Bayesian Decision Theory

- Bayesian Decision Theory is a fundamental statistical approach that quantifies the tradeoffs between various decisions using probabilities and costs that accompany such decisions.
- First, we will assume that all probabilities are known.
- Then, we will study the cases where the probabilistic structure is not completely known.

Fish Sorting Example Revisited

- State of nature is a random variable.
- Define w as the type of fish we observe (state of nature, *class*) where
  - $w = w_1$ for sea bass,
  - $w = w_2$ for salmon.
  - $P(w_1)$ is the *a priori probability* that the next fish is a sea bass.
  - $P(w_2)$ is the a priori probability that the next fish is a salmon.

Prior Probabilities

- Prior probabilities reflect our knowledge of how likely each type of fish will appear before we actually see it.
- How can we choose $P(w_1)$ and $P(w_2)$?
  - Set $P(w_1) = P(w_2)$ if they are equiprobable (*uniform priors*).
  - May use different values depending on the fishing area, time of the year, etc.
- Assume there are no other types of fish

$$P(w_1) + P(w_2) = 1$$

(exclusivity and exhaustivity).

Making a Decision

- How can we make a decision with only the prior information?

$$\text{Decide} \quad \begin{cases} w_1 & \text{if } P(w_1) > P(w_2) \\ w_2 & \text{otherwise} \end{cases}$$

- What is the *probability of error* for this decision?

$$P(error) = \min\{P(w_1), P(w_2)\}$$

Class-Conditional Probabilities

- Let's try to improve the decision using the lightness measurement x.
- Let x be a continuous random variable.
- Define $p(x|w_j)$ as the *class-conditional probability density* (probability of x given that the state of nature is w_j for $j = 1, 2$).
- $p(x|w_1)$ and $p(x|w_2)$ describe the difference in lightness between populations of sea bass and salmon.

Class-Conditional Probabilities

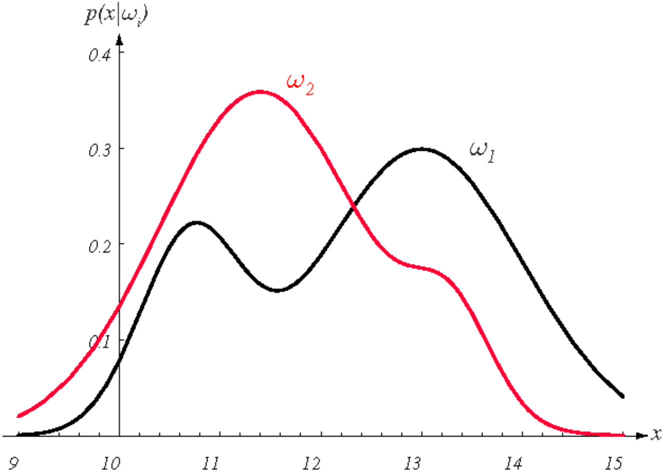

Figure 1: Hypothetical class-conditional probability density functions for two classes.

Posterior Probabilities

- Suppose we know $P(w_j)$ and $p(x|w_j)$ for $j = 1, 2$, and measure the lightness of a fish as the value x.
- Define $P(w_j|x)$ as the *a posteriori probability* (probability of the state of nature being w_j given the measurement of feature value x).
- We can use the *Bayes formula* to convert the prior probability to the posterior probability

$$P(w_j|x) = \frac{p(x|w_j)P(w_j)}{p(x)}$$

where $p(x) = \sum_{j=1}^{2} p(x|w_j)P(w_j)$.

Making a Decision

- $p(x|w_j)$ is called the *likelihood* and $p(x)$ is called the *evidence*.
- How can we make a decision after observing the value of x?

$$\text{Decide} \quad \begin{cases} w_1 & \text{if } P(w_1|x) > P(w_2|x) \\ w_2 & \text{otherwise} \end{cases}$$

- Rewriting the rule gives

$$\text{Decide} \quad \begin{cases} w_1 & \text{if } \frac{p(x|w_1)}{p(x|w_2)} > \frac{P(w_2)}{P(w_1)} \\ w_2 & \text{otherwise} \end{cases}$$

- Note that, at every x, $P(w_1|x) + P(w_2|x) = 1$.

