Exercise 4:
Part (a) and (b):
At stationary state the derivatives vanishes, thus giving us:

$$\frac{dx}{dt} = 0 = -x + ay + x^2y \tag{1}$$
$$\frac{dy}{dt} = 0 = b - ay - x^2y \tag{2}$$

Adding the above two equations

$$x = b \tag{3}$$
$$y = \frac{b}{a + b^2} \tag{4}$$

From eqn (1)

$$x = y(a + x^2) \tag{5}$$
$$y = \frac{b}{a + x^2} \tag{6}$$

Exercise 8:

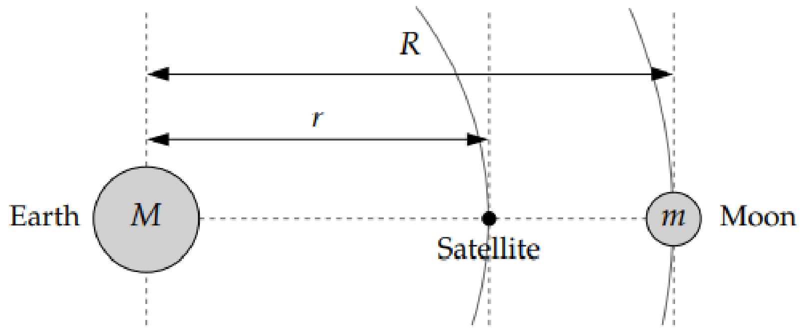


Figure 1: The schematic of the setup

The force equation of the satellite is:

$$\frac{GM_e m}{r^2} - \frac{GM_m m}{(R-r)^2} = m\omega^2 r \tag{7}$$

where $M_e \equiv$ Mass of Earth, $M_m \equiv$ Mass of Moon and $m \equiv$ Mass of Satellite. Thus we have to find the root of the equation;

$$\frac{GM_e}{r^2} - \frac{GM_m}{(R-r)^2} - \omega^2 r = 0 \tag{8}$$

Exercise 9:

Current flowing in is equal to the current going out from the node V_1

$$\frac{V_+ - V_1}{R_1} - I_0 e^{V/V_T} - 1 - \frac{V_1}{R_2} = 0 \tag{9}$$

Current going out from the node V_+ is equal to currently flowing into the node $V_0 = 0$

$$\frac{V_+ - V_1}{R_1} + \frac{V_+ - V_2}{R_3} - \frac{V_1}{R_2} - \frac{V_2}{R_4} = 0 \tag{10}$$

Thus we solve the above two system of non-linear equations using the Newton's method.

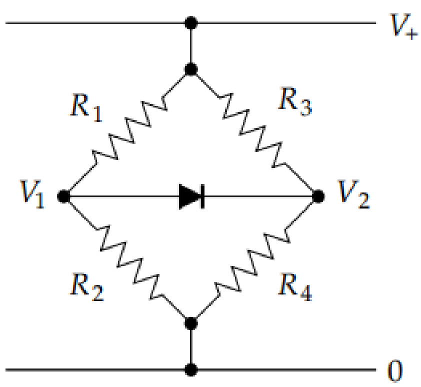


Figure 2: Circuit diagram

Exercise 10:
Part (a):
Writing the equations of current at each junction.

At junction V_1

$$\frac{V_1 - V_2}{R} + \frac{V_1 - V_3}{R} + \frac{V_1 - V_4}{R} + \frac{V_1 - V_+}{R} = 0 \tag{11}$$
$$4V_1 - V_2 - V_3 - V_4 = V_+ \tag{12}$$

At junction V_2

$$\frac{V_2 - V_1}{R} + \frac{V_2 - V_4}{R} + \frac{V_2 - 0}{R} = 0 \tag{13}$$
$$3V_2 - V_1 - V_4 = 0 \tag{14}$$

At junction V_3

$$\frac{V_3 - V_1}{R} + \frac{V_3 - V_4}{R} + \frac{V_3 - V_+}{R} = 0 \tag{15}$$
$$3V_3 - V_1 - V_4 = V_+ \tag{16}$$

At junction V_4

$$\frac{V_4 - V_1}{R} + \frac{V_4 - V_2}{R} + \frac{V_4 - V_3}{R} + \frac{V_4 - 0}{R} = 0 \tag{17}$$
$$4V_4 - V_1 - V_2 - V_3 = 0 \tag{18}$$

