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IMPORTANT PROPERTIES OF THE Z-TRANSFORM

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THE IMPORTANT PROPERTIES OF THE Z-TRANSFORM

- Linearity

$$Z\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 X_1(z) + a_2 X_2(z); \quad \text{ROC: } \text{ROC}_{x_1} \cap \text{ROC}_{x_2}$$

- Sample shifting

$$Z\left[x(n - n_o)\right] = z^{-n_o} X(z); \quad \text{ROC: } \text{ROC}_x$$

- Frequency shifting

$$Z\left[a^n x(n)\right] = X\left(\frac{z}{a}\right); \quad \text{ROC: } \text{ROC}_x \text{ scaled by } |a|$$

THE IMPORTANT PROPERTIES OF THE Z-TRANSFORM

- Folding

$$Z\left[x(-n)\right] = X(1/z); \quad \text{ROC: Inverted ROC}_x$$

- Complex conjugation

$$Z\left[x^*(n)\right] = X^*(z^*); \quad \text{ROC: ROC}_x$$

- Differentiation in the z-domain (multiplication-by-a-ramp property)

$$Z\left[nx(n)\right] = -z\frac{dX(z)}{dz}; \quad \text{ROC: ROC}_x$$

The Important Properties of the *Z*-Transform

- Multiplication

$$Z\left[x_1(n)x_2(n)\right]=\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv;$$

$$\text{ROC: }\ \text{ROC}_{x_1}\cap\ \text{Inverted ROC}_{x_2}$$

  - *C* is a closed contour that encloses the origin and lies in the common ROC

- Convolution

$$Z\left[x_1(n)*x_2(n)\right]=X_1(z)X_2(z);\quad \text{ROC: ROC}_{x_1}\cap\text{ROC}_{x_2}$$

EXAMPLE

- Let $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} +5z^{-2} +6z^{-3}$.
  - Determine $X_3(z) = X_1(z)\, X_2(z)$.

EXAMPLE

- From definition

$$X(z) \triangleq Z\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$
  - So

$$x_1(n) = \{2, 3, 4\} \qquad x_2(n) = \{3, 4, 5, 6\}$$

EXAMPLE

- $X_3(z) = X_1(z)\, X_2(z)$ ➔ Convolution

$$Z\left[x_1(n) * x_2(n)\right] = X_1(z)\, X_2(z)$$

- the convolution of these two sequences will give the coefficients of the required polynomial product

```
>> x1 = [2,3,4]; x2 = [3,4,5,6];
>> x3 = conv(x1,x2)
>> x3 = 6 17 34 43 38 24
```

- Hence

$$X_3(z) = 6+17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$$

EXAMPLE

- Let $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$.
    - Determine $X_3(z) = X_1(z)\, X_2(z)$.

EXAMPLE

```
function [y,ny] = conv_m(x,nx,h,nh)
% Modified convolution routine for signal processing
% --------------------------------------------------
% [y,ny] = conv_m(x,nx,h,nh)
% [y,ny] = convolution result
% [x,nx] = first signal
% [h,nh] = second signal
%
nyb = nx(1)+nh(1); nye = nx(length(x)) + nh(length(h));
ny = [nyb:nye]; y = conv(x,h);
```

EXAMPLE

- $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$

$$x_1(n) = \{1, \underset{\uparrow}{2}, 3\} \qquad x_2(n) = \{2, 4, \underset{\uparrow}{3}, 5\}$$

```
>> x1 = [1,2,3];   n1 = [-1:1];
>> x2 = [2,4,3,5]; n2 = [-2:1];
>> [x3,n3] = conv_m(x1,n1,x2,n2)
x3 = 2 8 17 23 19 15
n3 = -3 -2 -1 0 1 2
```

- Hence

$$X_3(z) = 2z^3 + 8z^2 + 17z + 23 + 19z^{-1} + 15z^{-2}$$

COMMON Z-TRANSFORM PAIRS

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall\, z$
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u(-n-1)$	$\dfrac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
$[a^n \sin \omega_0 n]\, u(n)$	$\dfrac{(a \sin \omega_0) z^{-1}}{1-(2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$[a^n \cos \omega_0 n]\, u(n)$	$\dfrac{1-(a \cos \omega_0) z^{-1}}{1-(2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-nb^n u(-n-1)$	$\dfrac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$