Probability of Error

- What is the probability of error for this decision?

$$P(error|x) = \begin{cases} P(w_1|x) & \text{if we decide } w_2 \\ P(w_2|x) & \text{if we decide } w_1 \end{cases}$$

- What is the average probability of error?

$$P(error) = \int_{-\infty}^{\infty} p(error, x)\, dx = \int_{-\infty}^{\infty} P(error|x)\, p(x)\, dx$$

- *Bayes decision rule* minimizes this error because

$$P(error|x) = \min\{P(w_1|x), P(w_2|x)\}.$$

Bayesian Decision Theory

- How can we generalize to
  - more than one feature?
    - replace the scalar x by the feature vector $\mathbf{x}$
  - more than two states of nature?
    - just a difference in notation
  - allowing actions other than just decisions?
    - allow the possibility of rejection
  - different risks in the decision?
    - define how costly each action is

Bayesian Decision Theory

- Let $\{w_1, \ldots, w_c\}$ be the finite set of c states of nature (*classes*, *categories*).
- Let $\{\alpha_1, \ldots, \alpha_a\}$ be the finite set of a possible *actions*.
- Let $\lambda(\alpha_i|w_j)$ be the *loss* incurred for taking action α_i when the state of nature is w_j.
- Let x be the d-component vector-valued random variable called the *feature vector*.

Bayesian Decision Theory

- $p(\mathbf{x}|w_j)$ is the class-conditional probability density function.
- $P(w_j)$ is the prior probability that nature is in state w_j.
- The posterior probability can be computed as

$$P(w_j|\mathbf{x}) = \frac{p(\mathbf{x}|w_j)P(w_j)}{p(\mathbf{x})}$$

where $p(\mathbf{x}) = \sum_{j=1}^{c} p(\mathbf{x}|w_j)P(w_j)$.

Conditional Risk

- Suppose we observe $\mathbf{x}$ and take action α_i.
- If the true state of nature is w_j, we incur the loss $\lambda(\alpha_i|w_j)$.
- The expected loss with taking action α_i is

$$R(\alpha_i|\mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i|w_j)P(w_j|\mathbf{x})$$

which is also called the *conditional risk*.

Minimum-Risk Classification

- The general *decision rule* $\alpha(\mathbf{x})$ tells us which action to take for observation $\mathbf{x}$.
- We want to find the decision rule that minimizes the overall risk

$$R = \int R(\alpha(\mathbf{x})|\mathbf{x})\, p(\mathbf{x})\, d\mathbf{x}.$$

- Bayes decision rule minimizes the overall risk by selecting the action α_i for which $R(\alpha_i|\mathbf{x})$ is minimum.
- The resulting minimum overall risk is called the *Bayes risk* and is the best performance that can be achieved.

Two-Category Classification

- Define
  - α_1: deciding w_1,
  - α_2: deciding w_2,
  - $\lambda_{ij} = \lambda(\alpha_i|w_j)$.
- Conditional risks can be written as

$$R(\alpha_1|\mathbf{x}) = \lambda_{11}\, P(w_1|\mathbf{x}) + \lambda_{12}\, P(w_2|\mathbf{x}),$$
$$R(\alpha_2|\mathbf{x}) = \lambda_{21}\, P(w_1|\mathbf{x}) + \lambda_{22}\, P(w_2|\mathbf{x}).$$

Two-Category Classification

- The *minimum-risk decision rule* becomes

$$\text{Decide } \begin{cases} w_1 & \text{if } (\lambda_{21} - \lambda_{11}) P(w_1|\mathbf{x}) > (\lambda_{12} - \lambda_{22}) P(w_2|\mathbf{x}) \\ w_2 & \text{otherwise} \end{cases}$$

- This corresponds to deciding w_1 if

$$\frac{p(\mathbf{x}|w_1)}{p(\mathbf{x}|w_2)} > \frac{(\lambda_{12} - \lambda_{22})}{(\lambda_{21} - \lambda_{11})} \frac{P(w_2)}{P(w_1)}$$

$\Rightarrow$ comparing the *likelihood ratio* to a threshold that is independent of the observation $\mathbf{x}$.

Minimum-Error-Rate Classification

- Actions are decisions on classes (α_i is deciding w_i).
- If action α_i is taken and the true state of nature is w_j, then the decision is correct if $i = j$ and in error if $i \neq j$.
- We want to find a decision rule that minimizes the probability of error.

