Computational Physics

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Differential equations for Astrophysics
Lund Observatory
FYSN33 - Lecture 2

Summary of this part of the course

- We will discuss more about the detail of smoothing functions
- We will introduce the full set of Navier Stokes equations to use for the projects
- We will discuss practical numerical aspects to take into account for the projects including how to integrate equations
- Initial conditions for the projects will be given

Smoothing function properties (1/2)

A central problem in meshfree methods is how to effectively construct smoothing functions, W, based on a set of particles scattered in an arbitrary manner.

The SPH method uses an integral representation of the smoothing function

The smoothing function (or kernel) is important as it

- determines the shape for the function approximation
- defines the dimensions of the support domain
- determines the accuracy of both the kernel and particle approximations

Different smoothing functions have been used in the literature.

Smoothing function properties (2/2)

- The smoothing function must be normalised over its support domain

$$\int_{\Omega} W(\mathbf{x}-\mathbf{x}', h)\,d\mathbf{x}' = 1$$

- The smoothing function are compactly supported (allows for sparse discrete matrices)

$$W(\mathbf{x}-\mathbf{x}') = 0 \text{ , for } |\mathbf{x}-\mathbf{x}'| > \kappa h$$

  The dimensions are defined by the smoothing length h and the scaling factor κ

- $W(\mathbf{x} - \mathbf{x}') \geq 0$ for any point at $\mathbf{x}'$ within the support domain of the particle $\mathbf{x}$, i.e. positive, to be physically meaningful
- The smoothing function should monotonically decrease with distance to represent decreasing interaction
- The smoothing function should satisfy the Dirac delta function as h approaches zero

$$\lim_{h \to 0} W(\mathbf{x}-\mathbf{x}', h) = \delta(\mathbf{x}-\mathbf{x}')$$

- Even function, i.e. symmetric to ensure inverse interactions have the same strength
- Should be sufficiently smooth to ensure that it and it's derivatives achieve reasonable accuracy

Table 3.1 Summary of smoothing functions (Defined for a point or particle at x; $R = \frac{r}{h} = \frac{|x - x'|}{h}$; r is the distance between two points or particles at x and x')

Name	Smoothing function $W(R,h)$
Quartic (Lucy, 1977)	$\alpha_d (1+3R)(1-R)^3 \quad R \le 1$
Guassian (Gingold and Monaghan, 1977)	$\alpha_d e^{-R^2}$
Piecewise cubic spline (Monaghan and Lattanzio, 1985)	$\alpha_d \begin{cases} \frac{2}{3} - R^2 + \frac{1}{2}R^3 & 0 \le R < 1 \\ \frac{1}{6}(2-R)^3 & 1 \le R \le 2 \end{cases}$
Piecewise quartic (Morris, 1996)	$\alpha_d \begin{cases} (R+2.5)^4 - 5(R+1.5)^4 + 10(R+0.5)^4 & 0 \le R < 0.5 \\ (2.5-R)^4 - 5(1.5-R)^4 & 0.5 \le R < 1.5 \\ (2.5-R)^4 & 1.5 \le R \le 2.5 \end{cases}$
Piecewise quintic (Morris, 1996)	$\alpha_d \begin{cases} (3-R)^5 - 6(2-R)^5 + 15(1-R)^5 & 0 \le R < 1 \\ (3-R)^5 - 6(2-R)^5 & 1 \le R < 2 \\ (3-R)^5 & 2 \le R \le 3 \end{cases}$
Quadratic (Johnson et al., 1996b)	$\alpha_d (\frac{3}{16}R^2 - \frac{3}{4}R + \frac{3}{4}) \quad 0 \le R \le 2$
Super Gaussian (Monaghan and Lattanzio, 1985)	$\alpha_d (\frac{3}{2} - R^2) e^{-R^2} \quad 0 \le R \le 2$
Dome-shaped quadratic (Hicks and Liebrock, 2000)	$\alpha_d (1 - R^2) \quad 0 \le R \le 1$
New quartic (Liu, Liu and Lam, 2002)	$\alpha_d (\frac{2}{3} - \frac{9}{8}R^2 + \frac{19}{24}R^3 - \frac{5}{32}R^4) \quad 0 \le R \le 2$

Summary of smoothing functions

Smoothing functions determine the shape, effective support domain and accuracy of the approximation.

Taylor series expansions can be used to derive conditions for the kernel approximation.

These conditions can then be used to systematically construct smoothing functions.

A few important points are listed below:

- The use of partially negative smoothing functions in SPH can lead to unphysical results for hydrodynamical problems where density or energy are used
- The centre peak value of the smoothing function plays an important role in determining the accuracy of the SPH kernel approximation
- Piecewise construction of smoothing functions are among the most popular and can provide greater flexibility and accuracy

Navier-Stokes equations

The Navier-Stokes are an explicit statement of the conservation of mass, momentum and energy. They generalise the conservation equations in the Lagrangian frame.

The continuity equation

$$\frac{D\rho}{Dt} = -\rho\frac{\partial v^{\beta}}{\partial x^{\beta}}$$

The momentum equation
add $+F^{\alpha}$ for external forces

$$\frac{Dv^{\alpha}}{Dt} = \frac{1}{\rho}\frac{\partial\sigma^{\alpha\beta}}{\partial x^{\beta}}$$

The energy equation

$$\frac{De}{Dt} = \frac{\sigma^{\alpha\beta}}{\rho}\frac{\partial v^{\alpha}}{\partial x^{\beta}}$$

Note: To include the coordinate indices properly we do double summation over the ε terms

$$\frac{De}{Dt} = -\frac{p}{\rho}\frac{\partial v^{\beta}}{\partial x^{\beta}} + \frac{\mu}{2\rho}\varepsilon^{\alpha\beta}\varepsilon^{\alpha\beta}$$

σ is the total stress tensor. It is made up of the isotropic pressure p and the viscous stress τ.

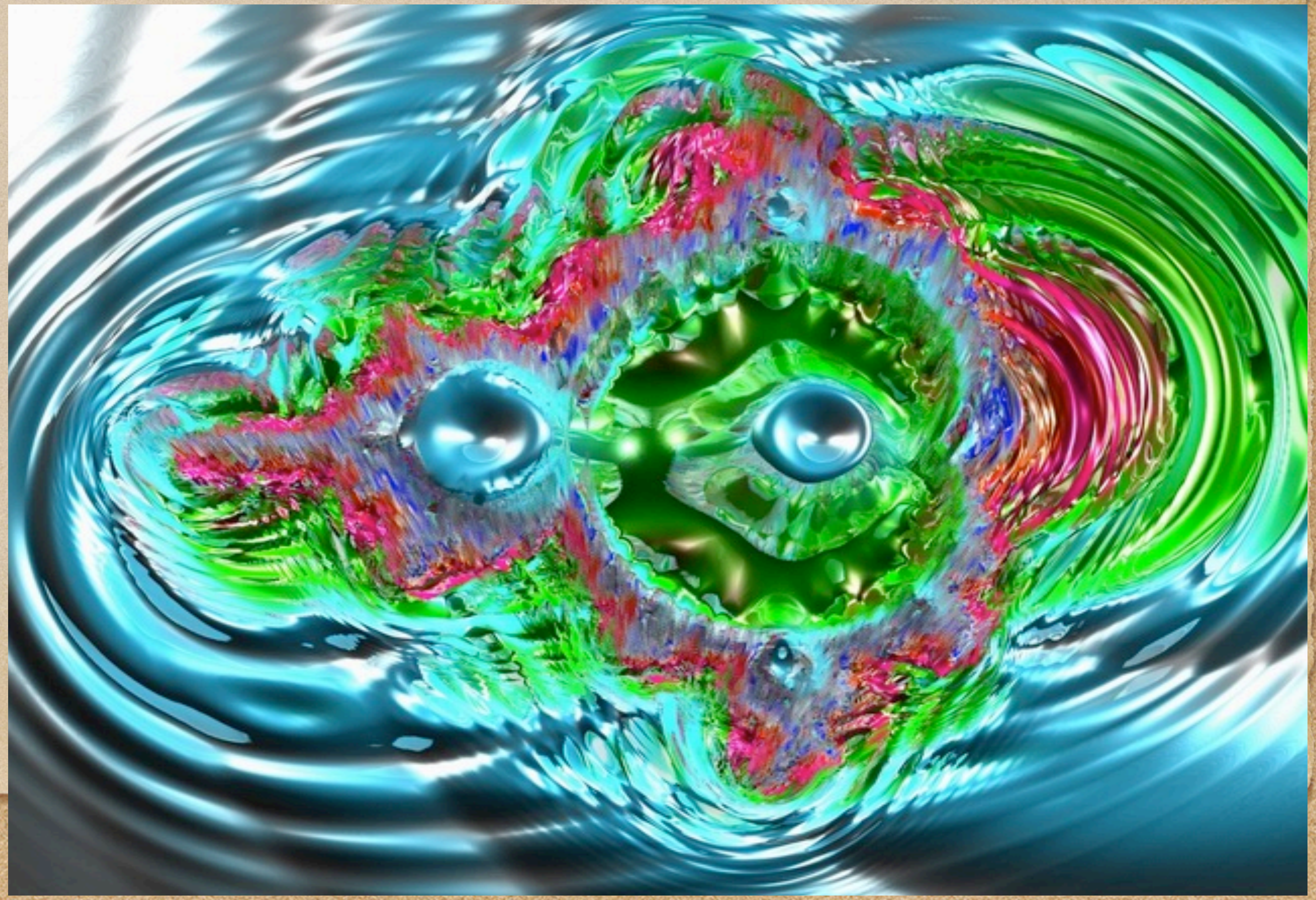
$$\sigma^{\alpha\beta} = -p\delta^{\alpha\beta} + \tau^{\alpha\beta}$$

The viscous stress should be proportional to the strain rate ε through the dynamic viscosity μ:

$$\tau^{\alpha\beta} = \mu\varepsilon^{\alpha\beta}$$

where

$$\varepsilon^{\alpha\beta} = \frac{\partial v^{\beta}}{\partial x^{\alpha}} + \frac{\partial v^{\alpha}}{\partial x^{\beta}} - \frac{2}{3}(\nabla\cdot v)\delta^{\alpha\beta}$$



Particle approximation for Navier-Stokes equations

We now apply the particle approximation to each of the Navier-Stokes equations.

We have already learnt how to do this is Lecture 1 so only the main results will be presented.

There are two approaches for density:

Summation Density

Continuity Density

Additionally, different forms of the equations are possible with slightly different properties, e.g. symmetric, particle differencing, normalisation, etc.

Summation density

This directly applies the SPH approximation to the density itself. For a given particle *i* we have

$$\rho_i = \sum_{j=1}^{N} m_j W_{ij} \qquad \text{alternatively} \qquad \rho_i = \frac{\sum_{j=1}^{N} m_j W_{ij}}{\sum_{j=1}^{N} \left(\frac{m_j}{\rho_j}\right) W_{ij}}$$

where N is the number of support domain particles and m_j is the mass of j.

W_{ij} is the smoothing function of particle i evaluated at j

$$W_{ij} = W(\mathbf{x}_i - \mathbf{x}_j, h) = W(\left|\mathbf{x}_i - \mathbf{x}_j\right|, h) = W(R_{ij}, h)$$

where $R_{ij} = \frac{r_{ij}}{h} = \frac{\left|\mathbf{x}_i - \mathbf{x}_j\right|}{h}$ is the relative distance of i and j, and r_{ij} is the scalar distance.

W_{ij} has units of inverse volume.

