Question 4

a. At step n, after computing the $\{d\}_{i=0}^{k}$ values, the Max-Lloyd algorithm will compute the optimal representation levels based on these decision levels according to the following formula (as seen in class): $r_i^n = \frac{\int_{d_{i-1}^n}^{d_i^n} xp(x)dx}{\int_{d_{i-1}^n}^{d_i^n} p(x)dx}$.

In the case of a uniform $p(x)$, we can simplify the expression:

$$r_i^n = \frac{\int_{d_{i-1}^n}^{d_i^n} xp(x)dx}{\int_{d_{i-1}^n}^{d_i^n} p(x)dx} = \frac{\frac{1}{b-a}\int_{d_{i-1}^n}^{d_i^n} xdx}{\frac{1}{b-a}\int_{d_{i-1}^n}^{d_i^n} dx} = \frac{\frac{1}{2}x^2 \Big|_{d_{i-1}^n}^{d_i^n}}{d_i^n - d_{i-1}^n} = \frac{1}{2}\frac{(d_i^n)^2 - (d_{i-1}^n)^2}{d_i^n - d_{i-1}^n}$$

$$= \frac{1}{2}\frac{(d_i^n + d_{i-1}^n)(d_i^n - d_{i-1}^n)}{d_i^n - d_{i-1}^n} = \frac{d_i^n + d_{i-1}^n}{2}$$

For $i = 0$, we take $r_0^n = a$, and for $i = k$ we take $r_k^n = b$.

We got, as expected, that every representation level is the average of the 2 decision levels next to it.

Now, the algorithm computes the d^{n+1} values:

$$d_i^{n+1} = \frac{r_i^n + r_{i+1}^n}{2} = \frac{\frac{d_{i-1}^n + d_i^n}{2} + \frac{d_i^n + d_{i+1}^n}{2}}{2} = \frac{d_{i-1}^n + 2d_i^n + d_{i+1}^n}{4}.$$

To finish step $(n+1)$, the algorithm computes the r^{n+1} values: (we saw before that the integral formula becomes just an average)

$$r_i^{n+1} = \frac{d_i^{n+1} + d_{i-1}^{n+1}}{2} = \frac{1}{2}\left(\frac{r_i^n + r_{i+1}^n}{2} + \frac{r_{i-1}^n + r_i^n}{2}\right) = \frac{r_{i-1}^n + 2r_i^n + r_{i+1}^n}{4}$$

$$= \frac{1}{4}\left(\frac{d_{i-1}^n + d_{i-2}^n}{2} + 2\frac{d_i^n + d_{i-1}^n}{2} + \frac{d_{i+1}^n + d_i^n}{2}\right)$$

$$= \frac{d_{i-2}^n + 3d_{i-1}^n + 3d_i^n + d_{i+1}^n}{8}$$

b. The uniform quantization will be invariant through the Max-Lloyd algorithm:

$$d_i^n = a + i\frac{(b-a)}{k}, \forall 0 \le i \le k$$

$$r_i^n = a + \left(i - \frac{1}{2}\right)\frac{(b-a)}{k}, \forall 1 \le i \le k$$

We will show that this quantization satisfies $r_i^n = r_i^{n+1}, d_j^n = d_j^{n+1}$.

$$r_i^{n+1} = \frac{d_{i-1}^{n+1} + d_i^{n+1}}{2} = \frac{1}{2}\left(\left(a + (i-1)\frac{b-a}{k}\right) + \left(a + i\frac{b-a}{k}\right)\right)$$

$$= \frac{1}{2}\left(a + i\frac{b-a}{k} - \frac{b-a}{k} + a + i\frac{b-a}{k}\right)$$

$$= \frac{1}{2}\left(2a + (2i-1)\frac{b-a}{k}\right) = a + \left(i - \frac{1}{2}\right)\frac{b-a}{k} = r_i^n$$

$$d_i^{n+1} = \frac{r_i^n + r_{i+1}^n}{2} = \frac{1}{2}\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{k} + a + \left(i + 1 - \frac{1}{2}\right)\frac{b-a}{k}\right)$$
$$= \frac{1}{2}\left(2a + \left(i + i + 1 - \frac{1}{2} - \frac{1}{2}\right)\frac{b-a}{k}\right) = \frac{1}{2}\left(2a + 2i\frac{b-a}{k}\right)$$
$$= a + i\frac{b-a}{k} = d_i^n$$

c. We will formulate $d_i^{n+1} = \frac{d_{i-1}^n + 2d_i^n + d_{i+1}^n}{4} = \frac{1}{4}d_{i-1}^n + \frac{1}{2}d_i^n + \frac{1}{4}d_{i+1}^n$ into matrix form:

$$d^{n+1} = Ad^n \to \begin{pmatrix} d_0^{n+1} \\ \vdots \\ d_k^{n+1} \end{pmatrix} = A\begin{pmatrix} d_0^n \\ \vdots \\ d_k^n \end{pmatrix}$$

The matrix A will be of size $(k+1) \times (k+1)$.

