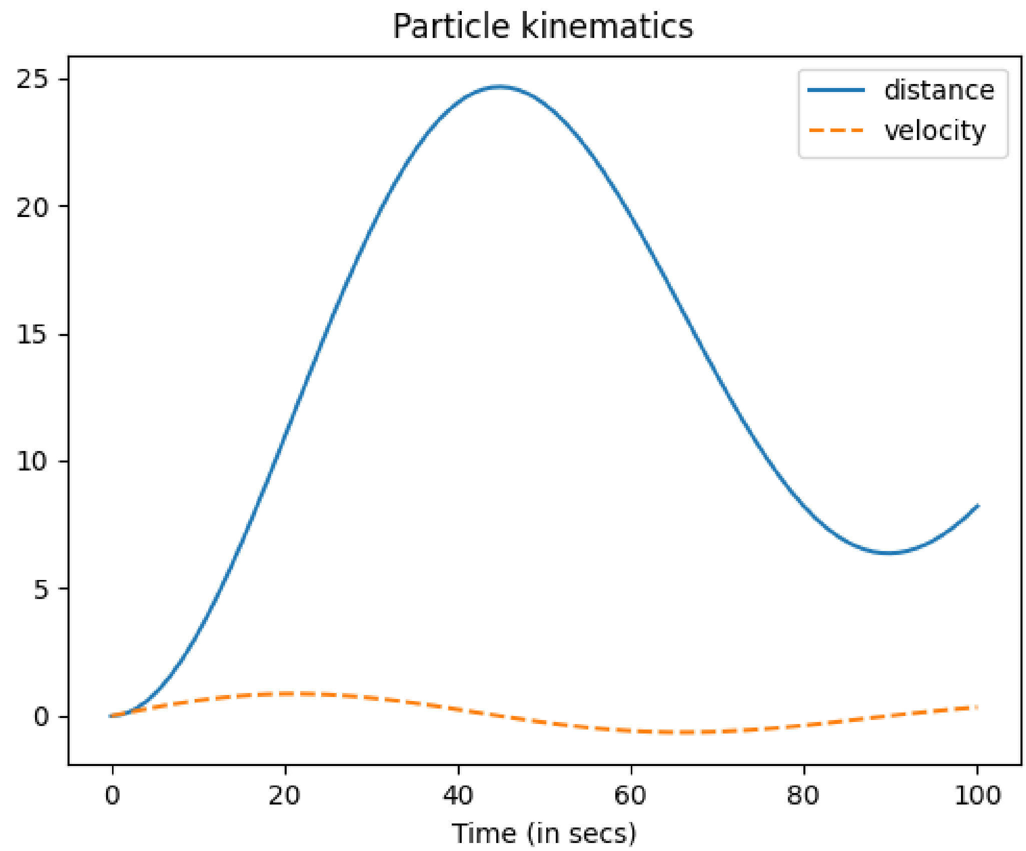
Exercise 1 :

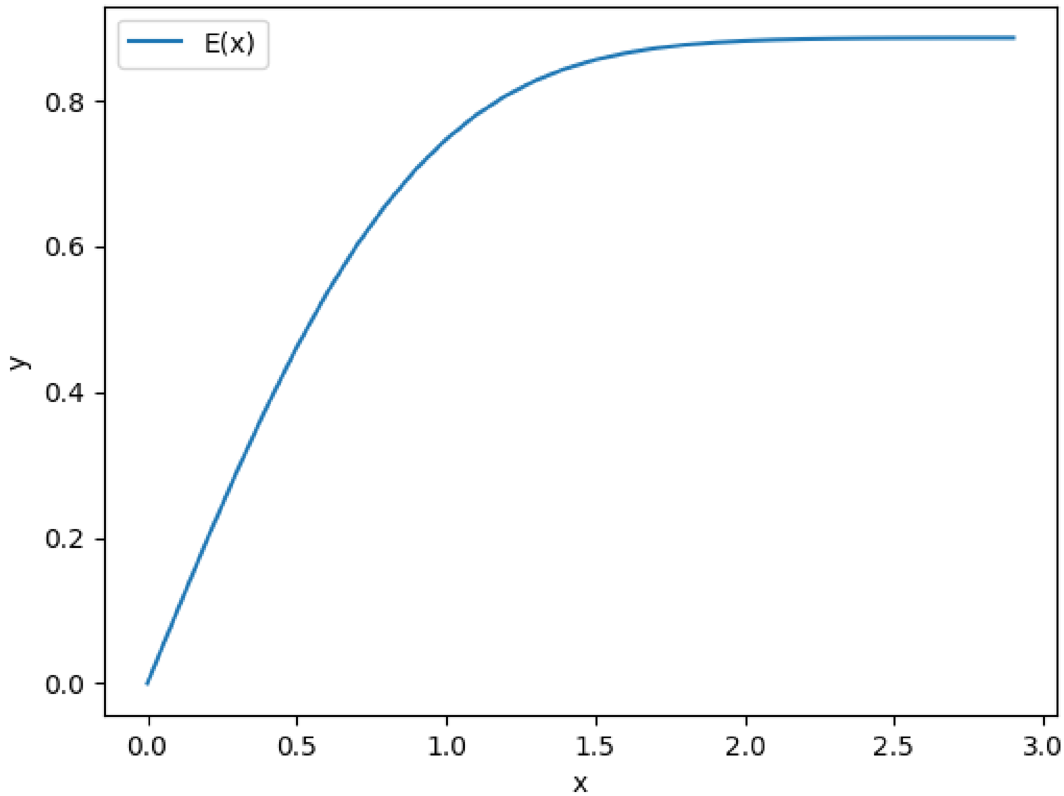


The plot showing distance and velocity as function of time

Exercise 2

```
The actual value of the integral is 4.4

 SIMPSON'S METHOD

The approximate value for the integral using the Simpson's method with 10 slices is
4.400426666666667
The fractional error on numerical integration (Simpson) with 10 slices is
9.696969696972666e-05
The approximate value for the integral using the Simpson's method with 100 slices
is  4.400000042666668
The fractional error on numerical integration (Simpson) with 100 slices is
9.696969893724372e-09
The approximate value for the integral using the Simpson's method with 1000 slices
is  4.400000000004267
The fractional error on numerical integration (Simpson) with 1000 slices is
9.697293473271367e-13

 TRAPEZOIDAL METHOD

The approximate value for the integral using the Trapezoidal method with 10 slices
is  4.50656
The fractional error on numerical integration (Trapezoidal) with 10 slices is
0.024218181818181812
The approximate value for the integral using the Trapezoidal method with 100 slices
is  4.401066656
The fractional error on numerical integration (Trapezoidal) with 100 slices is
0.00024242181818179273
The approximate value for the integral using the Trapezoidal method with 1000
slices is  4.4000106666656
The fractional error on numerical integration (Trapezoidal) with 1000 slices is
2.4242421817452255e-06

Thus we can see that for both methods as we increase the number of slices the
accuracy of the integral improves. Also for each every
numver of slices the error in the integral is very less in the case of Simpson's
method when compared to Trapezoidal method
```

Exercise 3 (a):
Using Simpson's method with 100 slices, the integral is computed.
Exercise 3 (b):



Plot of E(x)

Exercise 4 (a):

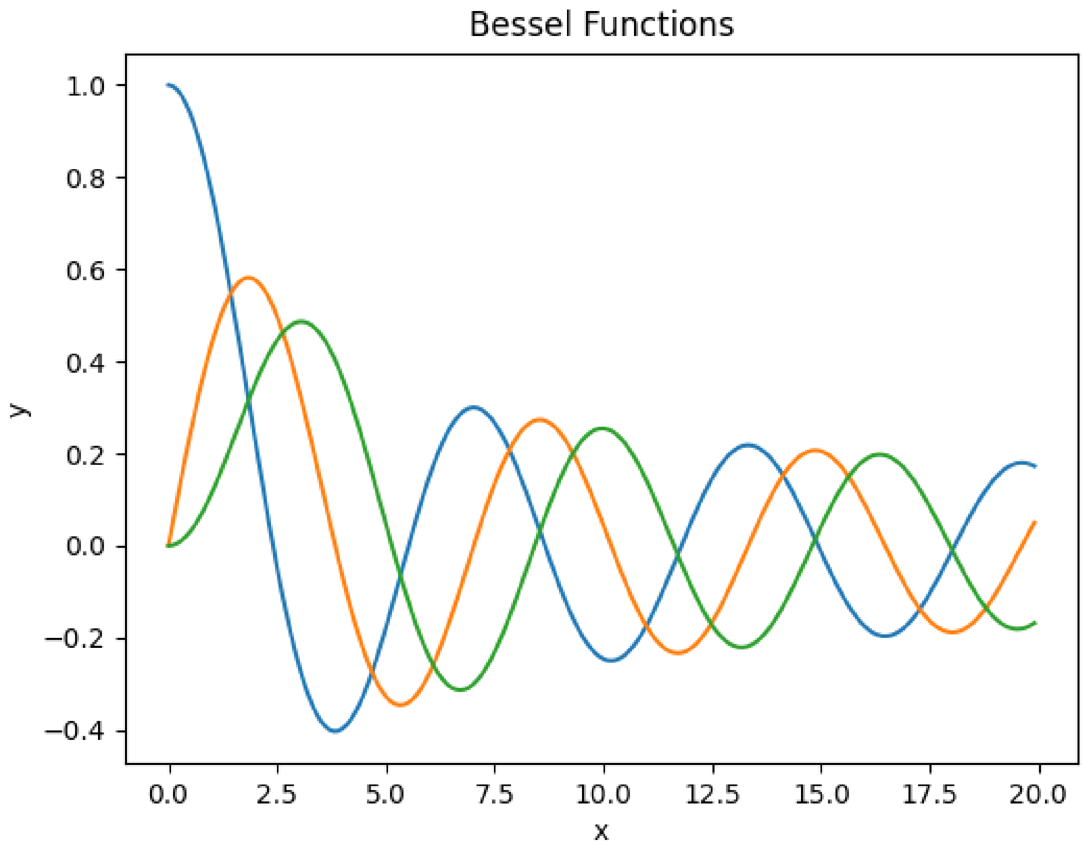


Figure 1: Plot of different Bessel functions

Exercise 4 (b):

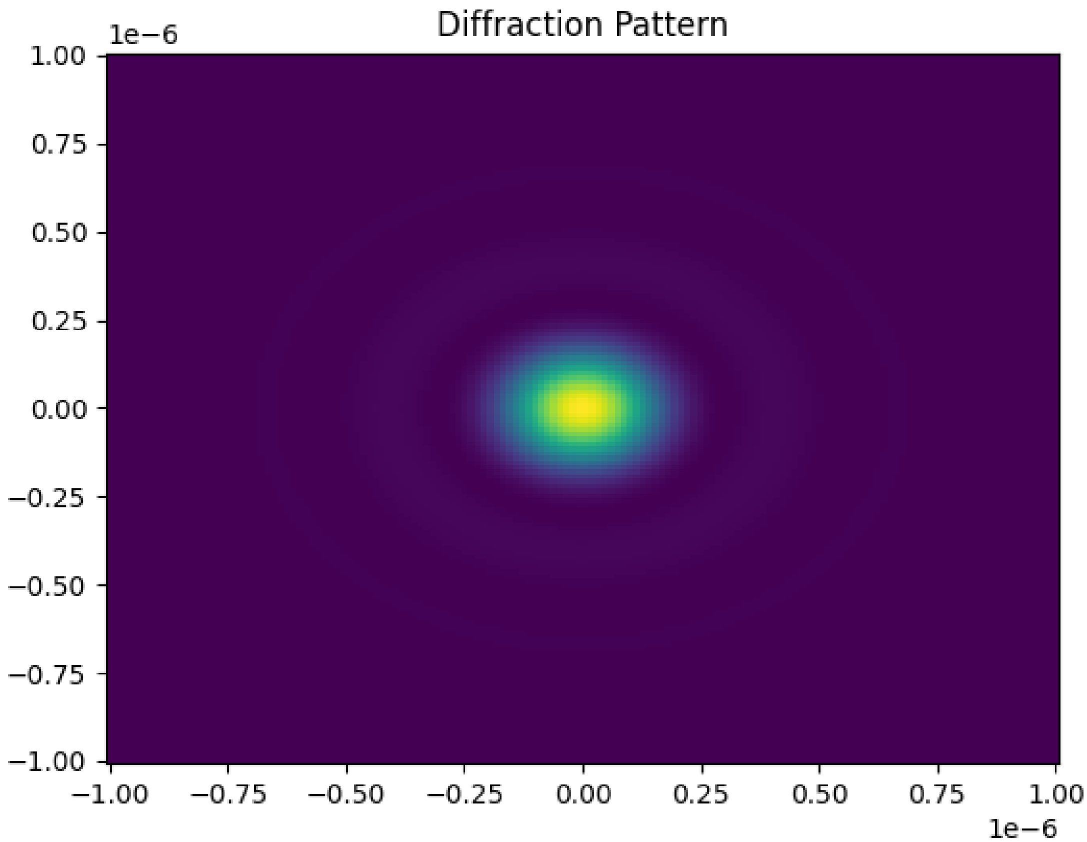


Figure 2: Density plot of the intensity of the circular diffraction pattern of a point light source