Continuity density

Here the density is obtained from the continuity equation, possible forms are

$$\frac{D\rho_i}{Dt} = -\rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} v_j^{\beta} \cdot \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

alternatively

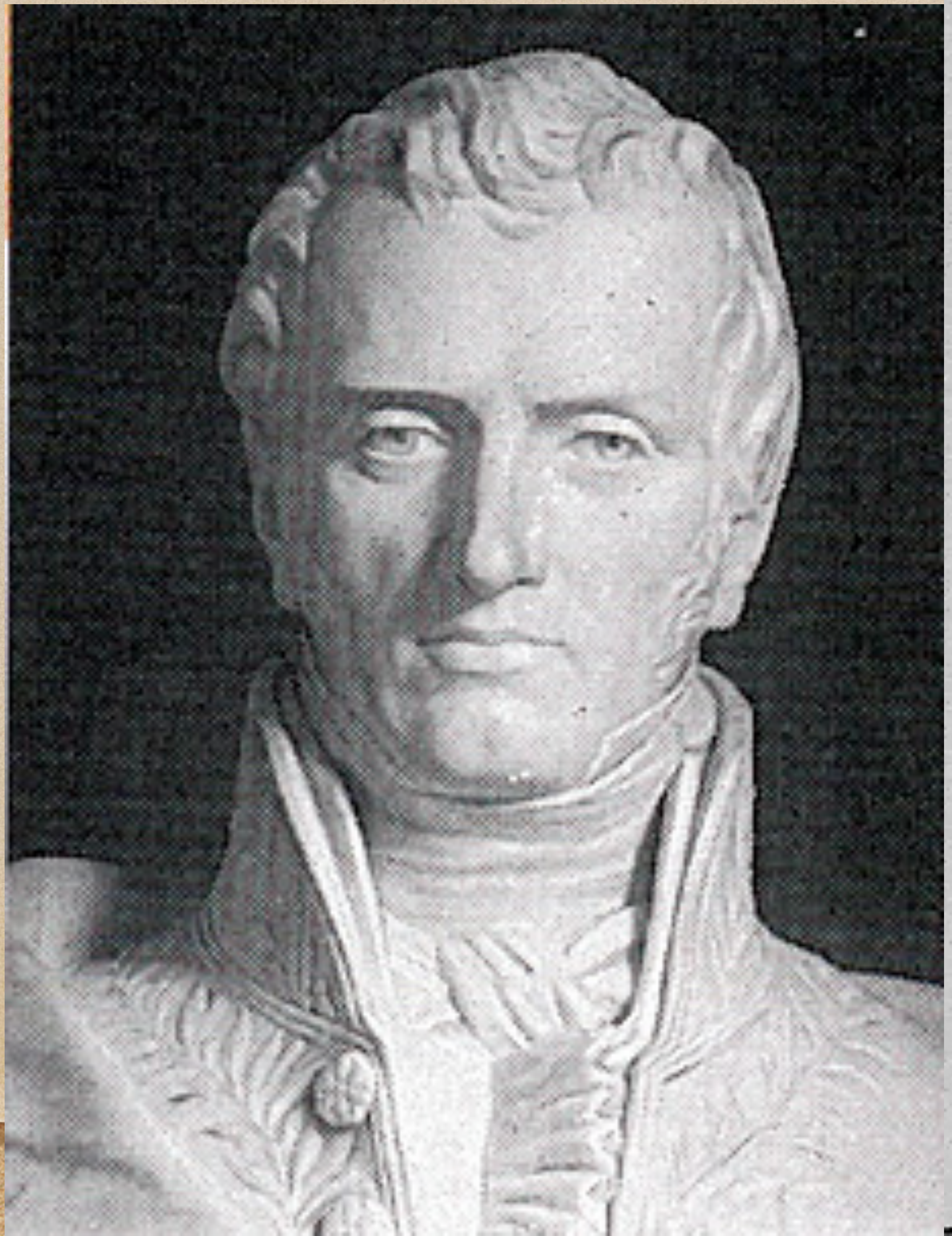
$$\frac{D\rho_i}{Dt} = \sum_{j=1}^{N} m_j v_{ij}^{\beta} \cdot \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

$$v_{ij}^{\beta} = (v_i^{\beta} - v_j^{\beta})$$

Similarly particle approximations for the other equations can be obtained, the most popular forms are summarised on the next slide.

Summary of Lagrange form Navier-Stokes equations

Claude-Louis Navier (10 February 1785 – 21 August 1836) was a French engineer and physicist who specialised in mechanics.



Conservation of mass

$$\rho_i = \sum_{j=1}^{N} m_j W_{ij}$$

$$\rho_i = \frac{\sum_{j=1}^{N} m_j W_{ij}}{\sum_{j=1}^{N} \left(\frac{m_j}{\rho_j}\right) W_{ij}}$$

$$\frac{D\rho_i}{Dt} = -\rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} v_j^{\beta} \cdot \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

$$\frac{D\rho_i}{Dt} = \rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} v_{ij}^{\beta} \cdot \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

$$\frac{D\rho_i}{Dt} = \sum_{j=1}^{N} m_j v_{ij}^{\beta} \cdot \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

Conservation of momentum

$$\frac{Dv_i^{\alpha}}{Dt} = \sum_{j=1}^{N} m_j \frac{\sigma_i^{\alpha\beta} + \sigma_j^{\alpha\beta}}{\rho_i \rho_j} \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

$$\frac{Dv_i^{\alpha}}{Dt} = \sum_{j=1}^{N} m_j \left(\frac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \frac{\sigma_j^{\alpha\beta}}{\rho_j^2}\right) \frac{\partial W_{ij}}{\partial x_i^{\beta}}$$

Conservation of energy

$$\frac{De_i}{Dt} = \frac{1}{2} \sum_{j=1}^{N} m_j \frac{p_i + p_j}{\rho_i \rho_j} v_{ij}^{\beta} \frac{\partial W_{ij}}{\partial x_i^{\beta}} + \frac{\mu_i}{2\rho_i} \varepsilon_i^{\alpha\beta} \varepsilon_i^{\alpha\beta}$$

$$\frac{De_i}{Dt} = \frac{1}{2} \sum_{j=1}^{N} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) v_{ij}^{\beta} \frac{\partial W_{ij}}{\partial x_i^{\beta}} + \frac{\mu_i}{2\rho_i} \varepsilon_i^{\alpha\beta} \varepsilon_i^{\alpha\beta}$$

Sir George Gabriel Stokes, (13 August 1819–1 February 1903), was a mathematician and physicist, who at Cambridge made important contributions to fluid dynamics (including the Navier–Stokes equations), optics, and mathematical physics. He was secretary, then president, of the Royal Society.

Artificial Viscosity

Special treatment is needed to model shock waves, to avoid the simulation developing unphysical oscillations.

Applying the conservation equations across a shock wave front requires a transformation of kinetic energy to heat energy.

This can be represented by a form of viscous dissipation which led to the development of the Neumann-Richtmyer viscosity is given by:

$$\Pi_1 = \begin{cases} a_1 \Delta x^2 \rho (\nabla \cdot \boldsymbol{v})^2 & \nabla \cdot \boldsymbol{v} < 0 \\ 0 & \nabla \cdot \boldsymbol{v} \geq 0 \end{cases}$$

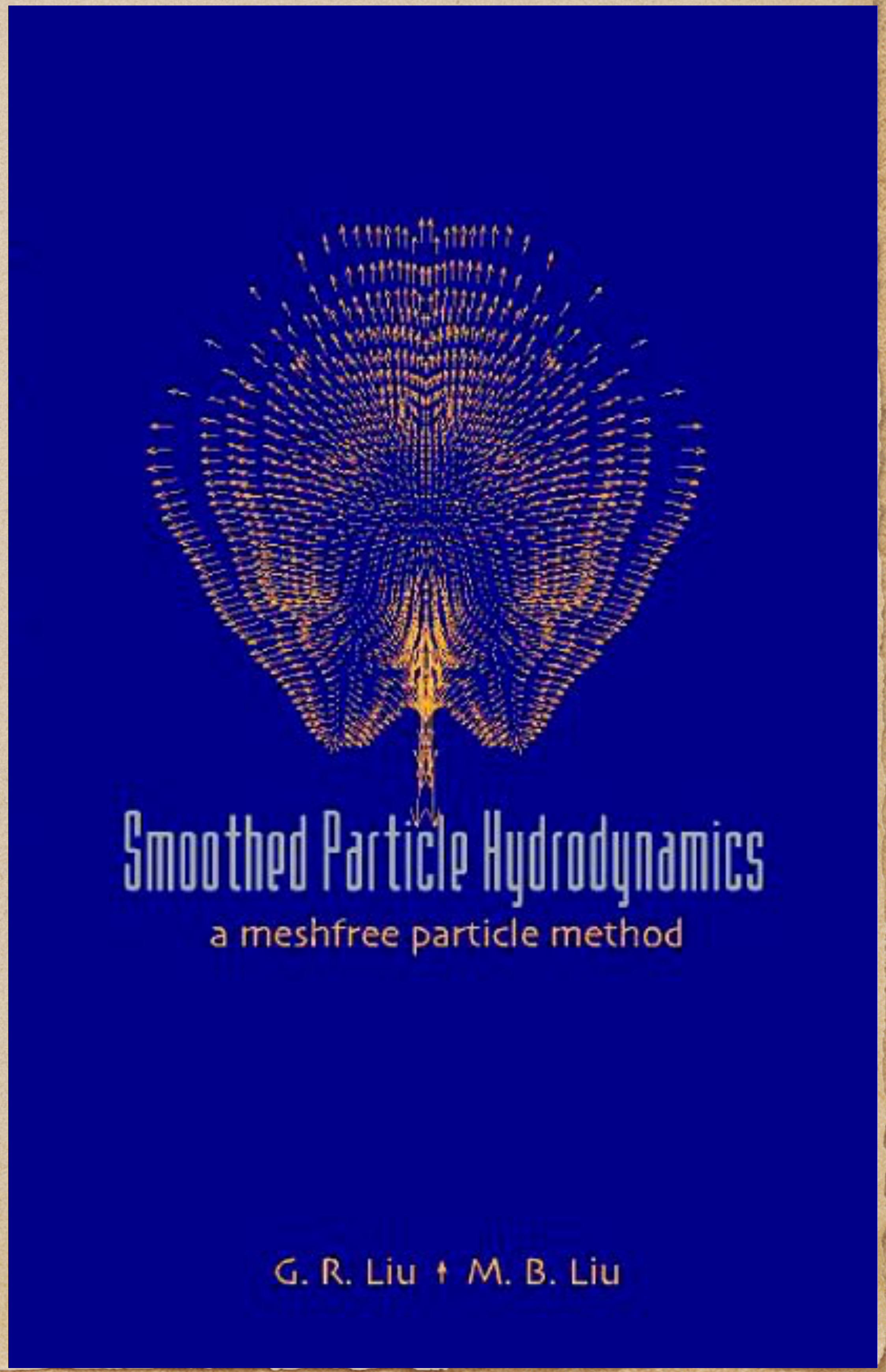
where this quadratic expression of velocity divergence, Π_1, needs only to be present during material compression, a_1 is an adjustable constant. $\Delta x = h$ i.e. the smoothing length.

More sophisticated models of viscosity exist, including effects such as artificial shear viscosity, etc. see Monaghan J.J., Smoothed Particle Hydrodynamics, Annual Review of Astronomy and Astrophysics, 30:543-574, (1992) for further details.

Artificial Heat

Excessive heating can also occur, for example, when a stream of gas is brought to rest against a wall.

Monaghan ref. [5] solved this by adding an artificial heat term, H_i, which can be added to the energy equation.



Finally how are they used in SPH?

Both of these and other effects can be modelled in SPH by including them in the appropriate Navier-Stokes equations. A popular form is:

$$
\begin{cases}
\dfrac{D\rho_i}{Dt} = \sum_{j=1}^{N} m_j v_{ij}^{\beta} \dfrac{\partial W_{ij}}{\partial x_i^{\beta}} \\[2ex]
\dfrac{Dv_i^{\alpha}}{Dt} = -\sum_{j=1}^{N} m_j \left(\dfrac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \dfrac{\sigma_j^{\alpha\beta}}{\rho_j^2} + \Pi_{ij} \right) \dfrac{\partial W_{ij}}{\partial x_i^{\beta}} + F_i^{\alpha} \\[2ex]
\dfrac{De_i}{Dt} = \dfrac{1}{2} \sum_{j=1}^{N} m_j \left(\dfrac{p_i}{\rho_i^2} + \dfrac{p_j}{\rho_j^2} + \Pi_{ij} \right) v_{ij}^{\beta} \dfrac{\partial W_{ij}}{\partial x_i^{\beta}} + \dfrac{\mu_i}{2\rho_i} \varepsilon_i^{\alpha\beta} \varepsilon_i^{\alpha\beta} + H_i \\[2ex]
\dfrac{Dx_i^{\alpha}}{Dt} = v_i^{\alpha}
\end{cases}
$$

Artificial viscosity

External forces

Heat term

Viscous stress, where ε is the strain rate and μ is the dynamic viscosity

Solving second order differential equations is done by reformulating the N second-order equations as a set of $2N$ coupled first-order equations. Any ordinary second-order differential equation of the form

$$A(x,t)\frac{d^2x}{dt^2} + B(x,t)\frac{dx}{dt} + C(x,t) = 0,$$

can be rewritten as a pair of first-order differential equations.

$$\frac{dx}{dt} = v(t)\,,$$
$$\frac{dv}{dt} = -\frac{B(x,t)}{A(x,t)}v(t) - \frac{C(x,t)}{A(x,t)}\,.$$

This is then in the form of the Navier Stokes equations on the previous slide:

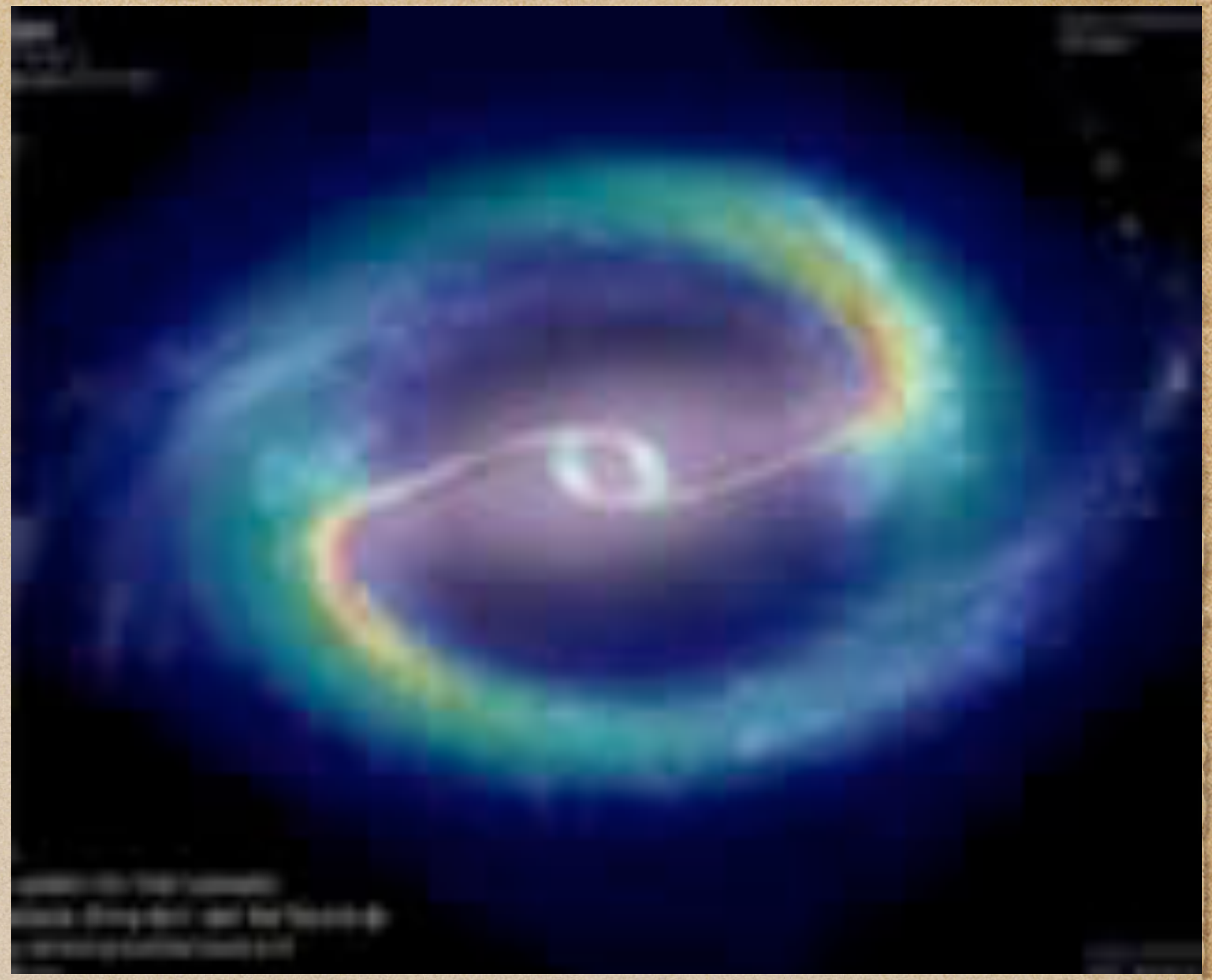
Artificial viscosity

External forces

$$\begin{cases} \dfrac{D\rho_i}{Dt} = \sum_{j=1}^{N} m_j v_{ij}^{\beta} \dfrac{\partial W_{ij}}{\partial x_i^{\beta}} \\ \dfrac{Dv_i^{\alpha}}{Dt} = -\sum_{j=1}^{N} m_j \left(\dfrac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \dfrac{\sigma_j^{\alpha\beta}}{\rho_j^2} + \Pi_{ij}\right) \dfrac{\partial W_{ij}}{\partial x_i^{\beta}} + F_i^{\alpha} \\ \dfrac{De_i}{Dt} = \dfrac{1}{2} \sum_{j=1}^{N} m_j \left(\dfrac{p_i}{\rho_i^2} + \dfrac{p_j}{\rho_j^2} + \Pi_{ij}\right) v_{ij}^{\beta} \dfrac{\partial W_{ij}}{\partial x_i^{\beta}} + \dfrac{\mu_i}{2\rho_i} \varepsilon_i^{\alpha\beta} \varepsilon_i^{\alpha\beta} + H_i \\ \dfrac{Dx_i^{\alpha}}{Dt} = v_i^{\alpha} \end{cases}$$

Heat term

Viscous stress, where ε is the strain rate and μ is the dynamic viscosity



If $\boldsymbol{w}_i = [\boldsymbol{x}_i, \boldsymbol{v}_i, \rho_i, e_i]$ represents the 8-D phase space coordinates of particle i, then the state of the N-body system is described by a $8N$ vector

$$\mathbf{W} \equiv [\mathbf{w}_1, \ldots, \mathbf{w}_N].$$

Evolution of the system takes the form

$$\frac{dW_l}{dt} = g_l(\mathbf{W}),$$

where the $8N$ functions g_l are given by the right hand sides of the Navier Stokes Equations. When the equations are in this form a single integration routine can be used to solve the system.

Differential equations like the Navier Stokes equations describe a continuous sequence of changes in response to an independent variable, t.

We need to specify $8N$ conditions ($\boldsymbol{x}_i$, $\boldsymbol{v}_i$, ρ_i, e_i) to solve our initial value problem.

The simplest solution to the problem comes from specifying a finite difference of the differential equation over an interval

$$h \equiv \Delta t \equiv t^{n+1} - t^n$$

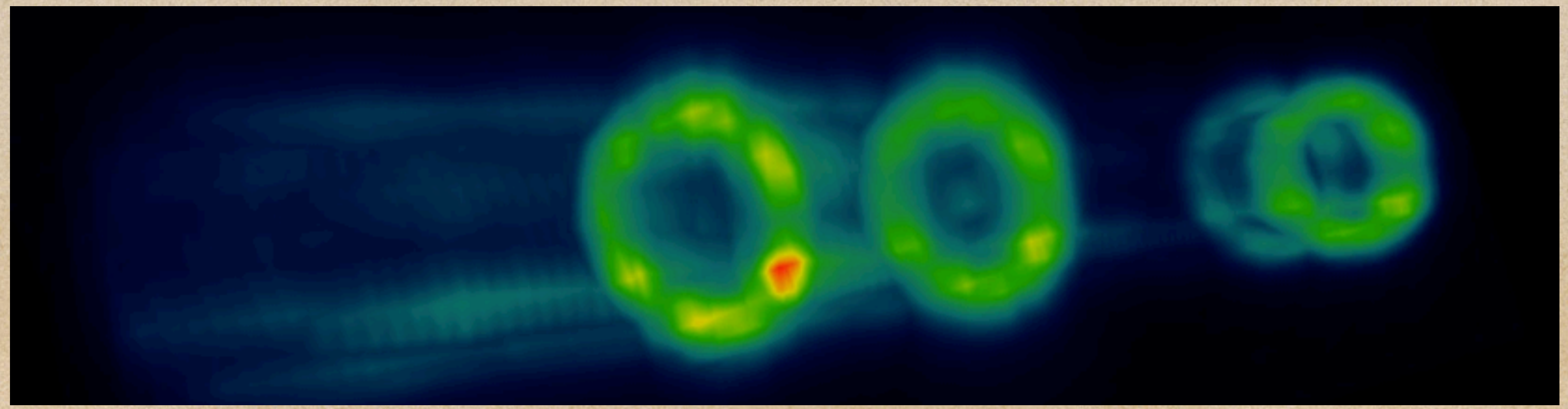
as

$$W_l^{n+1} = W_l^n + h\, g_l\left(W_1^n, \ldots, W_l^n, \ldots, W_{6N}^n\right) = W_l^n + h\, g_l(\mathbf{W}^n).$$

where the update variables, W_l^{n+1}, are calculated from information available at t^n.

By repeatedly applying this formula the position and velocity of the particles can be propagated into the future in time steps of h. This is know as Euler's method.

In practice this is not sufficiently stable for long integrations so more sophisticated methods are used but this simple form illustrates the basic principle well.



Euler's method is not good because of error accumulation. The exact equation is given by a Taylor series expansion.

$$W_l^{n+1} = W_l^n + \frac{h}{1!}\frac{dW_l}{dt} + \frac{h^2}{2!}\frac{d^2 W_l}{dt^2} + \cdots + \frac{h^n}{n!}\frac{d^n W_l}{dt^n} + \cdots$$

Euler's method is just the first two terms – it's a first order method.

The rest of the terms sum to give the total error.

Euler's method is asymmetric as the increment is based on the value of ***W*** at the beginning of the interval *h*.

$$h dW_l/dt \equiv h g_l(t^n, \mathbf{W^n})$$

Any non-linearity in ***g*** will result in inaccuracy in the updated values.

The key to improving Euler's method is to realise that ***g*** can be computed at any trial values (*t*, ***W***) so more refined estimates over *h* can be made.

One can construct weighted sums with *k* estimates of ***g*** and tuning the weights provides cancellation of error terms in the Taylor series up to order *k*+1.

Implementations such as this are known as Runge-Kutta methods (e.g. RK45).

The simplest Runge-Kutta scheme uses the Euler method to estimate values $W(t^n+h/2)$ at the halfway point.

$$\mathbf{W}\left(t^n + \frac{h}{2}\right) = \mathbf{W}_b = \mathbf{W}(t^n) + \frac{h}{2}\mathbf{g}(t^n, \mathbf{W}^n).$$

The values W_b provide a more accurate estimate of the slope in the interval h.

A 4th order Runge-Kutta method is commonly used:

$$\begin{aligned}
\mathbf{f}_a &= \mathbf{g}(t^n, \mathbf{W}^n) \\
\mathbf{W}_b &= \mathbf{W}^n + \frac{h}{2}\mathbf{f}_a \\
\mathbf{f}_b &= \mathbf{g}\left(t^n + \frac{h}{2}, \mathbf{W}_b\right) \\
\mathbf{W}_c &= \mathbf{W}^n + \frac{h}{2}\mathbf{f}_b \\
\mathbf{f}_c &= \mathbf{g}\left(t^n + \frac{h}{2}, \mathbf{W}_c\right) \\
\mathbf{W}_d &= \mathbf{W}^n + h\mathbf{f}_c \\
\mathbf{f}_d &= \mathbf{g}(t^n + h, \mathbf{W}_d) \\
\mathbf{W}^{n+1} &= \mathbf{W}^n + \frac{1}{6}h\mathbf{f}_a + \frac{1}{3}h\mathbf{f}_b + \frac{1}{3}h\mathbf{f}_c + \frac{1}{6}h\mathbf{f}_d.
\end{aligned}$$

