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Tut-5

Q1

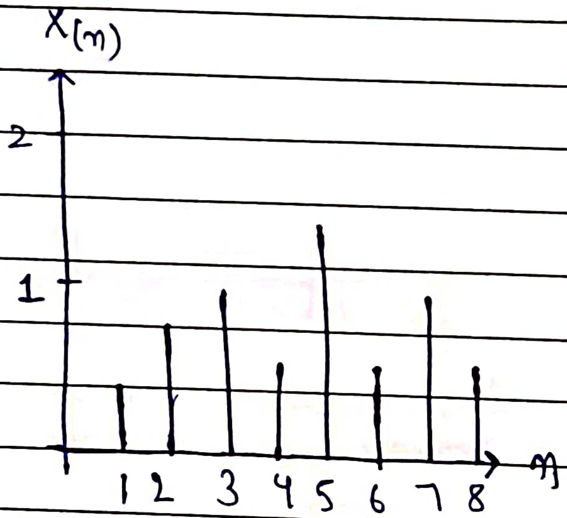
(a) Random process X_n is a discrete-time, continuous-valued process. The sample space is

$$S_X = \{ x : x \geq 0 \}$$

Index perimeter set is

$$I = \{1, 2, 3, \ldots\}$$

(b)



Q2

Male and female arrivals are independent poisson processes, with parameter $\frac{1}{2} \times 10 = 5$.

(a) Answer $= e^{-5} \cdot \frac{5^{10}}{10!}$

(b) $\sum_{K=10}^{\infty} P(K \text{ men entered}) = \sum_{K=10}^{\infty} e^{-5} \cdot \frac{5^K}{K!}$

$$= 1 - \sum_{K=0}^{9} e^{-5} \cdot \frac{5^K}{K!}$$

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Q3 6 customers can expected

$\Rightarrow \lambda = 6$

$$P(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\Rightarrow P(4 \text{ or fewer}) = \sum_{n=0}^{4} P(n;6)$$

$$= \frac{6^0 \cdot e^{-6}}{0!} + \frac{6^1 \cdot e^{-6}}{1!} + \frac{6^2 \cdot e^{-6}}{2!} + \frac{6^3 \cdot e^{-6}}{3!} + \frac{6^4 \cdot e^{-6}}{4!}$$

$$\approx 0.2851 = 28.5\%$$

—x———x———x———x———x—

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Tut-6

Q1 $x = u^2 - v^2$, $y = u^2 + v^2$

$$T = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv + 4uv$$

$$\Rightarrow \boxed{J = 8uv}$$

Q2

$g(x,y) \Rightarrow x+y = z$

$h(x,y) \Rightarrow x-y = w$

$x = \frac{z+w}{2}$, $y = \frac{z-w}{2}$

$$\Rightarrow J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{|J(x,y)|} = \frac{1}{2} f_{xy}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

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Since $N(0,1) \Rightarrow f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$$= \frac{1}{2\pi} e^{-1/2(x^2+y^2)}$$

As X and Y are independent,

$$f_{ZW}(z,w) = \frac{1}{2} f_X\left(\frac{z+w}{2}\right) \cdot f_Y\left(\frac{z-w}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} e^{-1/2\left[\left(\frac{z-w}{2}\right)^2 + \left(\frac{z-w}{2}\right)^2\right]}$$

$$= \frac{1}{4\pi} \cdot e^{-1/2\left(\frac{z^2+w^2}{4}\right)}, \quad \infty < z < +\infty$$

$$-\infty < w < +\infty$$

Q3. x, y have common Var. σ^2 and are 0 - mean independent R.V.

$\Rightarrow$ Derived R.V., $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x^2+y^2)/2\sigma^2}$$

$$Z = g(x,y) = \sqrt{x^2+y^2}$$

$$W = h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

If (x,y) is a soln., then so is $(-x,-y)$,

$$\frac{y}{x} = \tan w$$

$$\Rightarrow \quad y = x \tan w$$

Substituting in z, we get,

$$z = \sqrt{1+\tan^2 w} = x \sec w$$

$$\text{or} \quad x = z \cos w$$

$$\text{and} \quad y = x \tan w = z \sin w$$

$$\Rightarrow \quad x_1 = z \cos w, \quad y_1 = z \sin w$$
$$x_2 = -z \cos w, \quad y_2 = -z \sin w$$

$$\therefore \quad J(z,w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos w & -z \sin w \\ \sin w & z \cos w \end{vmatrix}$$

$$= z.$$

$f_{zw}(z,w) = z\left\{\left(f_{xy}(x_1, y_1)\right) + f_{xy}(x_2, y_2)\right)$

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$$= \frac{z}{\pi\sigma^2} e^{-z^2/2\sigma^2}, \quad 0 < z < \infty, \quad |w| < \pi/2$$

$$\therefore f_Z(z) = \int_{-\pi/2}^{\pi/2} f_{ZW}(z, w)\, dw$$

$$= \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} \quad \text{(Rayleigh's dist.)}$$

$$f_W(w) = \int_0^{\infty} f_{ZW}(z, w)\, dz$$

$$= \frac{1}{\pi} \quad \text{(uniform dist.)}$$

Q4 $\iint_R (X - 3Y)\, dA, \quad R \rightarrow (0,0), (2,1), (1,2)$

$$x = 2u + v$$
$$y = u + 2v$$

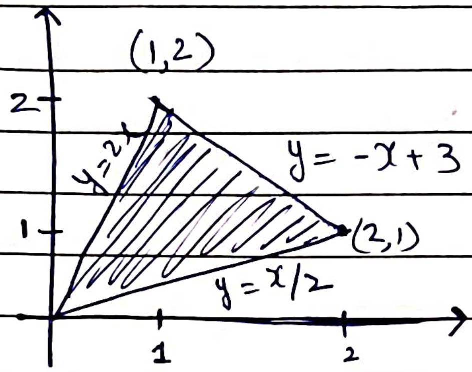
$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

