HW1-236200

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Question 1:

1. The function we want to minimize is the expected absolute-deviation:

$$E^1_{\epsilon_Q} = \int_{\phi_L}^{\phi_H} |x - Q(x)|p(x)dx = (*)$$

As for the inteval $[\phi_L, \phi_H]$we partitioning it using the given decision levels: $\{d_i\}_{i=0}^K$ creating K decisions regions: $D_i = [d_{i-1}, d_i), i = 1, ..., K$ s.t: $\phi_L = d_0 < d_1 < \cdots < d_K = \phi_H$,and for the mappsing function, we know that: $Q(x) = r_i | x \in D_i$ for representation levels $\{r_i\}_{i=1}^J$ s.t $r_1 < r_2 < \cdots < r_J$. and we get:

$$(*) = \sum_{i=1}^{K} \int_{d_{i-1}}^{d_i} |x - r_i|p(x)dx$$

and we get that the optimization problem we wish to solve is:

$$\underset{\{d_i\}_{i=0}^K \{r_i\}_{i=1}^J}{minimize} \sum_{i=1}^{K} \int_{d_{i-1}}^{d_i} |x - r_j|p(x)dx$$

2. Necessery condition for optimal representation level is given by:

$$\frac{\partial}{\partial r_j} E^1_{\epsilon_Q} = 0 \, for \, j = 1, ..., J$$

$$\Downarrow$$

$$\frac{\partial}{\partial r_j} E^1_{\epsilon_Q} = \frac{\partial}{\partial r_j} \sum_{i=1}^{K} \int_{d_{i-1}}^{d_i} |x - r_j|p(x)dx = \frac{\partial}{\partial r_j} \int_{d_{j-1}}^{d_j} |x - r_j|p(x)dx =$$

$$= \int_{d_{j-1}}^{d_j} \frac{\partial}{\partial r_j} |x - r_j|p(x)dx = - \int_{d_{j-1}}^{d_j} sign(x - r_j)p(x)dx =$$

$$= \int\limits_{d_{j-1}}^{r_j} p(x)dx - \int\limits_{r_j}^{d_j} p(x)dx \overset{demand}{=} 0$$

$$\Downarrow$$

$$\int\limits_{d_{j-1}}^{r_j} p(x)dx = \int\limits_{r_j}^{d_j} p(x)dx$$

From the optimality condition we can see that the optimal $r_j's$ devide the interval $[d_{j-1}, d_j)$ into two equal probabily regions, this can be interpereted as the probabilistic median of the interval for the given $p(x)$.

3. Necessery condition for optimal decision level is given by:

$$\frac{\partial}{\partial d_j} E^1_{\epsilon_Q} = 0 \, for \, j = 1, ..., K-1$$

for fixed $d_0 = \phi_L, d_K = \phi_H$

$$\Downarrow$$

$$\frac{\partial}{\partial d_j} E^1_{\epsilon_Q} = \frac{\partial}{\partial d_j} \sum_{i=1}^{K} \int\limits_{d_{i-1}}^{d_i} |x - r_j| p(x)dx =$$

$$= \frac{\partial}{\partial d_j} \int\limits_{d_{j-1}}^{d_j} |x - r_j| p(x)dx + \frac{\partial}{\partial d_j} \int\limits_{d_j}^{d_{j+1}} |x - r_{j+1}| p(x)dx =$$

$$= |d_j - r_j| p(d_j) - |d_j - r_{j+1}| p(d_j) \overset{demand}{=} 0$$

$$\Downarrow$$

$$|d_j - r_j| = |d_j - r_{j+1}|$$

If we open the absolute value we get two posibilities:

(a) $d_j - r_j = d_j - r_{j+1} \Longrightarrow r_j = r_{j+1}$ This answer is **impossible** as we know that for wach j: $r_j < r_{j+1}$.

(b) $d_j - r_j = -(d_j - r_{j+1}) \Longrightarrow d_j = \frac{r_j + r_{j+1}}{2}$ So given the representation level we get that the optimal decision levels are the mean of the representation levels.

4. the Max-Lloyd procedure for designing a *b*-bit quantizer that minimizes the expected absolute-deviation for a given input PDF *p*(*x*) can be described as:

(a) Initialization: Set arbitary decisions levels $\{d_i\}_{i=0}^{K}$.

(b) Compute th e optimal representation levels and set to-$\{r_i\}_{i=1}^{J}$ for the current decision levels $\{d_i\}_{i=0}^{K}$.

(c) Compute the optimal decisions levels and set to-$\{d_i\}_{i=0}^{K}$ for the current representation levels $\{r_i\}_{i=1}^{J}$.

(d) If stopping criteria has not met, return to (b).

Question 2

1. We need to formulate an optimal quantizier design that uses $b = 1$ bit

$$\Downarrow$$

we have $J = 2^b = 2$ representation levels, labled r_1, r_2 , and $J + 1 = 3$ decision levels labled $d_0 = -4a, d_1, d_2 = 4a$.

(a) As we learned in class we know that for given decision levels the optimal representation levles are:

$$r_j = \frac{\int_{d_{j-1}}^{d_j} xp(x)dx}{\int_{d_{j-1}}^{d_j} p(x)dx} for\, j = 1, 2$$

and fr given representation levels the optimal decision levels are:

$$d_j = \frac{r_j + r_{j+1}}{2} for\, j = 0, 1, 2$$

Now we guess the value of the decision level, $d_1 = 0$, and lets calculate the representation levels according to the current decision levels:

$$r_1 = \frac{\int_{-4a}^{-2a} \frac{x}{4a} dx}{\int_{d_{-4a}}^{d_{-2a}} \frac{1}{4a} dx} = \frac{\left.\frac{x^2}{8a}\right|_{-4a}^{-2a}}{\left.\frac{x}{4a}\right|_{-4a}^{-2a}} = -3a$$

$$r_2 = \frac{\int_{2a}^{3a} x\left(-\frac{1}{4a^2}x + \frac{7}{8a}\right)dx + \int_{3a}^{4a} x\left(\frac{1}{4a^2}x - \frac{5}{8a}\right)dx}{\int_{2a}^{3a} \left(-\frac{1}{4a^2}x + \frac{7}{8a}\right)dx + \int_{3a}^{4a} \left(\frac{1}{4a^2}x - \frac{5}{8a}\right)dx} = \cdots = 3a$$

Now lets calculate the decision level according to the calculated representation levels:

$$d_1 = \frac{r_1 + r_2}{2} = 0$$

The result didnt changed, that mean that the values we got are the optimal.

(b) The minimal squared error achived is:

$$E\left\{\epsilon^2\right\} = \int_{-4a}^{4a} p(x)\left(x - Q(x)\right)^2 dx =$$

$$= \int_{-4a}^{-2a} \frac{1}{4a}\left(x - (-3a)\right)^2 dx + \int_{2a}^{3a} \left(-\frac{1}{4a^2}x + \frac{7}{8a}\right)(x - 3a)^2 dx + \int_{3a}^{4a} \left(\frac{1}{4a^2}x - \frac{5}{8a}\right)(x - 3a)^2 dx =$$

$$= \cdots = \frac{a^2}{6} + \frac{5a^2}{48} - \frac{a^2}{48} = \frac{a^2}{4}$$

2. We need to formulate an optimal quantizier design that uses $b = 2$ bit

$$\Downarrow$$

we have $J = 2^b = 4$ representation levels, labled r_1, r_2, r_3, r_4 , and $J + 1 = 5$ decision levels labled d_0, d_1, d_2, d_3, d_4.