Thus the matrix form of the system of equations is ($V_+ = 5$);

$$\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_+ \\ 0 \\ V_+ \\ 0 \end{pmatrix} \tag{19}$$

Exercise 13:

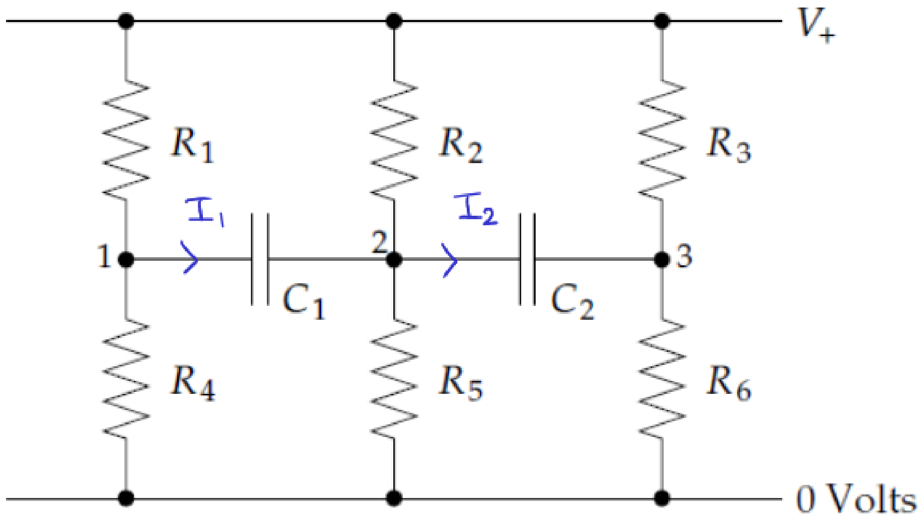


Figure 3: Circuit diagram

We analyse the points 1, 2 and 3.

Analysing V_1

$$\frac{V_1 - V_+}{R_1} + I_1 + \frac{V_1 - 0}{R_4} = 0 \tag{20}$$

Analysing V_2

$$\frac{V_2 - V_+}{R_2} - I_1 + I_2 + \frac{V_2 - 0}{R_5} = 0 \tag{21}$$

Analysing V_3

$$\frac{V_3 - V_+}{R_3} - I_2 + \frac{V_3 - 0}{R_6} = 0 \tag{22}$$

We assume the form of the potentials V_+ and V_i to be $x_+e^{i\omega t}$ and $x_ie^{i\omega t}$. Now consider the potential difference across the capacitors.

Potential difference across C_1

$$x_1 e^{i\omega t} - x_2 e^{i\omega t} = \frac{q_1}{C_1} \tag{23}$$

$$i\omega(x_1 e^{i\omega t} - x_2 e^{i\omega t}) = \frac{I_1}{C_1} \tag{24}$$

Potential difference across C_2

$$x_2 e^{i\omega t} - x_3 e^{i\omega t} = \frac{q_2}{C_2} \tag{25}$$

$$i\omega(x_2 e^{i\omega t} - x_3 e^{i\omega t}) = \frac{I_2}{C_2} \tag{26}$$

Finally substituting for V_i and I_i we get:

$$\frac{x_1 - x_+}{R_1} + iC_1\omega(x_1 - x_2) + \frac{x_1 - 0}{R_4} = 0 \tag{27}$$

$$\frac{x_2 - x_+}{R_2} - iC_1\omega(x_1 - x_2) + iC_2\omega(x_2 - x_3) + \frac{x_2 - 0}{R_5} = 0 \tag{28}$$

$$\frac{x_3 - x_+}{R_3} - iC_2\omega(x_2 - x_3) + \frac{x_3 - 0}{R_6} = 0 \tag{29}$$

After rearranging the terms in the above equations:

$$\big(\frac{1}{R_1} + \frac{1}{R_4} + i\omega C_1\big)x_1 - i\omega C_1 x_2 = \frac{x_+}{R_1} \tag{30}$$

$$\big(\frac{1}{R_2} + \frac{1}{R_5} + i\omega C_1 + i\omega C_2\big)x_2 - i\omega C_2 x_3 - i\omega C_1 x_1 = \frac{x_+}{R_2} \tag{31}$$

$$\big(\frac{1}{R_3} + \frac{1}{R_6} + i\omega C_2\big)x_3 - i\omega C_2 x_2 = \frac{x_+}{R_3} \tag{32}$$

Exercise 14:

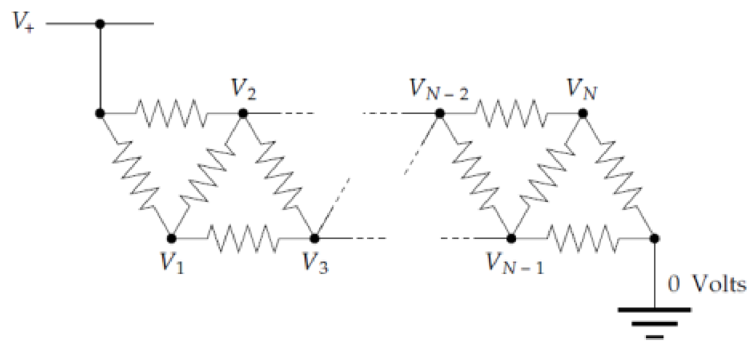


Figure 4: Circuit diagram

We perform nodal analysis on each point V_i

$$(V_1 - V_+) + (V_1 - V_2) + (V_1 - V_3) = 0 \tag{33}$$
$$3V_1 - V_2 - V_3 = V_+ \tag{34}$$
$$(V_2 - V_+) + (V_2 - V_1) + (V_2 - V_1) + (V_2 - V_4) = 0 \tag{35}$$
$$-V_1 + 4V_2 - V_3 - V_4 = V_+ \tag{36}$$
$$(V_i - V_{i-2}) + (V_i - V_{i-1}) + (V_i - V_{i+1}) + (V_i - V_{i+2}) = 0 \tag{37}$$
$$-V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} = 0 \tag{38}$$

Similarly

$$(V_N - 0) + (V_N - V_{N-2}) + (V_N - V_{N-1}) = 0 \tag{39}$$
$$-V_{N-1} - V_{N-2} + 3V_N = 0 \tag{40}$$

This can be represented generally as

$$\begin{pmatrix} 3 & -1 & -1 & -1 & \cdots & 0 \\ -1 & 4 & -1 & -1 & \cdots & 0 \\ -1 & -1 & 4 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} V_+ \\ V_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{41}$$

Exercise 16:
Part (a):
The wavefunction given is of the form:

$$\psi(x) = \sum_n \psi_n \sin(\frac{n\pi x}{L}) \tag{42}$$

Substituting this in the Schrodinger equation,

$$\sum_n \psi_n \hat{H} \sin(\frac{n\pi x}{L}) = \sum_n E\psi_n \sin(\frac{n\pi x}{L}) \tag{43}$$

Using the completeness of sines, we multiply by $\sin(\frac{m\pi x}{L})$ and integrate the equation:

$$\sum_n \psi_n \int_0^L \sin(\frac{m\pi x}{L})\hat{H}\sin(\frac{n\pi x}{L})dx = \sum_n E\psi_n \int_0^L \sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx \tag{44}$$

We know that $\int_0^L \sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx = \frac{L}{2}\delta_{mn}$

$$\sum_n \psi_n \int_0^L \sin(\frac{m\pi x}{L})\hat{H}\sin(\frac{n\pi x}{L})dx = E\psi_m \frac{L}{2} \tag{45}$$

By definition

$$H_{mn} := \frac{2}{L}\int_0^L \sin(\frac{m\pi x}{L})\hat{H}\sin(\frac{n\pi x}{L})dx \tag{46}$$

Hence we can rewrite the equation,

$$\sum_n \psi_n H_{mn} = E\psi_m \tag{47}$$

Thus considering H_{mn} as matrix elements of $\hat{H}$ thus we can rewrite the above equation as:

$$\hat{H}\vec{\psi} = E\vec{\psi} \tag{48}$$

where $\vec{\psi} \equiv (\psi_1, \cdots, \psi_n)$
Part (b):
We now compute the matrix element $H_m n$ for the potential $V(x) = \frac{ax}{L}$.

$$H_{mn} = \frac{2}{L}\int_0^L \sin(\frac{m\pi x}{L})\hat{H}\sin(\frac{n\pi x}{L})dx \tag{49}$$

$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{ax}{L}$

$$H_{mn} = \frac{2}{L}\int_0^L \sin(\frac{m\pi x}{L})(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{ax}{L})\sin(\frac{n\pi x}{L})dx \tag{50}$$

$$H_{mn} = \frac{2}{L}\int_0^L \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}\sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx + \frac{2}{L}\frac{a}{L}\int_0^L x\sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx \tag{51}$$