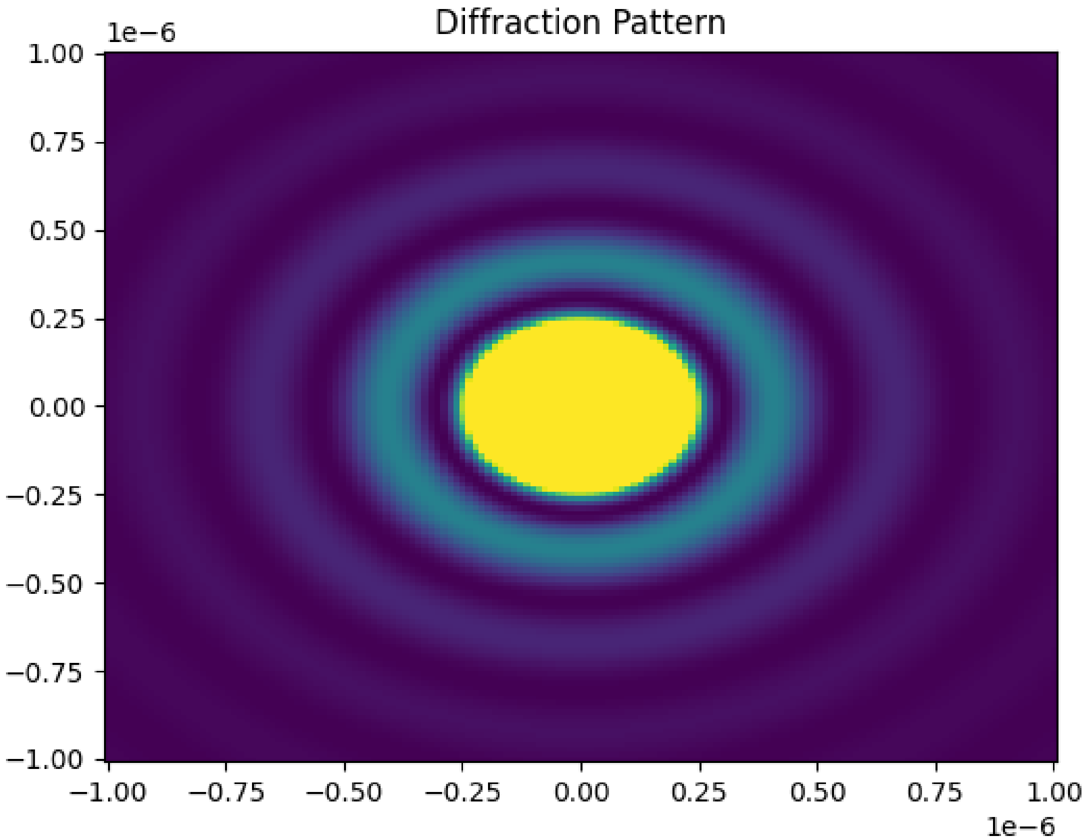


Figure 3: Density plot of the intensity of the circular diffraction pattern of a point light source, but using the vmax = 0.01 argument

Exercise 5

```
The value of the integral computed using the Trapezoidal method with 20 slices is:
4.426660000000001
The error computed using this method is: 0.026633333333333314
Direct computation error is 0.026660000000000572
```

The method suggested in this question uses tatylor expansion, where we only consider terms till power h^2 and omitting the higher order terms, whereas in the direct computation error we include all the terms in the expansion as we directly subtract the actual value from the computed value. Hence the two do not agree perfectly as we have omitted higher order terms in this method.Thus this error used in this problem is called truncation error.

Exercise 6

The value of the integral computed using the Simpson's method with 20 slices is:
4.400026666666667
The error computed using this method is: 2.666666666666373e-05
Direct computation error is 2.6666666666841365e-05

The method suggested in this question uses tatylor expansion, where we only consider terms till power h^4 (starts with this) and omitting the higher order terms, whereas in the direct computation error we include all the terms in the expansion as we directly subtract the actual value from the computed value. Hence the two do not agree perfectly as we have omitted higher order terms in this method.Thus this error used in this problem is called truncation error.

Thus the target accuracy is reached at 2048 with error 1.8960230755057002e-06

Exercise 7

```
Trapezoidal Method

The value of the integral for 1 slice is 0.147979484546652
The value of the integral for 2 slice is 0.3252319078064746 with error
0.05908414108660753
The value of the integral for 4 slice is 0.5122828507233315 with error
0.06235031430561896
The value of the integral for 8 slice is 0.40299744847824825 with error -
0.036428467415027734
The value of the integral for 16 slice is 0.43010336929474696 with error
0.009035306938832902
The value of the integral for 32 slice is 0.4484146657874698 with error
0.006103765497574 28
The value of the integral for 64 slice is 0.45391293121537596 with error
0.0018327551426353856
The value of the integral for 128 slice is 0.45534850437280205 with error
0.00047852438580869 85
The value of the integral for 256 slice is 0.45571126645324095 with error
0.00012092069347963141
The value of the integral for 512 slice is 0.4558021996516643 with error
3.031106614111619e-05
The value of the integral for 1024 slice is 0.45582494813241997 with error
7.582826918558124e-06
The value of the integral for 2048 slice is 0.4558306362016465 with error
1.8960230755057002e-06
The value of the integral for 4096 slice is 0.45583205827827056 with error
4.740255413563747e-07

Thus the target accuracy is reached at 2048 with error 1.8960230755057002e-06

Romberg Integration

0.3252319078064746
0.5122828507233315 0.5746331650289505
0.4029974484782483 0.3665689810632206 0.3526980354655053
0.43010336929474696 0.4391386762335798 0.44397665591160373 0.4454255229028117
0.4484146657874699 0.45451843128504427 0.4555437482884752 0.45572735292937794
0.45576775226281546
0.4539129312153758 0.4557456863580111 0.45582750336287553 0.45583200741167557
0.4558324178214101 0.4558324810330998
 The for 64 slices is 0.4558324944613787 with error 1.3428278877370225e-08 is the
final approximation
```

Exercise 8

```
The value of the integral for 2 slice is 0.38431604889308213
The value of the integral for 4 slice is 0.5746331650289502 with error
0.012687807742391206
The value of the integral for 8 slice is 0.36656898106322056 with error -
0.013870945597715312
The value of the integral for 16 slice is 0.4391386762335799 with error
0.004837979678023955
The value of the integral for 32 slice is 0.4545184312850443 with error
0.0010253170034309625
The value of the integral for 64 slice is 0.45574568635801105 with error
8.181700486444842e-05
The value of the integral for 128 slice is 0.45582702875861086 with error
5.422826706654357e-06
The value of the integral for 256 slice is 0.45583218714672064 with error
3.4389254065144335e-07

Thus the target accuracy is reached at 128 with error 5.422826706654357e-06

Thus we can see that the target accuracy is reached faster than trapezoidal method
with N = 128 but it is still larger than Romberg integration
which has N = 64.
```

Exercise 9 (b):

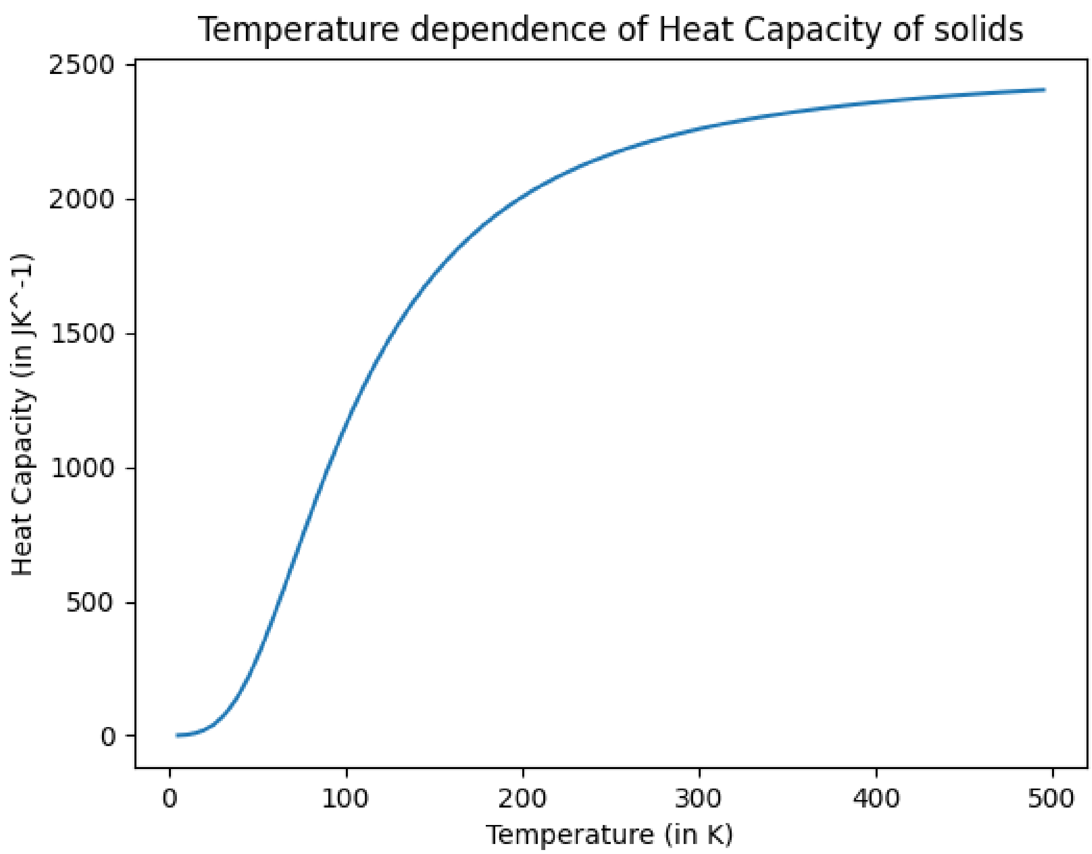


Figure 1: Plot of the temperature dependence of heat capacity of solids

Exercise 10 (a):
By energy conservation we have

$$E = T + V \tag{1}$$

where $T = \frac{1}{2}m(\frac{dx}{dt})^2$, i.e., the kinetic energy and V is the potential energy.

