

Exercise 4:**Part (a) and (b):**

At stationary state the derivatives vanishes, thus giving us:

$$\frac{dx}{dt} = 0 = -x + ay + x^2y \quad (1)$$

$$\frac{dy}{dt} = 0 = b - ay - x^2y \quad (2)$$

Adding the above two equations

$$x = b \quad (3)$$

$$y = \frac{b}{a + b^2} \quad (4)$$

From eqn (1)

$$x = y(a + x^2) \quad (5)$$

$$y = \frac{b}{a + x^2} \quad (6)$$

Exercise 8:

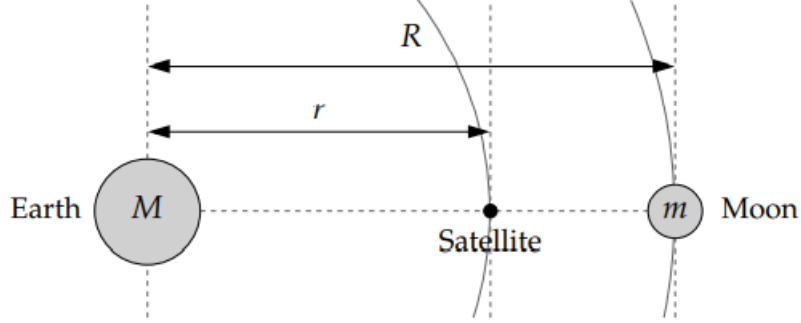


Figure 1: The schematic of the setup

The force equation of the satellite is:

$$\frac{GM_em}{r^2} - \frac{GM_m m}{(R-r)^2} = m\omega^2 r \quad (7)$$

where $M_e \equiv$ Mass of Earth, $M_m \equiv$ Mass of Moon and $m \equiv$ Mass of Satellite. Thus we have to find the root of the equation;

$$\frac{GM_e}{r^2} - \frac{GM_m}{(R-r)^2} - \omega^2 r = 0 \quad (8)$$

Exercise 9:

Current flowing in is equal to the current going out from the node V_1

$$\frac{V_+ - V_1}{R_1} - I_0 e^{V/V_T} - 1 - \frac{V_1}{R_2} = 0 \quad (9)$$

Current going out from the node V_+ is equal to currently flowing into the node $V_0 = 0$

$$\frac{V_+ - V_1}{R_1} + \frac{V_+ - V_2}{R_3} - \frac{V_1}{R_2} - \frac{V_2}{R_4} = 0 \quad (10)$$

Thus we solve the above two system of non-linear equations using the Newton's method.

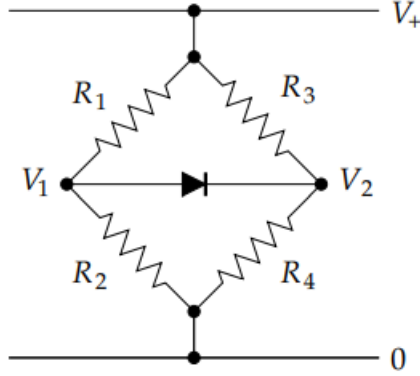


Figure 2: Circuit diagram

Exercise 10:**Part (a):**

Writing the equations of current at each junction.

At junction V_1

$$\frac{V_1 - V_2}{R} + \frac{V_1 - V_3}{R} + \frac{V_1 - V_4}{R} + \frac{V_1 - V_+}{R} = 0 \quad (11)$$

$$4V_1 - V_2 - V_3 - V_4 = V_+ \quad (12)$$

At junction V_2

$$\frac{V_2 - V_1}{R} + \frac{V_2 - V_4}{R} + \frac{V_2 - 0}{R} = 0 \quad (13)$$

$$3V_2 - V_1 - V_4 = 0 \quad (14)$$

At junction V_3

$$\frac{V_3 - V_1}{R} + \frac{V_3 - V_4}{R} + \frac{V_3 - V_+}{R} = 0 \quad (15)$$

$$3V_3 - V_1 - V_4 = V_+ \quad (16)$$

At junction V_4

$$\frac{V_4 - V_1}{R} + \frac{V_4 - V_2}{R} + \frac{V_4 - V_3}{R} + \frac{V_4 - 0}{R} = 0 \quad (17)$$

$$4V_4 - V_1 - V_2 - V_3 = 0 \quad (18)$$

Thus the matrix form of the system of equations is ($V_+ = 5$);

$$\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_+ \\ 0 \\ V_+ \\ 0 \end{pmatrix} \quad (19)$$

