236200 - HW2

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1. $E = L^2(\mathbb{C})$, $f \in E$, $F = Vec(\beta_1, ..., \beta_n)$

   (a) As we learned in class, we know that for orthonormal basis with n elements:

   $$f = \sum_{i=1}^{n} \langle f, \beta_i \rangle \beta_i$$

   therfore, the k-term approximation of f using the given vector is:

   $$f_k = \sum_{j=1}^{k} \langle f, \beta_{i_j} \rangle \beta_{i_j}$$

   The associated MSE is defined as:

   $$\langle f - f_k, f - f_k \rangle$$

   Lets define the set of $(n-k)$ increasing integers, $\left(1 \le \beta_{j_1} \le \cdots \le \beta_{j_{n-k}} \le n\right)$ s.t

   $$\left(\beta_{j_1}, ..., \beta_{j_{n-k}}\right) \bigcap \left(\beta_{i_1}, ..., \beta_{i_k}\right) = \emptyset$$

   $$\left(\beta_{j_1}, ..., \beta_{j_{n-k}}\right) \bigcup \left(\beta_{i_1}, ..., \beta_{i_k}\right) = \left(\beta_1, ..., \beta_n\right)$$

   than we get:

   $$f - f_k = \sum_{i=1}^{n} \langle f, \beta_i \rangle \beta_i - \sum_{j=1}^{k} \langle f, \beta_{i_j} \rangle \beta_{i_j} = \sum_{l=1}^{n-k} \langle f, \beta_{j_l} \rangle \beta_{j_l}$$

   and the associated MSE is:

   $$MSE = \langle f - f_k, f - f_k \rangle = \left(\sum_{l=1}^{n-k} \langle f, \beta_{j_l} \rangle \beta_{j_l}\right)^{*} \left(\sum_{l=1}^{n-k} \langle f, \beta_{j_l} \rangle \beta_{j_l}\right)$$

   $$=_{(1)} \sum_{l=1}^{n-k} \beta_{j_l}^{*} \beta_{j_l} \langle f, \beta_{j_l} \rangle^2 =_{(2)} \sum_{l=1}^{n-k} \langle f, \beta_{j_l} \rangle^2$$

   explanations:
   (1)+(2) : using the orthonormality of $\{\beta_i\}_{i=1}^{n}$

(b) As we can see in the term we got for the MSE, the best k-approximation of f is the one achieved by the k functions that have the **highest** inner-product with f, because then we will left with the (n-k) functions that have the lowest inner-product with f, thus we will achieve the lowest MSE possible. This value is not unique, as there might be two basis functions that their inner-product value with f is the same, and therefore the MSE will be the same.

2. Lets denote the approximation of f using $(\beta_1, ..., \beta_n)$ as f_1 and as f_2 the n-approximation we get using $\left(\tilde{\beta}_1, ..., \tilde{\beta}_1 \right)$.

   (a) The approximation of f_1, f_2 can be calculated as:

   $$f_1 = \sum_{i=1}^{n} \langle f, \beta_i \rangle \, \beta_i$$

   $$f_2 = \sum_{i=1}^{n} \left\langle f, \tilde{\beta}_i \right\rangle \tilde{\beta}_i$$

   From class we learned that for orthonormal basis $\{\beta\}_{i=1}^{n}$ the MSE is:

   $$MSE_\beta(n) = \int_0^1 f^2 - \sum_{i=1}^{n} \left(\langle f, \beta_i \rangle \right)^2$$

   Lets prove that for two orthonormal basis $\{\beta\}_{i=1}^{n}$ and $\left\{ \tilde{\beta} \right\}_{i=1}^{n}$:

   $$\sum_{i=1}^{n} \left(\langle f, \beta_i \rangle \right)^2 = \sum_{i=1}^{n} \left(\left\langle f, \tilde{\beta}_i \right\rangle \right)^2$$

   And from that we will derive that $MSE_\beta = MSE_{\tilde{\beta}}$.

   Since F is n dimension and both $\{\beta\}_{i=1}^{n}$ and $\left\{ \tilde{\beta} \right\}_{i=1}^{n}$ have n elements, it means the both of them are orthonormal basis to F, thus, as we know for orthonormal basis:

   $$\sum_{i=1}^{n} \left(\langle f, \beta_i \rangle \right)^2 = \sum_{i=1}^{n} \left(\left\langle f, \sum_{j=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle \tilde{\beta}_j \right\rangle \right)^2$$

   $$= \sum_{i,j=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle^2 \left\langle f, \tilde{\beta}_j \right\rangle^2 = \sum_{i=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle^2 \sum_{j=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle^2$$

   Now we prove that $\sum_{j=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle^2 = 1$ and from that we will get the wanted result.

   $$1 = \langle \beta_i, \beta_i \rangle =_{(1)} \left\langle \sum_{j=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle \tilde{\beta}_j, \sum_{l=1}^{n} \left\langle \beta_i, \tilde{\beta}_l \right\rangle \tilde{\beta}_l \right\rangle$$

$$= \sum_{j,l=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle \left\langle \beta_i, \tilde{\beta}_l \right\rangle \left\langle \tilde{\beta}_j, \tilde{\beta}_l \right\rangle =_{(2)} \sum_{j=1}^{n} \left\langle \beta_i, \tilde{\beta}_j \right\rangle^2 \ \blacksquare$$

(1) + (2) is based on properties of orthonormal basis. And from that we proved that $MSE_\beta = MSE_{\tilde{\beta}}$.

(b) As we seen the k-term approximation for an orthonormal basis $\{\beta\}_{i=1}^n$ is :

$f_k = \sum_{i=1}^{k} \langle f, \beta_i \rangle \beta_i$ and the corresponding MSE is:

$$MSE_\beta(k) = \int_0^1 f^2 - \sum_{i=1}^{k} \left(\langle f, \beta_i \rangle \right)^2$$

Thus, we can see that the MSE will be smaller for the basis which its first k elements will have higher inner-product with f.