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1. First, since $\{\alpha\}_{i=1}^{n}, \{\beta\}_{i=1}^{n}$ are orthonormal set, as we studied in class it means that:

$$\int_{0}^{1} \alpha_i(t)\alpha_j(t)dt = \delta_{ij}$$

$$\int_{0}^{1} \beta_i(t)\beta_j(t)dt = \delta_{ij}$$

As for the set :$\psi_i(t) = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha_i(x)\\ \beta_i(x)\end{bmatrix}$, first we will prove they are normalized using the inner-producr defined in the question:

$$\langle\psi_i, \psi_i\rangle = \int_{0}^{1} \left(\frac{1}{\sqrt{2}}\alpha_i(t)\frac{1}{\sqrt{2}}\alpha_i(t) + \frac{1}{\sqrt{2}}\beta_i(t)\frac{1}{\sqrt{2}}\beta_i(t)\right) dt$$

$$\frac{1}{2}\int_{0}^{1} \alpha_i(t)\alpha_i(t)dt + \frac{1}{2}\int_{0}^{1} \beta_i(t)\beta_i(t)dt =_{(1)} \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1$$

Explanations:
(1) : We used the orthogonality of $\{\alpha\}_{i=1}^{n}, \{\beta\}_{i=1}^{n}$ which means:

$$\int_{0}^{1} \alpha_i(t)\alpha_i(t)dt = 1$$

$$\int_{0}^{1} \beta_i(t)\beta_i(t)dt = 1$$

Now we prove they are orthogonal, for $i \neq j$:

$$\langle\psi_i, \psi_j\rangle = \int_{0}^{1} \left(\frac{1}{\sqrt{2}}\alpha_i(t)\frac{1}{\sqrt{2}}\alpha_j(t) + \frac{1}{\sqrt{2}}\beta_i(t)\frac{1}{\sqrt{2}}\beta_j(t)\right) dt$$

$$\frac{1}{2}\int_{0}^{1} \alpha_i(t)\alpha_j(t)dt + \frac{1}{2}\int_{0}^{1} \beta_i(t)\beta_j(t)dt =_{(1)} \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = 0$$

Explanations:
(1) : We used the orthogonality of $\{\alpha\}_{i=1}^{n}, \{\beta\}_{i=1}^{n}$ which means:

$$\int_{0}^{1} \alpha_i(t)\alpha_j(t)d = 0$$

$$\int_{0}^{1} \beta_i(t)\beta_j(t)dt = 0$$

2. $\psi_i(t) : [0,1) \to \mathbb{R}^2, \phi(t) : [0,1) \to \mathbb{R}^2$, therefore we can define $\psi_i(t) \triangleq \begin{bmatrix} \psi_i^1(t) \\ \psi_i^2(t) \end{bmatrix}, \phi(t) \triangleq \begin{bmatrix} \phi^1(t) \\ \phi^2(t) \end{bmatrix}$, than:

$$\widetilde{\phi}(t) = \sum_{i=1}^{N} \phi_i \psi_i(t) = \sum_{i=1}^{N} \phi_i \begin{bmatrix} \psi_i^1(t) \\ \psi_i^2(t) \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=1}^{N} \phi_i \psi_i^1(t) \\ \sum\limits_{i=1}^{N} \phi_i \psi_i^2(t) \end{bmatrix}$$

now we use this expression in order to calculate the error:

$$\phi(t) - \widetilde{\phi}(t) = \begin{bmatrix} \phi^1(t) \\ \phi^2(t) \end{bmatrix} - \begin{bmatrix} \sum\limits_{i=1}^{N} \phi_i \psi_i^1(t) \\ \sum\limits_{i=1}^{N} \phi_i \psi_i^2(t) \end{bmatrix} = \begin{bmatrix} \phi^1(t) - \sum\limits_{i=1}^{N} \phi_i \psi_i^1(t) \\ \phi^2(t) - \sum\limits_{i=1}^{N} \phi_i \psi_i^2(t) \end{bmatrix}$$

