Indian Institute of Technology Madras

Project 3

Suspension Control

ED5330

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Contents

Introduction

Suspension systems are a integral part in ride comfort and handling of a vehicle. Using this paper, we will try to answer the following questions:

- System modelling
  - Derive the equations of motion of this System
  - Calculate the natural frequencies of the system
  - State-Space representation of the system
- Open-loop Suspension Performance Analysis
  - Derive three transfer function
  - Bode plot of the transfer functions for multiple parameters
- Closed-loop Suspension Performance Analysis
  - Develop a Linear Quadratic Regulator (LQR)
  - Calculate the optimal regulator gain
  - plot the Bode diagram of each of the three transfer functions, with and without control.

Quarter Car Modelling

2.1 Governing Equations

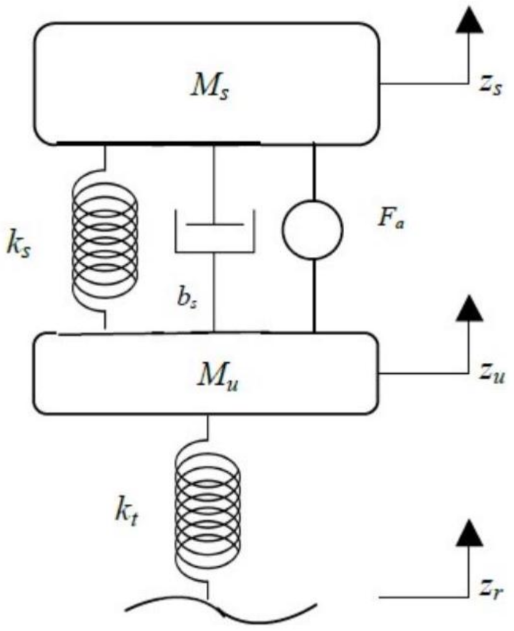


Figure 2.1: Quarter Model

Let the parameters be:

- M_s: Sprung Mass
- M_u: Unsprung mass
- k_s: Suspension stiffness
- k_t: Tyre stiffness
- b_s: Suspension damping

2.1.1 Equations of Motion

Considering the sprung mass, applying force balance

$$\begin{aligned} m_s\ddot{z}_s &= -k_s(z_s - z_u) - b_s(\dot{z}_s - \dot{z}_u) + F_a \\ m_s\ddot{z}_s + b_s(\dot{z}_s + k_s z_s &= F_a + b_s\dot{z}_u + k_s z_u \end{aligned} \tag{2.1}$$

Considering the unsprung mass, applying force balance

$$m_u\ddot{z_u} = k_s(z_s - z_u) + b_s(\dot{z_s} - \dot{z_u}) - F_a - k_t(z_u - z_r)$$
$$m_s\ddot{z_u} + b_s(\dot{z_u} + (k_s + k_t)z_u + F_a = b_s\dot{z_s} + k_s z_s + k_t z_r \tag{2.2}$$

The above equations can be written as:

$$\begin{bmatrix} m_s & 0 \\ 0 & m_u \end{bmatrix} \begin{bmatrix} \ddot{z_s} \\ \ddot{z_u} \end{bmatrix} + \begin{bmatrix} b_s & -b_s \\ -b_s & -b_s \end{bmatrix} \begin{bmatrix} \dot{z_s} \\ \dot{z_u} \end{bmatrix} + \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix} \begin{bmatrix} z_s \\ z_u \end{bmatrix} = \begin{bmatrix} 0 \\ k_t \end{bmatrix} z_r + \begin{bmatrix} 1 \\ -1 \end{bmatrix} F_a \tag{2.3}$$

$$M\ddot{z} + C\dot{z} + Kz = B_1 z_r + B_1 F_a \tag{2.4}$$

2.1.2 Natural Frequencies

Natural Frequencies of the system can be calculated using the following formula:

$$\left| -\omega_n^2 M + K \right| = 0 \tag{2.5}$$

$$\begin{vmatrix} -\omega_n^2 m_s + k_s & -k_s \\ -k_s & -\omega_n^2 m_u + k_s + k_t \end{vmatrix} = 0 \tag{2.6}$$

$$m_s m_u \omega_n^4 - (m_s(k_s + k_t) + m_u k_s)\omega_n^2 + k_s k_t = 0 \tag{2.7}$$

On solving for ω_n^2, we get:

$$\omega_n^2 = \frac{k_s + k_t}{2m_u} + \frac{k_s}{2m_s} \pm \left(\frac{\sqrt{(k_s + k_t)^2 m_s^2 + k_s^2 m_u^2 - 2(k_t - k_s)m_s}}{2m_s m_u} \right) \tag{2.8}$$

Substituting the values, we get

$$M = \begin{bmatrix} 300 & 0 \\ 0 & 40 \end{bmatrix} \text{and } K = \begin{bmatrix} 15000 & -15000 \\ -15000 & 165000 \end{bmatrix}$$

We get $\omega_n = 7.67, 77.07$ rad/s.