For every $0 \leq i \leq k$, we want $d_i^{n+1} = A_i\begin{pmatrix} d_0^n \\ \vdots \\ d_k^n \end{pmatrix}$, where A_i is the i'th row of A.

$$\to \frac{1}{4}d_{i-1}^n + \frac{1}{2}d_i^n + \frac{1}{4}d_{i+1}^n = \sum_{j=1}^{k+1} A_{ij}d_j^n \to A_{i,j-1} = \frac{1}{4}, A_{i,j} = \frac{1}{2}, A_{i,j+1} = \frac{1}{4}$$

We will now look at the edge cases ($i = 0 \; or \; i = k$):

$$i = 0 \to d_0^n = a \to d_0^n = d_0^{n-1} \to A_{i,0} = 1$$
$$i = k \to d_k^n = b \to d_k^n = d_k^{n-1} \to A_{i,k} = 1$$

This result finds the constant $\boldsymbol{c_1 = 1}$**.**

So, we know that A looks as follows: $\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Therefore, we also found the constant $\boldsymbol{c_2 = \frac{1}{4}}$.

Also, we found that A can be written as $A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B} & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$

Where $\tilde{B}$ is a tridiagonal Toeplitz matrix: $\boldsymbol{\tilde{B}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \mathbf{0} & \mathbf{0} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \vdots \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

d. We will expand the expression we found for d^n:

$$d^n = Ad^{n-1} = A(Ad^{n-2}) = A^2 d^{n-2} = \cdots = A^i d^{n-i} = \cdots = A^n d^0$$
$$\to \boldsymbol{d^n = A^n d^0}$$

e. Since $\tilde{B}$ has the same values in the diagonals above and below its main diagonal, it is symmetric, and therefore diagonalizable (every real symmetric matrix is diagonalizable).

f. We will start by showing that x_0 and x_1 are eigenvectors of A, associated to the eigenvalue 1:

$$Ax_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(k-1) \\ 2k \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot 0 \\ \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 \\ \vdots \\ \frac{1}{4} \cdot 2(j-2) + \frac{1}{2} \cdot 2(j-1) + \frac{1}{4} \cdot 2j \\ \vdots \\ \frac{1}{4} \cdot 2(k-2) + \frac{1}{2} \cdot 2(k-1) + \frac{1}{4} \cdot 2k \\ 1 \cdot 2k \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(j-1) \\ \vdots \\ 2(k-1) \\ 2k \end{pmatrix}$$

$$= 1 \cdot x_0$$

$$Ax_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2k \\ 2(k-1) \\ \vdots \\ 2 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 \cdot 2k \\ \frac{1}{4} \cdot 2k + \frac{1}{2} \cdot 2(k-1) + \frac{1}{4} \cdot 2(k-2) \\ \vdots \\ \frac{1}{4} \cdot 2(k-j+2) + \frac{1}{2} \cdot 2(k-j+1) + \frac{1}{4} \cdot 2(k-j) \\ \vdots \\ \frac{1}{4} \cdot 4 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 0 \\ 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2k \\ 2(k-1) \\ \vdots \\ 2(k-j+1) \\ \vdots \\ 2 \\ 0 \end{pmatrix} =$$

$$1 \cdot x_1$$

We will now show that x_0 and x_1 are linearly independent:

Let a, b be constants such that $ax_0 + bx_1 = \vec{0}$:

$$\to a\begin{pmatrix}0\\2\\\vdots\\2(k-1)\\2k\end{pmatrix}+b\begin{pmatrix}2k\\2(k-1)\\\vdots\\2\\0\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix}$$

$$\to\begin{pmatrix}0\cdot a+2bk\\2a+2b(k-1)\\\vdots\\2a(k-1)+2b\\2ak+0\cdot b\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix}$$

From the first row we get: $2bk=0\to b=0$.

From the last row we get : $2ak=0\to a=0$.