(a) We start with a guess for the decision levels: $d_0 = -4a, d_1 = -2a, d_2 = 2a, d_3 = 3a, d_4 = 4a$ and calculate the matching representation levels:

$$r_1 = -3a, r_2 = 0, r_3 = 2\frac{5}{12}a, r_4 = 3\frac{7}{12}a$$

Now we update the decision levels according to the calculated representation levels:

$$d_0 = -4a, d_1 = -1.5a, d_2 = 1\frac{5}{24}a, d_3 = 3a, d_4 = 4a$$

Now we update the representation levels:

$$r_1 = -3a, r_2 = 0, r_3 = 2\frac{5}{12}a, r_4 = 3\frac{7}{12}a$$

We got the same result as the previous calculation, that means that the algorithm converged, and the final answer is:

$$d_0 = -4a, d_1 = -1.5a, d_2 = 1\frac{5}{24}a, d_3 = 3a, d_4 = 4a$$

$$r_1 = -3a, r_2 = 0, r_3 = 2\frac{5}{12}a, r_4 = 3\frac{7}{12}a$$

(b) The minimal squared error achived is:

$$E\left\{\epsilon^2\right\} = \int\limits_{-4a}^{4a} p(x)\left(x - Q(x)\right)^2 dx =$$

$$= \int\limits_{-4a}^{-2a} \frac{1}{4a}\left(x - (-3a)\right)^2 dx + \int\limits_{2a}^{3a} \left(-\frac{1}{4a^2}x + \frac{7}{8a}\right)\left(x - 2\frac{5}{12}a\right)^2 dx + ...$$

$$... + \int\limits_{3a}^{4a} \left(\frac{1}{4a^2}x - \frac{5}{8a}\right)\left(x - 3\frac{7}{12}a\right)^2 dx = \cdots = 0.20466a^2$$

Question 3

1. $\int\limits_{t \in \Delta_i} (t-t_i)^k \, dt = \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} (t-t_i)^k \, dt = \left. \frac{(t-t_i)^{k+1}}{k+1} \right|_{\frac{i-1}{N}}^{\frac{i}{N}} =$

$$= \frac{\left(\frac{i}{N}-t_i\right)^{k+!}}{k+1} - \frac{\left(\frac{i-1}{N}-t_i\right)^{k=1}}{k+1} = \frac{\left(\frac{|\Delta_i|}{2}\right)^{k+1} - \left(\frac{-|\Delta_i|}{2}\right)^{k+1}}{k+1} =$$

$$= \frac{\left(1+(-1)^{k+2}\right)|\Delta_i|^{k+1}}{2^{k+1}(k+1)} = \begin{cases} 0, k-odd \\ \frac{|\Delta_i|^{k+1}}{2^k(k+1)}, k-even \end{cases}$$

2. As we studied in class:

$$MSE = \int\limits_0^1 \left(\phi(t) - \hat{\phi}(t)\right)^2 dt = \sum_{i=1}^N \overbrace{\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t) - a_i\left(t-t_i\right) - c_i\right)^2 dt}^{MSE_i}$$

Find the optimal a_i, c_i by requiring:

$$\frac{\partial}{\partial a_i} MSE_i = 0$$

$$\frac{\partial}{\partial c_i} MSE_i = 0$$

For a_i:

$$\frac{\partial}{\partial a_i} MSE_i = \frac{\partial}{\partial a_i} \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t) - a_i\left(t-t_i\right) - c_i\right)^2 dt =$$

$$= \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \frac{\partial}{\partial a_i} \left(\phi(t) - a_i\left(t-t_i\right) - c_i\right)^2 dt = -2 \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(t-t_i\right)\left(\phi(t) - a_i\left(t-t_i\right) - c_i\right) dt =$$

$$= -2 \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)\left(t-t_i\right) dt + 2a_i \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(t-t_i\right)^2 dt + 2c_i \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(t-t_i\right) dt$$

We calculate seperetly the first and second integral, based on the answer from previous section:

$$2a_i \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(t-t_i\right)^2 dt = 2a_i \frac{|\Delta_i|^3}{2^2 3} = \frac{a_i\,|\Delta_i|^3}{6}$$

$$2c_i \int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \left(t-t_i\right) dt = 0$$

By demending that $\frac{\partial}{\partial a_i} MSE_i = 0$ we get the equation:

$$-2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)\left(t-t_i\right) dt + \frac{a_i \left|\Delta_i\right|^3}{6} = 0$$

$$\Downarrow$$

$$a_i = \frac{12}{\left|\Delta_i\right|^3} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)\left(t-t_i\right) dt$$

Now we will optimize c_i:

$$\frac{\partial}{\partial c_i} MSE_i = \frac{\partial}{\partial c_i} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t) - a_i\left(t-t_i\right) - c_i\right)^2 dt =$$

$$= \int_{\frac{i-1}{N}}^{\frac{i}{N}} \frac{\partial}{\partial c_i} \left(\phi(t) - a_i\left(t-t_i\right) - c_i\right)^2 dt = -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t) - a_i\left(t-t_i\right) - c_i\right) dt =$$

$$= -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)dt + 2a_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t-t_i)dt + 2c_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} 1dt$$

Based on the answer from previous section we get that the middle integral equals zero, and:

$$2c_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} 1dt = 2c_i \left|\Delta_i\right|$$

By demanding that $\frac{\partial}{\partial c_i} MSE_i = 0$ we get that:

$$-2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)dt + 2c_i \left|\Delta_i\right| = 0$$

$$\Downarrow$$

$$c_i = \frac{1}{\left|\Delta_i\right|} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)dt$$

3. The optimal MSE is calculated by using the optimal a_i, c_i(that give us the optimal $\hat{\phi}(t)$):

$$MSE^{opt} = \int_0^1 \left(\phi(t) - \hat{\phi}(t)\right)^2 dt = \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t) - a_i^{opt}\left(t-t_i\right) - c_i^{opt}\right)^2 dt =$$

$$=\sum_{i=1}^{N}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \Big(\phi(t)^2 - 2a_i^{opt}\phi(t)\,(t-t_i) - 2c_i^{opt}\phi(t) + \left(a_i^{opt}\right)^2 (t-t_i)^2 + 2a_i^{opt}c_i^{opt}\,(t-t_i)$$

$$+\left(c_i^{opt}\right)^2 dt\Big)\,dt =$$

$$=\int\limits_0^1 (\phi(t))^2\,dt - 2\sum_{i=1}^{N} a_i^{opt}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)\,(t-t_i)\,dt - 2\sum_{i=1}^{N} c_i^{opt}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)dt +$$

$$+\sum_{i=1}^{N}\left(a_i^{opt}\right)^2\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} (t-t_i)^2\,dt + 2\sum_{i=1}^{N} a_i^{opt}c_i^{opt}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} (t-t_i)\,dt + |\Delta|\sum_{i=1}^{N}\left(c_i^{opt}\right)^2 = (**)$$