2.1.3 State-Space Representation of the system

Let the state vector be $X = \begin{bmatrix} z_s - z_u \\ \dot{z_s} \\ z_u - z_r \\ \dot{z_u} \end{bmatrix}$

$$\ddot{z_s} = \frac{F_a}{m_s} - \frac{b_s}{m_s}\dot{z_s} + \frac{b_s}{m_s}\dot{z_u} - \frac{k_s}{m_s}(z_s - z_u) \tag{2.9}$$

$$\ddot{z_u} = -\frac{F_a}{m_u} - \frac{b_s}{m_u}\dot{z_u} + \frac{b_s}{m_u}\dot{z_s} + \frac{k_s}{m_u}(z_s - z_u) - \frac{k_t}{m_u}(z_u - z_r) \tag{2.10}$$

Hence, the state-space representation of the system is:

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & -\frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} X + \begin{bmatrix} 0 \\ \frac{1}{m_s} \\ 0 \\ -\frac{1}{m_u} \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \dot{Z}_r \tag{2.11}$$

$$\dot{X} = AX + bF + I\dot{z}_r \tag{2.12}$$

2.2 Open Loop Suspension Analysis

2.2.1 Open Loop Equations

Acceleration Transfer Function

$$T_a(s) = \frac{L(\ddot{z_s}(t))}{L(\dot{z_r}(t))} \tag{2.13}$$

Since we are considering passive suspension, $F_a = 0$.

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & -\frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \dot{Z}_r \tag{2.14}$$

$$\ddot{z_r} = Y = \begin{bmatrix} -k_s/m_s & -b_s/m_s & 0 & b_s/m_s \end{bmatrix} X \tag{2.15}$$

Where we have $u = \dot{z}_r(t)$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & -\frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} -k_s/m_s & -b_s/m_s & 0 & b_s/m_s \end{bmatrix}, D = 0$$

Using MATLAB scripts, we get the following output.

```
k_s = 18000;
k_t = 160000;
m_s = 275;
m_u = 30;
b_s = 1000;
```

```
A = [0 1 0 -1;
    -k_s/m_s -b_s/m_s 0 b_s/m_s;
    0 0 0 1;
    k_s/m_u b_s/m_u -k_t/m_u -b_s/m_u];
B = [0;0;-1;0];
C1 = [-k_s/m_s -b_s/m_s 0 b_s/m_s];
D = 0;
```

```
[n,d] = ss2tf(A,B,C1,D)
tf1 = tf(n,d)
```

(a) MATLAB Parameters

```
tf1 =

          1.939e04 s^2 + 3.491e05 s - 2.254e-10
  ----------------------------------------------------
  s^4 + 36.97 s^3 + 5999 s^2 + 1.939e04 s + 3.491e05

Continuous-time transfer function.
```

(b) Acceleration Transfer Function

Figure 2.2: Acceleration Transfer Function calculation

$$T_a(s) = \frac{19390s^2 + 349100s - 2.254 * 10^{-10}}{s^4 + 36.97s^3 + 5999s^2 + 19390s + 349100} \tag{2.16}$$

Rattle Transfer Function:

$$T_r(s) = \frac{L(z_s(t) - z_u(t))}{L(\dot{z_r}(t))} \tag{2.17}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & -\frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, D = 0$$

Using MATLAB scripts, we get the following output.

```
A = [0 1 0 -1;
    -k_s/m_s -b_s/m_s 0 b_s/m_s;
    0 0 0 1;
    k_s/m_u b_s/m_u -k_t/m_u -b_s/m_u];
B = [0;0;-1;0];
C2 = [1 0 0 0];
D = 0;
```

```
[n,d] = ss2tf(A,B,C2,D)
tf2 = tf(n,d)
```

(a) MATLAB Parameters

```
tf2 =

                         -5333 s
  ---------------------------------------------------
  s^4 + 36.97 s^3 + 5999 s^2 + 1.939e04 s + 3.491e05

Continuous-time transfer function.
```

(b) Rattle Transfer Function

Figure 2.3: Rattle Transfer Function calculation

$$T_r(s) = \frac{-5333s}{s^4 + 36.97s^3 + 5999s^2 + 19390s + 349100} \tag{2.18}$$

Tyre Deflection Transfer Function

$$T_t(s) = \frac{L(z_u(t) - z_r(t))}{L(\dot{z}_r(t))} \tag{2.19}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & -\frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, D = 0$$

Using MATLAB scripts, we get the following output.

```
A = [0 1 0 -1;
    -k_s/m_s -b_s/m_s 0 b_s/m_s;
    0 0 0 1;
    k_s/m_u b_s/m_u -k_t/m_u -b_s/m_u];
B = [0;0;-1;0];
C3 = [0 0 1 0];
D = 0;
```

```
[n,d] = ss2tf(A,B,C3,D)
tf3 = tf(n,d)
```

(a) MATLAB Parameters

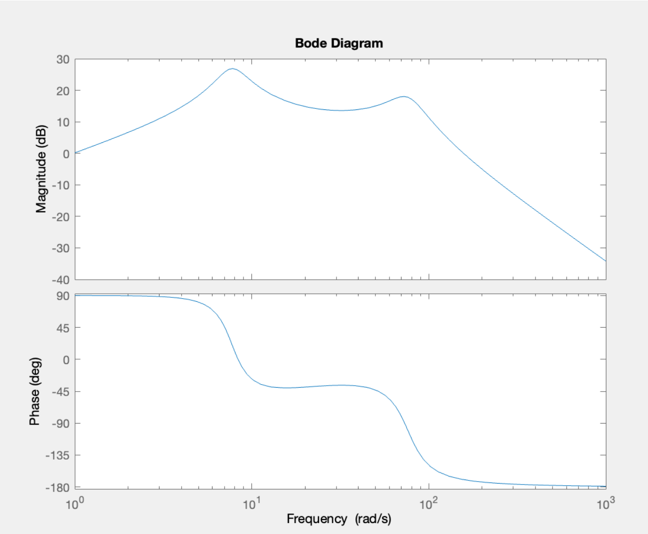
```
tf3 =

        -s^3 - 36.97 s^2 - 665.5 s - 1.824e-13
  ---------------------------------------------------
  s^4 + 36.97 s^3 + 5999 s^2 + 1.939e04 s + 3.491e05

Continuous-time transfer function.
```