EXAMPLE

- Using *z*-transform properties and the *z*-transform table, determine the *z*-transform of

$$x_1(n) = (n-2)(0,5)^{(n-2)}\cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

EXAMPLE

$$x_1(n) = (n-2)(0,5)^{(n-2)} \cos\left[\frac{\pi}{3}(n-2)\right] u(n-2)$$

- Applying the sample-shift property

$$Z\left[x(n-n_o)\right] = z^{-n_o} X(z)$$

$$\Downarrow$$

$$X(z) = Z\left[x(n)\right] = z^{-2} Z\left[n(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]$$

EXAMPLE

$$X(z) = Z\left[x(n)\right] = z^{-2} Z\left[n(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]$$

- Applying the multiplication by a ramp property

$$Z\left[nx(n)\right] = -z\frac{dX(z)}{dz}$$

$$\Downarrow$$

$$X(z) = z^{-2}\left\{-z\frac{dZ\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]}{dz}\right\}$$

EXAMPLE

$$X(z) = z^{-2}\left\{ -z \frac{dZ\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]}{dz} \right\}$$

- From table

$$Z\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right] = \frac{1-\left(0,5\cos\frac{\pi}{3}\right)z^{-1}}{1-2\left(0,5\cos\frac{\pi}{3}\right)z^{-1}+0,25z^{-1}}; \quad |z|>0,5$$

$$= \frac{1-0,25z^{-1}}{1-0,5z^{-1}+0,25\,z^{-2}}; \quad |z|>0,5$$

EXAMPLE

- Hence

$$X(z) = -z^{-1}\frac{d}{dz}\left\{\frac{1-0,25z^{-1}}{1-0,5z^{-1}+0,25\,\mathrm{z}^{-2}}\right\}; \quad |z|>0,5$$

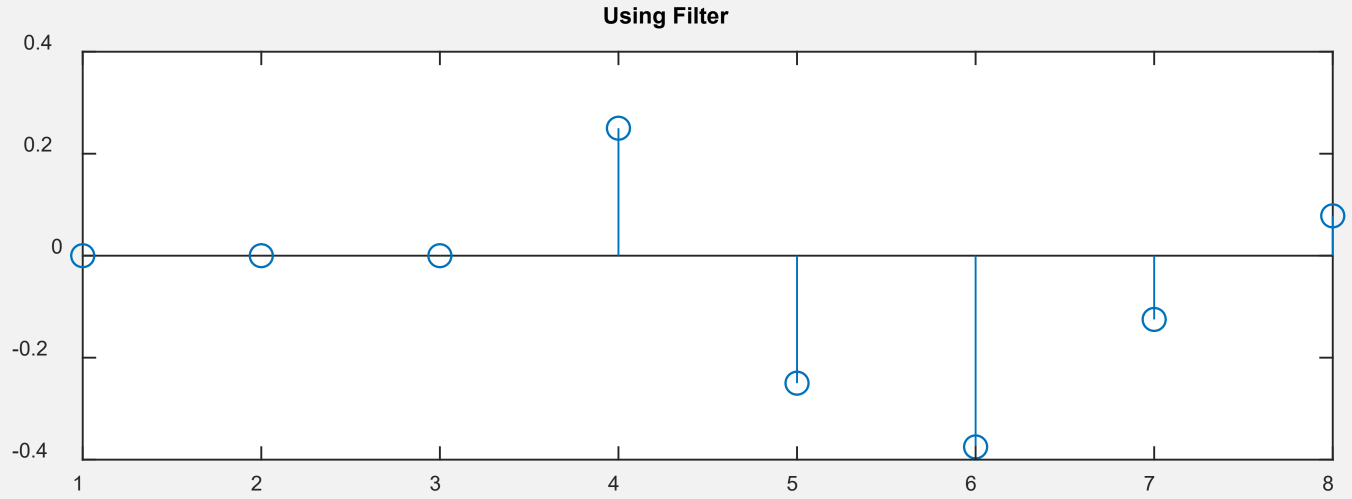
$$= -z^{-1}\left\{\frac{-0,25z^{-2}+0,5z^{-3}-0,0625z^{-4}}{1-z^{-1}+0,75z^{-2}-0,25z^{-3}+0,0625z^{-4}}\right\},$$