Using the outcomes from section 1 we get that:

$$\sum_{i=1}^{N}\left(a_i^{opt}\right)^2\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} (t-t_i)^2 = \sum_{i=1}^{N}\left(a_i^{opt}\right)^2\frac{|\Delta_i|^3}{2^2 3} \overset{(*)}{=} \frac{|\Delta|^3}{12}\sum_{i=1}^{N}\left(a_i^{opt}\right)^2$$

(*):All the intervals are in the same size:$|\Delta_i| = |\Delta| = \frac{1}{N}$.

$$2\sum_{i=1}^{N} a_i^{opt}c_i^{opt}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} (t-t_i)\,dt = 0$$

Using the expressions we got for optimal a_i, c_i we get:

$$-2\sum_{i=1}^{N} a_i^{opt}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)\,(t-t_i)\,dt = -2\sum_{i=1}^{N}\left(a_i^{opt}\right)^2\frac{|\Delta_i|^3}{12} =$$

$$= -\frac{|\Delta|^3}{6}\sum_{i=1}^{N}\left(a_i^{opt}\right)^2$$

$$-2\sum_{i=1}^{N} c_i^{opt}\int\limits_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t)dt = -2\sum_{i=1}^{N} c_i^{opt}\,|\Delta_i|\,c_i^{opt} = -2\,|\Delta|\sum_{i=1}^{N}\left(c_i^{opt}\right)^2$$

And finally we get:

$$(**) = \int\limits_0^1 (\phi(t))^2\,dt - \frac{|\Delta|^3}{6}\sum_{i=1}^{N}\left(a_i^{opt}\right)^2 - 2\,|\Delta|\sum_{i=1}^{N}\left(c_i^{opt}\right)^2 + \frac{|\Delta|^3}{12}\sum_{i=1}^{N}\left(a_i^{opt}\right)^2 + |\Delta|\sum_{i=1}^{N}\left(c_i^{opt}\right)^2 =$$

$$= \int\limits_0^1 (\phi(t))^2\,dt - \frac{|\Delta|^3}{12}\sum_{i=1}^{N}\left(a_i^{opt}\right)^2 - |\Delta|\sum_{i=1}^{N}\left(c_i^{opt}\right)^2 =$$

$|\Delta| = \frac{1}{N}$

$$=\int_0^1 (\phi(t))^2\, dt - \frac{1}{12N^3} \sum_{i=1}^{N} \left(a_i^{opt}\right)^2 - \frac{1}{N} \sum_{i=1}^{N} \left(c_i^{opt}\right)^2$$

4. As we saw in class the minimal MSE for picewise-constant approximation is:

$$MSE_{const} = \int_0^1 \phi^2(t)dt - \frac{1}{N} \sum_{i=1}^{N} \left(c_i^{opt}\right)^2$$

If we calculate the difference $MSE_{lin} - MSE_{const}$ we get:

$$MSE_{lin} - MSE_{const} = -\frac{1}{12N^3} \sum_{i=1}^{N} \left(a_i^{opt}\right)^2$$

And because $\forall i, (a_i)^2 \geq 0$ we get that:$-\frac{1}{12N^3} \sum_{i=1}^{N} \left(a_i^{opt}\right)^2 \leq 0$