Since we start at $x = a$ and we have zero initial velocity $E = V(a)$. We also know that the oscillator can reach $x = a$ to $x = 0$ in $T/4$ time, where T is the time period of oscillation. Thus:

$$\frac{2}{m}(E - V) = \frac{dx}{dt}^2 \tag{2}$$

$$dt = -\frac{dx}{\sqrt{\frac{2}{m}(E - V)}} \tag{3}$$

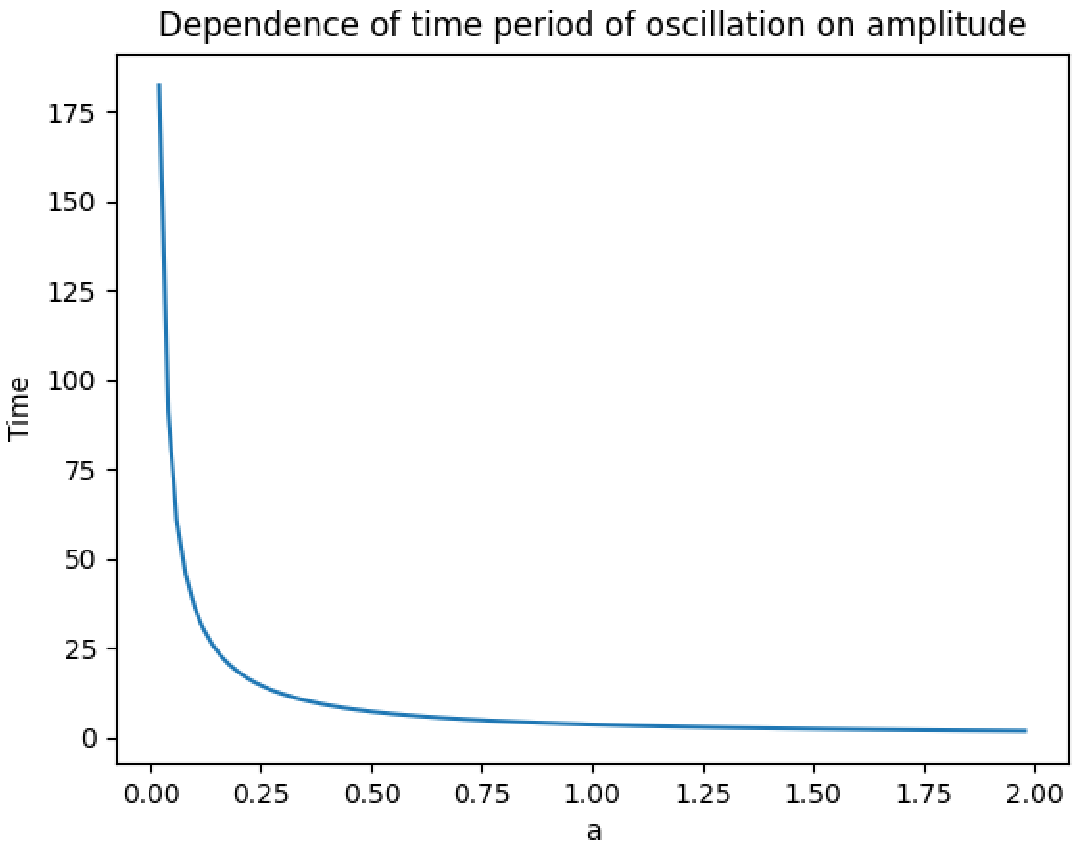
-ve sign is due to the fact that x goes from a to 0

$$\int_0^{\frac{T}{4}} dt = \int_a^0 -\frac{dx}{\sqrt{\frac{2}{m}(E - V)}} \tag{4}$$

$$\int_0^{\frac{T}{4}} dt = \int_0^a \frac{dx}{\sqrt{\frac{2}{m}(E - V)}} \tag{5}$$

$$\boxed{T = 4\int_0^a \frac{dx}{\sqrt{\frac{2}{m}(E - V)}}} \tag{6}$$

Exercise 10 (b):

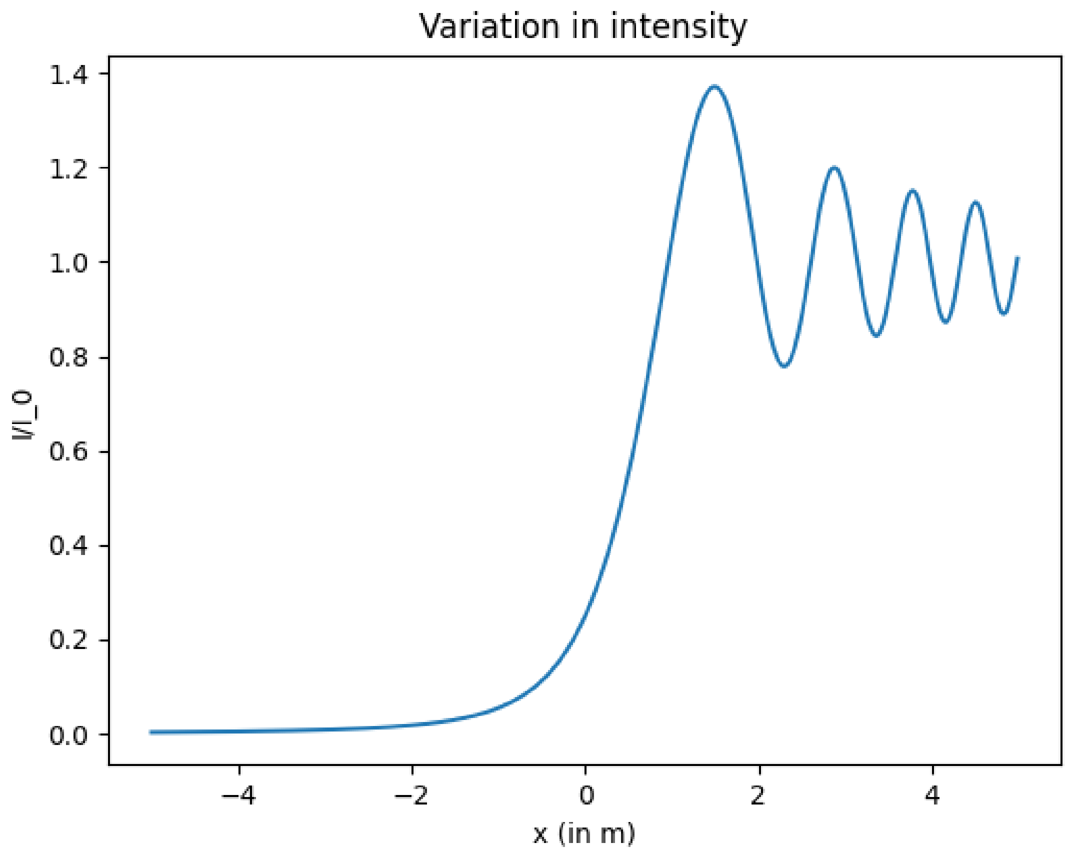


Plot of the dependence of time period on amplitude for anharmonic oscillator

Exercise 10 (c):

The restorative force grows as x^3 with distance x but the distance travelled is just amplitude a, so the particle is pulled back to $x = 0$ faster for larger a. Thus time decreases with a.

Exercise 11 :



Plot of relative intensity as a function of x

Exercise 12 (a):
We are given with the expression for the intensity as a function of ω:

$$I(\omega) = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} \tag{1}$$

Now we know that amount of thermal energy per second equals to $I(\omega)d\omega$. Thus:

$$dW = I(\omega)d\omega \tag{2}$$

$$W = \int_0^\infty I(\omega)d\omega \tag{3}$$

$$W = \int_0^\infty \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} d\omega \tag{4}$$

Now we perform change of variables $\frac{\hbar\omega}{k_B T} \to x \ \Rightarrow \frac{\hbar}{k_B T} d\omega \to dx$

$$W = \int_0^\infty \frac{\hbar}{4\pi^2 c^2} \frac{k_B^3 T^3}{\hbar^3} \frac{x^3}{e^x - 1} \frac{k_B T}{\hbar} dx \tag{5}$$

Thus we have the required equation:

$$\boxed{W = \frac{k_B^4 T^4}{4\pi^2 c^2 \hbar^3} \int_0^\infty \frac{x^3}{e^x - 1} dx} \tag{6}$$

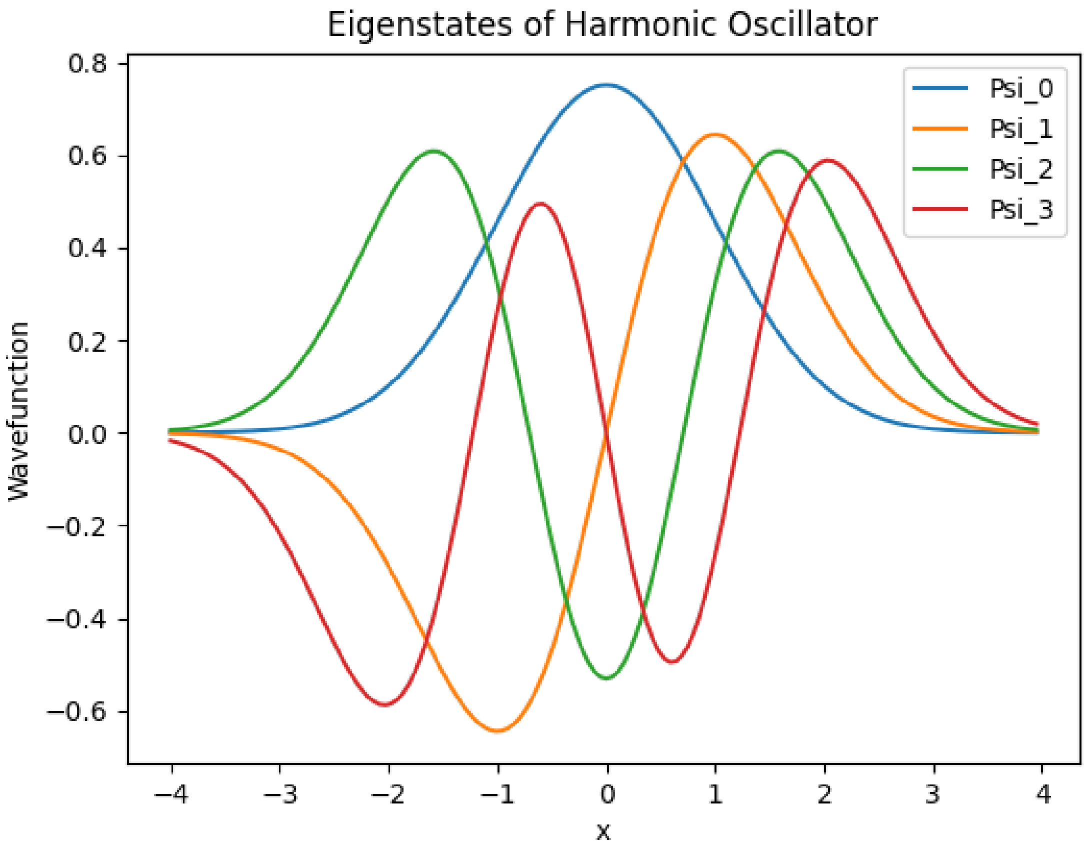
Part B

We have used the gaussian quadrature method to evaluate the integral and the number of sample points used for doing so is N = 100,
as it has high accuracy and converges to actual value very quickly, thus we can see the error to be very small, i.e., 4.432010314303625e-13

