

Advanced algorithms and data structures

Exact exponential algorithms and parameterized complexity

Jacob Holm (jaho@di.ku.dk)

December 16th 2024

Today's Lecture

Exact exponential algorithms and parameterized complexity

Introduction

Exact exponential algorithms

- Exact TSP via Dynamic Programming

- Dynamic Programming in general

- Exact MIS via Branching

Parameterized problems

- "Bar fight prevention" aka k -Vertex Cover

- Kernelization

- Bounded search tree

FPT vs XP

- Example: Vertex k -Coloring

- Example: k -Clique

- Example: Clique parameterized by Δ

Summary

Introduction

We usually want algorithms that

- 1) in polynomial time,
- 2) for all inputs,
- 3) find an exact solution.

Unfortunately some problems are hard, and we may have to settle for (at best) 2 out of 3. We call such algorithms

Exact exponential algorithms

if we relax 1) to allow using exponential time.

Parameterized algorithms

if we relax 2) to only inputs with small fixed values of some parameter.

Approximation algorithms

if we relax 3) to allow approximate solutions (final 2 lectures).

AADS Lecture on EE/FPT, Part 2

Exact exponential algorithms

Exact exponential algorithms

Recall that a *decision problem* is in NP if and only if there exists:

- ▶ A polynomial-time verifier $R(x, y)$; and
- ▶ a function $m(x) \in \mathcal{O}(\text{poly}|x|)$; such that
- ▶ for every problem instance x : x is a yes-instance if and only if there exists a certificate y of size $|y| \leq m(x)$ such that $R(x, y)$ is true.

Note: A certificate y is proof that x “has a solution”, but y does not have to *be* a solution. However, a solution is often the most natural certificate.

Note: Every optimization problem has a decision version. What is it?

Instead of asking for the “best” value z with some property, ask whether a value z given as part of the input has that property.

Every problem in NP has a simple brute-force algorithm of the following form: Given problem instance x , try all potential certificates y with $|y| \leq m(x)$ and check if $R(x, y)$ for any of them.

Since a potential certificate is just a bit string of length at most $m(x)$ there are at most $2^{m(x)}$ potential certificates to check, and each check takes $\mathcal{O}(\text{poly}|x|)$ time. Thus, if we assume $m(x)$ can be computed in $\mathcal{O}(\text{poly}|x|)$ time, the brute force running time is $\mathcal{O}(2^{m(x)} \cdot \text{poly}|x|)$.

Notation: $\mathcal{O}^*(\cdot)$

For any fixed $0 < a < b$, and $c, d \in \mathbb{R}$, we have $\mathcal{O}(a^n \cdot n^c) \subset \mathcal{O}(b^n \cdot n^d)$.

So when comparing exact exponential algorithms, the bases of the exponentials are much more important than the polynomial factors.

Define

$$f(n) \in \mathcal{O}^*(g(n)) \iff \exists c \in \mathbb{R} : f(n) \in \mathcal{O}(n^c \cdot g(n))$$

In other words, $\mathcal{O}^*(\cdot)$ is the same as $\mathcal{O}(\cdot)$ but ignores polynomial factors.

Notice that for all $0 < a < b$: $\mathcal{O}(a^n) \subset \mathcal{O}^*(a^n) \subset \mathcal{O}(b^n)$.

Using this notation, what is the running time for the simple brute-force algorithm? $\mathcal{O}^*(2^{m(x)})$

Size of a problem

What do we mean by the “size” n of a problem? Typically:

n , or $m + n$ for graphs with n vertices and m edges.

$|S|$ for problems involving some set S .

$\#variables$ for SAT-type problems.

This measure of “size” is usually sufficient to describe the running time of the natural brute-force algorithm and to show improvements in better algorithms.

