Information and Communication Theory (UEC-310)

Tutorial-3

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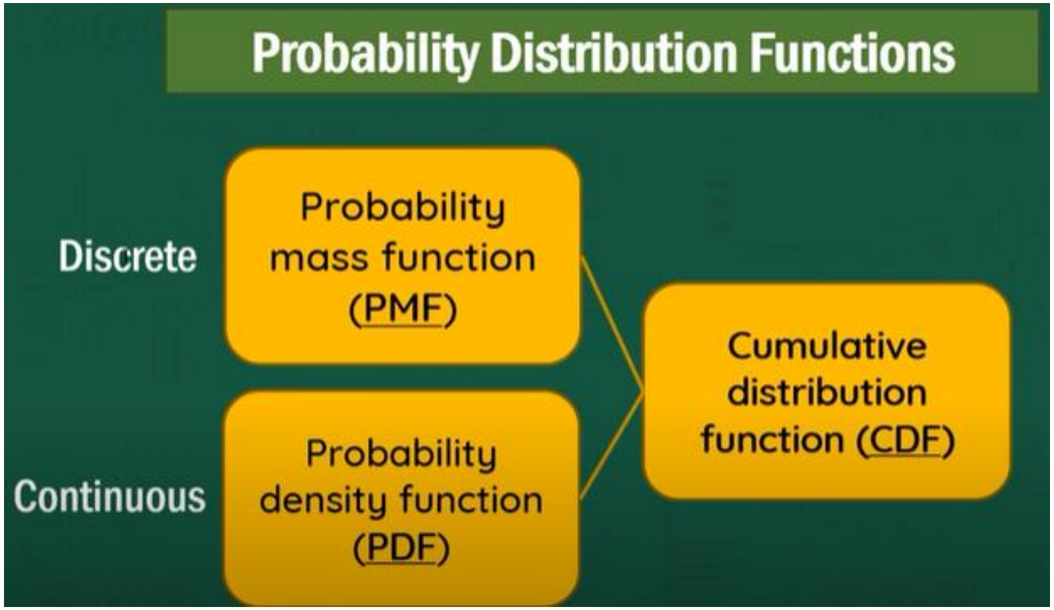


Probability Mass Function

Definition

Let X be a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$ (finite or countably infinite). The function

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, \ldots,$$

is called the *probability mass function (PMF)* of X.

Example:

I toss a fair coin twice, and let X be defined as the number of heads I observe. Find the range of X, R_X, as well as its probability mass function P_X.

Solution

Here, our sample space is given by

$$S = \{HH, HT, TH, TT\}.$$

The number of heads will be 0, 1 or 2. Thus

$$R_X = \{0, 1, 2\}.$$

Since this is a finite (and thus a countable) set, the random variable X is a discrete random variable.

Next, we need to find PMF of X. The PMF is defined as

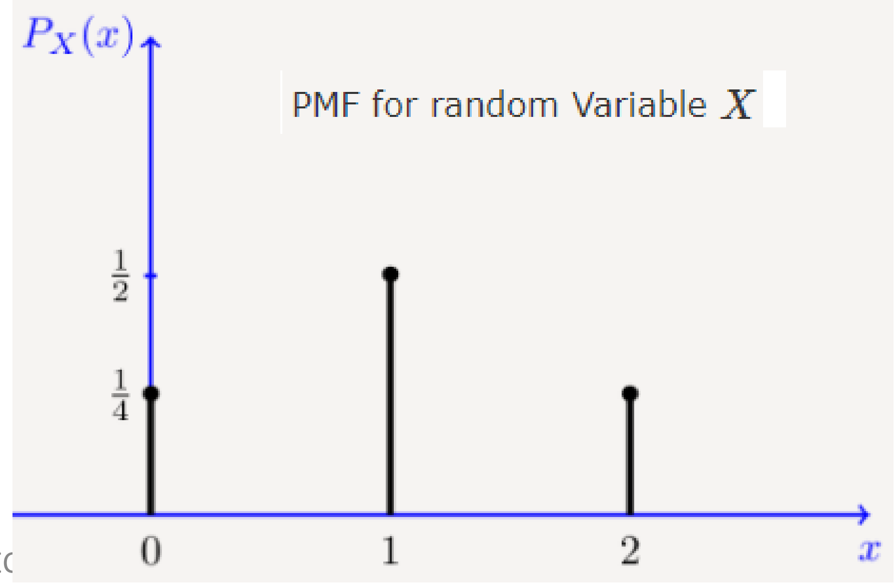
$$P_X(k) = P(X = k) \text{ for } k = 0, 1, 2.$$

We have

$$P_X(0) = P(X = 0) = P(TT) = \frac{1}{4},$$

$$P_X(1) = P(X = 1) = P(\{HT, TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$P_X(2) = P(X = 2) = P(HH) = \frac{1}{4}.$$



Question-1

Compute the mean, variance and standard deviation of the random variable X with the following table of values and probabilities.

value x	1	3	5
pmf $p(x)$	1/4	1/4	1/2

Solution-1

value x	1	3	5
pmf $p(x)$	1/4	1/4	1/2

$E(x) = \Sigma\, x * p(x)$

First we compute $E(X) = 7/2$.