Part C

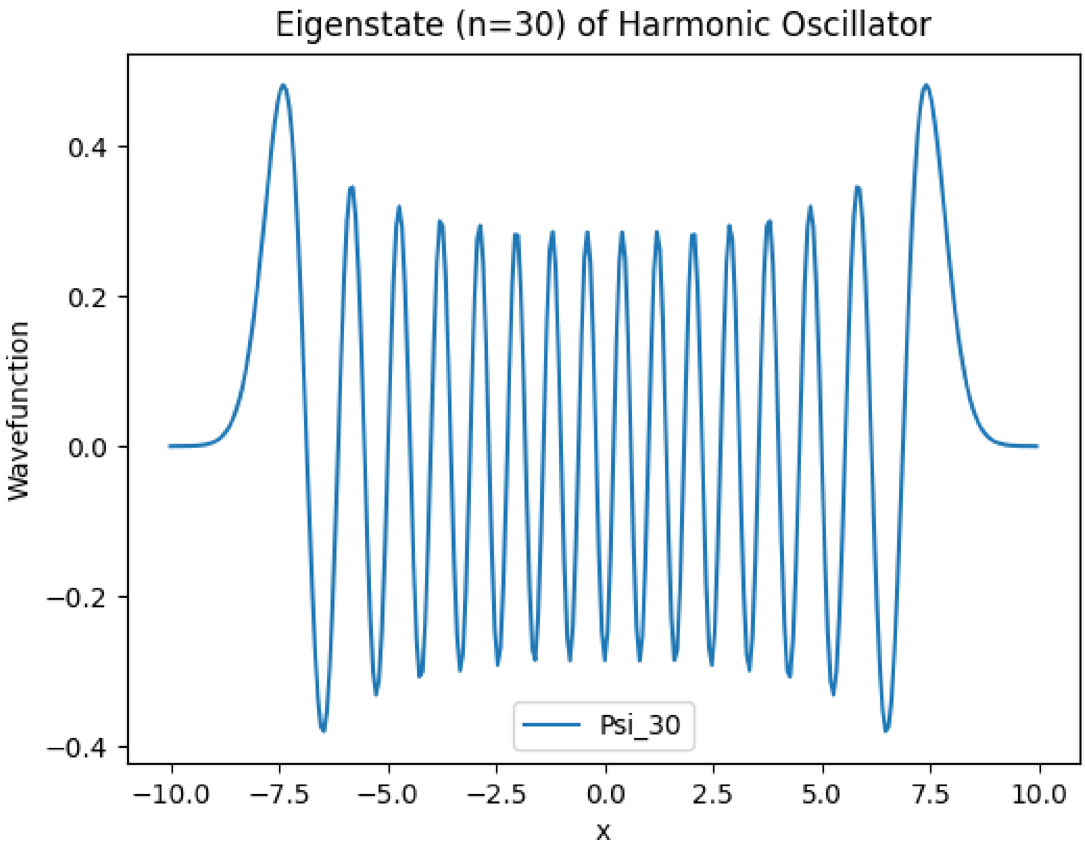
The value of the integral obtain using the gaussian quadrature method is: 6.493939402267271
The actual value calculated using Mathematica is: 6.493939402266828
Thus the error is: 4.432010314303625e-13
The value of the Stefan Boltzmann constant computed using the method given is: 5.670374419184816e-08
The literature value of the Stefan Boltzmann constant is: 5.670374419e-08
The error percentage in the computed value from literature value is: 3.25932571624752e-09

Exercise 13 (a):



Plot of wavefunctions of different states

Exercise 13 (b):



Wavefunction of state n=30

Note: The first time when I ran the program it took about 5mins to give the output because of my recursive implementation of the Hermite function. After the iterative loop implementation the runtime was reduced to couple of seconds

Exercise 13 (b):

The uncertainity in position for $n = 5$ is 2.34518960812129955. We have used the integration limits to be -5 to 5 as for large values of limits the exponential function in the wavefunction overflows and underflows, also for $\mid x \mid > 5$ the wavefunction goes to zero. Thus we can use the above limits

Exercise 14 (a):
Force experienced by a test particle of mass m at a point is given by (Using Newtonian Gravity):

$$\vec{F}_g = Gm \int d^3r' \rho(\vec{r'}) \frac{\hat{r} - \hat{r'}}{|\vec{r} - \vec{r'}|^2} \tag{1}$$

where $\vec{r} - \vec{r'}$ is the distance between the mass element and test particle.

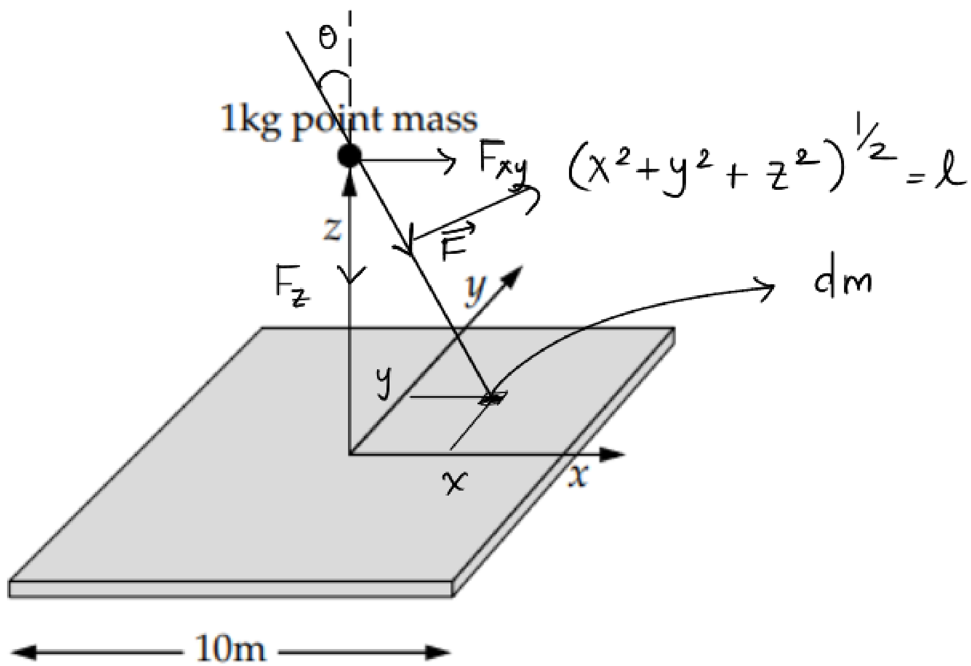


Figure 1: Force analysis of the problem

Before computing the integral we can see that the force $\vec{F}_g$ can be resolved into two components, one in z direction and another in xy plane. By symmetry of the surface we can for every point on the surface there exists another point on it for which the force along xy plane is same in magnitude but opposite in direction. Thus the net force along the xy plane vanishes and only the force along z direction adds up.

We use 1 to compute the force for this problem, using the same method but instead of volume density we have surface density $\rho(\vec{r'}) \to \sigma(\vec{r'}) \equiv \sigma$, and we integrate over the surface of the plate $d^3r' \to d^2r' = dxdy$:

$$\vec{F}_z = \vec{F}_g \cdot \hat{z} \tag{2}$$

$$\vec{F}_z = Gm \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma dxdy \frac{\hat{r} - \hat{r'}}{x^2 + y^2 + Z^2} \cdot \hat{z} \tag{3}$$

$$\vec{F}_z = Gm \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma dxdy \frac{cos(\theta)}{x^2 + y^2 + Z^2} \tag{4}$$

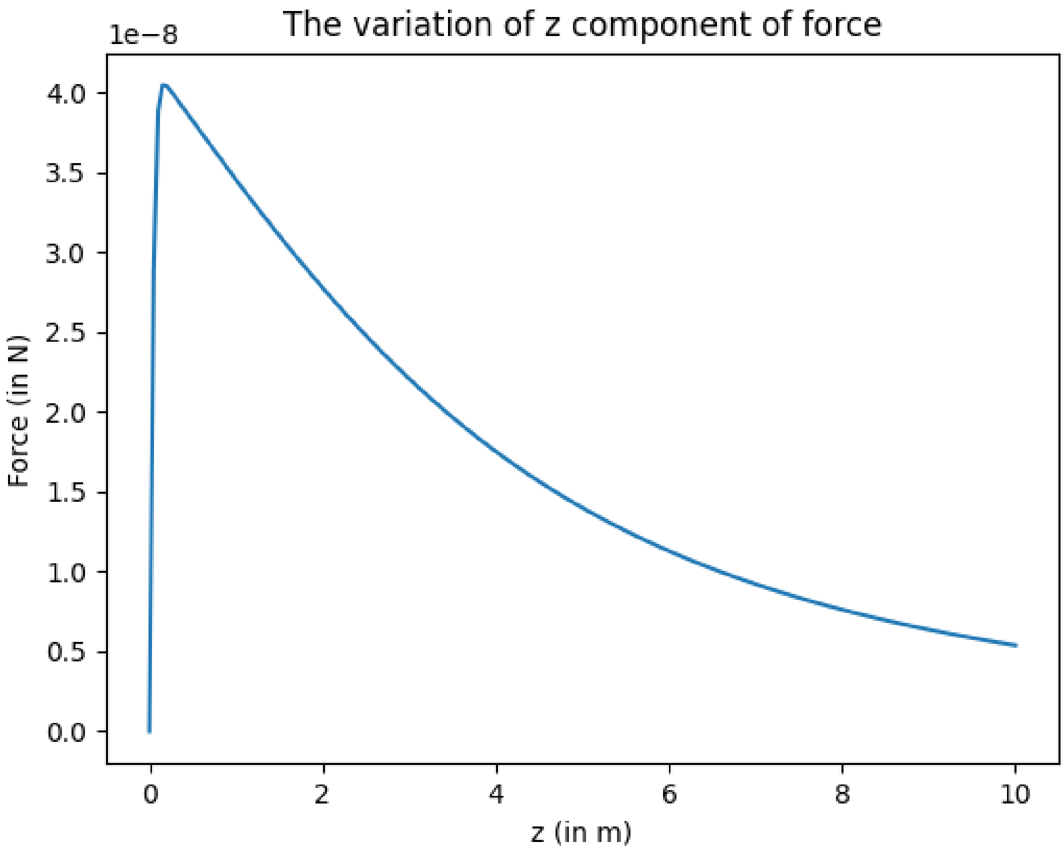
We can see from the figure that $cos(\theta) = \frac{z}{\sqrt{x^2+y^2+Z^2}}$

$$\vec{F}_z = Gmz \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma \frac{dxdy}{(x^2 + y^2 + Z^2)^{3/2}} \tag{5}$$

Thus for test mass, m=1kg:

$$\boxed{\vec{F}_z = G\sigma z \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{dxdy}{(x^2 + y^2 + Z^2)^{3/2}}} \tag{6}$$

Exercise 14 (b):

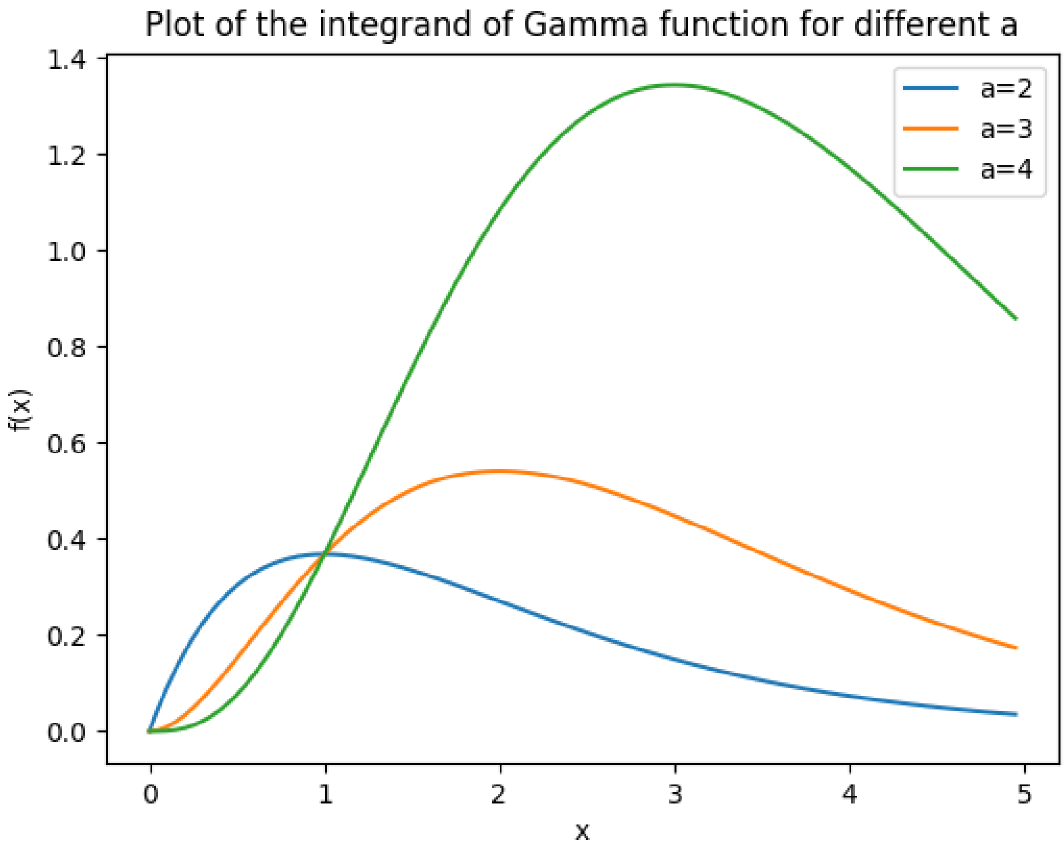


The z component of Force

Exercise 14 (c):

