Information and Communication Theory (UEC-310)

Tutorial-1

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Question-1

Multiplication Principle

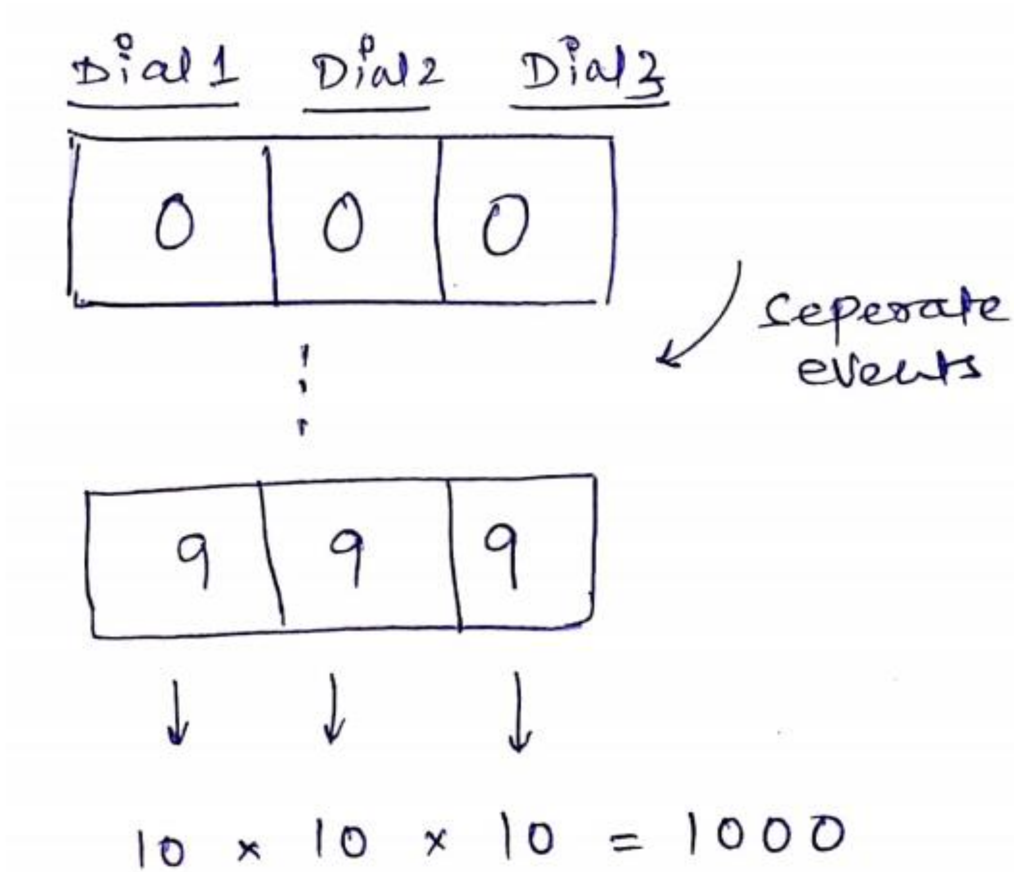
A. How many different combinations can be made for a briefcase that has a 3-dial lock, with each dial having numbers 0-9 available?

B. How many license plates can be made if the first 3 entries must be letters followed by 3 digits?

C. Suppose you take an MCQ exam that has 10 questions, each question has 5 answers. How many different ways could that exam be answered?

Solution-1

A. How many different combinations can be made for a briefcase that has a 3-dial lock, with each dial having numbers 0-9 available?



Continued…

B. How many license plates can be made if the first 3 entries must be letters followed by 3 digits?

A	B	C	1	2	3
↓	↓	↓	↓	↓	↓
26	26	26	10	10	10

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17576000$$

Continued…

C. Suppose you take an MCQ exam that has 10 questions, each question has 5 answers. How many different ways could that exam be answered?

Questions	Choices					
1	A	B	C	D	E	(5 choices)
2	–	–	–	–	–	
3						
4						
5	–	–	–	–	–	
6						
⋮						
10	A	B	C	D	E	(5 choices)

$$5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^{10}$$

$$= 9765625$$

Question-2

Permutations: Sampling without replacement but order matters!

A. Suppose 10 horses run a race; how many different ways could 1st place, 2nd place, and 3rd place occur? Please explain why this problem is of permutations.

B. Suppose there is a class of 20 students and elections are being held for class President and class V.P. Hoe many different ways could the candidates be picked?

C. A cricket team has 10 players. How many batting orders are possible if everyone gets to bat.

Solution-2

PERMUTATIONS:

Counting without replacement

Order matters!!

$$P(n, r) = \frac{n!}{(n-r)!}$$

$n =$ objects to choose from

$r =$ Pick 'r' of these objects

Link: https://web.ma.utexas.edu/users/parker/sampling/repl.htm

Solution-2

A. Suppose 10 horses run a race; how many different ways could 1st place, 2nd place, and 3rd place occur? Please explain why this problem is of permutations.