We found that the only a,b that satisfy $ax_0+bx_1=\vec{0}$ are $a=b=0$, and so x_0 and x_1 are linearly independent.

g. We will use the following observation:

If $\tilde{B}e_{\lambda_1}=\lambda_i e_{\lambda_1}$ then

$$A\begin{pmatrix}v1\\|\\e_{\lambda_1}\\|\\v_{k+1}\end{pmatrix}=\begin{pmatrix}1&0&0&\cdots&0\\\frac{1}{4}&\frac{1}{2}&\frac{1}{4}&\cdots&0\\\vdots&\ddots&\ddots&\ddots&0\\0&0&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}v1\\|\\e_{\lambda_1}\\|\\v_{k+1}\end{pmatrix}$$

$$=\begin{pmatrix}1&0&\cdots&0&0\\\frac{1}{4}&&&&\vdots\\0&&&&\vdots\\\vdots&&\tilde{B}&&\vdots\\0&&&&0\\\vdots&&&&\frac{1}{4}\\0&0&\cdots&0&1\end{pmatrix}\begin{pmatrix}v1\\|\\e_{\lambda_1}\\|\\v_{k+1}\end{pmatrix}=$$

$$\left(\begin{pmatrix}1&0&\cdots&0&0\\\frac{1}{4}&&&&\vdots\\0&&&&\vdots\\\vdots&&0&&\vdots\\0&&&&0\\\vdots&&&&\frac{1}{4}\\0&0&\cdots&0&1\end{pmatrix}+\begin{pmatrix}0&0&\cdots&0&0\\0&&&&\vdots\\0&&&&\vdots\\\vdots&&\tilde{B}&&\vdots\\0&&&&0\\\vdots&&&&0\\0&0&\cdots&0&0\end{pmatrix}\right)\begin{pmatrix}v1\\|\\e_{\lambda_1}\\|\\v_{k+1}\end{pmatrix}=$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & 0 & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} v1 \\ | \\ e_{\lambda_1} \\ | \\ v_{k+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B} & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} v1 \\ | \\ e_{\lambda_1} \\ | \\ v_{k+1} \end{pmatrix} nn$$

$$= \begin{pmatrix} v_1 \\ \frac{1}{4} v_1 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} v_{k+1} \\ v_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ | \\ \tilde{B} e_{\lambda_1} \\ | \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ \frac{1}{4} v_1 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} v_{k+1} \\ v_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ | \\ \lambda_i e_{\lambda_1} \\ | \\ 0 \end{pmatrix}$$

So, if we pick a vector with $v_1 = v_{k+1} = 0$, then for every eigenvector e_{λ_i} of $\tilde{B}$, , the vector $\begin{pmatrix} 0 \\ | \\ e_{\lambda_i} \\ | \\ 0 \end{pmatrix}$ is an eigenvector of A with the eigenvalue λ_i:

$$A \begin{pmatrix} 0 \\ | \\ e_{\lambda_i} \\ | \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \cdot 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} \cdot 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ | \\ \lambda_i e_{\lambda_1} \\ | \\ 0 \end{pmatrix} = \lambda_i \begin{pmatrix} 0 \\ | \\ e_{\lambda_1} \\ | \\ 0 \end{pmatrix}$$

In this way we found $(k-1)$ eigenvectors of A, where every the multiplicity of the eigenvector $\begin{pmatrix} 0 \\ | \\ e_{\lambda_i} \\ | \\ 0 \end{pmatrix}$ is the same as the multiplicity of e_{λ_i} as an eigenvector of $\tilde{B}$. We need to find 2 more eigenvectors. Fortunately, we found these 2 eigenvectors in section (f), which are not in the form $\begin{pmatrix} 0 \\ | \\ \bar{v} \\ | \\ 0 \end{pmatrix}$, so we know that these are the last 2 eigenvectors, and they share the eigenvalue of 1.

$\tilde{B}$ is diagonalizable and so the algebraic multiplicity of its eigenvalues equals to their geometric multiplicity. Let's have a look at the characteristic polynomial of A:

$$p_A(\lambda) = \det(\lambda I - A) = det\begin{pmatrix} \lambda - 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \lambda I - \tilde{B} & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & \lambda - 1 \end{pmatrix} =$$

$$(\lambda - 1) \cdot det\begin{pmatrix} & & & \vdots \\ & \lambda I - \tilde{B} & & \vdots \\ & & & \vdots \\ & & & 0 \\ & & & \frac{1}{4} \\ 0 & \cdots & 0 & 1 \end{pmatrix} =$$

$$\pm(\lambda - 1)(\lambda - 1) \cdot det\begin{pmatrix} & \lambda I - \tilde{B} & \end{pmatrix} = \pm(\lambda - 1)^2 \cdot p_{\tilde{B}}(\lambda)$$

From this polynomial we see that $\lambda = 1$ is an eigenvalue of algebraic multiplicity of 2, and we previously saw that it has 2 corresponding linearly independent eigenvectors, so its geometric multiplicity is also 2.
To conclude, we found out that the algebraic multiplicities of the eigenvalues of A equal to their geometric multiplicities, and so A is diagonalizable.