We know that in the limit as z approaches zero, the value of the force goes from a non-zero value ($4\pi G\sigma$) to zero and so there is a discontinuous as the mass density goes to zero at $z = 0$. Here we calculate the force using the gaussian quadrature and the function is computed at the grid, thus as theoretically there is a sharp peak at z = 0, so for values of z closer to 0 the value of force F_z gets more closer to the peak value ($4\pi G\sigma$). Hence there is a high probability that we might miss the peak as the peak is sharp for F_z. Thus the value obtained is less than the maximum value, and so we get values between 0 and $4\pi G\sigma$, which is indeed an artificial phenomena, thus giving us a line with a finite slope instead of slope infinity

Exercise 15 (a):



Plot of the integrand for different values of a

Exercise 15 (b):
The function given is

$$f(x) = x^{a-1}e^{-x} \tag{1}$$

To find the maxima point of the function it needs to satisfy

$$\left(\frac{df}{dx}\right)_{x=x_0} = 0 \text{ and } \left(\frac{d^2f}{dx^2}\right)_{x=x_0} < 0 \tag{2}$$

Using the first condition, $\left(\frac{df}{dx}\right)_{x=x_0} = 0$;

$$(a-1)x_0^{a-2}e^{-x_0} + (-1)x_0^{a-1}e^{-x_0} = 0 \tag{3}$$

$$\implies ((a-1) - x_0)x_0^{a-1}e^{-x_0} = 0 \tag{4}$$

$$\implies x_0 = a-1 \text{ or } x_0 = 0 \tag{5}$$

We now check the second condition, $\left(\frac{d^2f}{dx^2}\right)_{x=x_0} < 0$;

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = (a-1)(a-2)x_0^{a-3}e^{-x_0} - (a-1)x_0^{a-2}e^{-x_0} - (a-1)x_0^{a-2}e^{-x_0} + x_0^{a-1}e^{-x_0} \tag{6}$$

Thus if we substitute $x_0 = 0$ in 6 we get $\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = 0$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = ((a-1)(a-2) - 2(a-1)x_0 + x_0^2))x_0^{a-3}e^{-x_0} \tag{7}$$

Substituting $x_0 = a-1$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = ((a-1)(a-2) - 2(a-1)(a-1) + (a-1)^2))x_0^{a-3}e^{-x_0} \tag{8}$$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = ((a-1)(a-2) - (a-1)(a-1))(a-1)^{a-3}e^{-(a-1)} \tag{9}$$

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = -(a-1)^{a-2}e^{-(a-1)} \tag{10}$$

Thus for the given range of values of $a > 1$, we have $(a-1)^{a-2} > 0$. Hence:

$$\left(\frac{d^2f}{dx^2}\right)_{x=x_0} = -(a-1)^{a-2}e^{-(a-1)} < 0 \tag{11}$$

Thus the point $x_0 = a - 1$ is a maxima.

Exercise 15 (c):
We are given the following change of variable

$$z = \frac{x}{c + x} \tag{12}$$

We are given that $z = \frac{1}{2}$

$$\frac{1}{2} = \frac{x}{c + x} \tag{13}$$

$$x = c \tag{14}$$

As $x_0 = a - 1$, the value of c which puts the peak at $z = \frac{1}{2}$ is :

$$c = a - 1 \tag{15}$$

Exercise 15 (d):
We can see that for large values of x, x^{a-1} overflows and e^{-x} underflows thus giving us values with significant error. Whereas, if we write $x^{a-1} = e^{(a-1)\ln(x)}$, then $x^{a-1}e^{-x} = e^{(a-1)\ln(x)}e^{-x} = e^{(a-1)\ln(x)-x}$ thus for admissible values, greater than the limit on x used before the change of form of the expression, i.e., $x^{a-1}e^{-x}$, gives us the correct value. Hence we can compute the integral with far more accuracy.

Exercise 15 (e):

$$z = \frac{x}{x + c} \tag{16}$$

$$x = \frac{cz}{1 - z} \tag{17}$$

$$dx = \frac{c}{(1 - z)^2} dz \tag{18}$$

Thus as $c = a - 1$:

$$\boxed{\int_0^{\infty} x^{a-1} e^{-x} dx = \int_0^1 e^{(a-1)ln(\frac{cz}{1-z}) - \frac{cz}{1-z}} \frac{c}{(1 - z)^2} dz} \tag{19}$$

Part E

The user defined gamma function has passed the test for gamma(3/2):
0.8862269613087213

Part F

Gamma(3): 2.0000000000000013
Gamma(6): 119.99999999999997
Gamma(10): 362879.99999999977

Exercise 16 (a):

Let us make the following assumption that the slit width is s and the wall is of infinitesimally small length. From the given transmission function

$$q(u) = \sin^2(\alpha u) \tag{1}$$

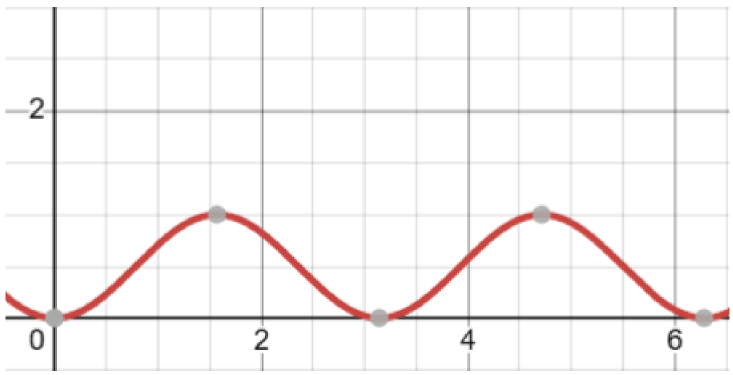


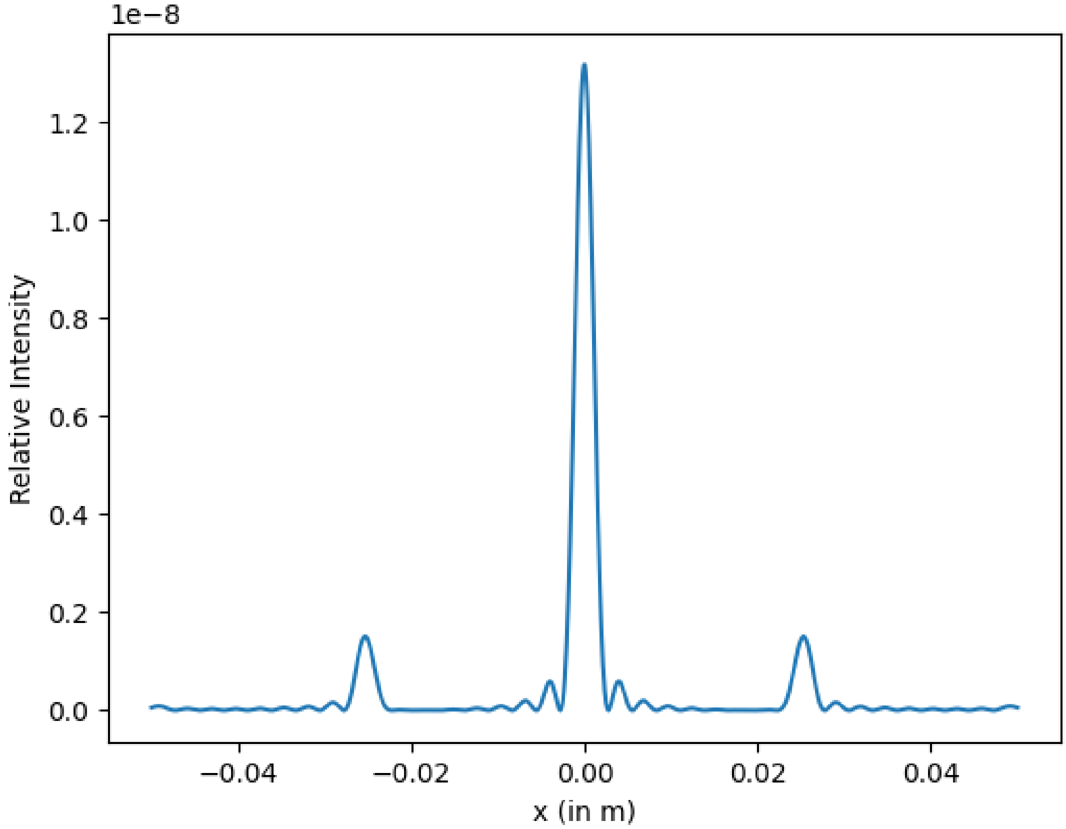
Figure 1: The $\sin^2(x)$ function

Following our assumption, we can conclude that the points at which $\sin^2(\alpha u) = 0$ is the wall. Thus the distance between two consecutive such points is $\pi = \alpha s$. Thus:

$$\boxed{\alpha = \frac{\pi}{s}} \tag{2}$$

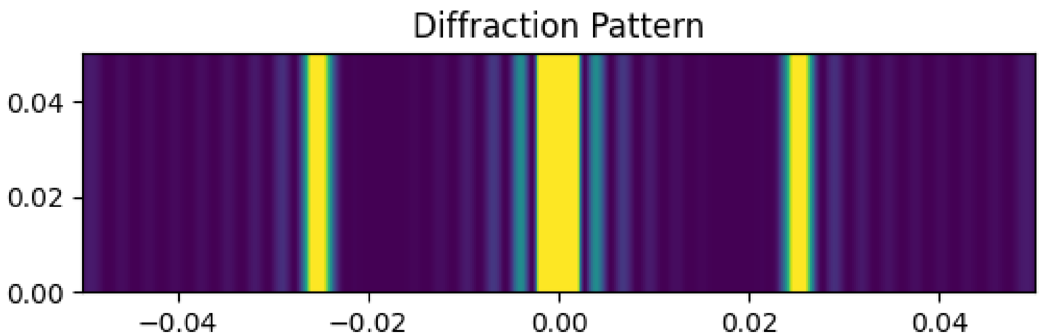
Exercise 16 (c):

The method used here to evaluate the integral is Gaussian quadrature with number sample points = 100. The reason is that the integral is evaluated to very high orders of x, thus N = 100 serves as a good and optimal approximate for the integral



