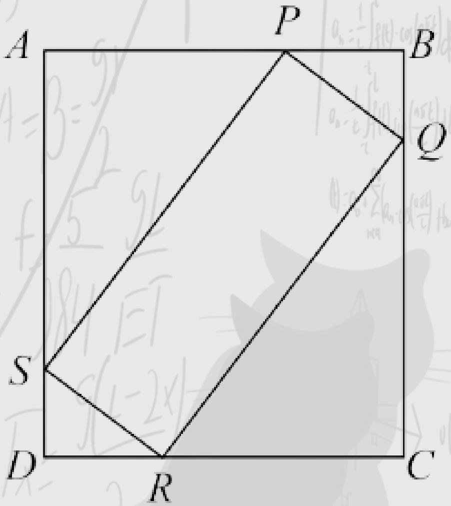
Properties of Triangles

By Aditya Gupta

Lets first look at the second diagram of the two rectangles:



Let BC = x, PB = b and BQ = a. In the question, we have been asked to find BC, which we are taking as x.

∵ BC is one the sides of the rectangle ABCD, the length of AD = length of BC and therefore AD = x

From the first diagram, we can see that the two rectangles have the same length and so, we can also say that PS = QR = BC = x

∵ BQ = a, QC = x - a

∵ AB = 718, and PB =b, AP = 718 - b

Let ∠BQP = θ

∵ PBQ is a right-angle triangle, ∠BPQ = 90 - θ

Properties of Triangles

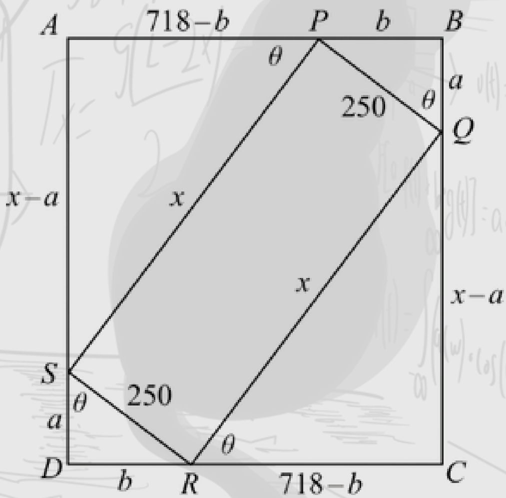
By Aditya Gupta

∵ ∠BQC is a straight angle, and ∠PQR = 90°, ∠RQC = 180 - 90 - θ = 90 - θ

Similarly, ∠APS = θ, because ∠APB is a straight angle and ∠SPQ = 90°

∵ △SAP and △QCR are both right-angled triangles, and have another angle in common with △PBQ, we can say these triangles are similar (by A-A similarity)

We then get this diagram:



∵∠SDR = 90°, and ∠DSR = θ, we can say that △RDS is also similar to the other three triangles.

∵ RS = PQ, we can say that △RDS is congruent to △PBQ (angle-side-angle). Similarly, △SAP is congruent to △QCR.

∵ △SAP and △PBQ are similar, SA/PB = AP/BQ = SP/PQ

Properties of Triangles

By Aditya Gupta

Thus, $\dfrac{x-a}{b} = \dfrac{718-b}{a} = \dfrac{x}{250}$

By the Pythagorean Theorem in $\triangle$PBQ, $a^2 + b^2 = 250^2$

By the Pythagorean Theorem in $\triangle$SAP, $x^2 = (x-a)^2 + (718-b)^2$

$$x^2 = x^2 - 2ax + a^2 + (718-b)^2$$

$$0 = -2ax + a^2 + (718-b)^2 \qquad \text{[Equation 1]}$$

Since $a^2 + b^2 = 250^2$, $a^2 = 250^2 - b^2$

Since $\dfrac{718-b}{a} = \dfrac{x}{250}$, we can say that $ax = 250(718-b)$

Substituting this into equation 1:

$0 = -2(250)(718-b) + 250^2 - b^2 + (718-b)^2$

$b^2 = 250^2 - 2(250)(718-b) + (718-b)^2$

$b^2 = (718 - b - 250)^2$

$b^2 = (468-b)^2$

b = 468 - b

2b = 468

b = 234

$\therefore a^2 = 250^2 - 234^2 = (250+234)(250-234) = 88^2$

$\therefore$ a = 88

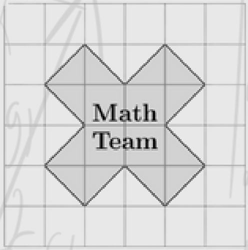
ax = 250(718 - b)

$\therefore$ x = (250(718 - b))/a = (250 x 484)/88 = 1375.

Answer: BC = 1375

Area of a Figure

By Anaya Meghani



The above figure has 4 squares colored completely. Since each square is 1 inch, that would mean the area of these 4 squares is 4 square inches. There are an additional 12 colored squares remaining. However, only half, or 0.5 inches, of each square is colored. So, to find out the total area of the remaining squares, we would have to multiply 12 by 0.5, which is equal to 6 square inches. Now, for the total area of the figure, we add the areas already calculated. 6 + 4 = 10 square inches.

