PENGOLAHAN SINYAL DIGITAL

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SYSTEM REPRESENTATION IN THE Z-DOMAIN

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THE SYSTEM FUNCTION

- The system function $H(z)$ is given by

$$H(z) \triangleq \sim Z\left[h(n)\right] = \sum_{-\infty}^{\infty} h(n) z^{-n}; \quad R_{h-} < |z| < R_{h+}$$

- Using the convolution property of the z-transform, the output transform $Y(z)$ is given by

$$Y(z) = H(z)X(z) \quad : \quad \mathrm{ROC}_y = \mathrm{ROC}_h \cap \mathrm{ROC}_x$$

THE SYSTEM FUNCTION

- Therefore a linear and time invariant system can be represented in the z-domain by

$$X(z) \;\rightarrow\; \boxed{H(z)} \;\rightarrow\; Y(z) = H(z)X(z)$$

SYSTEM FUNCTION FROM THE DIFFERENCE EQUATION REPRESENTATION

- LTI systems are described by a difference equation

$$y(n)+\sum_{k=1}^{N} a_k y(n-k)=\sum_{\ell=1}^{M} b_\ell x(n-\ell)$$

- System Function Representation

$$Y(z)+\sum_{k=1}^{N} a_k z^{-k} Y(z)=\sum_{\ell=0}^{M} b_\ell z^{-\ell} X(z)$$

$$H(z) \triangleq \frac{Y(z)}{X(z)}=\frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{1+\sum_{k=1}^{N} a_k z^{-k}}=\frac{B(z)}{A(z)}=\frac{b_o z^{-M}\left(z^{M}+\ldots+\frac{b_M}{b_o}\right)}{z^{-N}\left(z^{N}+\ldots+a_N\right)}$$

SYSTEM FUNCTION FROM THE DIFFERENCE EQUATION REPRESENTATION

- After factorization

$$H(z) = b_o z^{N-M} \frac{\prod_{\ell=1}^{N}(z - z_\ell)}{\prod_{k=1}^{N}(z - p_k)}$$

  - z_ℓ are the system zeros
  - pk are the system poles

- H(z) (an LTI system) can also be represented in the z-domain using a pole-zero plot

TRANSFER FUNCTION REPRESENTATION

- A frequency response function or transfer function $H(z)$ on the unit circle $z = e^{j\omega}$

$$H\left(e^{j\omega}\right) = b_o z e^{j(N-M)\omega} \frac{\prod_{\ell=1}^{N}\left(e^{j\omega} - z_\ell\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)}$$

- The factor $(e^{j\omega} - z_\ell)$ can be interpreted as a *vector* in the complex *z*-plane from a zero z to the unit circle at $z = e^{j\omega}$
- The factor $(e^{j\omega} - p_k)$ can be interpreted as a vector from a pole p_k to the unit circle at $z = e^{j\omega}$.

TRANSFER FUNCTION REPRESENTATION

- The magnitude response function

$$\left|H\left(e^{j\omega}\right)\right| = |b_o| \frac{\left|e^{j\omega} - z_1\right| \ldots \left|e^{j\omega} - z_M\right|}{\left|e^{j\omega} - p_1\right| \ldots \left|e^{j\omega} - p_N\right|}$$

  - a product of the lengths of vectors from zeros to the unit circle *divided* by the lengths of vectors from poles to the unit circle and *scaled* by $|b_o|$.

TRANSFER FUNCTION REPRESENTATION

- The phase response function

$$\angle H\left(e^{j\omega}\right)=\left[0 \text{ or } \pi\right]+\left[\left(N-M\right)\omega\right]+\sum_{1}^{M}\angle\left(e^{j\omega}-z_k\right)-\sum_{1}^{N}\angle\left(e^{j\omega}-p_k\right)$$

  - a sum of a constant factor, a linear-phase factor, and a nonlinear-phase factor (angles from the “zero vectors” *minus* the sum of angles from the “pole vectors”).

EXAMPLE

- Given a causal system

$$y(n) = 0.9y(n-1) + x(n)$$

  - Determine $H(z)$ and sketch its pole-zero plot.
  - Plot $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$.
  - Determine the impulse response $h(n)$.