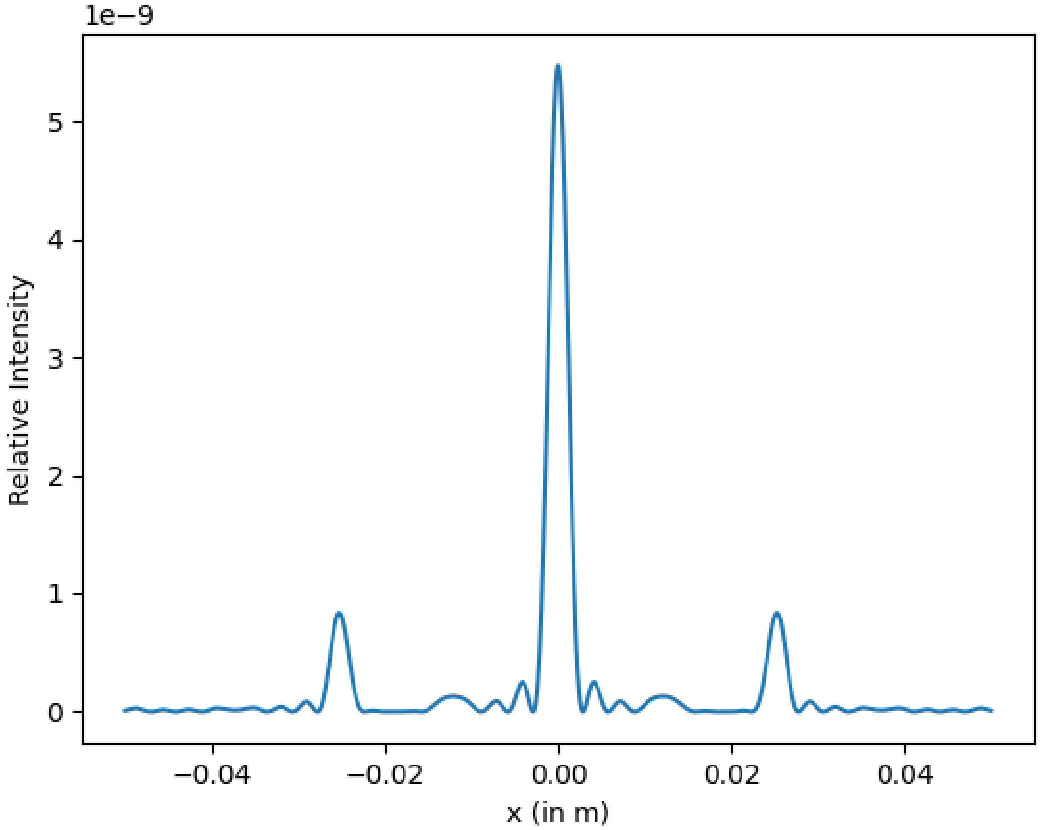
For $q(u) = \sin^2(\alpha u)$

Exercise 16 (d):

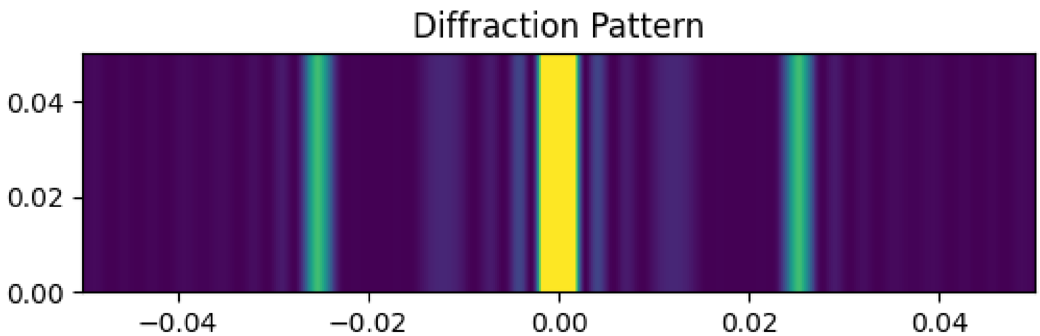


For $q(u) = \sin^2(\alpha u)$

Exercise 16 (e): (i)

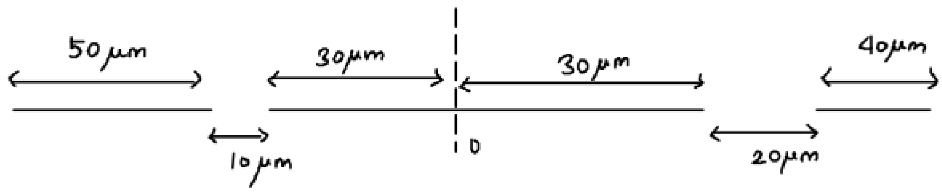


For $q(u) = \sin^2(\alpha u) \sin^2(\alpha u)$

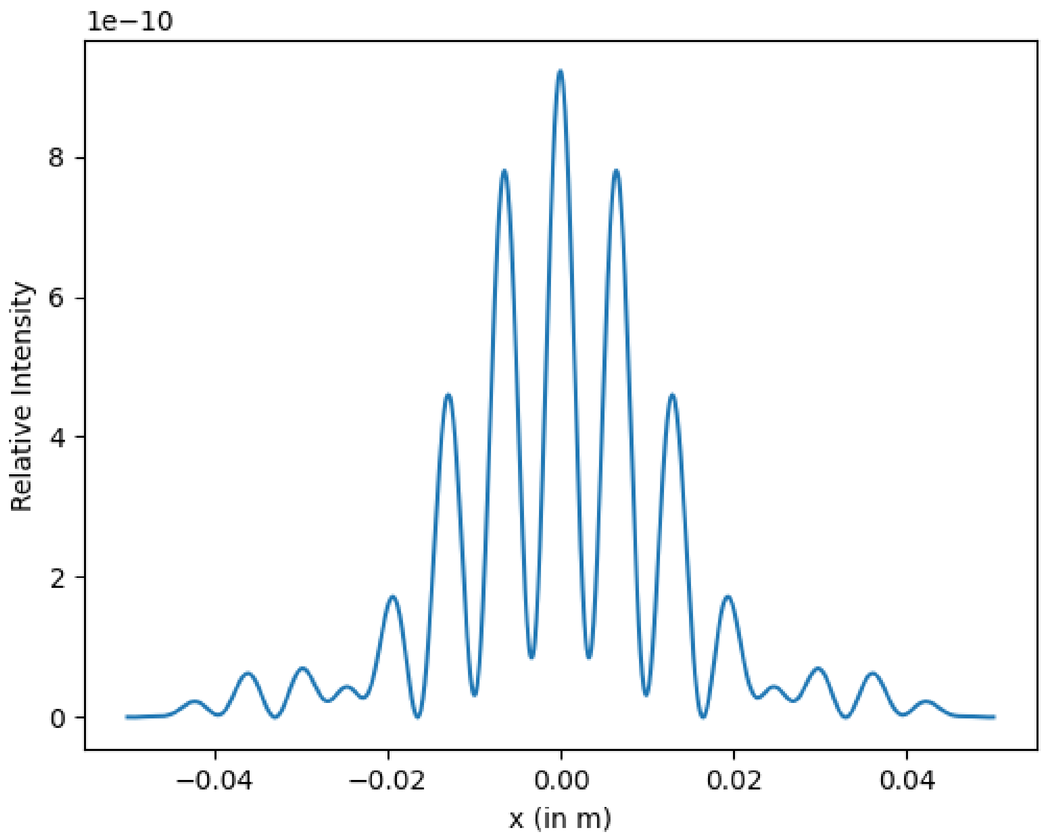


For $q(u) = \sin^2(\alpha u)\sin^2(\beta u)$

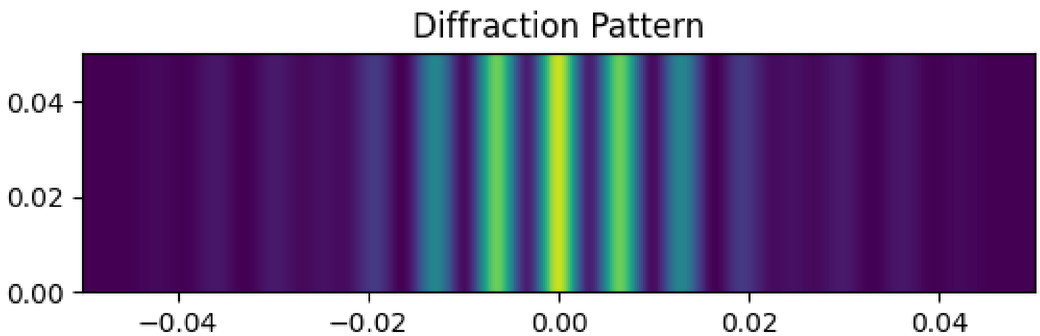
(i)