h. $B = 4\tilde{B} = 4\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & \cdots & 0 & 0 \\ 1 & 2 & 1 & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & 1 & 2 & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{pmatrix}.$

The eigenvectors of B are the same as those of $\tilde{B}$: $e_{\lambda_1}, \dots, e_{\lambda_{k-1}}$.
However the eigenvalues are slightly different: for every eigenvector e_{λ_i}, the eigenvalue of B is $4\lambda_i$:
$Be_{\lambda_i} = (4\tilde{B})e_{\lambda_i} = 4 \cdot (\tilde{B}e_{\lambda_i}) = 4 \cdot (\lambda_i e_{\lambda_i}) \rightarrow Be_{\lambda_i} = (4\lambda_i)e_{\lambda_i}$

i. $\chi B_{k-1}(X) = \det(B_{k-1} - XI) =$

$$\det(B - XI) = det\begin{pmatrix} 2 - X & 1 & \cdots & 0 & 0 \\ 1 & 2 - X & 1 & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & 1 & 2 - X & 1 \\ 0 & 0 & \cdots & 1 & 2 - X \end{pmatrix}_{(k-1)\times(k-1)} =$$

$$(2-X)det\begin{pmatrix} 2-X & 1 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ & 1 & 2-X & 1 \\ 0 & \cdots & 1 & 2-X \end{pmatrix}_{(k-2)\times(k-2)}$$

$$-det\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2-X & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 2-X & 1 \\ 0 & 0 & \cdots & 1 & 2-X \end{pmatrix}_{(k-2)\times(k-2)} =$$

$$(2-X)\cdot\chi B_{k-2}(X) - 1\cdot det\begin{pmatrix} 2-X & 1 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 2-X & 1 \\ 0 & \cdots & 1 & 2-X \end{pmatrix}_{(k-3)\times(k-3)} =$$

$$\boldsymbol{(2-X)\cdot\chi B_{k-2}(X)-\chi B_{k-3}(X)}$$

Now we will calculate $\chi B_0(X)$ and $\chi B_1(X)$:

$$\chi B_0(X) = \det(B_0 - XI) = \det\big((\ \ ) - XI_{0\times 0}\big) = \det\big((\ \ )\big) = 1$$

$$\chi B_1(X) = \det(B_1 - XI) = \det\big((2) - XI\big) = \det\big((2-X)\big) = 2-X$$

j. We can rewrite the recursive formula from the last section as: (for $i \in \{2, \dots, k-1\}$)

$$\chi B_i(X) = (2-X)\chi B_{i-1}(X) - \chi B_{i-2}(X)$$

So, if we will define $2Y = 2-2X \rightarrow 2-2Y = 2X$, then we can get that

$$\chi B_i(2-2X) = \chi B_i(2Y) = (2-2Y)\chi B_{i-1}(2Y) - \chi B_{i-2}(2Y) =$$
$$2X\cdot\chi B_{i-1}(2-2X) - \chi B_{i-2}(2-2X)$$

Therefore, if we will define $Q_i(X) = \chi B_i(2-2X)$, we will get the wanted recursive relation: (for every $i \in \{2, \dots, k-1\}$)

$$Q_i(X) = \chi B_i(2-2X) = 2X\cdot\chi B_{i-1}(2-2X) - \chi B_{i-2}(2-2X) =$$
$$2XQ_{i-1}(X) - Q_{i-2}(X)$$

$Q_i(X)$ and $\chi B_i(X)$ are easily expressed by one another:

- $\boldsymbol{Q_i(X) = \chi B_i(2-2X)}$
- $2-2X = Y \rightarrow 2X = 2-Y \rightarrow X = 1-\frac{Y}{2} \rightarrow \boldsymbol{\chi B_i(X) = Q_i(1-\frac{X}{2})}$

We will now find $Q_0(X)$ and $Q_1(X)$ using the relation we found:

$Q_0(X) = \chi B_0(2-2X) = 1\,\big|_{2-2X} = 1$

$Q_1(X) = \chi B_1(2-2X) = (2-X)\,\big|_{2-2X} = 2-(2-2X) = 2X$

k. Let's calculate the next few Q_i's:

$Q_2(X) = 2X\cdot Q_1(X) - Q_0(X) = 2X\cdot 2X - 1 = 4X^2 - 1$

$Q_3(X) = 2X\cdot Q_2(X) - Q_1(X) = 2X\cdot(4X^2-1) - 2X = 8X^3 - 4X$

The $Q_i(X)$ series is the **Chebyshev polynomials of the second kind**.

l. For every $i \in \{0, \dots, k-1\}$, the degree of $Q_i(X)$ is i. We will prove that by induction on i:

<u>Base</u>: For $i = 0$, we know that $Q_0(X) = 1$, which is of degree $i = 0$.

