Classical Mechanics and Electromagnetic Theory

Unit–II

1 Introduction

Recall that in the previous discussion we have been able to construct an interesting function known as Hamiltonian H using Lagrangian L as,

$$
\begin{aligned}
H &= \sum_{k=1}^{s} \dot{q}_k p_k - L(q_1, q_2, ..., q_k, ..., q_s; \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., \dot{q}_s; t) \\
&= H(q_1, q_2, ..., q_k, ..., q_s; \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., \dot{q}_s; t) \qquad (1.1)
\end{aligned}
$$

Clearly this is also a function of all the generalised co-ordinates q_k's and their time derivates $\dot{q}_k$'s. Now since we also know that we could define a canonical conjugate momenta p_k corresponding to each generalised q_k as

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = f(\dot{q}_k) \qquad (1.2)$$

for example for systems where V is a function of generalised co-ordinates only, $p_k = m\dot{q}_k$. This therefore implies that $\dot{q}_k$ could be scaled to be rewritten as p_k, and hence, the Hamiltonian becomes,

$$H(q_1, q_2, ..., q_k, ..., q_s; \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., \dot{q}_s; t) = H(q_1, q_2, ..., q_k, ..., q_s; p_1, p_2, ..., p_k, ..., p_s; t) \qquad (1.3)$$

Notice that here all p_k's will also be treated independently and therefore should be treated similat to q_k's. Now since the Hamiltonian does not involve any time derivative, equations of motion in terms of Hamiltonian are expected to be first-order differential equations in time unlike Lagrange equations of motion.

2 Hamilton's equation of Motion using Variational Principle

Since for monogenic systems,

$$\sum_{k=1}^{s} \dot{q}_k p_k - L\left(\{q_k\}; \{\dot{q}_k\}; t\right) \qquad (2.1)$$

According to variational method, the system will travel along a path for which the action integral,

$$
\begin{aligned}
\int_{t_1}^{t_2} \delta \left(L\left(\{q_k\};\{\dot{q}_k\};t\right)\right)\ dt = 0 = \int_{t_1}^{t_2} \delta \left(\sum_{k=1}^{s} \dot{q}_k p_k - H\left(\{q_k\};\{p_k\};t\right)\right)\ dt \\
\Longrightarrow \int_{t_1}^{t_2} \delta \left(\sum_{k=1}^{s} \dot{q}_k p_k - L\left(\{q_k\};\{\dot{q}_k\};t\right) - H\left(\{q_k\};\{p_k\};t\right)\right)\ dt \quad = \quad 0
\end{aligned}
\tag{2.2}
$$

As $dt \neq 0$, hence,

$$
\begin{aligned}
\delta \left(\sum_{k=1}^{s} \dot{q}_k p_k - L\left(\{q_k\};\{\dot{q}_k\};t\right) - H\left(\{q_k\};\{p_k\};t\right)\right) &= 0 \\
\text{or} \qquad & \\
\sum_{k=1}^{s} \delta\left(\dot{q}_k p_k\right) - \delta L\left(\{q_k\};\{\dot{q}_k\};t\right) - \delta H\left(\{q_k\};\{p_k\};t\right) &= 0 \qquad (2.3)
\end{aligned}
$$

The variation in Hamiltonian and Lagrangian are given as,

$$
\delta H\left(\{q_k\};\{p_k\};t\right) = \sum_{k=1}^{s} \left(\frac{\partial H}{\partial q_k}\delta q_k + \frac{\partial H}{\partial p_k}\delta p_k \right) + \frac{\partial H}{\partial t}\delta t \tag{2.4}
$$

and

$$
\begin{aligned}
\delta L\left(\{q_k\};\{\dot{q}_k\};t\right) &= \sum_{k=1}^{s} \left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k \right) + \frac{\partial L}{\partial t}\delta t \\
&= \sum_{k=1}^{s} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k \right] + \frac{\partial L}{\partial t}\delta t \\
&= \sum_{k=1}^{s} \left[\frac{d}{dt}\left(p_k\right)\delta q_k + p_k\delta \dot{q}_k \right] + \frac{\partial L}{\partial t}\delta t \\
&= \sum_{k=1}^{s} \left(\dot{p}_k\delta q_k + p_k\delta \dot{q}_k \right) + \frac{\partial L}{\partial t}\delta t \qquad (2.5)
\end{aligned}
$$

respectively, where in the second line we have used the Lagrange equations of motion $\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$ and in the third line we have used $\frac{\partial L}{\partial \dot{q}_k} = p_k$.

Using (2.4), (2.5) and $\delta\left(\dot{q}_k p_k\right) = \dot{q}_k\delta p_k + p_k\delta\dot{q}_k$ in (2.3), we write,

$$
\begin{aligned}
\sum_{k=1}^{s} \left[\dot{q}_k\delta p_k + p_k\delta\dot{q}_k - \left(\frac{\partial H}{\partial q_k}\delta q_k + \frac{\partial H}{\partial p_k}\delta p_k \right) - \left(\dot{p}_k\delta q_k + p_k\delta\dot{q}_k\right) \right] - \left(\frac{\partial L}{\partial t}\delta t + \frac{\partial H}{\partial t}\delta t \right) &= 0 \\
or \qquad & \\
\sum_{k=1}^{s} \left[\left(\dot{q}_k - \frac{\partial H}{\partial q_k} \right)\delta q_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right)\delta q_k \right] - \left(\frac{\partial L}{\partial t} + \frac{\partial H}{\partial t} \right)\delta t = 0 \quad (2.6)
\end{aligned}
$$

Since all the 2s+1 dimensions are independents and are non-zero, i.e. $p_k, q_k, t \neq 0$, all the brackets in the above equation will be independently zero. Therefore we obtain

$$\boxed{\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} \\ \frac{\partial L}{\partial t} &= -\frac{\partial H}{\partial t} \end{aligned}}$$

as the 2s+1 equations of motion which are known as Hamilton's equations of motion. It is to be noted that as expected these equations are first-order differential equation in time.

3 Steps to Construct Hamiltonian

- First construct Lagrangian $L(\{q_k\};\{\dot{q}_k\};t)$ by writting kinetic energy for each degrees-of freedom and potential of the system.
- Obtain canonical conjugate momenta $p_k = \frac{\partial L}{\partial \dot{q}_k}$ corresponding to all q_k. These will give the relations between $\dot{q}_k$ and p_k.
- Find out the Hamiltonian using $H(\{q_k\};\{\dot{q}_k\};t) = \sum_k \dot{q}_k p_k - L(\{q_k\};\{\dot{q}_k\};t)$.
- Transform $\dot{q}_k$ into p_k using the transformation laws given by the relations between $\dot{q}_k$ and p_k. This will finally give $H(\{q_k\};\{p_k\};t)$.

4 Conservation Laws Revisited

4.1 Conservation of Linear Momentum

If H is indepedendent of q_k, momentum corresponding to q_k will remain conserved.

Proof: As H is independent of q_k, therefore $\frac{\partial H}{\partial q_k} = 0$.

Using this is Hamilton's second equation, we write,

$$\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial q_k} = 0 \\ \implies p_k &= \text{constant} \end{aligned} \tag{4.1}$$

4.2 Conservation of Energy

If H is indepedendent of t, Total energy of the system will remain conserved.

Proof: As H is independent of t, therefore $\frac{\partial H}{\partial t} = 0$.

Therefore, the net rate of change of Hamiltonian with respect to time,

$$\frac{dH}{dt} = \sum_k \left(\frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right) + \frac{\partial H}{\partial t} \tag{4.2}$$

But according to Hamilton's equations, $\dfrac{\partial H}{\partial q_k} = -\dot{p}_k$ and $\dfrac{\partial H}{\partial p_k} = \dot{q}_k$ therefore, using these in the above equation, we obtain,

$$\begin{aligned} \frac{dH}{dt} &= \sum_k (-\dot{p}_k \dot{q}_k + \dot{q}_k \dot{p}_k) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t} = 0 \\ \Longrightarrow H &= \text{constant} \end{aligned} \tag{4.3}$$

Hence Energy of the system is conserved.

4.3 Conservation of Angular Momentum

If the Hamiltonian of the system is independent of rotation of the system around some axis, angular momentum of the system remains conserved with respect to that axis.

Suppose the object is rotating around an axis specified by $\hat{n}$ (see fig 1). Suppose q_j is a generalised co-ordinate which represents roatation and suppose the position of the ith particle at some instant is represented by the position vector $\vec{r}_i(q_j)$. Due to change of q_j to $q_j + dq_j$ let the new position becomes $\vec{r}_i(q_j + dq_j)$. The infinitesimal change in the position vector due to change in q_j is therefore,

$$\delta \vec{r}_i = (\hat{n} \times \vec{r}_i) \delta q_j, \tag{4.4}$$

$$\delta \vec{p}_i = (\hat{n} \times \vec{p}_i) \delta q_j \tag{4.5}$$

Thus the change in Lagrangian $H(r_i, p_i, t)$ due to change in q_j is

$$\delta H = \sum_{i=1}^{N} \left[\left(\frac{\partial H}{\partial r_i} \right) \left(\frac{dr_i}{dq_j} \right) \delta q_j + \left(\frac{\partial H}{\partial p_i} \right) \left(\frac{dp_i}{dq_j} \right) \delta q_j \right] \tag{4.6}$$

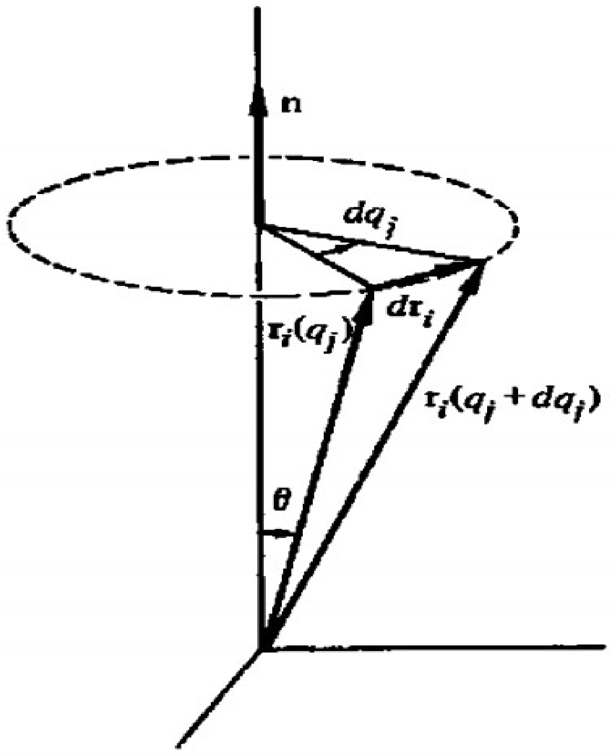

Figure 1: Change in position vector due to rotational motion.

