DMA: Induction

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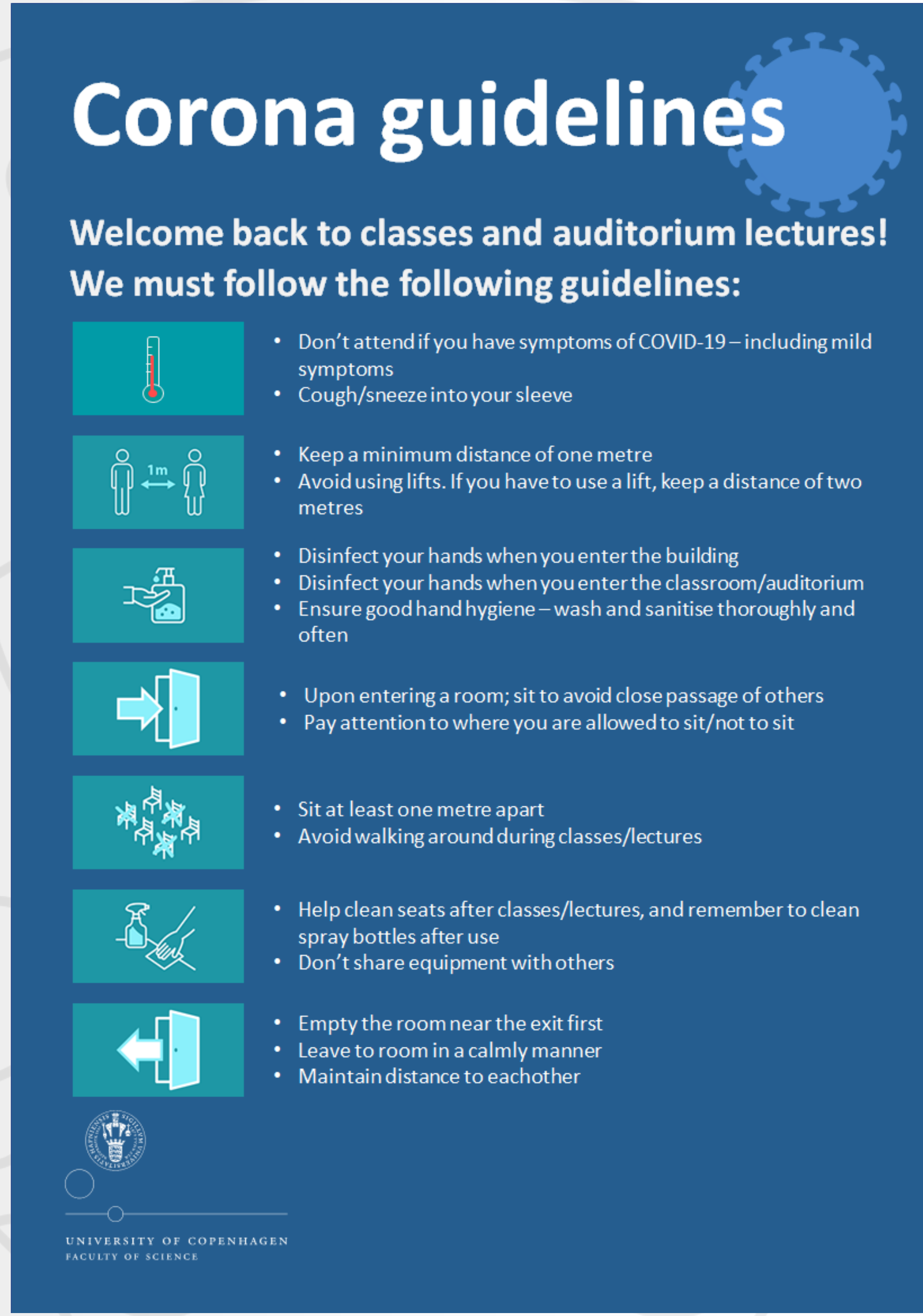


Plan for this class

- Induction as a proof technique
- Examples

Reading: Notes on Absalon (can also take a look Section 2.4 from KBR)

Suppose we want to prove a mathematical statement or formula

How do we show that

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Try different values of n:

$n = 1$ $\quad 1 \stackrel{?}{=} \frac{1(1+1)(2\cdot1+1)}{6}$ ✓

$n = 2$ $\quad 1 + 2^2 \stackrel{?}{=} \frac{2(2+1)(2\cdot2+1)}{6} = \frac{2\cdot3\cdot5}{6}$ ✓

$n = 3$ $\quad 1 + 2^2 + 3^2 \stackrel{?}{=} \frac{3(3+1)(2\cdot3+1)}{6} = \frac{3\cdot4\cdot7}{6} = 14$ ✓

$n = 4$ $\quad 1 + 2^2 + 3^2 + 4^2 \stackrel{?}{=} \frac{4(4+1)(2\cdot4+1)}{6} = \frac{4\cdot5\cdot9}{6} = 30$ ✓

Is it enough to check for only some values of n?

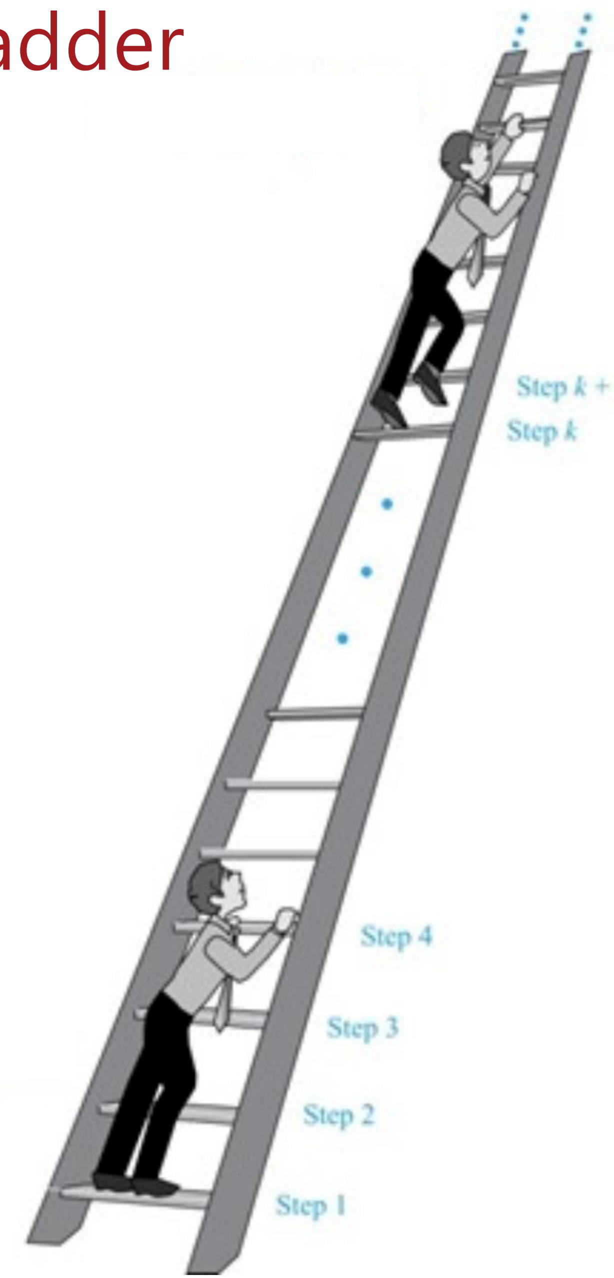
- Fermat conjectured that all numbers of the form $F_n = 2^{2^n} + 1$ are prime.
  - $F_1 = 2^{2^1} + 1 = 5$
  - $F_2 = 2^{2^2} + 1 = 2^4 + 1 = 17$
  - $F_3 = 2^{2^3} + 1 = 2^8 + 1 = 257$
  - Fermat checked up to $n = 4$
- 100 years later, Euler noticed that

  $$F_5 = 2^{32} + 1 = 4294967297 = 641 \times 6700417$$
- There are no known Fermat primes F_n with $n > 4$ (checked up to $n = 32$)

Mathematical induction

Analogy: climbing an infinite ladder

- Suppose you can reach the first rung
- If you are on a particular rung k you can get on the next rung $k+1$



The principle of mathematical induction

Let $P(n)$ be a predicate (statement) defined for integers $\{n_0, n_0 + 1, \dots\}$. If

a) $P(n_0)$ is true and

b) for any $n \geq n_0$, we have that $P(n)$ being true implies that $P(n+1)$ is true

then

- $P(n)$ is true for all integers $n \geq n_0$.

