6 The Laplace transform

6.1 Definition of the Laplace transform

Definition: Laplace transform

The Laplace transform of a function $f(t)$ (defined for $t \geq 0$) is defined as

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

The Laplace transform is defined for all values of s where the integral exists.

Note that f is a function of t, but $\mathcal{L}\{f\}$ is a function of s!

Computing some Laplace transforms

Example Compute the Laplace transform of $f(t) = 1, t \geq 0$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt$$

$$= \lim_{M\to\infty} \int_0^M e^{-st} dt$$

$$= \lim_{M\to\infty} \frac{-1}{s} e^{-st}\Big|_{t=0}^{M}$$

$$= \lim_{M\to\infty} \frac{-1}{s}\left(e^{-sM} - e^{0}\right)$$

$$= \frac{-1}{s}(0-1) \quad (\text{if } s>0)$$

$$\boxed{= \frac{1}{s}, \quad s>0}$$

Example Compute the Laplace transform of $f(t) = e^{at}, t \geq 0$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$$= \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$$

$$= \frac{1}{a-s}(0-1) \quad (a-s<0)$$

$$\boxed{= \frac{1}{s-a}, \quad s > a}$$

Example Compute the Laplace transform of

$$f(x) = \begin{cases} 1 & 0 \leq t < 1 \\ \frac{1}{2} & t = 1 \\ 0 & t > 1 \end{cases}$$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} \cdot 1\,dt$$

$$= \frac{-1}{s} e^{-st}\Big|_{t=0}^1$$

$$\boxed{= -\frac{1}{s}(e^{-s} - 1)}$$

Note: value at $t=1$ does not matter because integrals are not affected by single points.

Linearity of the Laplace transform

Theorem

If we know the Laplace transforms of f and g, then the Laplace transform of $c_1 f + c_2 g$ is

$$\begin{aligned}\mathcal{L}\{c_1 f + c_2 g\} &= c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}\\ &= c_1 F(s) + c_2 G(s).\end{aligned}$$

Example Compute the Laplace transform of $f(t) = 3 + e^{3t}$.

$$\mathcal{L}\{f\} = \mathcal{L}\{3 + e^{t}\}$$

$$= 3\mathcal{L}\{1\} + \mathcal{L}\{e^{t}\}$$

$$\boxed{= \frac{3}{s} + \frac{1}{s-3}, \quad s > 3}$$

Recall

Euler's formula: $e^{ibt} = \cos(bt) + i\sin(bt)$

Using this, we can see that

$$\cos(bt) = \frac{1}{2}(e^{ibt} + e^{-ibt})$$
$$\sin(bt) = \frac{1}{2i}(e^{ibt} - e^{-ibt})$$

Example Compute the Laplace transform of $\cos(t)$.

Assume that $\mathcal{L}\{e^{ibt}\} = \frac{1}{s-ib}$, $s > 0$.

$$\mathcal{L}\{\cos(t)\} = \frac{1}{2}\left(\mathcal{L}\{e^{ibt}\} + \mathcal{L}\{e^{-ibt}\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right) \quad (s>0)$$

$$= \frac{1}{2}\,\frac{s+ib+s-ib}{(s-ib)(s+ib)}$$

$$= \frac{1}{2}\,\frac{2s}{s^2-i^2b}$$

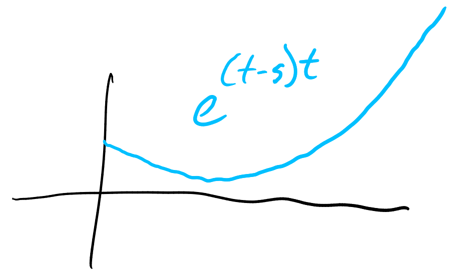
$$\boxed{= \frac{s}{s^2+b^2}, \quad s>0}$$

Does the Laplace transform always exist?

No.

Example Try to compute the Laplace transform of e^{t^2}.

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \int_0^\infty e^{\underbrace{(t-s)t}_{>0 \text{ for large } t}}\,dt$$



integrating a function that goes to infinity, so the integral diverges to ∞ (for all values of s). Therefore, the Laplace transform of e^{t^2} does not exist.

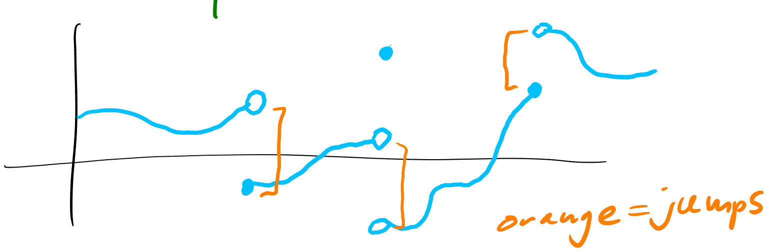
When does the Laplace transform exist?

Definition: Piecewise continuous

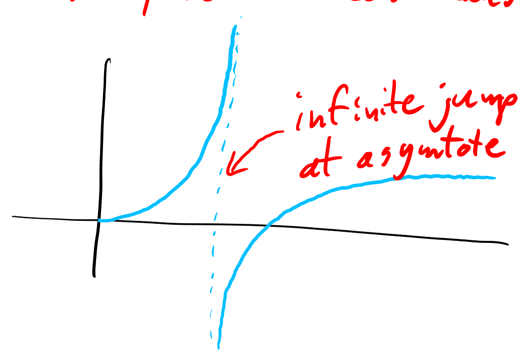
A function is said to be **piecewise continuous** if it is continuous except at a finite number of points where the function "jumps" a finite distance.

Examples

piecewise continuous



not piecewise continuous



Theorem

If f is piecewise continuous and there exists constants a and K such that

$$|f(t)| \leq Ke^{at},$$

then the Laplace transform exists.

i.e., the function f has to be "nice" (piecewise continuous) and can't grow faster than an exponential function.

Recall

Partial fraction decomposition

Note that partial fractions can only be used when the degree of the numerator is smaller than the degree of the denominator! If it's not, then you have to do polynomial long division first.[a]

In partial fraction decomposition, there are three cases:

1. **Simple roots:** If the denominator contains *one* factor $(x-a)$, then include a term with $x-a$ in the bottom.

$$\frac{3}{(x-2)(x+5)x} = \frac{A}{x-2} + \frac{B}{x+5} + \frac{C}{x}$$

2. **Irreducible quadratics:** If the denominator contains a quadratic that cannot be factored using real numbers, then include that factor with $Ax+B$ on top.

$$\frac{2x^2+3x-4}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3}$$

3. **Repeated roots:** If the denominator contains a k^{th} power of the previous two cases, then include all powers from 1 to k.

$$\frac{-2x-7}{(x-2)^3(x^2+5)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+5} + \frac{Fx+G}{(x^2+5)^2}$$

After you set up the correct form, then you can solve for the coefficients any way you want. A standard method is to multiply both sides by the bottom of the left hand side and then compare the coefficients on the x, x^2, x^3 terms and so on.