(b) Tyre Deflection Transfer Function

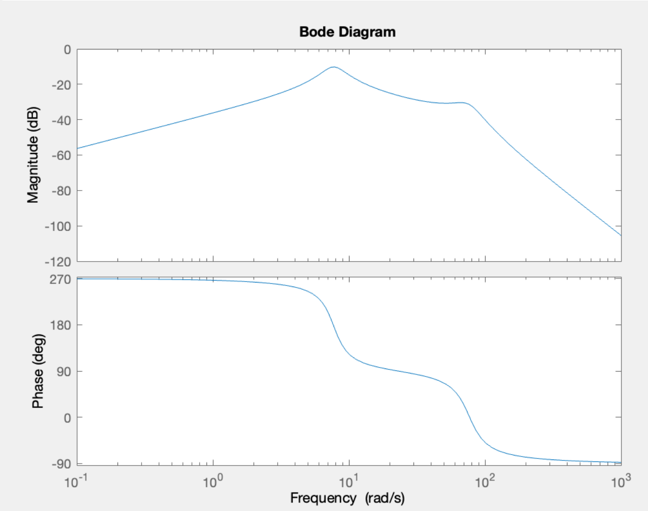
Figure 2.4: Tyre Transfer Function calculation

$$T_t(s) = \frac{-s^3 - 36.97s^2 - 665.5s - 1.824 * 10^{-13}}{s^4 + 36.97s^3 + 5999s^2 + 19390s + 349100} \tag{2.20}$$

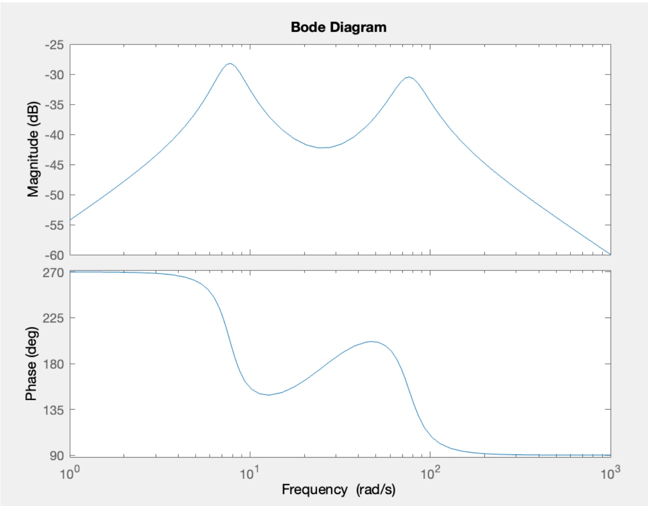
2.2.2 Bode Plots



(a) Acceleration Transfer Function Bode



(b) Rattle Transfer Function Bode



(c) Tyre Transfer Function

Figure 2.5: Bode Plots

Observations:

- All the three transfer functions have the same natural frequencies, which can be confirmed by observing the maxima of three plots. All maxima occur at the same frequencies.
- The first peak of the plot corresponds to the resonance of the input with the spring damper system of the suspension of the vehicle and the second peak corresponds to that of the tyre's effective spring.

2.2.3 Effect of Suspension Stiffness

For $k_s = 12000N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -43.6 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 400 & 33.3 & -5333.3 & -33.3 \end{bmatrix}$$

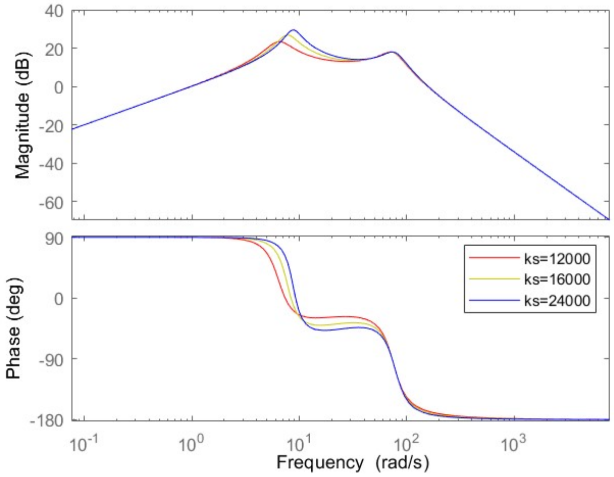
For $k_s = 18000N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -43.6 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 400 & 33.3 & -5333.3 & -33.3 \end{bmatrix}$$

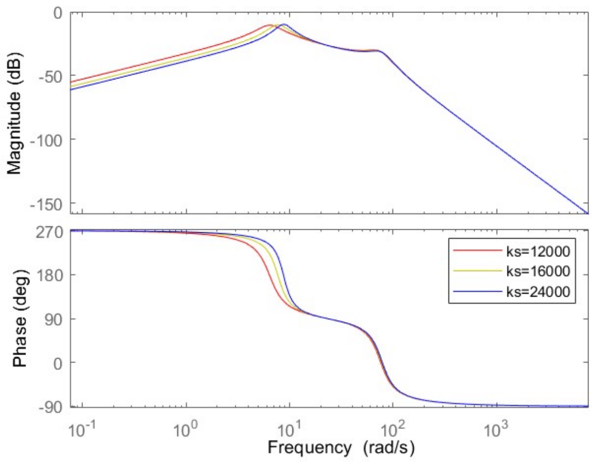
For $k_s = 24000 N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -43.6 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 400 & 33.3 & -5333.3 & -33.3 \end{bmatrix}$$

