

HW1-236200

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Question 1:

1. The function we want to minimize is the expected absolute-deviation:

$$E_{\epsilon_Q}^1 = \int_{\phi_L}^{\phi_H} |x - Q(x)| p(x) dx = (*)$$

As for the interval $[\phi_L, \phi_H]$ we partition it using the given decision levels: $\{d_i\}_{i=0}^K$ creating K decisions regions: $D_i = [d_{i-1}, d_i], i = 1, \dots, K$ s.t: $\phi_L = d_0 < d_1 < \dots < d_K = \phi_H$, and for the mapping function, we know that: $Q(x) = r_i | x \in D_i$ for representation levels $\{r_i\}_{i=1}^J$ s.t $r_1 < r_2 < \dots < r_J$. and we get:

$$(*) = \sum_{i=1}^K \int_{d_{i-1}}^{d_i} |x - r_i| p(x) dx$$

and we get that the optimization problem we wish to solve is:

$$\underset{\{d_i\}_{i=0}^K \{r_i\}_{i=1}^J}{\text{minimize}} \sum_{i=1}^K \int_{d_{i-1}}^{d_i} |x - r_i| p(x) dx$$

2. Necessary condition for optimal representation level is given by:

$$\frac{\partial}{\partial r_j} E_{\epsilon_Q}^1 = 0 \text{ for } j = 1, \dots, J$$

↓

$$\begin{aligned} \frac{\partial}{\partial r_j} E_{\epsilon_Q}^1 &= \frac{\partial}{\partial r_j} \sum_{i=1}^K \int_{d_{i-1}}^{d_i} |x - r_j| p(x) dx = \frac{\partial}{\partial r_j} \int_{d_{j-1}}^{d_j} |x - r_j| p(x) dx = \\ &= \int_{d_{j-1}}^{d_j} \frac{\partial}{\partial r_j} |x - r_j| p(x) dx = - \int_{d_{j-1}}^{d_j} \text{sign}(x - r_j) p(x) dx = \end{aligned}$$

$$\begin{aligned}
&= \int_{d_{j-1}}^{r_j} p(x)dx - \int_{r_j}^{d_j} p(x)dx \stackrel{\text{demand}}{=} 0 \\
&\Downarrow \\
&\int_{d_{j-1}}^{r_j} p(x)dx = \int_{r_j}^{d_j} p(x)dx
\end{aligned}$$

From the optimality condition we can see that the optimal r'_j 's devide the interval $[d_{j-1}, d_j]$ into two equal probabily regions, this can be interpereted as the probabilistic median of the interval for the given $p(x)$.

3. Necessery condition for optimal decision level is given by:

$$\frac{\partial}{\partial d_j} E_{\epsilon_Q}^1 = 0 \text{ for } j = 1, \dots, K-1$$

for fixed $d_0 = \phi_L, d_K = \phi_H$

$$\begin{aligned}
&\Downarrow \\
&\frac{\partial}{\partial d_j} E_{\epsilon_Q}^1 = \frac{\partial}{\partial d_j} \sum_{i=1}^K \int_{d_{i-1}}^{d_i} |x - r_j| p(x) dx = \\
&= \frac{\partial}{\partial d_j} \int_{d_{j-1}}^{d_j} |x - r_j| p(x) dx + \frac{\partial}{\partial d_j} \int_{d_j}^{d_{j+1}} |x - r_{j+1}| p(x) dx = \\
&= |d_j - r_j| p(d_j) - |d_j - r_{j+1}| p(d_j) \stackrel{\text{demand}}{=} 0 \\
&\Downarrow \\
&|d_j - r_j| = |d_j - r_{j+1}|
\end{aligned}$$

If we open the absolute value we get two possibilities:

- (a) $d_j - r_j = d_j - r_{j+1} \Rightarrow r_j = r_{j+1}$ This answer is **impossible** as we know that for wach j: $r_j < r_{j+1}$.
- (b) $d_j - r_j = -(d_j - r_{j+1}) \Rightarrow d_j = \frac{r_j + r_{j+1}}{2}$ So given the representation level we get that the optimal decision levels are the mean of the representation levels.

4. the Max-Lloyd procedure for designing a b -bit quantizer that minimizes the expected absolute-deviation for a given input PDF $p(x)$ can be described as:

- (a) Initialization: Set arbitrary decisions levels $\{d_i\}_{i=0}^K$.
- (b) Compute th e optimal representation levels and set to- $\{r_i\}_{i=1}^J$ for the current decision levels $\{d_i\}_{i=0}^K$.
- (c) Compute the optimal decisions levels and set to- $\{d_i\}_{i=0}^K$ for the current representation levels $\{r_i\}_{i=1}^J$.
- (d) If stopping criteria has not met, return to (b).

Question 2

1. We need to formulate an optimal quantizer design that uses $b = 1$ bit

\Downarrow

we have $J = 2^b = 2$ representation levels, labeled r_1, r_2 , and $J + 1 = 3$ decision levels labeled $d_0 = -4a, d_1, d_2 = 4a$.

- (a) As we learned in class we know that for given decision levels the optimal representation levels are:

$$r_j = \frac{\int_{d_{j-1}}^{d_j} xp(x)dx}{\int_{d_{j-1}}^{d_j} p(x)dx} \text{ for } j = 1, 2$$

and for given representation levels the optimal decision levels are:

$$d_j = \frac{r_j + r_{j+1}}{2} \text{ for } j = 0, 1, 2$$

Now we guess the value of the decision level, $d_1 = 0$, and let's calculate the representation levels according to the current decision levels:

$$\begin{aligned} r_1 &= \frac{\int_{-4a}^{-2a} \frac{x}{4a} dx}{\int_{-4a}^{d_{-2a}} \frac{1}{4a} dx} = \frac{\frac{x^2}{8a} \Big|_{-4a}^{-2a}}{\frac{x}{4a} \Big|_{-4a}^{d_{-2a}}} = -3a \\ r_2 &= \frac{\int_{2a}^{3a} x \left(-\frac{1}{4a^2}x + \frac{7}{8a}\right) dx + \int_{3a}^{4a} x \left(\frac{1}{4a^2}x - \frac{5}{8a}\right) dx}{\int_{2a}^{3a} \left(-\frac{1}{4a^2}x + \frac{7}{8a}\right) dx + \int_{3a}^{4a} \left(\frac{1}{4a^2}x - \frac{5}{8a}\right) dx} = \dots = 3a \end{aligned}$$

Now let's calculate the decision level according to the calculated representation levels:

$$d_1 = \frac{r_1 + r_2}{2} = 0$$

The result didn't change, that means that the values we got are the optimal.

- (b) The minimal squared error achieved is:

$$\begin{aligned} E\{\epsilon^2\} &= \int_{-4a}^{4a} p(x) (x - Q(x))^2 dx = \\ &= \int_{-4a}^{-2a} \frac{1}{4a} (x - (-3a))^2 dx + \int_{2a}^{3a} \left(-\frac{1}{4a^2}x + \frac{7}{8a}\right) (x - 3a)^2 dx + \int_{3a}^{4a} \left(\frac{1}{4a^2}x - \frac{5}{8a}\right) (x - 3a)^2 dx = \\ &= \dots = \frac{a^2}{6} + \frac{5a^2}{48} - \frac{a^2}{48} = \frac{a^2}{4} \end{aligned}$$

2. We need to formulate an optimal quantizer design that uses $b = 2$ bit

\Downarrow

we have $J = 2^b = 4$ representation levels, labeled r_1, r_2, r_3, r_4 , and $J + 1 = 5$ decision levels labeled d_0, d_1, d_2, d_3, d_4 .

- (a) We start with a guess for the decision levels: $d_0 = -4a, d_1 = -2a, d_2 = 2a, d_3 = 3a, d_4 = 4a$ and calculate the matching representation levels:

$$r_1 = -3a, r_2 = 0, r_3 = 2\frac{5}{12}a, r_4 = 3\frac{7}{12}a$$

Now we update the decision levels according to the calculated representation levels:

$$d_0 = -4a, d_1 = -1.5a, d_2 = 1\frac{5}{24}a, d_3 = 3a, d_4 = 4a$$

Now we update the representation levels:

$$r_1 = -3a, r_2 = 0, r_3 = 2\frac{5}{12}a, r_4 = 3\frac{7}{12}a$$

We got the same result as the previous calculation, that means that the algorithm converged, and the final answer is:

$$d_0 = -4a, d_1 = -1.5a, d_2 = 1\frac{5}{24}a, d_3 = 3a, d_4 = 4a$$

$$r_1 = -3a, r_2 = 0, r_3 = 2\frac{5}{12}a, r_4 = 3\frac{7}{12}a$$

- (b) The minimal squared error achieved is:

$$\begin{aligned} E \{ \epsilon^2 \} &= \int_{-4a}^{4a} p(x) (x - Q(x))^2 dx = \\ &= \int_{-4a}^{-2a} \frac{1}{4a} (x - (-3a))^2 dx + \int_{-2a}^{3a} \left(-\frac{1}{4a^2}x + \frac{7}{8a} \right) \left(x - 2\frac{5}{12}a \right)^2 dx + \dots \\ &\dots + \int_{3a}^{4a} \left(\frac{1}{4a^2}x - \frac{5}{8a} \right) \left(x - 3\frac{7}{12}a \right)^2 dx = \dots = 0.20466a^2 \end{aligned}$$

Question 3

$$\begin{aligned}
1. \int_{t \in \Delta_i} (t - t_i)^k dt &= \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i)^k dt = \frac{(t - t_i)^{k+1}}{k+1} \Big|_{\frac{i-1}{N}}^{\frac{i}{N}} = \\
&= \frac{\left(\frac{i}{N} - t_i\right)^{k+1}}{k+1} - \frac{\left(\frac{i-1}{N} - t_i\right)^{k+1}}{k+1} = \frac{\left(\frac{|\Delta_i|}{2}\right)^{k+1} - \left(\frac{-|\Delta_i|}{2}\right)^{k+1}}{k+1} = \\
&= \frac{(1 + (-1)^{k+2}) |\Delta_i|^{k+1}}{2^{k+1}(k+1)} = \begin{cases} 0, k - odd \\ \frac{|\Delta_i|^{k+1}}{2^k(k+1)}, k - even \end{cases}
\end{aligned}$$

2. As we studied in class:

$$MSE = \overbrace{\int_0^1 (\phi(t) - \hat{\phi}(t))^2 dt}^{\text{MSE}_i} = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} (\phi(t) - a_i(t - t_i) - c_i)^2 dt$$

Find the optimal a_i, c_i by requiring:

$$\frac{\partial}{\partial a_i} MSE_i = 0$$

$$\frac{\partial}{\partial c_i} MSE_i = 0$$

For a_i :

$$\begin{aligned}
\frac{\partial}{\partial a_i} MSE_i &= \frac{\partial}{\partial a_i} \int_{\frac{i-1}{N}}^{\frac{i}{N}} (\phi(t) - a_i(t - t_i) - c_i)^2 dt = \\
&= \int_{\frac{i-1}{N}}^{\frac{i}{N}} \frac{\partial}{\partial a_i} (\phi(t) - a_i(t - t_i) - c_i)^2 dt = -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i) (\phi(t) - a_i(t - t_i) - c_i) dt = \\
&= -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) (t - t_i) dt + 2a_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i)^2 dt + 2c_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i) dt
\end{aligned}$$

We calculate seperately the first and second integral, based on the answer from previous section:

$$2a_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i)^2 dt = 2a_i \frac{|\Delta_i|^3}{2^2 3} = \frac{a_i |\Delta_i|^3}{6}$$

$$2c_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i) dt = 0$$

By demanding that $\frac{\partial}{\partial a_i} MSE_i = 0$ we get the equation:

$$-2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) (t - t_i) dt + \frac{a_i |\Delta_i|^3}{6} = 0$$

$$\Downarrow$$

$$a_i = \frac{12}{|\Delta_i|^3} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) (t - t_i) dt$$