Minimum-Error-Rate Classification

- Define the *zero-one loss function*

$$\lambda(\alpha_i|w_j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \qquad i, j = 1, \ldots, c$$

  (all errors are equally costly).
- Conditional risk becomes

$$\begin{aligned} R(\alpha_i|\mathbf{x}) &= \sum_{j=1}^{c} \lambda(\alpha_i|w_j)\, P(w_j|\mathbf{x}) \\ &= \sum_{j \neq i} P(w_j|\mathbf{x}) \\ &= 1 - P(w_i|\mathbf{x}). \end{aligned}$$

Minimum-Error-Rate Classification

- Minimizing the risk requires maximizing $P(w_i|\mathbf{x})$ and results in the *minimum-error decision rule*

$$\text{Decide } w_i \text{ if } P(w_i|\mathbf{x}) > P(w_j|\mathbf{x}) \quad \forall j \neq i.$$

- The resulting error is called the *Bayes error* and is the best performance that can be achieved.

Minimum-Error-Rate Classification

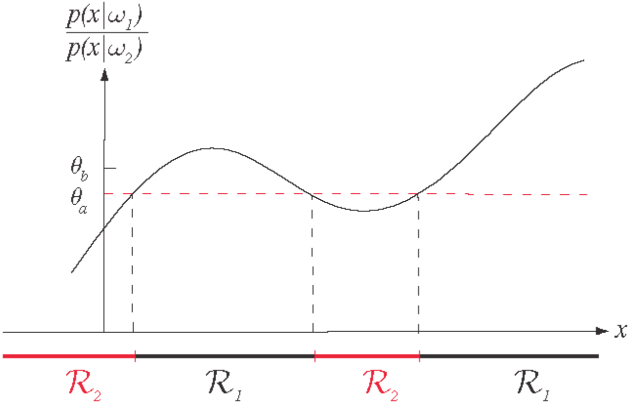

Figure 2: The likelihood ratio $p(\mathbf{x}|w_1)/p(\mathbf{x}|w_2)$. The threshold θ_a is computed using the priors $P(w_1) = 2/3$ and $P(w_2) = 1/3$, and a zero-one loss function. If we penalize mistakes in classifying w_2 patterns as w_1 more than the converse, we should increase the threshold to θ_b.

Discriminant Functions

- A useful way of representing classifiers is through *discriminant functions* $g_i(\mathbf{x}), i = 1, \ldots, c$, where the classifier assigns a feature vector $\mathbf{x}$ to class w_i if

$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \quad \forall j \neq i.$$

- For the classifier that minimizes conditional risk

$$g_i(\mathbf{x}) = -R(\alpha_i|\mathbf{x}).$$

- For the classifier that minimizes error

$$g_i(\mathbf{x}) = P(w_i|\mathbf{x}).$$

Discriminant Functions

- These functions divide the feature space into c *decision regions* ($\mathcal{R}_1, \ldots, \mathcal{R}_c$), separated by *decision boundaries*.
- Note that the results do not change even if we replace every $g_i(\mathbf{x})$ by $f(g_i(\mathbf{x}))$ where $f(\cdot)$ is a monotonically increasing function (e.g., logarithm).
- This may lead to significant analytical and computational simplifications.

The Gaussian Density

- Gaussian can be considered as a model where the feature vectors for a given class are continuous-valued, randomly corrupted versions of a single typical or prototype vector.
- Some properties of the Gaussian:
  - Analytically tractable.
  - Completely specified by the 1st and 2nd moments.
  - Has the maximum entropy of all distributions with a given mean and variance.
  - Many processes are asymptotically Gaussian (Central Limit Theorem).
  - Linear transformations of a Gaussian are also Gaussian.
  - Uncorrelatedness implies independence.

Univariate Gaussian

- For $x \in \mathbb{R}$:

$$\begin{aligned} p(x) &= N(\mu, \sigma^2) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \end{aligned}$$

where

$$\begin{aligned} \mu &= E[x] = \int_{-\infty}^{\infty} x\, p(x)\, dx, \\ \sigma^2 &= E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2\, p(x)\, dx. \end{aligned}$$

