Erasmus
School of
Economics

Statistics for Data Science
Lecture 2

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Erasmus University Rotterdam

Before next time

Assignment for next week

- Read
  - Chapter 1 (this week's material)
  - Chapter 2 (next week)
- Try some examples in the book yourself (see here for data and code)
- Finish today's assignments (1.1 - 1.4)
- Continue to practice using own data (or the housing data)
  - Create simple plots
  - Calculate summary statistics
  - Inspect distributions of some variables
    (also consider transformations of variables)
  - Visualize relations between variables
- Optional: Exercise 2 (Volkswagen prices)

Today

- Distributions
- Getting ready for statistical hypothesis testing...
  - Quantifying uncertainty (standard errors)
  - Confidence intervals

Distribution functions

Some theory on distributions

- Distributions describe the probabilities of events related to random variables.
- Many "standard" distributions exist
  - Bernoulli
  - Binomial
  - Negative Binomial
  - Normal
  - Poisson
  - ...
- What is a distribution?
  $\rightarrow$ Mathematical functions to summarize "probabilities" of events

We need to distinguish between

- Discrete random variables
- Continuous random variables

Distributions for discrete random variables

Consider a discrete random variable X

- outcomes in set: eg. $\{0, 1, 2, \ldots, K\}$
  (K may be infinite)
- cumulative distribution function (often written as $F(x)$ or $P(x)$):
  function that gives probability that $X \leq x$ for any x
- probability mass function: (often written as $f(x)$ or $p(x)$)
  function that gives probability that $X = x$ for any x

Note

- X is the random variable
- x is a particular value/outcome

Example: tossing a dice

x=outcome	prob. mass function	cumulative distr. function
0	0	0
1	1/6	1/6
2	1/6	2/6
$\vdots$	$\vdots$	$\vdots$
6	1/6	1

$$\text{Prob. mass function} = \Pr[X = x] = f(x) = \begin{cases} \frac{1}{6} & \text{if } x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Cum. distr. function} = \Pr[X \leq x] = F(x) = \begin{cases} 0 & \text{if } \quad x < 1 \\ \frac{1}{6} & \text{if } 1 \leq x < 2 \\ \frac{2}{6} & \text{if } 2 \leq x < 3 \\ \vdots & \\ 1 & \text{if } 6 \leq x \end{cases}$$

Distributions for continuous random variables

Consider a continuous random variable X

- outcomes in interval (a, b)
  (a may be $-\infty$ and/or b may be $+\infty$)
- cumulative distribution function (cdf):
  function $F(x)$ that gives probability that $X \leq x$ for any x
  (same as before)
- Probability that X equals x (eg. $x = 1.335221$)?
  $\rightarrow$ this equals 0 exactly!
- probability density function (pdf):
  function $f(x)$ such that areas under the curve correspond to probabilities

$$\Pr[a < X \leq b] = \int_a^b f(x)dx = F(b) - F(a)$$

$\rightarrow$ Note: $f(x)$ is the derivative of $F(x)$

Interpretation probability mass/density function

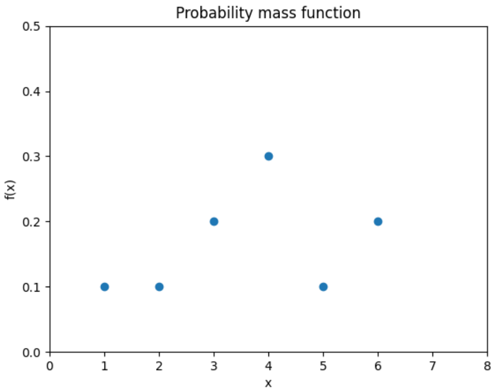
Consider the function at some value x:

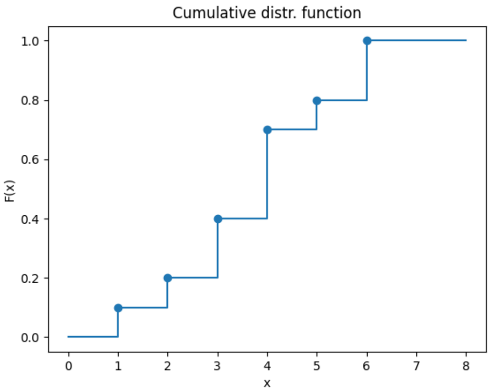
- Discrete variables:
  - Clear interpretation
  - Probability of outcome x