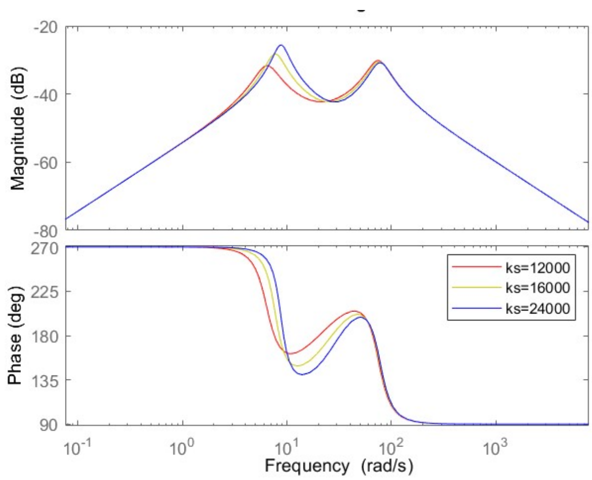
Bode Plots:



(a) Acceleration Transfer Function Bode



(b) Rattle Transfer Function Bode



(c) Tyre Transfer Function

Figure 2.6: Bode plots of transfer functions for different suspension stiffness

Observations:

- On observing the plots, the maxima of the plot occurs at higher frequencies as the suspension stiffness increases. So, natural frequencies increases with increase in suspension stiffness.

- As the stiffness increases, the ripples in the functions have steeper increase and decrease, indicating that the vehicle "jolts" more with increase in suspension stifiness.

- The decrease in relative stiffness between the suspension and the tyre as suspension stiffness increases, results ni an increased tyre deflection for higher values of suspension stiffness

2.2.4 Effect of Suspension Damping

For $b_s = 600N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -43.6 & -2.2 & 0 & 2.2 \\ 0 & 0 & 0 & 1 \\ 400 & 20.0 & -5333.3 & -20.0 \end{bmatrix}$$

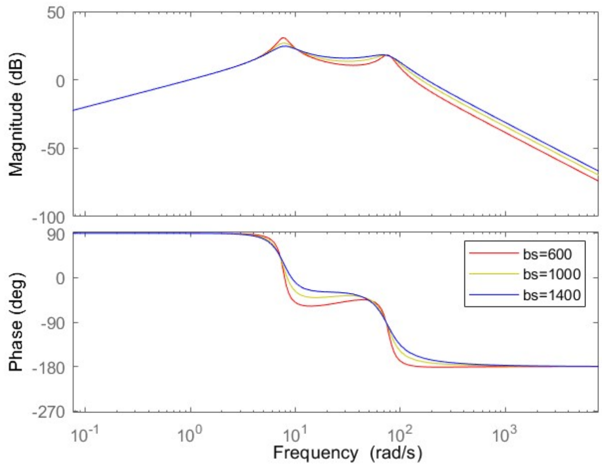
For $b_s = 1000N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -43.6 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 400 & 23.33 & -5333.3 & -33.33 \end{bmatrix}$$

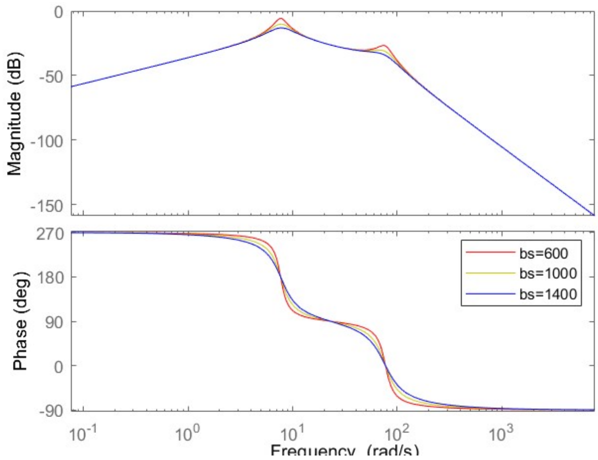
For $b_s = 1400N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -43.6 & -5.1 & 0 & 5.1 \\ 0 & 0 & 0 & 1 \\ 400 & 46.7 & -5333.3 & -46.7 \end{bmatrix}$$

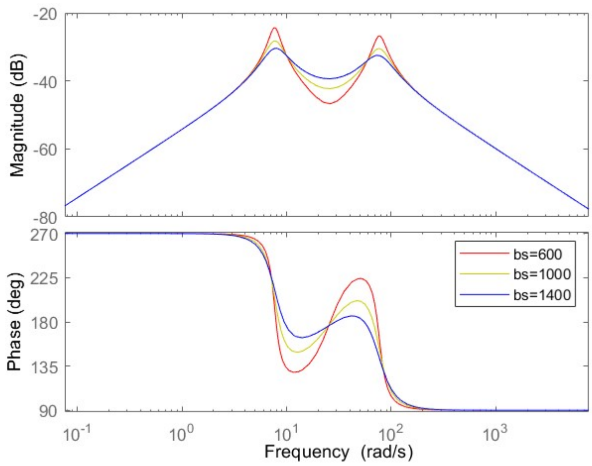
Bode Plots:



(a) Acceleration Transfer Function Bode



(b) Rattle Transfer Function Bode



(c) Tyre Transfer Function

Figure 2.7: Bode plots of transfer functions for different suspension damping

Observations:

- As the peaks ofthe Bode plots occur at the same frequencies, the natural frequency of the system remains unaffected with the increase in damping values.