Then we extend the table to include $(X - 7/2)^2$.

value x	1	3	5
$p(x)$	1/4	1/4	1/2
$(x - 7/2)^2$	25/4	1/4	9/4

$\mathrm{Var}(X) = E((X - \mu)^2)$ $\quad$ $Var\ (x) = \Sigma\, p(x) * (x - \mu)2$

Now the computation of the variance is similar to that of expectation:

$$\mathrm{Var}(X) = \frac{25}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{9}{4} \cdot \frac{1}{2} = \frac{11}{4}.$$

Taking the square root we have the standard deviation $\sigma = \sqrt{11/4}$.

Question-2

For each random variable X, Y, Z, and W plot the pmf and compute the mean and variance.

(i)

value x	1	2	3	4	5
pmf $p(x)$	1/5	1/5	1/5	1/5	1/5

(ii)

value y	1	2	3	4	5
pmf $p(y)$	1/10	2/10	4/10	2/10	1/10

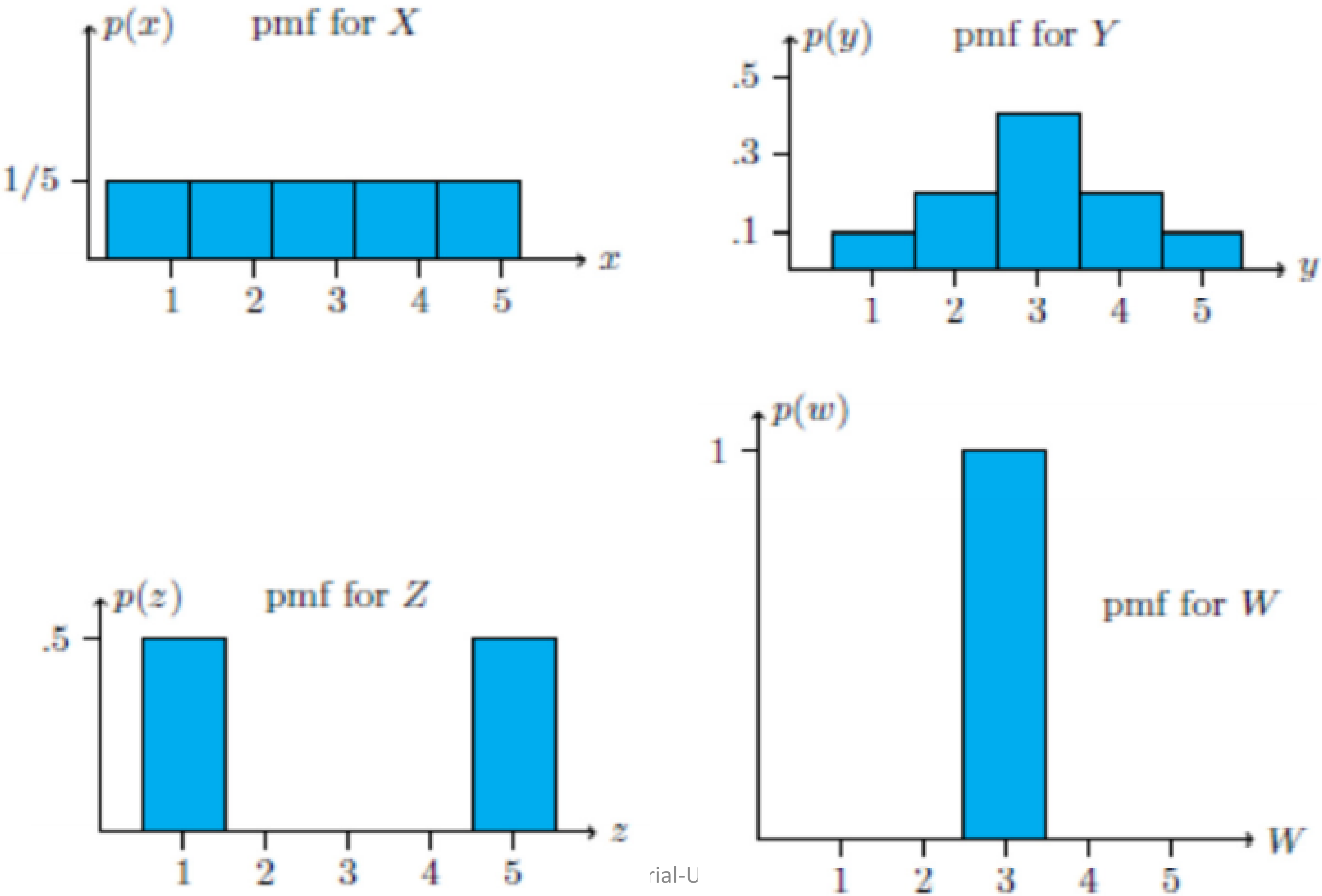
(iii)