Exercise 13:

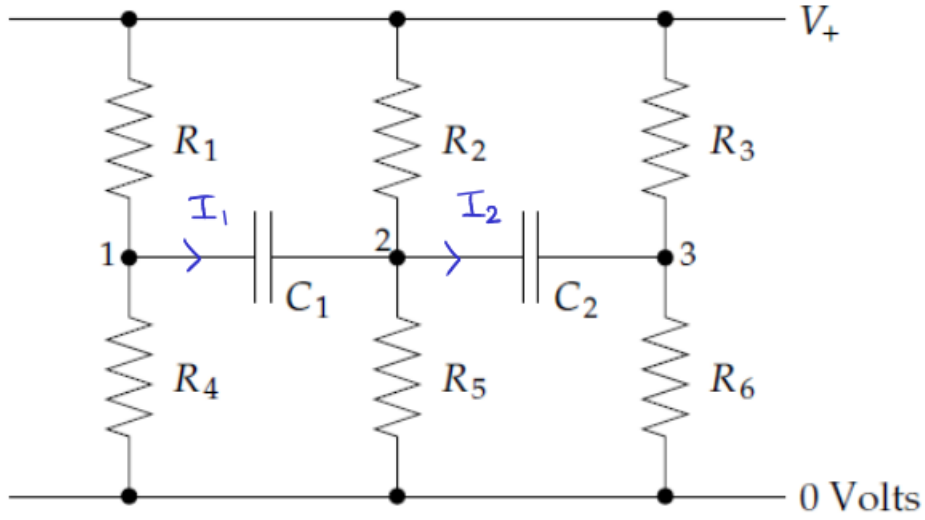


Figure 3: Circuit diagram

We analyse the points 1, 2 and 3.

Analysing V_1

$$\frac{V_1 - V_+}{R_1} + I_1 + \frac{V_1 - 0}{R_4} = 0 \quad (20)$$

Analysing V_2

$$\frac{V_2 - V_+}{R_2} - I_1 + I_2 + \frac{V_2 - 0}{R_5} = 0 \quad (21)$$

Analysing V_3

$$\frac{V_3 - V_+}{R_3} - I_2 + \frac{V_3 - 0}{R_6} = 0 \quad (22)$$

We assume the form of the potentials V_+ and V_i to be $x_+ e^{i\omega t}$ and $x_i e^{i\omega t}$. Now consider the potential difference across the capacitors.

Potential difference across C_1

$$x_1 e^{i\omega t} - x_2 e^{i\omega t} = \frac{q_1}{C_1} \quad (23)$$

$$i\omega(x_1 e^{i\omega t} - x_2 e^{i\omega t}) = \frac{I_1}{C_1} \quad (24)$$

Potential difference across C_2

$$x_2 e^{i\omega t} - x_3 e^{i\omega t} = \frac{q_2}{C_2} \quad (25)$$

$$i\omega(x_2 e^{i\omega t} - x_3 e^{i\omega t}) = \frac{I_2}{C_2} \quad (26)$$

Finally substituting for V_i and I_i we get:

$$\frac{x_1 - x_+}{R_1} + iC_1\omega(x_1 - x_2) + \frac{x_1 - 0}{R_4} = 0 \quad (27)$$

$$\frac{x_2 - x_+}{R_2} - iC_1\omega(x_1 - x_2) + iC_2\omega(x_2 - x_3) + \frac{x_2 - 0}{R_5} = 0 \quad (28)$$

$$\frac{x_3 - x_+}{R_3} - iC_2\omega(x_2 - x_3) + \frac{x_3 - 0}{R_6} = 0 \quad (29)$$

After rearranging the terms in the above equations:

$$\left(\frac{1}{R_1} + \frac{1}{R_4} + i\omega C_1\right)x_1 - i\omega C_1 x_2 = \frac{x_+}{R_1} \quad (30)$$

$$\left(\frac{1}{R_2} + \frac{1}{R_5} + i\omega C_1 + i\omega C_2\right)x_2 - i\omega C_2 x_3 - i\omega C_1 x_1 = \frac{x_+}{R_2} \quad (31)$$

$$\left(\frac{1}{R_3} + \frac{1}{R_6} + i\omega C_2\right)x_3 - i\omega C_2 x_2 = \frac{x_+}{R_3} \quad (32)$$