But according to Hamilton equation of motion,

$$\frac{\partial H}{\partial r_i} = -\dot{p}_i, \tag{4.7}$$
$$\frac{\partial H}{\partial p_i} = -\dot{r}_i \tag{4.8}$$
$$\tag{4.9}$$

Using eqn (4.5), (4.6), (4.8) and (4.9) in (4.7), we write,

$$
\begin{aligned}
\delta H &= \sum_{i=1}^{N}\left[-\dot{\vec{p}}_i\cdot(\hat{n}\times\vec{r}_i)\delta q_j+\dot{\vec{r}}_i\cdot(\hat{n}\times\vec{p}_i)\delta q_j\right] \\
&= \sum_{i=1}^{N}\left[-\hat{n}\cdot(\vec{r}_i\times\dot{\vec{p}}_i)-\hat{n}\cdot(\dot{\vec{r}}_i\times\vec{p}_i)\right]\delta q_j \\
&= -\sum_{i=1}^{N}\left[\hat{n}\cdot\frac{d}{dt}(\vec{r}_i\times\vec{p}_i)\right]\delta q_j \\
&= -\sum_{i=1}^{N}\left(\hat{n}\cdot\frac{d\vec{\tau}_i}{dt}\right)\delta q_j \\
&= -\left(\hat{n}\cdot\frac{d\sum_{i=1}^{N}\vec{\tau}_i}{dt}\right)\delta q_j \\
&= -\left(\hat{n}\cdot\frac{d\vec{\tau}}{dt}\right)\delta q_j \\
&= -\frac{d\tau_n}{dt}\delta q_j
\end{aligned} \tag{4.10}
$$

where, $\vec{\tau}_i=\vec{r}_i\times\vec{p}_i$ is the angular momentum vector of i^{th} particle. $\vec{\tau}=\sum_{i=1}^{N}\vec{\tau}_i$ is th toal angular momentum vector of the system and $\tau_n=\hat{n}\cdot\vec{\tau}$ is the total angular momentum of the sytem with respect to $\hat{n}$ axis.

Now as the Hamiltonian does not change due to change in q_j i.e. for $\delta q_j\neq 0$, $\delta L=0$, therefore,

$$
\begin{aligned}
\frac{d\tau_n}{dt} &= 0 \\
\tau_n &= \text{constant}
\end{aligned} \tag{4.11}
$$

Hence angular momentum of the system is constant.

5 Applications of Hamiltonian Mechanics

5.1 One dimensional Harmonic Osciallator

Recall the Lagangian for HO of length l and mass m

$$L(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta \tag{5.1}$$

The canonical conjugate momentum corresponding to θ is

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \\ \Longrightarrow \dot{\theta} &= \frac{p_\theta}{ml^2} \end{aligned} \tag{5.2}$$

Therefore Hamiltonian of the Harmonic oscillator is

$$\begin{aligned} H &= \dot{\theta} p_\theta - L \\ &= \dot{\theta}\left(ml^2\dot{\theta}\right) + \left(\frac{1}{2} ml^2 \dot{\theta}^2 + mgl\cos\theta\right) \\ &= ml^2\dot{\theta}^2 - \left(\frac{1}{2} ml^2\dot{\theta}^2 + mgl\cos\theta\right) \\ &= \frac{1}{2} ml^2 \dot{\theta}^2 - mgl\cos\theta \\ &= \frac{p_\theta^2}{2ml^2} - mgl\cos\theta = T + V \end{aligned} \tag{5.3}$$

The equations of motion are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Longrightarrow \dot{p}_\theta = -mgl\sin\theta \tag{5.4}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Longrightarrow \dot{\theta} = \frac{p_\theta}{ml^2} \tag{5.5}$$

5.2 The moving charge in an electromagnetic field

Recall the Lagrangian for the particle of mass m, charge q, moving with a velocity $\vec{v}$ in an electromagnetic-field specified by scalar and vector potentials ϕ and $\vec{A}$ respectively,

$$L = T - V = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\left(\phi - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z\right) \tag{5.6}$$

Therefore the canonical conjugate momenta corresponding to x, y, z are,

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x, \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z \end{aligned} \tag{5.7}$$

Notice here that the canonical conjugate momentum is not just mass times speed, rather it has some contribution due to the effects of field as well. The above equation thus imply

$$\begin{aligned} \dot{x} &= \frac{p_x - qA_x}{m}, \\ \dot{y} &= \frac{p_y - qA_y}{m}, \\ \dot{z} &= \frac{p_z - qA_z}{m} \end{aligned} \tag{5.8}$$

Therefore the Hamiltonian becomes

$$\begin{aligned} H = \dot{x}p_x + \dot{y}p_z + \dot{z}p_z - L \\ = \dot{x}(m\dot{x} + qA_x) + \dot{y}(m\dot{y} + qA_y) + \dot{z}(m\dot{z} + qA_z) \\ - \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\left(\phi - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z\right)\right] \\ = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + q\phi \\ = \frac{\left(\vec{p} - q\vec{A}\right) \cdot \left(\vec{p} - q\vec{A}\right)}{2m} + q\phi \end{aligned} \tag{5.9}$$

6 The Central Force Problem

Here we will discuss problems involving potentials which only depends upon positions only as we may be able to obtain a complete solution of these.

6.0.1 Construction of Lagrangian

In order to construct the Lagrangian of the problem, let us consider two mass points m_1 and m_2 having their positions specified by vectors $\vec{r}_1$ and $\vec{r}_2$ as shown in Fig 2, Therefore the Lagrangian for the system is

$$L = T - V = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|) \tag{6.1}$$

This problem could be reduced to one body problem in the Centre-of-mass

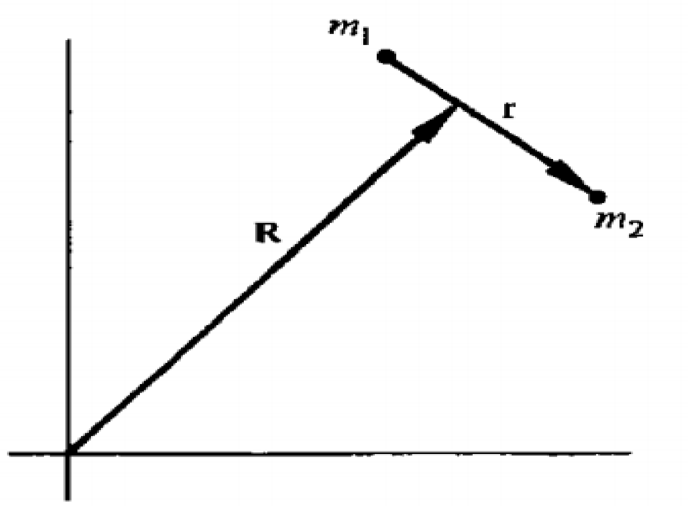

Figure 2: The two body problem. (PC: Goldstein)

frame defined by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{6.2}$$

with new co-ordinates with respect to CM being,

$$\vec{r}\,'_{1,2} = \vec{r}_{1,2} - \vec{R} \tag{6.3}$$

Clearly the difference of positions of the particles in the Lab frame and CM frame will remain same, i.e.

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{6.4}$$

$$= \vec{r}\,'_1 - \vec{r}\,'_2 = \vec{r}\,'. \tag{6.5}$$

Multiplying (6.4) by m_2 and adding it into (6.2) and similarly multiplying (6.4) by m_1 and substracting it from (6.2) we obtain $\vec{r}_{1,2}$ in terms of $\vec{r}$ and $\vec{R}$ as,

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2}\vec{r} \tag{6.6}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2}\vec{r} \tag{6.7}$$

respectively. Using these in (6.3), we get,

$$\vec{r}\,'_1 = \frac{m_2}{m_1+m_2}\vec{r} = \frac{\mu}{m_1}\vec{r} \tag{6.8}$$

$$\vec{r}\,'_2 = -\frac{m_1}{m_1+m_2}\vec{r} = -\frac{\mu}{m_2}\vec{r} \tag{6.9}$$

with

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{6.10}$$

$$M = m_1 + m_2 \tag{6.11}$$

μ being the reduced mass and M combined mass of the system.