Now we will optimize c_i :

$$\begin{aligned} \frac{\partial}{\partial c_i} MSE_i &= \frac{\partial}{\partial c_i} \int_{\frac{i-1}{N}}^{\frac{i}{N}} (\phi(t) - a_i (t - t_i) - c_i)^2 dt = \\ &= \int_{\frac{i-1}{N}}^{\frac{i}{N}} \frac{\partial}{\partial c_i} (\phi(t) - a_i (t - t_i) - c_i)^2 dt = -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (\phi(t) - a_i (t - t_i) - c_i) dt = \\ &= -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) dt + 2a_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i) dt + 2c_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} 1 dt \end{aligned}$$

Based on the answer from previous section we get that the middle integral equals zero, and:

$$2c_i \int_{\frac{i-1}{N}}^{\frac{i}{N}} 1 dt = 2c_i |\Delta_i|$$

By demanding that $\frac{\partial}{\partial c_i} MSE_i = 0$ we get that:

$$\begin{aligned} -2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) dt + 2c_i |\Delta_i| &= 0 \\ \Downarrow \\ c_i &= \frac{1}{|\Delta_i|} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) dt \end{aligned}$$

3. The optimal MSE is calculated by using the optimal a_i, c_i (that give us the optimal $\hat{\phi}(t)$):

$$MSE^{opt} = \int_0^1 \left(\phi(t) - \hat{\phi}(t) \right)^2 dt = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t) - a_i^{opt} (t - t_i) - c_i^{opt} \right)^2 dt =$$

$$\begin{aligned}
&= \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\phi(t)^2 - 2a_i^{opt} \phi(t) (t - t_i) - 2c_i^{opt} \phi(t) + (a_i^{opt})^2 (t - t_i)^2 + 2a_i^{opt} c_i^{opt} (t - t_i) \right. \\
&\quad \left. + (c_i^{opt})^2 dt \right) dt = \\
&= \int_0^1 (\phi(t))^2 dt - 2 \sum_{i=1}^N a_i^{opt} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) (t - t_i) dt - 2 \sum_{i=1}^N c_i^{opt} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) dt + \\
&\quad + \sum_{i=1}^N (a_i^{opt})^2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i)^2 dt + 2 \sum_{i=1}^N a_i^{opt} c_i^{opt} \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i) dt + |\Delta| \sum_{i=1}^N (c_i^{opt})^2 = (**)
\end{aligned}$$

Using the outcomes from section 1 we get that:

$$\sum_{i=1}^N (a_i^{opt})^2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i)^2 dt = \sum_{i=1}^N (a_i^{opt})^2 \frac{|\Delta_i|^3}{2^2 3} \stackrel{(*)}{=} \frac{|\Delta|^3}{12} \sum_{i=1}^N (a_i^{opt})^2$$

(*): All the intervals are in the same size: $|\Delta_i| = |\Delta| = \frac{1}{N}$.

$$2 \sum_{i=1}^N a_i^{opt} c_i^{opt} \int_{\frac{i-1}{N}}^{\frac{i}{N}} (t - t_i) dt = 0$$

Using the expressions we got for optimal a_i, c_i we get:

$$\begin{aligned}
-2 \sum_{i=1}^N a_i^{opt} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) (t - t_i) dt &= -2 \sum_{i=1}^N (a_i^{opt})^2 \frac{|\Delta_i|^3}{12} = \\
&= -\frac{|\Delta|^3}{6} \sum_{i=1}^N (a_i^{opt})^2 \\
-2 \sum_{i=1}^N c_i^{opt} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \phi(t) dt &= -2 \sum_{i=1}^N c_i^{opt} |\Delta_i| c_i^{opt} = -2 |\Delta| \sum_{i=1}^N (c_i^{opt})^2
\end{aligned}$$

And finally we get:

$$\begin{aligned}
(**) &= \int_0^1 (\phi(t))^2 dt - \frac{|\Delta|^3}{6} \sum_{i=1}^N (a_i^{opt})^2 - 2 |\Delta| \sum_{i=1}^N (c_i^{opt})^2 + \frac{|\Delta|^3}{12} \sum_{i=1}^N (a_i^{opt})^2 + |\Delta| \sum_{i=1}^N (c_i^{opt})^2 = \\
&= \int_0^1 (\phi(t))^2 dt - \frac{|\Delta|^3}{12} \sum_{i=1}^N (a_i^{opt})^2 - |\Delta| \sum_{i=1}^N (c_i^{opt})^2 =
\end{aligned}$$

$$|\Delta| = \frac{1}{N}$$

$$= \int_0^1 (\phi(t))^2 dt - \frac{1}{12N^3} \sum_{i=1}^N (a_i^{opt})^2 - \frac{1}{N} \sum_{i=1}^N (c_i^{opt})^2$$

4. As we saw in class the minimal MSE for picewise-constant approximation is:

$$MSE_{const} = \int_0^1 \phi^2(t) dt - \frac{1}{N} \sum_{i=1}^N (c_i^{opt})^2$$

If we calculate the difference $MSE_{lin} - MSE_{const}$ we get:

$$MSE_{lin} - MSE_{const} = -\frac{1}{12N^3} \sum_{i=1}^N (a_i^{opt})^2$$

And because $\forall i, (a_i)^2 \geq 0$ we get that: $-\frac{1}{12N^3} \sum_{i=1}^N (a_i^{opt})^2 \leq 0$

↓

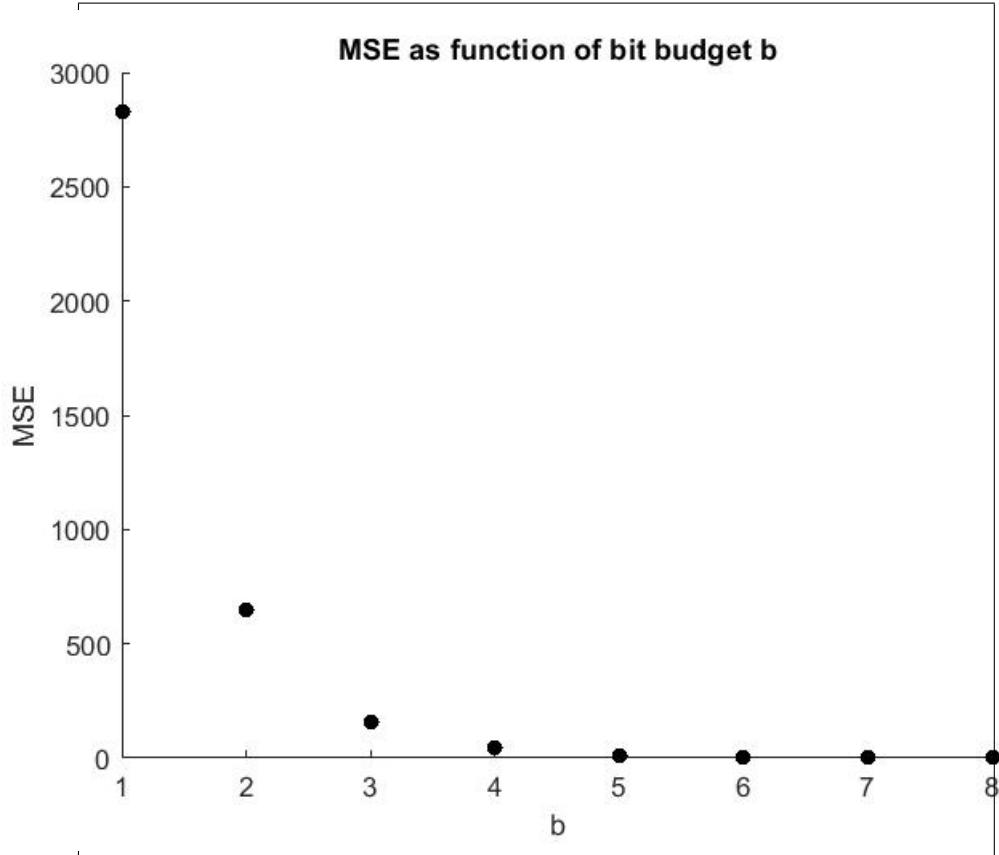
$$MSE_{lin} \leq MSE_{const}$$

Therefore the MSE using picewise linear approximation is lower(or equal) to the MSE achieved by using picewise-constant approximation.

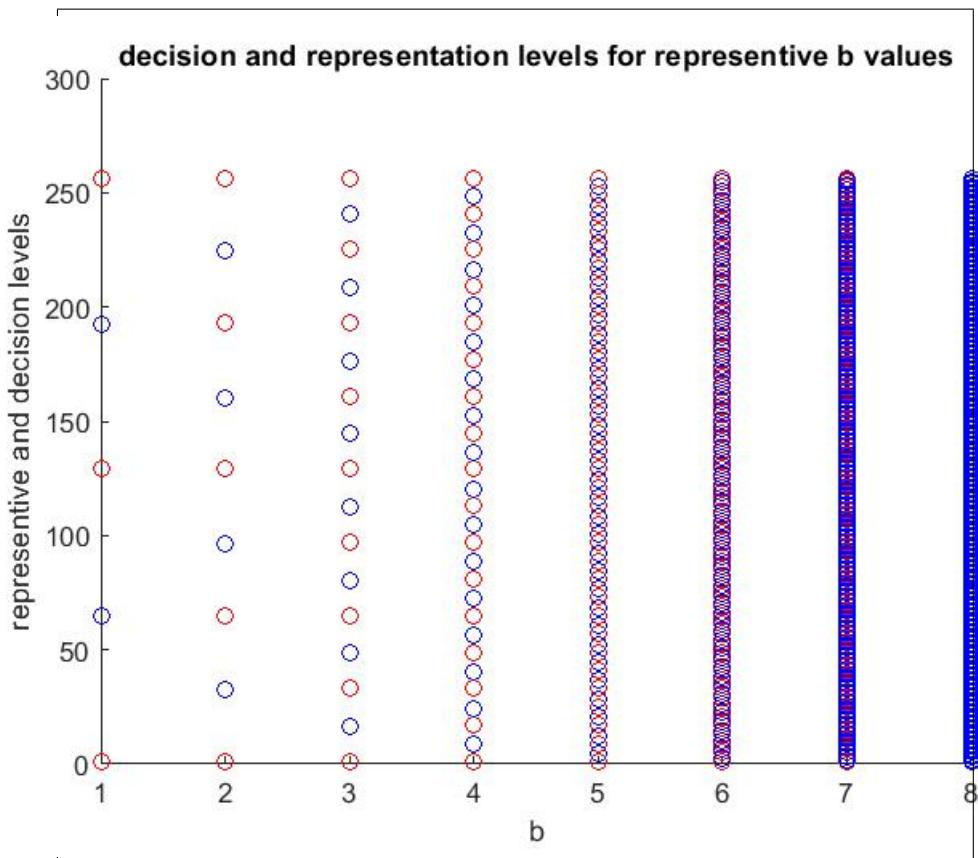
Part 2 - Matlab

Question 2 - uniform quantizer:

1. MSE as a function of the bit budget for $b=1,\dots,8$:



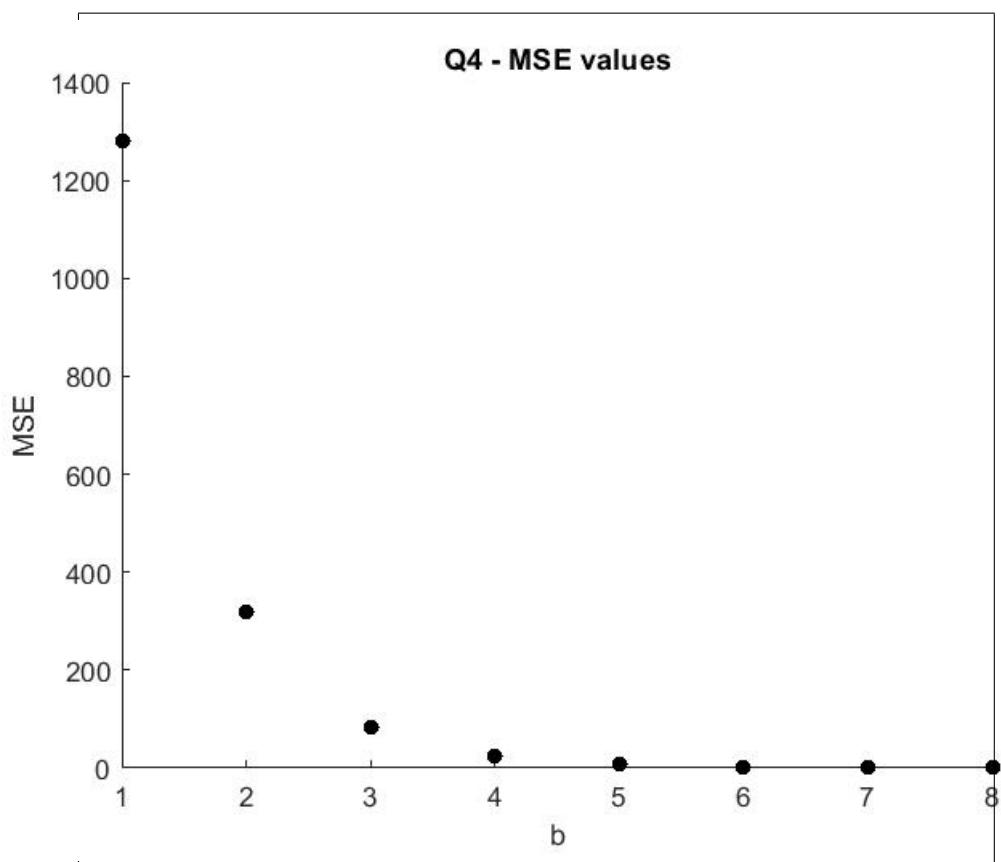
2. The decision and representation levels for representative b values:



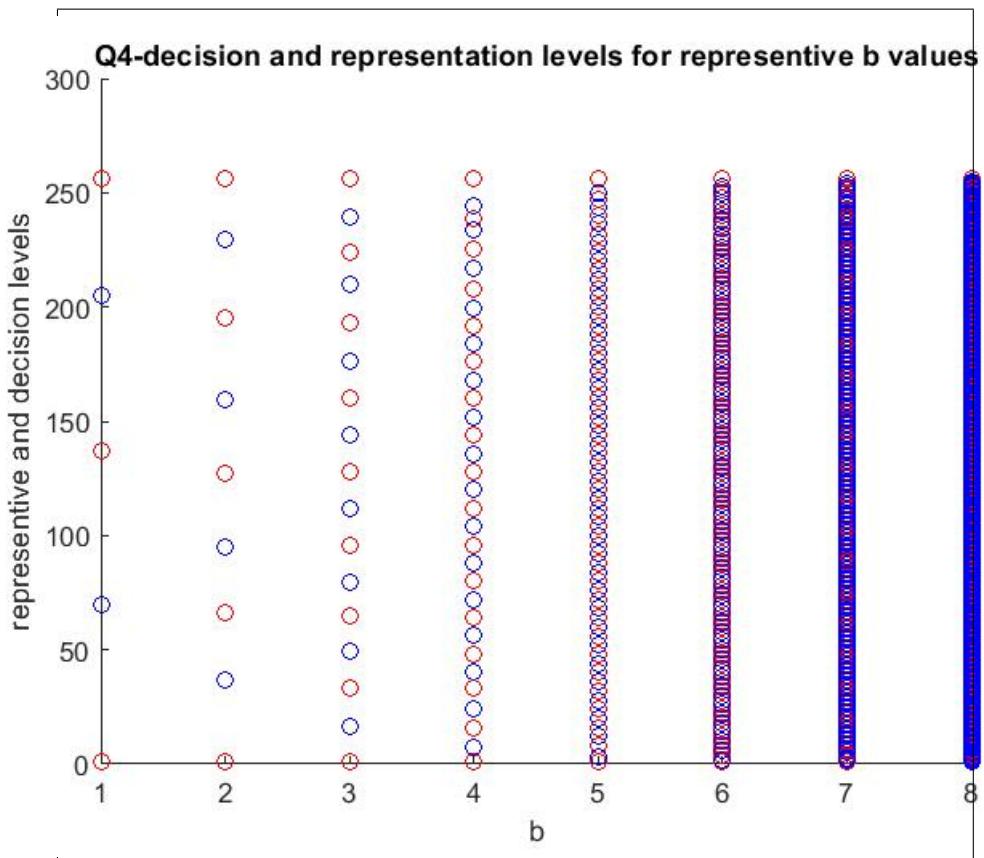
red-decision blue-representation 0.1: Figure

Question 4 - Max-Lloyd uniform quantization:

1. MSE as a function of the bit budget for $b=1,\dots,8$:



2. The decision and representation levels for representative b values:

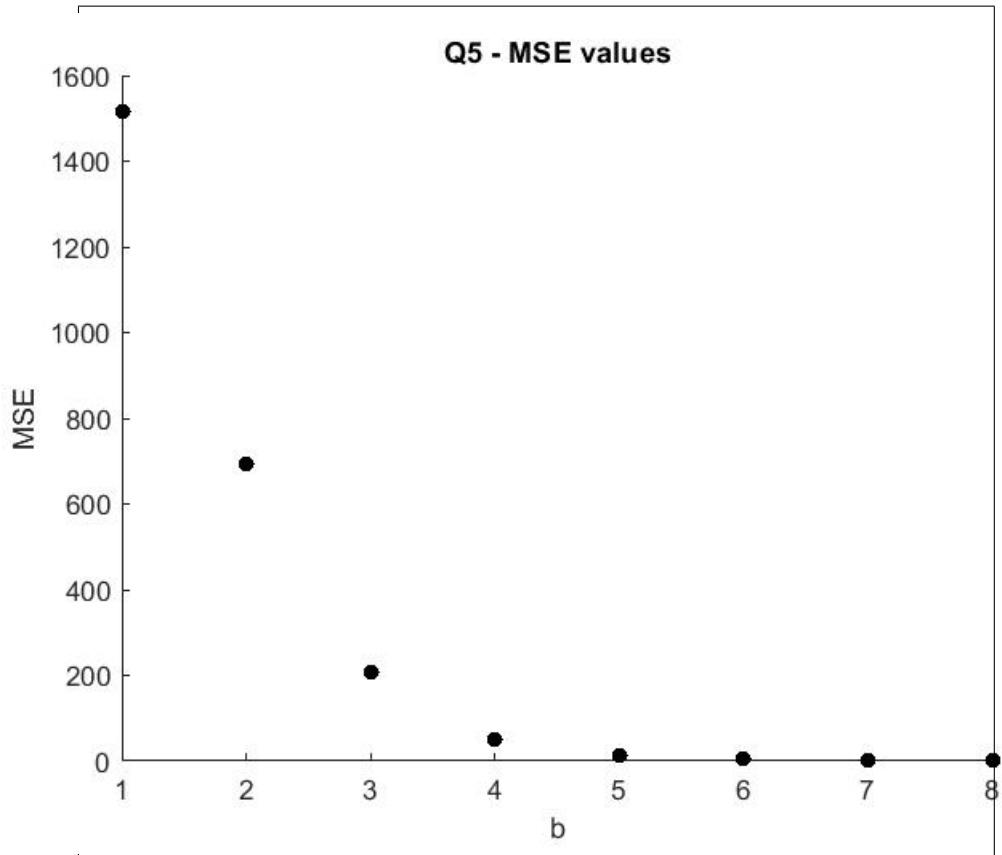


red-decision blue-representation 0.2: Figure

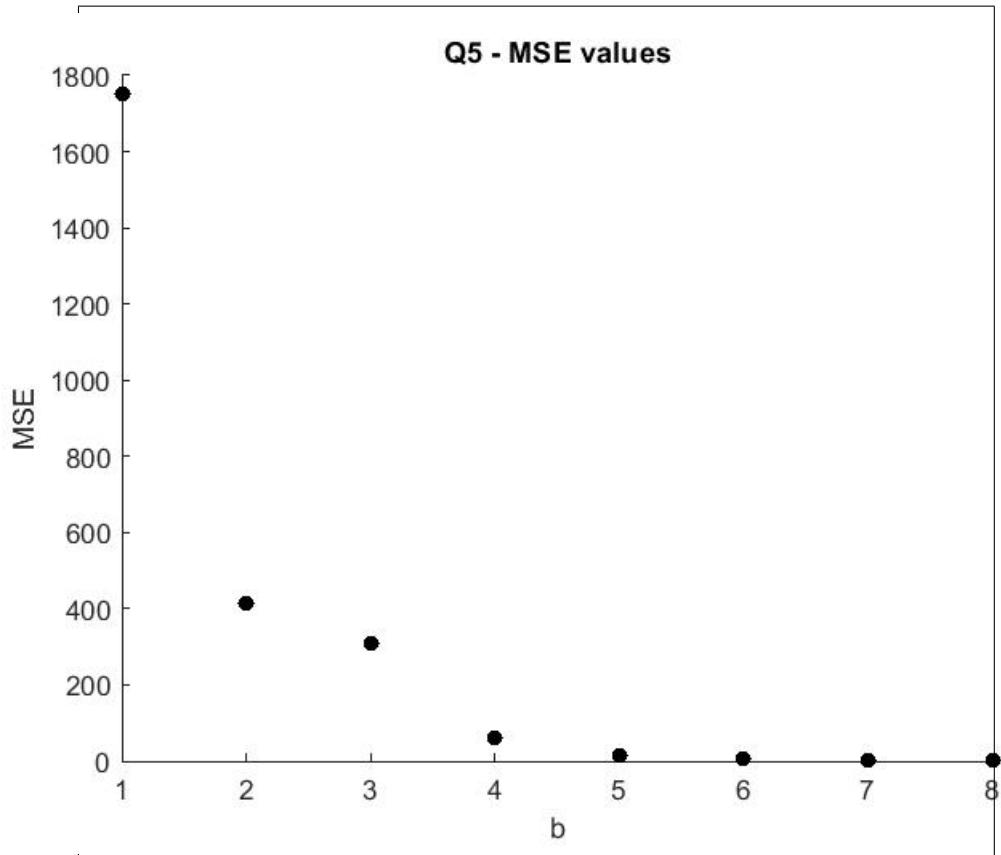
- As we could expected, the MSE values obtained using Max-Lloyd are lower than the MSE values obtained by the uniform quantization, as the Max-Lloyd algorithm should converge to the best representation and decision levels. And as we can see, the decision levels in the uniform quantization remain uniform, whereas in the Max-Lloyd algorithm they do not.