Exercise 14:

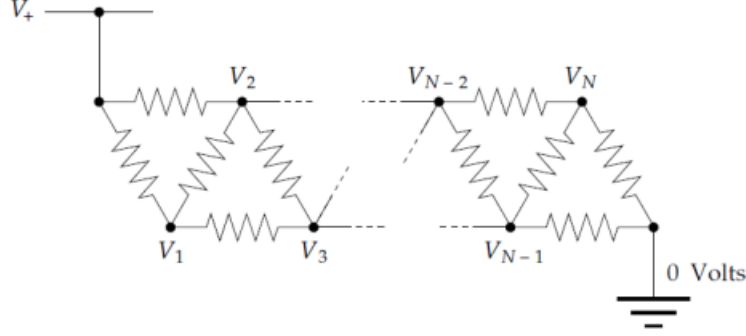


Figure 4: Circuit diagram

We perform nodal analysis on each point V_i

$$(V_1 - V_+) + (V_1 - V_2) + (V_1 - V_3) = 0 \quad (33)$$

$$3V_1 - V_2 - V_3 = V_+ \quad (34)$$

$$(V_2 - V_+) + (V_2 - V_1) + (V_2 - V_1) + (V_2 - V_4) = 0 \quad (35)$$

$$-V_1 + 4V_2 - V_3 - V_4 = V_+ \quad (36)$$

$$(V_i - V_{i-2}) + (V_i - V_{i-1}) + (V_i - V_{i+1}) + (V_i - V_{i+2}) = 0 \quad (37)$$

$$-V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} = 0 \quad (38)$$

Similarly

$$(V_N - 0) + (V_N - V_{N-2}) + (V_N - V_{N-1}) = 0 \quad (39)$$

$$-V_{N-1} - V_{N-2} + 3V_N = 0 \quad (40)$$

This can be represented generally as

$$\begin{pmatrix} 3 & -1 & -1 & -1 & \cdots & 0 \\ -1 & 4 & -1 & -1 & \cdots & 0 \\ -1 & -1 & 4 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} V_+ \\ V_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (41)$$

Exercise 16:**Part (a):**

The wavefunction given is of the form:

$$\psi(x) = \sum_n \psi_n \sin\left(\frac{n\pi x}{L}\right) \quad (42)$$

Substituting this in the Schrodinger equation,

$$\sum_n \psi_n \hat{H} \sin\left(\frac{n\pi x}{L}\right) = \sum_n E \psi_n \sin\left(\frac{n\pi x}{L}\right) \quad (43)$$

Using the completeness of sines, we multiply by $\sin(\frac{m\pi x}{L})$ and integrate the equation:

$$\sum_n \psi_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx = \sum_n E \psi_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \quad (44)$$

We know that $\int_0^L \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L}) dx = \frac{L}{2} \delta_{mn}$

$$\sum_n \psi_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx = E \psi_m \frac{L}{2} \quad (45)$$

By definition

$$H_{mn} := \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx \quad (46)$$

Hence we can rewrite the equation,

$$\sum_n \psi_n H_{mn} = E \psi_m \quad (47)$$

Thus considering H_{mn} as matrix elements of \hat{H} thus we can rewrite the above equation as:

$$\hat{H} \vec{\psi} = E \vec{\psi} \quad (48)$$

where $\vec{\psi} \equiv (\psi_1, \dots, \psi_n)$

Part (b):

We now compute the matrix element H_{mn} for the potential $V(x) = \frac{ax}{L}$.

$$H_{mn} = \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx \quad (49)$$

$$\hat{H} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{ax}{L}$$

$$H_{mn} = \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{ax}{L} \right) \sin\left(\frac{n\pi x}{L}\right) dx \quad (50)$$

$$H_{mn} = \frac{2}{L} \int_0^L \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L^2} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \frac{a}{L} \int_0^L x \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \quad (51)$$