Problem	certificate size	brute-force time	this lecture
SAT, MIS	$m(x) = n$	$T(n) \in \mathcal{O}^*(2^n)$	$\mathcal{O}^*(1.45^n)$
TSP	$m(x) = \log_2(n!)$	$T(n) \in \mathcal{O}^*(n!)$	$\mathcal{O}^*(2^n)$
k -Vertex Cover	$m(x) = k \log_2(n)$	$T(n) \in \mathcal{O}(n^k \cdot \text{poly} x)$	$\mathcal{O}_k(m + n)$
Vertex k -coloring	$m(x) = \log_2(k^n)$	$T(n) \in \mathcal{O}^*(k^n)$?

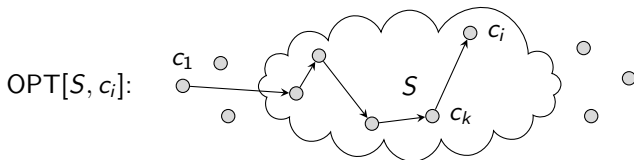
TSP via Dynamic Programming (Bellman-Held-Karp)

Problem: Given cities c_1, \dots, c_n , and distances $d_{ij} = d(c_i, c_j)$, find tour of minimal length, visiting all cities exactly once. Equivalently, find permutation π minimizing $d(c_{\pi(n)}, c_{\pi(1)}) + \sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)})$.

Idea: For all $S \subseteq \{c_2, \dots, c_n\}$ and $c_i \in S$ define

$\text{OPT}[S, c_i] :=$ minimum length of any path that starts in c_1 , visits all of S once without leaving S , and ends in c_i .

Then $\min\{\text{OPT}[\{c_2, \dots, c_n\}, c_i] + d(c_i, c_1) \mid c_i \in \{c_2, \dots, c_n\}\}$ is the length of the minimal tour.



Lemma

$$\text{OPT}[S, c_i] = \begin{cases} d(c_1, c_i) & \text{if } S = \{c_i\} \\ \min\{\text{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i) \mid c_k \in S \setminus \{c_i\}\} & \text{if } \{c_i\} \subset S \end{cases}$$

Proof.

Let $e = (c_k, c_i)$ be the last edge on such a path. If $k = 1$ we are done. If $k \neq 1$ the shortest length through e must be $\text{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i)$. The shortest such path must use the minimum over all $c_k \in S \setminus \{c_i\}$. \square

TSP via Dynamic Programming (Bellman-Held-Karp)

We can solve TSP by computing all $\text{OPT}[S, c_i]$ values in order of increasing size of S .

```
1: function TSP( $\{c_1, \dots, c_n\}, d$ )
2:   for  $i \leftarrow 2 \dots n$  do
3:      $\text{OPT}[\{c_i\}, c_i] \leftarrow d(c_1, c_i)$ 
4:   for  $j \leftarrow 2 \dots n-1$  do
5:     for  $S \subseteq \{c_2, \dots, c_n\}$  with  $|S| = j$  do
6:       for  $c_i \in S$  do
7:          $\text{OPT}[S, c_i] \leftarrow \min\{\text{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i) \mid c_k \in S \setminus \{c_i\}\}$ 
8:   return  $\min\{\text{OPT}[\{c_2, \dots, c_n\}, c_i] + d(c_i, c_1) \mid c_i \in \{c_2, \dots, c_n\}\}$ 
```

Lemma

The above procedure solves TSP by computing $\mathcal{O}(n^2 \cdot 2^n)$ shortest paths.

Proof.

The number of path lengths computed in line 7 is

$$\sum_{j=2}^{n-1} \binom{n-1}{j} \sum_{i=1}^j (j-1) \leq n^2 \sum_{j=0}^n \binom{n}{j} = n^2 \cdot 2^n$$

And lines 3 and 8 compute only $n-1$ more path lengths each. □

What is the running time of the algorithm? $\mathcal{O}^*(2^n)$ if we assume additions take at most polynomial time in n . Much better than $\mathcal{O}^*(n!)$.

Dynamic Programming in general

Similar to “Divide and Conquer” in that it requires “Optimal Substructure” but subproblems may be overlapping.