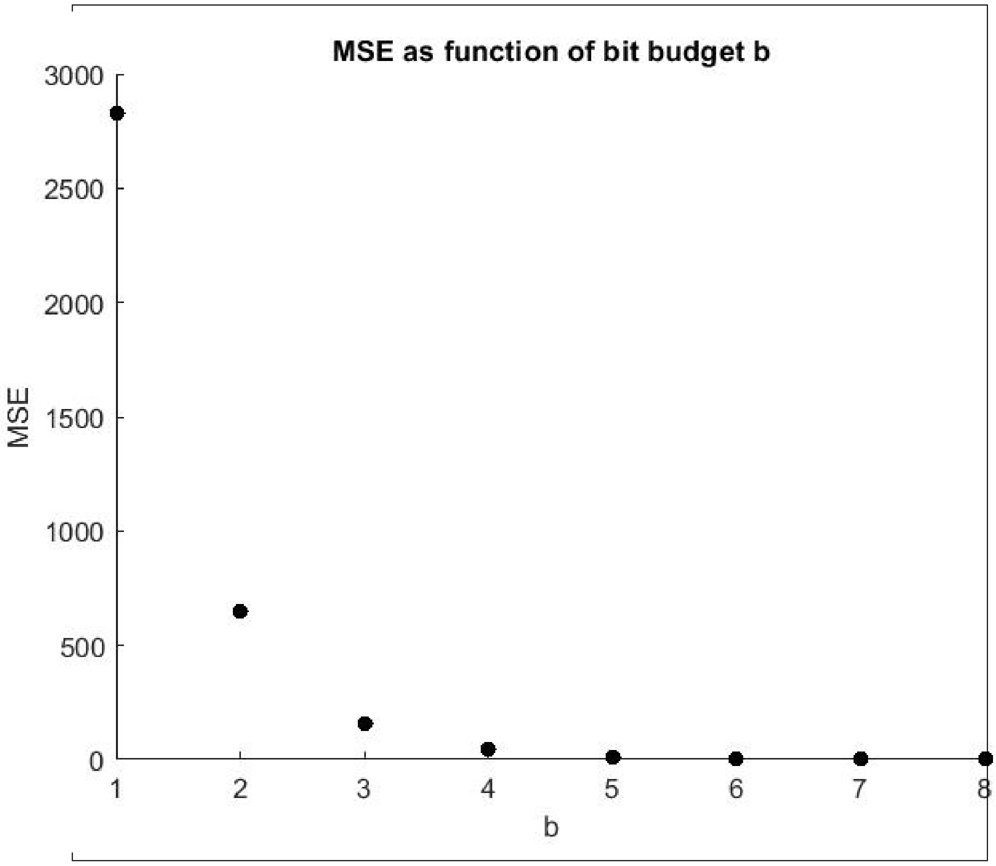
$$\Downarrow$$

$$MSE_{lin} \leq MSE_{const}$$

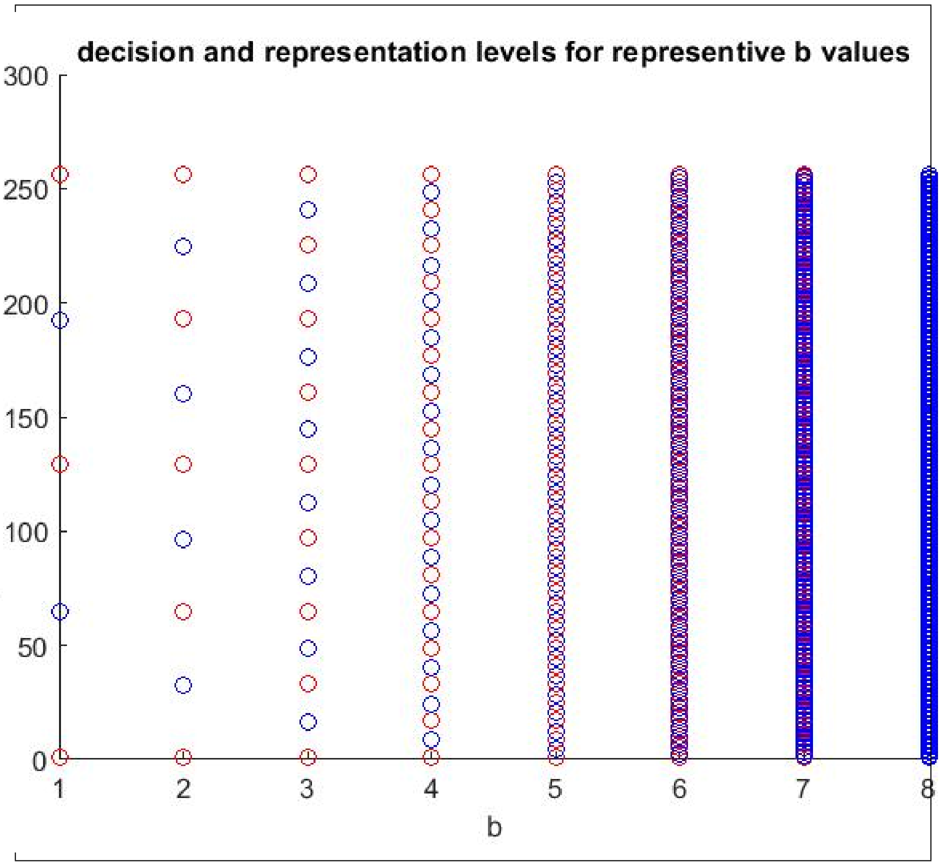
Therfore the MSE using picewise linear aaproximation is lower(or equal) to the MSE achived by using picewise-constant approximation.

Part 2 - Matlab
Question 2 - uniform quantizier:

1. MSE as a funcion of the bit budget for b=1,...,8:



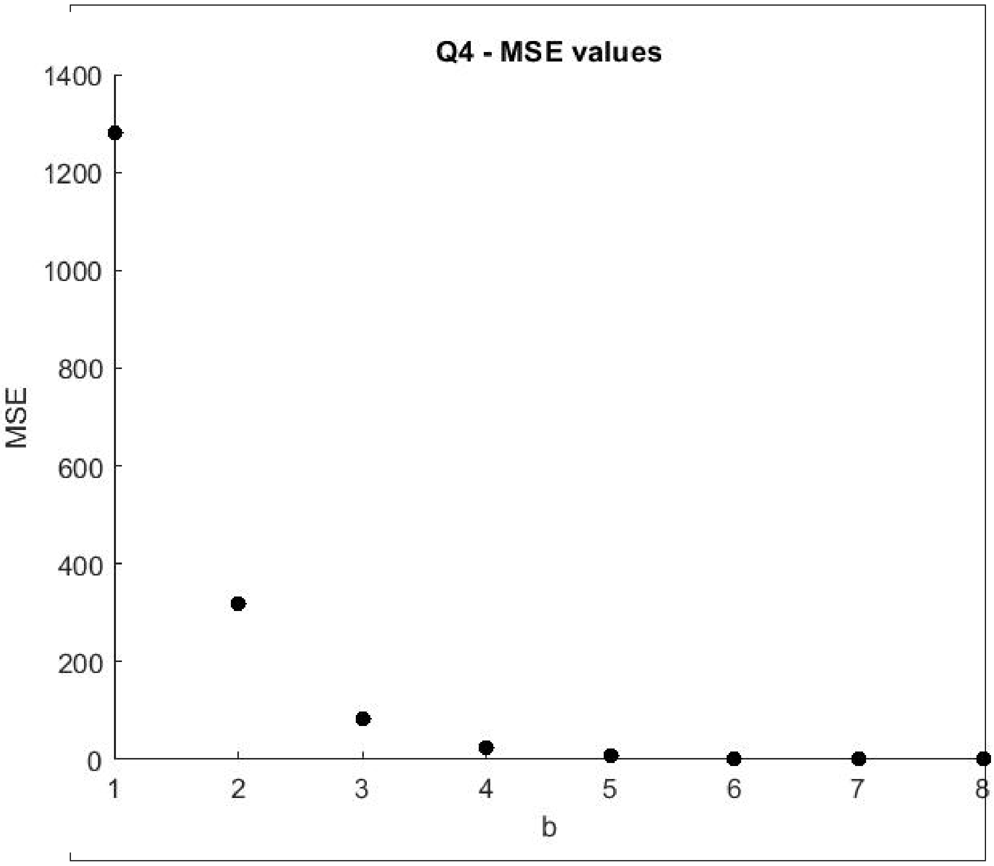
2. The decision and representation levels for representive b values:



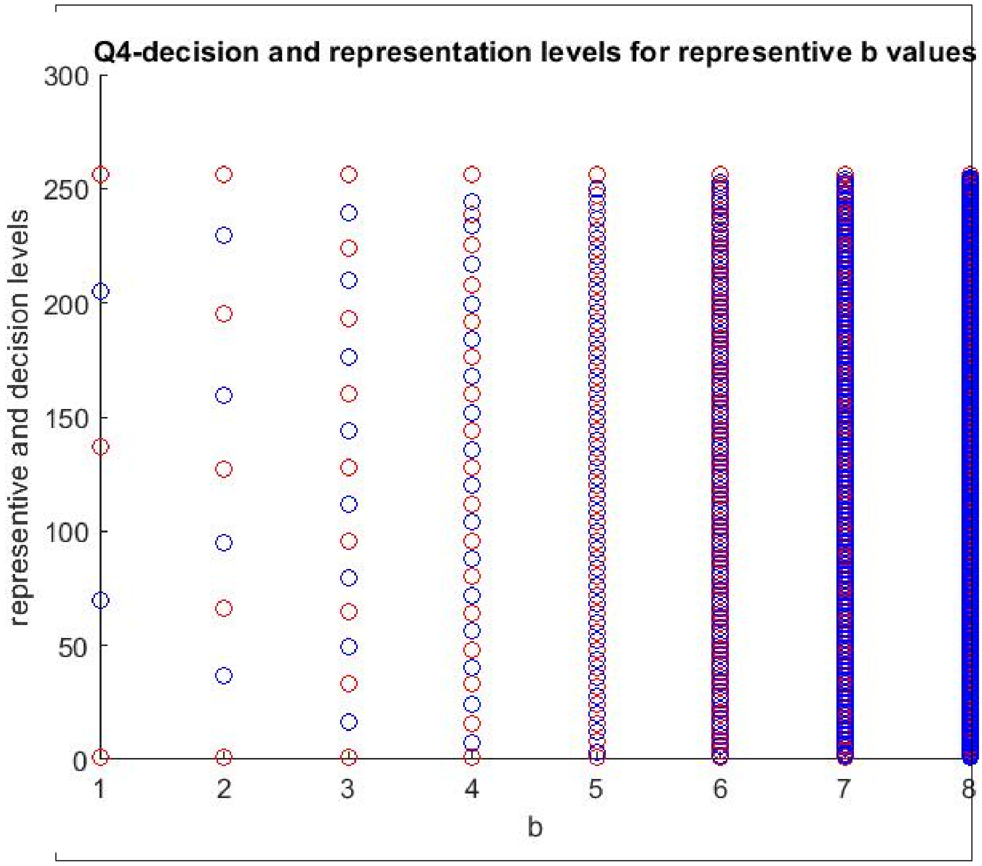
red-decision blue-representation 0.1: Figure