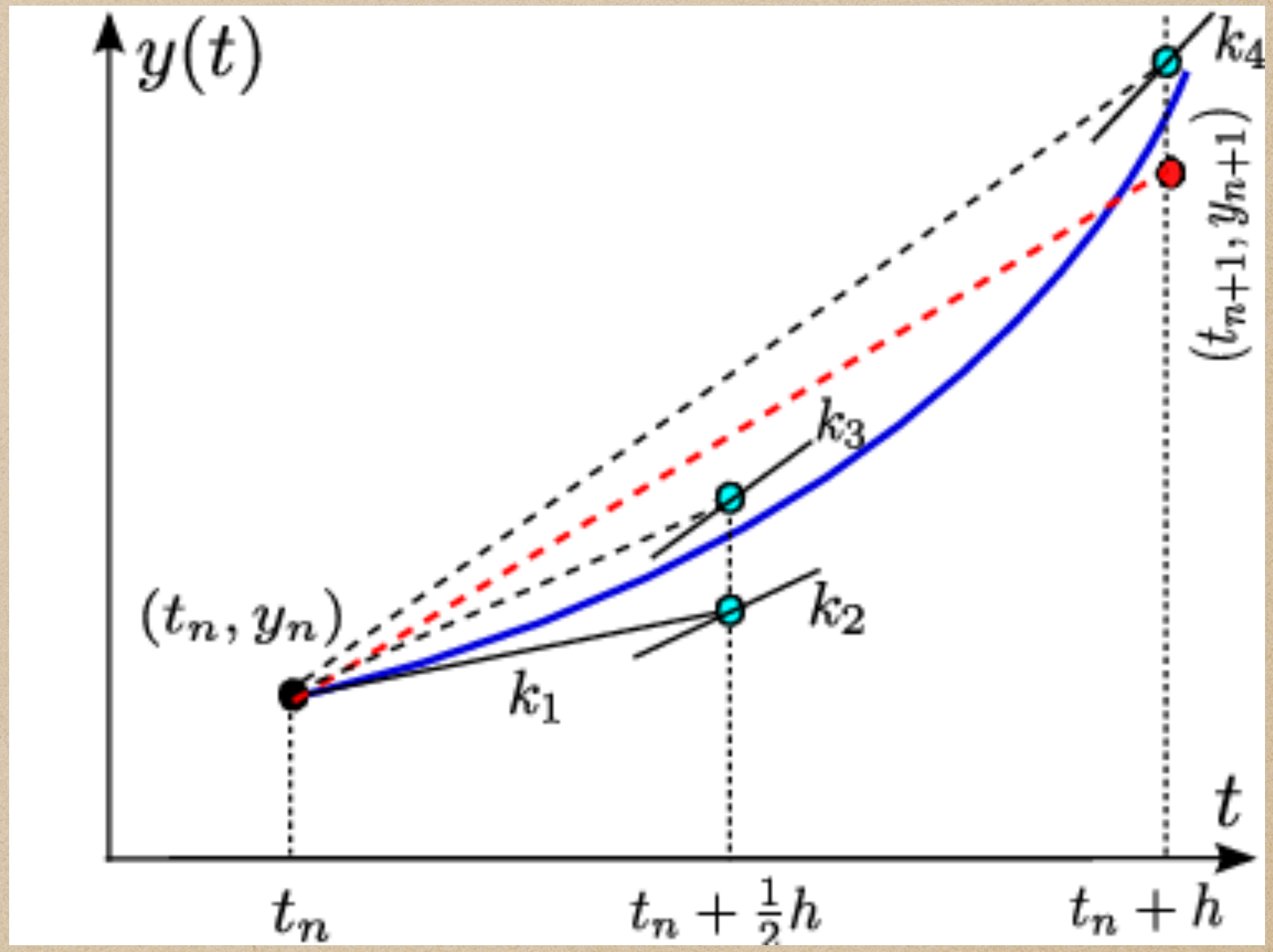
A single iteration evaluates

- the initial slope at the start
- two midpoint slopes
- the slope at the end

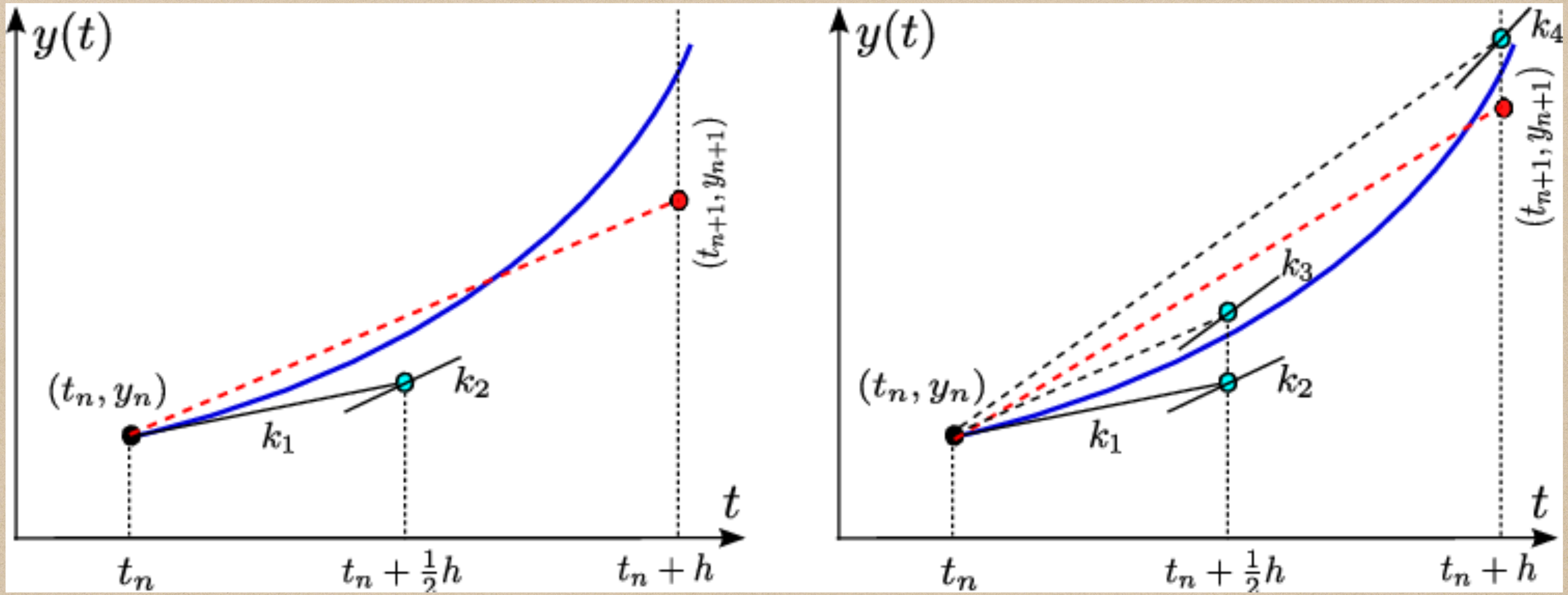
The four slopes are then combined with weights

$\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$

to produce a weighted average slope which accurately spans



4th order



Runge-Kutta method illustration. On the left the second-order and on the right the fourth-order Runge-Kutta method.

Variable smoothing length

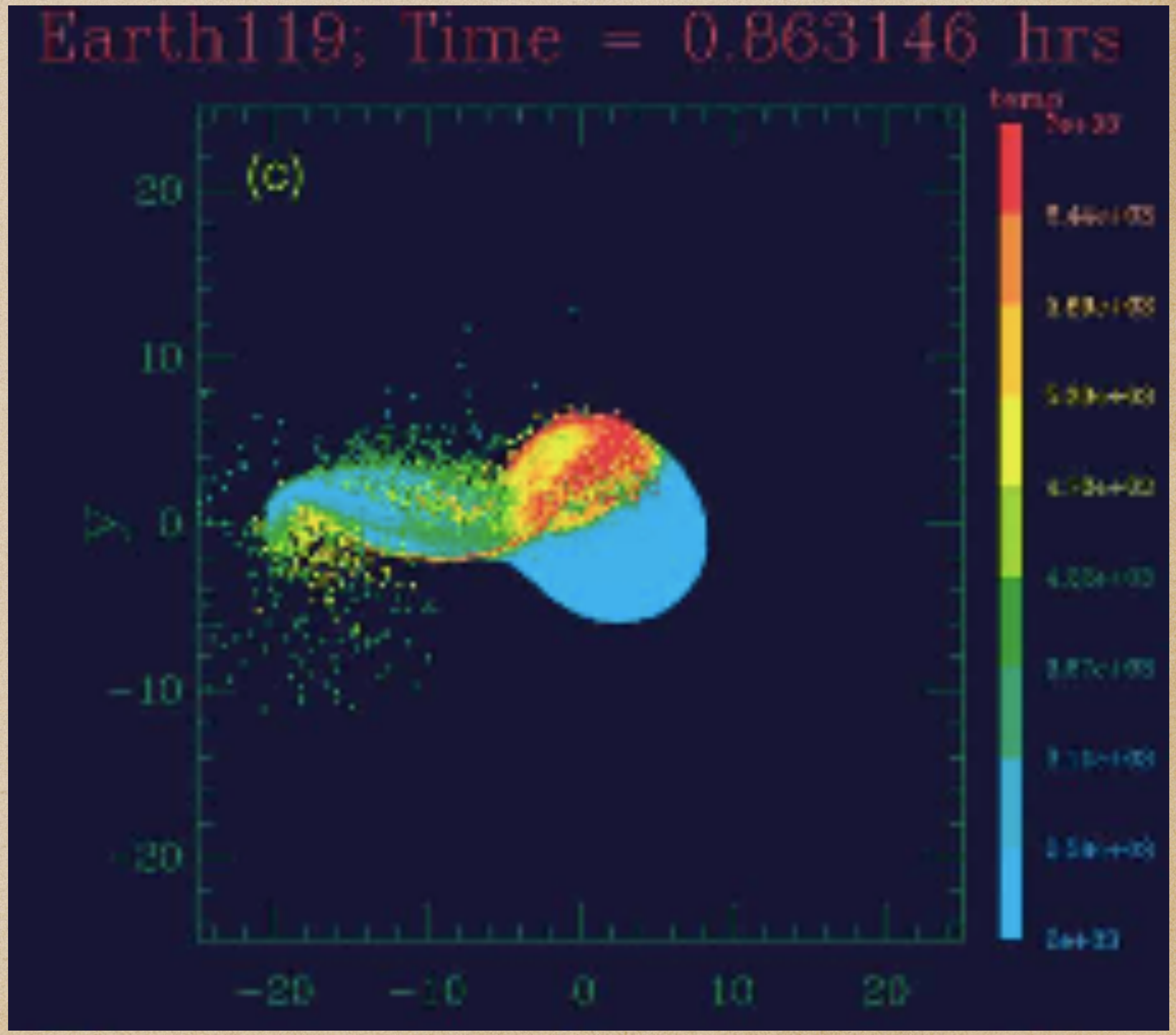
The smoothing length h has a direct impact on the accuracy of the solution and the size of the support domain κh.

We should have about 5, 21 or 57 particles in the support domain in 1-, 2- and 3-D respectively.

The smoothing length may need to be adapted in both space and time so h can evolve dynamically.

A simple approach is to update the smoothing length, in d-dimensions, according to the mass and density, where η = 1.2 to 1.5, as

$$h_i = \eta \left(\frac{m_i}{\rho_i} \right)^{\frac{1}{d}}$$



Canup, 2004, Icarus **168**:433

Symmetrisation of particle interactions

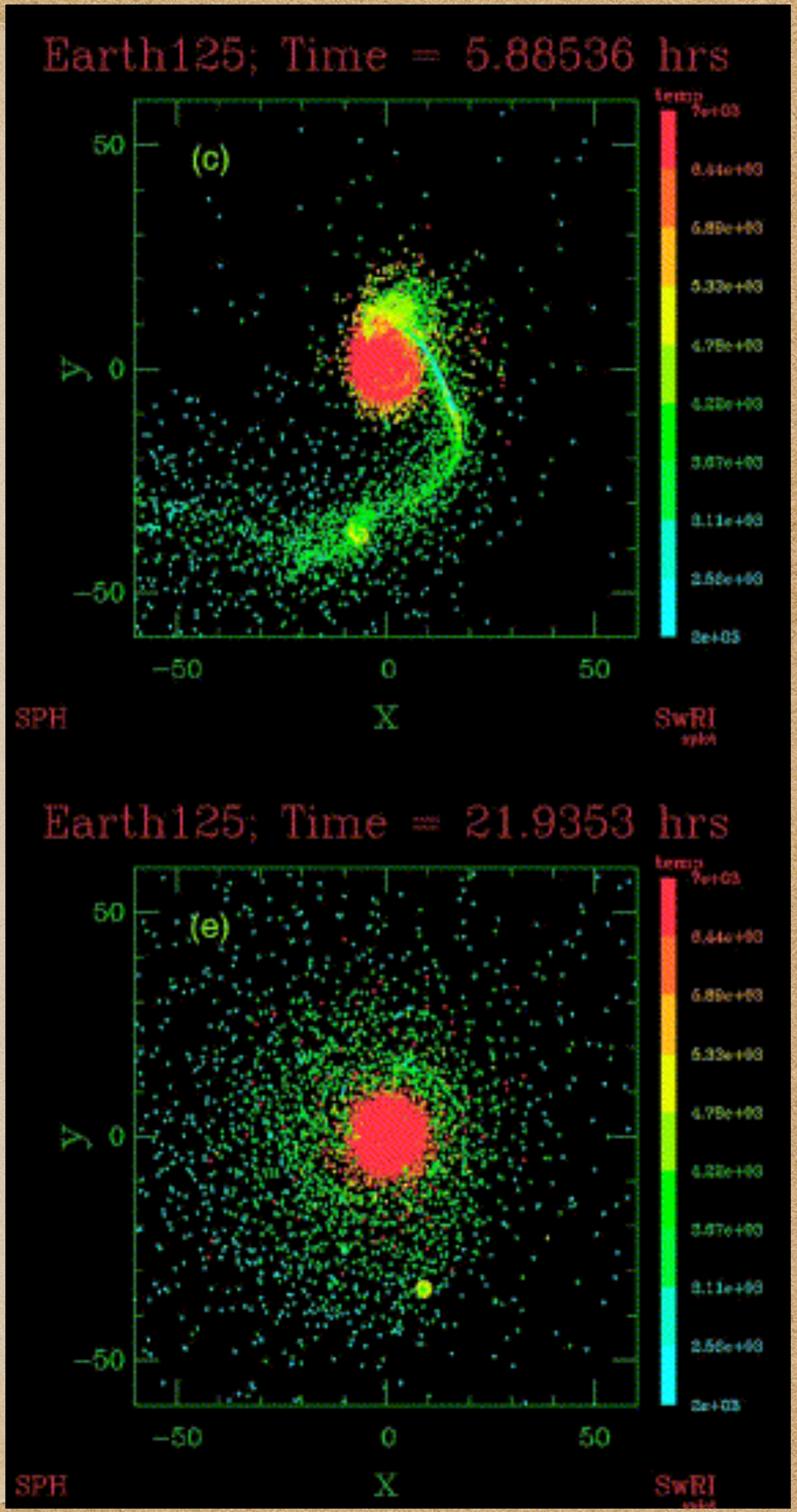
If each particle has its own smoothing length, h_i is not equal to h_j.

Therefore particle i could exert a force on particle j but not vice versa, which would violate Newton's third law.

To overcome this one can use averages or geometric means of smoothing lengths for pairs of interacting particles.

Two examples are:

$$h_{ij} = \frac{h_i + h_j}{2} \qquad h_{ij} = \frac{2h_ih_j}{h_i + h_j}$$



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Implementing SPH computer code

The figure shows a typical procedure for SPH simulations.