[a] But we won't have to do that in this class.

Inverse Laplace transforms

When solving ODEs with Laplace transforms, we will need to take both the Laplace transform and the inverse Laplace transform. You take the inverse transform by "going backwards" on the table. You often need to use partial fractions.

Examples Find the inverse Laplace transform of the following.

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n \quad (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

(a) $F(s) = \dfrac{6}{(s-2)^4}$

$$= \frac{3!}{(s-2)^{3+1}}$$

$$\boxed{f(t) = e^{2t}t^3}$$

(b) $F(s) = \dfrac{2}{s^2 - 9}$

$$= \frac{2}{(s+3)(s-3)} = \frac{A}{s+3} + \frac{B}{s-3}$$

$$2 = A(s-3) + B(s+3)$$

At $s=3$: $2 = 6B \Rightarrow B = 1/3$

At $s=-3$: $2 = -6A \Rightarrow A = -1/3$

$$F(s) = \underbrace{-\frac{1}{3}\left(\frac{1}{s+3}\right)}_{\mathcal{L}\{e^{-3t}\}} + \underbrace{\frac{1}{3}\left(\frac{1}{s-3}\right)}_{\mathcal{L}\{e^{3t}\}}$$

$$\boxed{f(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{3t}}$$

(c) $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2+4)} = \dfrac{As+B}{s^2+4} + \dfrac{C}{s}$

$$8s^2 - 4s + 12 = (As+B)s + C(s^2+4)$$

$$= (A+C)s^2 + Bs + 4C$$

$8 = A + C \qquad -4 = B \qquad 12 = 4C$

$\Rightarrow C = 3$

$8 = A + 3$

$\Rightarrow A = 5$

$$F(s) = \frac{5s-4}{s^2+4} + \frac{3}{s}$$

$$= 5\underbrace{\frac{s}{s^2+4}}_{\mathcal{L}\{\cos(2t)\}} - 2\underbrace{\left(\frac{2}{s^2+4}\right)}_{\mathcal{L}\{\sin(2t)\}} + 3\underbrace{\left(\frac{1}{s}\right)}_{\mathcal{L}\{1\}}$$

$$\boxed{f(t) = 5\cos(2t) - 2\sin(2t) + 3}$$

(d) $F(s) = \dfrac{2s+4}{s^2+4s+7}$

$$s = \frac{-4 \pm \sqrt{16-28}}{2}$$

complex roots, so irreducible quadratic

$$= \frac{2s+4}{(s^2+2)^2 - 4 + 7}$$

$$= \frac{2(s+2)}{(s^2+2)^2 + 3}$$

$$= 2\mathcal{L}\{e^{-2t}\cos(\sqrt{3}t)\}$$

$$\boxed{f(t) = 2e^{-2t}\cos(\sqrt{3}t)}$$

(e) $F(s) = \dfrac{1-2s}{s^2+4s+5}$

$$s = \frac{-4 \pm \sqrt{16-20}}{2}$$

complex roots, so irreducible quadratic

$$= \frac{1-2s}{(s+2)^2-4+5}$$

$$= \frac{-2(s+2)+4+1}{(s+2)^2+1}$$

$$= -2\underbrace{\frac{s+2}{(s+2)^2+1}}_{\mathcal{L}\{e^{-2t}\cos(t)\}} + 5\underbrace{\frac{1}{(s+2)^2+1}}_{\mathcal{L}\{e^{-2t}\sin(t)\}}$$

$$\boxed{f(t) = -2e^{-2t}\cos(t) + 5e^{-2t}\sin(t)}$$

6.2 Solution of initial value problems

Laplace transforms of derivatives

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -se^{-st}\,dt$ $\quad v = f(t)$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= \underbrace{\lim_{t\to\infty} e^{-st} f(t)}_{\substack{=0 \text{ as long as } f \\ \text{doesn't grow faster} \\ \text{than exponential functions}}} - e^0 f(0) + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{=\mathcal{L}\{f\}}$$

Theorem: Laplace transform of f'

Let f be differentiable. Suppose that f' is piecewise continuous and that there exist constants a and K such that $|f(t)| \le Ke^{at}$ for $t \ge 0$.[a] Then, $\mathcal{L}\{f'\}$ exists for $s > a$ and

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

$$= sF(s) - f(0).$$

[a] Gray text is technical details that we won't worry about in this class.

Theorem: Laplace transform of $f^{(n)}$

Let f be n-times differentiable. Suppose that $f^{(n)}$ is piecewise continuous and that there exist constants a and K such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, ..., and $|f^{(n-1)}| \le Ke^{at}$ for $t \ge 0$.[a] Then, $\mathcal{L}\{f^{(n)}\}$ exists for $s > a$ and

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

[a] Gray text is technical details that we won't worry about in this class.