EXAMPLE

- The difference equation can be put in the form

$$y(n) - 0.9y(n - 1) = x(n)$$

$$H(z) = b_o z^{N-M} \frac{\prod_{\ell=1}^{N} (z - z_\ell)}{\prod_{k=1}^{N} (z - p_k)}$$

$$H(z) = \frac{1}{1 - 0,9z^{-1}}; \quad |z| > 0,9$$

  - since the system is causal.
  - There is one pole at 0.9 and one zero at the origin

EXAMPLE

- illustrate using the zplane function

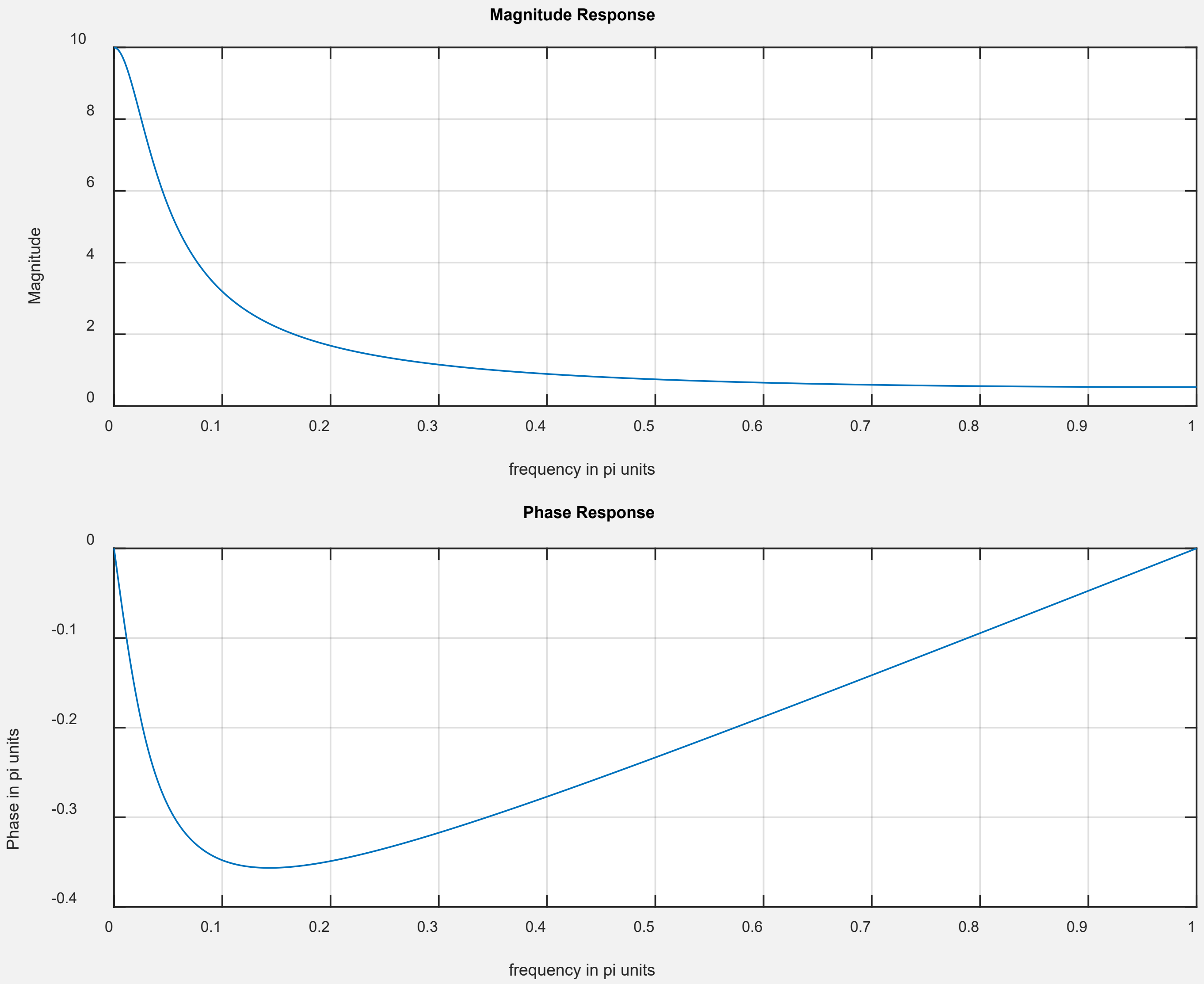
```
>> b = [1, 0]; a = [1, -0.9]; zplane(b,a)
```

EXAMPLE

- Determine the magnitude and phase of $H(e^{j\omega})$ using freqz function.
- Take 100 points along the upper half of the unit circle

```
[H,w] = freqz(b,a,100); magH = abs(H); phaH = angle(H);
subplot(2,1,1);plot(w/pi,magH);grid
xlabel('frequency in pi units'); ylabel('Magnitude');
title('Magnitude Response')
subplot(2,1,2);plot(w/pi,phaH/pi);grid
xlabel('frequency in pi units'); ylabel('Phase in pi units');
title('Phase Response')
```