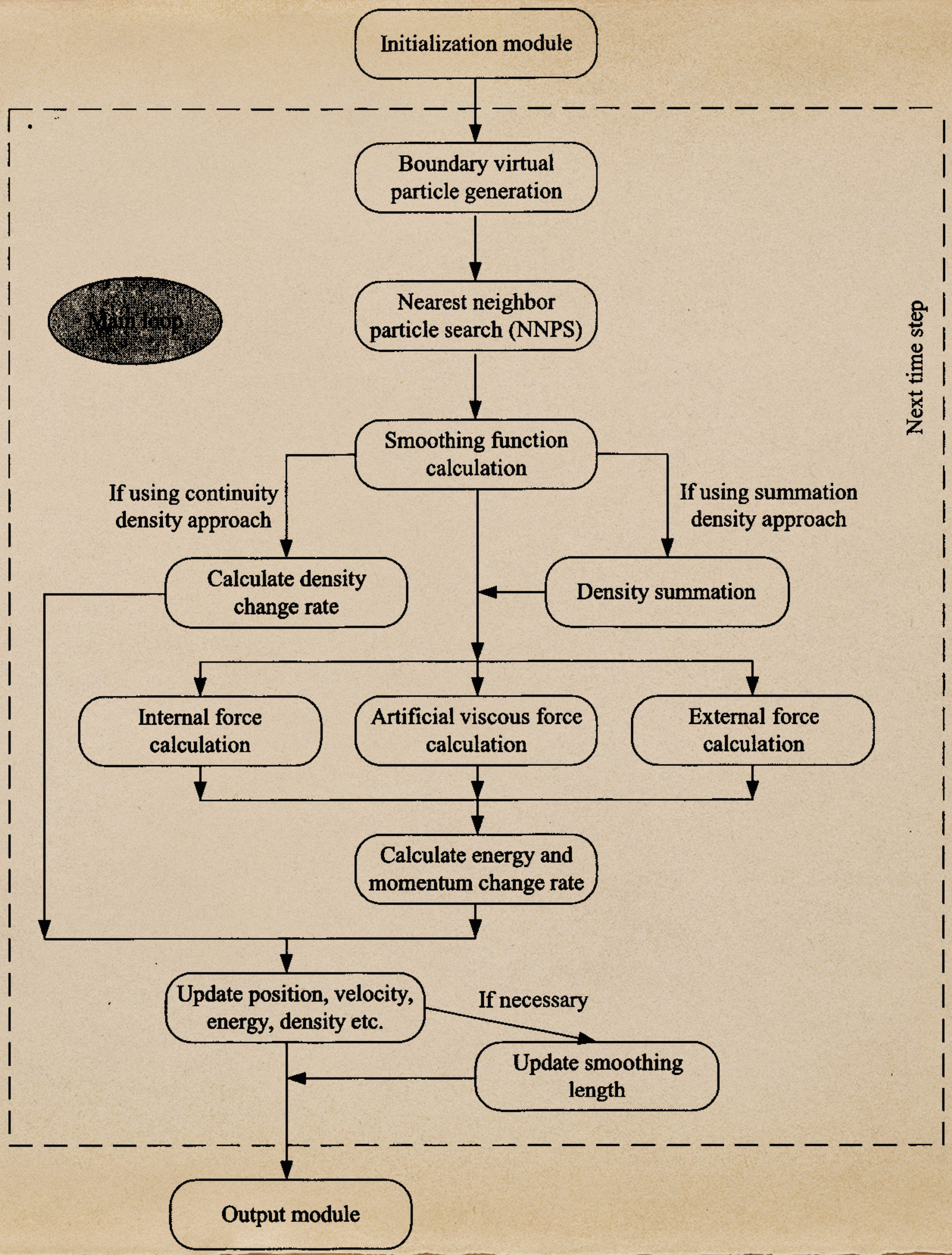
Initialisation

- input of problem geometry
- input of standard parameters

Main SPH process - time integration by predictor corrector or Runge-Kutta.

All sub-functions are called inside the main loop.

Output - save updated data to files, for making videos this could be called at each video output step.



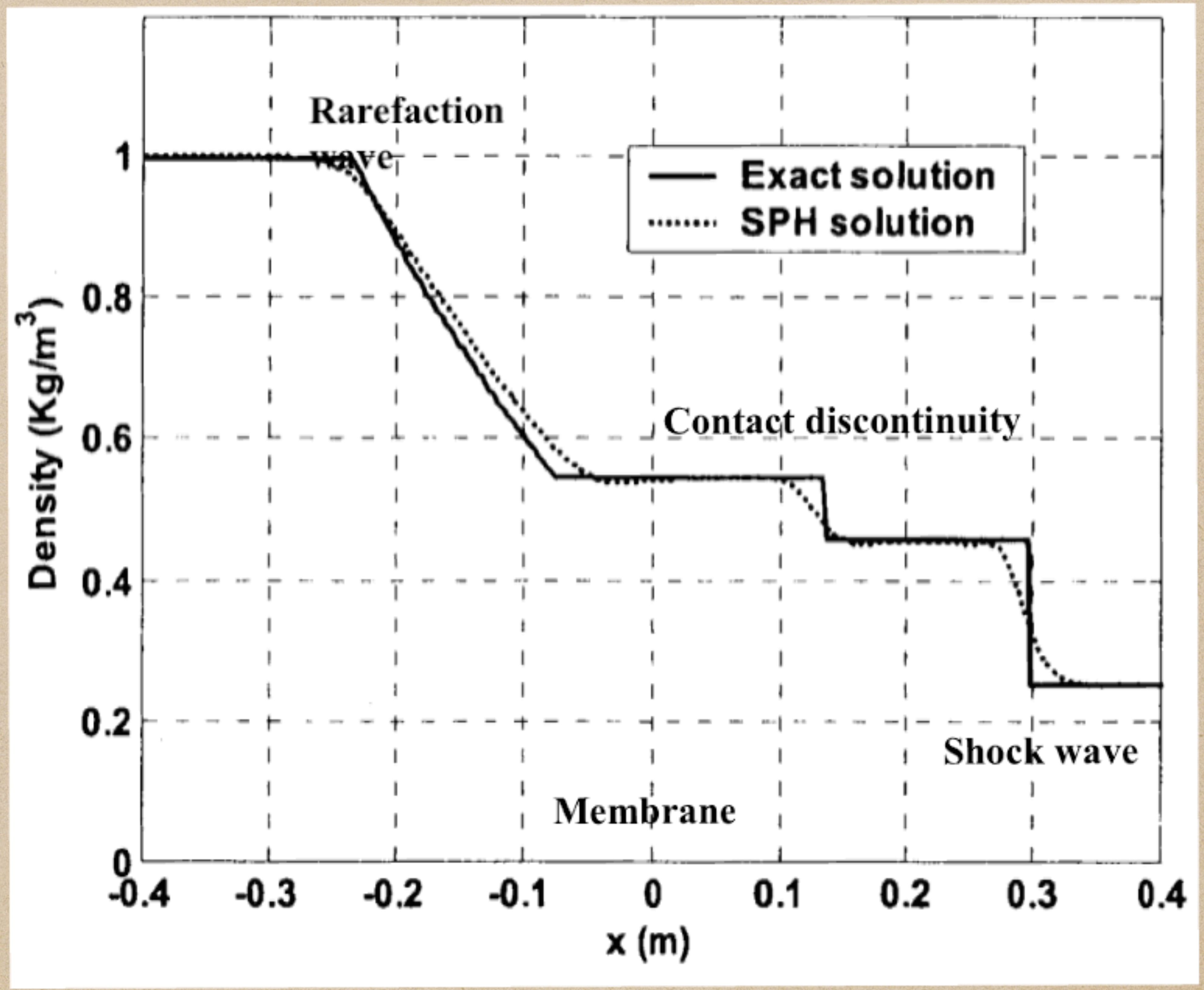
Project 1: The Shock Tube Problem

Shock tube in 1-D is a common problem for people starting in SPH simulations.

Long straight tube filled with gas, which is separated by a membrane in two parts with different pressures and densities but are individually in thermodynamic equilibrium.

When the membrane is taken away the following are produced

- a shock wave - moves into the region of lower density
- a rarefaction wave (reduction in density) - moves into the region of high density
- a contact discontinuity - forms in the centre and travels into the low density region behind the shock



Problem: The shock tube

The equations are listed without viscous stress, heat and external forces.

$$
\begin{cases}
\dfrac{D\rho_i}{Dt} = \displaystyle\sum_{j=1}^{N} m_j \cancel{(\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot \nabla_i W_{ij}} \text{ or } \boxed{\rho_i = \displaystyle\sum_{j=1}^{N} m_j W_{ij}} \\
\dfrac{D\boldsymbol{v}_i}{Dt} = -\displaystyle\sum_{j=1}^{N} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij}\right) \nabla_i W_{ij} \\
\dfrac{De_i}{Dt} = \dfrac{1}{2} \displaystyle\sum_{j=1}^{N} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij}\right) (\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot \nabla_i W_{ij} \\
\dfrac{D\boldsymbol{x}_i}{Dt} = \boldsymbol{v}_i \qquad \text{use} \quad \Pi_{ij} = 0
\end{cases}
$$

In this example we use

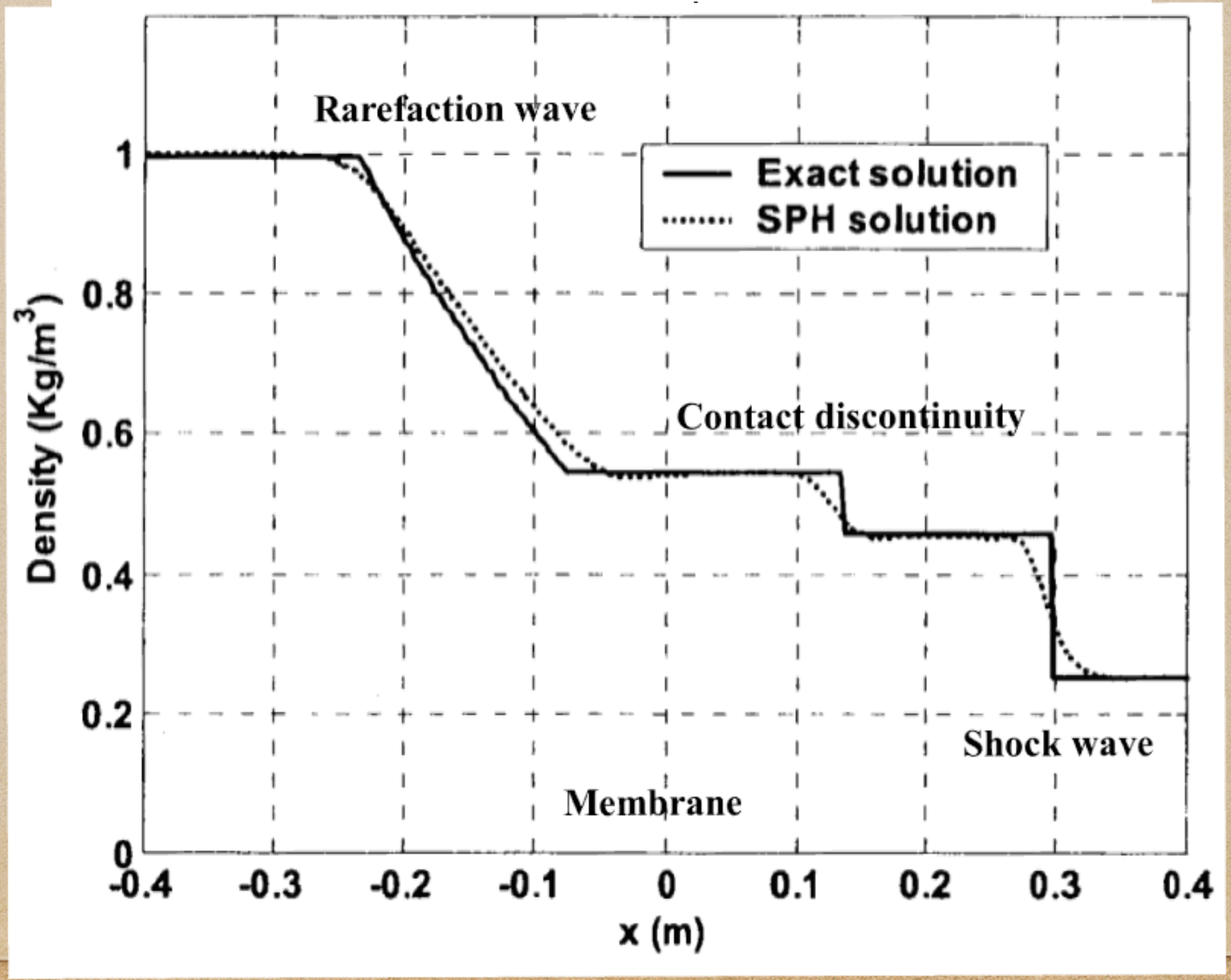
$$
\begin{array}{llllll}
x \le 0 & \rho = 1 & \boldsymbol{v} = 0 & e = 2.5 & p = 1 & \Delta x = 0.001875 \\
x > 0 & \rho = 0.25 & \boldsymbol{v} = 0 & e = 1.795 & p = 0.1795 & \Delta x = 0.0075
\end{array}
$$

where ρ, p, e and $\boldsymbol{v}$ are the density, pressure, internal energy, and velocity respectively.

Δx is the particle spacing.

With 400 particles of the same mass m_i=0.001875.

320 particles evenly distributed in the high density region [-0.6,0.0] and 80 in the low density region [0.0,0.6].