General idea for solving IVPs with Laplace transform

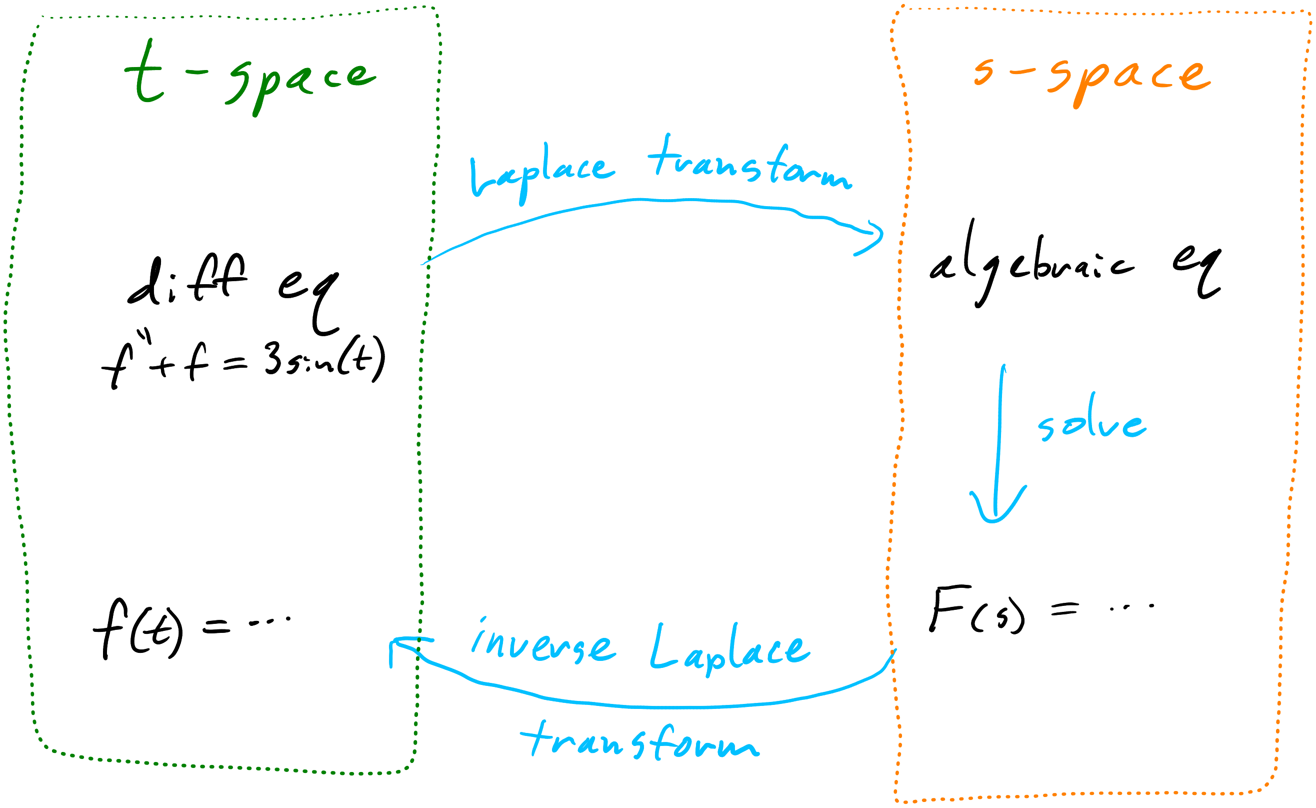


Table of some Laplace transforms

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n $(n = 1, 2, ...)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$ $(n = 1, 2, ...)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

only these ones will be given on the test. The other properties of the Laplace transform, you will need to memorize.

Example $y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$

Take Laplace transform of both sides:

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

$$s^2 Y(s) - s\,y(0) - y'(0) - (s\,Y(s) - y(0)) - 2\,Y(s) = 0$$

solve for $Y(s)$:

$$s^2 Y(s) - s - 0 - s\,Y(s) + 1 - 2\,Y(s) = 0$$

$$(s^2 - s - 2)\,Y(s) = s - 1$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

Take the inverse Laplace transform to find $y(t)$:

Use partial fractions to convert $Y(s)$ into a form that we've seen before.

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\Rightarrow s - 1 = A(s+1) + B(s-2)$$

at $s = -1$: $-2 = 0 - 3B \Rightarrow B = 2/3$

at $s = 2$: $1 = 3A \Rightarrow A = 1/3$

So,

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

Comparing with our table of Laplace transforms, we see

$$Y(s) = \frac{1}{3}\mathcal{L}\{e^{2t}\} + \frac{2}{3}\mathcal{L}\{e^{-t}\}$$

$$= \mathcal{L}\left\{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}\right\}$$

Take the inverse transform of both sides:

$$\boxed{y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$$

Example $y'' + y = \sin(2t), \qquad y(0) = 2, \qquad y'(0) = 1.$

Take Laplace transform of both sides:

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

Solve for $Y(s)$:

$$(s^2+1) Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

Take the inverse Laplace transform to find $y(t)$:

$$Y(s) = \underbrace{\frac{2}{(s^2+1)(s^2+4)}}_{\text{do partial fractions}} + \frac{2s+1}{s^2+1}$$

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$$2 = (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

Compare coefficients on both sides:

$A + C = 0 \Rightarrow A = -C$

$A + 4C = 0 \Rightarrow 3C = 0$

$\Rightarrow C = 0 \Rightarrow A = 0$

$B + D = 0 \Rightarrow B = -D$

$B + 4D = 2 \Rightarrow 3D = 2$

$D = \frac{2}{3} \Rightarrow B = -\frac{2}{3}$

$$Y(s) = \frac{-2/3}{s^2+4} + \frac{2/3}{s^2+1} + \frac{2s+1}{s^2+1}$$

$$= \frac{-1}{3}\left(\frac{2}{s^2+4}\right) + \frac{5}{3}\left(\frac{1}{s^2+1}\right) + 2\left(\frac{s}{s^2+1}\right)$$

$$= \frac{-1}{3}\mathcal{L}\{\sin(2t)\} + \frac{5}{3}\mathcal{L}\{\sin(t)\} + 2\mathcal{L}\{\cos(t)\}.$$

$$\boxed{y(t) = -\frac{1}{3}\sin(2t) + \frac{5}{3}\sin(t) + 2\cos(t)}$$

Example $y^{(4)} - y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0.$

Take Laplace transform of both sides:

$$\mathcal{L}\{y^{(4)} - y\} = 0$$

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0$$

Solve for $Y(s)$:

$$(s^4 - 1) Y(s) - s^2 = 0$$

$$Y(s) = \frac{s^2}{s^4 - 1}$$

Take the inverse Laplace transform to find $y(t)$:

$$Y(s) = \frac{s^2}{(s^2+1)(s^2-1)} = \frac{s^2}{(s^2+1)(s+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-1}$$

$$s^2 = (As+B)(s+1)(s-1) + C(s^2+1)(s-1) + D(s^2+1)(s+1)$$

At $s = 1$: $\quad 1 = 4D \Rightarrow D = 1/4$

At $s = -1$: $\quad 1 = -4C \Rightarrow C = -1/4$

At $s = 0$: $\quad 0 = -B + \frac{1}{4} + \frac{1}{4} \Rightarrow B = 1/2$

At $s = 2$: $\quad 4 = (2A + \frac{1}{2})(3) - \frac{5}{4} + \frac{15}{4}$

$$4 = 6A + \frac{3}{2} + \frac{10}{4}$$

$$4 = 6A + 4 \Rightarrow A = 0$$

$$Y(s) = \frac{1}{2}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{1}{4}\left(\frac{1}{s-1}\right)$$

$$= \frac{1}{2}\mathcal{L}\{\sin(t)\} - \frac{1}{4}\mathcal{L}\{e^{-t}\} + \frac{1}{4}\mathcal{L}\{e^{t}\}.$$

$$\boxed{y(t) = \frac{1}{2}\sin(t) - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}}$$