EXAMPLE


Magnitude Response
10
8
6
4
2
0
Magnitude
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units
Phase Response
0
-0.1
-0.2
-0.3
-0.4
Phase in pi units
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units

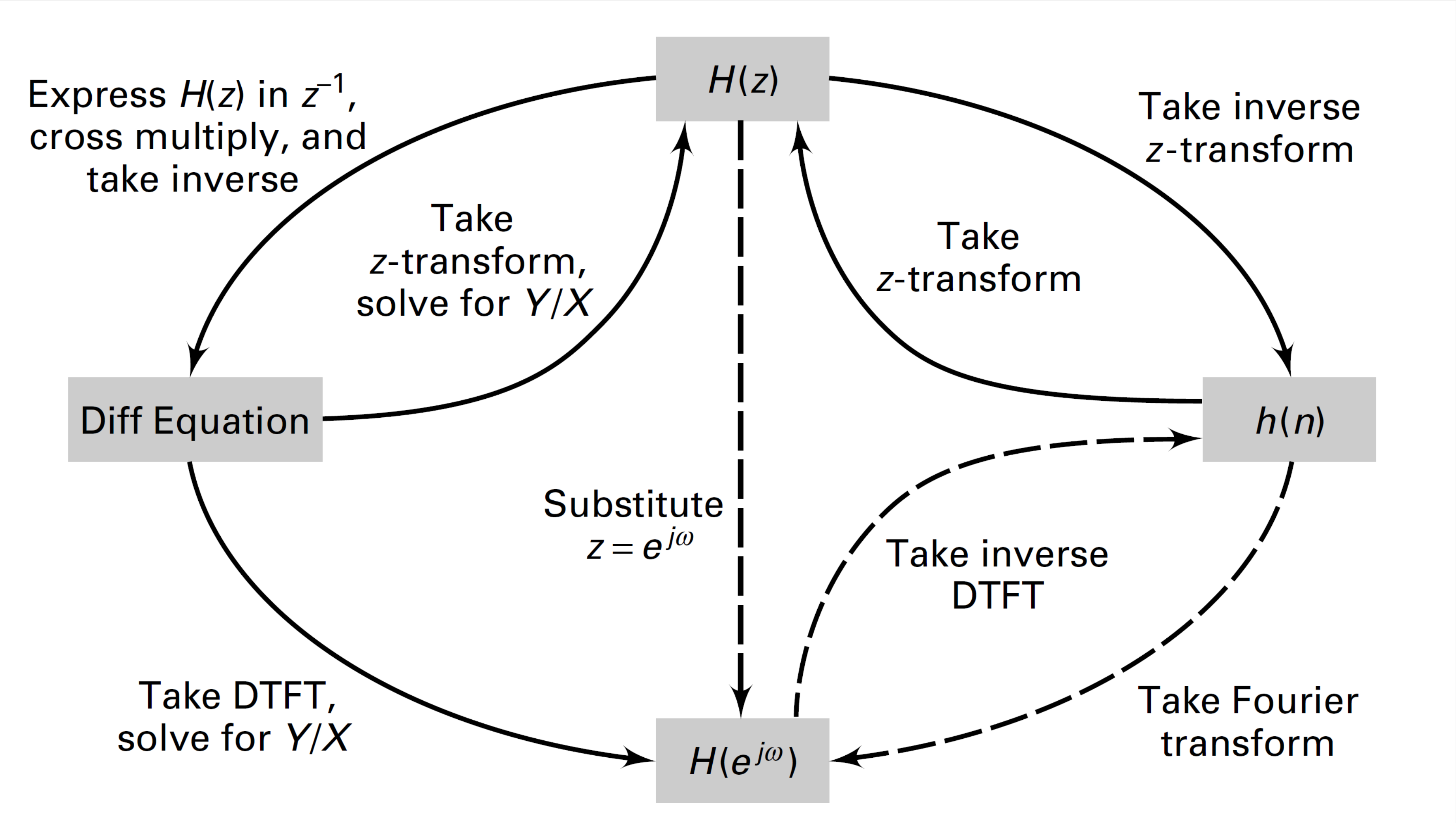
EXAMPLE

- From the *z*-transform in Table

$$h(n) = Z^{-1}\left[\frac{1}{1-0,9z^{-1}}; \quad |z| > 0,9\right]$$

$$= (0,9)^n u(n)$$

RELATIONSHIPS BETWEEN SYSTEM REPRESENTATIONS



Z-DOMAIN LTI STABILITY

- An LTI system is stable if and only if the unit circle is in the ROC of H(z).