Question 5-Max-Lloyd random quantization:

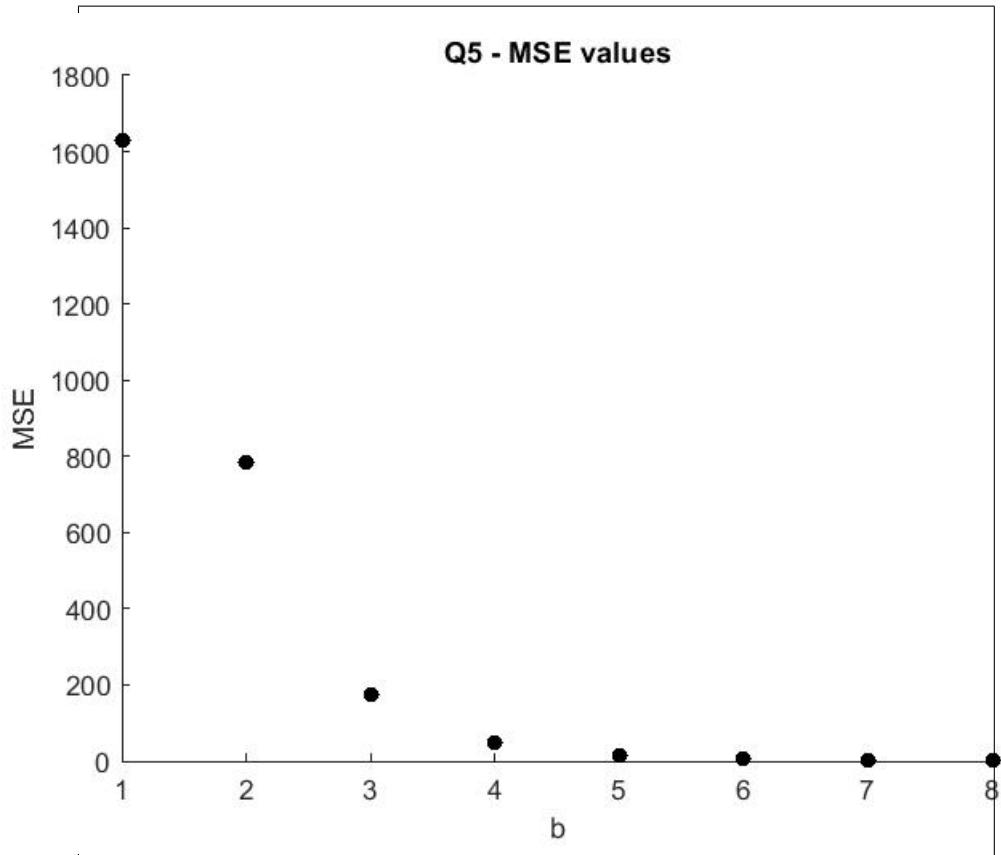
- MSE as a function of the bit budget for $b=1,\dots,8$ for 5 random quantizations:



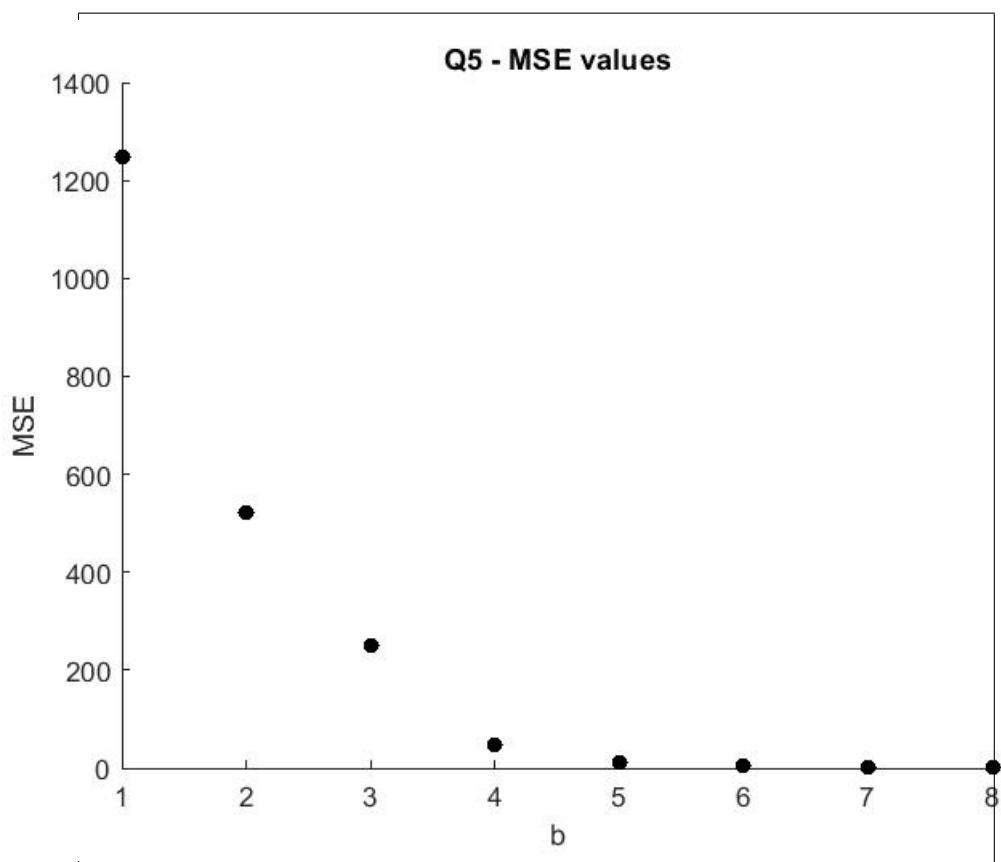
run no.1 0.3: Figure



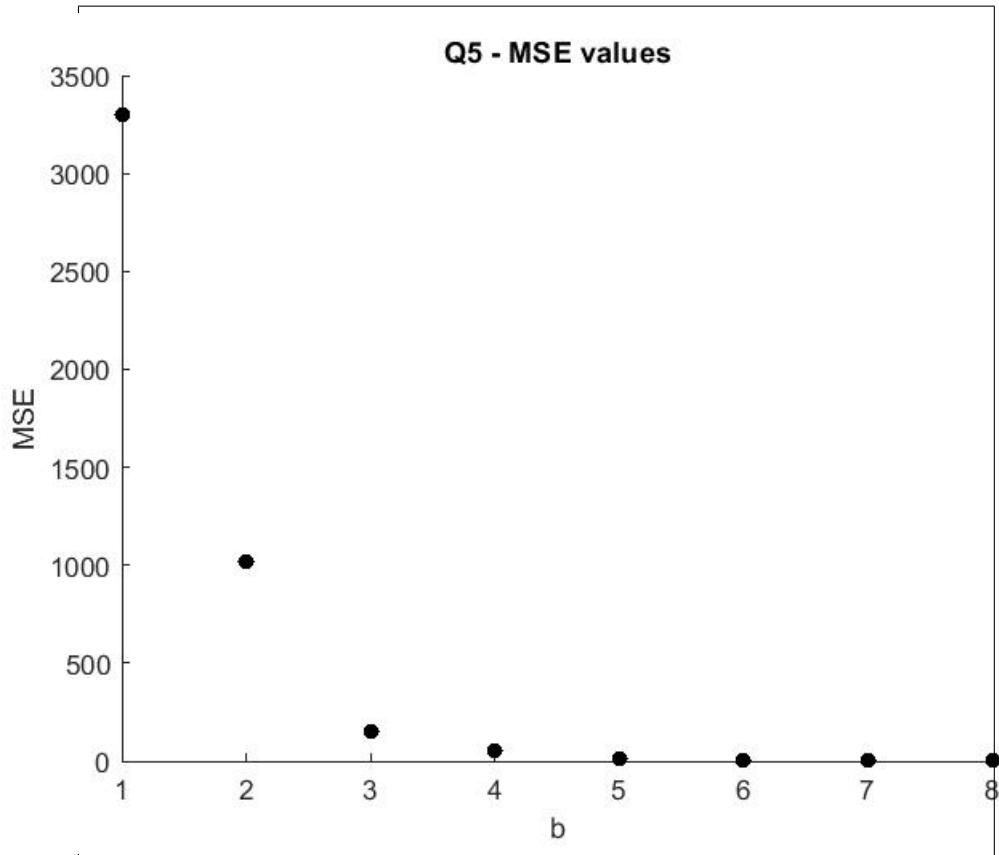
run no.2 0.4: Figure



run no.3 0.5: Figure

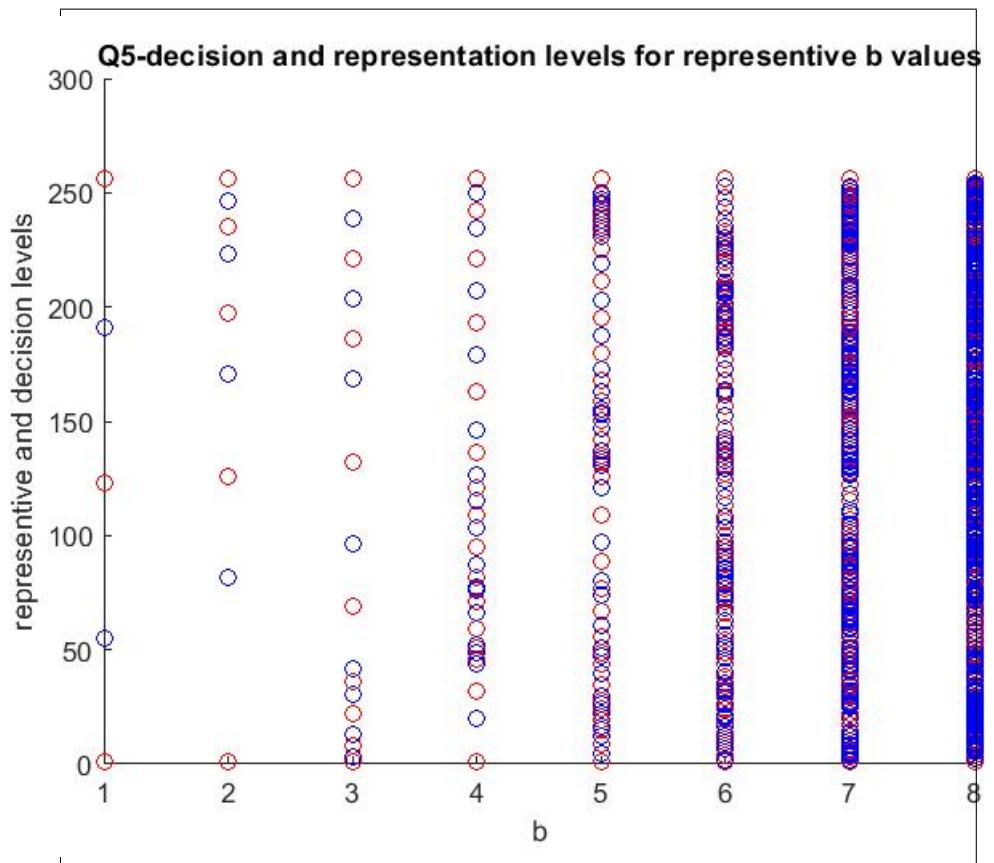


run no.4 0.6: Figure

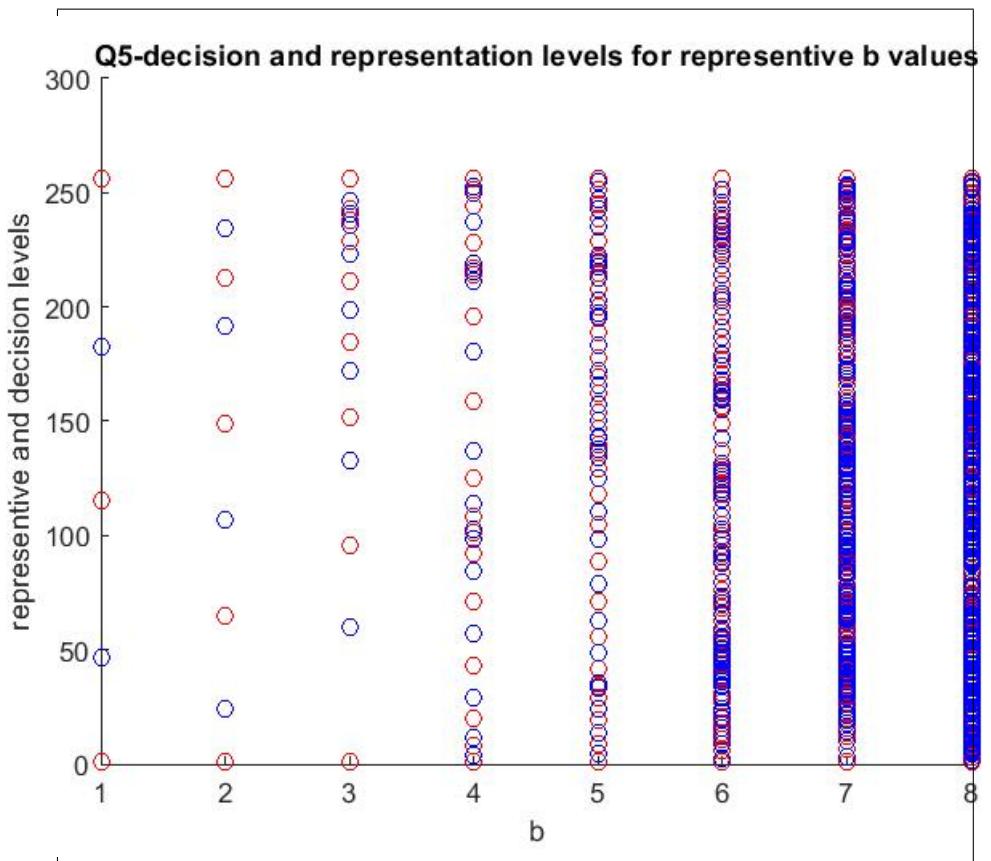


run no.5 0.7: Figure

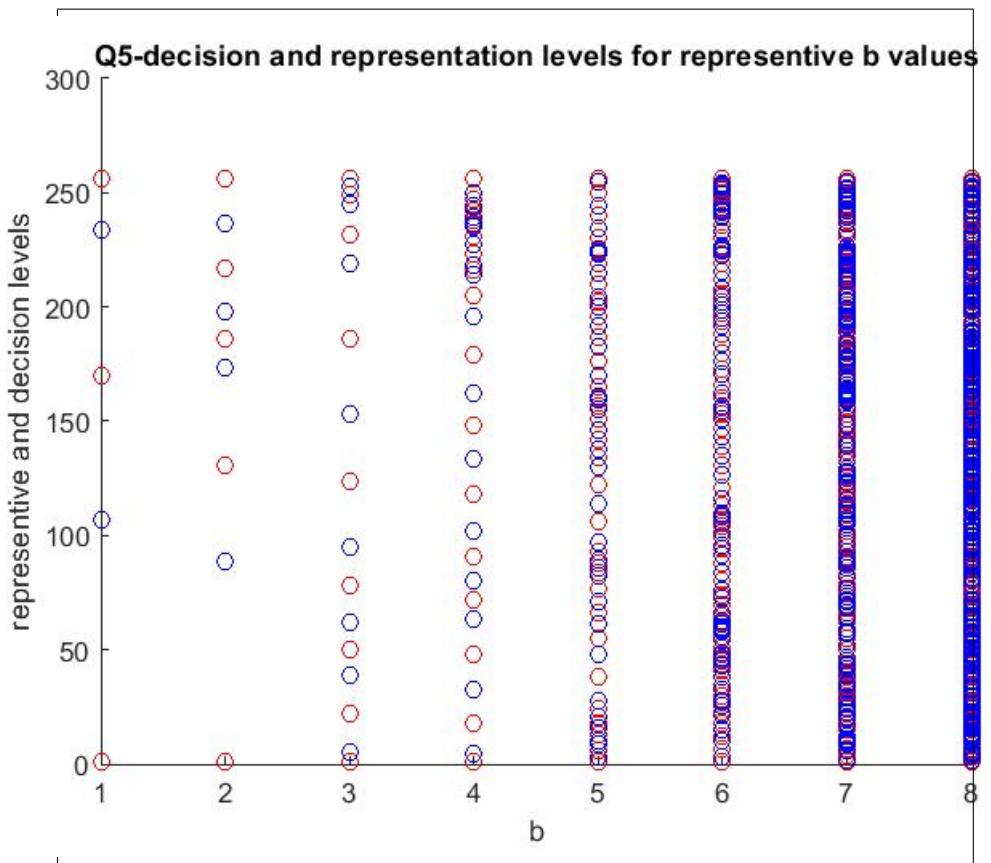
2. The decision and representation levels for representative b values, for 5 random quantizations:



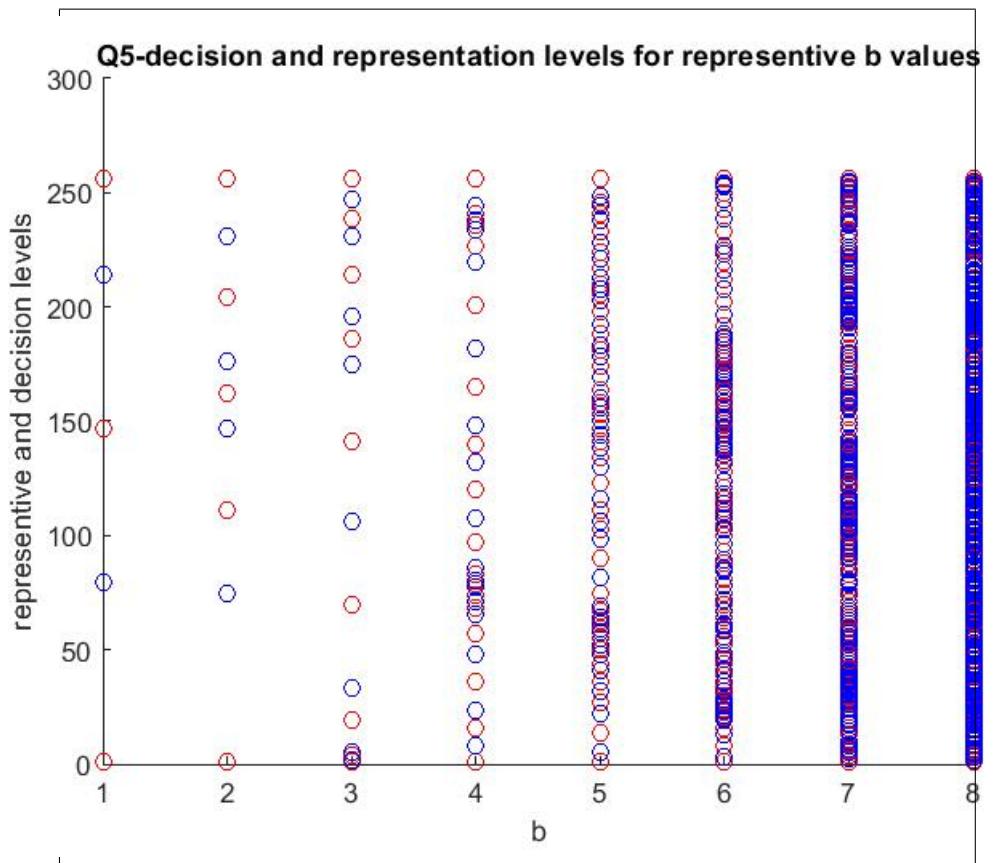
run no.1 0.8: Figure



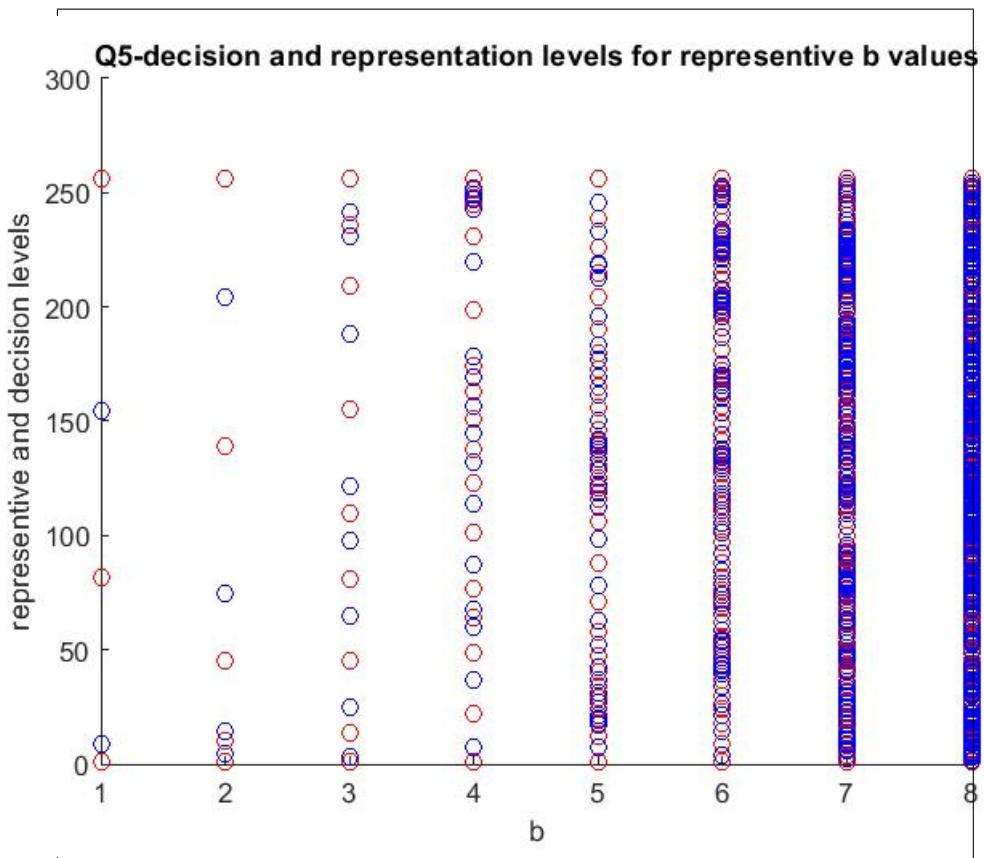
run no.2 0.9: Figure



run no.3 0.10: Figure



run no.4 0.11: Figure



run no.5 0.12: Figure

3. As we can see, the Max-Lloyd algorithm using uniform quantization obtained better results rather than starting with random quantization, as the MSE converged to zero in better performance.

Question 4

- a. At step n , after computing the $\{d\}_{i=0}^k$ values, the Max-Lloyd algorithm will compute the optimal representation levels based on these decision levels

according to the following formula (as seen in class): $r_i^n = \frac{\int_{d_{i-1}^n}^{d_i^n} xp(x)dx}{\int_{d_{i-1}^n}^{d_i^n} p(x)dx}$.

In the case of a uniform $p(x)$, we can simplify the expression:

$$\begin{aligned} r_i^n &= \frac{\int_{d_{i-1}^n}^{d_i^n} xp(x)dx}{\int_{d_{i-1}^n}^{d_i^n} p(x)dx} = \frac{\frac{1}{b-a} \int_{d_{i-1}^n}^{d_i^n} x dx}{\frac{1}{b-a} \int_{d_{i-1}^n}^{d_i^n} dx} = \frac{\frac{1}{2} x^2 \Big|_{d_{i-1}^n}^{d_i^n}}{d_i^n - d_{i-1}^n} = \frac{1}{2} \frac{(d_i^n)^2 - (d_{i-1}^n)^2}{d_i^n - d_{i-1}^n} \\ &= \frac{1}{2} \frac{(d_i^n + d_{i-1}^n)(d_i^n - d_{i-1}^n)}{d_i^n - d_{i-1}^n} = \frac{d_i^n + d_{i-1}^n}{2} \end{aligned}$$

For $i = 0$, we take $r_0^n = a$, and for $i = k$ we take $r_k^n = b$.

We got, as expected, that every representation level is the average of the 2 decision levels next to it.