The schematic of the grating



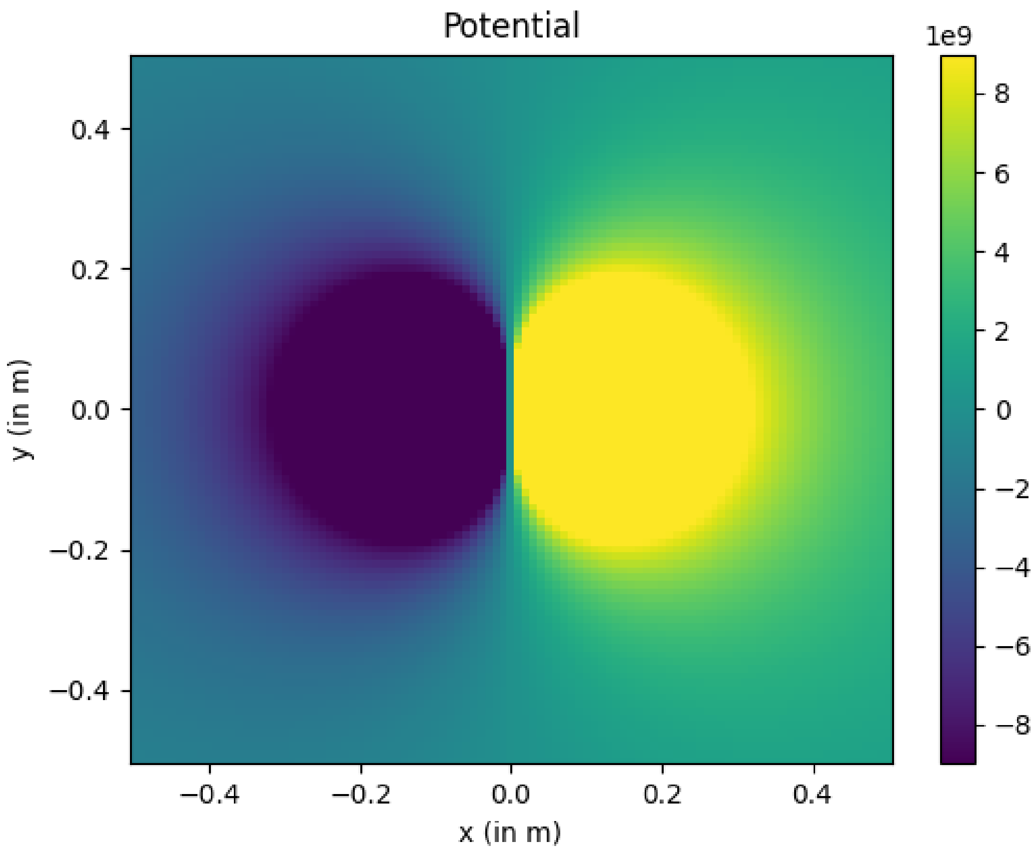
For $q(u)$ = Piecewise 0 and 1



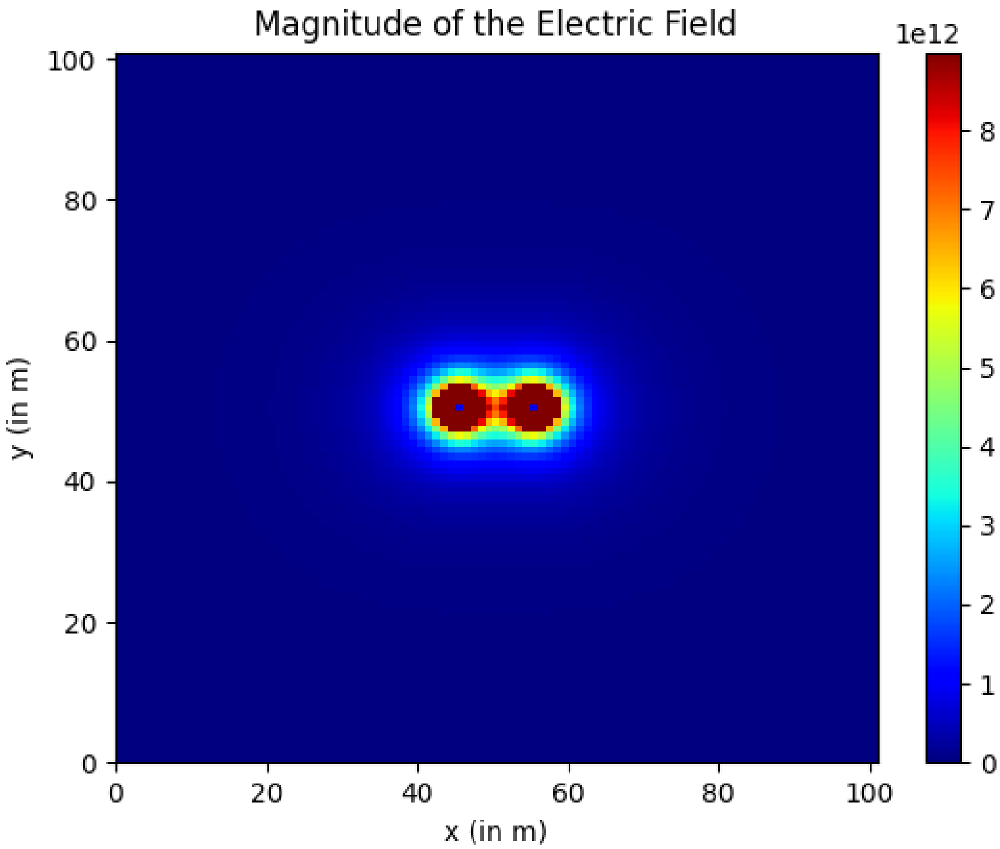
For $q(u)$ = Piecewise 0 and 1

Exercise 17 (b):

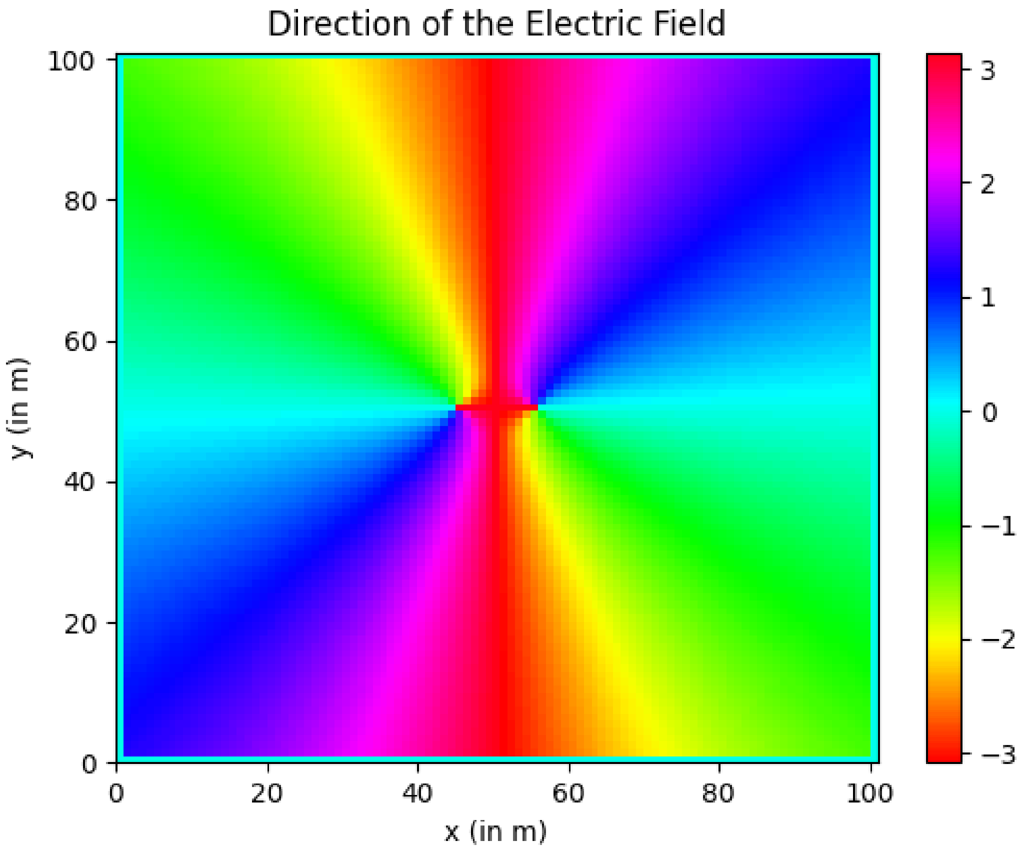
$$V(x,y) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-0.01)^2+y^2}} - \frac{1}{\sqrt{(x+0.01)^2+y^2}}\right) \tag{1}$$



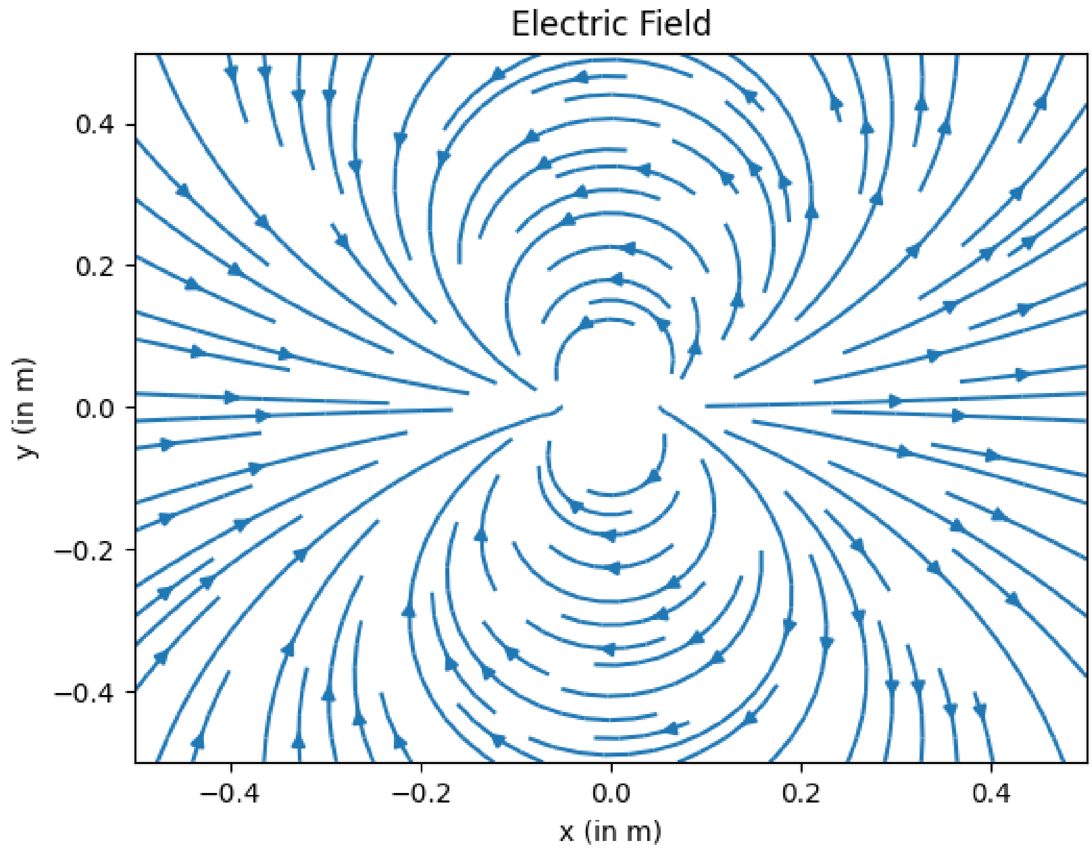
Plot of the Potential on a 1x1 m^2 surface



Plot of the magnitude of electric field on a 1x1 m^2 surface



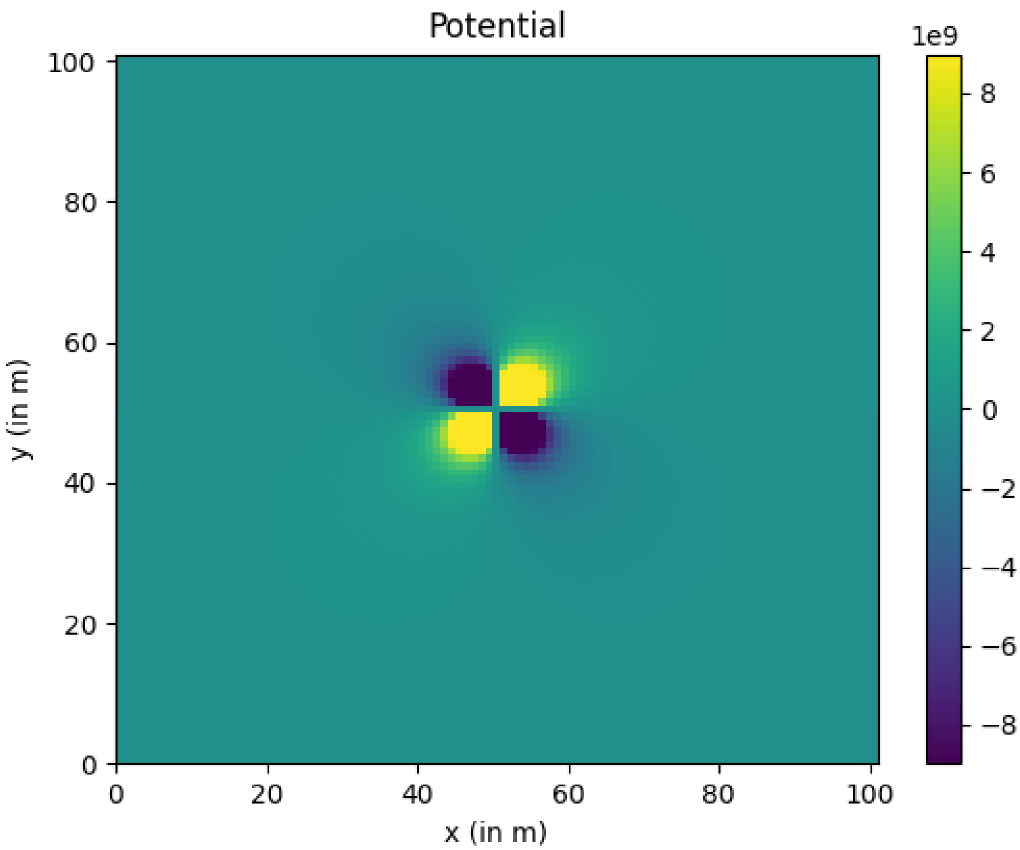
Plot of the direction of electric field on a 1x1 m^2 surface