Consider the case $m \neq n$ and m and n are both even or odd

$$H_{mn} = \frac{2}{L} \int_0^L \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L^2} \frac{L}{2} \delta_{mn} + 0 \quad (52)$$

$$H_{mn} = 0 \quad (53)$$

Consider the case $m \neq n$ one even and one odd

$$H_{mn} = 0 + \frac{-2}{L} \frac{a}{L} \frac{(2L)^2}{\pi^2} \frac{mn}{(m^2 - n^2)^2} \quad (54)$$

Consider the case $m = n$

$$H_{mn} = \frac{2}{L} \int_0^L \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L^2} \frac{L}{2} \delta_{mn} + \frac{2}{L} \frac{a}{L} \frac{L^2}{4} \quad (55)$$

Hence for H_{nm} we can see that for the case $m = n$ it is trivial. For the case $m \neq n$ and m and n are both even or odd, $H_{mn} = 0$. Finally for the case $m \neq n$ one even and one odd we can see that

$$-\frac{(2L)^2}{\pi^2} \frac{mn}{(m^2 - n^2)^2} = -\frac{(2L)^2}{\pi^2} \frac{nm}{(n^2 - m^2)^2} = -\frac{(2L)^2}{\pi^2} \frac{nm}{(-1)^2(m^2 - n^2)^2} = -\frac{(2L)^2}{\pi^2} \frac{nm}{(m^2 - n^2)^2} \quad (56)$$

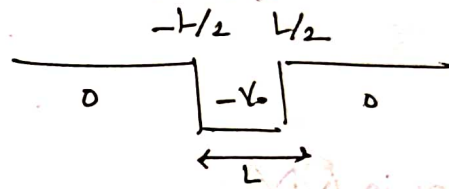
Hence $H_{mn} = H_{nm}$ is symmetric for all the cases.

Exercise 6

Potential Well

$$\Rightarrow E < 0 \quad \text{if} \quad E > -V_0$$

↳ Bound state



Outside the well $\psi(x) = e^{\pm qx/\hbar}$

$$E = -\frac{q^2}{2m} < 0.$$

$$\times \frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = \times \frac{q^2}{2m} \psi \Rightarrow \psi = e^{\pm qx/\hbar}$$

as at $x > 0$ we can normalize

$$\therefore \psi(x) = \begin{cases} e^{-qx/\hbar} & x > L/2 \\ e^{qx/\hbar} & x < -L/2 \end{cases}$$

as at $x < 0$ we can normalize.

Assume

$$\psi(x) = 1 \cdot e^{-qx/\hbar} \quad x > L/2$$

$$a e^{qx/\hbar} \quad x < L/2$$

Inside the well:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - V_0 \psi = E \psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = - \frac{(E + V_0) 2m}{\hbar^2} \psi$$

$$\Rightarrow \psi = b_1 e^{ipx/\hbar} + b_2 e^{-ipx/\hbar} \quad -\frac{L}{2} < x < \frac{L}{2}$$

$$\Rightarrow \psi = \begin{cases} e^{-qx/\hbar} & x > L/2 \\ b_1 e^{ipx/\hbar} + b_2 e^{-ipx/\hbar} & -\frac{L}{2} < x < \frac{L}{2} \\ a e^{qx/\hbar} & x < -L/2 \end{cases} \quad \text{--- (4)}$$

Lecture-10

Unknowns: a, b_1, b_2, E .

4 - eqns: $\psi_{-e}(L/2) = \psi_e(L/2); \left(\frac{\partial \psi}{\partial x}\right)_{L/2-e} = \left(\frac{\partial \psi}{\partial x}\right)_{L/2+e}$
 $\psi_{-e}(-L/2) = \psi_e(-L/2); \left(\frac{\partial \psi}{\partial x}\right)_{-L/2-e} = \left(\frac{\partial \psi}{\partial x}\right)_{-L/2+e}$

Symmetry in QM:

Suppose that there is a symmetry $\rightarrow \exists$ a unitary transformation U

such that $U H U^{-1} = H$.

$$\psi \longrightarrow U \psi$$

Unitary transformation: U

Any operator O

$$O \longrightarrow O' = U O U^{-1}$$

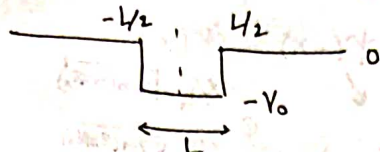
Any wave fn ψ

$$\psi \longrightarrow \psi' = U \psi$$

$$O \psi \longrightarrow O' \psi' = U O U^{-1} U \psi = U(O \psi) \rightarrow$$

If $UOU^{-1} = O \Rightarrow O$ is invariant under unitary transformation U .