6.3 Step functions

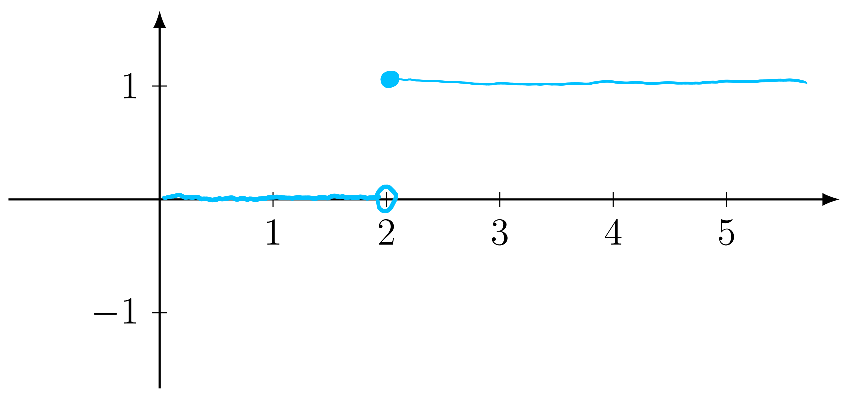
Definition: Unit step function

The **unit step function** (or Heaviside step function) is

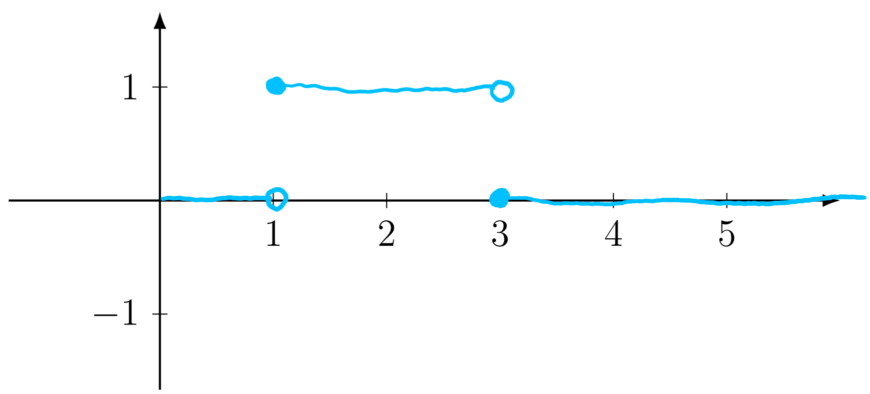
$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Examples Sketch the following functions.

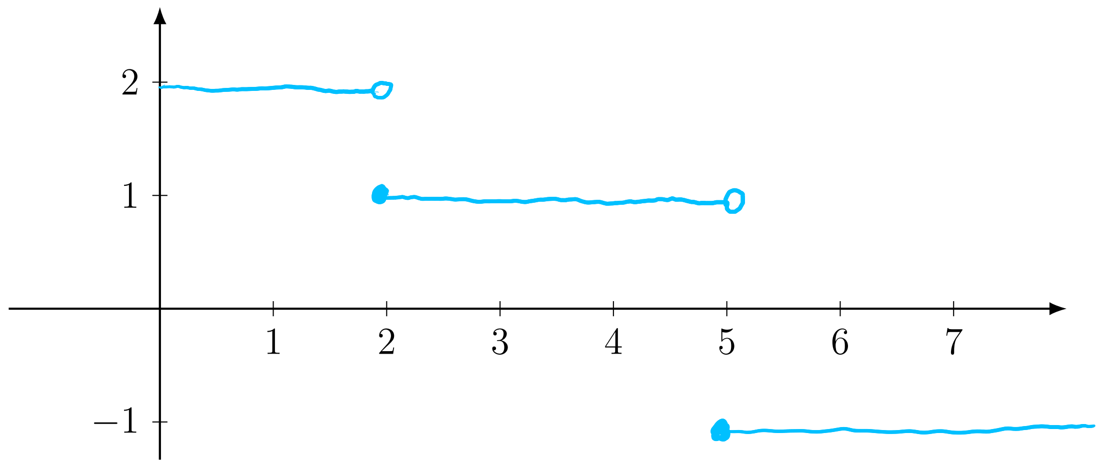
(a) $u_2(t)$



(b) $u_1(t) - u_3(t)$



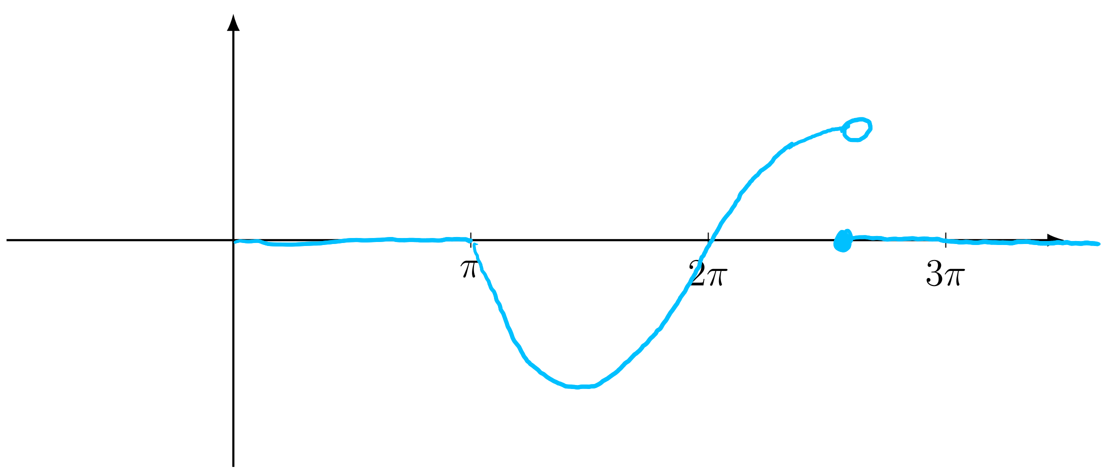
(c) $2 - u_2(t) - 2u_5(t)$



Turing functions on and off

Step functions are often used to turn a function "on" and "off". This is useful for modeling external forces on objects (for example, the spring-mass system).

Example Turn on the function $\sin(t)$ at $t = \pi$ and turn it off again at $t = 5\pi/2$.