Z-DOMAIN CAUSAL LTI STABILITY

- A causal LTI system is stable if and only if the system function $H(z)$ has all its poles inside the unit circle.

EXAMPLE

- A causal LTI system is described by the following difference equation:

$$y(n) = 0.81y(n-2) + x(n) - x(n-2)$$

- Determine
  - the system function $H(z)$,
  - the unit impulse response $h(n)$,
  - the unit step response $v(n)$, that is, the response to the unit step $u(n)$, and
  - the frequency response function $H(e^{j\omega})$, and plot its magnitude and phase over $0 \le \omega \le \pi$.

EXAMPLE

- Since the system is causal, the ROC will be outside a circle with radius equal to the largest pole magnitude.
- Taking the *z*-transform of both sides of the difference equation

$$H(z) = \frac{1 - z^{-2}}{1 - 0,81z^{-2}} = \frac{1 - z^{-2}}{\left(1 + 0,9z^{-1}\right)\left(1 - 0,9z^{-1}\right)}, \quad |z| > 0,9$$

EXAMPLE

- Using the MATLAB script for the partial fraction expansion

$$H(z) = \frac{1 - z^{-2}}{1 - 0,81z^{-2}} = \frac{1 - z^{-2}}{\left(1 + 0,9z^{-1}\right)\left(1 - 0,9z^{-1}\right)}, \quad |z| > 0,9$$

```
>> b = [1,0,-1]; a = [1,0,-0.81]; [R,p,C] = residuez(b,a);
R = -0.1173 -0.1173
p = -0.9000 0.9000
C = 1.2346
```

$$H(z) = 1,2346 - 0,1173\frac{1}{1 + 0,9z^{-1}} - 0,1173\frac{1}{1 - 0,9z^{-1}}, \quad |z| > 0,9$$

TABLE 4.1 *Some common z-transform pairs*

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall z$
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u(-n-1)$	$\dfrac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
$[a^n \sin\omega_0 n]\, u(n)$	$\dfrac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$[a^n \cos\omega_0 n]\, u(n)$	$\dfrac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-nb^n u(-n-1)$	$\dfrac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$

EXAMPLE

$$H(z) = 1,2346 - 0,1173\frac{1}{1+0,9z^{-1}} - 0,1173\frac{1}{1-0,9z^{-1}}, \quad |z| > 0,9$$

- from Table

$$h(n) = 1,2346\,\delta(n) - 0,1173\left\{1+(-1)^n\right\}(0,9)^n\,u(n)$$

EXAMPLE

- From Table 4.1

$$Z\left[u(n)\right] = U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$\begin{aligned}
V(z) &= H(z)U(z) \\
&= \left[\frac{\left(1+z^{-1}\right)\left(1-z^{-1}\right)}{\left(1+0,9z^{-1}\right)\left(1-0,9z^{-1}\right)}\right]\left[\frac{1}{1-z^{-1}}\right], \quad |z| > 0,9 \cap |z| > 1 \\
&= \frac{1+z^{-1}}{\left(1+0,9z^{-1}\right)\left(1-0,9z^{-1}\right)}, \quad |z| > 0,9 \\
&= 1,0556\frac{1}{1-0,9z^{-1}} - 0,0556\frac{1}{1+0,9z^{-1}}, \quad |z| > 0,9
\end{aligned}$$

EXAMPLE

- Finally,

$$v(n) = \left[1,0556(0,9)^n - 0,556(-0,9)^n\right]u(n)$$

- There is a pole-zero cancellation at $z = 1$.
- This has two implications.
  - First, the ROC of $V(z)$ is still $\{|z| > 0.9\}$ and not $\{|z| > 0.9 \cap |z| > 1 = |z| > 1\}$
  - The step response $v(n)$ contains no steady-state term $u(n)$.