value z	1	2	3	4	5
pmf $p(z)$	5/10	0	0	0	5/10

(iv)

value w	1	2	3	4	5
pmf $p(w)$	0	0	1	0	0

Solution-2



p(x)
pmf for X
1/5
1 2 3 4 5
x
p(y)
pmf for Y
.5
.3
.1
1 2 3 4 5
y
p(z)
pmf for Z
.5
1 2 3 4 5
z
p(w)
1
pmf for W
1 2 3 4 5
W

Each random variable has the same mean 3, but the probability is spread out differently.

Next we'll verify our visual intuition by computing the variance of each of the variables.

All of them have mean $\mu = 3$.

Since the variance is defined as an expected value, we can compute it using the tables.

(i)

value x	1	2	3	4	5
pmf $p(x)$	1/5	1/5	1/5	1/5	1/5
$(X-\mu)^2$	4	1	0	1	4

$$\text{Var}(X) = E((X-\mu)^2) = \tfrac{4}{5} + \tfrac{1}{5} + \tfrac{0}{5} + \tfrac{1}{5} + \tfrac{4}{5} = \boxed{2}.$$

(ii)

value y	1	2	3	4	5
$p(y)$	1/10	2/10	4/10	2/10	1/10
$(Y-\mu)^2$	4	1	0	1	4

$$\mathrm{Var}(Y) = E((Y-\mu)^2) = \tfrac{4}{10} + \tfrac{2}{10} + \tfrac{0}{10} + \tfrac{2}{10} + \tfrac{4}{10} = \boxed{1.2}.$$

(iii)

value z	1	2	3	4	5
pmf $p(z)$	5/10	0	0	0	5/10
$(Z-\mu)^2$	4	1	0	1	4

$$\mathrm{Var}(Z) = E((Z-\mu)^2) = \tfrac{20}{10} + \tfrac{20}{10} = \boxed{4}.$$

(iv)

value w	1	2	3	4	5
pmf $p(w)$	0	0	1	0	0
$(W-\mu)^2$	4	1	0	1	4

$\mathrm{Var}(W) = \boxed{0}$. Note that W doesn't vary, so it has variance 0!

Question-3

Bernoulli random variables are fundamental, so we should know their variance.

If $X \sim \text{Bernoulli}(p)$ then

$$\text{Var}(X) = p(1 - p).$$

Solution-3

Proof: We know that $E(X) = p$.

We compute $\text{Var}(X)$ using a table.

values X	0	1
pmf $p(x)$	$1-p$	p
$(X-\mu)^2$	$(0-p)^2$	$(1-p)^2$

$$\text{Var}(X) = (1-p)p^2 + p(1-p)^2 = (1-p)p(1-p+p)$$
$$= \boxed{(1-p)p.}$$

As with all things Bernoulli, you should remember this formula.

Question-4

Suppose X and Y are independent and $\mathrm{Var}(X) = 3$ and $\mathrm{Var}(Y) = 5$.

Find:

(i) $\mathrm{Var}(X + Y)$,

(ii) $\mathrm{Var}(3X + 4)$,

(iii) $\mathrm{Var}(X + X)$,

(iv) $\mathrm{Var}(X + 3Y)$.

Solution-4

To compute these variances we make use of Properties .

Property-1

Since X and Y are independent,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Property-2

$$\mathrm{Var}(aX) = a^2\mathrm{Var}(X))$$

(i) Since X and Y are independent,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$
$$= 8.$$

(ii) $\mathrm{Var}(3X+4) = \mathrm{Var}(3X) + \mathrm{Var}(4)$

Property-2 $\mathrm{Var}(aX) = a^2\mathrm{Var}(X))$

Using Property 2,

$$= 9\,\mathrm{Var}(X) + \mathrm{Var}(4)$$

Variance of any constant is zero.

$$\mathrm{Var}(4) = \mathbf{0}$$

$$\mathrm{Var}(3X+4) = 27.$$

(iii) Don't be fooled!

Property 1 fails since X is certainly not independent of itself.

$$\mathrm{Var}(X + X) = \mathrm{Var}(2X)$$

We can use Property 2:

$$\mathrm{Var}(2X) = 4 \cdot \mathrm{Var}(X)$$

$$= 12.$$

(Note: if we mistakenly used Property 1, we would the wrong answer of 6.)

