PENGOLAHAN SINYAL DIGITAL

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THE Z-TRANSFORM

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FOURIER TRANSFORM SHORTCOMINGS

- There are *two* shortcomings to the Fourier transform approach
  - Many useful signals $u(n)$ and $nu(n)$ does not exist in the discrete-time Fourier transform
  - The transient response of a system due to initial conditions or due to changing inputs cannot be computed using the discrete-time Fourier transform approach

THE Z-TRANSFORM

- An essential tool for the analysis of discrete-time systems
- A transformation that maps or transforms a discrete-time signal $x(n)$ into a function $X(z)$ of a complex variable z.

$$X(z) = Z\left[x(n)\right]$$

THE Z-TRANSFORM

- The difference-equation can be converted to a simple algebraic equation which is readily solved for the Z-transform of the output $Y(z)$

- Important qualitative features of discrete-time systems also can be obtained with the help of the Z-transform.

- A discrete-time system is *stable* if and only if every bounded input signal is guaranteed to produce a bounded output signal
  - Stability is an essential characteristic of practical digital filters, and the easiest way to establish stability is with the Z-transform

THE Z-TRANSFORM

- The Z-transform is a powerful tool that is useful for analyzing and solving linear discrete-time systems.

  - Its bilateral (or two-sided) version provides another domain in which a larger class of sequences and systems can be analysed

  - Its unilateral (or one-sided) version can be used to obtain system responses with initial conditions or changing inputs.

THE BILATERAL Z-TRANSFORM

- The *Z-transform* of a discrete-time signal $x(n)$ is a function $X(z)$ of a complex variable *z* defined

$$X(z) \triangleq Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z = Ae^{j\phi} = A \cdot (\cos\phi + j\sin\phi)$$

  - A is the magnitude of z
  - j is the imaginary unit
  - ϕ is the complex argument (also referred to as angle or phase) in radians.

- The set of *z* values for which $X(z)$ exists is called the *region of convergence* (*ROC*)

$$R_{x-} < |z| < R_{x+}$$

REGION OF CONVERGENCE

- The region of convergence (ROC) is the set of points in the complex plane for which the Z-transform summation converges.

$$\mathrm{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \right\}$$

THE BILATERAL Z-TRANSFORM

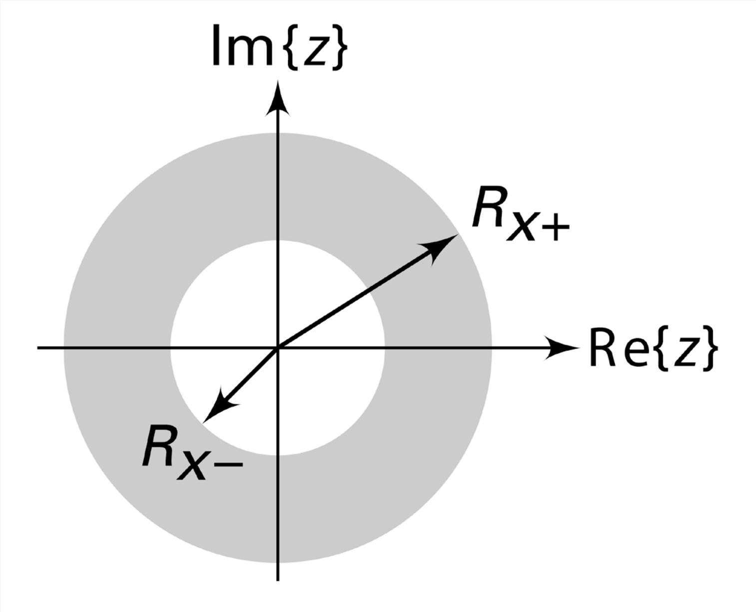
- The inverse *z*-transform of a complex function $X(z)$ is given by

$$x(n) \triangleq Z^{-1}\left[X(z)\right] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

  - C is a **counterclockwise contour** encircling the origin and lying in the ROC.

THE BILATERAL Z-TRANSFORM

- The complex variable z is called the *complex frequency* given by $z = |z|e^{j\omega}$, where $|z|$ is the magnitude and ω is the real frequency
- The shape of the ROC is an open ring



- R_{x-} may be equal to zero and/or R_{x+} could possibly be ∞.

THE BILATERAL Z-TRANSFORM

- If $R_{x+} < R_{x-}$, then the ROC is a *null space* and the *z*-transform *does not exist*.
- The function *|z|* = 1 (or $z = e^{j\omega}$) is a circle of unit radius in the *z*-plane and is called the *unit circle*.
- If the ROC contains the unit circle, then we can evaluate $X(z)$ on the unit circle.

$$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega} = \mathcal{F}\left[x(n)\right]$$

  - the discrete-time Fourier transform $X(e^{j\omega})$ may be viewed as a special case of the *z*-transform $X(z)$