Problem: Equation of State

We use the equation of state for the ideal gas

$$p = (\gamma - 1)\rho e$$

and the speed of sound is

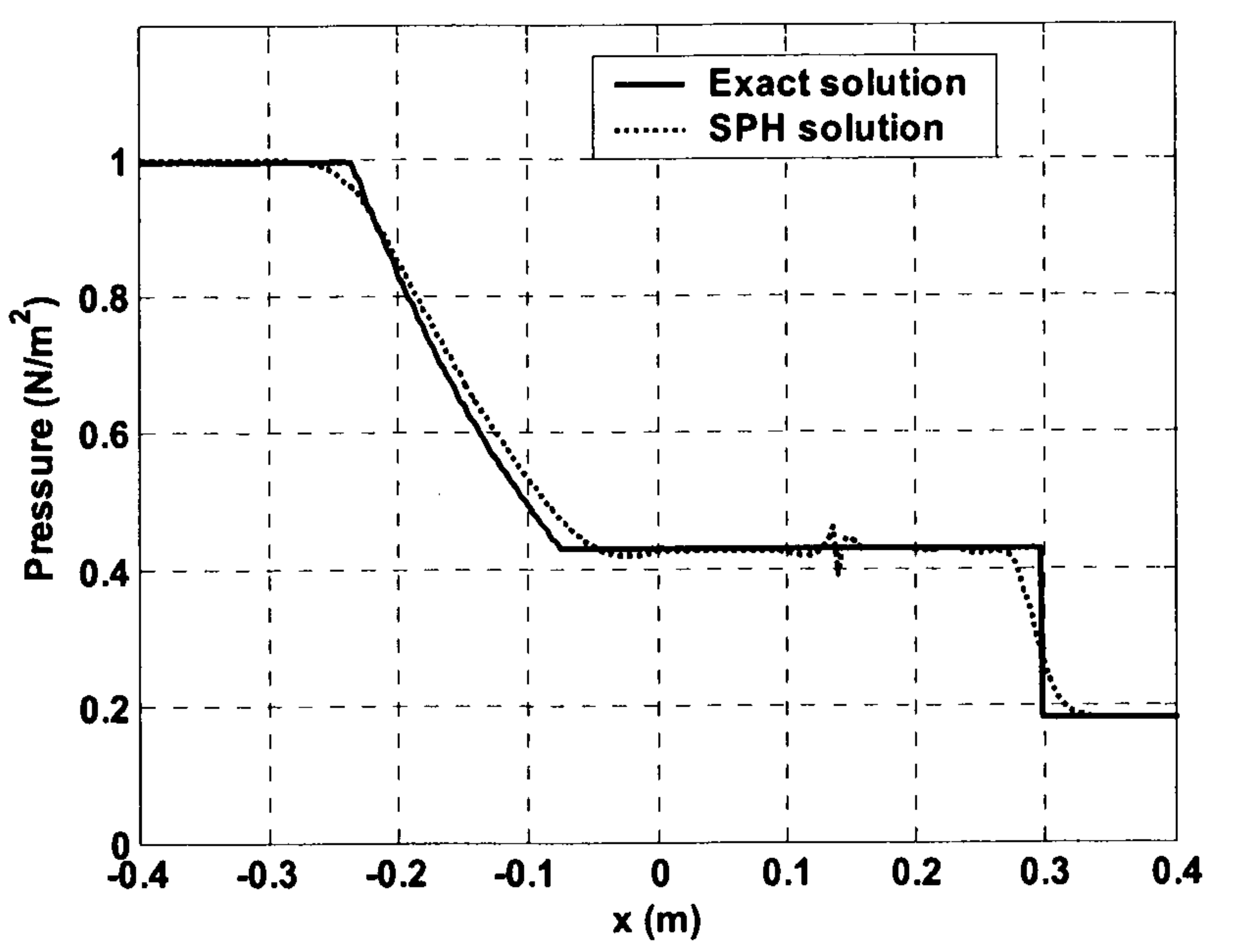
$$c = \sqrt{(\gamma - 1)e}$$

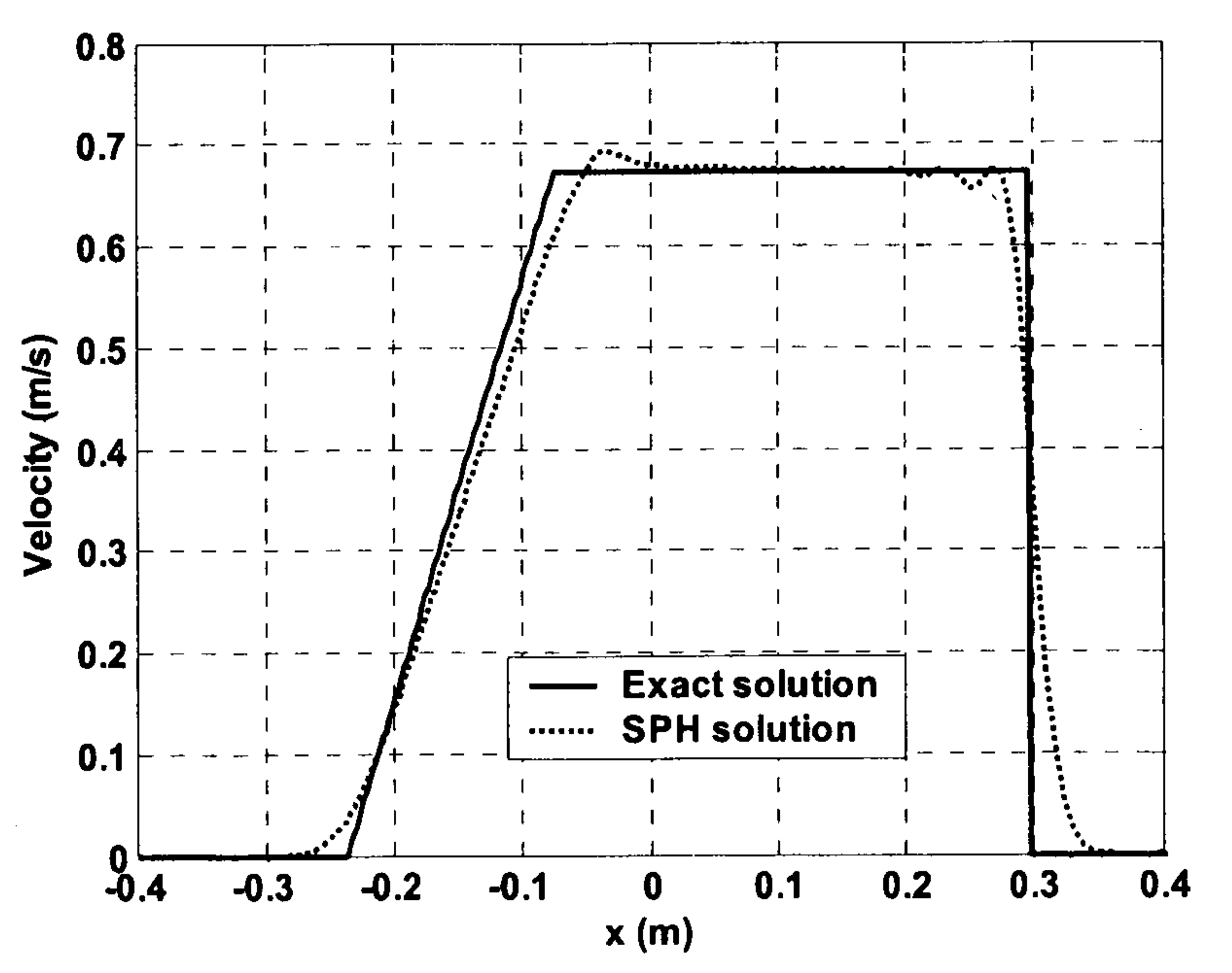
γ = Cp / Cv = 1.4 is the ratio of specific heat capacities. Cp and Cv are the heat absorbed per unit mass at constant pressure and volume respectively.

Set the time step to 0.005 and run the simulation for 40 time steps.

No special treatment is used for the boundary as the shock wave has not yet reached it.

Heat capacity the amount of heat required to change a substance's temperature by a given amount





Project assignment

- Implement an SPH code from scratch to solve the shock wave problem (don't include artificial viscosity yet).
- Extend the basic implementation to include artificial viscosity of Monaghan ref. [5]. (α_Π=1, β_Π=1 and $\varphi = 0.1h_{ij}$ to avoid divergence):
- Reproduce similar results shown for ρ, p, e and v.
- Write a project report describing, theory, implementation, results.

$$\Pi_{ij} = \begin{cases} \dfrac{-\alpha_\Pi \bar{c}_{ij}\phi_{ij} + \beta_\Pi \phi_{ij}^2}{\bar{\rho}_{ij}} & \boldsymbol{v}_{ij}\cdot\boldsymbol{x}_{ij} < 0 \\ 0 & \boldsymbol{v}_{ij}\cdot\boldsymbol{x}_{ij} \geq 0 \end{cases}$$

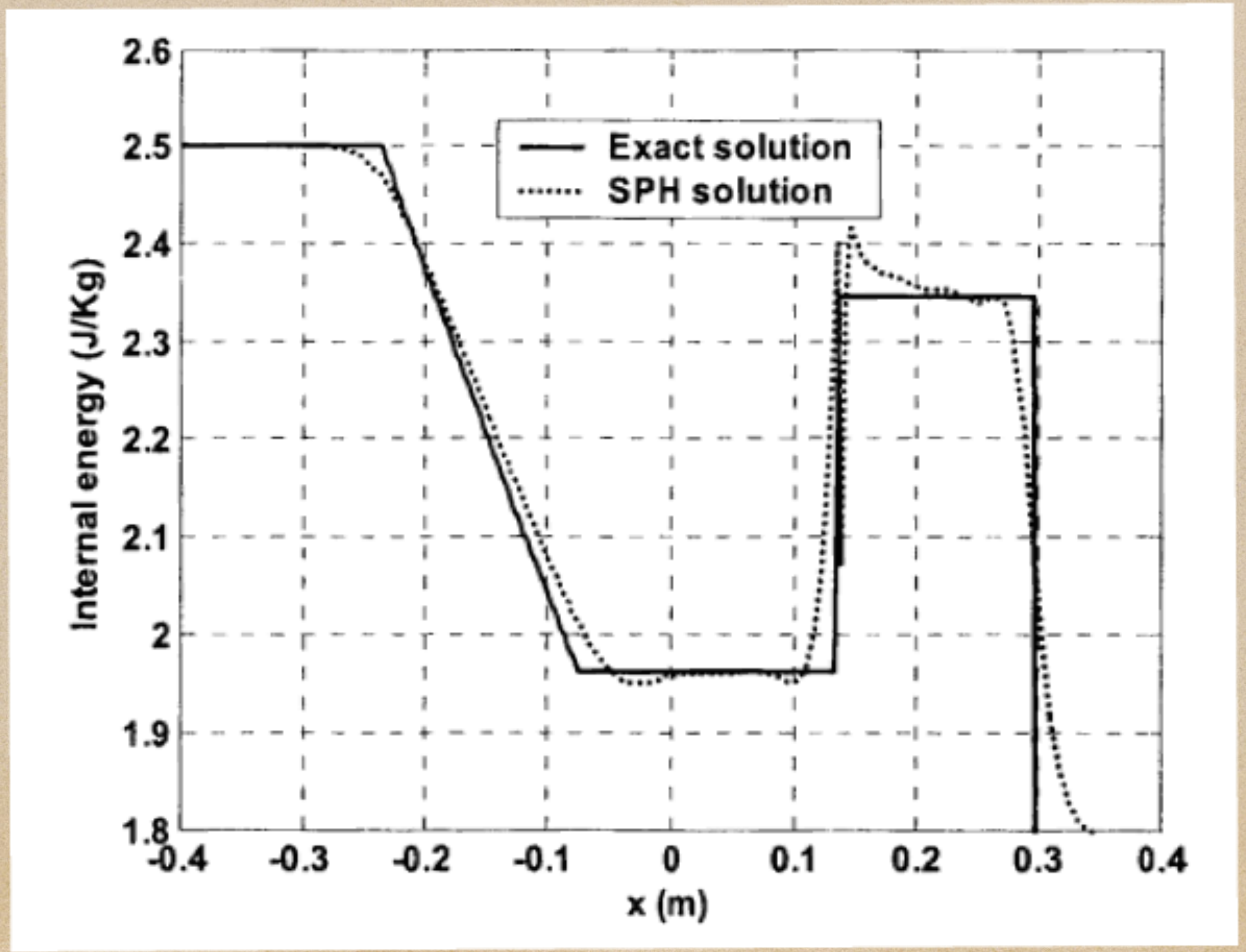
$$\phi_{ij} = \frac{h_{ij}\boldsymbol{v}_{ij}\cdot\boldsymbol{x}_{ij}}{\left|\boldsymbol{x}_{ij}\right|^2 + \varphi^2}$$