and the approximation MSE is:

$$MSE = \left\langle \phi(t) - \widetilde{\phi}(t), \phi(t) - \widetilde{\phi}(t) \right\rangle$$

$$= \left\langle \begin{bmatrix} \phi^1(t) - \sum\limits_{i=1}^{N} \phi_i \psi_i^1(t) \\ \phi^2(t) - \sum\limits_{i=1}^{N} \phi_i \psi_i^2(t) \end{bmatrix}, \begin{bmatrix} \phi^1(t) - \sum\limits_{i=1}^{N} \phi_i \psi_i^1(t) \\ \phi^2(t) - \sum\limits_{i=1}^{N} \phi_i \psi_i^2(t) \end{bmatrix} \right\rangle$$

$$= \int_0^1 \left(\left(\phi^1(t) - \sum_{i=1}^{N} \phi_i \psi_i^1(t) \right)^2 + \left(\phi^2(t) - \sum_{i=1}^{N} \phi_i \psi_i^2(t) \right)^2 \right) dt$$

$$= \int_0^1 \left(\phi^1(t) - \sum_{i=1}^{N} \phi_i \psi_i^1(t) \right)^2 dt + \int_0^1 \left(\phi^2(t) - \sum_{i=1}^{N} \phi_i \psi_i^2(t) \right)^2 dt$$

The optimal coefficients in MSE sense will achived by demending:

$$\frac{\partial MSE}{\partial \phi_j} = 0$$

Lets calulate the derivative:

$$\frac{\partial MSE}{\partial \phi_j} = \frac{\partial}{\partial \phi_j} \int_0^1 \left(\phi^1(t) - \sum_{i=1}^{N} \phi_i \psi_i^1(t) \right)^2 dt + \frac{\partial}{\partial \phi_j} \int_0^1 \left(\phi^2(t) - \sum_{i=1}^{N} \phi_i \psi_i^2(t) \right)^2 dt$$

$$= \int_0^1 \frac{\partial}{\partial \phi_j} \left(\phi^1(t) - \sum_{i=1}^{N} \phi_i \psi_i^1(t) \right)^2 dt + \int_0^1 \frac{\partial}{\partial \phi_j} \left(\phi^2(t) - \sum_{i=1}^{N} \phi_i \psi_i^2(t) \right)^2 dt$$

$$= -2 \int_0^1 \psi_j^1(t) \left(\phi^1(t) - \sum_{i=1}^{N} \phi_i \psi_i^1(t) \right) dt - 2 \int_0^1 \psi_j^2(t) \left(\phi^2(t) - \sum_{i=1}^{N} \phi_i \psi_i^2(t) \right) dt$$

$$= -2\int_0^1 \psi_j^1(t)\phi^1(t)dt + 2\sum_{i=1}^N \phi_i \int_0^1 \psi_j^1(t)\psi_i^1(t)dt - 2\int_0^1 \psi_j^2(t)\phi^2(t)dt + 2\sum_{i=1}^N \phi_i \int_0^1 \psi_j^2(t)\psi_i^2(t)dt$$

$$-2\int_0^1 \psi_j^1(t)\phi^1(t)dt - 2\int_0^1 \psi_j^2(t)\phi^2(t)dt + 2\sum_{i=1}^N \phi_i \left(\int_0^1 \psi_j^1(t)\psi_i^1(t)dt + \int_0^1 \psi_j^2(t)\psi_i^2(t)dt\right)$$

$$=_{(1)} -2\int_0^1 \psi_j^1(t)\phi^1(t)dt - 2\int_0^1 \psi_j^2(t)\phi^2(t)dt + 2\phi_j$$

(1) : $\psi_i(t) : [0, 1) \to \mathbb{R}^2$ is orthonormal set and therefore:

$$\int_0^1 \psi_j^1(t)\psi_i^1(t)dt + \int_0^1 \psi_j^2(t)\psi_i^2(t)dt = \delta_{ij}$$