- The acceleration gains decrease as the damping increases, indicating the dampening out of irregularities. Also, the variations in the displacement functions are smoother with higher damping, meaning better smoothening of bumps.

- Higher suspension damping results in lesser variation in phase in case of the tyre deflection plot.

2.2.5 Effect of Tyre Stiffness

For $k_t = 100000 N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -65.5 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 600 & 33.33 & -3333.3 & -33.33 \end{bmatrix}$$

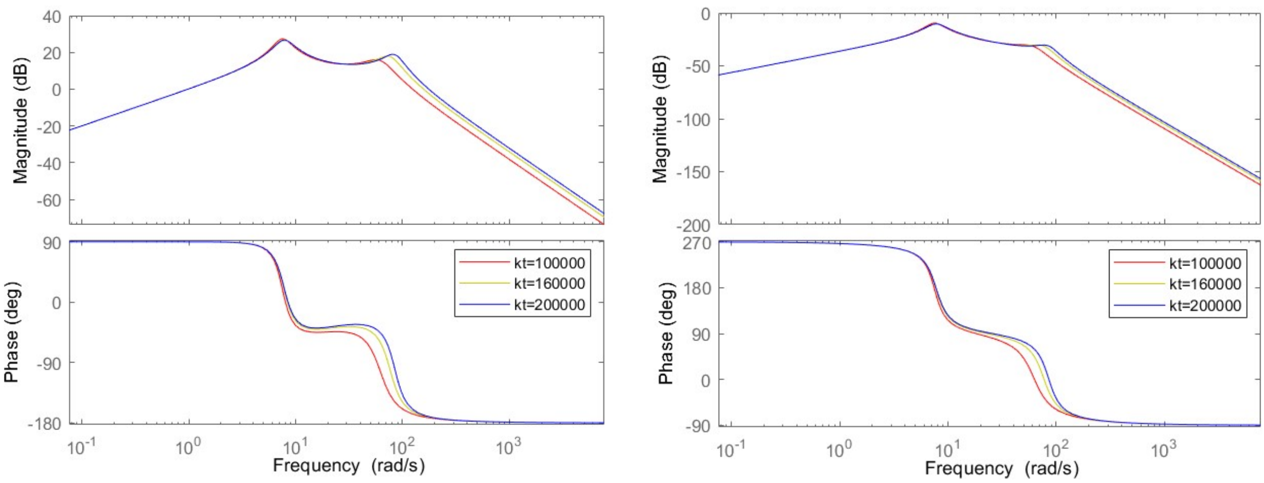
For $k_t = 160000 N/m$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -65.5 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 600 & 33.33 & -5333.3 & -33.33 \end{bmatrix}$$

For $k_t = 200000 N/m$

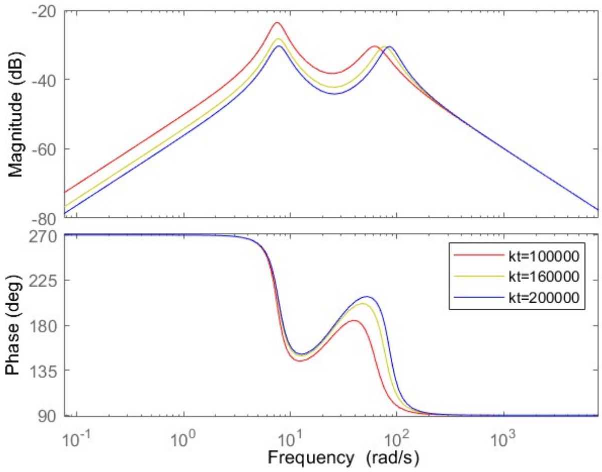
$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -65.5 & -3.6 & 0 & 3.6 \\ 0 & 0 & 0 & 1 \\ 600 & 33.33 & -6666.7 & -33.33 \end{bmatrix}$$

Bode Plots:



(a) Acceleration Transfer Function Bode

(b) Rattle Transfer Function Bode



(c) Tyre Transfer Function

Figure 2.8: Bode plots of transfer functions for different suspension damping

Observations:

- The closed loop system reaches the steady state value much faster than that of open loop system.
- Also, for rectangular input in case of open loop system, the oscillation doesnot dies out, it keeps propagating as bundle of oscillations.
- The oscillations are decreased for the sprung mass case in closed loop system as compared to the open loop system and the ripples are smoothened as well to a larger extent
- The closed loop system is more stable.

Half Car Model

3.1 Governing Equations for Half Car Model

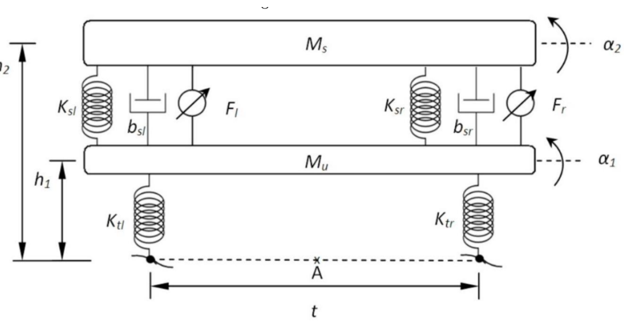


Figure 3.1: Half Car Model

- $M_s = 1200\,\text{kg}$: Sprung Mass
- $M_u = 120\,\text{kg}$: Unsprung mass
- $k_{sl} = k_{sr} = 50\,000\,\text{N/m}$: Suspension stiffness
- $k_{tl} = k_{tr} = 350000N/m$: Tyre stiffness
- $b_{sl} = b_{sr} = 3000Ns/m$: Suspension damping
- $t = 2.5m$: Track width
- $h_1 = 1m$: Height of unsprung mass
- $h_2 = 2.2m$: Height of sprung mass
- $I_s = 2000kg(m)^2$: Sprung mass moment of inertia
- $I_u = 200kg(m)^2$: Unsprung mass moment of inertia