Using (6.6) and (6.7) in (6.1) we could transform Lagrangian in CM frame as follows,

$$L = \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{\mu}{m_1}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{\mu}{m_2}\dot{\vec{r}}\right)^2 - V(|\vec{r}|) \tag{6.12}$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\dot{\vec{r}}^2 - V(r)$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \tag{6.13}$$

Since velocity of the CM $\dot{\vec{R}}$ is contant, therefore, the first term in the Lagrangian will be constant. Thus using the property that equations of motion do not change under the addition or substraction of constant terms in the Lagrangian, we may simply drop it and we therefore finally obtain, the Lagrangian to be,

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r)$$

$$= \frac{1}{2}\mu\left[\dot{r}^2 + (r\dot{\theta})^2\right] - V(r) \tag{6.14}$$

where we have used the fact that $\delta\vec{r} = r\hat{r} + r\delta\theta\hat{\theta} \underset{\rightarrow}{+} \delta z\hat{z}$ and assumed to be around the z-axis, so that $\delta z = 0$ and hence, $\delta r = r\hat{r} + r\delta\theta\hat{\theta}$ Eqn (6.14) is the desired Lagrangian for the central force problems.

6.0.2 Construction of Hamiltonian

Since the canonical conjugate momenta corresponding to generalised co-ordinates r and θ as,

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r} \tag{6.15}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} \tag{6.16}$$

The Hamiltonian of the system will therefore be,

$$\begin{aligned} H &= \dot{r} p_r + \dot{\theta} p_\theta - L \\ &= \frac{1}{2}\mu \left[\dot{r}^2 + (r\dot{\theta})^2 \right] + V(r) \\ &= \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + V(r) \end{aligned} \tag{6.17}$$

6.0.3 Symmetries and Conservation Theorems of Hamiltonian

As θ is a cyclic coordinate, p_θ will be conserved, i.e.

$$p_\theta = \mu r^2 \dot{\theta} = \text{ constant } = l(say) \tag{6.18}$$

Also, as the Hamiltonian does not explicitly depends upon time, hence total energy of the system is constant i.e.

$$\begin{aligned} H &= \frac{p_r^2}{2\mu} + \frac{l^2}{2\mu r^2} + V(r) = \text{ constant } = E(say) \\ \Longrightarrow \frac{p_r^2}{2\mu} &+ V_{eff} = E \end{aligned} \tag{6.19}$$

with

$$V_{eff} = V(r) + \frac{l^2}{2\mu r^2} \tag{6.20}$$

is known as the effective potential of the system.

6.0.4 Equation of Orbits

Using equation (6.17), we obtain the following equations of motion,

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Longrightarrow \dot{p}_\theta = 0 \Longrightarrow p_\theta = constant = l \tag{6.21}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Longrightarrow \dot{\theta} = \frac{p_\theta}{\mu r^2} = \frac{l}{\mu r^2} \tag{6.22}$$

$$\Longrightarrow \frac{d}{dt} = \left(\frac{l}{\mu r^2} \right) \frac{d}{d\theta} \tag{6.23}$$

$$
\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} \implies \dot{r} = \frac{p_r}{\mu} = \frac{l}{\mu} && (6.24)\\
\dot{p}_r &= -\frac{\partial H}{\partial r} \implies \dot{p}_r = \frac{p_r^2}{\mu r^3} - \frac{\partial V}{\partial r} = \frac{l^2}{\mu r^3} + F(r) && (6.25)
\end{aligned}
$$

or using $p_r = \mu\dot{r}$ from eqn (6.24), (6.25) could be written as,

$$
\mu\ddot{r} - \frac{l^2}{\mu r^3} = F(r) \qquad (6.26)
$$

Furthermore, using (6.23), the above equation could be written as ***differential equation of orbit*** as,

$$
\mu\left(\frac{l}{\mu r^2}\right)\frac{d}{d\theta}\left\{\left(\frac{l}{\mu r^2}\right)\frac{dr}{d\theta}\right\} - \frac{l^2}{\mu r^3} = F(r) \qquad (6.27)
$$

Or substituting $u = 1/r$ and $du = -1/r^2 dr$, we write,

$$
\begin{aligned}
-\frac{l^2}{\mu}u^2\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) - \frac{l^2}{\mu}u^3 &= F(\frac{1}{u})\\
\text{or} \quad \frac{d^2u}{d\theta^2} + u &= -\left(\frac{\mu}{l^2u^2}\right)F(\frac{1}{u}) && (6.28)\\
\text{or} \quad \frac{d^2u}{d\theta^2} + u &= -\left(\frac{\mu}{l^2}\right)\frac{\partial}{\partial u}\left\{V(\frac{1}{u})\right\} && (6.29)
\end{aligned}
$$

This is second order differential equation of orbits. Alternatively, using $p_r = \mu\dot{r}$ in (6.19), we may write,

$$
\begin{aligned}
\frac{1}{2}\mu\dot{r}^2 + V_{eff} &= E && (6.30)\\
\text{or} \quad \int_{t_0}^{t} dt &= \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left[E - V_{eff}(r)\right]}} && (6.31)
\end{aligned}
$$

or using (6.22), we may obtain the ***intergal equation of orbit*** as,

$$
\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{l dr/r^2}{\sqrt{2\mu\left[E - V_{eff}(r)\right]}} \qquad (6.32)
$$

or substituting $u = 1/r$ and $du = -1/r^2 dr$ and using (6.20), we get,

$$
\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu V}{l^2} - u^2}} \qquad (6.33)
$$

Now considering a general form of power law,

$$V \propto r^{n+1} \tag{6.34}$$

it could be shown that eqn. (6.33) is integrable for very limited values of n, which are $n = 1, -2, -3$ for trignometric functions and $n = 5, 3, 0, -4, -5, -7$ for elliptic functions respectively.

6.1 Nature of Orbits

Recall that we have

$$\begin{aligned} V_{eff}(r) &= V(r) + \frac{l^2}{2\mu r^2} && (6.35)\\ \implies F_{eff}(r) &= F(r) + \frac{l^2}{\mu r^3} && (6.36)\\ \text{and } E &= V_{eff}(r) + \frac{1}{2}\mu\dot{r}^2 && (6.37) \end{aligned}$$

where $F_{eff}(r) = -\frac{\partial V_{eff}}{\partial r}$ and $F(r) = -\frac{\partial V}{\partial r}$.
Thus depending upon the type of $V(r)$, the orbits may or may not be closed. This is discussed for inverse-square law $V = -\frac{k}{r}$, using Figs. 3, 4, 5, 6, a fourth power attractive force $V = -\frac{k}{r^4}$ Fig. 7 and a linear restoring force $V = \frac{1}{2}kr^2$as Fig. 8.

6.2 Stability analysis: Condition for closed orbit

For a closed orbit corresponding to an attractive potential V_{eff} need to extremum. However for an orbit to be stable the potential must have a minima, i.e. for an orbit to be stable the following conditions must satisfy at $r = r_0$

$$F_{eff}(r_0) = \left.\frac{\partial V_{eff}}{\partial r}\right|_{r_0} = 0 \implies F(r_0) = -\frac{l^2}{\mu r_0^3} \tag{6.38}$$

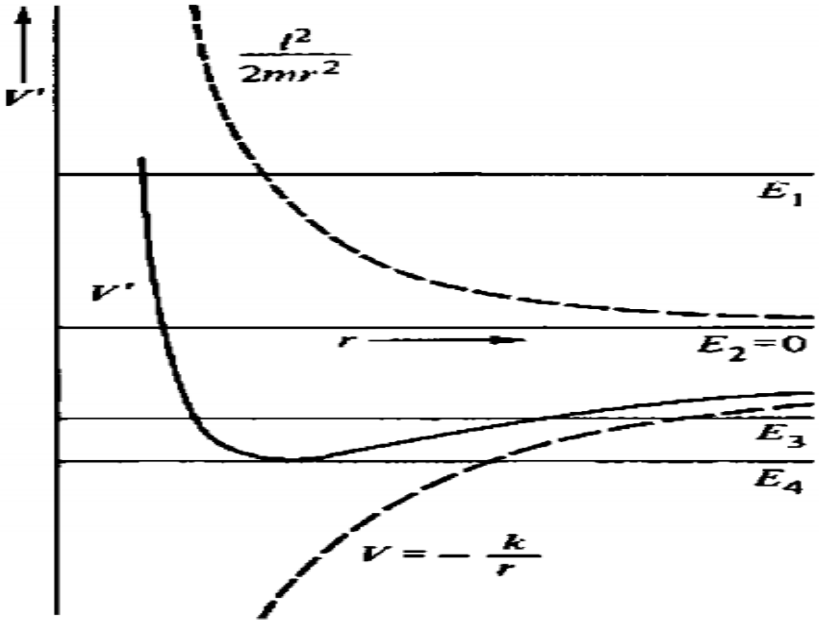

Figure 3: Inverse-square potential. (PC: Goldstein)

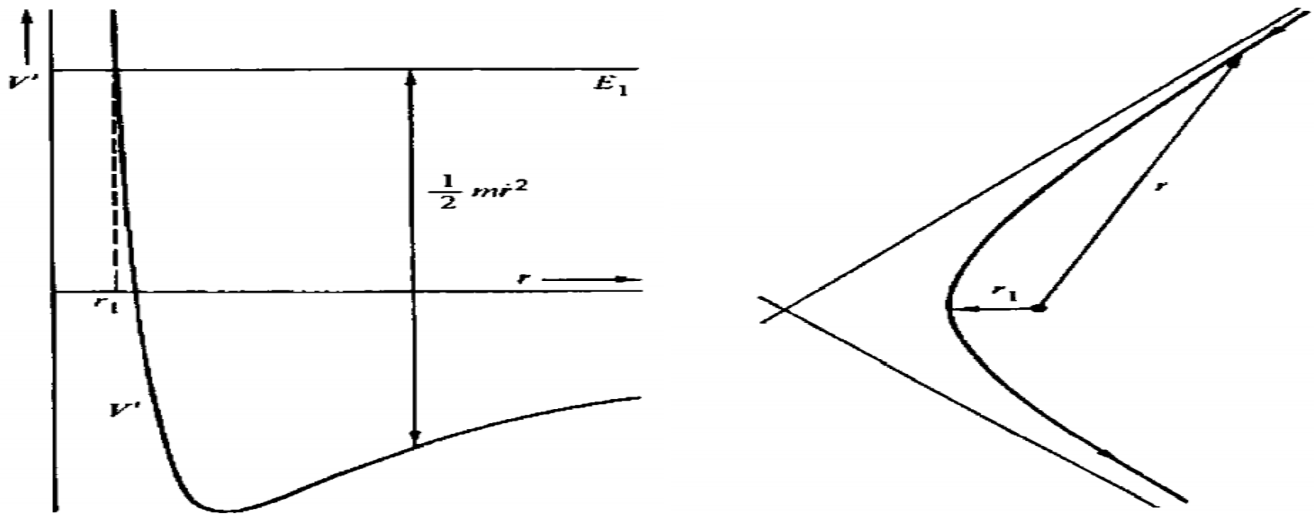

Figure 4: Inverse-square potential and expected shape of orbits for $E = E_1 > V$. (PC: Goldstein)

