York University

Lassonde School of Engineering

Math 1090 A

Final Examination
December 18, 2013

NAME (print):______________________________________
(Family) (Given)

SIGNATURE:______________________________________

STUDENT NUMBER:______________________________________

Instructions:

1. Please read these instructions before you start writing.
2. You have 3 hours. CLOSED BOOK. The axiom sheet is attached to the back of the exam. You may detach it and it is not to be handed in with the exam.
3. There are 12 questions on 12 pages.
4. Answer all questions.
5. Your work must justify the answer you give.
6. If you need more space, you may write on the backs of the pages. Clearly label each page as to which question you are answering.
7. You are free to use without proof, but with proper reference, any theorems and metatheorems that were proved in class or in the textbook. If you are unsure whether some statement was really proved, I suggest that your prove it just to be sure.

Question	**Points**	**Marks**
1	5	
2	4	
3	6	
4	5	
5	6	
6	4	
7	3	
8	3	
9	3	
10	3	
11	3	
12	5	
Total	50	

1. (5 points) Which of the following substitutions are defined? If undefined, explain why. If defined, give the answer after applying substitution.

   - $(p \vee q) \to \top[\bot := r]$
   - $((p \equiv r) \wedge (\neg q))[p := \top]$
   - $((\forall y)(\forall x) f(x) = z)[z := g(x)]$
   - $((\forall y)(\forall x)(\forall z) f(x) = z)[z := g(x)]$
   - $((\forall x)p)[p \backslash f(x) = y]$

2. (4 points) Prove (p) is not a well-formed formula.
   Hint. Analyze (Boolean) formula-calculations.

3. (6 points) Consider the string $(\forall y)(x_1 = y_1 \rightarrow ((\forall z)f(z) = x_1 \equiv (\forall z)f(z) = y_1))$.

   (a) Show this string is a well-formed formula. *Hint.* Use bottom-up or top-down parsing.

   (b) Is this well-formed formula a tautology? Show all work and remember that to show that a schema is not a tautology we must identify an instance of it that is not a tautology.

   (c) Can you prove $\vdash (\forall y)(x_1 = y_1 \rightarrow ((\forall z)f(z) = x_1 \equiv (\forall z)f(z) = y_1))$ in predicate logic? If so, give a proof, if not, explain why.

4. (5 points)

   (a) Give the abstraction of $((\forall x)x = y \to \bot) \to (((\forall x)x = y \to p) \to ((\forall x)x = y \to \bot \wedge p))$

   (b) Use the definition of tautological implication to show

$$((\forall x)x = y \to \bot), ((\forall x)x = y \to p) \models_{taut} ((\forall x)x = y \to \bot \wedge p).$$

5. (6 points) Prove $\vdash ((A \vee B \vee \neg C) \wedge (A \to B)) \to (C \to B)$
   You are expected to give two different solutions as follows:

   (a) Give a proof by resolution.

   (b) Give a Hilbert-style proof.

6. (4 points) Is

$$\vdash X \rightarrow Y \equiv Y \rightarrow X$$

an absolute theorem schema?

- if you think 'yes', then give a proof.
- if you think 'no', use the appropriate tool (formulate and name the tool) to give a counterexample for appropriately chosen X and Y.

7. (3 points) You must use the technique of the "auxiliary variable metatheorem" in the proof that you are asked to write here. Any other proof (if correct) will MAX at 1 MARK.
For any formulae A, B, C, show that $\vdash (\exists x)(A \wedge B) \rightarrow (\exists x)(A \rightarrow C \rightarrow B)$.

8. (3 points) Is $(\forall x)(A \vee B) \to (\forall x)A \vee (\forall x)B$ an absolute theorem schema?

- if you think 'yes', then give a proof.
- if you think 'no', construct a countermodel or prove the invalid strong generalization from it.

9. (3 points) Is $(\forall x)(\exists y)A \to (\exists y)(\forall x)A$ an absolute theorem schema?

   - if you think 'yes', then give a proof.
   - if you think 'no', construct a countermodel or prove the invalid strong generalization from it.

10. (3 points) Give an equational proof of $\vdash (\forall x)(A \to B) \equiv ((\exists x)A) \to B$.

11. (3 points) Is $(\forall x)(A \to B \wedge C) \to (\forall x)(A \to B)$ an absolute theorem schema?

- if you think 'yes', then give a proof.
- if you think 'no', construct a countermodel or prove the invalid strong generalization from it.

12. (5 points) Prove by induction on the complexity of A that if x is not free in A, then, for any term t, $A[x := t]$ is A.

The end