In potential well: $x \rightarrow -x$ (Symmetry).



$$P\hat{x}P^{-1} = -\hat{x}$$

$$P\hat{p}P^{-1} = -\hat{p}$$

$$P\hat{H}P^{-1} = \hat{H}$$

$$U\hat{x}U^{-1} = -\hat{x}$$

$$U\hat{p}U^{-1} = -\hat{p}$$

Parity / Inversion.

$$P \equiv U$$

$$P\psi(x) = \psi(-x)$$

If $\psi(x)$ satisfies $H\psi = E\psi$

$$P\hat{H}P^{-1}P\psi = EP\psi$$

$$\rightarrow P\hat{H}\psi = EP\psi$$

$$\hat{H}P\psi = EP\psi$$

$$\Rightarrow \hat{H}\psi(-x) = E\psi(-x)$$

$$\psi(x) = \lambda(\psi(-x))$$

indep. of x .

Assume that all energy levels are non-degenerate. For any allowed E , there is only wave fn. $\psi(x)$ which satisfies $H\psi = E\psi$.

$$P\psi(x) = \psi(-x) = \lambda\psi(x)$$

$$\Rightarrow \boxed{P\psi(x) = \lambda\psi(x)}$$

\rightarrow Given that $P\psi(x) = \psi(-x)$ and $\psi(x)$ is eigenstate of P , what are the allowed values of λ ?

$$P\psi(x) = \psi(-x) = \lambda\psi(x)$$

$$P^2\psi(x) = PP\psi(x) = P\psi(-x) = \psi(x) = \lambda^2\psi(x)$$

$$P^2\psi(x) = \lambda^2\psi(x) = \lambda^2\psi(x)$$

$$\Rightarrow \lambda^2 = 1 \quad \boxed{\lambda = \pm 1}$$

$$\Rightarrow \psi(-x) = \psi(x) \quad (0x) \\ = -\psi(x)$$

Now from (1)

Even fn. of x : $\psi(-x) = \psi(x)$

$$\Rightarrow a = 1 \quad \begin{cases} e^{-x/\hbar \sqrt{2m(E+V_0)}} & x > L/2 \\ e^{x/\hbar \sqrt{2m(E+V_0)}} & x < L/2 \end{cases}$$

$$\alpha \cos\left(\frac{x}{\hbar} \sqrt{2m(E+V_0)}\right) \quad -\frac{L}{2} < x < \frac{L}{2}$$

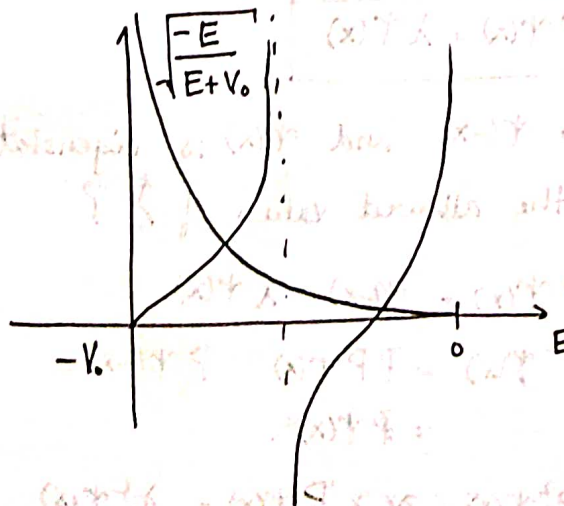
Match ψ & $\frac{\partial \psi}{\partial x}$ at $x = \frac{L}{2}$ (automatically matches at $x = -\frac{L}{2}$ by symmetry)

$$\begin{aligned} e^{-L/2\hbar} &= \alpha \cos\left(\frac{L}{2\hbar} P\right) \\ -\frac{q}{\hbar} e^{-L/2\hbar} &= -\frac{\alpha q}{\hbar} \sin\left(\frac{L}{2\hbar} P\right) \end{aligned} \quad \begin{aligned} q &= \sqrt{-2mE} \\ P &= \sqrt{2m(E+V_0)} \end{aligned}$$

$$\Rightarrow \frac{q}{P} \cos\left(\frac{L}{2\hbar} P\right) = \frac{q}{P} \sin\left(\frac{L}{2\hbar} P\right)$$

$$\frac{q}{P} = \tan\left(\frac{L}{2\hbar} P\right)$$

$$\sqrt{\frac{-E}{E+V_0}} = \tan\left(\frac{L}{2\hbar} \sqrt{2m(E+V_0)}\right)$$



$\Rightarrow \exists$ at least one E no matter the value $L \geq V_0$.