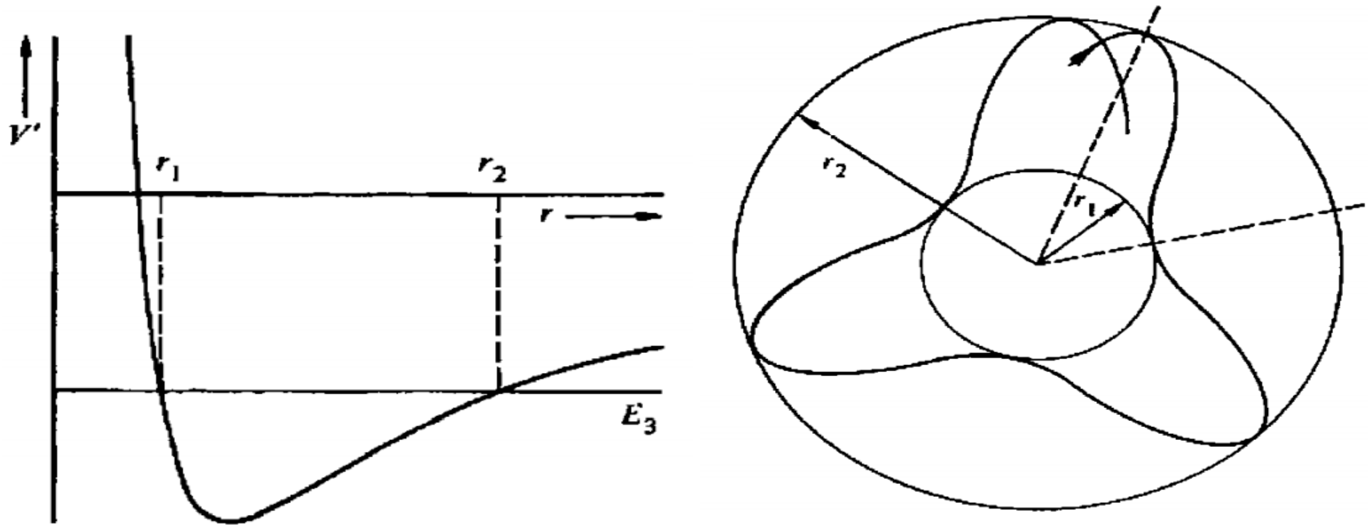

Figure 5: Inverse-square potential and expected shape of orbits for $E = E_3 < V$. (PC: Goldstein)

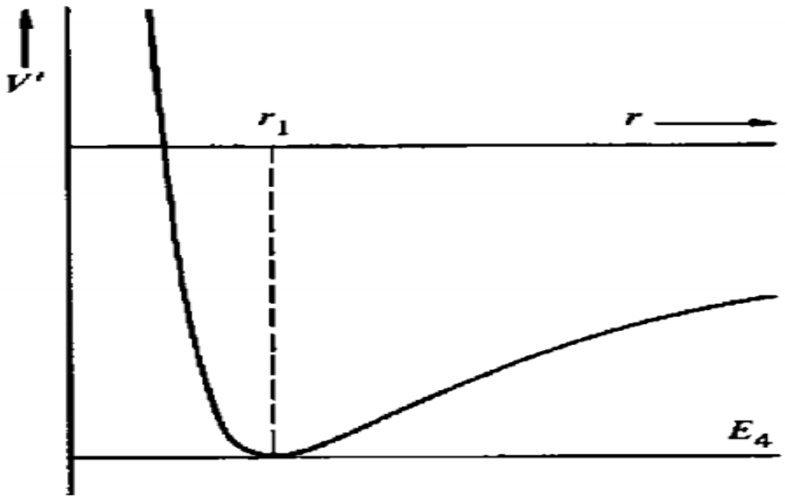

Figure 6: Inverse-square potential for $E_4 < E$.

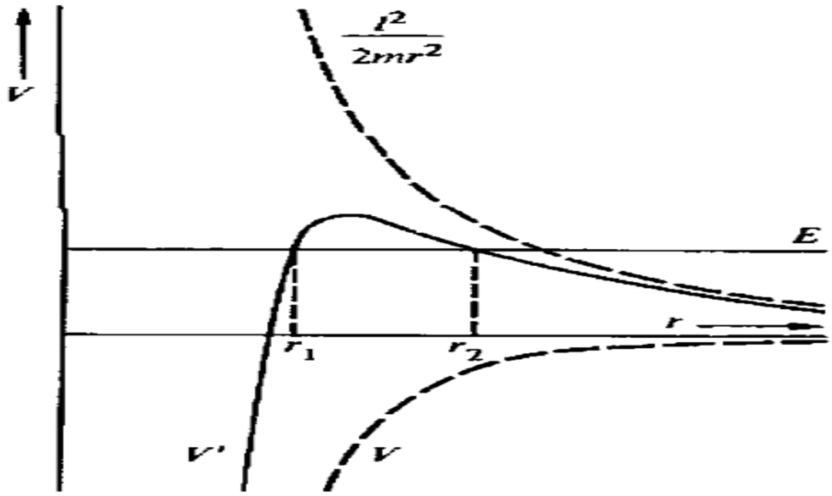

Figure 7: Fourth-power potential $V = -\frac{k}{r^4}$. (PC: Goldstein)

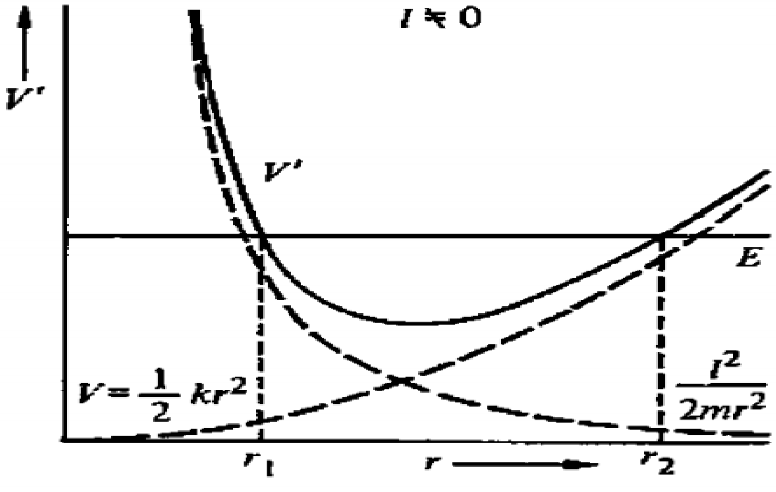

Figure 8: Linear restoring force, $V = \frac{1}{2}kr^2$. (PC: Goldstein)

and

$$
\begin{aligned}
-\frac{\partial F_{eff}}{\partial r}\bigg|_{r_0} &= \frac{\partial^2 V_{eff}}{\partial r^2}\bigg|_{r_0} > 0 && (6.39)\\
\implies -\frac{\partial F}{\partial r}\bigg|_{r_0} + \frac{3l^2}{\mu r_0^4} &> 0 && (6.40)\\
\implies \frac{\partial F}{\partial r}\bigg|_{r_0} &< \frac{3l^2}{\mu r_0^4} \\
\implies \frac{\partial F}{\partial r}\bigg|_{r_0} &< -\frac{3F(r_0)}{r_0} && (6.41)\\
\implies \frac{d\ln F}{d\ln r}\bigg|_{r_0} &> -3 && (6.42)
\end{aligned}
$$

For $F = -kr^n$ (6.40) gives

$$
\begin{aligned}
-nkr_0^{n-1} &< 3kr_0^{n-1}\\
\implies n &> -3 \qquad (6.43)
\end{aligned}
$$

The above condition is also referred as *Bertnard Theorem.*

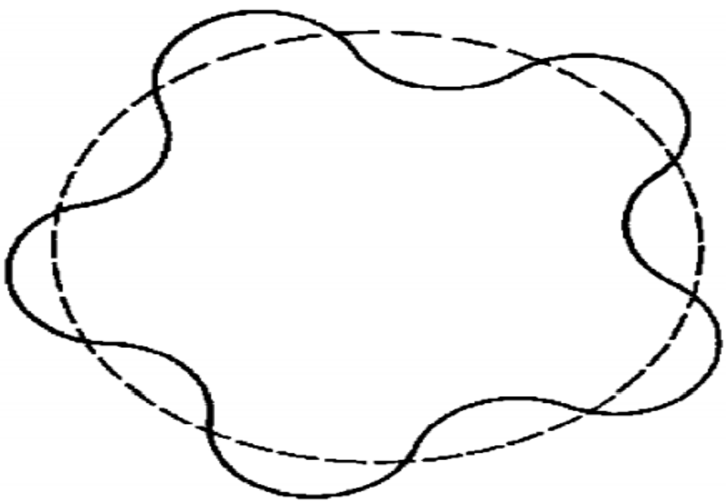

Figure 9: Shapes of a circular orbits for $\beta = 5$.(PC: Goldstein)

Rewritting (6.41) we may

$$
\frac{d\ln F}{d\ln r}\bigg|_{r_0} + 3 > 0 = \beta^2(say) \qquad (6.44)
$$

We obtain the force law as,

$$
F(r) \propto -\frac{1}{r^{3-\beta^2}} \qquad (6.45)
$$

and the equation of orbit as,

$$u = u_0 + acos\beta\theta \tag{6.46}$$

Clearly β represents how many loops the closed (circular) orbit has. This is clearly shown in Figure 9.