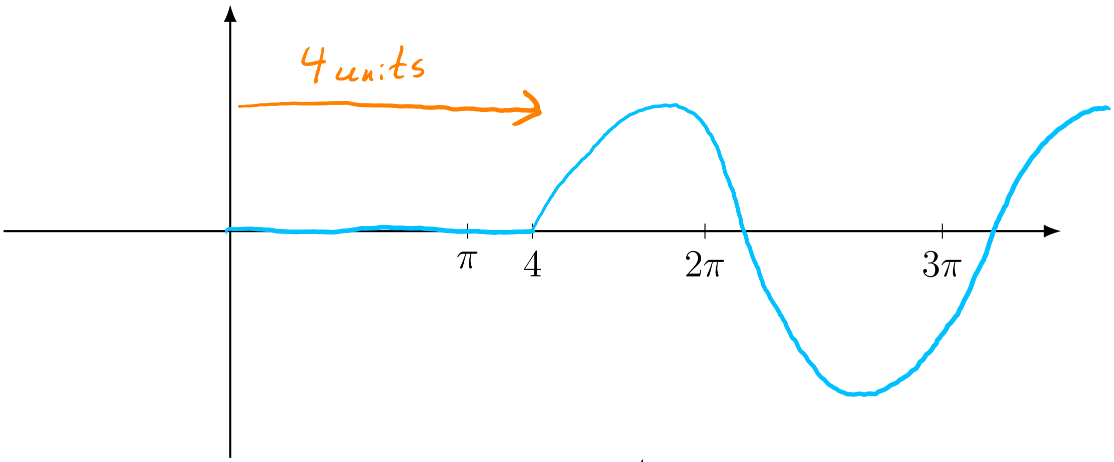


$u_\pi(t)\sin(t) - u_{\frac{5\pi}{2}}(t)\sin(t)$

Shifting functions

They can also be used to start a function at a later time than $t = 0$.

Example Shift the function $\sin(t)$ to the right 4 units and set the function equal to zero from $t = 0$ to $t = 4$.



$u_4(t)\sin(t-4)$

Laplace transforms of step functions

In order to solve IVPs with functions that get turned on and off, we need to be able to take their Laplace transforms.

Example Compute the Laplace transform of $u_c(t)$.

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt$$

$$= \int_c^\infty e^{-st} \cdot 1\,dt$$

$$= \frac{-1}{s} e^{-st}\Big|_c^\infty$$

$$= \frac{-1}{s}\left(0 - e^{-sc}\right) \quad (s>0)$$

$$\boxed{= \frac{e^{-sc}}{s}, \quad s>0}$$

Theorem: Laplace transform of u_c

$$\mathcal{L}\{u_c\} = \frac{e^{-sc}}{s}$$

Next, let's look at how the Laplace transform of a function f is affected when we multiply by a step function.

$$\mathcal{L}\{u_c(t)f(t)\} = \int_0^\infty e^{-st}u_c(t)f(t)\,dt$$

$$= \int_c^\infty e^{-st}f(t)\,dt \qquad u = t-c, \quad du = dt$$

$$= \int_0^\infty e^{-s(u+c)}f(u+c)\,du$$

$$= e^{-cs}\underbrace{\int_0^\infty e^{-su}f(u+c)\,du}_{\mathcal{L}\{f(t+c)\}}$$

Notice that if you plug in $f(t) = g(t-c)$, you get

$$\mathcal{L}\{u_c(t)g(t-c)\} = e^{-cs}\mathcal{L}\{g(t+c-c)\} = e^{-cs}\mathcal{L}\{g\}$$

$$\Rightarrow u_c(t)g(t-c) = \mathcal{L}^{-1}\{e^{-cs}G(s)\}.$$

Theorem: Laplace transform of shifts

If the Laplace transform of f exists for $s > a \geq 0$, and if $c > 0$, then

$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

Examples Find the Laplace transform of the following.

(a) $f(t) = \begin{cases} \sin(t) & 0 \le t < \frac{\pi}{4} \\ \sin(t) + \cos(t - \frac{\pi}{4}) & t \ge \frac{\pi}{4} \end{cases}$

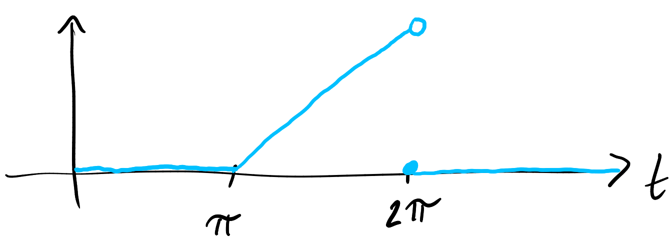
$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n $(n = 1, 2, ...)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$ $(n = 1, 2, ...)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

$$f(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \tfrac{\pi}{4})$$

$$F(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\mathcal{L}\{\cos(t + \tfrac{\pi}{4} - \tfrac{\pi}{4})\}$$

$$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}$$

(b) $f(t) = \begin{cases} 0 & t < \pi \\ t - \pi & \pi \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$



$$f(t) = (t - \pi)u_\pi(t) - (t - \pi)u_{2\pi}(t)$$

$$F(s) = e^{-\pi s}\mathcal{L}\{t - \pi + \pi\} - e^{-2\pi s}\mathcal{L}\{t - \pi + 2\pi\}$$

$$= e^{-\pi s}\mathcal{L}\{t\} - e^{-2\pi s}\mathcal{L}\{t + \pi\}$$

$$\boxed{= e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right)}$$

(c) $f(t) = \sin(t-2)u_2(t) - (t-2)u_3(t)$

$$= \sin(t-2)u_2(t) - (t-3+1)u_3(t)$$

$$= \sin(t-2)u_2(t) - (t-3)u_3(t) - u_3(t)$$

$$\boxed{F(s) = e^{-2s}\frac{1}{s^2+1} - e^{-3s}\frac{1}{s^2} - \frac{e^{-3s}}{s}}$$

Examples Find the inverse Laplace transform of the following.

(a) Find the inverse Laplace transform of $F(s) = \dfrac{1-e^{-2s}}{s^2}$.