Consider the case $m \neq n$ and m and n are both even or odd

$$H_{mn} = \frac{2}{L}\int_0^L \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}\frac{L}{2}\delta_{mn} + 0 \tag{52}$$

$$H_{mn} = 0 \tag{53}$$

Consider the case $m \neq n$ one even and one odd

$$H_{mn} = 0 + \frac{-2}{L}\frac{a}{L}\frac{(2L)^2}{\pi^2}\frac{mn}{(m^2-n^2)^2} \tag{54}$$

Consider the case $m = n$

$$H_{mn} = \frac{2}{L}\int_0^L \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}\frac{L}{2}\delta_{mn} + \frac{2}{L}\frac{a}{L}\frac{L^2}{4} \tag{55}$$

Hence for H_{nm} we can see that for the case $m = n$ it is trivial. For the case $m \neq n$ and m and n are both even or odd, $H_{mn} = 0$. Finally for the case $m \neq n$ one even and one odd we can see that

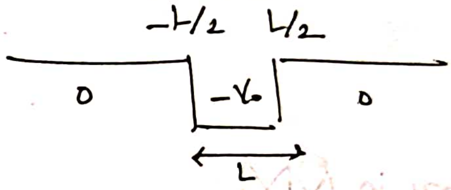
$$-\frac{(2L)^2}{\pi^2}\frac{mn}{(m^2-n^2)^2} = -\frac{(2L)^2}{\pi^2}\frac{nm}{(n^2-m^2)^2} = -\frac{(2L)^2}{\pi^2}\frac{nm}{(-1)^2(m^2-n^2)^2} = -\frac{(2L)^2}{\pi^2}\frac{nm}{(m^2-n^2)^2} \tag{56}$$

Hence $H_{mn} = H_{nm}$ is symmetric for all the cases.

Potential Well

$\Rightarrow E < 0$ & $E > -V_0$

↳ Bound state



Outside the well $\quad \psi(x) = e^{\pm qx/\hbar}$

$$E = -\frac{q^2}{2m} < 0 .$$

$$\cancel{-}\frac{\hbar^2}{\cancel{2m}}\frac{d^2\psi}{dx^2} = \cancel{-}\frac{q^2}{\cancel{2m}}\psi \Rightarrow \psi = e^{\pm q/\hbar\, x}$$

$$\therefore \quad \psi(x) = \begin{cases} e^{-qx/\hbar} & x > L/2 \\ e^{qx/\hbar} & x < -L/2 \end{cases}$$

$e^{-qx/\hbar} \rightarrow$ as at $x>0$ we can normalize

$e^{qx/\hbar} \rightarrow$ as at $x<0$ we can normalize.

Assume

$$\psi(x) = \begin{cases} 1 \cdot e^{-qx/\hbar} & x > L/2 \\ a\, e^{qx/\hbar} & x < L/2 \end{cases}$$

Inside the well.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} - V_0\psi = E\psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\overbrace{(E+V_0)\times 2m}^{p^2}}{\hbar^2}\psi$$

$$\Rightarrow \psi = b_1 e^{ipx/\hbar} + b_2 e^{-ipx/\hbar} \qquad -\frac{L}{2} < x < \frac{L}{2}$$

$$\Rightarrow \psi = \begin{cases} e^{-qx/\hbar} & x > L/2 \\ b_1 e^{ipx/\hbar} + b_2 e^{-ipx/\hbar} & -\frac{L}{2} < x < L/2 \\ a e^{qx/\hbar} & x < L/2 \end{cases} \quad \text{--- (4)}$$

Lecture-10

Unknowns: a, b_1, b_2, E.

4-eqns: $\psi_{-\epsilon}(L/2) = \psi_{\epsilon}(L/2)$; $\left(\frac{\partial\psi}{\partial x}\right)_{L/2-\epsilon} = \left(\frac{\partial\psi}{\partial x}\right)_{L/2+\epsilon}$

$\psi_{-\epsilon}(-L/2) = \psi_{+\epsilon}(-L/2)$; $\left(\frac{\partial\psi}{\partial x}\right)_{-L/2-\epsilon} = \left(\frac{\partial\psi}{\partial x}\right)_{-L/2+\epsilon}$

Symmetry in QM:

Suppose that there is a symmetry $\rightarrow$ $\exists$ a $\underset{\uparrow}{\overset{U}{}}$unitary transforma

such that $UHU^{-1} = H$.