Instead of recursively solving smaller disjoint subproblems, “Dynamic Programming” solves all smaller subproblems in order of increasing size.

A hybrid idea called “Memoization” (not “Memorization”!) does the same by using recursion, but caching results so each subproblem is only solved once.

In our TSP example the original problem does not have the optimal substructure property (A piece of an optimal tour does not have to be an optimal tour of some subgraph). *The trick is to notice that the problem of computing $\text{OPT}[\{c_2, \dots, c_n\}, c_i]$ does have the property, and that TSP can be solved once we know that for all c_i .*

Exact MIS via Branching

Problem: Given undirected graph (V, E) , find the maximum cardinality of $I \subseteq V$ so each edge has at most one endpoint in I .

Such a set I is called a *Maximum Independent Set (MIS)* for the graph.

Naive: Try all 2^n subsets (where $n = |V|$). This takes $\mathcal{O}^*(2^n)$ time.

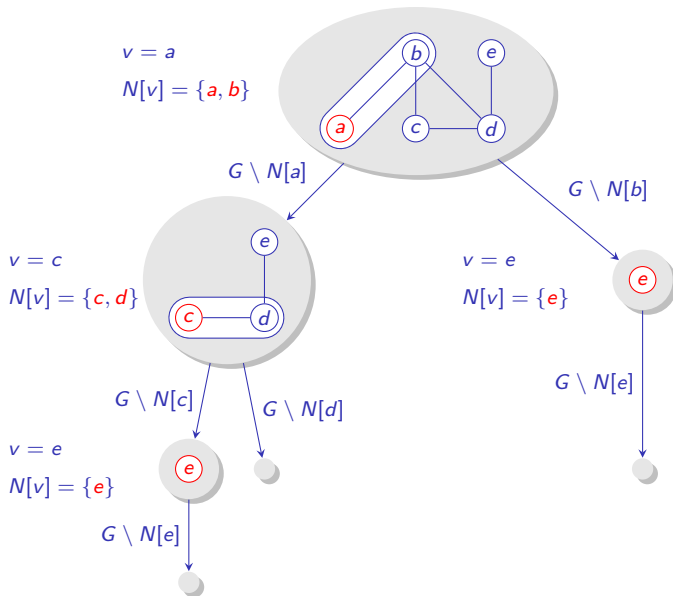
For $v \in V$ define $N[v] := \{v\} \cup \{w \in V \mid (v, w) \in E\}$. This is called the *closed neighborhood* of v .

Observation: $N[v] \cap I \neq \emptyset$ for all $v \in V$ and all MIS I .

Why? If $N[v] \cap I = \emptyset$ for some $v \in V$, $I \cup \{v\}$ would be a larger solution.

- 1: **function** MISsize($G = (V, E)$)
- 2: **if** $V = \emptyset$ **then return** 0
- 3: $v \leftarrow$ vertex in V of minimum degree.
- 4: **return** $1 + \max\{\text{MISsize}(G \setminus N[w]) \mid w \in N[v]\}$

Exact MIS via Branching



Exact MIS via Branching

Let $T(n)$ be the maximum number of subproblems considered by the branching algorithm on a graph with n vertices, then (very loosely):

$$T(0) = 1$$

$$T(n) \leq 1 + \sum_{w \in N[v]} T(n - (d(w) + 1))$$

$$\leq 1 + (d(v) + 1) \cdot T(n - (d(v) + 1)) \quad \text{since } T(\cdot) \text{ is nondecreasing}$$

$$= 1 + s \cdot T(n - s)$$

for some $v \in V$ of minimum degree

where $s = d(v) + 1$ and thus $s \in \{1, \dots, n\}$

Lemma

$$T(n) \in \mathcal{O}(3^{n/3}) \subset \mathcal{O}(1.44225^n)$$

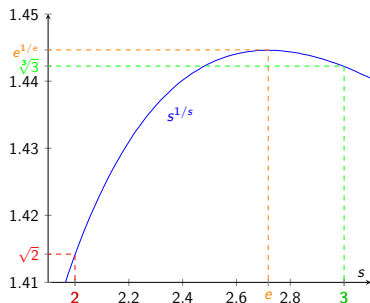
“Proof” (Error: s depends on v).