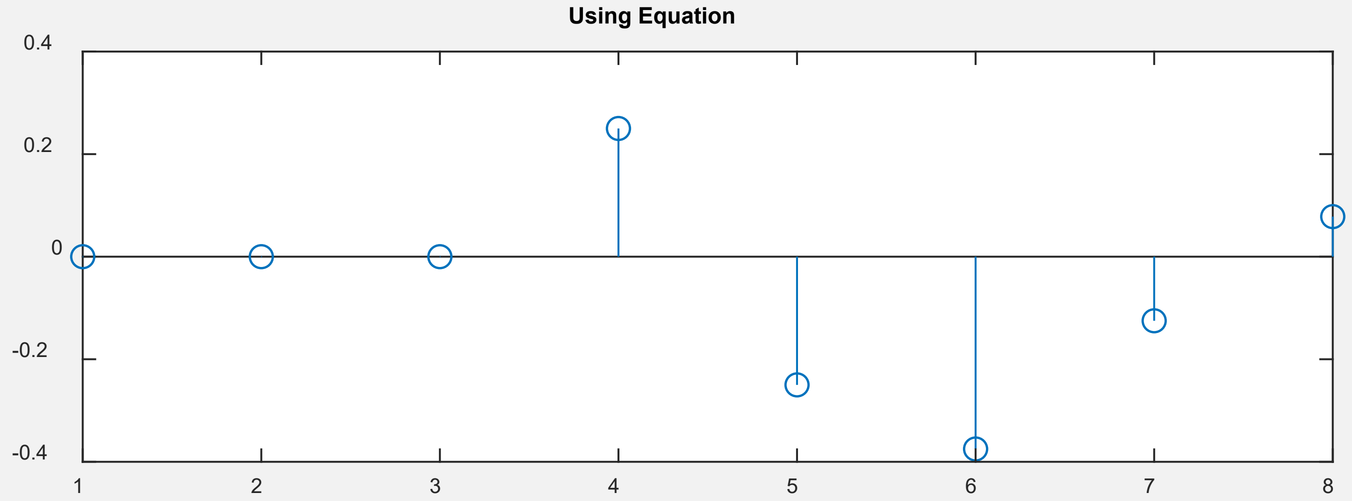
$$= \frac{0,25z^{-3}-0,5z^{-4}+0,0625z^{-5}}{1-z^{-1}+0,75z^{-2}-0,25z^{-3}+0,0625z^{-4}}, \qquad |z|>0,5$$

Example: MATLAB Verification

```
>> b = [0,0,0,0.25,-0.5,0.0625]; a = [1,-1,0.75,-0.25,0.0625];
>> [delta,n]=impseq(0,0,7)
>> x = filter(b,a,delta) % check sequence
>> x = [(n-2).*(1/2).^(n-2).*cos(pi*(n-2)/3)].*stepseq(2,0,7)
```

EXAMPLE: MATLAB VERIFICATION





THE INVERSION OF Z-TRANSFORM

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THE INVERSE Z-TRANSFORM

- The inverse *z*-transform of a complex function $X(z)$ is given by

$$x(n) \triangleq Z^{-1}\left[X(z)\right] = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz$$

- The inverse *z*-transform computation requires an evaluation of a complex contour integral
    - a complicated procedure
    - use the partial fraction expansion method

THE INVERSE Z-TRANSFORM IDEA

- $X(z)$ is a rational function of z^{-1}
  - can be expressed as a sum of simple factors using the partial fraction expansion

- The individual sequences corresponding to these factors can be written down using the *z*-transform table.

THE INVERSE Z-TRANSFORM PROCEDURE

- Given

$$X(z) = \frac{b_o + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}, \quad R_{x-} < |z| < R_{z+}$$

- express it as

$$X(z) = \frac{\tilde{b}_o + \tilde{b}_1 z^{-1} + \ldots + \tilde{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

Proper rational part

polynomial part if $M \geq N$

- Can be obtained by performing polynomial division if $M \geq N$ using the deconv function.