$$\psi \longrightarrow U\psi$$

Unitary transformation : U

Any operator O

$$O \longrightarrow O' = UOU^{-1}$$

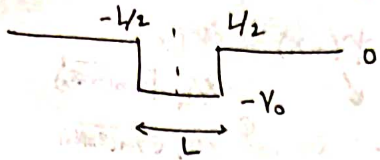
Any wave fn ψ

$$\psi \longrightarrow \psi' = U\psi.$$

$$O\psi \longrightarrow O'\psi' = UOU^{-1}U\psi$$
$$= U(O\psi) \rightarrow$$

If $UOU^{-1} = O$, $\Rightarrow$ O is invariant under unitary transformation U.

In potential well : $x \longrightarrow -x$ (Symmetry)



$$\left.\begin{array}{l} P\hat{x}P^{-1} = -\hat{x} \\ P\hat{p}P^{-1} = -\hat{p} \\ P\hat{H}P^{-1} = \hat{H} \end{array}\right\} \Leftarrow \begin{array}{l} U\hat{x}U^{-1} = -\hat{x} \\ U\hat{p}U^{-1} = -\hat{p} \end{array} \Bigg\} \text{ Parity / Inversion.}$$

$\hookrightarrow P \equiv U$

$$P\psi(x) = \psi(-x)$$

If $\psi(x)$ satisfies $H\psi = E\psi$

$$P\hat{H}P^{-1}P\psi = EP\psi$$

$$\cancel{\Rightarrow P\hat{H}\psi = EP\psi}$$

$$\hat{H}P\psi = EP\psi$$

$$\Rightarrow \hat{H}\psi(-x) = E\psi(-x)$$

$\psi(x) = \lambda(\psi(-x))$

$\hookrightarrow$ indep. of x.

Assume that all energy levels are non-degenerate. For any allowed E, there is only wave fn. $\psi(x)$ which satisfies $H\psi = E\psi$.

$$P\psi(x) = \psi(-x) = \lambda\psi(x)$$

$$\Rightarrow \boxed{P\psi(x) = \lambda\psi(x)}$$

$\rightarrow$ Given that $P(\psi(x)) = \psi(-x)$ and $\psi(x)$ is eigenstate of P, what are the allowed values of λ ?

$$P\psi(x) = \psi(-x) = \lambda\psi(x)$$

$$P^2\psi(x) = PP\psi(x) = P\psi(-x)$$

$$= P\psi(x).$$

$$P^2\psi(x) = \lambda P\psi(x) = \lambda^2\psi(x)$$

$$\Rightarrow \lambda^2 = 1 \quad \boxed{\lambda = \pm 1}$$

Any eigenstate ...

$$\Rightarrow \psi(-x) = \psi(x) \quad (\text{or})$$
$$= -\psi(x)$$

Now from (1)

Even fn. of x: $\psi(-x) = \psi(x)$.

$$\Rightarrow a = 1 \qquad \begin{cases} e^{-x/\hbar \sqrt{2m(E+V_0)}} & x > L/2 \\ e^{x/\hbar \sqrt{2m(E+V_0)}} & x < L/2 \\ \alpha \cos\left(\frac{x}{\hbar}\sqrt{2m(E+V_0)}\right) & -\frac{L}{2} < x < \frac{L}{2} \end{cases}$$

Match ψ & $\frac{\partial \psi}{\partial x}$ at $x = \frac{L}{2}$ (automatically matches at $x = -L/2$ by symmetry)

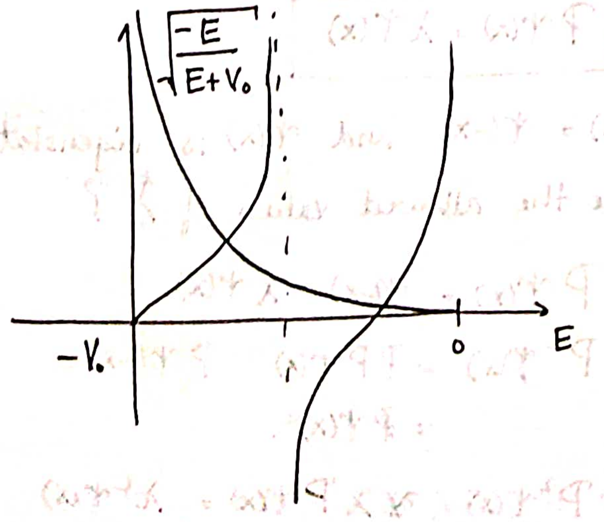
$$e^{-L/2\hbar\, q} = \alpha \cos\left(\frac{L}{2\hbar} p\right) \qquad q = \sqrt{-2mE}$$