Odd fn of x $\psi(-x) = -\psi(x)$

$$a = 1 \quad \begin{cases} e^{-x/\hbar^2} & x > L/2 \\ -e^{x/\hbar^2} & x < L/2 \end{cases}$$

$$\alpha' \sin\left(\frac{x p}{\hbar}\right) \quad -L/2 < x < L/2$$

Similarly to previous step/case:

$$e^{-L/2\hbar^2} = \alpha' \cos\left(\frac{L p}{2\hbar}\right)$$

$$-\frac{p}{\hbar} e^{-L/2\hbar^2} = \frac{p \alpha'}{\hbar} \sin\left(\frac{L p}{2\hbar}\right)$$

$$\Rightarrow \frac{p}{p} = -\cot\left(\frac{L p}{2\hbar}\right)$$

parameters if \exists a E which satisfies the eqn.

1.) For even fn of x \exists at least one Bound state

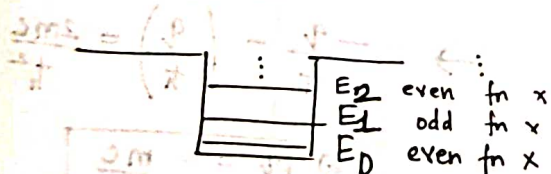
2.) — odd fn — there may not be —

3.) For even fn, the energy levels, E , $n\pi < \frac{L p}{2\hbar} < (n+1)\pi$

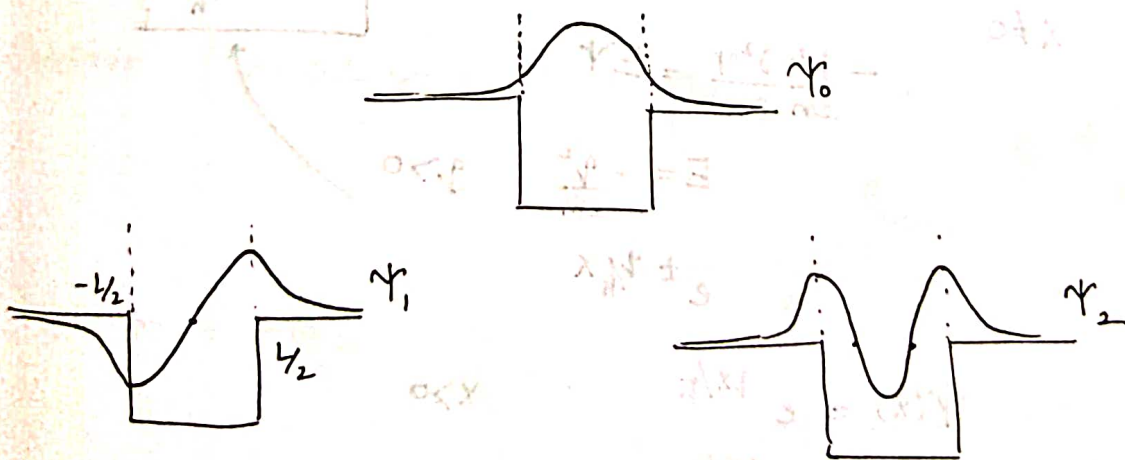
— odd fn .

$n \in \mathbb{N}$.

$$(n+1/2)\pi < \frac{L p}{2\hbar} < (n+1)\pi \quad n \in \mathbb{N}$$



E_n , $n \in \mathbb{N}$ has n nodes



Condition for a bound to appear at $E=0$

even fn: $\tan\left(\frac{L \sqrt{2mV_0}}{2\hbar}\right) = 0 \quad \frac{L}{2\hbar} \sqrt{2mV_0} = (n+1)\pi \quad n \in \mathbb{N}$

odd fn: $\cot\left(\frac{L \sqrt{2mV_0}}{2\hbar}\right) = 0 \quad \text{---} = (n+1/2)\pi$