- Continuous variables:
  - No direct formal interpretation
  - "Indication of relative frequency of outcomes *close to* x"
  - Area under curve gives probability

Graphical illustration – Discrete example

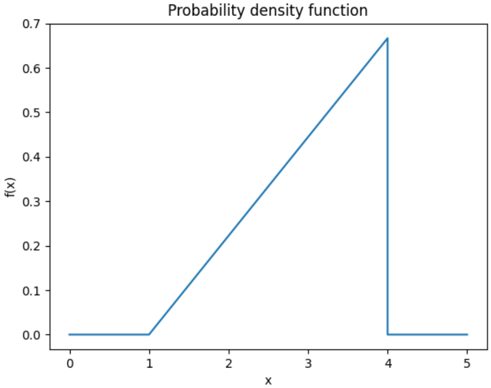
Discrete random variable

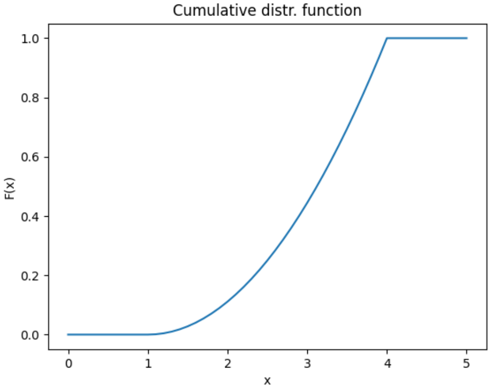




Graphical illustration – Continuous example

Continuous random variable





Graphical illustration – Continuous example

For the previous example we have

$$f(x) = \begin{cases} \frac{2}{9}(x-1) & \text{if } 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{9}(x^2 - 2x + 1) & \text{if } 1 \leq x \leq 4 \\ 1 & \text{if } x > 4 \end{cases}$$

Question/Assignment (Advanced/For fun/At home)

Check that $f(x)$ and $F(x)$ indeed match and that both satisfy the requirements for a density/distribution function

Distribution functions in Python

`scipy.stats` contains many many distributions

- Normal (`norm(..)`)
- Binomial (`binom(..)`)
  $\rightarrow$ number of successes in k trials, if each trial has success probability p
- Exponential (`expon(..)`)
- ...

Standard available functions

- probability density function (method `.pdf()`)
- probability mass function (method `.pmf()`)
- cumulative distribution function (method `.cdf()`)
- quantile function (inverse cdf) (method `.ppf()`=percent point function)
- generate random numbers (method `.rvs()`)

Examples:

Calculate cdf for standard normal distribution at 1:

`scipy.stats.norm.cdf(1, loc=0, scale=1)`

or

`scipy.stats.norm(loc=0, scale=1).cdf(1)` (this makes clear which parameters belong to which part)

If you use `from scipy.stats import norm` you can omit the `scipy.stats` part

Assignment

In-class assignment 2.1

1. Calculate the probability that...
   - a standard normally distributed variable is larger than 1.
   - a normally distributed variable with mean 20 and variance 10 is smaller than 15.
   - we get (exactly) 15 times head in 30 coin tosses.

   Hint: use for example `help(norm.cdf)` and `help(binom.pmf)` to find out how to use these functions
2. Suppose that a soccer club has a 60% probability of winning each match they play. What is the probability that they do **not** win any of the first four matches of the year?
3. Calculate a quantile
   - Suppose that the waiting time for the bus has an exponential distribution with scale 10. How many minutes does one have to wait at least on the 5% worst days?

→ Use inclass_2_1.py (on Canvas) to get started

Special distributions

Standard distributions

We will often use four special distributions

- Normal distribution: $X \sim N(\mu, \sigma^2)$ ($\sim$ means *"has distribution"*)
- F distribution: $X \sim F(d_1, d_2)$
- Chi-squared distribution: $X \sim \chi^2(k)$
- t distribution: $X \sim t_n$

(these 4 distributions are strongly related)

In Python use

- `norm`(μ,σ), `f`(d_1,d_2), `chi2(k)`, `t(n)` from `scipy.stats`

$\rightarrow$ see also scipy documentation, wikipedia and the distribution zoo for more information on distributions (including formulas and references)

Normal distribution

Normal distribution with mean μ and variance σ^2: $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