Answer: (A) 10

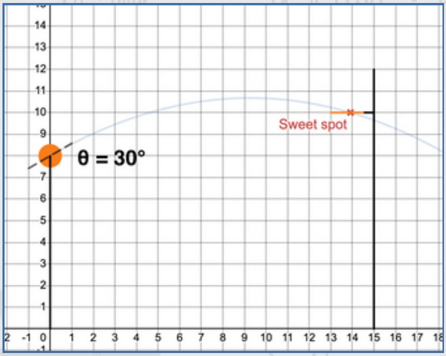
****

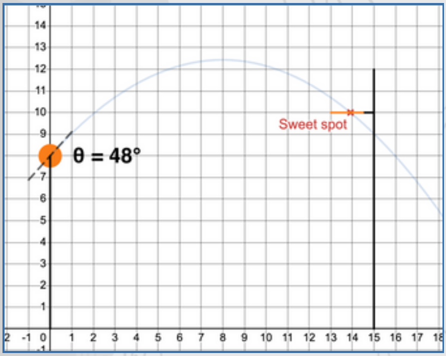
Math x Basketball: Shooting Angles

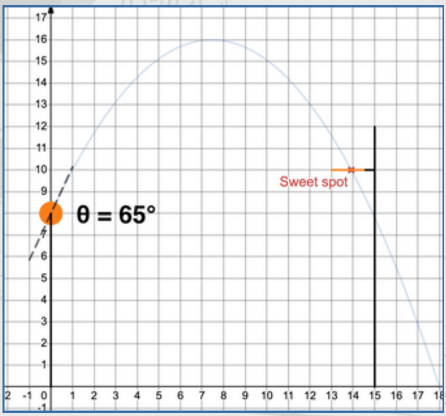
By Vaibhav Sanghai

Basketball might look like a sport of pure athleticism, but every successful shot in the game involves mathematics, specifically projectile motion. Math helps us to explain the geometry of the hoop, the probability of scoring and the angle required to make a particular shot. It explains why some shots go in and others miss.

The most important element is the **shooting angle or the entry angle**. Studies suggest that the optimal launch angle for a jump shot is between **45 & 52°**. A ball entering at a steeper angle (>52°) has a larger "effective target", that is, the space for the ball to pass through is greater, and vice-versa. The entry space for a shot changes depending on the ball's arc. (The diameter of a standard basketball hoop ≈ 18 inches.)







Math x Basketball: Shooting Angles

By Vaibhav Sanghai

For a free throw (approximately 15 feet from the basket), players must choose a speed that allows gravity to pull the ball into the hoop without overshooting. The above graphs showcase the angles at 30°, 48°, and 65° for free throw shots.

Angle (θ)	Required speed (v)	Effort Needed	Notes
30°	~ 8.00 m/s	High	Low Arc, Smaller Target
45°	~ 6.93 m/s	Low	Optimal Arc Range
65°	~ 7.70 m/s	High	High Arc, Larger Target

Thus, the least force is required in the range of $45° \le \theta \le 52°$ and is the easiest to shoot.

Thus, we must remember that the mathematics of shooting angles is crucial for maximising the success rate of shots with minimal effort. So, the next time you watch a game, remember that each swish is as much about numbers as it is about skill.

Aryabhatta

By Aditya Gupta

Born in 476 CE, Aryabhatta is regarded as one of the greatest if not the greatest minds of Ancient India. From calculating pi as 3.1416, to being the first person to use 0 as a placeholder in the decimal system, the list of his achievements does not end there.

His achievements go far beyond numbers, with him also having made significant contributions to Astronomy. He suggested many ideas, some of which were far ahead of their time. An example of this is his conclusion that the Earth is rotating on its axis. He explained eclipses as being caused by shadows of the Moon and the Earth, rather than being caused by mythical beings.

His most famous work, where a lot of his ideas were first mentioned, is the book "Aryabhatiya".

The book, originally written in Sanskrit, is a compilation of Aryabhatta's ideas and concepts, from a range of fields including arithmetic, algebra, trigonometry, and astronomy. In this book, he calculated the length of a year, which was off by just a few minutes when compared to modern measurements. He also developed trigonometric tables, another one of his ideas that shaped mathematics all over the world for the coming years.

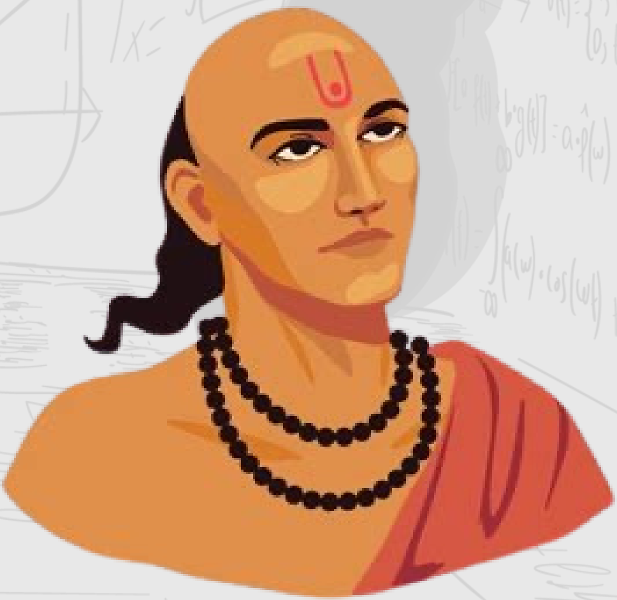
While the details about his life might still be a mystery, needless to say, his influence is everywhere. His theories shaped Science and Mathematics, not only in India but all across the world, leaving prominent marks on Islamic and European mathematics.

