NUMERICAL DIFFERENTIATION

- Using Newton's Forward difference:

$$f(x) \approx y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{\underline{|2}} + u(u-1)(u-2)\frac{\Delta^3 y_0}{\underline{|3}} + \dots$$

$$\approx y_0 + u\Delta y_0 + (u^2-u)\frac{\Delta^2 y_0}{\underline{|2}} + (u-1)(u^2-2u)\frac{\Delta^3 y_0}{\underline{|3}} + \dots$$

$$\approx y_0 + u\Delta y_0 + (u^2-u)\frac{\Delta^2 y_0}{\underline{|2}} + [u^3 - 3u^2 + 2u]\frac{\Delta^3 y_0}{\underline{|3}} + \dots$$

Differentiating $f(x)$ w.r.t x. $\left(u = \frac{x-x_0}{h}\right) \Rightarrow \frac{du}{dx} = \frac{1}{h}$.

$$f'(x) = \Delta y_0\left(\frac{1}{h}\right) + (2u-1)\frac{\Delta^2 y_0}{\underline{|2}}\cdot\frac{1}{h} + (3u^2-6u+2)\frac{\Delta^3 y_0}{\underline{|3}}\cdot\frac{1}{h} + \dots$$

$$= \frac{1}{h}\left[\Delta y_0 + (2u-1)\frac{\Delta^2 y_0}{\underline{|2}} + (3u^2-6u+2)\frac{\Delta^3 y_0}{\underline{|3}} + \dots\right]$$

If $x = x_0 \Rightarrow u = 0$.

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{\underline{|2}} + 2\frac{\Delta^3 y_0}{\underline{|3}} + \dots\right]$$ is the derivative ~~of~~ at the first point.

- Using Newton's Backward difference.

$$f(x) \approx \phi(x) = y_n + u\nabla y_n + u(u+1)\frac{\nabla^2 y_n}{\underline{|2}} + u(u+1)(u+2)\frac{\nabla^3 y_n}{\underline{|3}} + \dots$$

$$= y_n + u\nabla y_n + (u^2+u)\frac{\nabla^2 y_n}{\underline{|2}} + (u^3+3u^2+2u)\frac{\nabla^3 y_n}{\underline{|3}}$$

Diff. w.r.t x, $\left(u = \frac{x-x_n}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}\right)$

$$f'(x) = \left(\nabla y_n + (2u+1)\frac{\nabla^2 y_n}{\underline{|2}} + (3u^2+6u+2)\frac{\nabla^3 y_n}{\underline{|3}} + \dots\right)\cdot\frac{1}{h}$$

If $x = x_n$, then $u = 0$, ...$6236)] = 0.152$

$$\therefore f'(x_0) = \left(\nabla y_n + \frac{\nabla^2 y_n}{\lfloor 2} + 2\frac{\nabla^3 y_n}{\lfloor 3} + \cdots\right)\cdot\frac{1}{h}$$ is the derivative at x_n.

• Using Newton's divided difference.

$$f(x) \approx f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \cdots$$

$$= f(x_0) + (x-x_0) f(x_0, x_1) + (x^2 - (x_0+x_1)x + x_1 x_0) f(x_0, x_1, x_2) + \cdots$$

Diff. w.r.t x;

$$f'(x) = f(x_0, x_1) + (2x - (x_0 + x_1)) f(x_0, x_1, x_2) + \cdots$$

• Using Lagrange's interpolation formula:

$$L(x) = w(x) \sum_{i=0}^{n} \frac{f(x_i)}{(x-x_i) w'(x_i)}$$

$$L'(x) = w'(x) \sum_{i=0}^{n} \frac{f(x_i)}{(x-x_i) w'(x_i)} - w(x) \sum_{i=0}^{n} \frac{f(x_i)}{(x-x_i)^2 w'(x_i)}$$

Drawback: We cannot find the derivative of $L(x)$ at point $(x_0, x_1, x_2 \cdots x_n)$ as when substituted, the denominator of both terms become zero, thus making $L'(x)$ not defined.

Alternative:

$$L(x) = w(x) \sum_{\substack{i=0 \\ i\neq r}}^{n} \frac{f(x_i)}{(x-x_i) w'(x_i)} + f(x_r)\frac{(x-x_0)\cdots(x-x_{r-1})(x-x_{r+1})\cdots(x-x_n)}{(x_r-x_0)\cdots(x_r-x_{r-1})(x_r-x_{r+1})\cdots(x_r-x_n)}$$

If $x = x_r$

$$L'(x_r) = w'(x_r) \sum_{\substack{i=0 \\ i\neq r}}^{n} \frac{f(x_i)}{(x_r-x_i) w'(x_r)} + f(x_r) \sum_{\substack{i=0 \\ i\neq r}}^{n} \frac{1}{(x_r - x_i)}$$

Ex: Find $f'(1)$, $f''(1)$, $f'(5)$, $f''(5)$

x :	1	2	3	4	5
$f(x)$:	5	10	17	26	37

Ex: · Find $f'(3)$ by Lagrange's & Newton's D.D formula.

· Find $f''(3)$ by Newton's D.D formula.

x :	0	1	3	3.5	5.5	7
$f(x)$:	2	3	4.1	5.3	6	6.9