Now lets demand $\frac{\partial MSE}{\partial \phi_j} = 0$ and we get:

$$\phi_j^{opt} = \int_0^1 \psi_j^1(t)\phi^1(t)dt + \int_0^1 \psi_j^2(t)\phi^2(t)dt = \langle \psi_j, \phi(t)\rangle$$

for $j = 1, ..., N$

Now lets calulate expression for minimal approximation MSE:

$$MSE_{opt} = \int_0^1 \left(\phi^1(t) - \sum_{i=1}^N \phi_i^{opt}\psi_i^1(t)\right)^2 dt + \int_0^1 \left(\phi^2(t) - \sum_{i=1}^N \phi_i^{opt}\psi_i^2(t)\right)^2 dt$$

$$= \int_0^1 \left(\phi^1(t)\right)^2 - 2\int_0^1 \phi^1(t)\sum_{i=1}^N \phi_i^{opt}\psi_i^1(t)dt + \int_0^1 \left(\sum_{i=1}^N \phi_i^{opt}\psi_i^1(t)\right)^2 dt$$

$$+ \int_0^1 \left(\phi^2(t)\right)^2 - 2\int_0^1 \phi^2(t)\sum_{i=1}^N \phi_i^{opt}\psi_i^2(t)dt + \int_0^1 \left(\sum_{i=1}^N \phi_i^{opt}\psi_i^2(t)\right)^2 dt$$

$$= \int_0^1 \left(\phi^1(t)\right)^2 - 2\sum_{i=1}^N \phi_i^{opt}\int_0^1 \phi^1(t)\psi_i^1(t)dt + \sum_{i=1}^N\sum_{j=1}^N \phi_i^{opt}\phi_j^{opt}\int_0^1 \psi_i^1(t)\psi_j^1(t)dt$$

$$+ \int_0^1 \left(\phi_i^2(t)\right)^2 - 2\sum_{i=1}^N \phi_i^{opt}\int_0^1 \phi^2(t)\psi_i^2(t)dt + \sum_{i=1}^N\sum_{j=1}^N \phi_i^{opt}\phi_j^{opt}\int_0^1 \psi_i^2(t)\psi_j^2(t)dt$$

$$= \int_0^1 \left(\phi^1(t)\right)^2 + \int_0^1 \left(\phi^2(t)\right)^2 - 2\sum_{i=1}^N \phi_i^{opt}\left(\int_0^1 \phi^1(t)\psi_i^1(t)dt + \int_0^1 \phi^2(t)\psi_i^2(t)dt\right)$$

$$+ \sum_{i=1}^N\sum_{j=1}^N \phi_i^{opt}\phi_j^{opt}\left(\int_0^1 \psi_i^1(t)\psi_j^1(t)dt + \int_0^1 \psi_i^2(t)\psi_j^2(t)dt\right)$$

$$=_{(1)}\int\limits_0^1 \left(\phi^1(t)\right)^2+\int\limits_0^1 \left(\phi^2(t)\right)^2-2\sum_{i=1}^N \left(\phi_i^{opt}\right)^2+\sum_{i=1}^N \left(\phi_i^{opt}\right)^2$$

$$=\int\limits_0^1 \left(\phi^1(t)\right)^2+\int\limits_0^1 \left(\phi^2(t)\right)^2-\sum_{i=1}^N \left(\phi_i^{opt}\right)^2$$

Explanations:

(1) :As we showed : $\int\limits_0^1 \phi^1(t)\psi_i^1(t)dt+\int\limits_0^1 \phi^2(t)\psi_i^2(t)dt=\phi_i^{opt}$, and by inner-product definition:

$$\int\limits_0^1 \psi_i^1(t)\psi_j^1(t)dt+\int\limits_0^1 \psi_i^2(t)\psi_j^2(t)dt=\delta_{ij}$$

And by the given definition for inner-product we can write the expression we got as:

$$MSE_{opt}=\langle\phi(t),\phi(t)\rangle-\sum_{i=1}^N \left(\phi_i^{opt}\right)^2$$