PENGOLAHAN SINYAL DIGITAL

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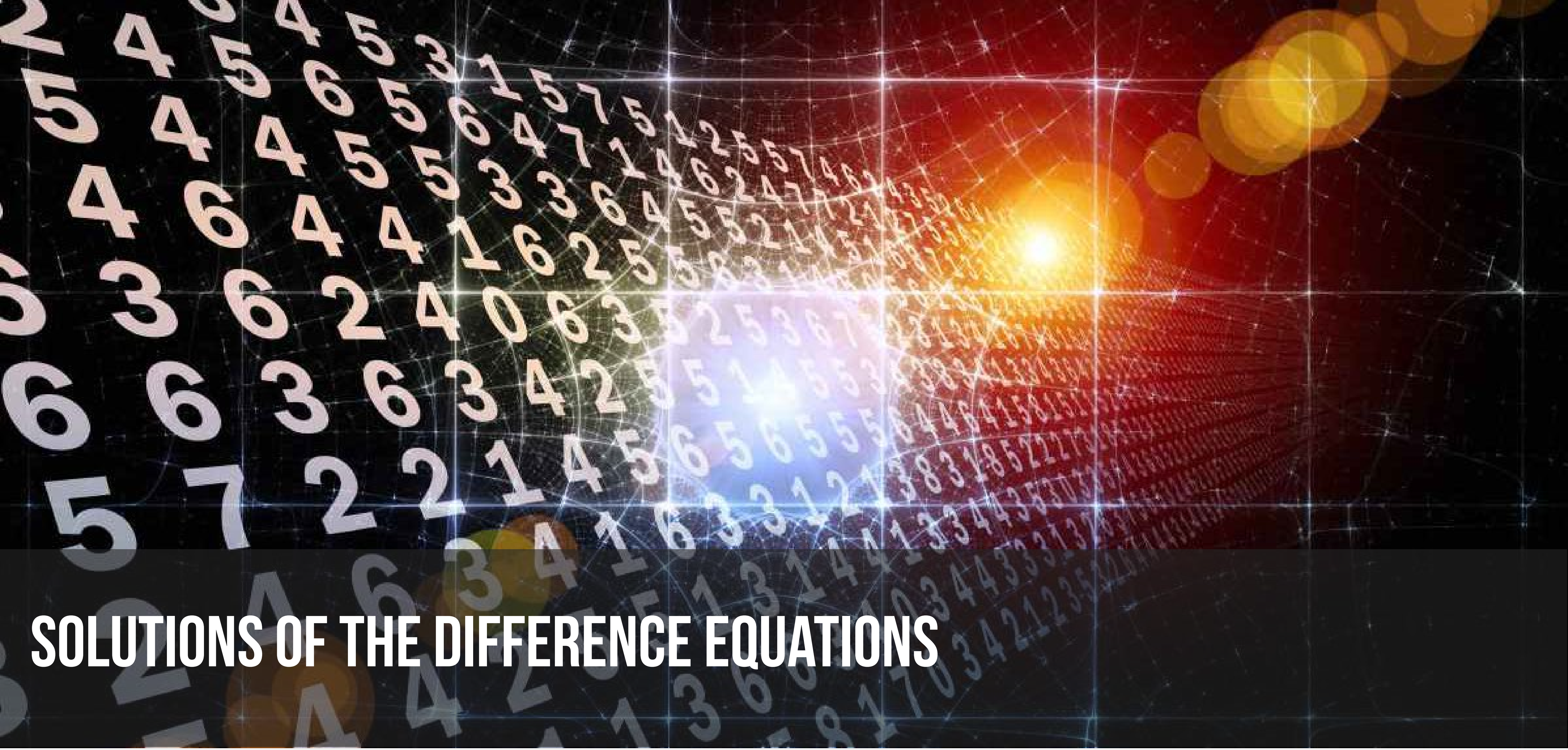

SOLUTIONS OF THE DIFFERENCE EQUATIONS

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ONE-SIDED Z-TRANSFORM

- The one-sided z-transform of a sequence x(n) is given by

$$Z^{+}\left[x(n)\right] \triangleq Z\left[x(n)u(n)\right] \triangleq X^{+}\left[z\right] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

