Exercise 14 (a):

Force experienced by a test particle of mass m at a point is given by (Using Newtonian Gravity):

$$\vec{F}_g = Gm \int d^3r' \rho(\vec{r'}) \frac{\hat{r} - \hat{r'}}{\mid \vec{r} - \vec{r'} \mid^2} \tag{1}$$

where $\vec{r} - \vec{r'}$ is the distance between the mass element and test particle.

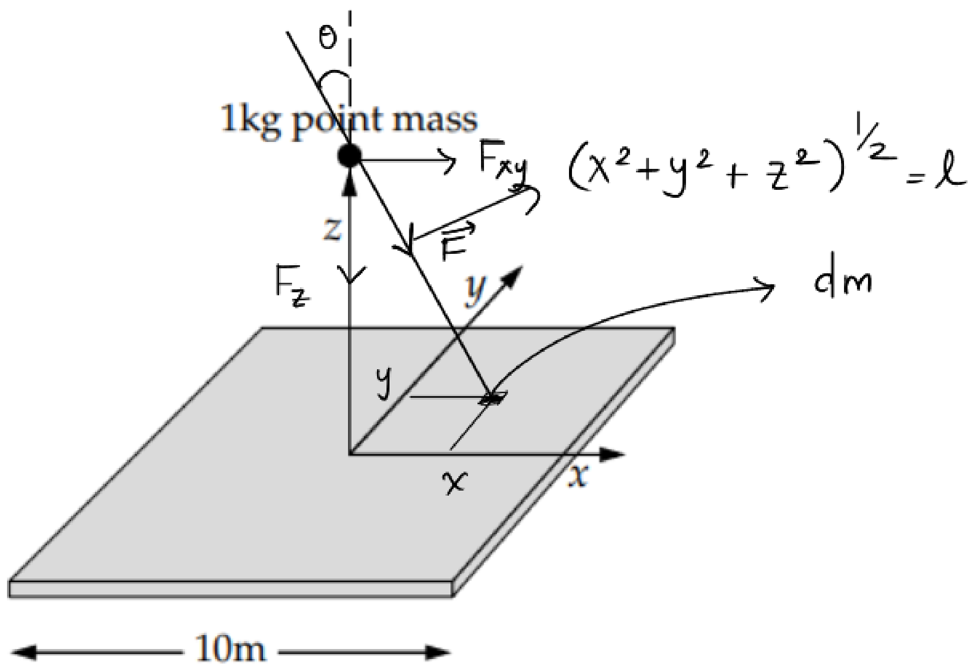


Figure 1: Force analysis of the problem

Before computing the integral we can see that the force $\vec{F}_g$ can be resolved into two components, one in z direction and another in xy plane. By symmetry of the surface we can for every point on the surface there exists another point on it for which the force along xy plane is same in magnitude but opposite in direction. Thus the net force along the xy plane vanishes and only the force along z direction adds up.

We use 1 to compute the force for this problem, using the same method but instead of volume density we have surface density $\rho(\vec{r'}) \rightarrow \sigma(\vec{r'}) \equiv \sigma$, and we integrate over the surface of the plate $d^3r' \rightarrow d^2r' = dxdy$:

$$\vec{F}_z = \vec{F}_g \cdot \hat{z} \tag{2}$$

$$\vec{F}_z = Gm \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma dxdy \frac{\hat{r} - \hat{r'}}{x^2 + y^2 + Z^2} \cdot \hat{z} \tag{3}$$

$$\vec{F}_z = Gm \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma dxdy \frac{cos(\theta)}{x^2 + y^2 + Z^2} \tag{4}$$