3.1.1 Equations of Motion

Moment equation of unsprung mass about the roll point A ,

$$\begin{split}-k_{tr}\alpha_1(t/2)^2 - k_{tl}\alpha_1(t/2)^2 - F_r(t/2) + b_{sr}(\dot{\alpha_2} - \dot{\alpha_1})(t/2)^2 \\ + F_l(t/2) + k_{sl}(\alpha_2 - \alpha_1)(t/2)^2 + b_{sl}(\dot{\alpha_2} - \dot{\alpha_1})(t/2)^2 - m_u a_y h_1 = (I_u + m_u h_1^2)\ddot{\alpha_1}\end{split} \quad (3.1)$$

Moment equation of sprung mass about the roll point A ,

$$\begin{split}F_r(t/2) - b_{sr}(\dot{\alpha_2} - \dot{\alpha_1})(t/2)^2 - k_{sr}(\alpha_2 - \alpha_1)(t/2)^2 - F_l(t/2) - k_{sl}(\alpha_2 - \alpha_1)(t/2)^2 \\ - b_{sl}(\dot{\alpha_2} - \dot{\alpha_1})(t/2)^2 - m_s a_y h_2 = (I_s + m_s h_2^2)\ddot{\alpha_2}\end{split} \quad (3.2)$$

3.2 State Space Model

$$\ddot{\alpha_1} = -\frac{(K_{sr}+k_{sl}+k_{tl}+k_{sl})}{I_u+m_u h_1^2}\alpha_1 + \frac{b_{sr}+b_{sl}}{I_u+m_u h_1^2}\dot{\alpha_1} + \frac{k_{sr}+k_{sl}}{I_u+m_u h_1^2}(t/2)^2\alpha_2 + \frac{b_{sr}+b_{sl}}{I_u+m_u h_1^2}(t/2)^2\dot{\alpha_1} + \frac{F_l}{I_u+m_u h_1^2}(t/2) - \frac{F_r}{I_u+m_u h_1^2}(t/2) - \frac{m_a a_y h_1}{I_u+m_u h_1^2} \quad (3.3)$$

$$\ddot{\alpha_2} = -\frac{(K_{sr}+k_{sl})}{I_s+m_s h_2^2}\alpha_1 + \frac{b_{sr}+b_{sl}}{I_s+m_s h_2^2}\dot{\alpha_1} + \frac{k_{sr}+k_{sl}}{I_s+m_s h_2^2}(t/2)^2\alpha_2 + \frac{b_{sr}+b_{sl}}{I_s+m_s h_2^2}(t/2)^2\dot{\alpha_2} + \frac{F_r}{I_s+m_s h_2^2}(t/2) - \frac{F_l}{I_s+m_s h_2^2}(t/2) - \frac{m_a a_y h_2}{I_s+m_s h_2^2} \quad (3.4)$$

Considering the following as state variables,

$$X = \begin{bmatrix}\alpha_1\\ \dot{\alpha_1}\\ \alpha_2\\ \dot{\alpha_2}\end{bmatrix}; f(t) = \begin{bmatrix}F_l\\ F_r\end{bmatrix} \quad (3.5)$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{(k_{sr}+k_{tr}+k_{tl}+k_{sl})}{I_u+m_u h_1^2}(t/2)^2 & -\frac{b_{sr}+b_{sl}}{I_u+m_u h_1^2}(t/2)^2 & \frac{k_{sr}+k_{sl}}{I_u+m_u h_1^2}(t/2)^2 & \frac{b_{sr}+b_{sl}}{I_u+m_u h_1^2}(t/2)^2\\ 0 & 0 & 0 & 1\\ \frac{(k_{sr}+k_{sl})}{I_s+m_s h_2^2}(t/2)^2 & \frac{b_{sr}+b_{sl}}{I_s+m_s h_2^2}(t/2)^2 & -\frac{k_{sr}+k_{sl}}{I_s+m_s h_2^2} & -\frac{b_{sr}+b_{sl}}{I_s+m_s h_2^2}(t/2)^2 \end{bmatrix} X + \begin{bmatrix} 0 & 0\\ -\frac{t/2}{I_u+m_u h_1^2} & \frac{t/2}{I_u+m_u h_1^2}\\ 0 & 0\\ \frac{t/2}{I_s+m_s h_2^2} & -\frac{t/2}{I_s+m_s h_2^2}\end{bmatrix} f(t) + \begin{bmatrix}0\\ -\frac{m_u h_1}{I_u+m_u h_1^2}\\ 0\\ \frac{m_s h_2}{I_s+m_s h_2^2}\end{bmatrix} a_y \quad (3.6)$$