  - Difference equations generally evolve in the positive n direction.
  - Time frame for these solutions will be *n* ≥ 0
  - One form involved finding the particular and the homogeneous solutions
  - The other form involved finding the zero-input (initial condition) and the zero-state responses

ONE-SIDED Z-TRANSFORM

- The sample shifting property is given by

$$Z^{+}\left[x(n-k)\right] = x(-1)z^{1-k} + x(-2)z^{2-k} + \ldots + x(-k) + z^{-k}X^{+}(z)$$

- The result can now be used to solve difference equations with nonzero initial conditions or with changing inputs

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m), \quad n \geq 0$$

- subject to these initial conditions:

$$\left\{y(i), i = -1, \ldots, -N\right\} \qquad \left\{x(i), i = -1, \ldots, -M\right\}$$

EXAMPLE

- Solve

$$y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = x(n), \quad n \geq 0$$

- where

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

- subject to $y(-1) = 4$ and $y(-2) = 10$.

EXAMPLE

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=x(n),\quad n\geq 0$$

- Taking the one-sided *z*-transform of both sides of the difference equation

$$Y^{+}(z)-\frac{3}{2}\left[y(-1)+z^{-1}Y^{+}(z)\right]+\frac{1}{2}\left[y(-2)+z^{-1}y(-1)+z^{-2}Y^{+}(z)\right]=\frac{1}{1-\frac{1}{4}z^{-1}}$$

- Substituting the initial conditions and rearranging

$$Y^{+}(z)\left[1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right]=\frac{1}{1-\frac{1}{4}z^{-1}}+\left(1-2z^{-1}\right)$$

EXAMPLE

$$Y^{+}(z)\left[1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right]=\frac{1}{1-\frac{1}{4}z^{-1}}+\left(1-2z^{-1}\right)$$

$$Y^{+}(z)=\frac{\dfrac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}+\frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

Using the partial fraction expansion

$$Y^{+}(z)=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{\frac{2}{3}}{1-z^{-1}}+\frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}$$

EXAMPLE

- After inverse transformation the solution is

$$y(n) = \left[\left(\frac{1}{2} \right)^n + \frac{2}{3} + \frac{1}{2} \left(\frac{1}{4} \right)^n \right] u(n)$$

EXAMPLE

- Homogeneous and particular parts

$$y(n) = \underbrace{\left[\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]u(n)}_{\text{Homogeneous part}} + \underbrace{\frac{1}{2}\left(\frac{1}{4}\right)^n u(n)}_{\text{Particular part}}$$

  - The homogeneous part is due to the *system poles*, and the particular part is due to the *input poles*.

EXAMPLE

- Transient and steady-state responses

$$y(n) = \underbrace{\left[\frac{1}{3}\left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^n\right] u(n)}_{\text{Transient response}} + \underbrace{\frac{2}{3} u(n)}_{\text{Steady-state response}}$$

- The transient response is due to poles that are *inside* the unit circle, whereas the steady-state response is due to poles that are *on* the unit circle.

EXAMPLE

- Zero-input (or initial condition) and zero-state responses

$$Y^{+}(z)=\frac{\dfrac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}+\frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$Y_{ZS}(z)=H(z)X(z) \qquad\qquad Y_{ZI}(z)=H(z)X_{IC}(z)$$

- $X_{IC}(z)$ can be thought of as an equivalent *initial-condition input* that generates the same output Y_{ZI} as generated by the initial conditions.

$$x_{IC}(n)=\left\{\underset{\uparrow}{1},-2\right\}$$

EXAMPLE

- taking the inverse z-transform of each part of

$$Y^{+}(z)=\frac{\frac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}+\frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

- The complete response as

$$y(n)=\underbrace{\left[\frac{1}{3}\left(\frac{1}{4}\right)^{n}-2\left(\frac{1}{2}\right)^{n}+\frac{8}{3}\right]u(n)}_{\text{Zero-state response}}+\underbrace{\left[3\left(\frac{1}{2}\right)^{n}-2\right]u(n)}_{\text{Zero-input response}}$$

