DMA 2021

– Week 12 –

Work instructions

This week we will discuss *relations*, which will be the foundation for almost all further math in DMA. Relations can be viewed in very different ways:

- as a predicate $P(a, b)$ regarding a pair of elements $a \in A$ and $b \in B$,
- as a subset of the product set $A \times B$,

and if A and B are finite sets,

- as a Boolean matrix,
- as a directed graph (digraph).

It might seem a little extreme to have four different descriptions of the same concept. However, it is useful to be able to represent relations in different ways, as then we have the freedom to choose the description best suited for the task at hand.

During our treatment of KBR Chapter 4, we will make a quick detour to KBR 5.1 in order to see that functions are relations of a special form and discuss the concepts of injective, surjective, and inverse functions. We will then focus on understanding properties of relations such as *reflexivity*, *symmetry*, and *transitivity*. These three properties together define an important family of *equivalence relations*. We focus on KBR Section 5.1 and Chapter 4 during the week. Section 4.6 contains known material from the algorithms part of DMA and Section 4.8 is less essential so these two sections will not be covered during the lectures.

Assigned reading

- KBR Chapter 4
- KBR 5.1

Lecture plan

Monday Dec. 6th 09:00–09:45

Relations and their representations. (KBR 4.1–4.2)

Tuesday Dec. 7th, 13:00–14:45

Paths in directed graphs. Functions as relations. Reflexivity, symmetry, and transitivity. (KBR 4.3–4.4 and 5.1).

Friday Dec. 10th, 09:00–09:45

Equivalence relations and operations. Examples. (KBR 4.5 and 4.7)

Exercise plan

Monday Dec. 6th, 10:15–12:00

- Solve KBR exercises 4.1.5, 4.1.10
- Solve KBR exercises 4.2.4, 4.2.9, 4.2.10, 4.2.23, 4.2.25, 4.2.29, 4.2.37
- A relation R from A to B can be represented as a list of pairs from the product set $A \times B$. Discuss the following two exercises among yourselves:
  - Write a pseudocode that, given a list as described above, finds the *range* $\operatorname{Ran} R$.
  - In the same spirit, write a pseudocode that finds the *domain* $\operatorname{Dom} R$.
- A relation R from A to B can also be represented as a Boolean matrix. How should the above algorithms be adapted to accommodate a matrix as input instead of a list of elements in $A \times B$?
- [*] *(Proof exercise)* Solve KBR 4.2.20. Time permitting, instructor presents a sample proof on the board.

Tuesday Dec. 7th, 15:15-17:00

- Solve KBR exercises 4.3.1–2, 4.3.4–8, 4.3.19
- Solve KBR exercises 4.4.1, 4.4.13, 4.4.31–32, 4.4.35
- Solve KBR exercises 5.1.1, 5.1.11, 5.1.30
- The instructor facilitates a discussion of *injective*, *surjective*, *bijective*, and *inverse* functions.
- Discuss the following among yourselves:
  - Write a pseudocode that determines whether a relation given as a list of elements from $A \times B$ is
    - reflexive
    - irreflexive
    - symmetric
    - asymmetric
    - antisymmetric
  - Write a pseudocode that does the same for a relation given as a Boolean matrix.

Friday Dec. 10, 10:15–12:00

- Solve KBR exercises 4.5.3, 4.5.4, 4.5.8, 4.5.12
- [*] *(Proof exercise)* Solve KBR 4.5.23. Time permitting, instructor presents a sample proof on the board at the end of the exercise session.
- Solve KBR exercises 4.7.2, 4.7.7, 4.7.12, 4.7.16–17, 4.7.19
- Finish any leftover exercises from Monday and Tuesday.

Extra exercises

(1) [***] Find a function $m : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that every time A is a $n \times n$-matrix with the property that

$$A_\odot^k = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

for some k, then $A_{\odot}^{m(n)} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$.