Univariate Gaussian

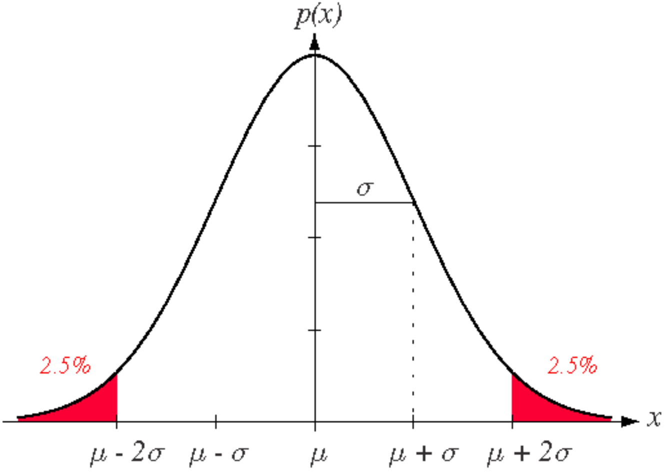

Figure 3: A univariate Gaussian distribution has roughly 95% of its area in the range $|x - \mu| \leq 2\sigma$.

Multivariate Gaussian

- For $\mathbf{x} \in \mathbb{R}^d$:

$$
\begin{aligned}
p(\mathbf{x}) &= N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \\
&= \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{\mu} &= E[\mathbf{x}] = \int \mathbf{x}\, p(\mathbf{x})\, d\mathbf{x}, \\
\boldsymbol{\Sigma} &= E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T] = \int (\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\, p(\mathbf{x})\, d\mathbf{x}.
\end{aligned}
$$

Multivariate Gaussian

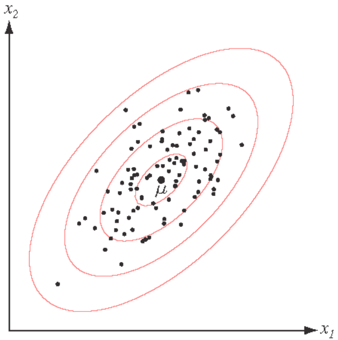

Figure 4: Samples drawn from a two-dimensional Gaussian lie in a cloud centered on the mean $\boldsymbol{\mu}$. The loci of points of constant density are the ellipses for which $(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is constant, where the eigenvectors of $\boldsymbol{\Sigma}$ determine the direction and the corresponding eigenvalues determine the length of the principal axes. The quantity $r^2 = (\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is called the squared *Mahalanobis distance* from $\mathbf{x}$ to $\boldsymbol{\mu}$.

Linear Transformations

- Recall that, given $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{A} \in \mathbb{R}^{d \times k}$, $\mathbf{y} = \mathbf{A}^T\mathbf{x} \in \mathbb{R}^k$, if $x \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $y \sim N(\mathbf{A}^T\boldsymbol{\mu}, \mathbf{A}^T\boldsymbol{\Sigma}\mathbf{A})$.
- As a special case, the *whitening transform*

$$\mathbf{A}_\mathbf{w} = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1/2}$$

  where
  - $\boldsymbol{\Phi}$ is the matrix whose columns are the orthonormal eigenvectors of $\boldsymbol{\Sigma}$,
  - $\boldsymbol{\Lambda}$ is the diagonal matrix of the corresponding eigenvalues,

  gives a covariance matrix equal to the identity matrix $\mathbf{I}$.

Discriminant Functions for the Gaussian Density

- Discriminant functions for minimum-error-rate classification can be written as

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}|w_i) + \ln P(w_i).$$

- For $p(\mathbf{x}|w_i) = N(\boldsymbol{\mu_i}, \boldsymbol{\Sigma_i})$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu_i})^T \boldsymbol{\Sigma_i}^{-1}(\mathbf{x}-\boldsymbol{\mu_i}) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma_i}| + \ln P(w_i).$$

Case 1: $\Sigma_i = \sigma^2 \mathbf{I}$

- Discriminant functions are

$$g_i(\mathbf{x}) = \mathbf{w}_{\boldsymbol{i}}^T \mathbf{x} + w_{i0} \quad \text{(linear discriminant)}$$

where

$$\mathbf{w}_{\boldsymbol{i}} = \frac{1}{\sigma^2} \boldsymbol{\mu}_{\boldsymbol{i}}$$
$$w_{i0} = -\frac{1}{2\sigma^2} \boldsymbol{\mu}_{\boldsymbol{i}}^T \boldsymbol{\mu}_{\boldsymbol{i}} + \ln P(w_i)$$

(w_{i0} is the threshold or bias for the i'th category).