`scipy.stats.norm(loc=mu,scale=sigma).pdf(x)` and

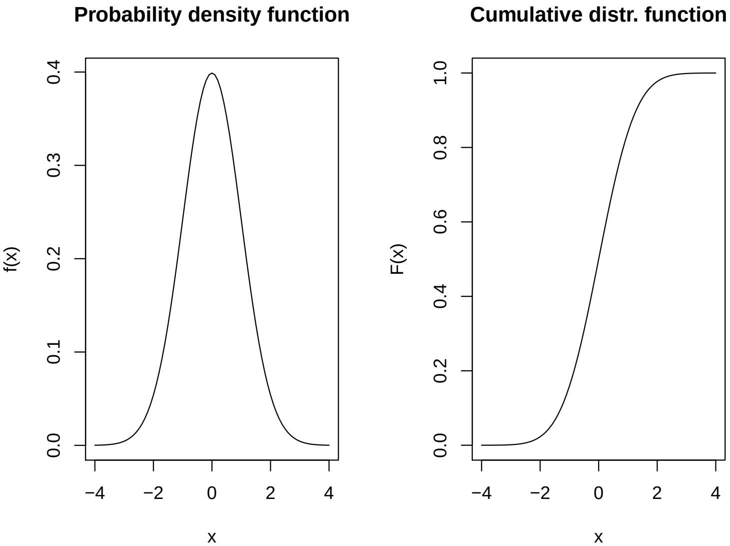
$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(z)dz \\ &= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}\frac{(z-\mu)^2}{\sigma^2})dz \end{aligned}$$

`scipy.stats.norm(loc=mu,scale=sigma).cdf(x)`

If $\mu = 0$, $\sigma = 1$

- standard normal
- Usual notation: $f(x) = \phi(x)$ and $F(x) = \Phi(x)$

Standard normal



Useful properties of the normal

Given $X \sim N(\mu, \sigma^2)$ and a, b numbers (not random)

- $a + X \sim N(a + \mu, \sigma^2)$
- $bX \sim N(b\mu, b^2\sigma^2)$
- $a + bX \sim N(a + b\mu, b^2\sigma^2)$
- $\frac{X-\mu}{\sigma} \sim N(0, 1)$

Given $Y \sim N(\alpha, \nu^2)$ independent of X

- $X + Y \sim N(\mu + \alpha, \sigma^2 + \nu^2)$

Definitions of distributions related to the normal

Suppose

- $X_1, \ldots, X_k \sim N(0,1)$
- $Y_1, \ldots, Y_n \sim N(0,1)$
- All X_i and Y_j independent

We have

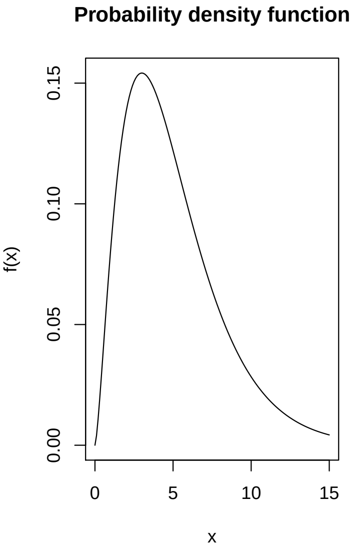
- For "sums of squares": $\sum_{i=1}^k X_i^2 \sim \chi^2(k)$ and $\sum_{i=1}^n Y_i^2 \sim \chi^2(n)$
- Ratio of "mean squares"

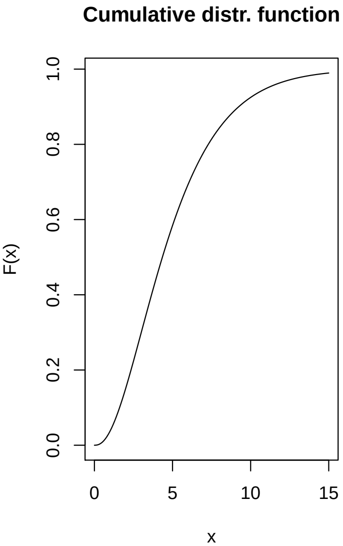
$$\frac{\sum_{i=1}^k X_i^2/k}{\sum_{i=1}^n Y_i^2/n} \sim F(k,n)$$

- Ratio of normal to "root mean squares"