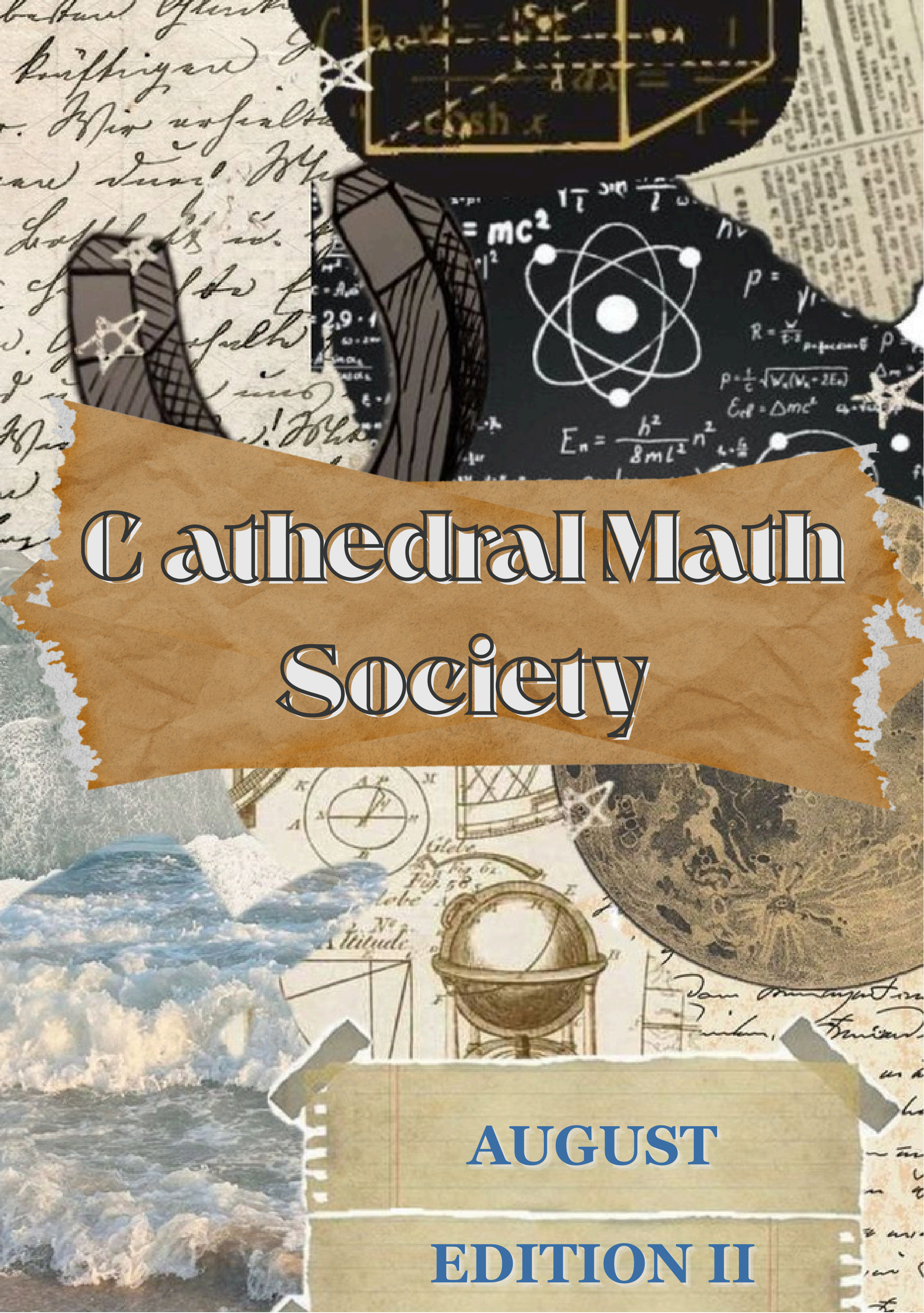
Aryabhatta

By Aditya Gupta

India's first satellite, which was launched in 1975, was named after Aryabhatta. Additionally, the Aryabhatta Research Institute of Observational Sciences, located in Uttarakhand, continues to keep Aryabhatta's legacy alive.

More than 1500 years later, we still use Aryabhatta's ideas, proving that his understanding of mathematics was far ahead of its time. This is why Aryabhatta is often regarded as the father of Indian Mathematics.

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