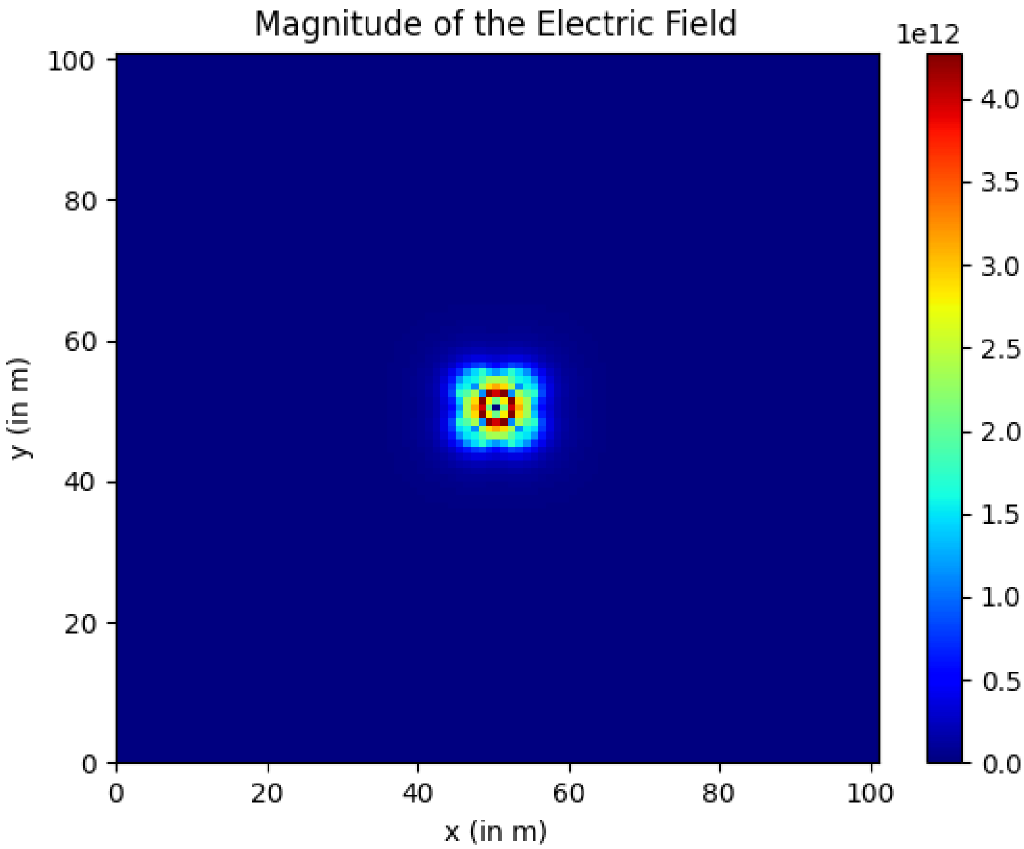
Plot of the electric field on a 1x1 m^2 surface

Exercise 17 (c):

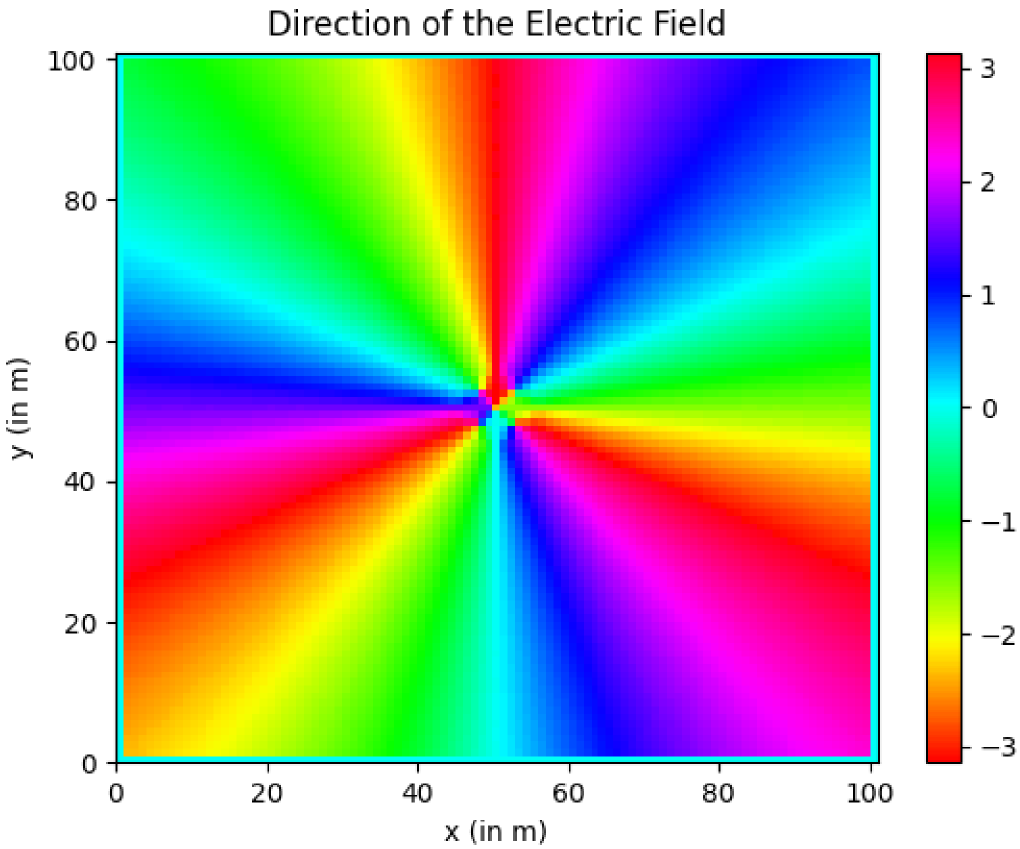
$$V(x,y) = \frac{1}{4\pi\epsilon_0}\int_{-L/2}^{L/2} dx' \int_{-L/2}^{L/2} dy' \frac{q_0 \sin(\frac{2\pi x'}{L})\sin(\frac{2\pi y'}{L})}{\sqrt{(x-x')^2+(y-y')^2}} \tag{2}$$



Plot of the Potential on a 1x1 m^2 surface



Plot of the magnitude of electric field on a 1x1 m^2 surface

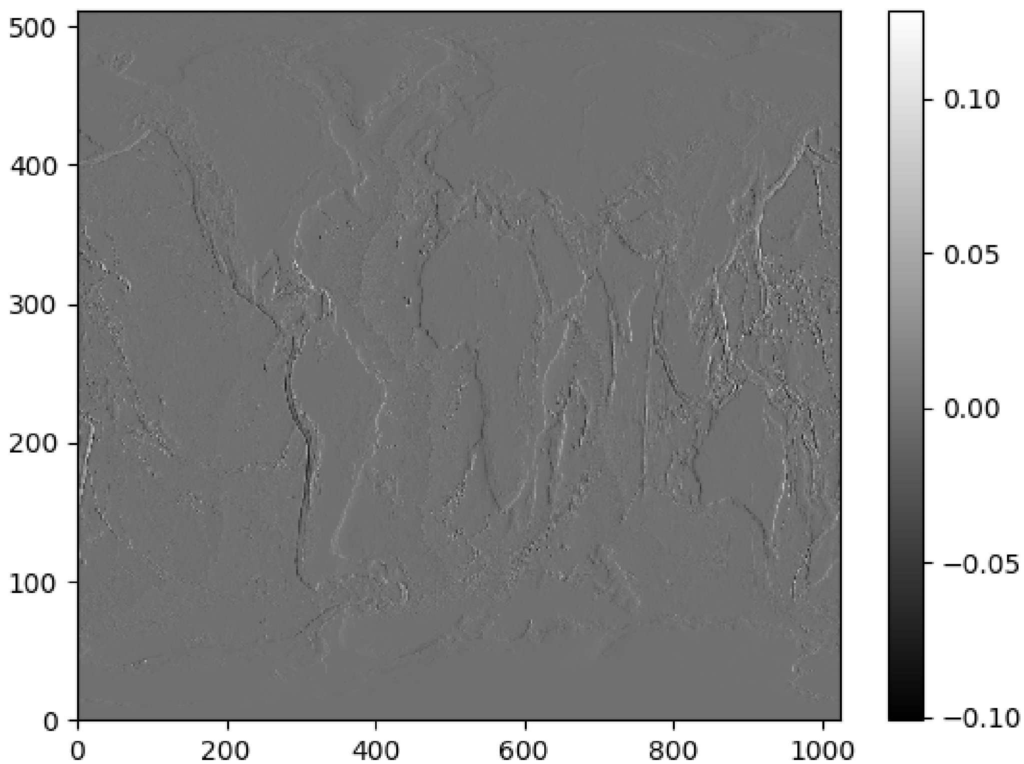


Plot of the direction of electric field on a 1x1 m^2 surface

Exercise 18

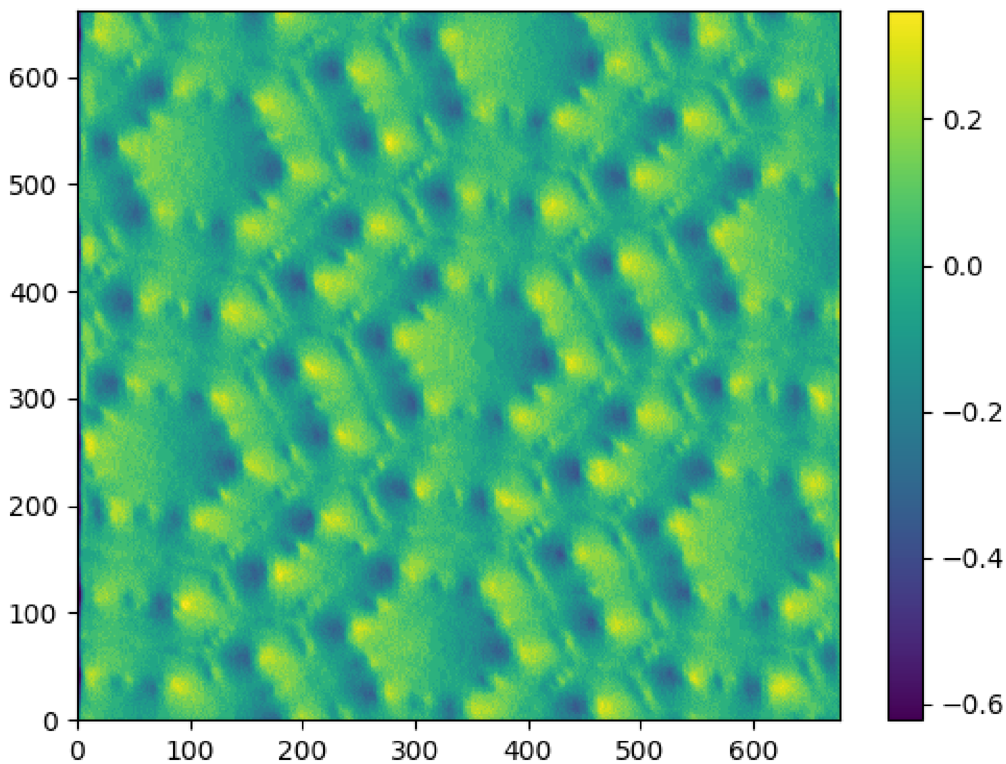
```
The mth derivative of the function for each m is as follows:
The 1th derivative at z = 0 is 1.9999999999999978
The 2th derivative at z = 0 is 3.9999999999999982
The 3th derivative at z = 0 is 7.999999999999985
The 4th derivative at z = 0 is 16.0
The 5th derivative at z = 0 is 32.00000000000001
The 6th derivative at z = 0 is 63.999999999999964
The 7th derivative at z = 0 is 127.99999999999869
The 8th derivative at z = 0 is 255.9999999999803
The 9th derivative at z = 0 is 511.999999999951
The 10th derivative at z = 0 is 1023.9999999991169
The 11th derivative at z = 0 is 2047.9999999591387
The 12th derivative at z = 0 is 4095.9999999328834
The 13th derivative at z = 0 is 8192.000002590277
The 14th derivative at z = 0 is 16383.999980568917
The 15th derivative at z = 0 is 32767.998795431035
The 16th derivative at z = 0 is 65535.997189001006
The 17th derivative at z = 0 is 131072.02920286346
The 18th derivative at z = 0 is 262143.04369028722
The 19th derivative at z = 0 is 524211.3389932175
The 20th derivative at z = 0 is 1048164.0241144444
```

Exercise 19 (b):



Map of the altitudes of points on Earth's surface

Exercise 19 (c):



Map of the altitudes of points on Earth's surface