Terminology

a) Is called basis step or base case

b) Is called induction step.

Using induction to prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbb{Z}^+$$

- Predicate $P(n)$: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$
- Base case: Check that $P(1)$ holds.

$$\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1(2)(3)}{6} = 1 = 1^2$$

- Induction step:

Assume that $P(n)$ holds for some $n \geq 1$ (Induction hypothesis)

Show that this implies that $P(n+1)$ holds.

Structure of an inductive proof

- (State that the proof uses induction.)
- Define an appropriate predicate $P(n)$. Identify n_0.
- Prove that $P(n_0)$ is true.
- Prove that $P(n)$ implies $P(n+1)$.
- Invoke the principle of mathematical induction.

Example: divisibility statement

Task: Show that $5|(6^n - 5n + 4)$ for any $n \in \mathbb{Z}^+$

Let $b_n = 6^n - 5n + 4$ where $n \in \mathbb{Z}^+$.

Let $P(n)$ be the statement that b_n is divisible by 5.

- Base case: Check that $P(1)$ holds.

$b_1 = 6 - 5 + 4 = 5$ is divisible by 5.

- Induction step:

Assume that $P(n)$ holds for some $n \geq 1$ (Induction hypothesis)

We want to show that $P(n + 1)$ holds.

Invariants

An invariant is a property that is preserved throughout a program or a procedure

- $xy = z$ after every iteration of a **for**-loop.
- Temperature of a nuclear reactor doesn’t exceed a critical value.

Invariants: Loop invariant (in the Euclidean algorithm)

```
GCD(X,Y)
    While (X ≠ Y)
        If  (X > Y) then
            X ← X - Y
        Else
            Y ← Y - X
    return X
```

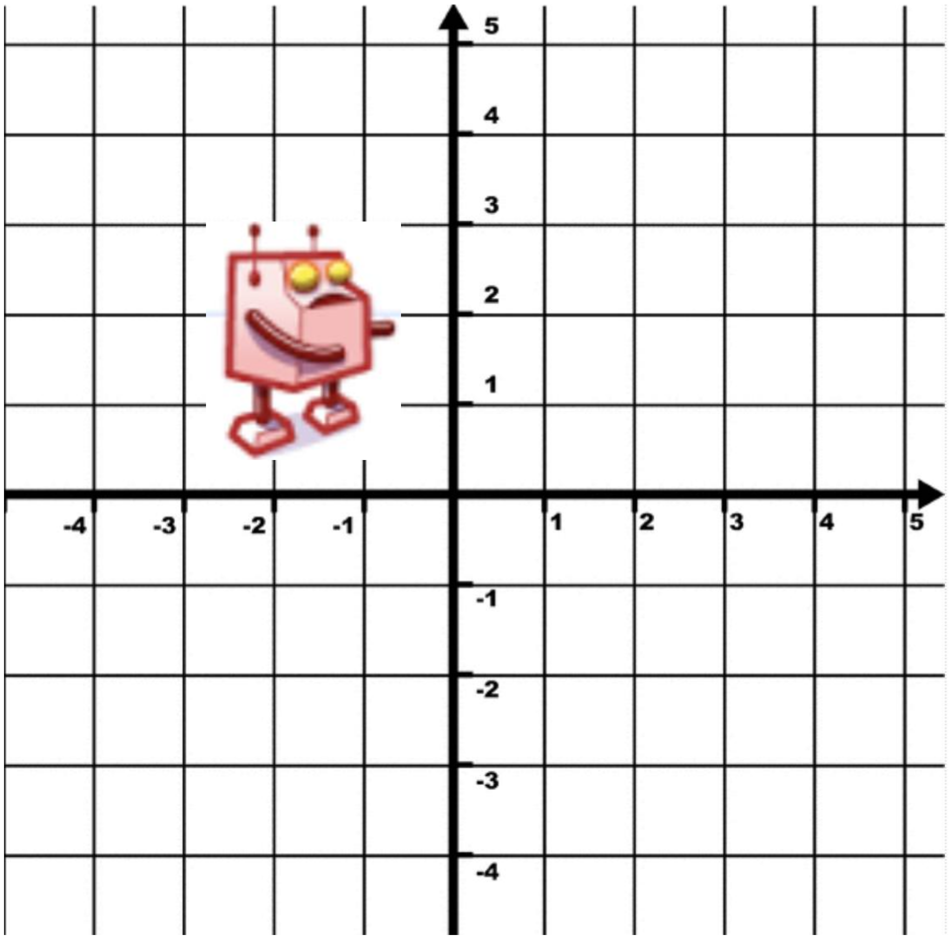
Let X_n and Y_n be the values of X and Y after n passes.
Then GCD(X_n,Y_n) = GCD(X,Y)