Speed of sound $\bar{c}_{ij} = \frac{1}{2}\left(c_i + c_j\right)$

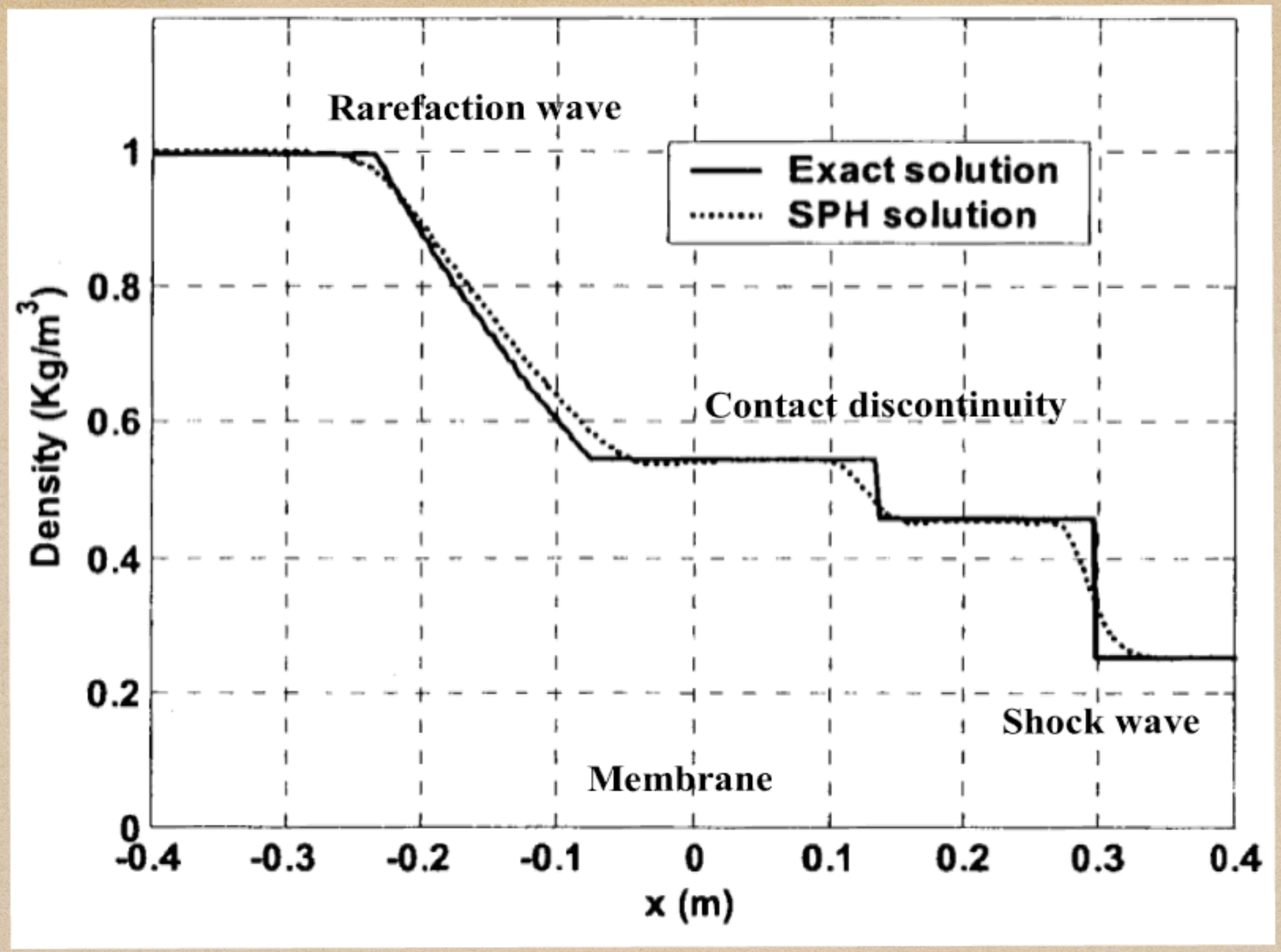
$$\bar{\rho}_{ij} = \frac{1}{2}\left(\rho_i + \rho_j\right)$$

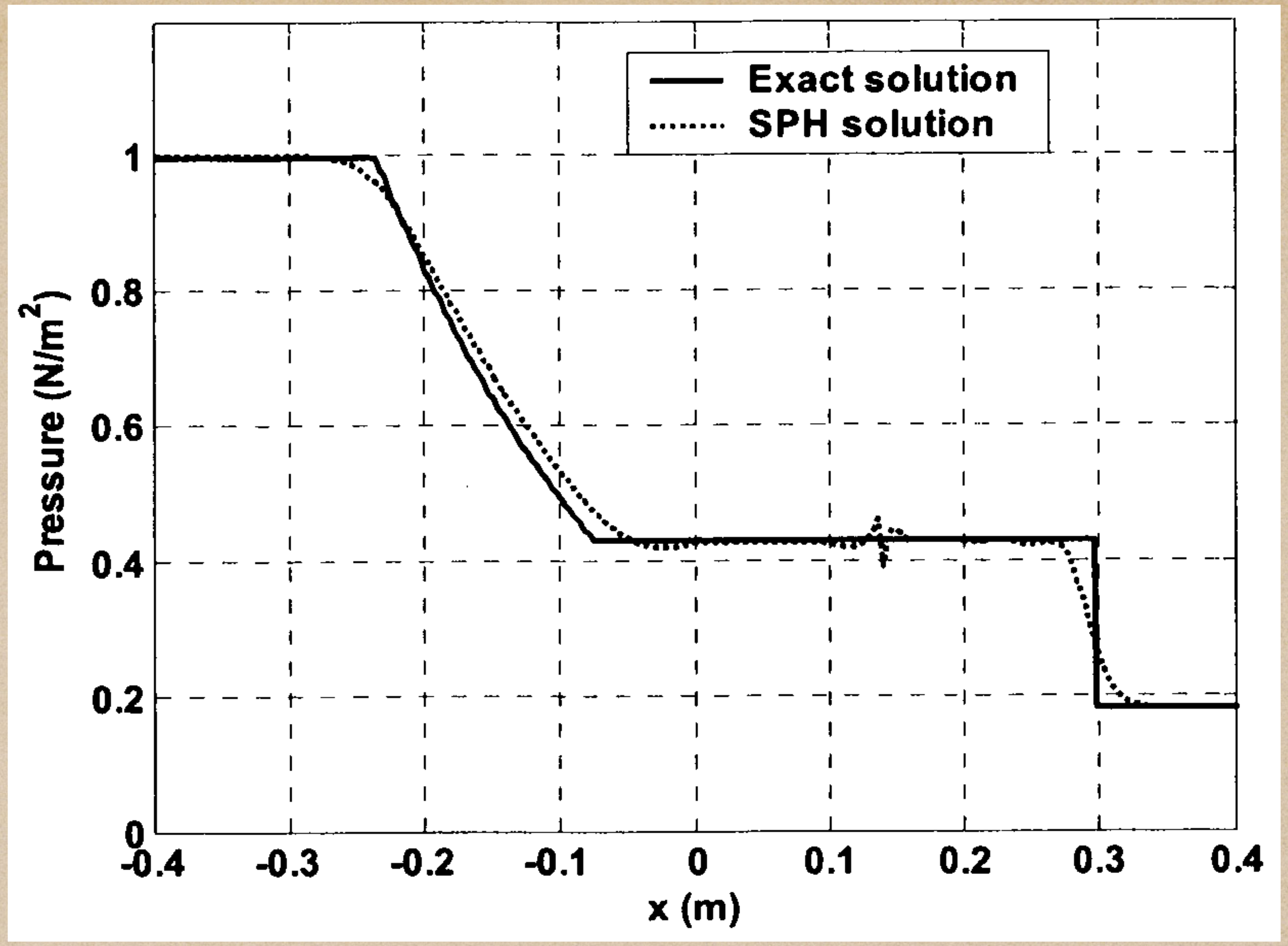
$$h_{ij} = \frac{1}{2}\left(h_i + h_j\right)$$

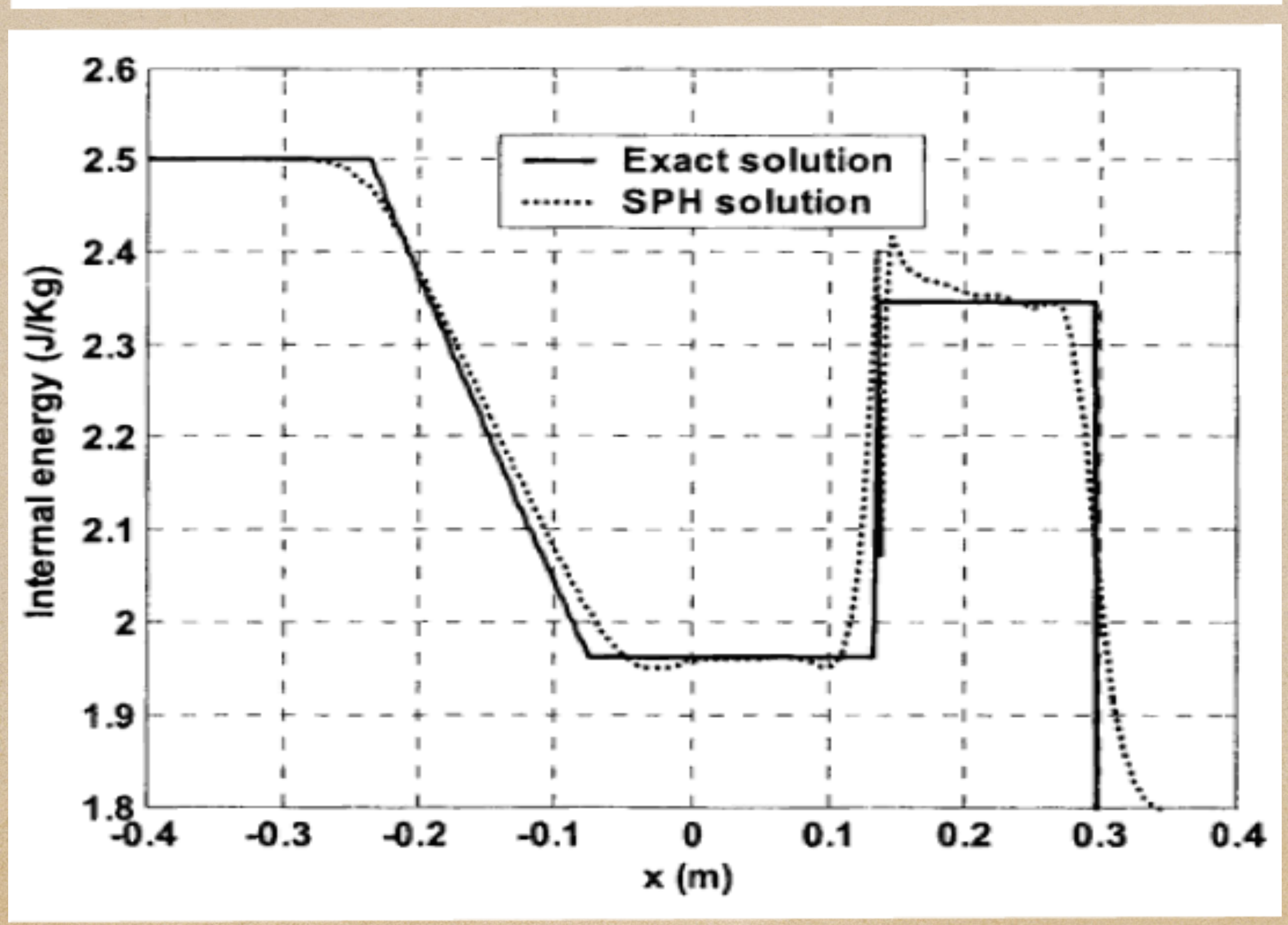
$$\boldsymbol{v}_{ij} = \boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{x}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$$

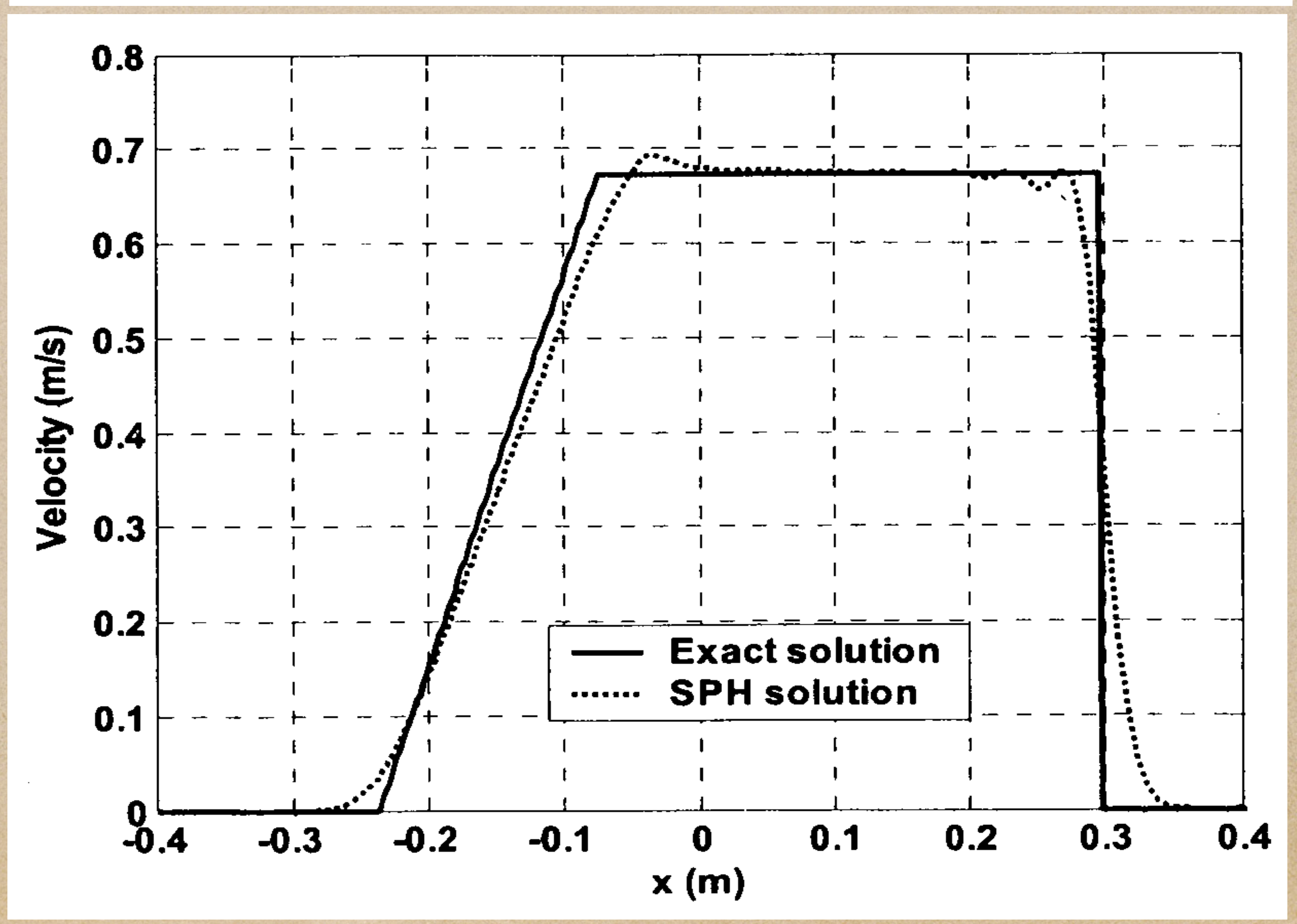


Help is available from me!