Horses : 10 Positions: Three (03)

I st, II nd, III rd

$$P(10, 3)$$

$$= \frac{10!}{7!} = 10 \times 9 \times 8$$

$$= 720$$

Permutation: Because it is counting without replacement.

$\because$ If a horse comes I^{st}, it can not come II^{nd} or III^{rd}.

Order matters: Because if horse A comes 1^{st}, B comes 2^{nd}, C comes 3^{rd}, then it will be a different outcomes if A comes 3^{rd} B comes I^{st} and C comes 2^{nd}.

Continued…

B. Suppose there is a class of 20 students and elections are being held for class President and class V.P. How many different ways could the candidates be picked?

Sampling without replacement.

Order matters!!

$$P(20, 2) = \frac{20!}{18!} = 380$$

Continued…

C. A cricket team has 10 players. How many batting orders are possible if everyone gets to bat.

Sampling without replacement

Order Matters!!

$$P(10, 10) = \frac{10!}{0!} = 10!$$

$$= 3628800$$

Combinations: **Sampling without replacement but order does not matters!**

$$^{n}C_{r} = \frac{n!}{(n-r)!\, r!}$$

Question-3

Combinations

A. Suppose 10 horses run a race. You would like to know in how many ways 3 horses can finish in 1^{st}, $2^{nd,}$ and 3^{rd} in any order. Explain how this problem is of combinations and not permutations.

B. On a test, a student must select 6 questions out of 10 for an attempt. In how many ways can this be done?

Solution-3

A. Suppose 10 horses run a race. You would like to know in how many ways 3 horses can finish in 1^{st}, $2^{nd,}$ and 3^{rd} in any order. Explain how this problem is of combinations and not permutations.

Horses : 10

How many ways:
3 Horses finish in I^{st}, II^{nd} and III^{rd}

$\{A, B, C\}$ $\{B, C, A\}$, ...

$$^{10}C_3 = \frac{10!}{7!\,3!} = 120$$

Continued…

B. On a test, a student must select 6 questions out of 10 for an attempt. In how many ways can this be done?

Number of Questions: 10

How many ways: 6 questions can be selected

$$^{10}C_6 = \frac{10!}{4!\,6!} = 210$$

Question-4

At a certain university, **4%** of men are over **6** feet tall and **1%** of women are over **6** feet tall. The total student population is divided in the ratio of **3:2** in favour of **women**.
If **a student** is selected at random from among all those over **6** feet tall, what is the **probability** that the student is a **woman**?

Bayes' theorem:

$P(A)$ represents the *a-priori probability* of the event A. Suppose B has occurred, and assume that A and B are not independent.

How can this new information be used to update our knowledge about A?

Bayes' rule take into account the new information ("B has occurred") and gives out the *a-posteriori probability* of A given B.

A more general version of Bayes' theorem is -

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)},$$

Example: If a "1" is received what is the probability that a zero was transmitted?

Solution-4

Let M={Student is Male}, F={Student is Female}.
Note that M and F partition the sample space of students.
Let T={Student is over 6 feet tall}.

We know that $\mathsf{P}(M) = 2/5$, $\mathsf{P}(F) = 3/5$, $\mathsf{P}(T|M) = 4/100$ and $\mathsf{P}(T|F) = 1/100$.

We require $\mathsf{P}(F|T)$. Using Bayes' theorem we have:

$$\begin{aligned}
\mathsf{P}(F|T) &= \frac{\mathsf{P}(T|F)\mathsf{P}(F)}{\mathsf{P}(T|F)\mathsf{P}(F) + \mathsf{P}(T|M)\mathsf{P}(M)} \\
&= \frac{\frac{1}{100} \times \frac{3}{5}}{\frac{1}{100} \times \frac{3}{5} + \frac{4}{100} \times \frac{2}{5}} \\
&= \frac{3}{11}
\end{aligned}$$

Question-5

A factory production line is manufacturing bolts using three machines, **A, B**, and **C** of the total output, machine A is responsible for **25%**, machine B for **35%**, and machine **C** for the **rest**. It is known from previous experience with the machines that **5%** of the output from machine **A** is defective, **4%** from machine **B**, and **2%** from machine **C**. A bolt is chosen at random from the production line and found to be defective. What is the **probability** that it came from

(a) machine **A** (b) machine **B** (c) machine **C**?

Solution-5

Let

D={bolt is defective},

A={bolt is from machine A},

B={bolt is from machine B},

C={bolt is from machine C}.

We know that $\mathsf{P}(A) = 0.25$, $\mathsf{P}(B) = 0.35$ and $\mathsf{P}(C) = 0.4$.