For $i = 1$, we know that $Q_1(X) = 2X$, which is of degree $i = 1$.

Step: Let i be in the range of $[2, k-1]$. We will assume that the degree of $Q_{i-1}(X)$ is $(i-1)$, and that the degree of $Q_{i-2}(X)$ is $(i-2)$, and then we will show that the degree of $Q_i(X)$ is i.
The degree of $2X \cdot Q_{i-1}(X)$ is then necessarily i, which is strictly larger than $(i-2)$. We will use the recursive definition of $Q_i(X)$:
$Q_i(X) = 2X \cdot Q_{i-1}(X) - Q_{i-2}(X)$, and therefore the degree of $Q_i(X)$ is the same as $2X \cdot Q_{i-1}(X)$, which is i.
According to a theorem from linear algebra, we know that since $Q_i(X)$ is of degree i, it has at most i roots.

m. We will prove that $Q_i(\cos(\theta)) = \frac{\sin((i+1)\theta)}{\sin(\theta)}$ by induction on i:

Base: For $i = 0$, we know that $Q_0(\cos(\theta)) = 1 = \frac{\sin(\theta)}{\sin(\theta)} = \frac{\sin((0+1)\theta)}{\sin(\theta)}$.

For $i = 1$, we know that $Q_1(\cos(\theta)) = 2\cos(\theta) = \frac{\sin(2\theta)}{\sin(\theta)} = \frac{\sin\left((1+1)\theta\right)}{\sin(\theta)}$.

(We used the following identity:
$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \rightarrow 2\cos(\theta) = \frac{\sin(2\theta)}{\sin(\theta)}$)

Step: We assume that $Q_{i-1}(\cos(\theta)) = \frac{\sin\left(((i-1)+1)\theta\right)}{\sin(\theta)} = \frac{\sin(i\theta)}{\sin(\theta)}$, and that $Q_{i-2}(\cos(\theta)) = \frac{\sin\left(((i-2)+1)\theta\right)}{\sin(\theta)} = \frac{\sin((i-1)\theta)}{\sin(\theta)}$. Now we will show that the statement holds for Q_i:

$$Q_i(\cos(\theta)) = 2\cos(\theta) \cdot Q_{i-1}(\cos(\theta)) - Q_{i-2}(\cos(\theta)) =$$
$$2\cos(\theta) \cdot \frac{\sin(i\theta)}{\sin(\theta)} - \frac{\sin\left((i-1)\theta\right)}{\sin(\theta)} = \frac{2\cos(\theta)\sin(i\theta) - \sin\left((i-1)\theta\right)}{\sin(\theta)} =$$
$$\frac{2\cos(\theta)\sin(i\theta) - \sin(i\theta - \theta)}{\sin(\theta)}$$
$$= \frac{2\cos(\theta)\sin(i\theta) - (\sin(i\theta)\cos(\theta) - \cos(i\theta)\sin(\theta))}{\sin(\theta)} =$$
$$\frac{2\sin(i\theta)\cos(\theta) - \sin(i\theta)\cos(\theta) + \sin(\theta)\cos(i\theta)}{\sin(\theta)} =$$
$$\frac{\sin(i\theta)\cos(\theta) + \sin(\theta)\cos(i\theta)}{\sin(\theta)} = \frac{\sin(i\theta + \theta)}{\sin(\theta)} = \frac{\sin((i+1)\theta)}{\sin(\theta)}$$

n. We will use the equivalent expression from the last section in order to find roots of $Q_i(X)$ in the range $[-1, 1]$: $\frac{\sin\left((i+1)\theta\right)}{\sin(\theta)} = 0$

The denominator can't be 0, so we can rule out $\theta = 0$ and $\theta = \pi$.
Now we are left with $\sin\left((i+1)\theta\right) = 0 \rightarrow (i+1)\theta = \pi m$ (for $m \in \mathbb{N}$)
$$\rightarrow \theta = \frac{m}{i+1}\pi$$
For every $m \in \{1, 2, \ldots, i\}$ we get a valid solution for θ which yields a unique $X = \cos(\theta)$ solution to $Q_i(X) = 0$. The values of $\cos(\theta)$ are unique for these