A POSITIVE-TIME SEQUENCE

- Let $x_1(n) = a^n u(n),\ 0 < |a| < \infty$.
- Then

$$X_1(z) = \sum_0^\infty a^n z^{-n} = \sum_0^\infty \left(\frac{a}{z}\right)^n = \frac{1}{1-az^{-1}}; \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z-a}, \quad |z| > |a| \Rightarrow \mathrm{ROC}_1 : \underbrace{|a|}_{R_{x-}} < |z| < \underbrace{\infty}_{R_{x+}}$$

- $X_1(z)$ is a rational function

A POSITIVE-TIME SEQUENCE

- $X_1(z)$ is a rational function

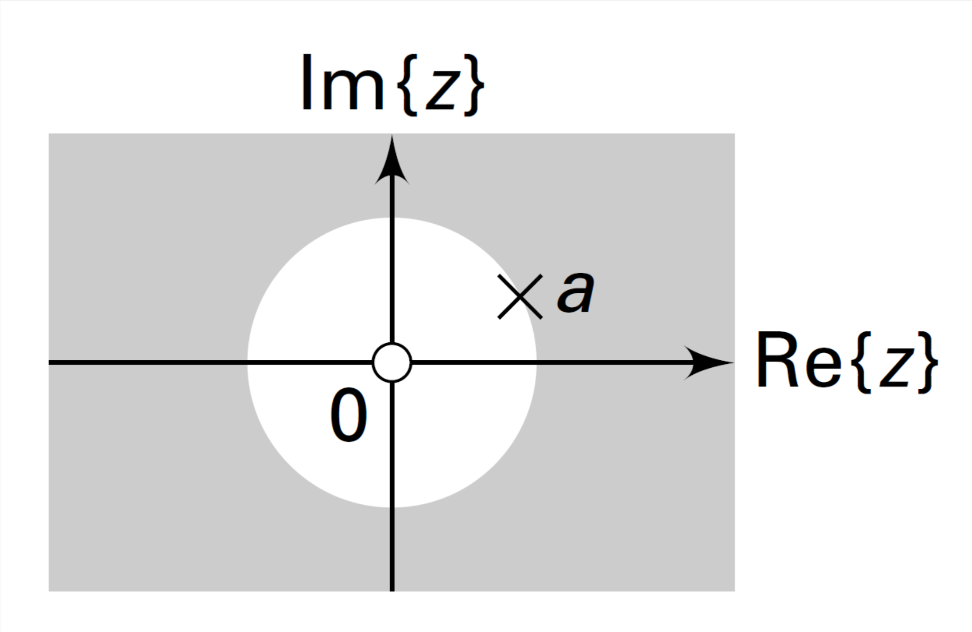
$$X_1(z) \triangleq \frac{B(z)}{A(z)} = \frac{z}{z-a}$$

  - $\mathrm{B(z) = z}$ is the numerator polynomial
  - $\mathrm{A(z) = z{-}a}$ is the denominator polynomial
- The roots of *B*(*z*) are called the *zeros* of *X*(*z*)
- The roots of *A*(*z*) are called the *poles* of *X*(*z*).

A POSITIVE-TIME SEQUENCE

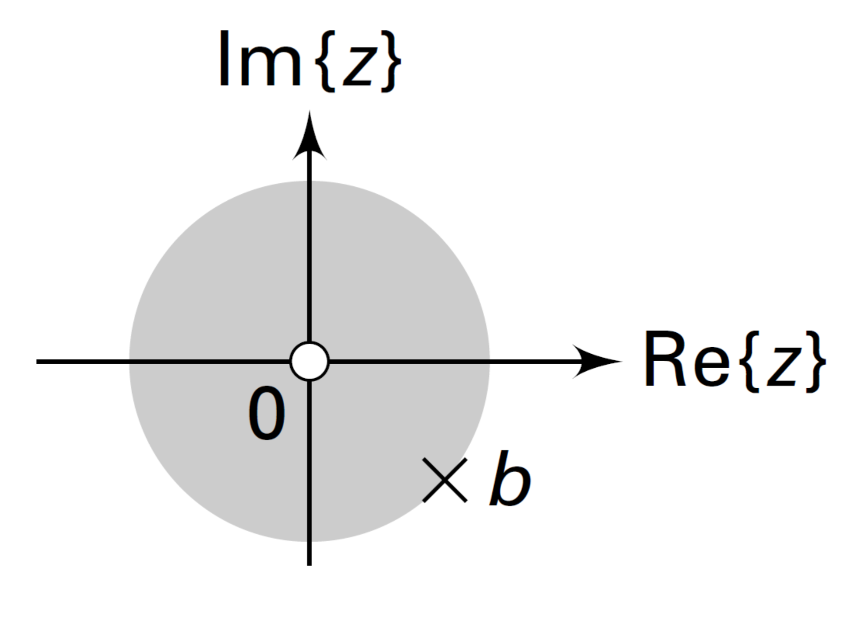
$$X_1(z) = \frac{z}{z-a}, \qquad |z| > |a| \Rightarrow \mathrm{ROC}_1 : \underbrace{|a|}_{R_{x-}} < |z| < \underbrace{\infty}_{R_{x+}}$$