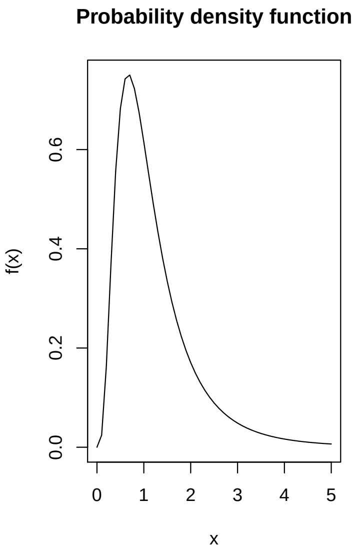
$$\frac{X_i}{\sqrt{\sum_{i=1}^n Y_i^2/n}} \sim t_n$$

Chi-squared(5)

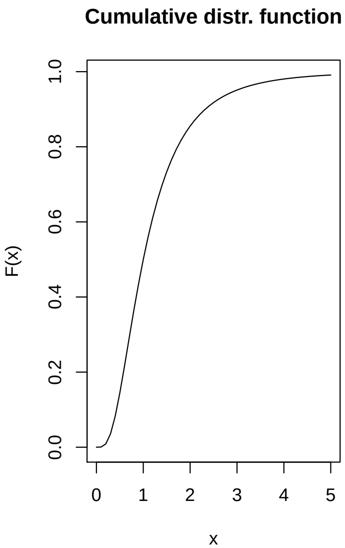




F(10,10)



Probability density function
f(x)
0.0
0.2
0.4
0.6
0 1 2 3 4 5
x



Cumulative distr. function
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0 1 2 3 4 5
x

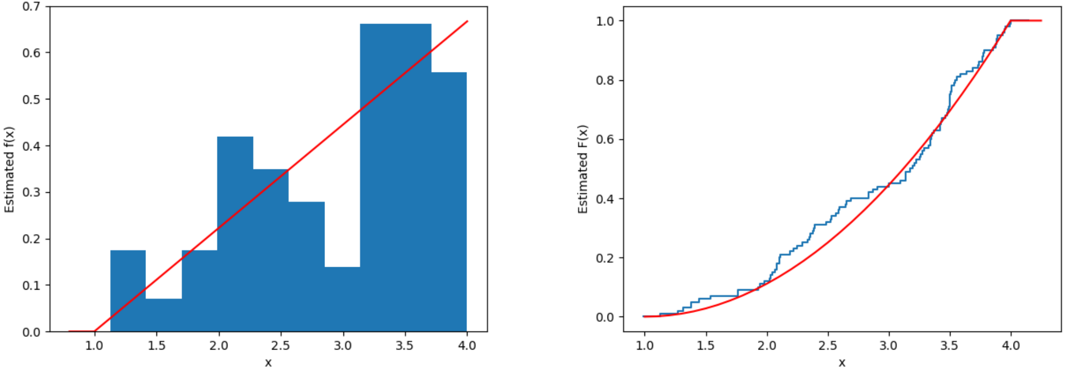
Estimating distributions

Histograms vs. densities

- Histogram $\rightarrow$ (discretized) estimated density (up to scaling)
  - Histogram $=$ estimate
  - Density $=$ population equivalent (often unknown)
- Continuous *density* estimators also exist (have seen this already)
- Can also estimate cumulative distribution function
  - Plot sorted data vs. index/n
    $\rightarrow$ `scipy.stats.ecdf(data).cdf.plot()`
  - Use the `scipy.stats.fit()` functions to fit a specific distribution

Histogram vs. true density (in red)

Given 100 observations:



Fitting distributions

The package `scipy.stats` can be used for

1. Plotting the empirical distribution
2. Fitting a *particular* distribution to data
3. Inspecting the fit
4. Comparing the fit of different distributions

Key functions

- `fit = stats.fit(stats.norm, data, bounds)`
  → fit a normal distribution to the data (or different distributions), parameters are within specified `bounds` → choose these sensibly based on your data.
  For example `bounds = {'loc': (-4,4), 'scale': (0,1)}`
- `fit.plot()` and `fit.plot(plottype=t)` with `t` one of "hist", "qq", "pp", "cdf"
- `fit.nllf()`
  → Negative log-likelihood → measures fit (lower=better)

Assignment

In-class assignment 2.2

1. Look at the example code in `inclass_2_2.py`
2. Load the houseprice data (from week 1)
3. Use `.hist()` to show the empirical density of lotsize
4. Fit a normal ("norm") and a log-normal ("lognorm") distribution to the lotsize and graphically inspect the fit of both
5. Which one fits better on the basis of the graphical inspection?