$$T(n) \leq 1 + s \cdot T(n - s)$$

$$\leq 1 + s + s^2 + \dots + s^{n/s}$$

$$= \frac{s^{1+n/s} - 1}{s - 1} < 2s^{n/s} \quad (\text{for } s \geq 2)$$

$$\in \mathcal{O}(s^{n/s}) \subseteq \mathcal{O}(3^{n/3}) \quad \square$$



AADS Lecture on EE/FPT, Part 3

Parameterized problems

“Bar fight prevention” aka k -Vertex Cover

Problem: Bouncer in a small city wants to block people at the door to prevent fights. Assume he knows everyone and knows which pairs of people would fight if they were both let in. Management only allows him to block $\leq k$ of the n people who wants in. Is that enough to prevent fights, and if so, who should be blocked?

Equivalent Problem: Given a graph (V, E) with $n = |V|$ vertices, is there a subset $C \subseteq V$ of size $|C| \leq k$ such that every edge has at least one endpoint in C ? Such a set C is called a *k -Vertex Cover* in the graph, and its complement $V \setminus C$ is an *Independent Set* of size $n - k$.

For concreteness in the following, suppose $n = 1000$ and $k = 10$.

Naive 1: Try all 2^n subsets of people. ($2^{1000} \approx 1.07 \cdot 10^{301}$ cases).

Naive 2: Use Exact MIS algorithm. ($2 \cdot 3^{1000/3} - 1 \approx 2.195 \cdot 10^{159}$ cases).

Better 1: Try all $\binom{n}{k}$ subsets of k people. ($\binom{1000}{10} \approx 2.63 \cdot 10^{23}$ cases).

“Bar fight prevention” via Kernelization

Consider the conflict graph $G = (V, E)$.

Idea: If $d(v) = 0$: let v in and drop v from G .

Why? **Safe because no conflicts.**

Idea: If $d(v) > k$: reject v , drop v from G , and decrease k .

Why? **Not rejecting v means rejecting $d(v) > k$ people.**

Note: If $d(v) \leq k$ for all v and $|E| > k^2$, there is no solution.

Why? **Each rejection resolves at most k conflicts.**

Better 2: The above ideas reduce to a graph H with $|V| \leq 2k^2$ vertices.

Why? $|V| = \sum_{v \in V} 1 \leq \sum_{v \in V} d(v) = 2|E| \leq 2k^2$

Now try all $\binom{2k^2}{k}$ subsets of k people. $\left(\binom{2 \cdot 10^2}{10} \approx 2.24 \cdot 10^{16}\right)$.

Idea: If $N[v] = \{v, w\}$: let v in, reject w , drop $N[v]$ from G , and decrease k .

Why? **In any solution that lets w in, we can let v in instead. Never worse.**

Better 3: The above ideas reduce to a graph H with $|V| \leq k^2$ vertices.

Why? $|V| = \sum_{v \in V} 1 = \frac{1}{2} \sum_{v \in V} 2 \leq \frac{1}{2} \sum_{v \in V} d(v) = |E| \leq k^2$

Now try all $\binom{k^2}{k}$ subsets of k people. $\left(\binom{10^2}{10} \approx 1.73 \cdot 10^{13}\right)$.