- $X_1(z)$ has a zero at the origin $z = 0$ and a pole at $z = a$
- Hence $x_1(n)$ can also be represented by a *pole-zero diagram* in the z-plane in which zeros are denoted by O and poles by ×



A NEGATIVE-TIME SEQUENCE

- Let $x_2(n) = -b^n u(-n-1),\ 0 < |b| < \infty$.
- Then



$$X_2(z) = -\sum_{-\infty}^{-1} b^n z^{-n} = -\sum_{-\infty}^{-1}\left(\frac{b}{z}\right)^n = -\sum_{1}^{\infty}\left(\frac{z}{b}\right)^n = 1-\sum_{0}^{\infty}\left(\frac{z}{b}\right)^n$$

$$= 1-\frac{1}{1-z/b} = \frac{z}{z-b}, \quad \mathrm{ROC}_2 : \underbrace{0}_{R_{x-}} < |z| < \underbrace{|b|}_{R_{x+}}$$

COMPARING

- If $b = a$ in both example, then $X_2(z) = X_1(z)$ except for their respective ROCs; that is, $ROC_1 = ROC_2$.
- This implies that the ROC is a distinguishing feature that guarantees the uniqueness of the z-transform

A TWO-SIDED SEQUENCE

- Let $x_3(n) = x_1(n) + x_2(n) = a^n u(n) - b^n u(-n-1)$
- Using the preceding two examples

$$X_3(z) = \sum_{n=0}^{\infty} a^n z^{-1} - \sum_{-\infty}^{-1} b^n z^{-1}$$

$$= \left\{ \frac{z}{z-a}, \mathrm{ROC}_1 : |z| > |a| \right\} + \left\{ \frac{z}{z-b}, \mathrm{ROC}_2 : |z| < |b| \right\}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}; \quad \mathrm{ROC}_3 : \ \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

A TWO-SIDED SEQUENCE

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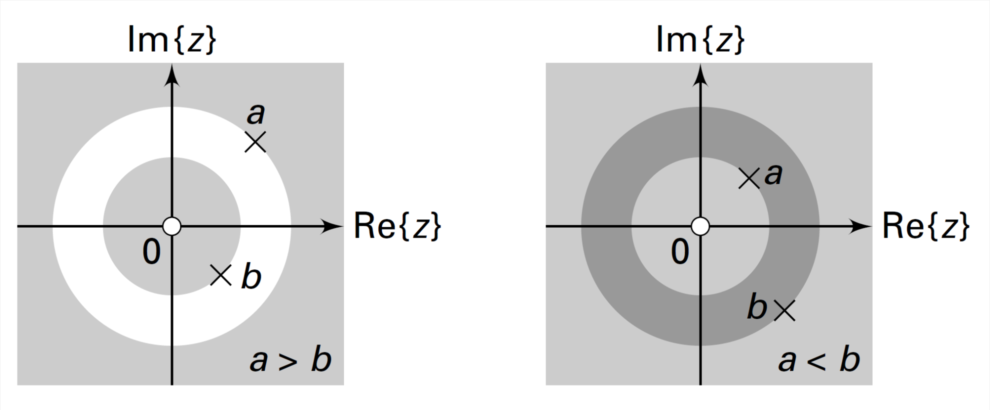
$$= \frac{z}{z-a} + \frac{z}{z-b}; \quad \mathrm{ROC}_3 : \ \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

- If |b| < |a|, than ROC3 is a null space, and $X_3(z)$ does not exist.
- If |a| < |b|, then the ROC3 is |a| < |z| < |b|, and $X_3(z)$ exists in this region



PROPERTIES OF THE ROC

- The ROC is **always bounded by a circle** since the convergence condition is on the magnitude $|z|$.
- The sequence $x_1(n) = a^n u(n)$ is a special case of a *right sided sequence*, defined as a sequence $x(n)$ that is zero for some $n < n_0$
  - the ROC for right-sided sequences is **always outside of a circle of radius R_{x-}**.
  - If $n_0 \geq 0$, then the right-sided sequence is also called a *causal* sequence.

PROPERTIES OF THE ROC

- The sequence $x_2(n) = -b^n u(-n-1)$ is a special case of a *left-sided* sequence, defined as a sequence $x(n)$ that is zero for some $n > n_0$.
  - If $n_0 \leq 0$, the resulting sequence is called an *anticausal* sequence.
  - the ROC for left-sided sequences is **always inside of a circle of radius R_{x+}**.

- The sequences that are zero for $n < n_1$ and $n > n_2$ are called *finite-duration sequences*.
  - The ROC for such sequences is **the entire *z*-plane**.
  - If $n_1 < 0$, then $z = \infty$ is not in the ROC.
  - If $n_2 > 0$, then $z = 0$ is not in the ROC.

PROPERTIES OF THE ROC

- The ROC cannot include a pole since $X(z)$ converges uniformly in there.
- There is at least one pole on the boundary of a ROC of a rational $X(z)$.
- The ROC is one contiguous region; that is, the ROC does not come in pieces.

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