Case 1: $\Sigma_i = \sigma^2 \mathbf{I}$

- Decision boundaries are the hyperplanes $g_i(\mathbf{x}) = g_j(\mathbf{x})$, and can be written as

$$\mathbf{w}^T(\mathbf{x} - \mathbf{x_0}) = 0$$

where

$$\begin{aligned}
\mathbf{w} &= \boldsymbol{\mu_i} - \boldsymbol{\mu_j} \\
\mathbf{x_0} &= \frac{1}{2}(\boldsymbol{\mu_i} + \boldsymbol{\mu_j}) - \frac{\sigma^2}{\|\boldsymbol{\mu_i} - \boldsymbol{\mu_j}\|^2} \ln \frac{P(w_i)}{P(w_j)} (\boldsymbol{\mu_i} - \boldsymbol{\mu_j}).
\end{aligned}$$

- Hyperplane separating $\mathcal{R}_i$ and $\mathcal{R}_j$ passes through the point $\mathbf{x_0}$ and is orthogonal to the vector $\mathbf{w}$.

Case 1: $\Sigma_i = \sigma^2 \mathbf{I}$

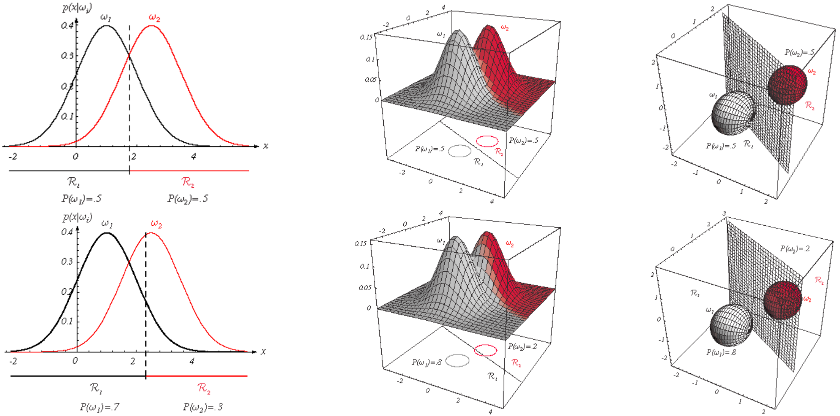

Figure 5: If the covariance matrices of two distributions are equal and proportional to the identity matrix, then the distributions are spherical in d dimensions, and the boundary is a generalized hyperplane of $d-1$ dimensions, perpendicular to the line separating the means. The decision boundary shifts as the priors are changed.

Case 1: $\Sigma_i = \sigma^2 \mathbf{I}$

- Special case when $P(w_i)$ are the same for $i = 1, \ldots, c$ is the *minimum-distance classifier* that uses the decision rule

$$\text{assign } \mathbf{x} \text{ to } w_{i^*} \text{ where } i^* = \arg\min_{i=1,\ldots,c} \|\mathbf{x} - \boldsymbol{\mu}_{\boldsymbol{i}}\|.$$

Case 2: $\Sigma_i = \Sigma$

- Discriminant functions are

$$g_i(\mathbf{x}) = \mathbf{w}_{\boldsymbol{i}}^T \mathbf{x} + w_{i0} \quad \text{(linear discriminant)}$$

where

$$\mathbf{w}_{\boldsymbol{i}} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_{\boldsymbol{i}}$$
$$w_{i0} = -\frac{1}{2} \boldsymbol{\mu}_{\boldsymbol{i}}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_{\boldsymbol{i}} + \ln P(w_i).$$

Case 2: $\Sigma_i = \Sigma$

- Decision boundaries can be written as

$$\mathbf{w}^T(\mathbf{x} - \mathbf{x_0}) = 0$$

  where

$$\begin{aligned}
\mathbf{w} &= \mathbf{\Sigma}^{-1}(\boldsymbol{\mu_i} - \boldsymbol{\mu_j}) \\
\mathbf{x_0} &= \frac{1}{2}(\boldsymbol{\mu_i} + \boldsymbol{\mu_j}) - \frac{\ln(P(w_i)/P(w_j))}{(\boldsymbol{\mu_i} - \boldsymbol{\mu_j})^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu_i} - \boldsymbol{\mu_j})}(\boldsymbol{\mu_i} - \boldsymbol{\mu_j}).
\end{aligned}$$

- Hyperplane passes through $\mathbf{x_0}$ but is not necessarily orthogonal to the line between the means.