Now, the algorithm computes the d^{n+1} values:

$$d_i^{n+1} = \frac{r_i^n + r_{i+1}^n}{2} = \frac{\frac{d_{i-1}^n + d_i^n}{2} + \frac{d_i^n + d_{i+1}^n}{2}}{2} = \frac{d_{i-1}^n + 2d_i^n + d_{i+1}^n}{4}.$$

To finish step $(n + 1)$, the algorithm computes the r^{n+1} values: (we saw before that the integral formula becomes just an average)

$$\begin{aligned} r_i^{n+1} &= \frac{d_i^{n+1} + d_{i-1}^{n+1}}{2} = \frac{1}{2} \left(\frac{r_i^n + r_{i+1}^n}{2} + \frac{r_{i-1}^n + r_i^n}{2} \right) = \frac{r_{i-1}^n + 2r_i^n + r_{i+1}^n}{4} \\ &= \frac{1}{4} \left(\frac{d_{i-1}^n + d_{i-2}^n}{2} + 2 \frac{d_i^n + d_{i-1}^n}{2} + \frac{d_{i+1}^n + d_i^n}{2} \right) \\ &= \frac{d_{i-2}^n + 3d_{i-1}^n + 3d_i^n + d_{i+1}^n}{8} \end{aligned}$$

- b. The uniform quantization will be invariant through the Max-Lloyd algorithm:

$$\begin{aligned} d_i^n &= a + i \frac{(b-a)}{k}, \forall 0 \leq i \leq k \\ r_i^n &= a + \left(i - \frac{1}{2}\right) \frac{(b-a)}{k}, \forall 1 \leq i \leq k \end{aligned}$$

We will show that this quantization satisfies $r_i^n = r_i^{n+1}$, $d_j^n = d_j^{n+1}$.

$$\begin{aligned} r_i^{n+1} &= \frac{d_{i-1}^{n+1} + d_i^{n+1}}{2} = \frac{1}{2} \left(\left(a + (i-1)\frac{b-a}{k}\right) + \left(a + i\frac{b-a}{k}\right) \right) \\ &= \frac{1}{2} \left(a + i\frac{b-a}{k} - \frac{b-a}{k} + a + i\frac{b-a}{k} \right) \\ &= \frac{1}{2} \left(2a + (2i-1)\frac{b-a}{k} \right) = a + \left(i - \frac{1}{2}\right) \frac{b-a}{k} = r_i^n \end{aligned}$$

$$\begin{aligned}
d_i^{n+1} &= \frac{r_i^n + r_{i+1}^n}{2} = \frac{1}{2} \left(a + \left(i - \frac{1}{2} \right) \frac{b-a}{k} + a + \left(i + 1 - \frac{1}{2} \right) \frac{b-a}{k} \right) \\
&= \frac{1}{2} \left(2a + \left(i + i + 1 - \frac{1}{2} - \frac{1}{2} \right) \frac{b-a}{k} \right) = \frac{1}{2} \left(2a + 2i \frac{b-a}{k} \right) \\
&= a + i \frac{b-a}{k} = d_i^n
\end{aligned}$$

- c. We will formulate $d_i^{n+1} = \frac{d_{i-1}^n + 2d_i^n + d_{i+1}^n}{4} = \frac{1}{4}d_{i-1}^n + \frac{1}{2}d_i^n + \frac{1}{4}d_{i+1}^n$ into matrix form:

$$d^{n+1} = Ad^n \rightarrow \begin{pmatrix} d_0^{n+1} \\ \vdots \\ d_k^{n+1} \end{pmatrix} = A \begin{pmatrix} d_0^n \\ \vdots \\ d_k^n \end{pmatrix}$$

The matrix A will be of size $(k+1) \times (k+1)$.

For every $0 \leq i \leq k$, we want $d_i^{n+1} = A_i \begin{pmatrix} d_0^n \\ \vdots \\ d_k^n \end{pmatrix}$, where A_i is the i 'th row of A .

$$\rightarrow \frac{1}{4}d_{i-1}^n + \frac{1}{2}d_i^n + \frac{1}{4}d_{i+1}^n = \sum_{j=1}^{k+1} A_{ij} d_j^n \rightarrow A_{i,j-1} = \frac{1}{4}, A_{i,j} = \frac{1}{2}, A_{i,j+1} = \frac{1}{4}$$

We will now look at the edge cases ($i = 0$ or $i = k$):

$$i = 0 \rightarrow d_0^n = a \rightarrow d_0^n = d_0^{n-1} \rightarrow A_{i,0} = 1$$

$$i = k \rightarrow d_k^n = b \rightarrow d_k^n = d_k^{n-1} \rightarrow A_{i,k} = 1$$

This result finds the constant $c_1 = 1$.

$$\text{So, we know that } A \text{ looks as follows: } \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, we also found the constant $c_2 = \frac{1}{4}$.

$$\text{Also, we found that } A \text{ can be written as } A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & \tilde{B} & & \vdots \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \mathbf{0} & \mathbf{0} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \vdots \\ \vdots & \ddots & & & \vdots \\ \mathbf{0} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Where \tilde{B} is a tridiagonal Toeplitz matrix: $\tilde{B} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \mathbf{0} & \mathbf{0} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \vdots \\ \vdots & \ddots & & & \vdots \\ \mathbf{0} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

- d. We will expand the expression we found for d^n :

$$d^n = Ad^{n-1} = A(Ad^{n-2}) = A^2d^{n-2} = \dots = A^i d^{n-i} = \dots = A^n d^0$$

$$\rightarrow \mathbf{d}^n = A^n \mathbf{d}^0$$

- e. Since \tilde{B} has the same values in the diagonals above and below its main diagonal, it is symmetric, and therefore diagonalizable (every real symmetric matrix is diagonalizable).
- f. We will start by showing that x_0 and x_1 are eigenvectors of A , associated to the eigenvalue 1:

$$Ax_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(k-1) \\ 2k \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 \\ \vdots \\ \frac{1}{4} \cdot 2(j-2) + \frac{1}{2} \cdot 2(j-1) + \frac{1}{4} \cdot 2j \\ \vdots \\ \frac{1}{4} \cdot 2(k-2) + \frac{1}{2} \cdot 2(k-1) + \frac{1}{4} \cdot 2k \\ 1 \cdot 2k \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(j-1) \\ 2(k-1) \\ 2k \end{pmatrix}$$

$$= 1 \cdot x_0$$

$$Ax_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2k \\ 2(k-1) \\ \vdots \\ 2 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{4} \cdot 2k + \frac{1}{2} \cdot 2(k-1) + \frac{1}{4} \cdot 2(k-2) \\ \vdots \\ \frac{1}{4} \cdot 2(k-j+2) + \frac{1}{2} \cdot 2(k-j+1) + \frac{1}{4} \cdot 2(k-j) \\ \vdots \\ \frac{1}{4} \cdot 4 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 0 \\ 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2k \\ 2(k-1) \\ \vdots \\ 2(k-j+1) \\ 2 \\ 0 \end{pmatrix} =$$

$$1 \cdot x_1$$

We will now show that x_0 and x_1 are linearly independent:

Let a, b be constants such that $ax_0 + bx_1 = \vec{0}$:

$$\begin{aligned} & \rightarrow a \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(k-1) \\ 2k \end{pmatrix} + b \begin{pmatrix} 2k \\ 2(k-1) \\ \vdots \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \\ & \rightarrow \begin{pmatrix} 0 \cdot a + 2bk \\ 2a + 2b(k-1) \\ \vdots \\ 2a(k-1) + 2b \\ 2ak + 0 \cdot b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

From the first row we get: $2bk = 0 \rightarrow b = 0$.

From the last row we get : $2ak = 0 \rightarrow a = 0$.

We found that the only a, b that satisfy $ax_0 + bx_1 = \vec{0}$ are $a = b = 0$, and so x_0 and x_1 are linearly independent.

g. We will use the following observation:

If $\tilde{B}e_{\lambda_1} = \lambda_i e_{\lambda_1}$ then

$$\begin{aligned} A \begin{pmatrix} v_1 \\ | \\ e_{\lambda_1} \\ | \\ v_{k+1} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ | \\ e_{\lambda_1} \\ | \\ v_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & & & 0 \\ 0 & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ | \\ e_{\lambda_1} \\ | \\ v_{k+1} \end{pmatrix} = \\ &\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & 0 & & & 0 \\ 0 & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & \vdots \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} v_1 \\ | \\ e_{\lambda_1} \\ | \\ v_{k+1} \end{pmatrix} = \end{aligned}$$

$$\begin{aligned}
& \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & v_1 \\ \vdots & 0 & & & e_{\lambda_1} \\ 0 & & 0 & & \vdots \\ \vdots & & 1 & & v_{k+1} \\ 0 & 0 & \cdots & 0 & \frac{1}{4} \\ \end{array} \right) \left(\begin{array}{c} v_1 \\ \vdots \\ e_{\lambda_1} \\ \vdots \\ v_{k+1} \end{array} \right) + \left(\begin{array}{cccc} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & & v_1 \\ 0 & 0 & \cdots & 0 & e_{\lambda_1} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \end{array} \right) \tilde{B} \left(\begin{array}{c} v_1 \\ \vdots \\ e_{\lambda_1} \\ \vdots \\ v_{k+1} \end{array} \right) nn \\
& = \left(\begin{array}{c} v_1 \\ \frac{1}{4}v_1 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}v_{k+1} \\ v_{k+1} \end{array} \right) + \left(\begin{array}{c} 0 \\ \vdots \\ \tilde{B}e_{\lambda_1} \\ \vdots \\ 0 \end{array} \right) = \left(\begin{array}{c} v_1 \\ \frac{1}{4}v_1 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}v_{k+1} \\ v_{k+1} \end{array} \right) + \left(\begin{array}{c} 0 \\ \vdots \\ \lambda_i e_{\lambda_1} \\ \vdots \\ 0 \end{array} \right)
\end{aligned}$$

So, if we pick a vector with $v_1 = v_{k+1} = 0$, then for every eigenvector e_{λ_i} of

\tilde{B} , the vector $\left(\begin{array}{c} 0 \\ e_{\lambda_i} \\ \vdots \\ 0 \end{array} \right)$ is an eigenvector of A with the eigenvalue λ_i :

$$A \left(\begin{array}{c} 0 \\ e_{\lambda_i} \\ \vdots \\ 0 \end{array} \right) = \left(\begin{array}{c} 0 \\ \frac{1}{4} \cdot 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} \cdot 0 \\ 0 \end{array} \right) + \left(\begin{array}{c} 0 \\ \vdots \\ \lambda_i e_{\lambda_1} \\ \vdots \\ 0 \end{array} \right) = \lambda_i \left(\begin{array}{c} 0 \\ e_{\lambda_1} \\ \vdots \\ 0 \end{array} \right)$$

In this way we found $(k - 1)$ eigenvectors of A , where every the multiplicity

of the eigenvector $\left(\begin{array}{c} 0 \\ e_{\lambda_i} \\ \vdots \\ 0 \end{array} \right)$ is the same as the multiplicity of e_{λ_i} as an

eigenvector of \tilde{B} . We need to find 2 more eigenvectors. Fortunately, we

found these 2 eigenvectors in section (f), which are not in the form $\left(\begin{array}{c} 0 \\ \bar{v} \\ \vdots \\ 0 \end{array} \right)$, so

we know that these are the last 2 eigenvectors, and they share the eigenvalue of 1.

\tilde{B} is diagonalizable and so the algebraic multiplicity of its eigenvalues equals to their geometric multiplicity. Let's have a look at the characteristic polynomial of A :

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 0 & \cdots & 0 & 0 \\ 1 & \frac{1}{4} & & & \vdots \\ 0 & & \ddots & & \vdots \\ \vdots & & & \lambda I - \tilde{B} & \vdots \\ 0 & & & & 0 \\ \vdots & & & & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & \lambda - 1 \end{pmatrix} =$$

$$(\lambda - 1) \cdot \det \begin{pmatrix} \lambda I - \tilde{B} & & & \vdots \\ & \ddots & & 0 \\ & & \lambda I - \tilde{B} & \vdots \\ 0 & \cdots & 0 & \frac{1}{4} \end{pmatrix} =$$

$$\pm(\lambda - 1)(\lambda - 1) \cdot \det \begin{pmatrix} \lambda I - \tilde{B} \end{pmatrix} = \pm(\lambda - 1)^2 \cdot p_{\tilde{B}}(\lambda)$$

From this polynomial we see that $\lambda = 1$ is an eigenvalue of algebraic multiplicity of 2, and we previously saw that it has 2 corresponding linearly independent eigenvectors, so its geometric multiplicity is also 2.