Quantifying estimation uncertainty

Estimation uncertainty

Last week:

- Moments of random variables (mean, variance, etc)
- Remember that we have
  1. theoretical moments: eg. expected value $\mathrm{E}[X]$
  2. sample moments: eg. sample mean $\frac{1}{n}\sum_i X_i$

Remember:

- Moments are (in principle) unknown
- Given a sample $X_1, \ldots, X_n$:
  - we **estimate** the population mean and
  - **estimate** the population variance!

Question

How big can estimation uncertainty be?

Illustration of estimation uncertainty

Visualizing estimation uncertainty

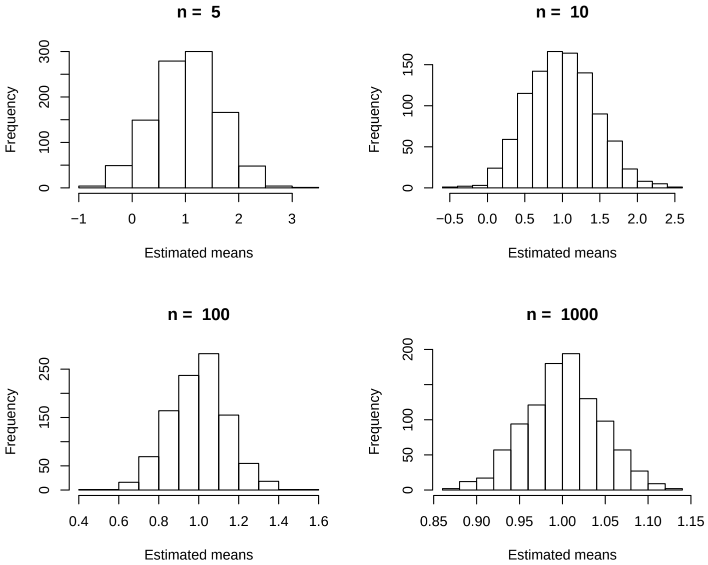
- Generate some data with known expectation
- Calculate mean
- Compare mean to expectation

$\rightarrow$ Repeat many times!

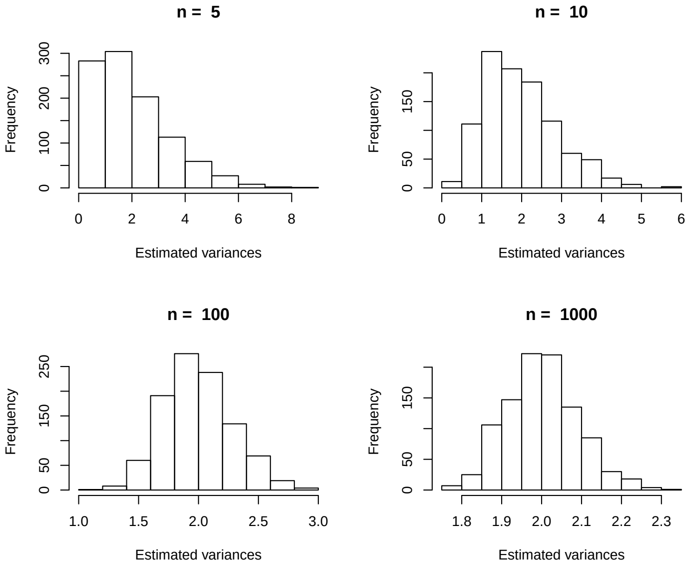
Interesting questions

- How bad can things get?
  $\rightarrow$ look at min/max
- How bad are things on average?
  $\rightarrow$ look at variance of found means over data sets

Example with expectation=1, variance=2: Sample mean



Example with expectation=1, variance=2: Sample variance



Estimation uncertainty

In statistics there will always be estimation uncertainty
$\rightarrow$ Quantifying the uncertainty is important!

How uncertain is an estimate of the population mean with n observations?
$\rightarrow$ Calculate the variance of the estimator!