(iv) We use both Properties 1 and 2.

$$\begin{aligned}\mathrm{Var}(X+3Y) &= \mathrm{Var}(X)+\mathrm{Var}(3Y)\\ &= 3+9\cdot 5\\ &= 48.\end{aligned}$$

Question-5

Suppose $X \sim \text{binomial}(n, p)$. Since X is the sum of *independent* Bernoulli(p) variables and each Bernoulli variable has variance $p(1-p)$ we have

$$X \sim \text{binomial}(n, p) \;\Rightarrow\; \text{Var}(X) = np(1-p).$$

Solution-5

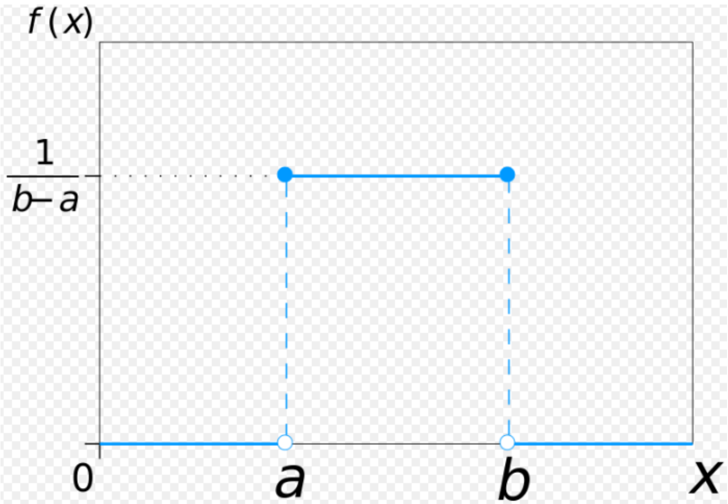
Completed in class lecture.

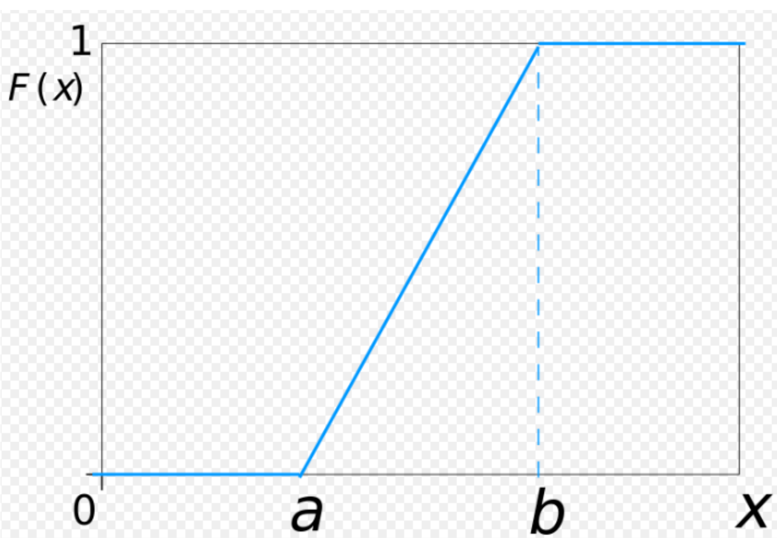
Question-6

Let $X \sim \text{uniform}(0, 1)$.

Find $E(X)$.

Uniform Distribution





PDF	$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
CDF	$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$
Mean	$\frac{1}{2}(a+b)$
Median	$\frac{1}{2}(a+b)$
Mode	any value in (a,b)
Variance	$\frac{1}{12}(b-a)^2$
Skewness	0
Ex. kurtosis	$-\frac{6}{5}$
Entropy	$\ln(b-a)$
MGF	$\begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$

Solution-6

X has range $[0, 1]$ and density $f(x) = 1$.

Therefore,

$$E(X) = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \boxed{\frac{1}{2}}.$$

Not surprisingly the mean is at the midpoint of the range.

Question-7

Let $X \sim \text{uniform}(0, 1)$.
Find $\text{Var}(X)$ and σ_X.

Solution-7

We know that

$$E(X) = \mu = 1/2.$$

Next we compute

$$\mathrm{Var}(X) = E((X-\mu)^2)$$

$$= \int_0^1 (x - 1/2)^2\, dx$$

$$= \boxed{\frac{1}{12}}.$$

Question-8

Let $X \sim \mathrm{N}(\mu, \sigma^2)$.

Show $\mathrm{Var}(X) = \sigma^2$.

Solution-8 **We know that** $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

This is an exercise in change of variables.

Letting $z = (x-\mu)/\sigma$, we have

$$\mathrm{Var}(X) = E((X-\mu)^2)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 \mathrm{e}^{-z^2/2}\,dz$$

$$= \sigma^2.$$

Thanks !