Comparing the the above equation 3.6, we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{(k_{sr}+k_{tr}+k_{tl}+k_{sl})}{I_u+m_u h_1^2}(t/2)^2 & -\frac{b_{sr}+b_{sl}}{I_u+m_u h_1^2}(t/2)^2 & \frac{k_{sr}+k_{sl}}{I_u+m_u h_1^2}(t/2)^2 & \frac{b_{sr}+b_{sl}}{I_u+m_u h_1^2}(t/2)^2\\ 0 & 0 & 0 & 1\\ \frac{(k_{sr}+k_{sl})}{I_s+m_s h_2^2}(t/2)^2 & \frac{b_{sr}+b_{sl}}{I_s+m_s h_2^2}(t/2)^2 & -\frac{k_{sr}+k_{sl}}{I_s+m_s h_2^2} & -\frac{b_{sr}+b_{sl}}{I_s+m_s h_2^2}(t/2)^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -3906.2 & -29.3 & 488.3 & 29.3\\ 0 & 0 & 0 & 1\\ 20 & 1.2 & -12.8 & -1.2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0\\ -\frac{t/2}{I_u+m_u h_1^2} & \frac{t/2}{I_u+m_u h_1^2}\\ 0 & 0\\ \frac{t/2}{I_s+m_s h_2^2} & -\frac{t/2}{I_s+m_s h_2^2}\end{bmatrix} = \begin{bmatrix} 0 & 0\\ -0.0039 & 0.0039\\ 0 & 0\\ 0.0002 & -0.0002\end{bmatrix}$$

$$d = \begin{bmatrix}0\\ -\frac{m_u h_1}{I_u+m_u h_1^2}\\ 0\\ \frac{m_s h_2}{I_s+m_s h_2^2}\end{bmatrix} = \begin{bmatrix}0\\ 0.0038\\ 0\\ 2.0613\end{bmatrix} \times 10^7$$

3.3 Performance Index

We define the performance index as the following:

$$J = \int_0^{t_f} \left(\rho_1\alpha_1^2 + \rho_2\dot{\alpha_1}^2 + \rho_3\alpha_2^2 + \rho_4\dot{\alpha_2}^2 + \rho_4F_r^2 + \rho_4F_l^2\right) dt \tag{3.7}$$

This can be written as:

$$J = \int_0^{\infty} \left(x^TQx + 2x^Tnf + f^TRf\right) dt \tag{3.8}$$

where Q is the state cost matrix, R is the input cost matrix, n is the input cost vector, and x is the state vector.

Given weights $\begin{bmatrix}\rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6\end{bmatrix} = \begin{bmatrix}16 & 16 & 40000 & 16 & 10^{-9} & 10^{-9}\end{bmatrix}$

$$Q = \begin{bmatrix}\rho_1 & 0 & 0 & 0\\ 0 & \rho_2 & 0 & 0\\ 0 & 0 & \rho_3 & 0\\ 0 & 0 & 0 & \rho_4\end{bmatrix}; R = \begin{bmatrix}\rho_5 & 0\\ 0 & \rho_6\end{bmatrix}; n = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

Calculating using MATLAB, we get

```
K = 2×4
10^6 ×
   -3.8418   -0.0996    6.4490    0.6711
    0.0000    0.0000   -0.0000   -0.0000

S = 4×4
10^3 ×
    0.5922   -0.0023   -0.7071   -0.0800
   -0.0023    0.0000    0.0075    0.0006
   -0.7071    0.0075    4.3279    0.2222
   -0.0800    0.0006    0.2222    0.0184

P = 4×1 complex
10^2 ×
  -0.0965 + 0.1611i
  -0.0965 - 0.1611i
  -0.2014 + 0.0000i
  -4.8741 + 0.0000i
```

Figure 3.2: LQR Parameters

3.4 Active and Passive Suspension Comparison

3.4.1 Passive Suspension

For passive suspension, $\dot{X} = AX + da_y$

$$\alpha_1(s) = \frac{38400s^2 + 6.039e08s + 1.007e10}{38400s^2 + 6.039e08s + 1.007e10} \tag{3.9}$$

$$\alpha_2(s) = \frac{2.061e07s^2 + 6.039e08s + 8.052e10}{2.061e07s^2 + 6.039e08s + 8.052e10} \tag{3.10}$$