Assumptions/notation:

1. The n observations are independent
2. The n observations all have the same distribution
   $\rightarrow$ mean & variance are constant
3. Population mean and variance are called μ and σ^2
4. Sample is denoted by $X_1, \ldots, X_n$

1\. and 2. are abbreviated as *iid* (=independent and identically distributed)

Estimation uncertainty

Estimator for population mean: $\bar{X} = \frac{1}{n}\sum_i X_i$

Measure uncertainty by variance

$$\begin{aligned} \text{Var}[\bar{X}] &= \text{Var}[\frac{1}{n}\sum_i X_i] = \frac{1}{n^2}\text{Var}[\sum_i X_i] = \frac{1}{n^2}\sum_i \text{Var}[X_i] \\ &= \frac{1}{n^2}\sum_i \sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{1}{n}\sigma^2 \end{aligned}$$

(Q: which assumption is used where?)

- Variance of estimator decreases with factor n
- In practice: σ^2 also unknown!
  $\rightarrow$ replace σ^2 by estimator s^2 (see last week)

Standard error of the mean

$\sqrt{\frac{1}{n}s^2}$

- Standard error of the mean
- Measure of uncertainty in estimated mean

$\rightarrow$ Standard error $=$ most important measure of uncertainty

Terminology

- Standard deviation:
  1. square root of theoretical variance *or*
  2. square root of sample variance
- Standard error
  1. square root of *estimated* variance of an estimator

Confidence intervals

Confidence intervals

Question:
How to transform the standard error into something interpretable?
$\rightarrow$ Random samples come with random variation, what range of variation can we expect?

Assume $X_1, \ldots, X_n$ iid from $N(\mu, \sigma^2)$

- $\sum_i X_i \sim N(n\mu, n\sigma^2)$
- $\bar{X} \sim N(\mu, \frac{1}{n}\sigma^2)$
- $(\bar{X} - \mu) \sim N(0, \frac{1}{n}\sigma^2)$

If we know $\sigma^2 \rightarrow$ we have an idea how large $\bar{X} - \mu$ can be!

Confidence intervals

Working things out further

$$\frac{(\bar{X}-\mu)}{\sqrt{\frac{1}{n}\sigma^2}} \sim N(0,1)$$

Therefore

$$\Pr[-1.96 < \frac{\bar{X}-\mu}{\sqrt{\frac{1}{n}\sigma^2}} < 1.96] = 0.95$$

`stats.norm.ppf(0.025)` `stats.norm.ppf(0.975)`

So

$$\Pr[\mu - 1.96\sqrt{\frac{1}{n}\sigma^2} < \bar{X} < \mu + 1.96\sqrt{\frac{1}{n}\sigma^2}] = 0.95$$

Confidence intervals

Given

$$\Pr[\mu - 1.96\sqrt{\frac{1}{n}\sigma^2} < \bar{X} < \mu + 1.96\sqrt{\frac{1}{n}\sigma^2}] = 0.95$$

Knowing μ and σ^2:

$\underbrace{\text{sample mean}}_{\uparrow\ \text{Random}}$ falls in the $\underbrace{\text{interval } \mu \pm 1.96\sqrt{\tfrac{1}{n}\sigma^2}}_{\uparrow\ \text{Non-random}}$ with probability 95%

$\rightarrow$ Is this useful in practice?

Confidence intervals

Without knowing $\mu \rightarrow \mu$ and $\bar{X}$ swap places

- the interval $\bar{X} \pm 1.96\sqrt{\frac{1}{n}\sigma^2}$ contains true value μ with probability 95%

Note: the interval is now the random variable!

What if:

1. σ^2 not known
2. X_i not normally distributed

Challenge I: Unknown σ^2

Usually variance σ^2 is not known!
$\rightarrow$ replace σ^2 by estimate s^2

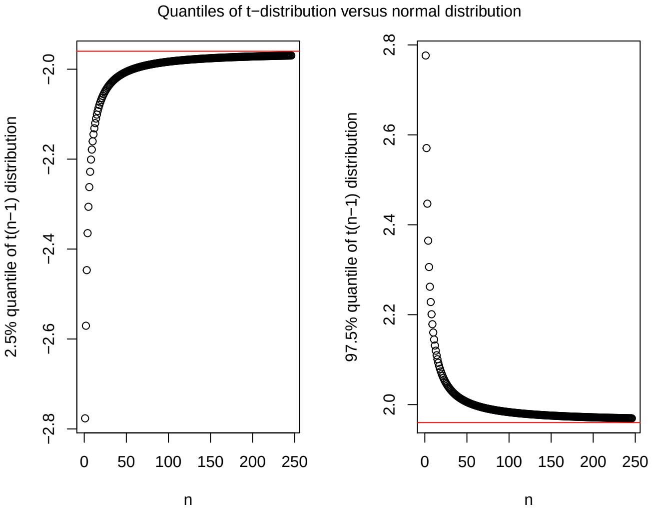
Consequences

- $\frac{(\bar{X}-\mu)}{\sqrt{\frac{1}{n}s^2}}$ in general is NOT standard normal
- In fact