Rarefaction wave
Exact solution
SPH solution
Contact discontinuity
Shock wave
Membrane
Density (Kg/m^3)
1
0.8
0.6
0.4
0.2
0
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
x (m)


Exact solution
SPH solution
Pressure (N/m^2)
1
0.8
0.6
0.4
0.2
0
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
x (m)


Exact solution
SPH solution
Internal energy (J/Kg)
2.6
2.5
2.4
2.3
2.2
2.1
2
1.9
1.8
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
x (m)


Exact solution
SPH solution
Velocity (m/s)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
x (m)

The smoothing function (kernel) and its derivative

Use the cubic spline form

$$W(R,h) = \alpha_d \times \begin{cases} \frac{2}{3} - R^2 + \frac{1}{2}R^3 & 0 \le R < 1 \\ \frac{1}{6}(2-R)^3 & 1 \le R < 2 \\ 0 & R \ge 2 \end{cases}$$

where α_d is $1/h$, $15/(7\pi h^2)$, and $3/(2\pi h^3)$ and $\kappa=2$ in 1-, 2-, and 3-D respectively.
If we take the derivative of this we have

$$W'(R,h) = \begin{cases} \alpha_d \times \left(-2 + \frac{3}{2}R\right)\frac{dx}{h^2} & 0 \le R < 1 \\ -\alpha_d \times \frac{1}{2}(2-R)^2\frac{dx}{hr} & 1 \le R < 2 \\ 0 & R \ge 2 \end{cases}$$

and

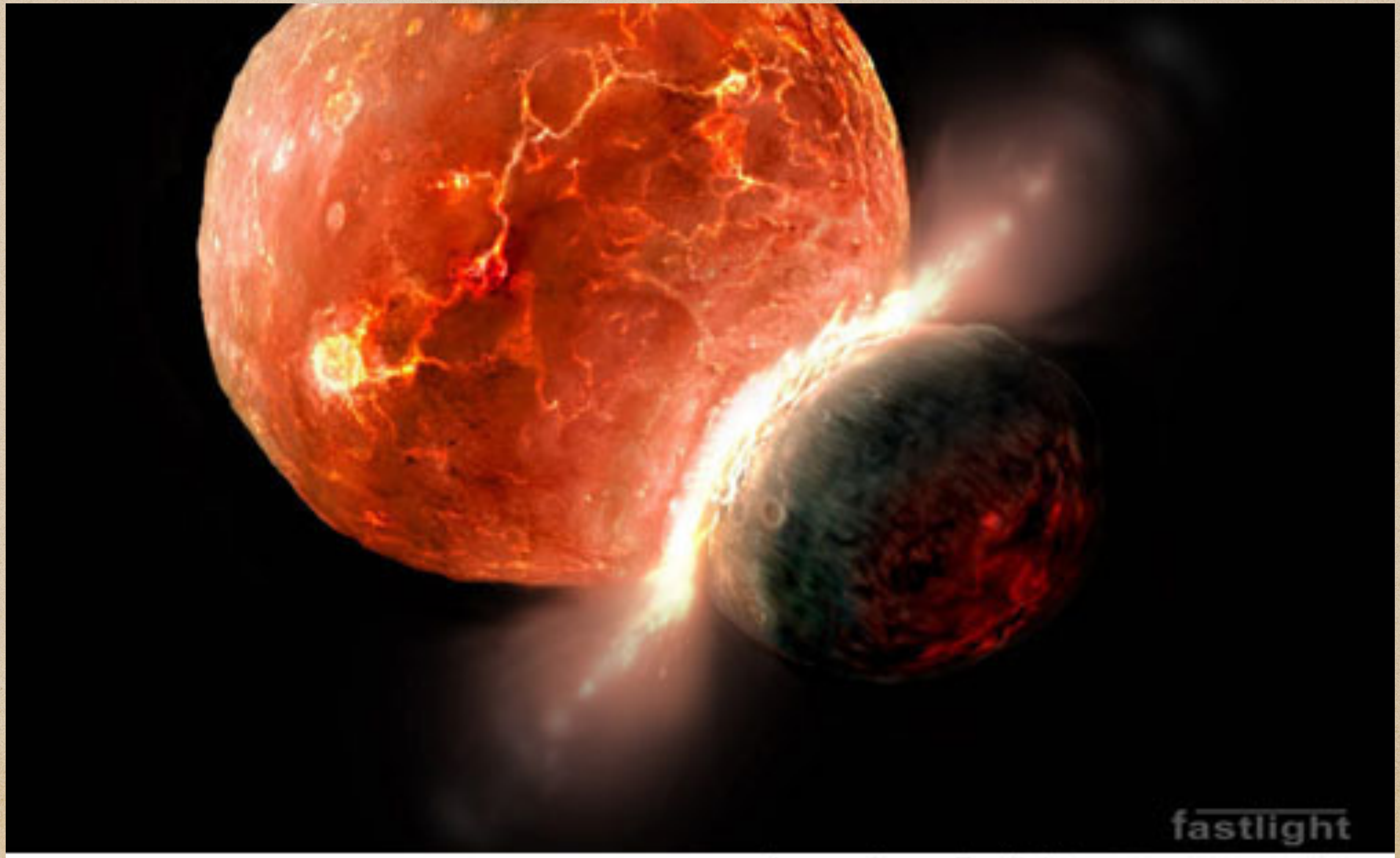
$$R_{ij} = \frac{r_{ij}}{h} = \frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{h} = \frac{|d\boldsymbol{x}|}{h}$$

Project 2: Colliding Jupiter like Planets

You can implement this project yourself using the input file: planet300.dat

- First, generalise your code to 3D, ***x*** and ***v*** become vectors and the kernel needs additions.
- Implement a gravitational potential in a self consistent manner.

(P. Cossins, Chapter 3, Smoothed Particle Hydrodynamics, Ph.D. Thesis, Leicester 2010)



James Garry, Fastlight Used with permission.

Gravitational Potential

Use the following equations from the thesis: (P. Cossins, Chapter 3, Smoothed Particle Hydrodynamics, Ph.D. Thesis, Leicester 2010) (Note notation, they use x we use R). For constant smoothing length the following simplified formula is valid:

$$\boxed{\frac{\partial \phi(r,h)}{\partial r} = \begin{cases} \frac{1}{h^2}\left(\frac{4}{3}R - \frac{6}{5}R^3 + \frac{1}{2}R^4\right) & 0 \le R \le 1, \\ \frac{1}{h^2}\left(\frac{8}{3}R - 3R^2 + \frac{6}{5}R^3 - \frac{1}{6}R^4 - \frac{1}{15R^2}\right) & 1 \le R \le 2, \\ \frac{1}{r^2} & R \ge 2. \end{cases}}$$

$$\boxed{\left(\frac{d\boldsymbol{v}}{dt}\right)^{i}_{\text{Gravity}} = -\frac{G}{2}\sum_j m_j \left(\nabla_i \phi_{ij}(h_i) + \nabla_i \phi_{ij}(h_j)\right)}$$

$$R_{ij} = \frac{r_{ij}}{h} = \frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{h} = \frac{|d\boldsymbol{x}|}{h}$$

Making a Planet Spin

Making a planet spin can be done using the equation

$v = \omega \times r.$

For example you can give each planet an angular velocity of

$$\omega = (0,0,\omega_z)$$

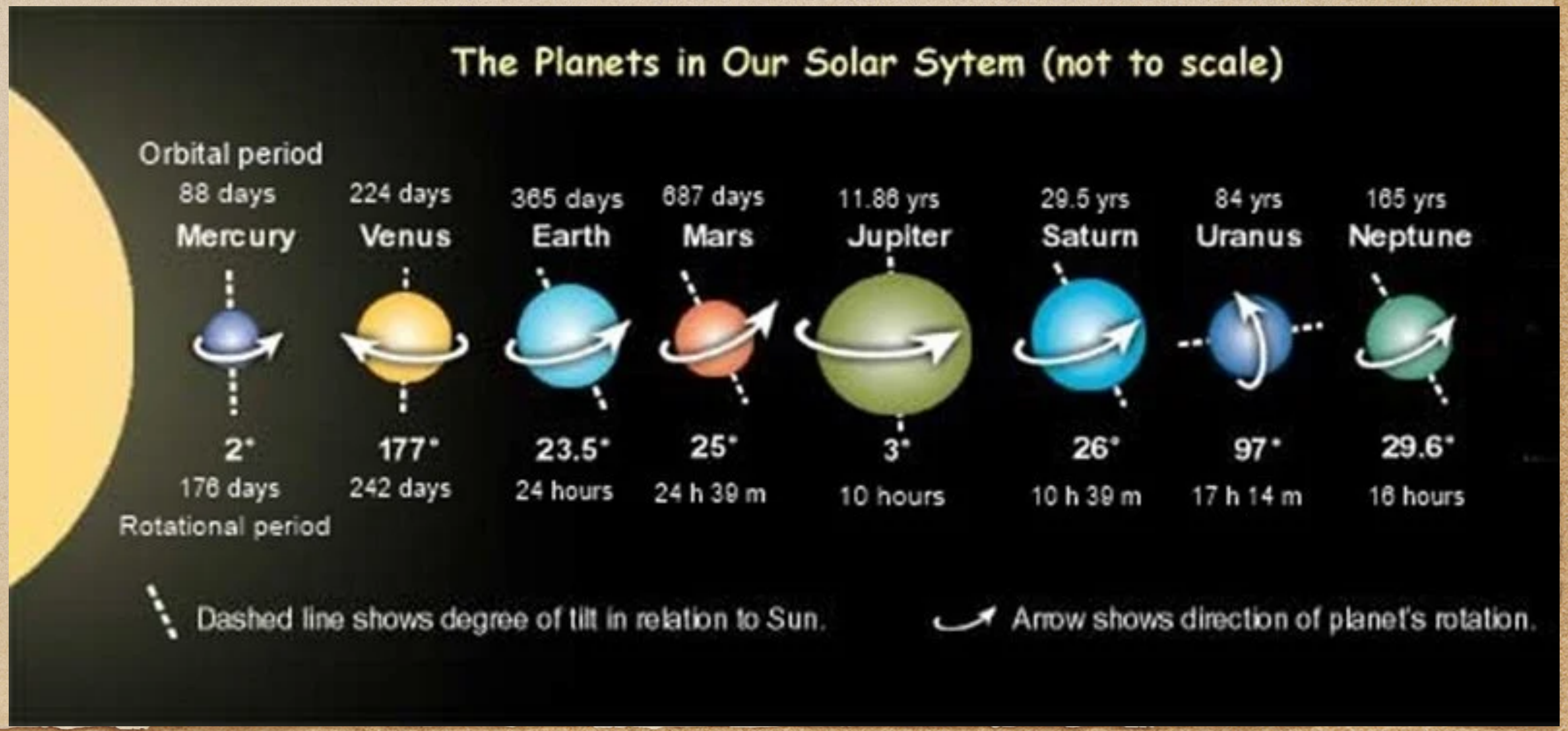
where

$$\omega_z = 2\pi/T$$

and T is the period of rotation of the planet.

Rotational angular momentum (not needed) is given by

$L = (4\pi m r^2)/(5T)$



Advanced SPH code

There are some free advanced versions of SPH available on the internet which have been developed for astronomy purposes. These codes are state of the art and require some effort to install and use.

The following links will take you to more tools but not all have SPH included:

https://en.wikipedia.org/wiki/Computational_astrophysics

A widely used SPH tool is Gadget-4 which is a massively parallel cosmological code:

https://wwwmpa.mpa-garching.mpg.de/gadget4/

Conclusions

Hopefully, this has given you a well rounded introduction into the main components of SPH techniques.