Information and Communication Theory (UEC-310)

Tutorial-4

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Question-1

Let X be a discrete random variable with $P_X(k) = \frac{1}{5}$.

For k = -1, 0, 1, 2,3.

Let $Y = 2|X|$.

Find the range and probability mass function of Y.

Solve the problem using Law of Unconscious Statistician (LOTUS) also.

Solution-1

Range of X

$$R_x = \{-1, 0, 1, 2, 3\}$$

$$P_x(-1) = P_x(0) = P_x(1) = P_x(2) = P_x(3) = \frac{1}{5}$$

$$Y = 2|X|$$

Range of Y

$$R_Y = \{2, 0, 2, 4, 6\}$$

As 2 is repeated.

$$R_Y = \{0, 2, 4, 6\}$$

four possible values of Y.

$$P_Y(0) = P(Y=0) = P(2|X|=0) = P(X=0)$$
$$= \frac{1}{5}$$

$$P_Y(2) = P(Y=2) = P(2|X|=2)$$
$$= P(X=1 \text{ or } X=-1)$$
$$= P_X(1) + P_X(-1)$$
$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$P_Y(4) = P(Y=4) = P(2|X|=4) = P(X=2 \text{ or } X=-2)$$

But $P_X(2) = \frac{1}{5}$ and $P_X(-2) = 0$

$$\therefore P_Y(4) = P_X(2) = \frac{1}{5}$$

Similarly

$$P_Y(6) = \frac{1}{5}$$

$$\therefore \quad P_Y(k) = \begin{cases} \frac{1}{5} & \text{for } k = 0, 4, 6 \\ \frac{2}{5} & \text{for } k = 2 \end{cases}$$

Law of the unconscious statistician (LOTUS)

Law of the unconscious statistician (LOTUS) says that the expected value of a transformed RV can be found without finding the distribution of the transformed RV, simply by applying the probability weights of the original RV to the transformed values.

Discrete X with pmf p_X: $\quad \mathrm{E}[g(X)] = \sum_x g(x) p_X(x)$

Continuous X with pdf f_X: $\quad \mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

Evaluate E[X] and E[Y] using Law of the unconscious statistician (LOTUS)

Let x be a random variable $P_x(k) = \frac{1}{5}$

for $k = -1, 0, 1, 2, 3$

$$E[x] = \sum g(x_k) \cdot P_x(x_k)$$

$$= -1 \times \frac{1}{5} + 0 \times \frac{1}{5} + 1 \times \frac{1}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5}$$

$$= \frac{1}{5}\left[-1 + 0 + 1 + 2 + 3\right]$$

$$= 1$$

$$Y = 2|X|$$

Then

$$E[Y] = E[2|X|]$$

$$= \sum_k g(x_k)\, P_X(x_k)$$

$$= (2 \times |-1|) \times \frac{1}{5} + (2 \times |0|) \times \frac{1}{5} + (2 \times |1|) \times \frac{1}{5}$$

$$+ (2 \times |2|) \times \frac{1}{5} + (2 \times |3|) \times \frac{1}{5}$$

$$= \frac{1}{5}\left[2 + 0 + 2 + 4 + 6\right]$$

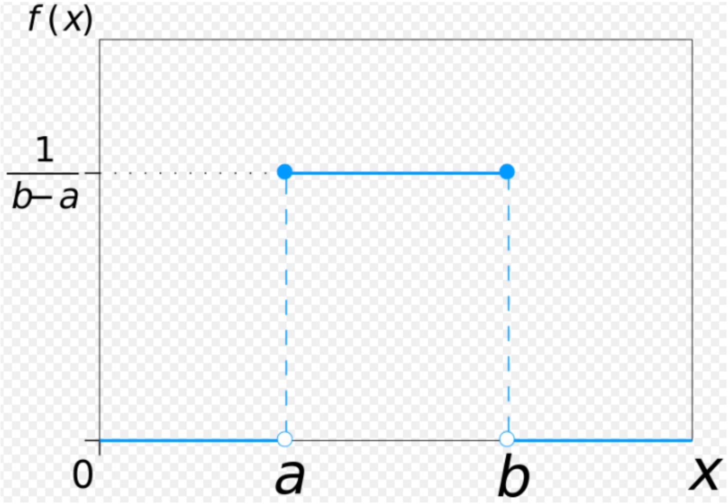
$$= \frac{14}{5}$$

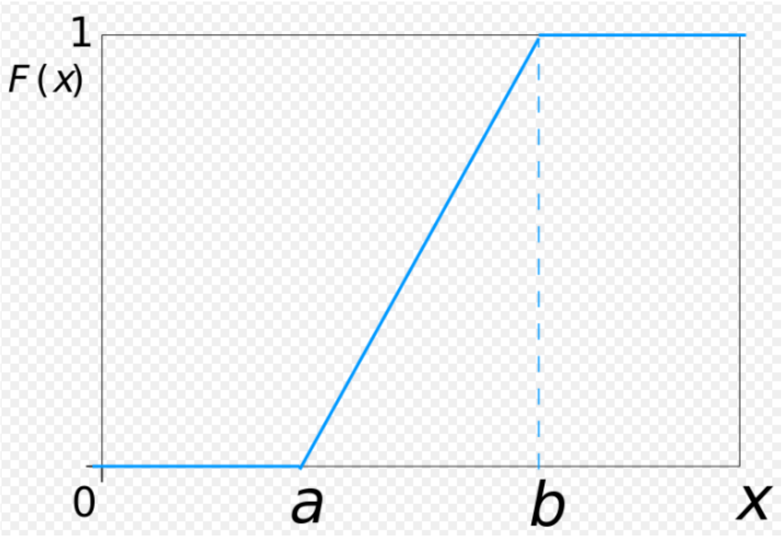
Question-2

Let X be a random variable with uniform distribution [0,1] and let $Y = e^X$.

i. Find CDF of Y.
ii. Find PDF of Y.
iii. Find E[Y].