6.3 Virial Theorem

This theorem is useful in estimating the average kinetic energy of a system. This could be derived as follows,

Consider a N-particle system with forces given by $\dot{\vec{p}_i} = \vec{F}_i$. Now assume th following function

$$\begin{aligned} G &= \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i \qquad (6.47) \\ \implies \frac{dG}{dt} &= \sum_{i=1}^{N} \vec{p}_i \cdot \dot{\vec{r}_i} + \sum_{i=1}^{N} \dot{\vec{p}_i} \cdot \vec{r}_i \\ &= \sum_{i=1}^{N} \vec{p}_i \cdot \frac{\vec{p}_i}{m_i} + \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i \\ &= 2\sum_{i=1}^{N} T_i + \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i \\ &= 2T + \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i \qquad (6.48) \end{aligned}$$

The time avergae of the above equation over the whole period τ will therefore be,

$$\begin{aligned} \frac{1}{\tau}\overline{\frac{dG}{dt}} &= \frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt} dt \\ &= 2\overline{T} + \overline{\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i} \\ \frac{1}{\tau}\left(G(\tau) - G(0)\right) &= 2\overline{T} + \overline{\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i} \qquad (6.49) \end{aligned}$$

For a closed orbit, i.e. for $G(\tau) = G(0)$ the above equation gives,

$$
\begin{aligned}
2\overline{T} + \overline{\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i} &= 0 \\
\overline{T} &= -\frac{1}{2}\overline{\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i} \qquad (6.50)
\end{aligned}
$$

The above equation is known as virial theorem.

6.3.1 Application 1 (Conservative Forces)

For conservative forces,

$$
\begin{aligned}
\vec{F}_i &= -\vec{\nabla}_i V \\
\Longrightarrow \overline{T} &= \frac{1}{2}\overline{\sum_{i=1}^{N} \vec{\nabla}_i V \cdot \vec{r}_i} \\
\Longrightarrow \overline{T} &= \frac{1}{2}\overline{r\frac{\partial V}{\partial r}} \qquad (6.51)
\end{aligned}
$$

For $V \propto r^{n+1}$ gives,

$$
\begin{aligned}
\overline{T} &= \frac{1}{2} r \frac{\partial V}{\partial r} \\
&\propto r(n+1)r^{n} \\
&\propto (n+1)r^{n+1} \\
&= \frac{n+1}{2}\overline{V} \qquad (6.52)
\end{aligned}
$$

This for inverse square law, $V = -kr^{-1}$ gives, $n = -2$ and hence,

$$
T = -\frac{1}{2}\overline{V}.
$$

6.3.2 Application 2 (Equation of State of a Gas)

Consider the Gas of N particle occupies a volume V and has a pressure P at temperature T. Therefore,

$$
\overline{T} = \frac{3}{2}NkT \qquad (6.53)
$$

Now the force by the i^{th} particle on an area δA normal to the surface

$$
\begin{aligned}
\delta\vec{F}_i &= -P\delta\vec{A} = -P\delta A\hat{n} \\
\Longrightarrow \frac{1}{2}\vec{F}_i \cdot \vec{r}_i &= \frac{1}{2}\int \delta\vec{F}_i \cdot \vec{r}_i \\
&= -\frac{P}{2}\int \hat{n} \cdot \vec{r}_i dA
\end{aligned}
$$

Applying Gauss's theorem the above integral becomes,

$$\frac{1}{2}\vec{F}_i \cdot \vec{r}_i = -\frac{P}{2}\int \vec{\nabla} \cdot \vec{r} dV = -\frac{3PV}{2} \tag{6.54}$$

Thus using virial theorem

$$\begin{aligned} \overline{T} &= -\overline{\frac{1}{2}\vec{F}_i \cdot \vec{r}_i} \\ \frac{3}{2}NkT &= \frac{3PV}{2} \\ \Longrightarrow PV = NkT \end{aligned} \tag{6.55}$$

7 The Keplar's Problem

In Kepler's problem, $V = -\frac{k}{r}$ with $k = GM_S m$ where M_S and m represent mass of the Sun and Planet respectively which are at a distance r from each other. All the result of the Central-problem will be valid in this case. Let us discuss Keplar's laws one by one.

7.1 First Law

Using $V = -k/r = -ku$ in the integral equation of orbit (6.33) we write,

$$\begin{aligned}
\theta - \theta_0 &= -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu V}{l^2} - u^2}} \\
&= -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \left[\frac{2\mu k}{l^2}u - u^2\right]}} \\
&= -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \left[\left(\frac{\mu k}{l^2}\right)^2 - \left(u - \frac{\mu k}{l^2}\right)^2\right]}} \\
&= -\int_{u_0}^{u} \frac{du}{\sqrt{\left[\sqrt{\frac{2\mu E}{l^2} + \left(\frac{\mu k}{l^2}\right)^2}\right]^2 - \left(u - \frac{\mu k}{l^2}\right)^2}} \\
&= \cos^{-1}\left[\frac{\left(u - \frac{\mu k}{l^2}\right)}{\sqrt{\frac{2\mu E}{l^2} + \left(\frac{\mu k}{l^2}\right)^2}}\right] \\
&= \cos^{-1}\left[\frac{\frac{l^2}{\mu k}u - 1}{\sqrt{\frac{2El^2}{\mu k^2} + 1}}\right] \\
&= \cos^{-1}\left[\frac{\frac{\left(\frac{l^2}{\mu k}\right)}{r} - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}}\right] \\
\implies \frac{\left(\frac{l^2}{\mu k}\right)}{r} &= 1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos(\theta - \theta_0)
\end{aligned}$$

$$\frac{L}{r} = 1 + \epsilon\cos(\theta - \theta_0) \tag{7.1}$$

which is the equation of an ellipse with Latus rectum L and eccentricity ϵ given by,

$$L = \frac{l^2}{\mu k} = \frac{b^2}{a} \tag{7.2}$$

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} = \sqrt{1 - \frac{b^2}{a^2}} < 0 \quad (\text{as} < 0)$$

$$\implies \frac{2El^2}{\mu k^2} = -\frac{b^2}{a^2} \tag{7.3}$$

where a and b represent semi-major and semi-minor axis of the ellipse. Thus

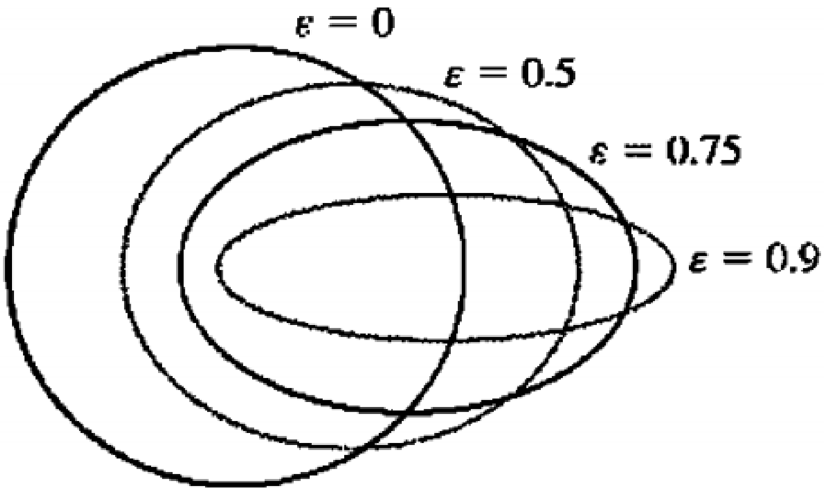

Figure 10: Shape of closed orbits for inverse-square law case with varying eccentricity. (PC: Goldstein)

the plant orbits around the Sun in an elliptical orbit as shown in Fig 10. Which is Kepler's first law.

Eliminating l^2 between (7.2) and (7.3) we obtain,

$$E = -\frac{k}{2a} = -\frac{GM_S m}{2a} < 0 \tag{7.4}$$

7.2 Second Law

Recall the Hamiltonian

$$H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + V(r) \tag{7.5}$$

As θ is cyclic, therefore,

$$p_\theta = \text{constant} \implies \mu r^2\dot{\theta} = l(say) \tag{7.6}$$

Now the areal velocity,

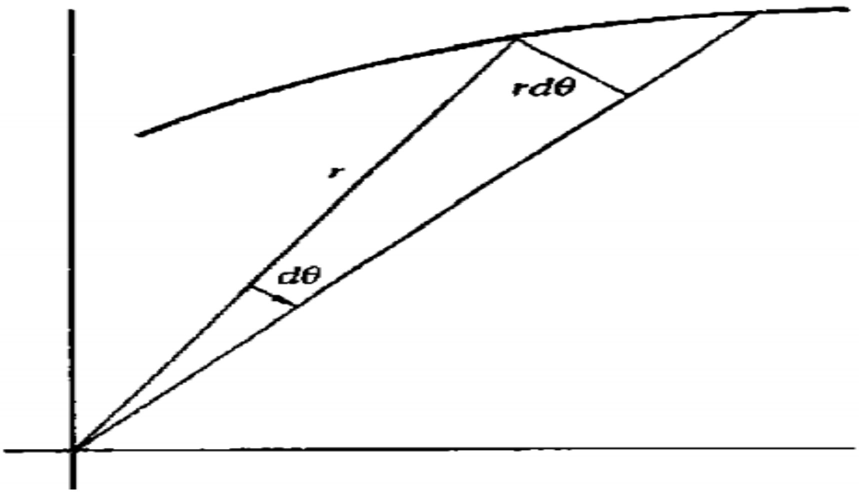

Figure 11: Area swept by a planet around the Sun in infinitesimal time. (PC: Goldstein)

$$\begin{aligned} \frac{dA}{dt} &= \lim_{\delta t \to 0} \frac{\delta A}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{1}{2} r \cdot r \frac{\delta \theta}{\delta t} \\ &= \frac{1}{2} r^2 \dot{\theta} \\ &= \frac{l}{2\mu} = \text{constant} \end{aligned} \tag{7.7}$$

Thus the areal velocity remains constant, which is the second law.