MATLAB IMPLEMENTATION

y = filter(b,a,x,xic)

- xic is an equivalent initial-condition input array

MATLAB IMPLEMENTATION

```
n = [0:7]; x = (1/4).^n; xic = [1, -2];
format long; y1 = filter(b,a,x,xic)
y2 = (1/3)*(1/4).^n+(1/2).^n+(2/3)*ones(1,8)
```

MATLAB IMPLEMENTATION

```
xic = filtic(b,a,Y,X)
```

- to determine $x_{IC}(n)$ analytically
- b and a are the filter coefficient arrays and Y and X are the initialcondition arrays from the initial conditions on $y(n)$ and $x(n)$

```
>> Y = [4, 10]; xic = filtic(b,a,Y)
```

EXAMPLE

- Solve the difference equation

$$y(n) = \frac{1}{3}\left[x(n) + x(n-1) + x(n-2) + 0{,}95y(n-1) - 0{,}9025y(n-2)\right]$$

  - where $x(n) = \cos(\pi n/3)u(n)$ and $y(-1) = -2,\ y(-2) = -3;\ x(-1) = 1,\ x(-2) = 1$

- First determine the solution analytically and then by using MATLAB

EXAMPLE

- Taking a one-sided *z*-transform of the difference equation

$$Y^+(z) = \frac{1}{3}\left[X^+(z)x(-1) + z^{-1}X^+(z) + x(-2) + z^{-1}x(-1) + z^{-2}X^+(z)\right]$$
$$+0,95\left[y(-1) + z^{-1}Y^+(z)\right] - 0,9025\left[y(-2) + z^{-1}y(-1) + z^{-2}Y^+(z)\right]$$

- and substituting the initial conditions

$$Y^+(z) = \frac{\frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2}}{1 - 0,95z^{-1} + 0,9025z^{-2}}X^+(z) + \frac{1,4742 + 2,1383z^{-1}}{1 - 0,95z^{-1} + 0,9025z^{-2}}$$

- Clearly, $x_{IC}(n)$ = [1,4742, 2,1383].

EXAMPLE

```
% This simplification and further partial fraction expansion can be
done using MATLAB.


b = [1,1,1]/3; a = [1,-0.95,0.9025];
Y = [-2,-3]; X = [1,1]; xic=filtic(b,a,Y,X)
bxplus = [1,-0.5]; axplus = [1,-1,1]; % X(z) transform coeff.
ayplus = conv(a,axplus) % Denominator of Yplus(z)
byplus = conv(b,bxplus)+conv(xic,axplus)
[R,p,C] = residuez(byplus,ayplus)
Mp = abs(p), Ap = angle(p)/pi
```

EXAMPLE

- Substituting $X^+(z)$

$$X^{+}\left(z\right)=\frac{1-0,5z^{-1}}{1-z^{-1}+z^{-2}}$$

- obtain $Y^+(z)$ as a rational function

$$Y^{+}\left(z\right)=\frac{0,0584+j3,9468}{1-e^{-j\pi/3}z^{-1}}+\frac{0,0584-j3,9468}{1-e^{j\pi/3}z^{-1}}$$
$$+\frac{0,8453+j2,0311}{1-0,95e^{j\pi/3}z^{-1}}+\frac{0,8453-j2,0311}{1-0,95e^{-j\pi/3}z^{-1}}$$

EXAMPLE

- From Table

$$y(n) = 0,1169\cos(\pi n/3) + 7,8937\sin(\pi n/3)$$
$$+(0,95)^n\left[1,6906\cos(\pi n/3) - 4,0623\sin(\pi n/3)\right], \quad n \geq 0$$

EXAMPLE – MATLAB VERIFICATION

```
>> n = [0:7]; x = cos(pi*n/3); y = filter(b,a,x,xic)
% Matlab Verification
>> A=real(2*R(1)); B=imag(2*R(1)); C=real(2*R(3)); D=imag(2*R(4));
>> y=A*cos(pi*n/3)+B*sin(pi*n/3)+((0.95).^n).*(C*cos(pi*n/3)+
D*sin(pi*n/3))
```

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