To conclude, we found out that the algebraic multiplicities of the eigenvalues of A equal to their geometric multiplicities, and so A is diagonalizable.

$$h. B = 4\tilde{B} = 4 \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & \cdots & 0 & 0 \\ 1 & 2 & 1 & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 1 & 2 & 1 & \\ 0 & 0 & \cdots & 1 & 2 \end{pmatrix}.$$

The eigenvectors of B are the same as those of \tilde{B} : $e_{\lambda_1}, \dots, e_{\lambda_{k-1}}$.

However the eigenvalues are slightly different: for every eigenvector e_{λ_i} , the eigenvalue of B is $4\lambda_i$:

$$Be_{\lambda_i} = (4\tilde{B})e_{\lambda_i} = 4 \cdot (\tilde{B}e_{\lambda_i}) = 4 \cdot (\lambda_i e_{\lambda_i}) \rightarrow Be_{\lambda_i} = (4\lambda_i)e_{\lambda_i}$$

i. $\chi B_{k-1}(X) = \det(B_{k-1} - XI) =$

$$\det(B - XI) = \det \begin{pmatrix} 2 - X & 1 & \cdots & 0 & 0 \\ 1 & 2 - X & 1 & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 1 & 2 - X & 1 & \\ 0 & 0 & \cdots & 1 & 2 - X \end{pmatrix}_{(k-1) \times (k-1)} =$$

$$\begin{aligned}
& (2 - X) \det \begin{pmatrix} 2 - X & 1 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 1 & 2 - X & 1 \\ 0 & \cdots & 1 & 2 - X \end{pmatrix}_{(k-2) \times (k-2)} \\
& - \det \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 - X & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 - X & 1 \\ 0 & 0 & \cdots & 1 & 2 - X \end{pmatrix}_{(k-2) \times (k-2)} = \\
& (2 - X) \cdot \chi B_{k-2}(X) - 1 \cdot \det \begin{pmatrix} 2 - X & 1 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 2 - X & 1 \\ 0 & \cdots & 1 & 2 - X \end{pmatrix}_{(k-3) \times (k-3)} = \\
& (2 - X) \cdot \chi B_{k-2}(X) - \chi B_{k-3}(X)
\end{aligned}$$

Now we will calculate $\chi B_0(X)$ and $\chi B_1(X)$:

$$\chi B_0(X) = \det(B_0 - XI) = \det((\) - XI_{0 \times 0}) = \det((\)) = 1$$

$$\chi B_1(X) = \det(B_1 - XI) = \det((2) - XI) = \det((2 - X)) = 2 - X$$

- j. We can rewrite the recursive formula from the last section as:

(for $i \in \{2, \dots, k-1\}$)

$$\chi B_i(X) = (2 - X)\chi B_{i-1}(X) - \chi B_{i-2}(X)$$

So, if we will define $2Y = 2 - 2X \rightarrow 2 - 2Y = 2X$, then we can get that

$$\begin{aligned}
\chi B_i(2 - 2X) &= \chi B_i(2Y) = (2 - 2Y)\chi B_{i-1}(2Y) - \chi B_{i-2}(2Y) = \\
&= 2X \cdot \chi B_{i-1}(2 - 2X) - \chi B_{i-2}(2 - 2X)
\end{aligned}$$

Therefore, if we will define $Q_i(X) = \chi B_i(2 - 2X)$, we will get the wanted recursive relation: (for every $i \in \{2, \dots, k-1\}$)

$$\begin{aligned}
Q_i(X) &= \chi B_i(2 - 2X) = 2X \cdot \chi B_{i-1}(2 - 2X) - \chi B_{i-2}(2 - 2X) = \\
&= 2XQ_{i-1}(X) - Q_{i-2}(X)
\end{aligned}$$

$Q_i(X)$ and $\chi B_i(X)$ are easily expressed by one another:

- $\mathbf{Q}_i(X) = \chi B_i(2 - 2X)$

- $2 - 2X = Y \rightarrow 2X = 2 - Y \rightarrow X = 1 - \frac{Y}{2} \rightarrow \chi B_i(X) = \mathbf{Q}_i(1 - \frac{X}{2})$

We will now find $Q_0(X)$ and $Q_1(X)$ using the relation we found:

$$Q_0(X) = \chi B_0(2 - 2X) = 1 \Big|_{2-2X} = 1$$

$$Q_1(X) = \chi B_1(2 - 2X) = (2 - X) \Big|_{2-2X} = 2 - (2 - 2X) = 2X$$

- k. Let's calculate the next few Q_i 's:

$$Q_2(X) = 2X \cdot Q_1(X) - Q_0(X) = 2X \cdot 2X - 1 = 4X^2 - 1$$

$$Q_3(X) = 2X \cdot Q_2(X) - Q_1(X) = 2X \cdot (4X^2 - 1) - 2X = 8X^3 - 4X$$

The $Q_i(X)$ series is the **Chebyshev polynomials of the second kind**.

- l. For every $i \in \{0, \dots, k-1\}$, the degree of $Q_i(X)$ is i . We will prove that by induction on i :

Base: For $i = 0$, we know that $Q_0(X) = 1$, which is of degree $i = 0$.

For $i = 1$, we know that $Q_1(X) = 2X$, which is of degree $i = 1$.

Step: Let i be in the range of $[2, k - 1]$. We will assume that the degree of $Q_{i-1}(X)$ is $(i - 1)$, and that the degree of $Q_{i-2}(X)$ is $(i - 2)$, and then we will show that the degree of $Q_i(X)$ is i .

The degree of $2X \cdot Q_{i-1}(X)$ is then necessarily i , which is strictly larger than $(i - 2)$. We will use the recursive definition of $Q_i(X)$:

$Q_i(X) = 2X \cdot Q_{i-1}(X) - Q_{i-2}(X)$, and therefore the degree of $Q_i(X)$ is the same as $2X \cdot Q_{i-1}(X)$, which is i .

According to a theorem from linear algebra, we know that since $Q_i(X)$ is of degree i , it has at most i roots.

m. We will prove that $Q_i(\cos(\theta)) = \frac{\sin((i+1)\theta)}{\sin(\theta)}$ by induction on i :

Base: For $i = 0$, we know that $Q_0(\cos(\theta)) = 1 = \frac{\sin(\theta)}{\sin(\theta)} = \frac{\sin((0+1)\theta)}{\sin(\theta)}$.

For $i = 1$, we know that $Q_1(\cos(\theta)) = 2 \cos(\theta) = \frac{\sin(2\theta)}{\sin(\theta)} = \frac{\sin((1+1)\theta)}{\sin(\theta)}$.

(We used the following identity:

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta) \rightarrow 2 \cos(\theta) = \frac{\sin(2\theta)}{\sin(\theta)}$$

Step: We assume that $Q_{i-1}(\cos(\theta)) = \frac{\sin(((i-1)+1)\theta)}{\sin(\theta)} = \frac{\sin(i\theta)}{\sin(\theta)}$, and that

$Q_{i-2}(\cos(\theta)) = \frac{\sin(((i-2)+1)\theta)}{\sin(\theta)} = \frac{\sin((i-1)\theta)}{\sin(\theta)}$. Now we will show that the statement holds for Q_i :

$$\begin{aligned} Q_i(\cos(\theta)) &= 2 \cos(\theta) \cdot Q_{i-1}(\cos(\theta)) - Q_{i-2}(\cos(\theta)) = \\ 2 \cos(\theta) \cdot \frac{\sin(i\theta)}{\sin(\theta)} - \frac{\sin((i-1)\theta)}{\sin(\theta)} &= \frac{2 \cos(\theta) \sin(i\theta) - \sin((i-1)\theta)}{\sin(\theta)} = \\ \frac{2 \cos(\theta) \sin(i\theta) - \sin(i\theta - \theta)}{\sin(\theta)} &= \\ \frac{2 \cos(\theta) \sin(i\theta) - (\sin(i\theta) \cos(\theta) - \cos(i\theta) \sin(\theta))}{\sin(\theta)} &= \\ \frac{2 \sin(i\theta) \cos(\theta) - \sin(i\theta) \cos(\theta) + \sin(\theta) \cos(i\theta)}{\sin(\theta)} &= \\ \frac{\sin(i\theta) \cos(\theta) + \sin(\theta) \cos(i\theta)}{\sin(\theta)} &= \frac{\sin(i\theta + \theta)}{\sin(\theta)} = \frac{\sin((i+1)\theta)}{\sin(\theta)} \end{aligned}$$

n. We will use the equivalent expression from the last section in order to find roots of $Q_i(X)$ in the range $[-1, 1]$: $\frac{\sin((i+1)\theta)}{\sin(\theta)} = 0$

The denominator can't be 0, so we can rule out $\theta = 0$ and $\theta = \pi$.

Now we are left with $\sin((i+1)\theta) = 0 \rightarrow (i+1)\theta = \pi m$ (for $m \in \mathbb{N}$)

$$\rightarrow \theta = \frac{m}{i+1}\pi$$

For every $m \in \{1, 2, \dots, i\}$ we get a valid solution for θ which yields a unique $X = \cos(\theta)$ solution to $Q_i(X) = 0$. The values of $\cos(\theta)$ are unique for these

solutions because the θ solutions are unique, and are in the range

$[\frac{1}{i+1}\pi, \frac{i}{i+1}\pi]$, and $\cos(\theta)$ is strictly decreasing in that range.

We found i unique roots of $Q_i(X)$, which is the upper bound to the number of roots, and so we know that there are no other roots.

To conclude, for every $m \in \{1, 2, \dots, i\}$, $X = \cos(\frac{m}{i+1}\pi)$ is a root of multiplicity 1 of the polynomial $Q_i(X)$.

From the relation $\chi B_i(X) = Q_i\left(1 - \frac{X}{2}\right)$, we know that for every root

$X = \cos(\frac{m}{i+1}\pi)$ of $Q_i(X)$, $1 - \frac{X}{2} = 1 - \frac{1}{2}\cos(\frac{m}{i+1}\pi)$ is a root of χB_i .

Since $f(X) = 1 - \frac{X}{2}$ is a one-to-one function, we get that there are i unique roots of χB_i , each of multiplicity 1: $1 - \frac{1}{2}\cos(\frac{m}{i+1}\pi)$ for every $m \in \{1, \dots, i\}$.

- o. The eigenvalues of B are the roots of the characteristic polynomial χB_{k-1} .

So, the eigenvalues of B are $(1 - \frac{1}{2}\cos(\frac{m}{k}\pi))$ for every $m \in \{1, \dots, k-1\}$.

In section (h) we showed that for every eigenvalue λ of \tilde{B} , 4λ is an eigenvalue of B , therefore, for every eigenvalue λ of B , $\frac{1}{4}\lambda$ is an eigenvalue of \tilde{B} :

For every $m \in \{1, \dots, k-1\}$, $\frac{1}{4}\left(1 - \frac{1}{2}\cos\left(\frac{m}{k}\pi\right)\right) = \frac{1}{4} - \frac{1}{8}\cos\left(\frac{m}{k}\pi\right)$ is an eigenvalue of \tilde{B} .

In section (g) we saw that the eigenvalues of A are 1, with multiplicity 2, and all the eigenvalues of \tilde{B} , with multiplicity 1.

- p. If we remember the definition of A , it is the matrix transformation of $d^{n+1} = Ad^n$, and in section (d) we reshaped it to $d^{n+1} = A^n d^0$.