7.3 Third Law

Assuming τ to be the time period of the planet, by integrating eqn(7.3) with respect to time over one time period, we could calculate area of the orbit as,

$$\begin{aligned} A &= \int_0^\tau \frac{dA}{dt} dt \\ &= \frac{l}{2\mu} \int_0^\tau dt \\ &= \frac{l}{2\mu} \tau \end{aligned} \tag{7.8}$$

but since the orbit is an ellipse, its area $A = \pi ab$ where a and $b = a\sqrt{1-\epsilon^2}$ represent the semi-major and semi-minor axis respectively and epsilon is the eccentricity of the elipse. Therefore the above equation yields,

$$\begin{aligned}
\pi ab &= \frac{l}{2\mu}\tau \\
\Longrightarrow \tau &= 2\pi ab\frac{\mu}{l} \\
&= 2\pi a\sqrt{(l^2/\mu k)\,a}\frac{\mu}{l} \\
&= 2\pi a^{3/2}\sqrt{\frac{\mu}{k}} \\
&= 2\pi a^{3/2}\sqrt{\frac{\mu}{GM_S m}} \\
&= 2\pi\frac{a^{3/2}}{\sqrt{G(M_S+m)}} \\
&= 2\pi\frac{a^{3/2}}{\sqrt{GM}} \\
&\propto a^{3/2} \qquad (7.9)
\end{aligned}$$

where $M = M_S + m \simeq M_S$. This proves the third law.

8 The Scattering Problem

Scattering is the phenomenon of influencing one object by another. This may or may not be due to direct hitting.

The phenomenon of scattering is also a central force phenomenon involing two bodies namely the target and the incident beam called projectile. The equivalent one body problem is the scattering of particles by a centre of force. To formulate this let us consider a uniform beam of intensity I having particles of same mass and energy is incident upon a centre of force. The force is assumed to be vanishing at large distances so as to ensure that that the centre-of-force does not influence the particles at large distances. As a particles approaches the

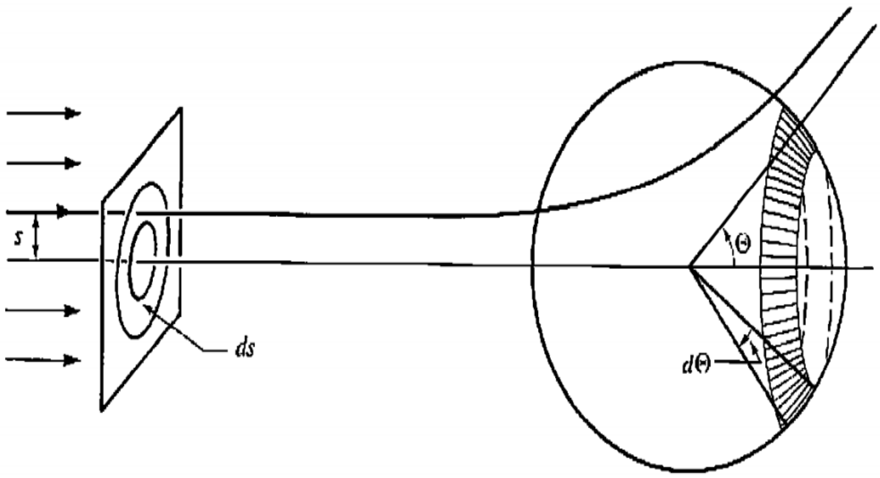

Figure 12: Scattering Phenomenon in Lab frame. (PC: Goldstein)

cente-of-force, it will deviate from its incident (straight line) trajectory due to attraction or repulsion. Once it has crossed the centre-of-force, the trajectories will again return back to straight line though the direction will now be different from the direction of the incident beam which will be determined by the nature and stregth of the centre-of-force.

The scattering cross-section (or differential cross-section) is given by the following formula

$$\sigma(\Omega)d\Omega = \frac{\text{Number of particles scattered in solid angle } d\Omega \text{ per unit time}}{\text{Incident Intensity}} \tag{8.1}$$

where $d\Omega = 2\pi d\cos\Theta = 2\pi \sin\Theta d\Theta$ is the line element of solid angle Ω in its direction and incident intensity I is the number of incident particles per unit area normal to the incident beam per unit time. The angle Θ is the angle between the direction of incident beam to the scattered beam and is therefore reffered as the *scattering angle*.

The total cross-section could be obtained by integrating the differential cross-section with respect to whole solid angle, i.e.

$$\sigma_T = \int_\Omega \sigma(\Omega) d\Omega = 2\pi \int \sigma(\Theta) \sin\Theta d\Theta \tag{8.2}$$

Assume that a beam of incident particles at a distance s normal to the centre-of-force and incident velocity and thickness ds scatters into the solid angle $d\Omega$. Since the incident flux through the strip between s and $s + ds$ must be equal to the flux of the scattered beam between Ω and $Omega + d\Omega$, i.e.

$$\begin{aligned} 2\pi s|ds|I &= \sigma(\Omega) d\Omega I \\ &= 2\pi \sin\Theta d\Theta \sigma(\Theta) I \\ \implies \sigma(\Theta) &= \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \end{aligned} \tag{8.3}$$

If m, v_0 and E are the mass, velocity and energy of the particles of incident beam, then angular momentum,

$$l = mv_0 s = s\sqrt{2mE} = \text{constant} \tag{8.4}$$

The distance s is also referred as the impact parameter and is obviously given by

$$s = s(\Theta, E) \tag{8.5}$$

Clearly our aim must to first find the relation (8.5), then to obtain the differential cross-section using equation (8.3) and finally the total cross-section through the equation (8.2).

9 Ruthorford Scattering

In this case the target is a nucleus and the projectile is a beam of α- particles. The force between them is given by $F(r) = \frac{k}{r^2}$ with $k = -\frac{(Ze)(Z'e)}{4\pi\epsilon_0}$, where Ze and $Z'e$ represent charges of target and the projectile.

This problem could be analogous to the Kepler's Problem except for the fact that now instead the particle is bound to move in a closed orbit between r_1 and r_2, it starts at a distance $r_1 = r_m$ and then flies off to infinity, i.e. $r_2 = \infty$. The scattering angle Θ could be carefully thought to be related with the orbital angle Ψ as shown in Figure 13.

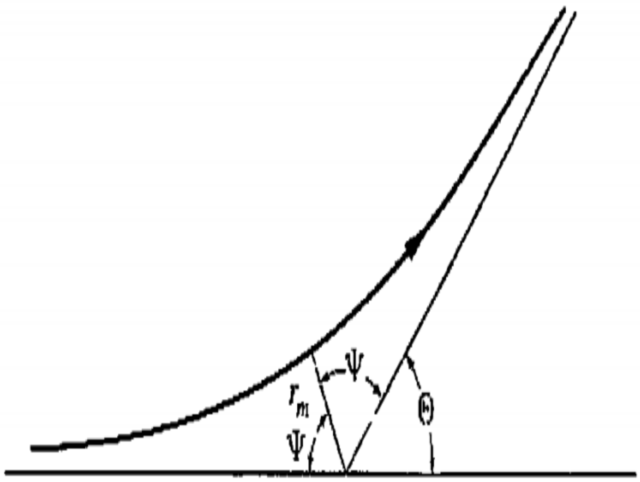

Figure 13: Display of various angles used in scattering phenomenon. (PC: Goldstein)

$$\begin{aligned} \Psi + \Psi + \Theta &= \pi \\ \implies \Theta &= \pi - 2\Psi \end{aligned} \tag{9.1}$$

Thus we may have using the integral equation of motion (6.32),

$$\int_{\pi-\Psi}^{\pi} d\theta = \int_{r_m}^{\infty} \frac{dr/r^2}{\sqrt{2\mu E/l^2 - 2\mu V/l^2 - 1/r^2}} \tag{9.2}$$

Putting $l = s\sqrt{2\mu E}$,

$$\begin{aligned} \Psi &= \int_{r_m}^{\infty} \frac{dr/r^2}{\sqrt{(1 - V/E)s^2 - 1/r^2}} \\ &= \int_{r_m}^{\infty} \frac{sdr/r}{\sqrt{r^2(1 - V/E) - s^2}} \end{aligned} \tag{9.3}$$

Putting the value of Ψ from above equation in (9.1) and using $r = 1/u$,

$$\Theta = \pi - 2\int_{0}^{u_m} \frac{sdu}{\sqrt{(1 - V/E) - s^2u^2}} \tag{9.4}$$