Uniform Distribution





PDF	$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
CDF	$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$
Mean	$\frac{1}{2}(a+b)$
Median	$\frac{1}{2}(a+b)$
Mode	any value in (a,b)
Variance	$\frac{1}{12}(b-a)^2$
Skewness	0
Ex. kurtosis	$-\frac{6}{5}$
Entropy	$\ln(b-a)$
MGF	$\begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$

Solution-2

First, note that we already know the CDF and PDF of X. In particular,

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

It is a good idea to think about the range of Y before finding the distribution. Since e^x is an increasing function of x and $R_X = [0, 1]$, we conclude that $R_Y = [1, e]$.

So we immediately know that

$$F_Y(y) = P(Y \leq y) = 0, \qquad \text{for } y < 1,$$

$$F_Y(y) = P(Y \leq y) = 1, \qquad \text{for } y \geq e.$$

a. To find $F_Y(y)$ for $y \in [1, e]$, we can write

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(e^X \leq y)
\end{aligned}
$$

$$
\begin{aligned}
&= P(X \leq \ln y) && \text{since } e^x \text{ is an increasing function} \\
&= F_X(\ln y) = \ln y && \text{since } 0 \leq \ln y \leq 1.
\end{aligned}
$$

To summarize

$$
F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ \ln y & \text{for } 1 \leq y < e \\ 1 & \text{for } y \geq e \end{cases}
$$

b. The above CDF is a continuous function, so we can obtain the PDF of Y by taking its derivative. We have

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{y} & \text{for } 1 \leq y \leq e \\ 0 & \text{otherwise} \end{cases}$$

Note that the CDF is not technically differentiable at points 1 and e, but as we mentioned earlier we do not worry about this since this is a continuous random variable and changing the PDF at a finite number of points does not change probabilities.

c. To find the EY, we can directly apply LOTUS,

$$\begin{aligned} E[Y] = E[e^X] &= \int_{-\infty}^{\infty} e^x f_X(x) dx \\ &= \int_0^1 e^x dx \\ &= e - 1. \end{aligned}$$

For this problem, we could also find EY using the PDF of Y,

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_1^e y \frac{1}{y} dy \\ &= e - 1. \end{aligned}$$

Note that since we have already found the PDF of Y it did not matter which method we used to find $E[Y]$.

However, if the problem only asked for $E[Y]$ without asking for the PDF of Y, then using LOTUS would be much easier.

Question-3

Let X and Y be two random variables. X and Y are related to a third random variable Z with the following equation:

$$Z = X + Y.$$

i. Find CDF of Z
ii. Find PDF of Z.

Solution-3

Attempt yourself.

Question-4

Let X and Y have joint density

$$f(x, y) = cxy, \quad 0 \leqslant x, y \leqslant 1.$$

What is c?

Solution-4

We know that $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$$

We know $f \geqslant 0$, so $c \geqslant 0$.

$$\int_0^1\int_0^1 cxy\,dx\,dy = 1$$

$$= c\int_0^1 x\,dx\int_0^1 y\,dy$$

$$= \frac{c}{4} = 1$$

Hence $c = 4$.

Question-5

Let X and Y have joint density

$$f(x, y) = cxy, \qquad 0 \leqslant x \leqslant 1, \quad x \leqslant y \leqslant 1$$

What is c?

Solution-5

We know that $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$$

For $c \geqslant 0$

$$\int_0^1\int_x^1 cxy\,dy\,dx = 1$$

$$= \int_0^1 cx\int_x^1 y\,dy\,dx$$

$$= \int_0^1 \frac{cx}{2}(1 - x^2)dx$$

$$= \frac{c}{8} = 1$$

$$c = 8$$

Question-6

A pair of dice bear the numbers 1, 2, 3 twice each, on pairs of opposite faces.

Both dice are rolled, yielding the scores X and Y respectively.

Obviously

$$p(j, k) = \tfrac{1}{9}, \quad \text{for } 1 \leqslant (j, k) \leqslant 3.$$

Now suppose we roll these dice again, and consider the difference between their scores, denoted by U, and the sum of their scores, denoted by V.

Calculate the joint distribution of U and V by running over all possible outcomes.

Solution-6

$$p(0,\,4) = \mathrm{P}(U = 0,\; V = 4)$$
$$= \mathrm{P}(\{2,\,2\}) = \tfrac{1}{9}.$$

$$\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix}$$

Eventually we produce the following array of probabilities:

V						
6	0	0	$\frac{1}{9}$	0	0	
5	0	$\frac{1}{9}$	0	$\frac{1}{9}$	0	
4	$\frac{1}{9}$	0	$\frac{1}{9}$	0	$\frac{1}{9}$	
3	0	$\frac{1}{9}$	0	$\frac{1}{9}$	0	
2	0	0	$\frac{1}{9}$	0	0	
	-2	1	0	1	2	U

We could write this algebraically, but it is more informative and appealing as shown.

V						
6	0	0	$\frac{1}{9}$	0	0	
5	0	$\frac{1}{9}$	0	$\frac{1}{9}$	0	
4	$\frac{1}{9}$	0	$\frac{1}{9}$	0	$\frac{1}{9}$	
3	0	$\frac{1}{9}$	0	$\frac{1}{9}$	0	
2	0	0	$\frac{1}{9}$	0	0	
	-2	1	0	1	2	U

In practice we rarely display probabilities as an array, as the functional form is usually available and is of course much more compact.

Thanks !