$$\frac{(\bar{X}-\mu)}{\sqrt{\frac{1}{n}s^2}} \sim t_{n-1}$$

  $\rightarrow$ t_{n-k} in general (if k parameters are estimated)
- For large n: $t_{n-k} \approx N(0,1)$

Quantiles t_{n-1}-distribution for different n



Confidence interval when σ^2 is unknown

- Now use $\frac{(\bar{X}-\mu)}{\sqrt{\frac{1}{n}s^2}} \sim t_{n-k}$:
- Interval

$$\bar{X} \pm t_{n-k}^{0.975}\sqrt{\frac{1}{n}s^2}$$

  contains μ with probability 95%
- Note $t_{n-k}^{0.025} = -t_{n-k}^{0.975}$ (Use: `stats.t(n-k).ppf(.)`)

Assignment

In-class assignment 2.3

1. Load the houseprice data
2. Use to get the mean lotsize
3. Calculate the mean lotsize directly using the right function
4. Calculate the standard error of the mean (see slide 32)
5. Use the formula on slide 39 to calculate a 95% confidence interval around the mean (You should obtain the interval [4967.998, 5332.533])

Challenge II: non-normal distribution

The normality assumption is needed to get

$$\frac{(\bar{X} - \mu)}{\sqrt{\frac{1}{n}s^2}} \sim t_{n-k}$$

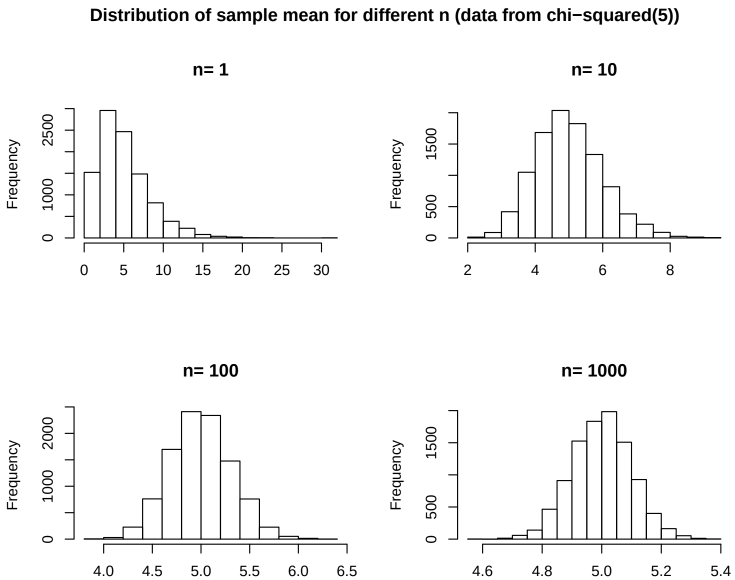
$\rightarrow$ What if X_i not normal?

Solution: use Central limit theorem

Central Limit Theorem

If $n \rightarrow \infty$: $\sqrt{n}\times$sample mean converges in distribution to a normal distribution (some regularity conditions are required)

Illustration CLT



Use of central limit theorem

Given sample $X_1, \dots, X_n$ and n large enough

$$\bar{X} \overset{approx}{\sim} N(\mu, \frac{1}{n}\sigma^2)$$

Implication:

- Confidence intervals can be made based on the normal distribution (Note $t_{n-k} \approx N(0,1)$ for large n)

What if n not large?

If n is (too) small **and** data is non-normal

- CLT is not useful
- Confidence intervals based on t-distr. not correct!

Solutions:

- If distribution is known: work out new formulas
- Use non-parametric methods
  - non-parametric tests
  - bootstrap methodology (later lecture)

Before next time

Assignment for next week

- Finish/Reread Chapter 2
- Read Chapter 3 on testing (skip ANOVA and Multi-Arm Bandits)
- Reconsider/finish the in-class assignments
- Look at examples in book
- You can already start working on the "final" assignment (will be on Canvas early next week)