Thus anologous to Kepler's problem, this will also yield the following equation of orbit,

$$\begin{aligned} -\frac{\frac{l^2}{\mu k}}{r} &= 1 + \epsilon\cos(\pi - \theta) \\ \implies \frac{\frac{4\pi\epsilon_0 l^2}{\mu ZZ'e^2}}{r} &= \epsilon\cos\theta - 1 \end{aligned} \tag{9.5}$$

with eccentricity

$$\begin{aligned}
\epsilon &= \sqrt{1+\frac{2El^2}{\mu k^1}} \\
&= \sqrt{1+\frac{2El^2}{\mu(ZZ'e^2/4\pi\epsilon_0)^2}} \\
&= \sqrt{1+\left(\frac{8\pi\epsilon_0 Es}{ZZ'e^2}\right)^2} > 1 \qquad (9.6)
\end{aligned}$$

As for $r\to\infty$, $\theta\to\Psi$, therefore, the equation of orbit becomes,

$$\begin{aligned}
\epsilon\cos\Psi - 1 &= 0 \\
\implies \cos\Psi &= \frac{1}{\epsilon} \\
\implies \sec\Psi &= \epsilon \\
\implies \tan\Psi &= \sqrt{\sec^2\Psi - 1} = \sqrt{\epsilon^2-1} = \frac{2Es}{k} \\
\implies s &= \frac{k}{2E}\tan\Psi \\
\text{or } s &= \frac{k}{2E}\cot\left(\frac{\Theta}{2}\right) \qquad (9.7)
\end{aligned}$$

and hence

$$\left|\frac{ds}{d\Theta}\right| = \frac{k}{E}cosec^2\left(\frac{\Theta}{2}\right) \qquad (9.8)$$

Thus,

$$\begin{aligned}
\sigma(\Theta) &= \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \\
&= \frac{\frac{k}{2E}\cot\left(\frac{\Theta}{2}\right)}{2\sin(\Theta/2)\cos(\Theta/2)} \times \frac{k}{E}cosec^2\left(\frac{\Theta}{2}\right) \\
&= \left(\frac{k}{2E}\right)^2 cosec^4\left(\frac{\Theta}{2}\right) \\
&= \left(\frac{ZZ'e^2}{8\pi\epsilon_0 E}\right)^2 cosec^4\left(\frac{\Theta}{2}\right) \qquad (9.9)
\end{aligned}$$

Finally the total cross-section

$$\begin{aligned}
\sigma_T &= 2\pi\int\sigma(\Theta)\sin\Theta d\Theta \\
&= 4\pi\left(\frac{ZZ'e^2}{8\pi\epsilon_0 E}\right)^2\int_0^\pi cosec^4\left(\frac{\Theta}{2}\right)\sin(\Theta/2)\cos(\Theta/2)\,d\Theta \\
&= 8\pi\left(\frac{ZZ'e^2}{8\pi\epsilon_0 E}\right)^2\int_0^{\frac{\pi}{2}} cosec^2\alpha\cot\alpha d\alpha \qquad (9.10)
\end{aligned}$$

where we have used $\Theta/2 = \alpha$. Integrating equation (9.10) we obtain,

$$\sigma_T = \infty \tag{9.11}$$

This makes sense as the coulamb interaction is a long range force, i.e. it vanishes at infinity.

10 Transformation of Scattering Phenomenon into the Lab Frame

Although we have solved the scattering phenomenon into the cm frame, we need to revert back to the lab frame as ultimately we would be observing it in the lab frame. To do so, first let us try to understand how the scattering angle in the cm frame Θ relates with the scattering angle in the lab frame θ.

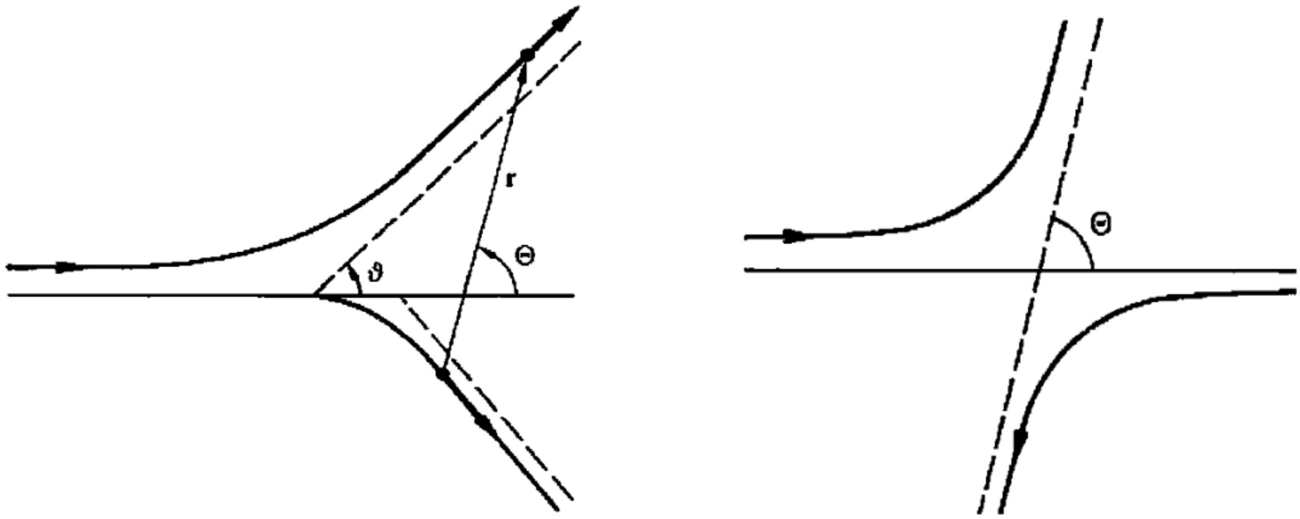

Figure 14: Scattering Phenomenon when viewed from (a) Lab frame, (b) Centre-of-mass frame.(PC: Goldstein)

It is to be noted that in the CM frame the target and projectile moves in opposite direction and the scattering angle between them is obtained with respect to the direction of the incident beam to the scattered beam. In the Lab frame, if the target is stationary, $\Theta = \theta$. However since the target which was initially at rest starts moving due to scattering, it will be different as shown in the figure 14(a). In order to understand this let us first find the transformation equations between the CM and Lab frames for the position and velocities.

To do so, let us assume that m_1 and m_2 are the massess of the incident particle and the target respectively. Also let $\vec{v}_0$ and $\vec{v}$ represent velocity of the incident particle and relative velocity before scattering in the *Lab frame*, $\vec{r}_1, \vec{v}_1$ represent the position and velocity of the incident particle after scattering in the *Lab* *frame* $\vec{r}_1', \vec{v}_1'$ represent the position and velocity of the incident particle after scattering, in the *CM frame* $\vec{R}, \vec{V}$ represent the position and velocity of the centre-of-mass. Therefore,

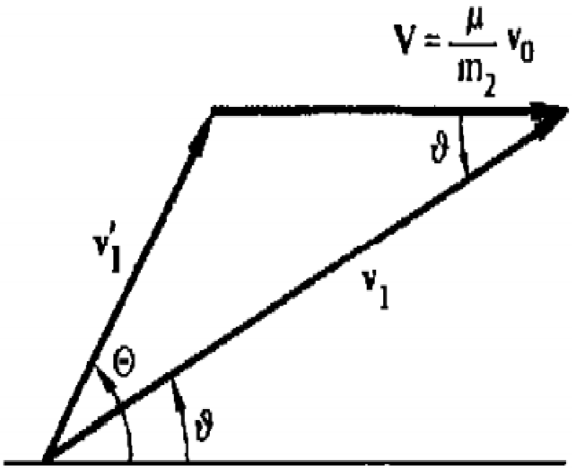

Figure 15: Relating scattering phenomenon in Lab frame with centre-of-mass frame.(PC: Goldstein)

$$\begin{aligned} \vec{r}_1' &= \vec{r}_1 - \vec{R} && (10.1) \\ \vec{v}_1' &= \vec{v}_1 - \vec{V} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1 + m_2} = \frac{\mu}{m_1}\vec{v} && (10.2) \end{aligned}$$

$$\begin{aligned} m_1\vec{v}_0 + m_2 \times \vec{0} &= (m_1 + m_2)\vec{V} \\ \implies \vec{V} &= \frac{m_1}{m_1 + m_2}\vec{v}_0 = \frac{\mu}{m_2}\vec{v}_0 && (10.3) \end{aligned}$$

Also resolving the components of velocities along horizontal and normal directions in Figure 15, we write,

$$\begin{aligned} v_1 \cos\theta &= {v'}_1 \cos\Theta + V && (10.4) \\ v_1 \sin\theta &= {v'}_1 \sin\Theta && (10.5) \end{aligned}$$

Squaring (10.5) and (10.6) and adding and then taking square-root,

$$\begin{aligned} v_1 &= \sqrt{{v_1'}^2 + 2Vv_1' \cos\Theta + V^2} \\ &= v_1'\sqrt{1 + 2\rho\cos\Theta + \rho^2} && (10.6) \end{aligned}$$

with

$$\begin{aligned}
\rho &= \frac{V}{v_1'} \\
&= \left(\frac{\mu}{m_2}\right)\left(\frac{v_0}{v_1'}\right) \qquad \text{using (10.3)} \\
&= \left(\frac{m_1}{m_2}\right)\left(\frac{v_0}{v}\right) \qquad \text{using (10.2)}
\end{aligned} \tag{10.7}$$

Using the value of v_1 from (10.7) in (10.5) we get,

$$\cos\theta = \frac{\cos\Theta + \rho}{\sqrt{1 + 2\rho\cos\Theta + \rho^2}} \tag{10.8}$$