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

$$\left(\frac{1}{s^2} = \frac{1!}{s^{1+1}} = \mathcal{L}\{t\}\right)$$

$$\boxed{f(t) = t - u_2(t)(t-2)}$$

(b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2} = e^{-2s}\left(\dfrac{1}{\underbrace{(s+2)(s-1)}_{\text{do partial fractions}}}\right)$

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$\Rightarrow 1 = A(s-1) + B(s+2)$$

$s=1$: $1 = 3B$, $B = 1/3$ $\qquad$ $s=-2$: $1 = -3A$, $A = -1/3$

$$F(s) = -\frac{1}{3}\frac{e^{-2s}}{s+2} + \frac{1}{3}\frac{e^{-2s}}{s-1}$$

$$= -\frac{1}{3}u_2(t)\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}_{t\to t-2} + \frac{1}{3}u_2(t)\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}_{t\to t-2}$$

$$\boxed{= -\frac{1}{3}u_2(t)e^{-2(t-2)} + \frac{1}{3}u_2(t)e^{t-2}}$$

(c) $F(s) = \dfrac{2(s-1)e^{-2s} + e^{-s}}{s^2 - 2s + 2}$

$$= 2e^{-2s}\frac{(s-1)}{(s-1)^2+1} + e^{-s}\frac{1}{(s-1)^2+1}$$

$$= 2u_2(t)\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+1}\right\}_{t\to t-2} + u_1(t)\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}_{t\to t-1}$$

$$\boxed{= 2u_2(t)e^{t-2}\cos(t-2) + u_1(t)e^{t-1}\sin(t-1)}$$

Inverse Laplace transform of shifts

We can also consider the opposite direction: What if the Laplace transform is shifted? Then what is the inverse transform?

$$F(s-c) = \int_0^\infty e^{-(s-c)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left(e^{ct} f(t)\right) dt$$

$$= \mathcal{L}\{e^{ct} f(t)\}.$$

Theorem: Inverse Laplace transform of shift

If $F(s)$ exists for $s > a \geq 0$, and c is a constant, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$$

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Example Using the above theorem and the fact that $\mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s^2+b^2}}_{F(s)}\right\} = \underbrace{\cos(bt)}_{f(t)}$, derive the formula for $\mathcal{L}^{-1}\left\{\underbrace{\frac{s-a}{(s-a)^2+b^2}}_{F(s-a)}\right\}$.

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)}$$

6.4 Differential equations with discontinuous forcing functions

We will look at 2nd order homogeneous linear ODEs

$$P(t)y'' + Q(t)y' + R(t)y = g(t)$$

where the right hand side (i.e., $g(t)$) is discontinuous.

Note: The nonhomogeneous term $g(t)$ is often called the **forcing function**.

Example Find the solution to the initial value problem

$$2y'' + y' + 2y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where

$$\cancel{g(t) = \begin{cases} 1 & 5 \le t < 20 \\ 0 & \text{otherwise} \end{cases}}. \qquad g(t) = \begin{cases} 0 & \text{if } t < 5 \\ 1 & \text{if } t \ge 5 \end{cases}$$

$$g(t) = 1 - u_5(t)$$

Laplace transform:

$$2\left(s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)}\right) + sY(s) - \cancel{y(0)} + 2Y(s) = \frac{1}{s} - \frac{e^{-5s}}{s}$$

Solve for $Y(s)$:

$$(2s^2 + s + 2)Y(s) = (1 - e^{-5s})\frac{1}{s}$$

$$Y(s) = (1 - e^{-5s})\frac{1}{s(2s^2+s+2)}$$

Inverse transform:

$$\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$1 = A(2s^2+s+2) + (Bs+C)s$$

$$1 = 2As^2 + As + 2A + Bs^2 + Cs$$

$$1 = \underbrace{(2A+B)}_{=0}s^2 + \underbrace{(A+C)}_{=0}s + \underbrace{2A}_{=1}$$

$$2\left(\tfrac{1}{2}\right) + B = 0 \qquad C = -\tfrac{1}{2} \qquad A = \tfrac{1}{2}$$

$$B = -1$$

$$Y(s) = \left(1 - e^{-5s}\right)\left(\frac{1/2}{s} + \frac{-s-1/2}{2s^2+s+2}\right)$$

$$= \left(1 - e^{-5s}\right)\left(\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s+1/2}{s^2+\frac{1}{2}s+1}\right)$$

$$= \left(1 - e^{-5s}\right)\left(\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s+1/2}{(s+\frac{1}{4})^2+\frac{15}{16}}\right)$$

$$= \left(1 - e^{-5s}\right)\left(\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s+1/4}{(s+\frac{1}{4})^2+\frac{15}{16}} - \frac{1}{2}\,\frac{1/4}{(s+\frac{1}{4})^2+\frac{15}{16}}\right)$$

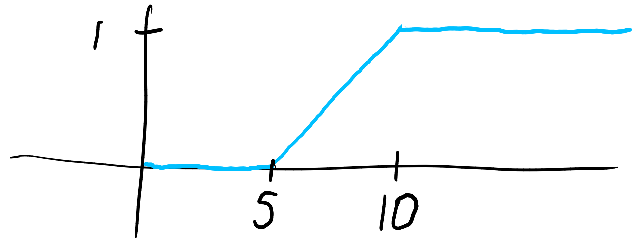
$$= \left(1 - e^{-5s}\right)\left(\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s+1/4}{(s+1/4)^2+\frac{15}{16}} - \frac{1}{2\sqrt{15}}\,\frac{\frac{\sqrt{15}}{4}}{(s+1/4)^2+\frac{15}{16}}\right)$$

$$= \left(1 - e^{-5s}\right)\left(\frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{e^{-t/4}\cos(\tfrac{\sqrt{15}}{4}t)\} - \frac{1}{2\sqrt{15}}\mathcal{L}\{e^{-t/4}\sin(\tfrac{\sqrt{15}}{4}t)\}\right)$$

$$\boxed{\begin{aligned} &= \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right) \\ &\quad - u_5(t)\left[\frac{1}{2} - \frac{1}{2}e^{-\frac{1}{4}(t-5)}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) - \frac{1}{2\sqrt{15}}e^{-\frac{1}{4}(t-5)}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right.\end{aligned}}$$

Example Solve the initial value problem

$$y'' + 4y = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1 & t \ge 10 \end{cases}$$



with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$y'' + 4y = \frac{1}{5} u_5(t)(t-5) - \frac{1}{5} u_{10}(t)(t-5) + u_{10}(t)$$

Laplace transform:

$$s^2 Y(s) + 4Y(s) = \frac{1}{5} e^{5s} \mathcal{L}\{t\} - \frac{1}{5} e^{10s} \mathcal{L}\{t+5\} + e^{10s} \cdot \frac{1}{s}$$

$$= \frac{1}{5} e^{5s} \frac{1}{s^2} - \frac{1}{5} e^{10s}\left(\frac{1}{s^2} + \cancel{\frac{5}{s}}\right) + \cancel{e^{10s} \frac{1}{s}}$$

$$(s^2+4) Y(s) = \frac{1}{5}\left(e^{-5s} - e^{-10s}\right)\frac{1}{s^2}$$

$$Y(s) = \frac{1}{5}\left(e^{-5s} - e^{-10s}\right) \underbrace{\frac{1}{s^2(s^2+4)}}_{\text{partial fractions}}$$