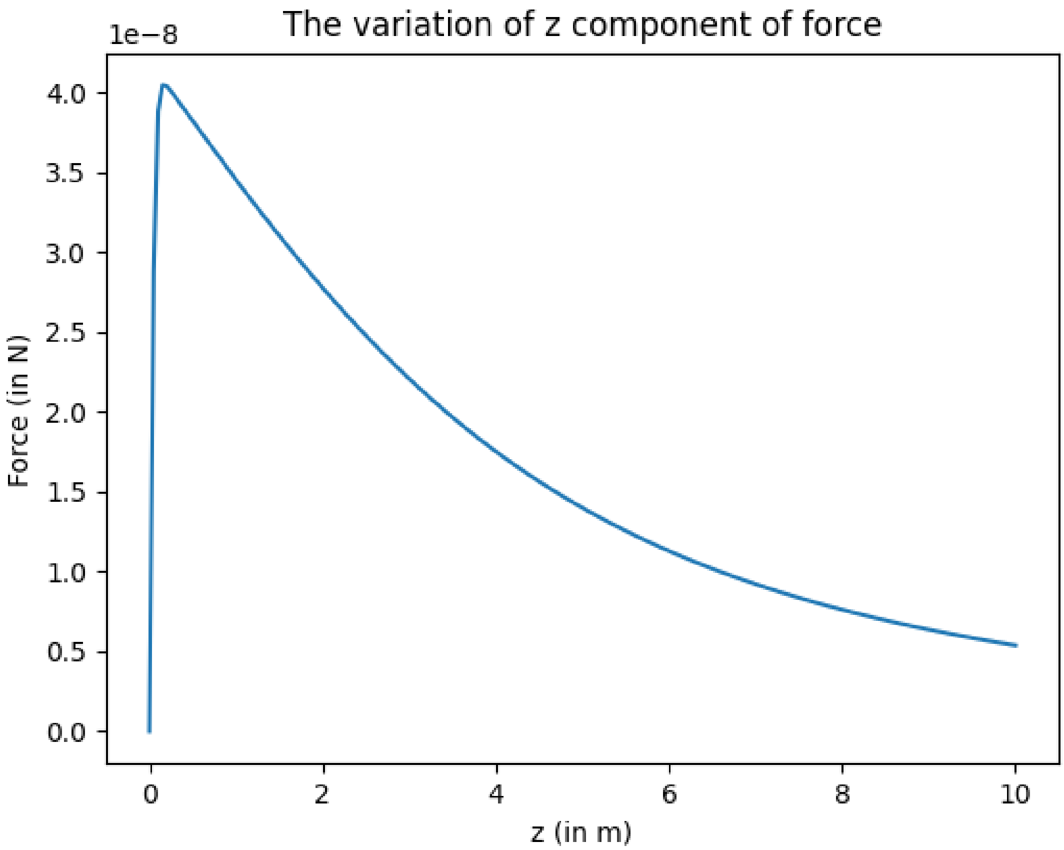
We can see from the figure that $cos(\theta) = \frac{z}{\sqrt{x^2+y^2+Z^2}}$

$$\vec{F}_z = Gmz \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma \frac{dxdy}{(x^2 + y^2 + Z^2)^{3/2}} \tag{5}$$

Thus for test mass, m=1kg:

$$\boxed{\vec{F}_z = G\sigma z \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{dxdy}{(x^2 + y^2 + Z^2)^{3/2}}} \tag{6}$$

Exercise 14 (b):



The z component of Force

Exercise 14 (c):

We know that in the limit as z approaches zero, the value of the force goes from a non-zero value ($4\pi G\sigma$) to zero and so there is a discontinuous as the mass density goes to zero at $z = 0$. Here we calculate the force using the gaussian quadrature and the function is computed at the grid, thus as theoretically there is a sharp peak at z = 0, so for values of z closer to 0 the value of force F_z gets more closer to the peak value ($4\pi G\sigma$). Hence there is a high probability that we might miss the peak as the peak is sharp for F_z. Thus the value obtained is less than the maximum value, and so we get values between 0 and $4\pi G\sigma$, which is indeed an artificial phenomena, thus giving us a line with a finite slope instead of slope infinity