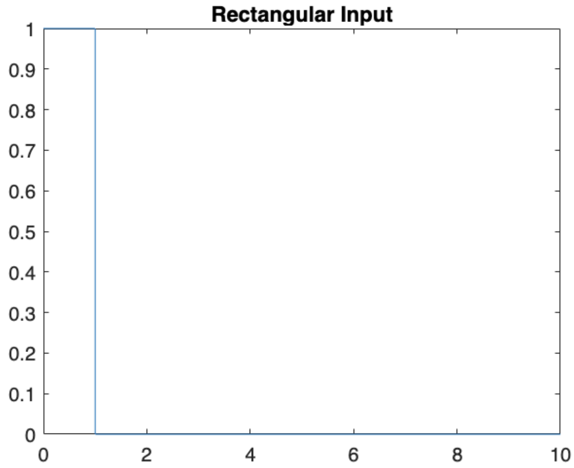
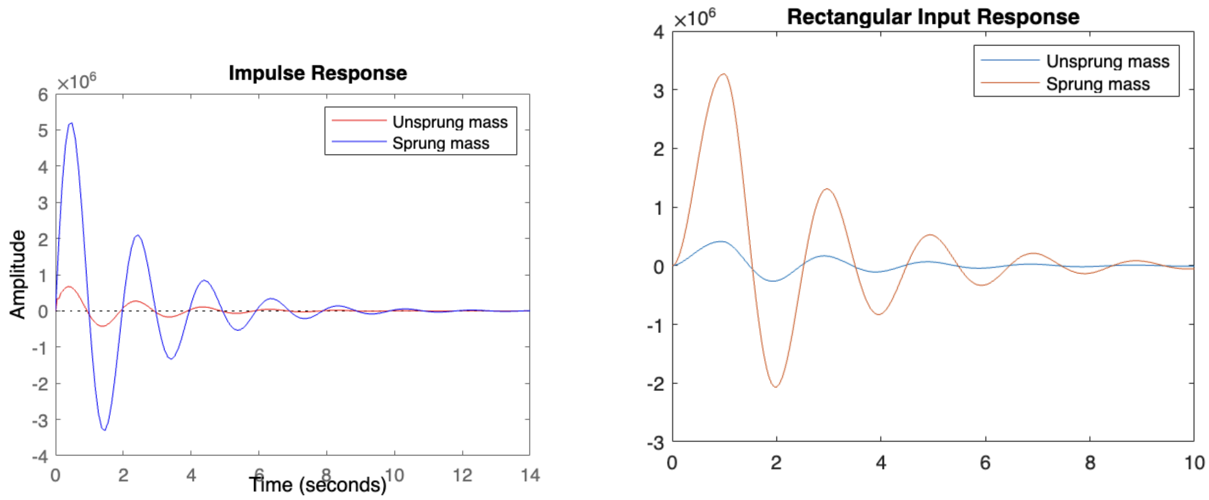


Figure 3.3: Step Input



(a) Passive Suspension Impulse Response   (b) Passive Suspension Step Response

Figure 3.4: Passive Suspension Responses

3.5 Active Suspension

For active suspension, $\dot{X} = AX + Bf(t) + da_y$

From LQR Control, we obtain K. Hence, $f = -KX$. Therefore, $\dot{X} = AX + B(-KX) + da_y = (A - BK)X + da_y$

Hence, $A_{cl} = A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1891.3 & -41.8 & 2568 & 265.1 \\ 0 & 0 & 0 & 1 \\ 63.5 & 1.7 & -104.5 & -10.9 \end{bmatrix}$

The new transfer function will be,

$$\alpha_1(s) = \frac{3.84e04s^2 + 5.464e10s + 5.294e11}{s^4 + 526.8s^3 + 1.996e04s^2 + 3.683e05s + 3.461e06} \tag{3.11}$$

$$\alpha_2(s) = \frac{2.061e07s^2 + 8.621e09s + 3.899e11}{2.061e07s^2 + 8.621e09s + 3.899e11} \tag{3.12}$$

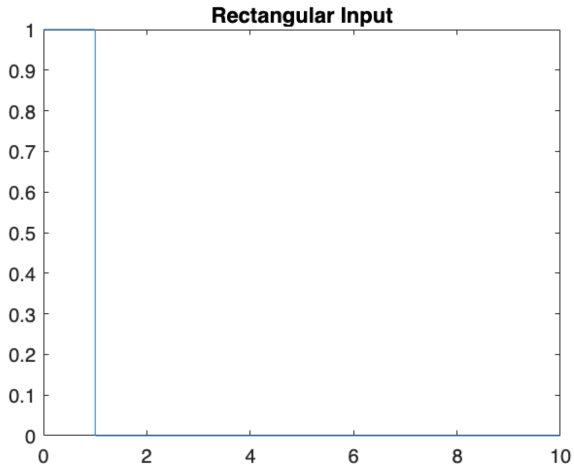
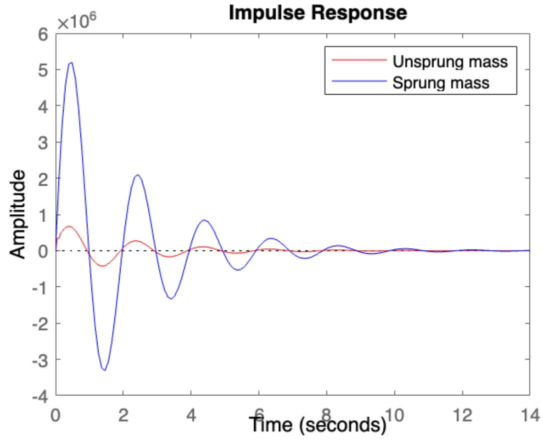
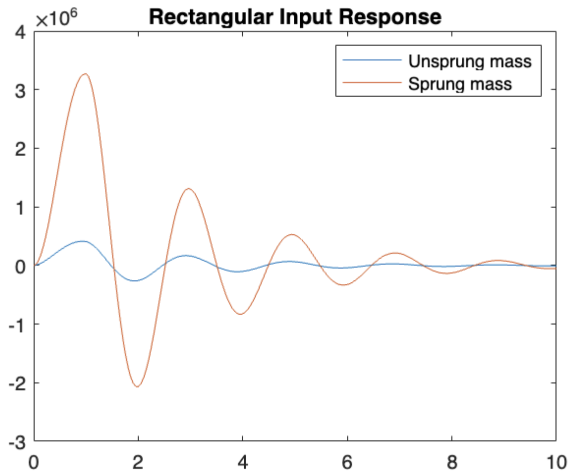


Figure 3.5: Step Input



(a) Active Suspension Impulse Response



(b) Active Suspension Step Response

Figure 3.6: Active Suspension Responses

Conclusion

We have successfully derived the dynamic bicycle model using state space representation and converted it into transfer function. We have also designed a controller for the system using root locus method. We have also seen the effect of longitudinal speed on the system.