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$1 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2$$

$$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2$$

$$1 = \underbrace{(A+C)}_{=0} s^3 + \underbrace{(B+D)}_{=0} s^2 + \underbrace{4A}_{=0} s + \underbrace{4B}_{=1}$$

$C = 0$ $\quad D = -1/4$ $\quad A = 0$ $\quad B = 1/4$

$$Y(s) = \frac{1}{5}\left(e^{-5s} - e^{-10s}\right)\left(\frac{1}{4}\left(\frac{1}{s^2}\right) - \frac{1}{4}\left(\frac{1}{s^2+4}\right)\right)$$

$$= \frac{1}{5}\left(e^{-5s} - e^{-10s}\right)\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{2}\,\frac{2}{s^2+4}\right)$$

$$= \frac{1}{20}\left(e^{-5s} - e^{-10s}\right)\left(\mathcal{L}\{t\} - \frac{1}{2}\mathcal{L}\{\sin(2t)\}\right)$$

$$\boxed{\begin{aligned} y(t) = \frac{1}{20}\Big[& u_5(t)\left((t-5) - \tfrac{1}{2}\sin(2(t-5))\right) \\ & - u_{10}(t)\left((t-10) - \tfrac{1}{2}\sin(2(t-10))\right)\Big] \end{aligned}}$$

6.5 Impulse functions

Dirac delta function

Imagine that we hit a ball with a force that only last a very short time. How can we model this force as the time it acts goes to zero?

total impulse $= \int F(t)\,dt$

$$F_n(t) = \begin{cases} n & \frac{-1}{2n} < t < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

total impulse of all these forces is $n\left(\frac{1}{2n} - \frac{-1}{2n}\right) = 1$.

As n increases, the time interval gets shorter and the max force gets larger, but the impulse stays the same.

What if we take the limit as $n \to \infty$?

Infinite force that acts for one instant $(t=0)$, but still has impulse $= 1$. We call this the Dirac delta function $\delta(t)$.

Definition: Dirac delta function

The **Dirac delta function** $\delta(t)$ is defined to have the following properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0, \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

Properties of the delta function

Shifted delta function

If we want a unit impulse at $t = t_0$, instead of $t = 0$, then we can use

$\delta(t-t_0)$ (i.e., shift by t_0)

Integration against a function

What is $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,\mathrm{d}t$?

$\tau = t - t_0$
$d\tau = dt$

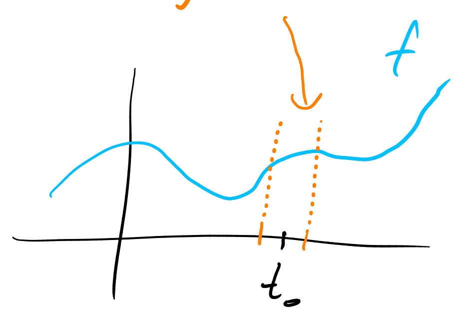
$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \int_{-\infty}^{\infty} \delta(\tau) f(\tau + t_0)\, d\tau$$

$$= \int_{-\infty}^{\infty} \lim_{n\to\infty} F_n(\tau)\, f(\tau+t_0)\, d\tau$$

$$= \int_{-1/2n}^{1/2n} n\, \underbrace{f(\tau+t_0)}\, d\tau$$

if f is continuous and n is large, then $f(\tau+t_0) \approx f(t_0)$

if we don't go far from t_0, f doesn't change much



becomes = as $n\to\infty$

$$\approx \int_{-1/2n}^{1/2n} n\, f(t_0)\, d\tau$$

$$= f(t_0) \underbrace{\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\, d\tau}_{=1}$$

$$= f(t_0).$$

Laplace transform of the delta function

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt$$

$$= e^{-st_0}. \qquad (t_0 \geq 0)$$

Theorem

The Laplace transform of the delta function is

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}.$$

Example Suppose we have a spring and mass system with mass $m = 2\,\mathrm{kg}$, spring constant $k = 6\,\mathrm{N/m}$, and damping coefficient $\gamma = 4\,\mathrm{N\,s/m}$. Suppose the system is at rest at equilibrium until we hit it with a hammer at time $t = 5\,\mathrm{s}$ with an impulse of $1\,\mathrm{N\,s}$. Determine the movement of the mass. The corresponding initial value problem is

$$2u'' + 4u' + 6u = \delta(t-5), \qquad u(0) = 0, \qquad u'(0) = 0.$$

Laplace transform:

$$2\left(s^2 U(s) - s\cancel{u(0)} - \cancel{u'(0)}\right) + 4\left(sU(s) - \cancel{u(0)}\right) + 6U(s) = e^{-5s}$$

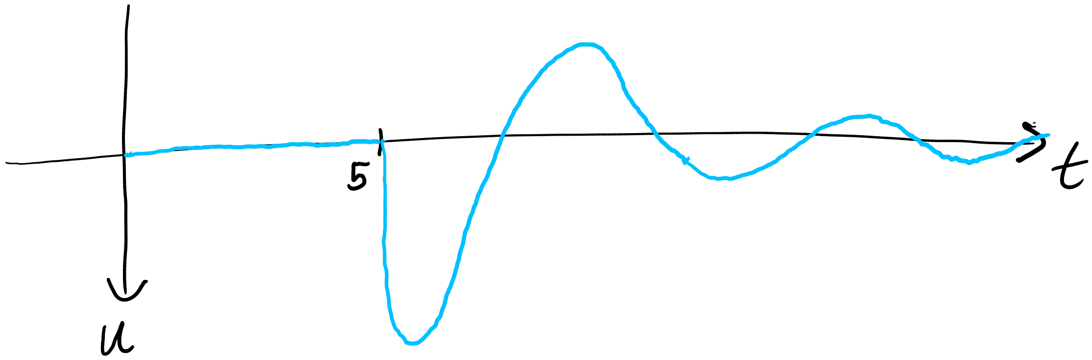
Solve for $U(s)$:

$$(2s^2 + 4s + 6)U(s) = e^{-5s}$$

$$U(s) = \frac{e^{-5s}}{2(s^2+2s+3)} = \frac{1}{2\sqrt{2}} e^{-5s} \frac{\sqrt{2}}{(s+1)^2 + 2}$$

Inverse transform:

$$\boxed{u(t) = \frac{1}{2\sqrt{2}} u_5(t) e^{-(t-5)} \sin\left(\sqrt{2}(t-5)\right)}$$