solutions because the θ solutions are unique, and are in the range $[\frac{1}{i+1}\pi, \frac{i}{i+1}\pi]$, and $\cos(\theta)$ is strictly decreasing in that range.
We found i unique roots of $Q_i(X)$, which is the upper bound to the number of roots, and so we know that there are no other roots.
To conclude, for every $m \in \{1,2,\dots,i\}$, $X = \cos(\frac{m}{i+1}\pi)$ is a root of multiplicity 1 of the polynomial $Q_i(X)$.
From the relation $\chi B_i(X) = Q_i\left(1 - \frac{X}{2}\right)$, we know that for every root $X = \cos(\frac{m}{i+1}\pi)$ of $Q_i(X)$, $1 - \frac{X}{2} = 1 - \frac{1}{2}\cos(\frac{m}{i+1}\pi)$ is a root of χB_i.
Since $f(X) = 1 - \frac{X}{2}$ is a one-to-one function, we get that there are i unique roots of χB_i , each of multiplicity 1: $1 - \frac{1}{2}\cos(\frac{m}{i+1}\pi)$ for every $m \in \{1,\dots,i\}$.

o. The eigenvalues of B are the roots of the characteristic polynomial χB_{k-1}.
So, the eigenvalues of B are $(1 - \frac{1}{2}\cos\left(\frac{m}{k}\pi\right))$ for every $m \in \{1,\dots,k-1\}$.
In section (h) we showed that for every eigenvalue λ of $\tilde{B}$, 4λ is an eigenvalue of B, therefore, for every eigenvalue λ of B, $\frac{1}{4}\lambda$ is an eigenvalue of $\tilde{B}$:
For every $m \in \{1,\dots,k-1\}$, $\frac{1}{4}\left(1 - \frac{1}{2}\cos\left(\frac{m}{k}\pi\right)\right) = \frac{1}{4} - \frac{1}{8}\cos(\frac{m}{k}\pi)$ is an eigenvalue of $\tilde{B}$.
In section (g) we saw that the eigenvalues of A are 1, with multiplicity 2, and all the eigenvalues of $\tilde{B}$, with multiplicity 1.

p. If we remember the definition of A, it is the matrix transformation of $d^{n+1} = Ad^n$, and in section (d) we reshaped it to $d^{n+1} = A^n d^0$.
In section (g) we proved that A is diagonalizable, and so we can express it by its eigendecomposition: $A = QSQ^{-1}$.

Where $S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4} - \frac{1}{8}\cos(\frac{1}{k}\pi) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{4} - \frac{1}{8}\cos(\frac{k-1}{k}\pi) \end{pmatrix}$

(a diagonal matrix with eigenvalues at the diagonal).

And $Q = \begin{pmatrix} 0 & 2k & & & \\ 2 & 2(k-1) & | & & | \\ \vdots & \vdots & v_1 & \cdots & v_{k-1} \\ 2(k-1) & 2 & | & & | \\ 2k & 0 & & & \end{pmatrix}$ where v_i ($i \in \{1,\dots,k-1\}$) is the eigenvector of the eigenvalue $\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi)$.
As said, we can express A as $A = QSQ^{-1} \rightarrow A^n = QS^nQ^{-1}$.
S is a diagonal matrix and therefore S^n is:

$$S^n = \begin{pmatrix} 1^n & 0 & 0 & \cdots & 0 \\ 0 & 1^n & 0 & \cdots & 0 \\ 0 & 0 & \left(\frac{1}{4} - \frac{1}{8}\cos\left(\frac{1}{k}\pi\right)\right)^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left(\frac{1}{4} - \frac{1}{8}\cos\left(\frac{k-1}{k}\pi\right)\right)^n \end{pmatrix}$$

We will look at every eigenvalue of the form $(\frac{1}{4} - \frac{1}{8}\cos\left(\frac{i}{k}\pi\right))$, since $|\cos(?)| \le 1$, we know that $\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi) \in [\frac{1}{4} - \frac{1}{8}, \frac{1}{4} + \frac{1}{8}]$ and in particular $|\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi)| < 1$, and so we know that $\lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi)\right)^n = 0$.

Therefore we know that $\lim_{n\to\infty} S^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Let's denote $Q^{-1} = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_{k+1} & - \end{pmatrix}$.

$$\to \lim_{n\to\infty} A^n = Q \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} Q^{-1}$$

$$= \left(Q \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} Q^{-1} \right)$$

$$= \begin{pmatrix} 0 & 2k & 0 & \cdots & 0 \\ 2 & 2(k-1) & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & 0 \\ 2(k-1) & 2 & \vdots & \ddots & \vdots \\ 2k & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & 0 & - \\ & \vdots & \\ - & 0 & - \end{pmatrix}$$

$$= \begin{pmatrix} - & 2k \cdot r_2 & - \\ - & 2r_1 + 2(k-1) \cdot r_2 & - \\ & \vdots & \\ - & 2(k-1) \cdot r_1 + 2 \cdot r_2 & - \\ - & 2k \cdot r_1 & - \end{pmatrix}$$

We don't know the exact values of r_1 and r_2, but we proved that A^n converges as $n \to \infty$.