EXAMPLE

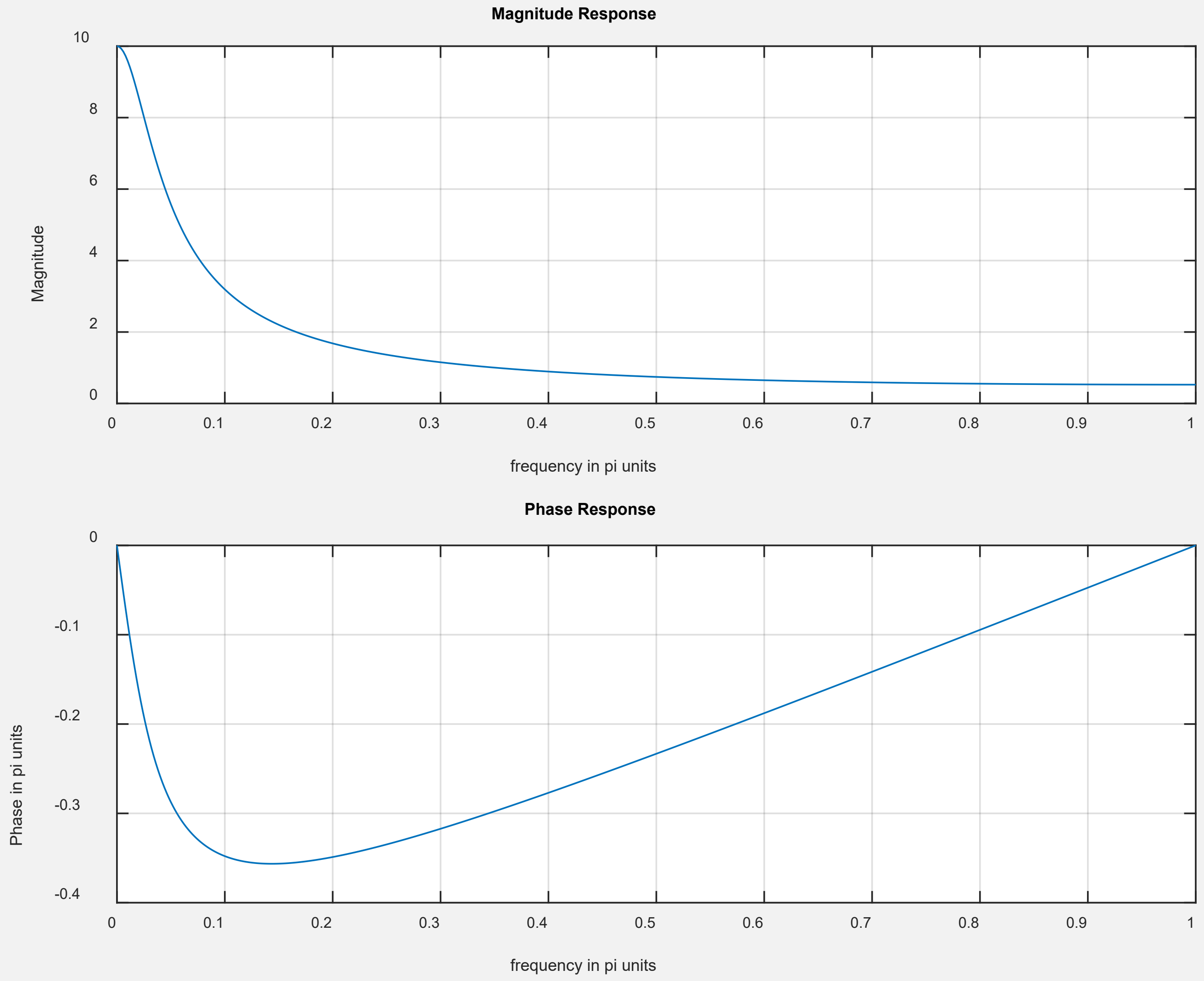
- The frequency response function $H(e^{j\omega})$
- Substituting $z = e^{j\omega}$ in $H(z)$,

$$H\left(e^{j\omega}\right) = \frac{1 - e^{-j2\omega}}{1 - 0,81e^{-j2\omega}}$$

EXAMPLE

```
% MATLAB script to compute and plot responses
w = [0:1:500]*pi/500; H = freqz(b,a,w);
magH = abs(H); phaH = angle(H);
subplot(2,1,1); plot(w/pi,magH); grid
xlabel(‘frequency in pi units’); ylabel(’Magnitude’)
title(’Magnitude Response’)
subplot(2,1,2); plot(w/pi,phaH/pi); grid
xlabel(’frequency in pi units’); ylabel(’Phase in pi units’)
title(’Phase Response’)
```

EXAMPLE


Magnitude Response
Magnitude
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units
Phase Response
Phase in pi units
0
-0.1
-0.2
-0.3
-0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units

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