Case 2: $\Sigma_i = \Sigma$

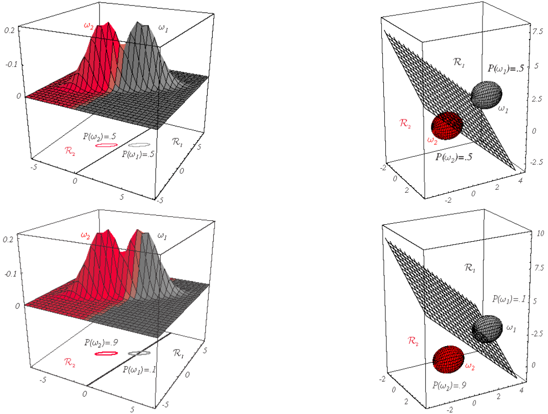

Figure 6: Probability densities with equal but asymmetric Gaussian distributions. The decision hyperplanes are not necessarily perpendicular to the line connecting the means.

Case 3: Σ_i = arbitrary

- Discriminant functions are

$$g_i(\mathbf{x}) = \mathbf{x}^T \mathbf{W}_{\boldsymbol{i}} \mathbf{x} + \mathbf{w}_{\boldsymbol{i}}^T \mathbf{x} + w_{i0} \quad \text{(quadratic discriminant)}$$

  where

$$\begin{aligned} \mathbf{W}_{\boldsymbol{i}} &= -\frac{1}{2} \boldsymbol{\Sigma}_{\boldsymbol{i}}^{-1} \\ \mathbf{w}_{\boldsymbol{i}} &= \boldsymbol{\Sigma}_{\boldsymbol{i}}^{-1} \boldsymbol{\mu}_{\boldsymbol{i}} \\ w_{i0} &= -\frac{1}{2} \boldsymbol{\mu}_{\boldsymbol{i}}^T \boldsymbol{\Sigma}_{\boldsymbol{i}}^{-1} \boldsymbol{\mu}_{\boldsymbol{i}} - \frac{1}{2} \ln |\boldsymbol{\Sigma}_{\boldsymbol{i}}| + \ln P(w_i). \end{aligned}$$

- Decision boundaries are hyperquadrics.

Case 3: $\Sigma_i =$ arbitrary

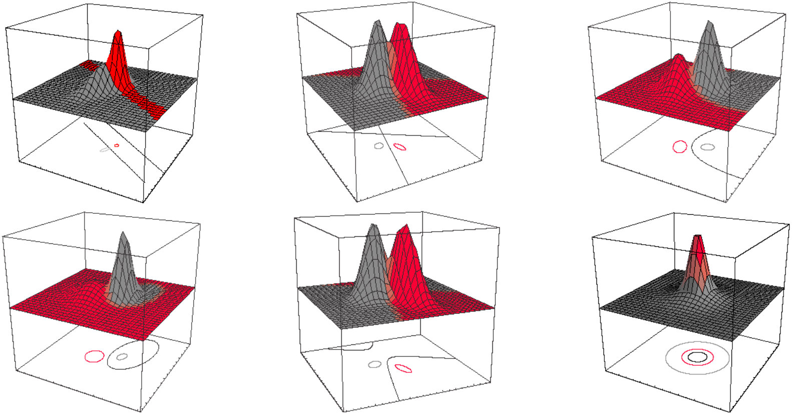

Figure 7: Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics.

Case 3: Σ_i = arbitrary

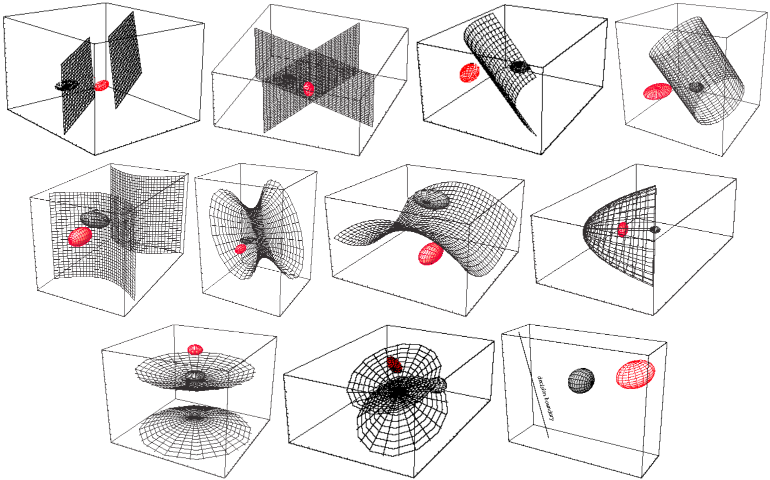

Figure 8: Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics.