“Bar fight prevention” via Kernelization

```
1: function BarFightPrevention( $k, G$ )
2:    $k', H, C \leftarrow \text{BFP-Kernel}(k, G)$ 
3:   if  $H$  has  $\leq (k')^2$  edges and  $\text{BFP-Brute-Force}(k', H)$  returns a solution  $C'$  then
4:     return  $C \cup C'$ 
5:   return “No solution”

6: function BFP-Kernel( $k, G$ )
7:    $k' \leftarrow k, H \leftarrow G, C \leftarrow \emptyset$ 
8:   loop
9:     if Some  $v$  has  $d(v) = 0$  then
10:       $H \leftarrow H \setminus \{v\}$ 
11:     elseif  $k' > 0$  and some  $v$  has  $d(v) > k'$  then
12:       $H \leftarrow H \setminus \{v\}, C \leftarrow C \cup \{v\}, k' \leftarrow k' - 1$ 
13:     elseif  $k' > 0$  and some  $v$  has  $N[v] = \{v, w\}$  for some  $w$  then
14:       $H \leftarrow H \setminus N[v], C \leftarrow C \cup \{w\}, k' \leftarrow k' - 1$ 
15:     else
16:       return  $k', H, C$ 

17: function BFP-Brute-Force( $k, G = (V, E)$ )
18:   for every subset  $C \subseteq V$  of size  $k$  do
19:     if  $C$  is a vertex cover of  $G$  then
20:       return  $C$ 
21:   return “No solution”
```

Kernelization

The subgraph H we reduced to before brute-forcing is called a *Kernel* for the Bar Fight Prevention problem, and the process of finding such a kernel is called *Kernelization*.

The general idea is to use the parameter k to quickly reduce to a smaller problem, whose size ideally depends only on k and not on n .

For the bar fight prevention problem we have just shown that:

- ▶ If there is a solution for a given k and a given graph G with n vertices and m edges, then we can find a kernel H with at most k^2 vertices.
- ▶ Furthermore, such a kernel can be found in $\mathcal{O}(m + n)$ time, and checking if a given subset of size at most k is a solution can be done in $\mathcal{O}(k^2)$ time.
- ▶ Thus, for any fixed k , the total running time of this algorithm is $\mathcal{O}(m + n + \binom{k^2}{k} k^2) \subseteq \mathcal{O}(m + n + k^{2k+2}) \subseteq \mathcal{O}_k(m + n)$.

“Bar fight prevention” via Bounded Search Tree

Note: For each edge $(u, v) \in E$, at least one of u, v must be rejected.

Idea: Pick arbitrary edge (u, v) , and recursively try with u rejected and with v rejected.

```
1: function BFP-Bounded-Search( $k, G$ )
2:   if  $G$  has no edges then
3:     return  $\emptyset$ 
4:   if  $k > 0$  then
5:     Let  $(u, v)$  be an arbitrary edge of  $G$ 
6:     for  $w \in \{u, v\}$  do
7:       if BFP-Bounded-Search( $k - 1, G \setminus \{w\}$ ) returns a solution  $C$  then
8:         return  $C \cup \{w\}$ 
9:   return “No solution”
```

This recursive procedure has depth at most k .

Thus the total number of subproblems considered at most 2^k .

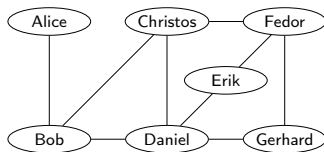
If we start by rejecting all vertices of degree $d(v) > k$ (like in the kernelization approach), the resulting graph has at most

$|E| = \frac{1}{2} \sum_{v \in V} d(v) \leq \frac{1}{2} nk$ edges, so constructing each subproblem can be done in $\mathcal{O}(nk)$ time.

The total running time is then $\mathcal{O}(m + nk \cdot 2^k)$. $(1000 \cdot 10 \cdot 2^{10} \approx 10^7)$

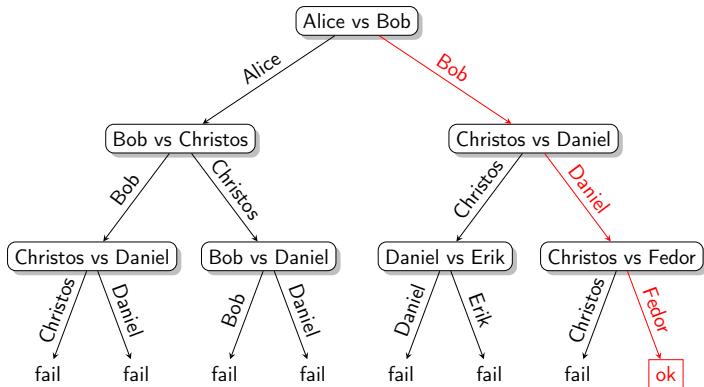
Part of Assignment 5 asks you to improve this.