THE INVERSE Z-TRANSFORM PROCEDURE

- Perform a partial fraction expansion on the proper rational part of $X(z)$ to obtain

$$X(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

  - p_k is the kth pole of $X(z)$ and R_k is the residue at p_k
- The poles are distinct for which the residues are given by

$$R_k = \frac{\tilde{b}_o + \tilde{b}_1 z^{-1} + \ldots + \tilde{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \left(1 - p_k z^{-1}\right)\Big|_{z=p_k}$$

THE INVERSE Z-TRANSFORM PROCEDURE

- If a pole p_k has multiplicity r, then its expansion is given by

$$\sum_{\ell-1}^{r} \frac{R_{k,\ell} z^{-(\ell-1)}}{\left(1-p_k z^{-1}\right)} = \frac{R_{k,1}}{1-p_k z^{-1}} + \frac{R_{k,2} z^{-1}}{\left(1-p_k z^{-1}\right)^2} + \ldots + \frac{R_{k,\mathrm{r}} z^{-(r-1)}}{\left(1-p_k z^{-1}\right)^r}$$

- the residues R_k, are computed using a more general formula

THE INVERSE Z-TRANSFORM PROCEDURE

- write $x(n)$ as

$$x(n)=\sum_{k=1}^{N}R_k Z^{-1}\left[\frac{1}{1-p_k z^{-1}}\right]+\sum_{k=0}^{M-N}C_k\delta(n-k)$$

- finally, use the relation from Table to complete $x(n)$

$$Z^{-1}\left[\frac{z}{z-p_k}\right]=\begin{cases} p_k^n u(n) & |z_k|\le R_{x-} \\ -p_k^n u(-n-1) & |z_k|\ge R_{x+} \end{cases}$$

EXAMPLE

- Find the inverse *z*-transform of

$$x(z) = \frac{z}{3z^2 - 4z + 1}$$

EXAMPLE

$$x(z)=\frac{z}{3z^2-4z+1}$$

- Write

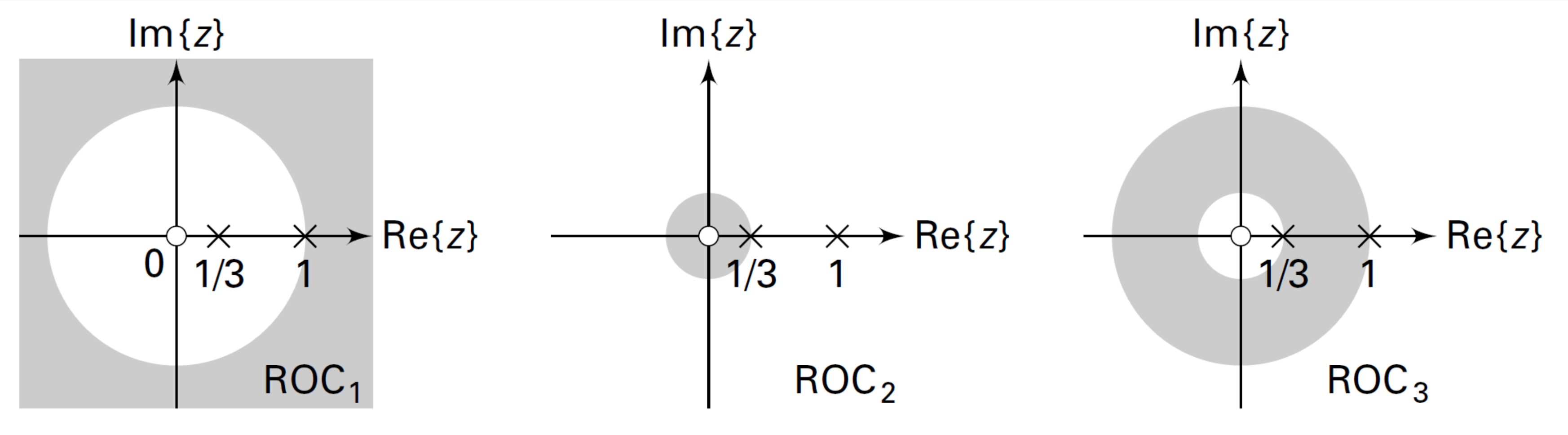
$$X(z)=\frac{z}{3\left(z^2-\frac{4}{3}z+\frac{1}{3}\right)}=\frac{\frac{1}{3}z^{-1}}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}}$$

$$=\frac{\frac{1}{3}z^{-1}}{\left(1-z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}=\frac{\frac{1}{2}}{1-z^{-1}}-\frac{\frac{1}{2}}{1-\frac{1}{3}z^{-1}}$$

$$=\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right)-\frac{1}{2}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)$$