Question 4 - Max-Lloyd unifrom quantization:

1. MSE as a funcion of the bit budget for b=1,...,8:



2. The decision and representation levels for representive b values:

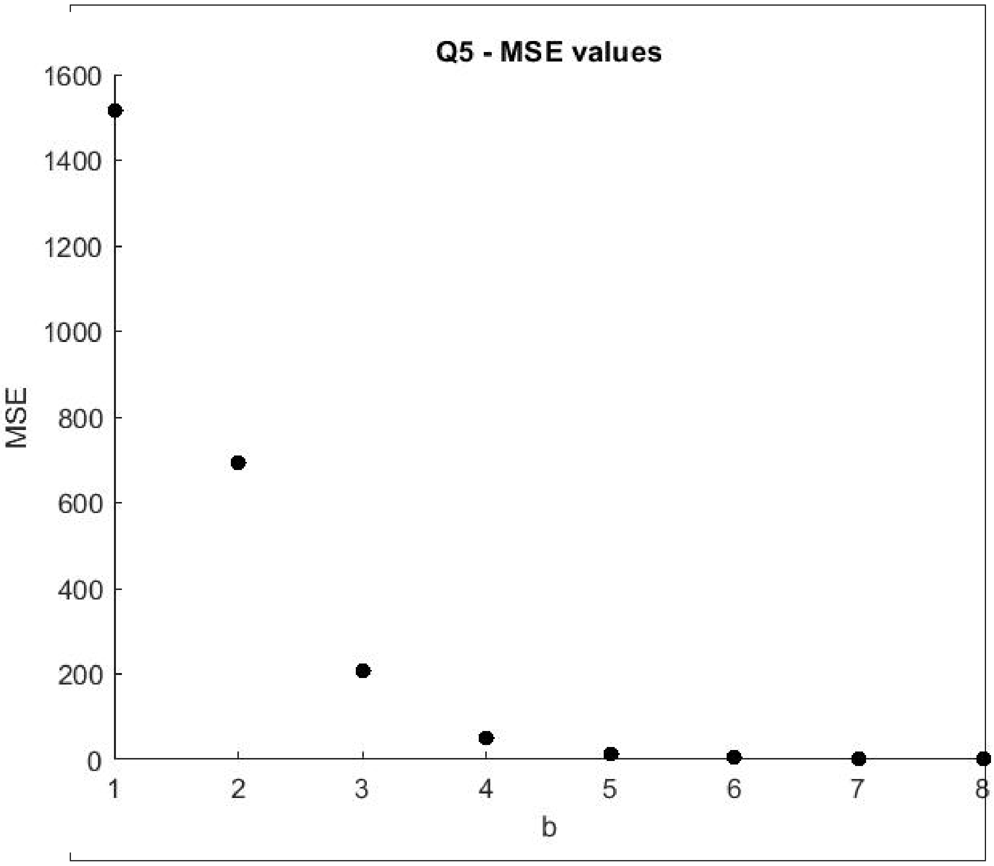


red-decision blue-representation 0.2: Figure

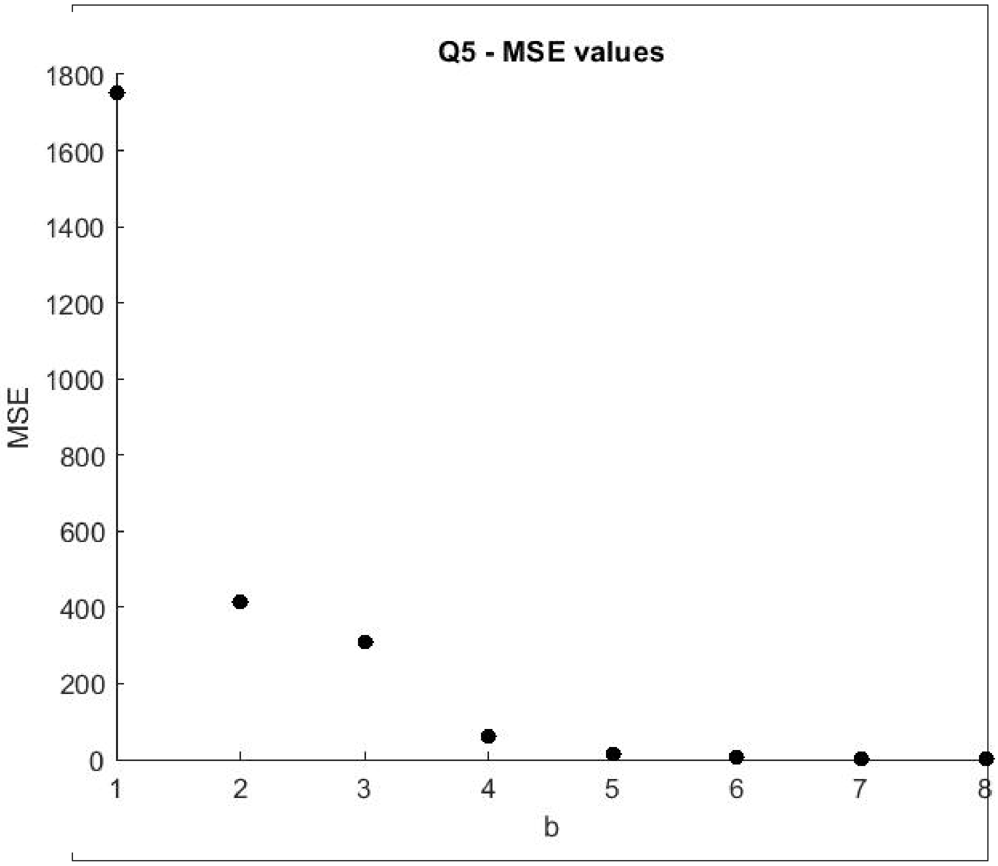
3. As we could expected, the MSE values obtained using Max-Lloyd are lower than the MSE values obtained by the unifrom quantization, as the Max-Lloyd algorithm should converge to the best representation and decision levels. And as we can see, the decision levels in the uniform quntization ramain uniform, where as in the Max-Lloyd algorithm they do not.