Invariants: Diagonally moving robot on a plane

Robot starts at (0,0)

If the robot is at (x, y) after step k, then in step $k + 1$ it can go to either

- $(x + 1, y + 1)$ or
- $(x + 1, y - 1)$ or
- $(x - 1, y + 1)$ or
- $(x - 1, y - 1)$



Can the robot be at (1,0) after some number of steps?

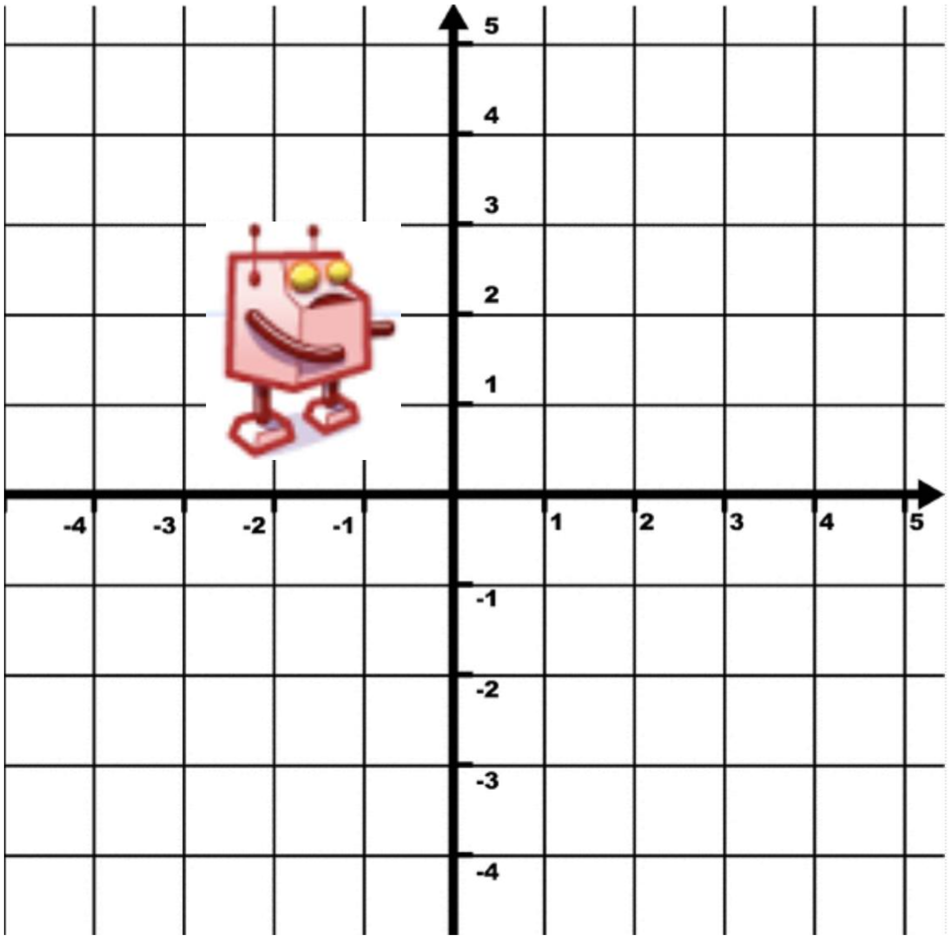
Invariants: Diagonally moving robot on a plane

Robot starts at (0,0)

If the robot is at (x, y) after step k, then in step $k+1$ it can go to either

- $(x+1, y+1)$ or
- $(x+1, y-1)$ or
- $(x-1, y+1)$ or
- $(x-1, y-1)$

Can the robot reach (1,0)?



Claim. After any number of steps $n \geq 0$ robot's position (x, y) is such that $x + y$ is even.