To conclude,

$$\lim_{n\to\infty} d^n = \lim_{n\to\infty} A^n\, d^0 = \begin{pmatrix} - & 2k\cdot r_2 & - \\ - & 2r_1 + 2(k-1)\cdot r_2 & - \\ & \vdots & \\ - & 2(k-1)\cdot r_1 + 2\cdot r_2 & - \\ - & 2k\cdot r_1 & - \end{pmatrix} d^0.$$

q. We are looking for vectors such that $vA = v \to A^t v^t = v^t$.

Let's examine A^t: $A^t = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B} & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & \frac{1}{4} & \cdots & 0 & 0 \\ 0 & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B}^t & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & \frac{1}{4} & 1 \end{pmatrix}$

Since $\tilde{B}$ is symmetric, we get that $A^t = \begin{pmatrix} 1 & \frac{1}{4} & \cdots & 0 & 0 \\ 0 & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B} & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & \frac{1}{4} & 1 \end{pmatrix}$.

We will denote a wanted eigenvector as $v^t = (v_1 \quad v_2 \quad \cdots \quad v_k \quad v_{k+1})^t$.

$$\to A^t v^t = v^t \to \begin{pmatrix} 1 & \frac{1}{4} & \cdots & 0 & 0 \\ 0 & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B}^t & & \vdots \\ 0 & & & & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \end{pmatrix}$$

$$\to \begin{pmatrix} v_1 + \frac{1}{4} v_2 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} v_k + v_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{B} \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_k \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \end{pmatrix}$$

We can subtract $\begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \\ v_{k+1} \end{pmatrix}$ from both sides and get:

$$\begin{pmatrix} \frac{1}{4}v_2 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}v_k \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{B}\begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_k \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_k \\ 0 \end{pmatrix}$$

Now we know that $v_2 = v_k = 0$ (from elementwise equality).

So we can substitute these values and get:

$$\begin{pmatrix} \frac{1}{4}\cdot 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}\cdot 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{B}\begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{B}\begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \\ 0 \end{pmatrix}$$

Again, from elementwise equality we can deduce that $\tilde{B}\begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix}$

We can look at it as looking for an eigenvector of $\tilde{B}$ with eigenvalue of 1.

We already know that 1 isn't an eigenvalue of $\tilde{B}$, so the only solution is $\begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$. So, the wanted eigenvector is of the shape $\begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \\ v_{k+1} \end{pmatrix}$, where v_1 and v_{k+1} are degrees of freedom. We will pick them such that we get 2 linearly independent eigenvectors: $\overline{v_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \overline{v_2} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

r. We have the eigendecomposition $A = U\Sigma U^{-1}$ where the columns of U are the right eigenvectors of A, and the rows of U^{-1} are the left eigenvectors of A, and Σ is a diagonal matrix with the elements sorted in decreasing order. From the structure of the eigendecomposition, we know that we can assume that the values of Σ are the eigenvalues of A, sorted in decreasing order. From our deductions in section (p), we know that:

$\lim_{n\to\infty} \Sigma^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. Therefore:

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} U\Sigma^n U^{-1} = U\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}U^{-1} =$$

$$\begin{pmatrix}| & | & & |\\ R_1 & R_2 & \cdots & R_{k+1}\\ | & | & & |\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}- & L_1 & -\\ - & L_2 & -\\ & \vdots & \\ - & L_{k+1} & -\end{pmatrix} =$$

$$\begin{pmatrix}| & | & & |\\ R_1 & R_2 & \cdots & R_{k+1}\\ | & | & & |\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}- & L_1 & -\\ - & L_2 & -\\ & \vdots & \\ - & L_{k+1} & -\end{pmatrix}$$

$$= \begin{pmatrix}| & | & | & & |\\ R_1 & R_2 & 0 & \cdots & 0\\ | & | & | & & |\end{pmatrix}\begin{pmatrix}- & L_1 & -\\ - & L_2 & -\\ - & 0 & -\\ & \vdots & \\ - & 0 & -\end{pmatrix} =$$

$$\begin{pmatrix}|\\ R_1\\ |\end{pmatrix}\begin{pmatrix}- & L_1 & -\end{pmatrix} + \begin{pmatrix}|\\ R_2\\ |\end{pmatrix}\begin{pmatrix}- & L_2 & -\end{pmatrix} = R_1L_1 + R_2L_2$$