Question 5-Max-Lloyd random quantization:

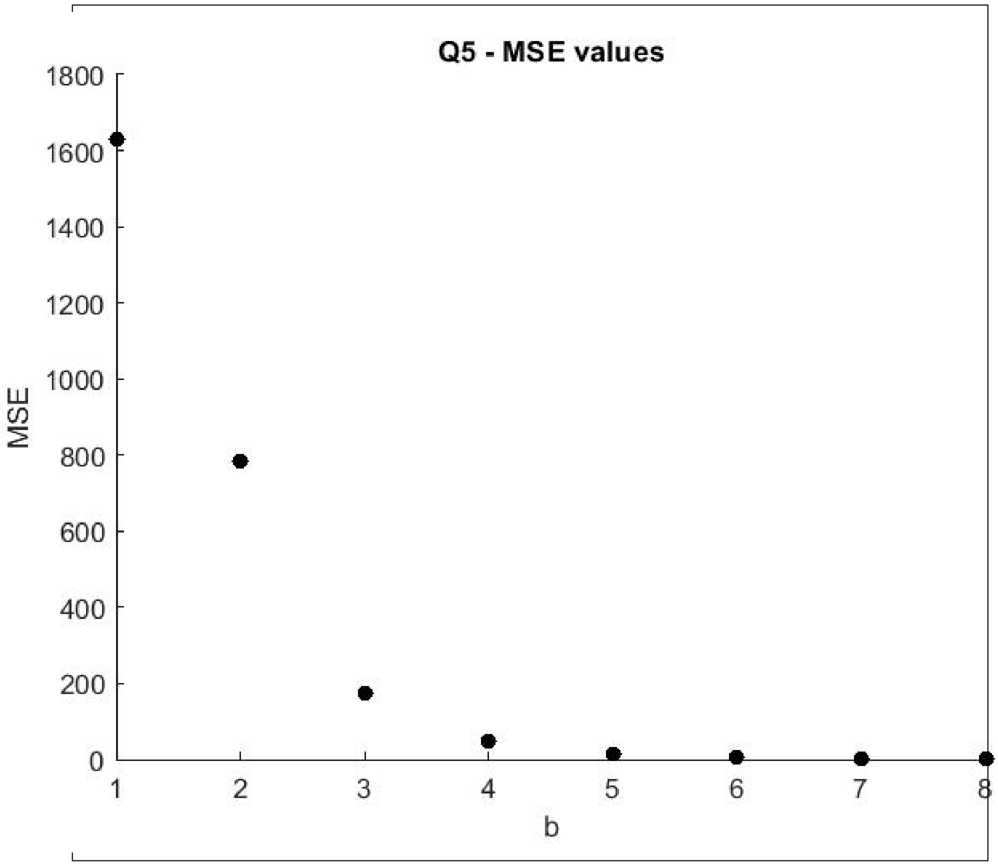
1. MSE as a funcion of the bit budget for b=1,...,8 for 5 random quantizations:



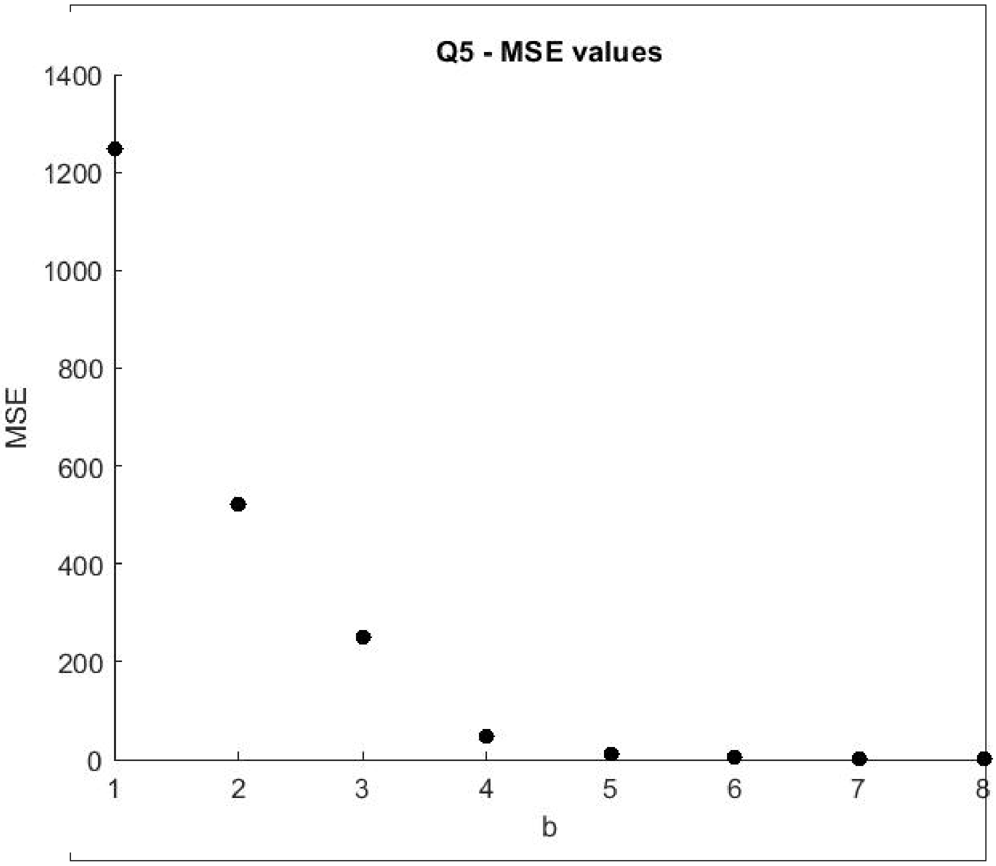
run no.1 0.3: Figure



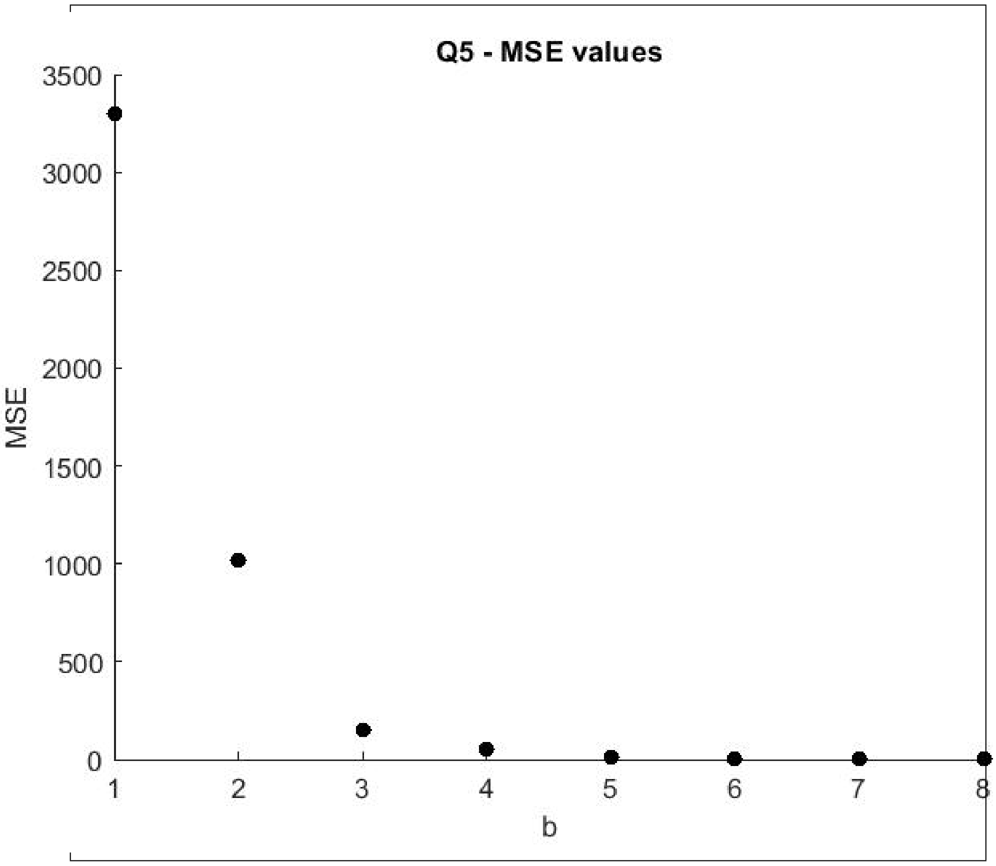
run no.2 0.4: Figure



run no.3 0.5: Figure

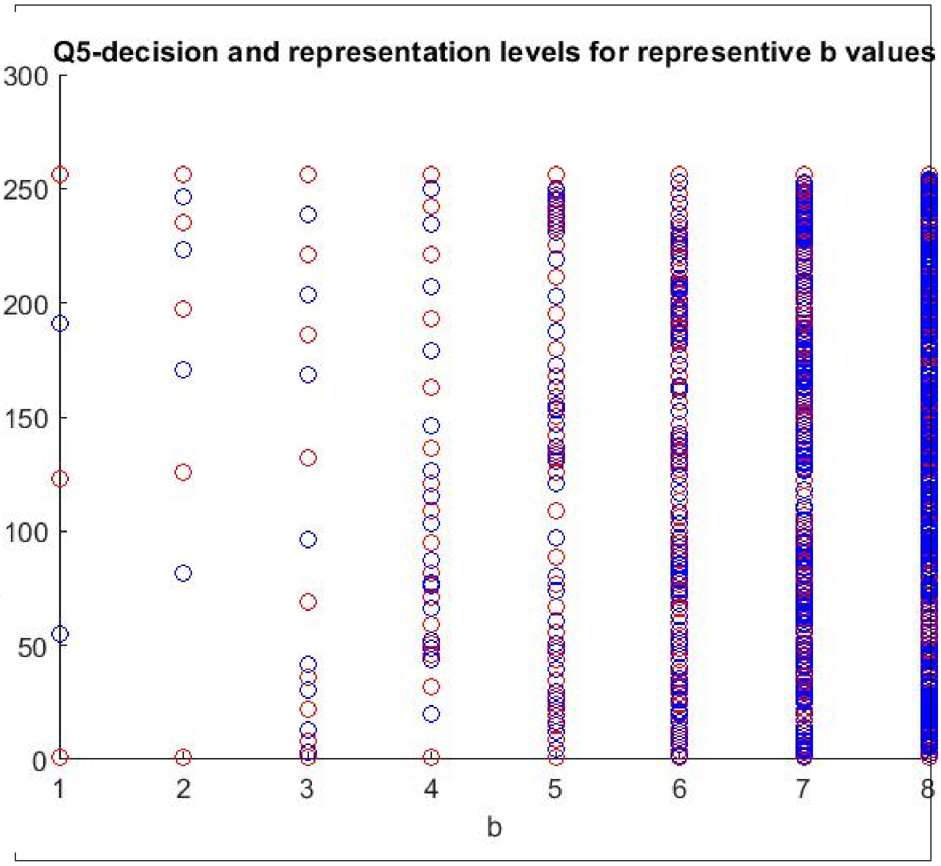


run no.4 0.6: Figure

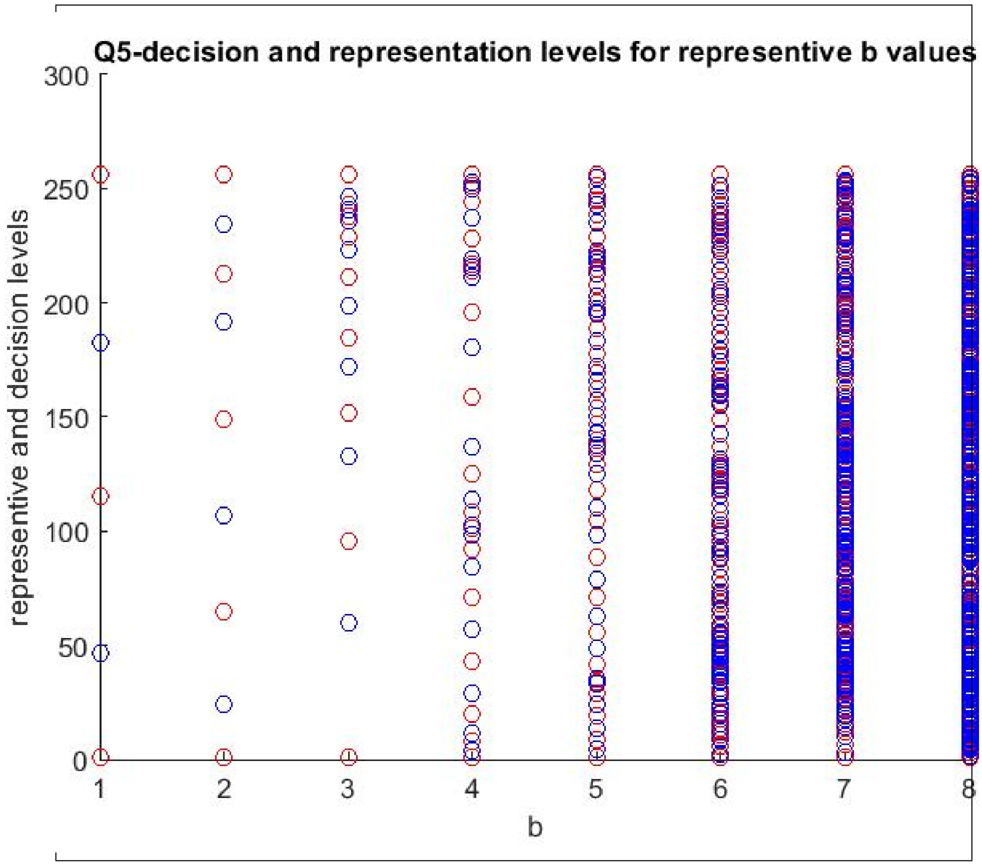


run no.5 0.7: Figure

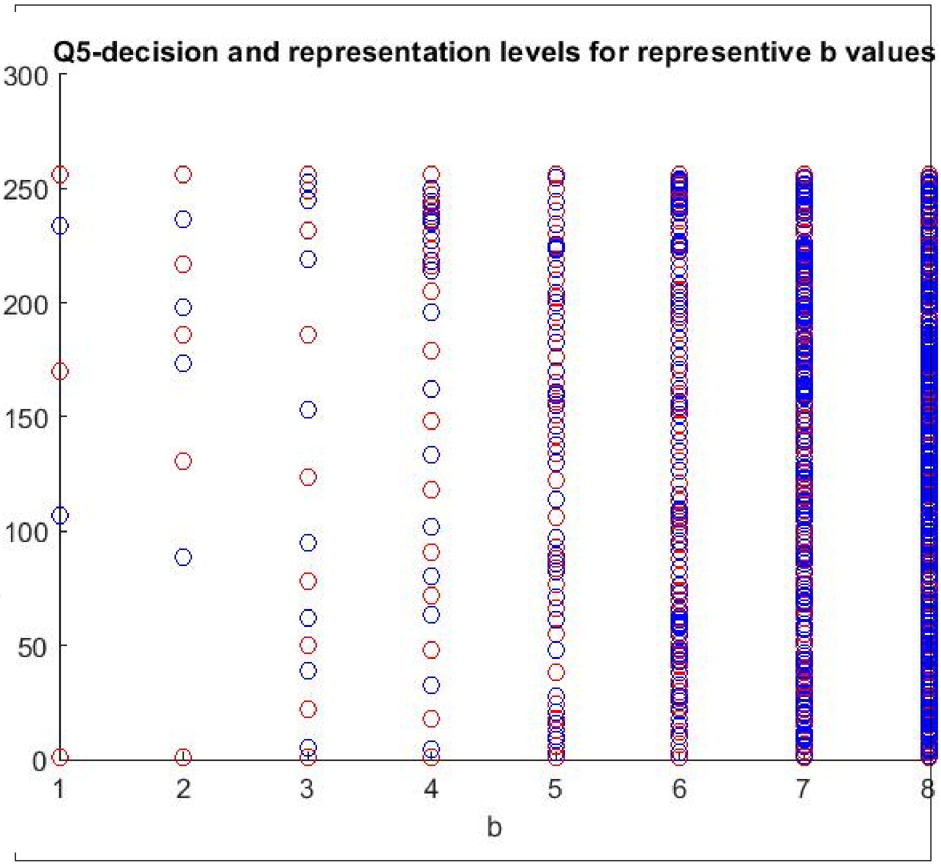
2. The decision and representation levels for representive b values, for 5 random quantizations:



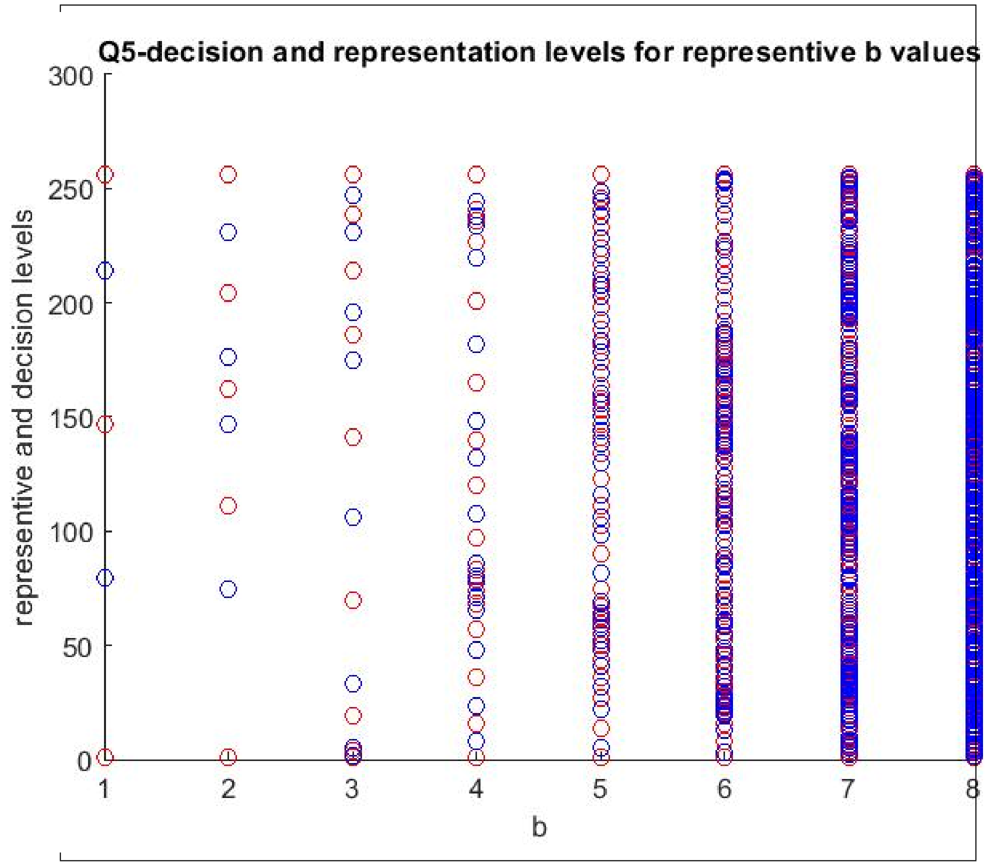
run no.1 0.8: Figure



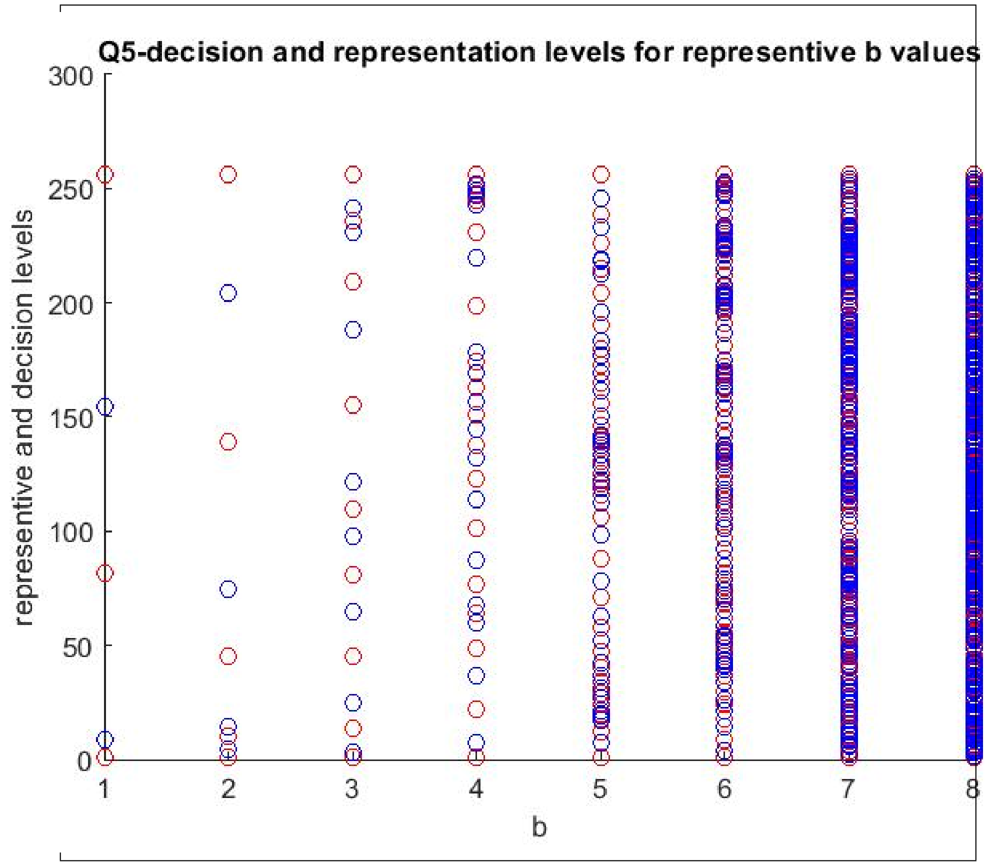
run no.2 0.9: Figure



run no.3 0.10: Figure



run no.4 0.11: Figure



run no.5 0.12: Figure

3. As we can see, the Max-Lloyd algorithm using uniform quantization obtained better results rather then starting with random quantization, as the MSE converged to zero in better performance.