EXAMPLE

- $X(z)$ has two poles: $z_1 = 1$ and $z_2 = 1/3$
- there are *three* possible ROCs



EXAMPLE

1. ROC_1: $1 < |z| < \infty$.

   Both poles are on the interior side of the ROC_1

   $|z_1| \leq R_{x-} = 1$ and $|z_2| \leq 1$

   a right-sided sequence.

$$x_1(n) = \frac{1}{2}u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE

2. ROC$_2$: $0 < |z| < \frac{1}{3}$.

   both poles are on the exterior side of the ROC$_2$

   $|z_1| \geq R_{x+} = \frac{1}{3}$ *and* $|z_2| \geq \frac{1}{3}$

$$x_2(n) = \frac{1}{2}\{-u(-n-1)\} - \frac{1}{2}\left\{-\left(\frac{1}{3}\right)^n u(-n-1)\right\}$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2}u(-n-1)$$

a left-sided sequence.

EXAMPLE

3. ROC$_3$: $\frac{1}{3} < |z| < 1$.

   pole z_1 is on the exterior side of the ROC$_3$: $|z_1| \geq R_{x+} = 1$

   pole z_2 is on the interior side of the ROC$_3$: $|z_2| \leq \frac{1}{3}$

$$x_3(n) = -\frac{1}{2}u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

a two-sided sequence.

MATLAB IMPLEMENTATION

- A MATLAB function residuez is available to compute the residue part and the direct (or polynomial) terms of a rational function in z^{-1}.
- A rational function in which the numerator and the denominator polynomials are in *ascending* powers of z^{-1}

$$X(z) = \frac{b_o + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_o + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{B(z)}{A(z)}$$

$$= \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

MATLAB IMPLEMENTATION

```
[R,p,C]=residuez(b,a)
```

- Computes the residues, poles, and direct terms of $X(z)$ in which two polynomials $B(z)$ and $A(z)$ are given in two vectors b and a
  - column vector R contains the residues
  - column vector p contains the pole locations
  - row vector C contains the direct terms

MATLAB IMPLEMENTATION

- If p(k) = ... = p(k+r-1) is a pole of multiplicity r, then the expansion includes the term of the form

$$\frac{R_k}{1-p_k z^{-1}}+\frac{R_{k+1}}{\left(1-p_k z^{-1}\right)^2}+\ldots+\frac{R_{k+r-1}}{\left(1-p_k z^{-1}\right)^r}$$

MATLAB IMPLEMENTATION

```
[b,a]=residuez(R,p,C)
```

- Three input arguments and two output arguments
- Converts the partial fraction expansion back to polynomials with coefficients in row vectors b and a.

EXAMPLE - RESIDUE CALCULATIONS

- Consider the rational function

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

EXAMPLE - RESIDUE CALCULATIONS

- Consider the rational function

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

- Rearrange $X(z)$ so that it is a function in ascending powers of z^{-1}.

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

EXAMPLE - RESIDUE CALCULATIONS

- using the MATLAB script

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

```
>> b = [0,1]; a = [3,-4,1]; [R,p,C] = residuez(b,a)
R =
0.5000
-0.5000
p =
1.0000
0.3333
c =
[]
```

$$X(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3} z^{-1}}$$

EXAMPLE - RESIDUE CALCULATIONS

- convert back to the rational function form

$$X(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}$$

```
>> [b,a] = residuez(R,p,C)
b =
0.0000
0.3333
a =
1.0000
-1.3333
0.3333
```

$$X(z) = \frac{0 + \frac{1}{3}z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z}{3z^2 - 4z + 1}$$

EXAMPLE

- Compute the inverse *z*-transform of

$$X(z) = \frac{1}{\left(1 - 0,9z^{-1}\right)^2 \left(1 + 0,9z^{-1}\right)}, \quad |z| > 0,9$$

EXAMPLE

- Evaluate the denominator polynomial as well as the residues using the MATLAB script

$$X(z)=\frac{1}{\left(1-0,9z^{-1}\right)^{2}\left(1+0,9z^{-1}\right)},\quad |z|>0,9$$

```
>> b = 1; a = poly([0.9,0.9,-0.9])
a = 1.0000 -0.9000 -0.8100 0.7290
>> [R,p,C]=residuez(b,a)
R = 0.2500 0.5000 0.2500
p = 0.9000 0.9000 -0.9000
c = []
```