$$\to \lim_{n\to\infty} d^n = \lim_{n\to\infty} A^n d^0 = \lim_{n\to\infty} A^n d^0 = U\left(\lim_{n\to\infty} \Sigma^n\right)U^{-1}d^0$$

$$= (R_1L_1 + R_2L_2)d^0$$

From the structure of U and U^{-1} in the eigendecomposition, we know that R_1, R_2, L_1, L_2 are the eigenvectors associated with the 2 largest eigenvalues, which are both 1. We have already found left and right eigenvectors associated with the eigenvalue 1, so we know R_1, R_2, L_1, L_2 (the order between each pair doesn't matter since they are orthogonal), up to some constant (on one the matrices U or U^{-1}):

$$R_1 = c_1\begin{pmatrix}0\\ 2\\ \vdots\\ 2(k-1)\\ 2k\end{pmatrix}, R_2 = c_1\begin{pmatrix}2k\\ 2(k-1)\\ \vdots\\ 2\\ 0\end{pmatrix}, L_1 = \begin{pmatrix}1\\ 0\\ \vdots\\ 0\\ 0\end{pmatrix}^t, \ L_2 = \begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{pmatrix}^t$$

$$\to \lim_{n\to\infty} d^n = (R_1L_1 + R_2L_2)d^0 = c_1\begin{pmatrix}0 & 0 & \cdots & 0 & 2k\\ 2 & 0 & \cdots & 0 & 2(k-1)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 2(k-1) & 0 & \cdots & 0 & 2\\ 2k & 0 & \cdots & 0 & 0\end{pmatrix}d^0$$

We can denote $d^0 = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \\ d_k \end{pmatrix}$ and get the following expression:

$$\lim_{n\to\infty} d^n = c_1 \begin{pmatrix} 0 & 0 & \cdots & 0 & 2k \\ 2 & 0 & \cdots & 0 & 2(k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k-1) & 0 & \cdots & 0 & 2 \\ 2k & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \\ d_k \end{pmatrix} =$$

$$c_1 \begin{pmatrix} 2k \cdot d_k \\ 2 \cdot d_0 + 2(k-1) \cdot d_k \\ \vdots \\ 2(k-1) \cdot d_0 + 2 \cdot d_k \\ 2k \cdot d_0 \end{pmatrix}$$

The d^n vector is the vector of decision levels for our quantization, so we know that for every n, the first and last values are the boundaries: $d_0 = a\,, d_k = b$.

$$\to d = c_1 \begin{pmatrix} 2k \cdot d_k \\ 2 \cdot d_0 + 2(k-1) \cdot d_k \\ \vdots \\ 2(k-1) \cdot d_0 + 2 \cdot d_k \\ 2k \cdot d_0 \end{pmatrix} = c_1 \begin{pmatrix} 2kb \\ 2a + 2(k-1)b \\ \vdots \\ 2(k-1)a + 2b \\ 2ka \end{pmatrix}$$

We can see that d is only dependent on a, b, k (c_1 is a constant that we can find), and in particular d is independent from the choice of d^0.

s. What we are looking for is the value of c_1. We already said that for every d^n, the first and last values are the bounds, and that also include d. Therefore we can find c_1 by requiring the first and last values to be a and b:

$$c_1 \cdot 2kb = b \to c_1 = \frac{1}{2k}$$

$$c_1 \cdot 2ka = a \to c_1 = \frac{1}{2k}$$

$$\to d = \frac{1}{2k} \begin{pmatrix} 2kb \\ 2a + 2(k-1)b \\ \vdots \\ 2(k-1)a + 2b \\ 2ka \end{pmatrix} = \begin{pmatrix} b \\ \frac{a + (k-1)b}{k} \\ \vdots \\ \frac{(k-1)a + b}{k} \\ a \end{pmatrix}$$

$$\to \forall i \in \{0, \dots, k\}: d_i = \frac{(k-i) \cdot a + i \cdot b}{k} = \frac{ka - ia + ib}{k} = a + i\frac{b-a}{k}$$

Also, we know from section (p) that the diagonal values of Σ (other than the first 2) are less than 1 in their absolute value, and therefore they converge to 0 when raised to a power, and that is why the algorithm will converge in exponential time.

t. What we have shown so far is that the Max-Lloyd converges to the uniform decision levels. We learned in class that this algorithm converges to a local minimum, but since we found that for every initial d^0 it converges to the same point, we know that all local minima are the same point, and so it must be a single global minimum of the loss function. After we have the optimal decision levels, the representation levels are chosen optimally as the mid-point in every decision level (we saw in the lectures).