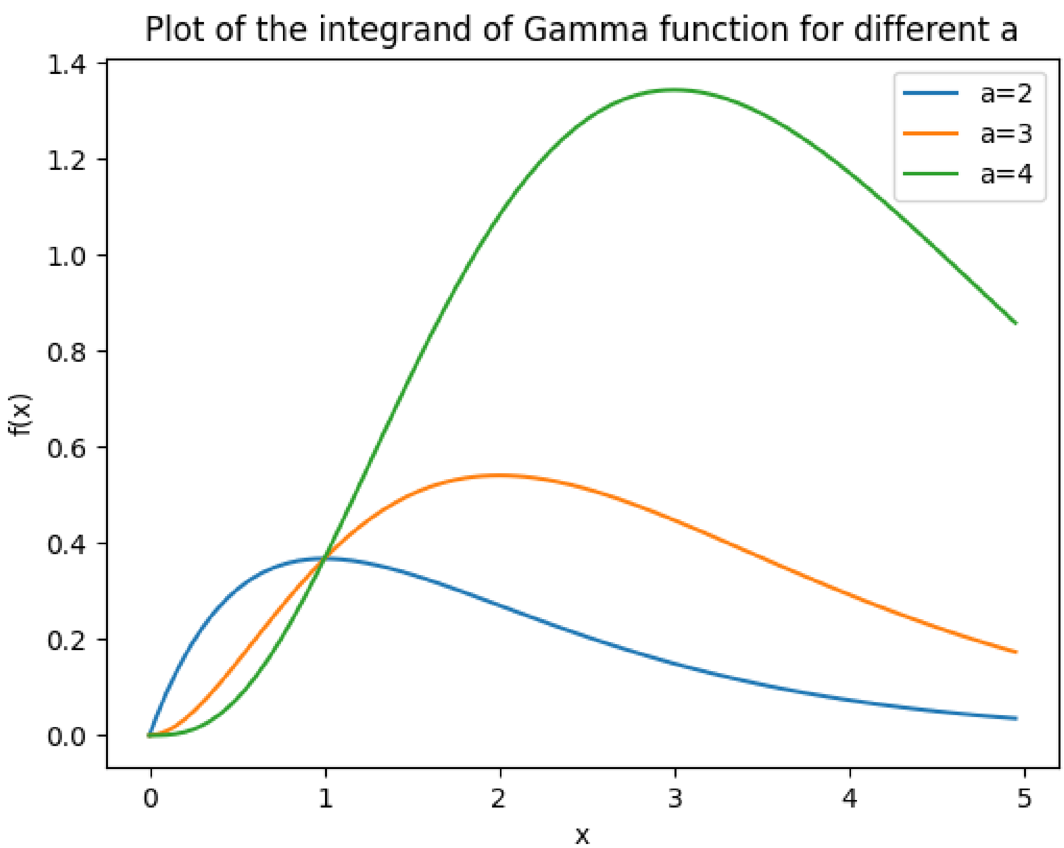
Exercise 15 (a):



Plot of the integrand for different values of a

Exercise 15 (b):
The function given is

$$f(x) = x^{a-1}e^{-x} \tag{1}$$

To find the maxima point of the function it needs to satisfy

$$\left(\frac{df}{dx}\right)_{x=x_0} = 0 \text{ and } \left(\frac{d^2f}{dx^2}\right)_{x=x_0} < 0 \tag{2}$$

Using the first condition, $\left(\frac{df}{dx}\right)_{x=x_0} = 0$;

$$(a-1)x_0^{a-2}e^{-x_0} + (-1)x_0^{a-1}e^{-x_0} = 0 \tag{3}$$

$$\implies ((a-1) - x_0)x_0^{a-1}e^{-x_0} = 0 \tag{4}$$

$$\implies x_0 = a-1 \text{ or } x_0 = 0 \tag{5}$$

We now check the second condition, $\left(\frac{d^2f}{dx^2}\right)_{x=x_0} < 0$;

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = (a-1)(a-2)x_0^{a-3}e^{-x_0} - (a-1)x_0^{a-2}e^{-x_0} - (a-1)x_0^{a-2}e^{-x_0} + x_0^{a-1}e^{-x_0} \tag{6}$$

Thus if we substitute $x_0 = 0$ in 6 we get $\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = 0$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = ((a-1)(a-2) - 2(a-1)x_0 + x_0^2))x_0^{a-3}e^{-x_0} \tag{7}$$

Substituting $x_0 = a-1$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = ((a-1)(a-2) - 2(a-1)(a-1) + (a-1)^2))x_0^{a-3}e^{-x_0} \tag{8}$$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = ((a-1)(a-2) - (a-1)(a-1))(a-1)^{a-3}e^{-(a-1)} \tag{9}$$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = -(a-1)^{a-2}e^{-(a-1)} \tag{10}$$

Thus for the given range of values of $a > 1$, we have $(a-1)^{a-2} > 0$. Hence:

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = -(a-1)^{a-2}e^{-(a-1)} < 0 \tag{11}$$

Thus the point $x_0 = a - 1$ is a maxima.

Exercise 15 (c):
We are given the following change of variable

$$z = \frac{x}{c + x} \tag{12}$$

We are given that $z = \frac{1}{2}$

$$\frac{1}{2} = \frac{x}{c + x} \tag{13}$$
$$x = c \tag{14}$$

As $x_0 = a - 1$, the value of c which puts the peak at $z = \frac{1}{2}$ is :

$$c = a - 1 \tag{15}$$

Exercise 15 (d):
We can see that for large values of x, x^{a-1} overflows and e^{-x} underflows thus giving us values with significant error. Whereas, if we write $x^{a-1} = e^{(a-1)\ln(x)}$, then $x^{a-1}e^{-x} = e^{(a-1)\ln(x)}e^{-x} = e^{(a-1)\ln(x)-x}$ thus for admissible values, greater than the limit on x used before the change of form of the expression, i.e., $x^{a-1}e^{-x}$, gives us the correct value. Hence we can compute the integral with far more accuracy.

Exercise 15 (e):

$$z = \frac{x}{x + c} \tag{16}$$
$$x = \frac{cz}{1 - z} \tag{17}$$
$$dx = \frac{c}{(1 - z)^2} dz \tag{18}$$

Thus as $c = a - 1$:

$$\boxed{\int_0^\infty x^{a-1} e^{-x} dx = \int_0^1 e^{(a-1)ln(\frac{cz}{1-z}) - \frac{cz}{1-z}} \frac{c}{(1 - z)^2} dz} \tag{19}$$