Error Probabilities and Integrals

- For the two-category case

$$
\begin{aligned}
P(error) &= P(\mathbf{x} \in \mathcal{R}_2, w_1) + P(\mathbf{x} \in \mathcal{R}_1, w_2) \\
&= P(\mathbf{x} \in \mathcal{R}_2 | w_1) P(w_1) + P(\mathbf{x} \in \mathcal{R}_1 | w_2) P(w_2) \\
&= \int_{\mathcal{R}_2} p(\mathbf{x}|w_1)\, P(w_1)\, d\mathbf{x} + \int_{\mathcal{R}_1} p(\mathbf{x}|w_2)\, P(w_2)\, d\mathbf{x}.
\end{aligned}
$$

Error Probabilities and Integrals

- For the multicategory case

$$
\begin{aligned}
P(error) &= 1 - P(correct) \\
&= 1 - \sum_{i=1}^{c} P(\mathbf{x} \in \mathcal{R}_i, w_i) \\
&= 1 - \sum_{i=1}^{c} P(\mathbf{x} \in \mathcal{R}_i | w_i) P(w_i) \\
&= 1 - \sum_{i=1}^{c} \int_{\mathcal{R}_i} p(\mathbf{x}|w_i)\, P(w_i)\, d\mathbf{x}.
\end{aligned}
$$

Error Probabilities and Integrals

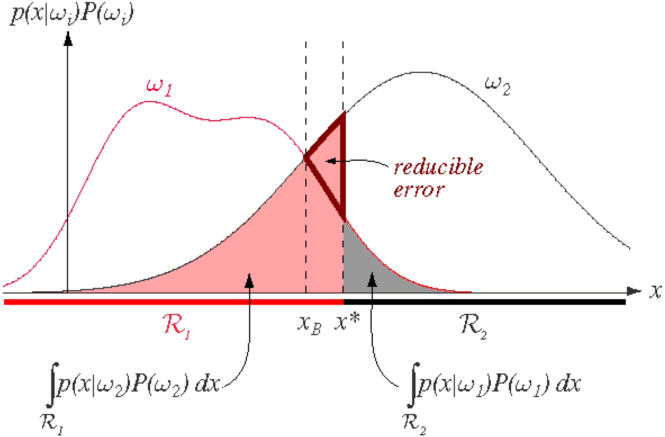

Figure 9: Components of the probability of error for equal priors and the non-optimal decision point x^*. The optimal point x_B minimizes the total shaded area and gives the Bayes error rate.

Receiver Operating Characteristics

- Consider the two-category case and define
  - w_1: target is present,
  - w_2: target is not present.

Table 1: *Confusion matrix*.

		Assigned w_1	Assigned w_2
True	w_1	correct detection	mis-detection
True	w_2	false alarm	correct rejection

- Mis-detection is also called false negative or Type I error.
- False alarm is also called false positive or Type II error.

Receiver Operating Characteristics

- If we use a parameter (e.g., a threshold) in our decision, the plot of these rates for different values of the parameter is called the *receiver operating characteristic* (ROC) curve.

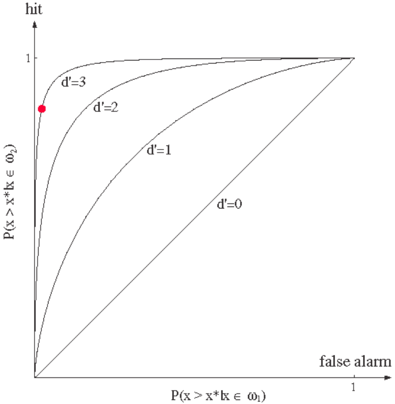

Figure 10: Example receiver operating characteristic (ROC) curves for different settings of the system.

Summary

- To minimize the overall risk, choose the action that minimizes the conditional risk $R(\alpha|\mathbf{x})$.
- To minimize the probability of error, choose the class that maximizes the posterior probability $P(w_j|\mathbf{x})$.
- If there are different penalties for misclassifying patterns from different classes, the posteriors must be weighted according to such penalties before taking action.
- Do not forget that these decisions are the optimal ones under the assumption that the "true" values of the probabilities are known.