$$-\frac{q}{\hbar} e^{-L/2\hbar\, q} = -\frac{\alpha p}{\hbar} \sin\left(\frac{L}{2\hbar} p\right) \qquad p = \sqrt{2m(E+V_0)}$$

$$\Rightarrow \cancel{-}\frac{q\alpha}{\cancel{\hbar}} \cos\left(\frac{L}{2\hbar} p\right) = \cancel{-}\frac{\alpha}{\cancel{\hbar}} p \sin\left(\frac{L}{2\hbar} p\right)$$

$$\frac{q}{p} = \tan\left(\frac{L}{2\hbar} p\right)$$

$$\sqrt{\frac{-E}{E+V_0}} = \tan\left(\frac{L}{2\hbar}\sqrt{2m(E+V_0)}\right)$$



$\Rightarrow \exists$ at least one E no matter the value L & V_0.

Odd fn of x $\psi(-x) = -\psi(x)$

$$a = 1 \qquad e^{-x/\hbar\, q} \qquad x > L/2$$

$$-e^{x/\hbar\, q} \qquad x < L/2$$

$$\alpha' \sin\left(\frac{xp}{\hbar}\right) \qquad -L/2 < x < L/2$$

Similarly to previous step/case:

$$e^{-L/2\hbar\, q} = \alpha' \cancel{\cos} \sin\left(\frac{Lp}{2\hbar}\right)$$

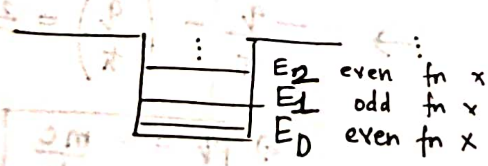
$$-\frac{q}{\hbar} e^{-L/2\hbar\, q} = \frac{p\alpha'}{\hbar} \cos\left(\frac{Lp}{2\hbar}\right)$$

$$\Rightarrow \quad \frac{q}{p} = -\cot\left(\frac{Lp}{2\hbar}\right)$$ → It depends ~~of~~ on the parameters if $\exists$ a E which satisfies the eqn.

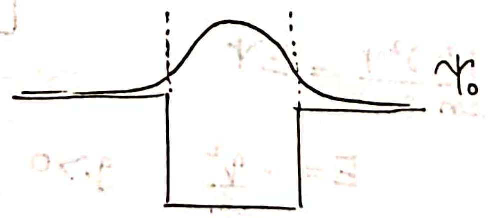
1.) For even fn of x $\exists$ at least one Bound state
2.) —11— odd fn —11— there may not by —11—
3.) For even fn, the energy levels, E, $n\pi < \frac{Lp}{2\hbar} < (n+\frac{1}{2})\pi$ $\quad n \in \mathbb{N}$.

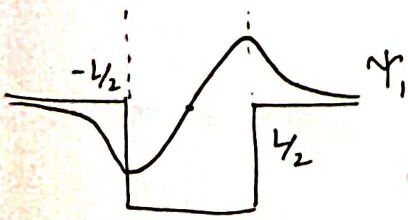
—11— odd fn.

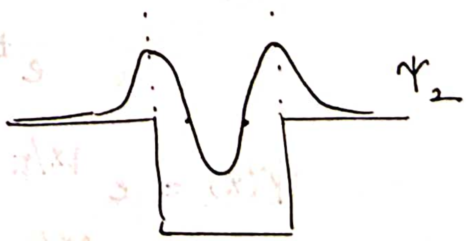
$$(n+\tfrac{1}{2})\pi < \frac{L}{2\hbar}p < (n+1)\pi \qquad n \in \mathbb{N}.$$



E_n, $n \in \mathbb{N}$ has n nodes







Condition for a bound to appear at $E = 0$

even fn: $\tan\left(\frac{L\sqrt{2mV_0}}{2\hbar}\right) = 0 \qquad \frac{L}{2\hbar}\sqrt{2mV_0} = (n+1)\pi \quad n \in \mathbb{N}$

odd fn: $\cot\left(\frac{L\sqrt{2mV_0}}{2\hbar}\right) = 0 \qquad$ —11— $= (n+\frac{1}{2})\pi$.