Example Solve the initial value problem

$$y'' + 4y' + 4y = \delta(t-2) + 2u_1(t), \qquad y(0) = 0, \qquad y'(0) = 2.$$

$$s^2 Y(s) - \cancel{sy(0)} - y'(0) + 4(sY(s) - \cancel{y(0)}) + 4Y(s) = e^{-2s} + 2\frac{e^{-s}}{s}$$

$$(s^2 + 4s + 4)Y(s) - 2 = e^{-2s} + 2\frac{e^{-s}}{s}$$

$$Y(s) = \frac{2 + e^{-2s}}{(s+2)^2} + 2e^{-s}\frac{1}{s(s+2)^2}$$

$$\frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + Bs(s+2) + Cs$$

$$1 = As^2 + 4As + 4A + Bs^2 + 2Bs + Cs$$

$$1 = \underbrace{(A+B)}_{=0}s^2 + \underbrace{(4A+2B+C)}_{=0}s + \underbrace{4A}_{=1}$$

$$B = \frac{-1}{4} \qquad 4\left(\tfrac{1}{4}\right) + 2\left(\tfrac{-1}{4}\right) + C = 0 \qquad A = \frac{1}{4}$$

$$C = -\tfrac{1}{2}$$

$$Y(s) = 2\left(\frac{1}{(s+2)^2}\right) + e^{-2s}\left(\frac{1}{(s+2)^2}\right) + 2e^{-s}\left(\frac{1/4}{s} - \frac{1/4}{s+2} - \frac{1/2}{(s+2)^2}\right)$$

$$= 2\mathcal{L}\{te^{-2t}\} + e^{-2s}\mathcal{L}\{te^{-2t}\} + e^{-s}\left(\tfrac{1}{2}\mathcal{L}\{1\} - \tfrac{1}{2}\mathcal{L}\{e^{-2t}\} - \mathcal{L}\{te^{-2t}\}\right)$$

$$\boxed{y(t) = 2te^{-2t} + u_2(t)(t-2)e^{-2(t-2)} + u_1(t)\left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2(t-1)} - (t-1)e^{-2(t-1)}\right)}$$

6.6 The convolution theorem

Definition: Convolution

The **convolution of f and g** is defined to be

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Product of Laplace transforms

If we know the inverse Laplace transform of $F(s)$ and $G(s)$, then is there an easy way to find the inverse Laplace transform of $F(s)G(s)$?

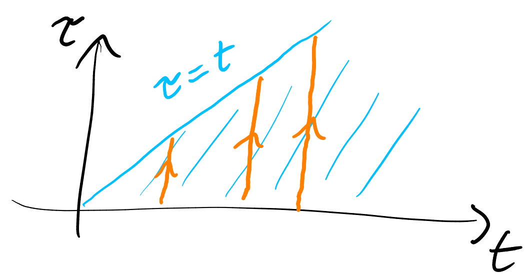
$$F(s)G(s) = \int_0^\infty e^{-sw} f(w)\,dw \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$$

$$= \int_0^\infty g(\tau) \int_0^\infty e^{-s(w+\tau)} f(w)\,dw\,d\tau$$

$t = w + \tau$

$dt = dw$

$$= \int_0^\infty g(\tau) \int_\tau^\infty e^{-st} f(t-\tau)\,dt\,d\tau$$



change order of integration

$$= \int_0^\infty \int_0^t e^{-st} f(t-\tau) g(\tau)\,d\tau\,dt$$

$$= \int_0^\infty e^{-st} \underbrace{\int_0^t f(t-\tau) g(\tau)\,d\tau}_{(f*g)(t)}\,dt$$

Theorem: Convolution theorem

If f and g are the inverse Laplace transforms of F and G, then

$$F(s)G(s) = \mathcal{L}\{(f * g)(t)\}.$$

Example Find the inverse Laplace transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

$$= \underbrace{\left(\frac{1}{s^2}\right)}_{\mathcal{L}\{t\}}\underbrace{\left(\frac{a}{s^2+a^2}\right)}_{\mathcal{L}\{\sin(at)\}}$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\} = (t * \sin(at))(t)$$

$$= \int_0^t (t-\tau)\sin(a\tau)\,d\tau \qquad u = t-\tau \quad dv = \sin(a\tau)\,d\tau \qquad du = -d\tau \quad v = \frac{-1}{a}\cos(a\tau)$$

$$= \frac{-1}{a}(t-\tau)\cos(a\tau)\Big|_{\tau=0}^{t} + \int_0^t \frac{-1}{a}\cos(a\tau)\,d\tau$$

$$= \frac{-1}{a}(0 - t\cos(0)) - \frac{1}{a}\frac{1}{a}\sin(a\tau)\Big|_{\tau=0}^{t}$$

$$= \frac{1}{a}t - \frac{1}{a^2}(\sin(at) - \sin(0))$$

$$\boxed{= \frac{1}{a}t - \frac{1}{a^2}\sin(at)}$$

Example Find the Laplace transform of $f(t) = \int_0^t e^{-2(t-\tau)} \cos(\tau)\,\mathrm{d}\tau$.

$$= \left(e^{-2t} * \cos(t)\right)(t)$$

$$\mathcal{L}\{f\} = \mathcal{L}\{e^{-2t} * \cos(t)\}$$

$$= \mathcal{L}\{e^{-2t}\}\mathcal{L}\{\cos(t)\}$$

$$\boxed{= \frac{1}{s+2} \cdot \frac{s}{s^2+1}}$$

Example Solve the Volterra integral equation

$$f(t) = 3 + \underbrace{\int_0^t \sin(2t - 2\tau) f(\tau)\,\mathrm{d}\tau}_{(\sin(2t) * f)(t)}.$$

Laplace transform both sides.

$$F(s) = \frac{3}{s} + \frac{2}{s^2+4} F(s)$$

$$\left(1 - \frac{2}{s^2+4}\right) F(s) = \frac{3}{s}$$

$$\frac{s^2+2}{s^2+4} F(s) = \frac{3}{s}$$

$$F(s) = \frac{3}{s} \frac{s^2+4}{s^2+2} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

$$3s^2+12 = A(s^2+2)+(Bs+C)s$$

$$= As^2+2A+Bs^2+Cs$$

$$= \underbrace{(A+B)}_{3}s^2+\underbrace{C}_{0}s+\underbrace{2A}_{12}$$

$A+B=3$ $\quad C=0 \quad$ $2A=12$

$A=6$

$6+B=3$

$B=-3$

$$F(s) = \frac{6}{s} - 3\frac{s}{s^2+2}$$

Inverse transform.

$$\boxed{f(t) = 6 - 3\cos(\sqrt{2}t)}$$