In section (g) we proved that A is diagonalizable, and so we can express it by its eigendecomposition: $A = QSQ^{-1}$.

$$\text{Where } S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4} - \frac{1}{8}\cos\left(\frac{1}{k}\pi\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{4} - \frac{1}{8}\cos\left(\frac{k-1}{k}\pi\right) \end{pmatrix}$$

(a diagonal matrix with eigenvalues at the diagonal).

$$\text{And } Q = \begin{pmatrix} 0 & 2k & & & \\ 2 & 2(k-1) & | & & \\ \vdots & \vdots & v_1 & \cdots & v_{k-1} \\ 2(k-1) & 2 & | & & \\ 2k & 0 & & & \end{pmatrix} \text{ where } v_i$$

($i \in \{1, \dots, k-1\}$) is the eigenvector of the eigenvalue $\frac{1}{4} - \frac{1}{8}\cos\left(\frac{i}{k}\pi\right)$.

As said, we can express A as $A = QSQ^{-1} \rightarrow A^n = QS^nQ^{-1}$.

S is a diagonal matrix and therefore S^n is:

$$S^n = \begin{pmatrix} 1^n & 0 & 0 & \cdots & 0 \\ 0 & 1^n & 0 & \cdots & 0 \\ 0 & 0 & \left(\frac{1}{4} - \frac{1}{8}\cos\left(\frac{1}{k}\pi\right)\right)^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left(\frac{1}{4} - \frac{1}{8}\cos\left(\frac{k-1}{k}\pi\right)\right)^n \end{pmatrix}$$

We will look at every eigenvalue of the form $(\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi))$, since $|\cos(\theta)| \leq 1$, we know that $\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi) \in [\frac{1}{4} - \frac{1}{8}, \frac{1}{4} + \frac{1}{8}]$ and in particular $|\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi)| < 1$, and so we know that $\lim_{n \rightarrow \infty} \left(\frac{1}{4} - \frac{1}{8}\cos(\frac{i}{k}\pi)\right)^n = 0$.

Therefore we know that $\lim_{n \rightarrow \infty} S^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Let's denote $Q^{-1} = \begin{pmatrix} -r_1 & - \\ -r_2 & - \\ \vdots & \\ -r_{k+1} & - \end{pmatrix}$.

$$\begin{aligned} \rightarrow \lim_{n \rightarrow \infty} A^n &= Q \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} Q^{-1} \\ &= \left(Q \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} Q^{-1} \right) \\ &= \begin{pmatrix} 0 & 2k & 0 & \cdots & 0 \\ 2 & 2(k-1) & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & 0 \\ 2(k-1) & 2 & \vdots & \ddots & \vdots \\ 2k & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} -r_1 & - \\ -r_2 & - \\ \vdots & \\ -0 & - \end{pmatrix} \\ &= \begin{pmatrix} -2k \cdot r_2 & - \\ -2r_1 + 2(k-1) \cdot r_2 & - \\ \vdots & \\ -2(k-1) \cdot r_1 + 2 \cdot r_2 & - \\ -2k \cdot r_1 & - \end{pmatrix} \end{aligned}$$

We don't know the exact values of r_1 and r_2 , but we proved that A^n converges as $n \rightarrow \infty$.

To conclude,

$$\lim_{n \rightarrow \infty} d^n = \lim_{n \rightarrow \infty} A^n d^0 = \begin{pmatrix} - & 2k \cdot r_2 & - \\ - & 2r_1 + 2(k-1) \cdot r_2 & - \\ & \vdots & \\ - & 2(k-1) \cdot r_1 + 2 \cdot r_2 & - \\ - & 2k \cdot r_1 & - \end{pmatrix} d^0.$$

q. We are looking for vectors such that $vA = v \rightarrow A^t v^t = v^t$.

$$\text{Let's examine } A^t: A^t = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & \tilde{B} & & \vdots \\ 0 & & & 0 & \frac{1}{4} \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & \frac{1}{4} & \cdots & 0 & 0 \\ 0 & 0 & & & \vdots \\ 0 & 0 & & & \vdots \\ \vdots & \tilde{B}^t & & & 0 \\ 0 & 0 & & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{4} & 1 \end{pmatrix}$$

Since \tilde{B} is symmetric, we get that $A^t = \begin{pmatrix} 1 & \frac{1}{4} & \cdots & 0 & 0 \\ 0 & 0 & & & \vdots \\ 0 & 0 & & & \vdots \\ \vdots & \tilde{B} & & & 0 \\ 0 & 0 & & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{4} & 1 \end{pmatrix}$.

We will denote a wanted eigenvector as $v^t = (v_1 \ v_2 \ \cdots \ v_k \ v_{k+1})^t$.

$$\rightarrow A^t v^t = v^t \rightarrow \begin{pmatrix} 1 & \frac{1}{4} & \cdots & 0 & 0 \\ 0 & 0 & & & \vdots \\ 0 & 0 & & & \vdots \\ \vdots & \tilde{B}^t & & & 0 \\ 0 & 0 & & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} v_1 + \frac{1}{4}v_2 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}v_k + v_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \end{pmatrix}$$

We can subtract $\begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \\ v_{k+1} \end{pmatrix}$ from both sides and get:

$$\begin{pmatrix} \frac{1}{4}v_2 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}v_k \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ \vdots \\ v_k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_k \\ 0 \end{pmatrix}$$

Now we know that $v_2 = v_k = 0$ (from elementwise equality).

So we can substitute these values and get:

$$\begin{pmatrix} \frac{1}{4} \cdot 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} \cdot 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix}$$

$$\text{Again, from elementwise equality we can deduce that } \tilde{B} \begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix}$$

We can look at it as looking for an eigenvector of \tilde{B} with eigenvalue of 1.

We already know that 1 isn't an eigenvalue of \tilde{B} , so the only solution is

$$\begin{pmatrix} 0 \\ v_3 \\ \vdots \\ v_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \text{ So, the wanted eigenvector is of the shape } \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \\ v_{k+1} \end{pmatrix}, \text{ where}$$

v_1 and v_{k+1} are degrees of freedom. We will pick them such that we get 2

$$\text{linearly independent eigenvectors: } \overline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \overline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

- r. We have the eigendecomposition $A = U\Sigma U^{-1}$ where the columns of U are the right eigenvectors of A , and the rows of U^{-1} are the left eigenvectors of A , and Σ is a diagonal matrix with the elements sorted in decreasing order. From the structure of the eigendecomposition, we know that we can assume that the values of Σ are the eigenvalues of A , sorted in decreasing order.

From our deductions in section (p), we know that:

$$\lim_{n \rightarrow \infty} \Sigma^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \text{ Therefore:}$$

$$\lim_{n \rightarrow \infty} A^n = \lim_{n \rightarrow \infty} U \Sigma^n U^{-1} = U \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} U^{-1} =$$

$$\begin{pmatrix} | & | & & | \\ R_1 & R_2 & \cdots & R_{k+1} \\ | & | & & | \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} - & L_1 & - \\ - & L_2 & - \\ \vdots & & \\ - & L_{k+1} & - \end{pmatrix} =$$

$$\begin{pmatrix} | & | & & | \\ R_1 & R_2 & \cdots & R_{k+1} \\ | & | & & | \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} - & L_1 & - \\ - & L_2 & - \\ \vdots & & \\ - & L_{k+1} & - \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | & & | \\ R_1 & R_2 & 0 & \cdots & 0 \\ | & | & | & & | \end{pmatrix} \begin{pmatrix} - & L_1 & - \\ - & L_2 & - \\ - & 0 & - \\ \vdots & & \\ - & 0 & - \end{pmatrix} =$$

$$\begin{pmatrix} | \\ R_1 \\ | \end{pmatrix} (-L_1) + \begin{pmatrix} | \\ R_2 \\ | \end{pmatrix} (-L_2) = R_1 L_1 + R_2 L_2$$

$$\rightarrow \lim_{n \rightarrow \infty} d^n = \lim_{n \rightarrow \infty} A^n d^0 = \lim_{n \rightarrow \infty} A^n d^0 = U \left(\lim_{n \rightarrow \infty} \Sigma^n \right) U^{-1} d^0 = (R_1 L_1 + R_2 L_2) d^0$$

From the structure of U and U^{-1} in the eigendecomposition, we know that R_1, R_2, L_1, L_2 are the eigenvectors associated with the 2 largest eigenvalues, which are both 1. We have already found left and right eigenvectors associated with the eigenvalue 1, so we know R_1, R_2, L_1, L_2 (the order between each pair doesn't matter since they are orthogonal), up to some constant (on one the matrices U or U^{-1}):

$$R_1 = c_1 \begin{pmatrix} 0 \\ 2 \\ \vdots \\ 2(k-1) \\ 2k \end{pmatrix}, R_2 = c_1 \begin{pmatrix} 2k \\ 2(k-1) \\ \vdots \\ 2 \\ 0 \end{pmatrix}, L_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t, L_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}^t$$

$$\rightarrow \lim_{n \rightarrow \infty} d^n = (R_1 L_1 + R_2 L_2) d^0 = c_1 \begin{pmatrix} 0 & 0 & \cdots & 0 & 2k \\ 2 & 0 & \cdots & 0 & 2(k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k-1) & 0 & \cdots & 0 & 2 \\ 2k & 0 & \cdots & 0 & 0 \end{pmatrix} d^0$$

We can denote $d^0 = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \\ d_k \end{pmatrix}$ and get the following expression:

$$\lim_{n \rightarrow \infty} d^n = c_1 \begin{pmatrix} 0 & 0 & \cdots & 0 & 2k \\ 2 & 0 & \cdots & 0 & 2(k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(k-1) & 0 & \cdots & 0 & 2 \\ 2k & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \\ d_k \end{pmatrix} =$$

$$c_1 \begin{pmatrix} 2k \cdot d_k \\ 2 \cdot d_0 + 2(k-1) \cdot d_k \\ \vdots \\ 2(k-1) \cdot d_0 + 2 \cdot d_k \\ 2k \cdot d_0 \end{pmatrix}$$

The d^n vector is the vector of decision levels for our quantization, so we know that for every n , the first and last values are the boundaries:

$$d_0 = a, d_k = b.$$

$$\rightarrow d = c_1 \begin{pmatrix} 2k \cdot d_k \\ 2 \cdot d_0 + 2(k-1) \cdot d_k \\ \vdots \\ 2(k-1) \cdot d_0 + 2 \cdot d_k \\ 2k \cdot d_0 \end{pmatrix} = c_1 \begin{pmatrix} 2kb \\ 2a + 2(k-1)b \\ \vdots \\ 2(k-1)a + 2b \\ 2ka \end{pmatrix}$$

We can see that d is only dependent on a, b, k (c_1 is a constant that we can find), and in particular d is independent from the choice of d^0 .

- s. What we are looking for is the value of c_1 . We already said that for every d^n , the first and last values are the bounds, and that also include d . Therefore we can find c_1 by requiring the first and last values to be a and b :

$$c_1 \cdot 2kb = b \rightarrow c_1 = \frac{1}{2k}$$

$$c_1 \cdot 2ka = a \rightarrow c_1 = \frac{1}{2k}$$

$$\rightarrow d = \frac{1}{2k} \begin{pmatrix} 2kb \\ 2a + 2(k-1)b \\ \vdots \\ 2(k-1)a + 2b \\ 2ka \end{pmatrix} = \begin{pmatrix} b \\ \frac{a + (k-1)b}{k} \\ \vdots \\ \frac{(k-1)a + b}{k} \\ a \end{pmatrix}$$

$$\rightarrow \forall i \in \{0, \dots, k\}: d_i = \frac{(k-i) \cdot a + i \cdot b}{k} = \frac{ka - ia + ib}{k} = a + i \frac{b-a}{k}$$

Also, we know from section (p) that the diagonal values of Σ (other than the first 2) are less than 1 in their absolute value, and therefore they converge to 0 when raised to a power, and that is why the algorithm will converge in exponential time.

- t. What we have shown so far is that the Max-Lloyd converges to the uniform decision levels. We learned in class that this algorithm converges to a local minimum, but since we found that for every initial d^0 it converges to the same point, we know that all local minima are the same point, and so it must be a single global minimum of the loss function. After we have the optimal decision levels, the representation levels are chosen optimally as the mid-point in every decision level (we saw in the lectures).