EXAMPLE

- From the residue calculations and using the order of residues

$$X(z) = \frac{0,25}{1-0,9z^{-1}} + \frac{0,5}{(1-0,9z^{-1})^2} + \frac{0,25}{1+0,9z^{-1}}, \quad |z| > 0,9$$

$$= \frac{0,25}{1-0,9z^{-1}} + \frac{0,5}{0,9} z \frac{0,9z^{-1}}{(1-0,9z^{-1})^2} + \frac{0,25}{1+0,9z^{-1}}, \quad |z| > 0,9$$

- Using table and the *z*-transform property of time-shift

$$x(n) = 0,25(0,9)^n u(n) + \frac{5}{9}(n+1)(0,9)^{n+1} u(n+1) + 0,25(-0,9)^n u(n)$$

$$= 0,75(0,9)^n u(n) + 0,5n(0,9)^n u(n) + 0,25(-0,9)^n u(n)$$

EXAMPLE

- MATLAB verification

```
>> [delta,n] = impseq(0,0,7); x = filter(b,a,delta) % check sequence
x =
Columns 1 through 4
1.00000000000000 0.90000000000000 1.62000000000000 1.45800000000000
Columns 5 through 8
1.96830000000000 1.77147000000000 2.12576400000000 1.91318760000000
>> x = (0.75)*(0.9).^n + (0.5)*n.*(0.9).^n + (0.25)*(-0.9).^n % answer sequence
x =
Columns 1 through 4
1.00000000000000 0.90000000000000 1.62000000000000 1.45800000000000
Columns 5 through 8
1.96830000000000 1.77147000000000 2.12576400000000 1.91318760000000
```

EXAMPLE

- Determine the inverse *z*-transform of

$$X(z)=\frac{1+0,4\sqrt{2}z^{-1}}{1-0,8\sqrt{2}z^{-1}+0,64z^{-2}}$$

- so that the resulting sequence is causal and contains no complex numbers

EXAMPLE

- have to find the poles of $X(z)$ in the polar form to determine the ROC of the causal sequence

```
>> b = [1,0.4*sqrt(2)]; a=[1,-0.8*sqrt(2),0.64];
>> [R,p,C] = residuez(b,a)
R =
0.5000 - 1.0000i
0.5000 + 1.0000i
p =
0.5657 + 0.5657i
0.5657 - 0.5657i
C = []
>> Mp=(abs(p))’ % pole magnitudes
Mp = 0.8000 0.8000
>> Ap=(angle(p))’/pi % pole angles in pi units
Ap = 0.2500 -0.2500
```

$$X(z)=\frac{1+0,4\sqrt{2}z^{-1}}{1-0,8\sqrt{2}z^{-1}+0,64z^{-2}}$$

EXAMPLE

$$X(z) = \frac{1 + 0,4\sqrt{2}z^{-1}}{1 - 0,8\sqrt{2}z^{-1} + 0,64z^{-2}}$$

- From these calculations

- Using table

$$X(z) = \frac{0,5 - j}{1 - 0,8e^{+j\frac{\pi}{4}}z^{-1}} + \frac{0,5 + j}{1 - 0,8e^{+j\frac{\pi}{4}}z^{-1}}, \quad |z| > 0,8$$

$$\begin{aligned} x(n) &= (0,5 - j)0,8^n e^{+j\frac{\pi}{4}n}u(n) + (0,5 + j)0,8^n e^{-j\frac{\pi}{4}n}u(n) \\ &= 0,8^n \left[0,5\left\{ e^{+j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} \right\} - j\left\{ e^{+j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} \right\} \right] u(n) \\ &= 0,8^n \left[\cos\left(\frac{\pi n}{4}\right) + 2\sin\left(\frac{\pi n}{4}\right) \right] u(n) \end{aligned}$$

EXAMPLE

$$X(z)=\frac{1+0,4\sqrt{2}z^{-1}}{1-0,8\sqrt{2}z^{-1}+0,64z^{-2}}$$

- MATLAB verification

```
>> [delta, n] = impseq(0,0,6);
>> x = filter(b,a,delta) % check sequence
>> x = ((0.8).^n).*(cos(pi*n/4)+2*sin(pi*n/4))
```

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