"Bar fight prevention" via Bounded Search Tree



$k = 3$

$C = \{\text{Fedor, Daniel, Bob}\}$



AADS Lecture on EE/FPT, Part 4

FPT vs XP

FPT vs XP

An important feature of the Bar Fight Prevention problem is the existence of the *parameter* k . The problem of finding the minimum k that works is NP-complete, but for any fixed constant k we have just seen two linear-time algorithms!

We say the problem is *parameterized* by the *parameter* k . In this case k is the maximum solution size, but other problems may have different parameters (and may have more than one).

Definition: A parameterized problem is *Fixed Parameter Tractable (FPT)* if it has an algorithm with running time $f(k) \cdot n^c$ for some function f and some constant $c \in \mathbb{R}$.

Definition: A parameterized problem is *Slice-wise Polynomial (XP)* if it has an algorithm with running time $f(k) \cdot n^{g(k)}$ for some functions f, g .

Note: $\text{FPT} \subset \text{XP}$, why? Simply set $g(k) = c$.

Example: Vertex k -Coloring

Problem: For an integer k , given a graph G does G have a proper vertex coloring with k colors?

Lemma

Unless $P = NP$, this problem is not XP and therefore not FPT .

Proof sketch.

The problem is NP-hard even for $k = 5$, so unless $P = NP$ there can be no algorithm for general k with running time $f(k) \cdot n^{g(k)}$. □

Example: k -Clique

Problem: For an integer k , given a graph G does G have a clique of size k ?

Lemma

k -clique is XP.

Proof.

A simple brute-force algorithm is to check every k -subset of the vertices. There are $\binom{n}{k} \leq n^k$ such subsets, and we can check in $\mathcal{O}(k^2)$ time whether a given subset forms a clique. Thus the running time of this algorithm is $\mathcal{O}(k^2 \cdot n^k)$ which proves the problem is in XP. \square

It is unknown whether k -clique is FPT, but it is widely believed that $\mathcal{O}(n^k)$ is optimal which would prove it is not.

Example: Clique parameterized by Δ

Problem: For integer Δ , given a graph G with maximum degree Δ and an integer k , does G have a clique of size k ?

Lemma

Clique is FPT when parameterized by the maximum degree Δ .

Proof.

If $k > \Delta + 1$ the answer is just no. Otherwise a naive algorithm is for each vertex to try all subsets of its neighbors. There are at most $n \cdot 2^\Delta$ such subsets and each can be checked in $\mathcal{O}(\Delta^2)$ time. The total time is thus $\mathcal{O}((2^\Delta \cdot \Delta^2) \cdot n)$, which proves the problem is FPT. \square

In fact, we can easily improve this algorithm to run in $\mathcal{O}(\binom{\Delta}{k-1} \cdot k^2 \cdot n)$ time when $k \leq \Delta + 1$.

There are often many possible choices of parameter. Choosing the right one for a specific problem is an art.

Summary

Today's topics were “Exact exponential algorithms” and “Parameterized Complexity”. We have covered

- ▶ The natural brute force algorithm for problems in NP.
- ▶ An exact $\mathcal{O}^*(2^n)$ -time dynamic programming algorithm for TSP.
- ▶ An exact $\mathcal{O}^*(3^{n/3})$ -time branching algorithm for MIS.
- ▶ A kernelization for the “Bar Fight Prevention” problem, a.k.a. k -vertex cover.
- ▶ A bounded search tree algorithm for k -vertex cover.
- ▶ Definitions of parameterized complexity, FPT and XP.
- ▶ Examples of problems in FPT, XP but not FPT, and not XP.
- ▶ Next time: van Emde Boas trees / predecessor search.