Dividing (10.6) by (10.5) we get,

$$\tan\theta = \frac{\sin\Theta}{\cos\Theta + \rho} \tag{10.9}$$

Since the flux in both the frames should remain the same, therefore,

$$\begin{aligned}
I\sigma(\Omega)|d\Omega| &= I\sigma_l(\Omega_l)|d\Omega_l| \\
2\pi I\sigma(\Theta)|\sin\Theta d\Theta| &= 2\pi I\sigma(\theta)|\sin\theta d\theta| \\
\implies \sigma_l(\theta) &= \sigma(\Theta)\left|\frac{\sin\Theta d\Theta}{\sin\theta d\theta}\right| \\
\implies \sigma_l(\theta) &= \sigma(\Theta)\left|\frac{d\cos\Theta}{d\cos\theta}\right|
\end{aligned} \tag{10.10}$$

Differentiating (10.8) with respect to $\cos\Theta$,

$$\begin{aligned}
\frac{d\cos\theta}{d\cos\Theta} &= \frac{1}{(1 + 2\rho\cos\Theta + \rho^2)^{1/2}} - \frac{(\cos\Theta + \rho)\rho}{(1 + 2\rho\cos\Theta + \rho^2)^{3/2}} \\
&= \frac{1 + \rho\cos\Theta}{(1 + 2\rho\cos\Theta + \rho^2)^{3/2}}
\end{aligned}$$

Hence

$$\frac{d\cos\Theta}{d\cos\theta} = \frac{(1 + 2\rho\cos\Theta + \rho^2)^{3/2}}{1 + \rho\cos\Theta} \tag{10.11}$$

Using this in (10.10) we get,

$$\sigma_l(\theta) = \sigma(\Theta)\left|\frac{(1 + 2\rho\cos\Theta + \rho^2)^{3/2}}{1 + \rho\cos\Theta}\right| \tag{10.12}$$

10.1 Eleastic and Inelastic Scattering

Scattering phenomenon is of two types – one in which the total kinetic energy of the projectile and target remain unaltered is called *elastic scattering* whereas the one in which the total kinetic energy is changed is known as *inelastic scattering.*

Inelastic Scattering

Let us now examine the case of inelastic scattering first.

Suppose the energy of the incident particle in the lab frame is $E = \frac{1}{2}m_1v_0^2$, and Q is amount of enery lost during scattering therefore,

$$\begin{aligned}
\frac{1}{2}\mu v_0^2 &= \frac{1}{2}\mu v^2 + Q \\
\frac{v}{v_0} &= \sqrt{1 + \frac{2Q}{\mu v_0^2}} \\
&= \sqrt{1 + \frac{2Q}{\mu(2E/m_1)}} \\
&= \sqrt{1 + \frac{Q}{m_2/(m_1+m_2)E}} \\
&= \sqrt{1 + \left(1 + \frac{m_2}{m_1}\right)\frac{Q}{E}} \qquad (10.13)
\end{aligned}$$

Using this in (10.7), we write,

$$\rho = \frac{m_1}{m_2\sqrt{1 + \left(1 + \frac{m_2}{m_1}\right)\frac{Q}{E}}} \qquad (10.14)$$

Finally using (10.7), we write,

$$v_1' = \frac{\mu}{m_2\rho}v_0 \qquad (10.15)$$

Using this in (10.6) we get,

$$\frac{v_1}{v_0} = \left(\frac{\mu}{m_2\rho}\right)\sqrt{1 + 2\rho\cos\Theta + \rho^2} \qquad (10.16)$$

Therefore the ratio of kinetic energies of the incident particle after and before scattering in the lab frame will be

$$\begin{aligned}
\frac{E_1}{E_0} &= \left(\frac{v_1}{v_0}\right)^2 \\
&= \left(\frac{\mu}{m_2\rho}\right)^2 (1 + 2\rho\cos\Theta + \rho^2) \qquad (10.17)
\end{aligned}$$

Elastic Scattering

As the energy is neither gained nor lost during the elastic scattering, the results for elastic scattering could be obtained by setting $Q = 0$ in the above equations. Thr value of ρ would be,

$$\rho = \frac{m_1}{m_2} \tag{10.18}$$

and therefore the reduced mass could be written as,

$$\begin{aligned} \mu &= \frac{m_1 m_2}{m_1 + m_2} \\ &= \frac{m_2}{1 + m_2/m_1} \\ &= \frac{m_2}{1 + 1/\rho} \\ &= \frac{m_2 \rho}{1 + \rho} \end{aligned} \tag{10.19}$$

Therefore,

$$\begin{aligned} \frac{E_1}{E_0} &= \left(\frac{\mu}{m_2 \rho}\right)^2 (1 + 2\rho \cos\Theta + \rho^2) \\ &= \frac{1 + 2\rho \cos\Theta + \rho^2}{(1+\rho)^2} \end{aligned} \tag{10.20}$$

Case 1: If the particles are identical then $m_1 = m_2$, therefore,

$$\rho = 1 \tag{10.21}$$

This means

$$\begin{aligned} \cos\theta &= \frac{\cos\Theta + 1}{1 + 2\cos\Theta + 1} \\ &= \frac{\cos\Theta + 1}{2(1 + \cos\Theta)} \\ &= \sqrt{\frac{\cos\Theta + 1}{2}} \\ &= \cos(\Theta/2) \\ \implies \theta &= \frac{\Theta}{2} \end{aligned} \tag{10.22}$$

Also,

$$\begin{aligned}
\sigma_l(\theta) &= \sigma(\Theta)\frac{(1+2\cos\Theta+1)^{3/2}}{1+\cos\Theta} \\
&= \sigma(\Theta)2^{3/2}(1+\cos\Theta)^{1/2} \\
&= 4\sigma(\Theta)\cos(\Theta/2) \\
&= 4\sigma(\Theta)\cos\theta
\end{aligned} \tag{10.23}$$

Now since $\Theta \leq \pi$ therefore $\theta \leq \frac{\pi}{2}$.

Finally,

$$\begin{aligned}
\frac{E_1}{E_0} &= \frac{1+2\rho\cos\Theta+\rho^2}{(1+\rho)^2} \\
&= \frac{1+\cos\Theta}{2} \\
&= \cos^2(\Theta/2) \\
&= \cos^2\theta \leq 1
\end{aligned} \tag{10.24}$$

Case 2: If the target is much heavier than the projectile i.e. $m_2 >> m_1$, therefore,

$$\rho \simeq 0 \tag{10.25}$$

therefore using (10.8)

$$\begin{aligned}
\cos\theta &\simeq \cos\Theta \\
\Longrightarrow \theta &\simeq \Theta \leq \pi
\end{aligned} \tag{10.26}$$

and using (10.12),

$$\sigma_l(\theta) \simeq 4\,\sigma(\Theta) \tag{10.27}$$

Finally

$$\frac{E_1}{E_0} \simeq 1 \tag{10.28}$$

11 Scattering by a Hard Sphere

Suppose the projectile hits the hard sphere (target) of radius R at a distance s normal to its centre and scatters off at an angle Θ with respect to its direction of incidence.

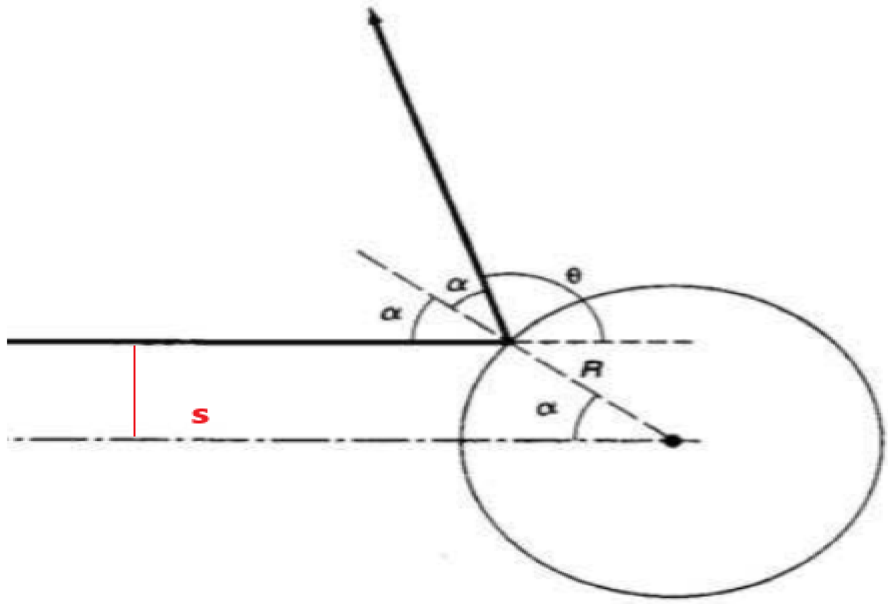

Figure 16: Scattering due to a hard sphere.

Assuming α to be the angle made by a line drawn from the point of scattering to the centre. Therefore from geometry,

$$s = R \sin \alpha \tag{11.1}$$

with

$$\begin{aligned} \alpha + \alpha + \theta &= \pi \\ \implies \alpha &= \pi/2 - \theta/2 \end{aligned} \tag{11.2}$$

Thus s becomes,

$$\begin{aligned} s &= R \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\ &= R \cos \frac{\theta}{2} \end{aligned} \tag{11.3}$$

Differentiating s with respect to θ,

$$\frac{ds}{d\theta} = -\frac{R}{2} \sin \frac{\theta}{2} \tag{11.4}$$

Therefore the differential cross-section,

$$\begin{aligned}\sigma(\theta) &= \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right| \\ &= \frac{R\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\frac{R}{2}\sin\frac{\theta}{2} \\ &= \frac{R^2}{2} \qquad (11.5)\end{aligned}$$

Therefore total cross-section is

$$\begin{aligned}\sigma_T &= 2\pi\int_0^\pi \sigma(\theta)\sin\theta d\theta \\ &= \frac{\pi R^2}{2}\int_0^\pi \sin\theta d\theta \\ &= \pi R^2 \qquad (11.6)\end{aligned}$$

Thus the scattering cross-section would be equal to the area of a circle of the same radius.