Also

$$\mathsf{P}(D|A) = 0.05, \mathsf{P}(D|B) = 0.04, \mathsf{P}(D|C) = 0.02.$$

A statement of Bayes' theorem for three events A, B and C is

$$\begin{aligned} \mathsf{P}(A|D) &= \frac{\mathsf{P}(D|A)\mathsf{P}(A)}{\mathsf{P}(D|A)\mathsf{P}(A) + \mathsf{P}(D|B)\mathsf{P}(B) + \mathsf{P}(D|C)\mathsf{P}(C)} \\ &= \frac{0.05 \times 0.25}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\ &= 0.362 \end{aligned}$$

Continued…

Similarly

$$
\begin{aligned}
\mathsf{P}(B|D) &= \frac{0.04 \times 0.35}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.406 \\
\mathsf{P}(C|D) &= \frac{0.02 \times 0.4}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.232
\end{aligned}
$$

Question-6

An engineering company advertises a job in three newspapers, **A, B**, and **C**. It is known that these papers attract undergraduate engineering readerships in the proportions **2:3:1**. The probabilities that an engineering undergraduate sees and replies to the job advertisement in these papers are **0.002, 0.001**, and **0.005** respectively. Assume that the undergraduate sees only one job advertisement.

(a) If the engineering company receives only one reply to it advertisements, calculate the probability that the applicant has seen the job advertised in newspaper:
(i)**A**, (ii) **B**, (iii) **C**.

(b)If the company receives two replies, what is the probability that both applicants saw the job advertised in paper **A**?

Solution-6

Let

$$A = \{\text{Person is a reader of paper } A\},$$
$$B = \{\text{Person is a reader of paper } B\},$$
$$C = \{\text{Person is a reader of paper } C\},$$
$$R = \{\text{Reader applies for the job}\}.$$

We have the probabilities

(a)

$$\begin{aligned} \mathsf{P}(A) &= 1/3 \qquad \mathsf{P}(R|A) = 0.002 \\ \mathsf{P}(B) &= 1/2 \qquad \mathsf{P}(R|B) = 0.001 \\ \mathsf{P}(C) &= 1/6 \qquad \mathsf{P}(R|C) = 0.005 \\ \mathsf{P}(A|R) &= \frac{\mathsf{P}(R|A)\mathsf{P}(A)}{\mathsf{P}(R|A)\mathsf{P}(A) + \mathsf{P}(R|B)\mathsf{P}(B) + \mathsf{P}(R|C)\mathsf{P}(C)} = \frac{1}{3} \end{aligned}$$

Similarly

$$\mathsf{P}(B|R) = \frac{1}{4} \quad \text{and} \quad \mathsf{P}(C|R) = \frac{5}{12}$$

Continued…

(b) Now, assuming that the replies and readerships are independent

$$
\begin{aligned}
\text{P(Both applicants read paper}A) &= \text{P}(A|R) \times \text{P}(A|R) \\
&= \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{9}
\end{aligned}
$$

Question-7

A biased coin (with a probability of obtaining a Head equal to $p > 0$) is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even-numbered toss.

Solution-7

Define:

- sample space Ω to consist of all possible infinite binary sequences of coin tosses;
- event H_1 - head on **first** toss;
- event E - first head on even numbered toss.

We want $P(E)$: using the Theorem of Total Probability, and the partition of Ω given by $\{H_1, H_1'\}$

$$P(E) = P(E|H_1)P(H_1) + P(E|H_1')P(H_1').$$

Now clearly, $P(E|H_1) = 0$ (given H_1, that a head appears on the first toss, E cannot occur) and also $P(E|H_1')$ can be seen to be given by

$$P(E|H_1') = P(E') = 1 - P(E),$$

Continued…

(given that a head does **not** appear on the first toss, the required conditional probability is merely the probability that the sequence concludes after a further **odd** number of tosses, that is, the probability of E'). Hence $P(E)$ satisfies

$$P(E) = 0 \times p + (1 - P(E)) \times (1 - p) = (1 - p)(1 - P(E)),$$

so that

$$P(E) = \frac{1 - p}{2 - p}.$$

Alternatively, consider the partition of E into $E_1, E_2, \ldots$ where E_k is the event that the first head occurs on the $2k$th toss. Then $E = \bigcup_{k=1}^{\infty} E_k$, and

$$P(E) = P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k).$$

Continued…

Now $P(E_k) = (1-p)^{2k-1}p$ (that is, $2k-1$ tails, then a head), so

$$\begin{aligned} P(E) &= \sum_{k=1}^{\infty}(1-p)^{2k-1}p \\ &= \frac{p}{1-p}\sum_{k=1}^{\infty}(1-p)^{2k} = \frac{p}{1-p}\frac{(1-p)^2}{1-(1-p)^2} \\ &= \frac{1-p}{2-p}. \end{aligned}$$

Thanks !