In double integration,

$$x = 2u + v \quad , \quad y = u + 2v$$

$$x - 3y = 2u + v - 3(u + 2v)$$
$$= 2u + v - 3u - 6v$$
$$= -u - 5v$$

Region $\longrightarrow$



In u-v plane

$$y = 2x$$
$$\Rightarrow u + 2v = 2(2u + v)$$
$$\Rightarrow u + \cancel{2v} = 4u + \cancel{2v}$$

$$\boxed{u = 0}.$$

$y = -x + 3$

$$u + 2v = -(2u + v) + 3$$
$$3u + 3v = 3$$

$$u + v = 1$$
$$\Rightarrow \boxed{v = 1 - u}$$

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$$y = \frac{1}{2}x$$

$$u + 2v = \frac{2u + v}{2}$$

$$2u + 4v = 2u + v$$

$$\Rightarrow \boxed{v = 0}$$

$$\therefore \iint_{uv} (-u - 5v)\,|3|\,du\,dv$$

$$= 3\int_0^1\int_0^{1-u} (-u - 5v)\,du\,dv$$

$$= -3\int_0^1 \left. uv + \frac{5v^2}{2} \right|_0^{1-u} du$$

$$= -3\int_0^1 \left(\frac{5}{2} - 4u + \frac{3}{2}u^2\right) du$$

$$= -3\left|\frac{5u}{2} - 2u^2 + \frac{u^3}{3}\right|_0^1$$

$$= -3\left(\frac{5}{2} - 2 + \frac{1}{2}\right) = -3 \text{ Ans}$$

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Q5 $S_{XY} = \{(x,y) : x^2+y^2 \leq 1\}$

(a) Joint PDF, $f_{XY}(x,y)$

Since double integral should be 1 and area $S = \pi r^2 = \pi$,

$$f_{XY}(x,y) = \begin{cases} 1/\pi, & x^2+y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $P(A) = ?$ if $A = \{(x,y) : u \geq 0, v \geq 0\}$

The region is a ~~greater~~ quarter circle $\Rightarrow P(A) = \frac{1}{4}$

(c) If $r \in [0,1]$, $x^2+y^2 \leq r^2$ is a disc of radius 'r' contained in 'S'.

$\Rightarrow$ Area of intersecting region $= \pi r^2$

$\therefore P[X^2+Y^2 \leq r] = r^2, \forall\ 0 \leq r \leq 1$

If $r > 1$, region contains the full 'S'.

$\Rightarrow P[X^2+Y^2 \leq r] = 1 \quad \forall\ r > 1.$

(d) Marginal PDF, $f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \\ = \int_{\sqrt{1-x^2}}^{\sqrt{1+x^2}} \frac{1}{\pi}\, dy \\ = \frac{2\sqrt{1-x^2}}{\pi} \quad \forall\ |x| \leq 1 \\ 0, \text{ otherwise} \end{cases}$

(e) conditional PDF, $f(y|x)$

$$= \begin{cases} \frac{1/\pi}{2\sqrt{1-x^2}/\pi} = \frac{1}{2\sqrt{1-x^2}}; \quad y \in [-\sqrt{1+x^2}, \sqrt{1+x^2}] \\ 0, \text{ otherwise} \end{cases}$$

Q6 $X = \begin{cases} 1; & \text{if 'one' shows} \\ 0, & \text{else} \end{cases}$

$Y = \begin{cases} 1; & \text{if 'two' shows} \\ 0, & \text{else} \end{cases}$

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(a) marginal PDF,

$$P_X(x) = P_Y(y) = \frac{1}{6}$$

(b) $E[X] = E[Y] = \frac{1}{6}$

$Var(X) = E(X^2) - E(X)^2 = p - p^2 = \frac{5}{36}.$

$Var(Y) = E(Y^2) - E(Y)^2 = p - p^2 = \frac{5}{36}.$

(c) $P_{XY}(0,0) = \frac{4}{6}$, $P_{XY}(0,1) = \frac{1}{6}$

$P_{XY}(1,0) = \frac{1}{6}$, $P_{XY}(1,1) = 0$

(d) $Cov(X, Y) = E(XY) - E(X).E(Y)$

$E(XY) = 0$ $\because$ two numbers can't come up at the same time on a dice

$\Rightarrow Cov(X, Y) = 0 - \frac{1}{36} = \frac{-1}{36}.$

(e) $\rho = \frac{Cov(X, Y)}{\sqrt{Var(X).Var(Y)}} = \frac{-1/36}{\sqrt{5/36 \cdot 5/36}} = \frac{-1}{5}.$