PENGOLAHAN SINYAL DIGITAL

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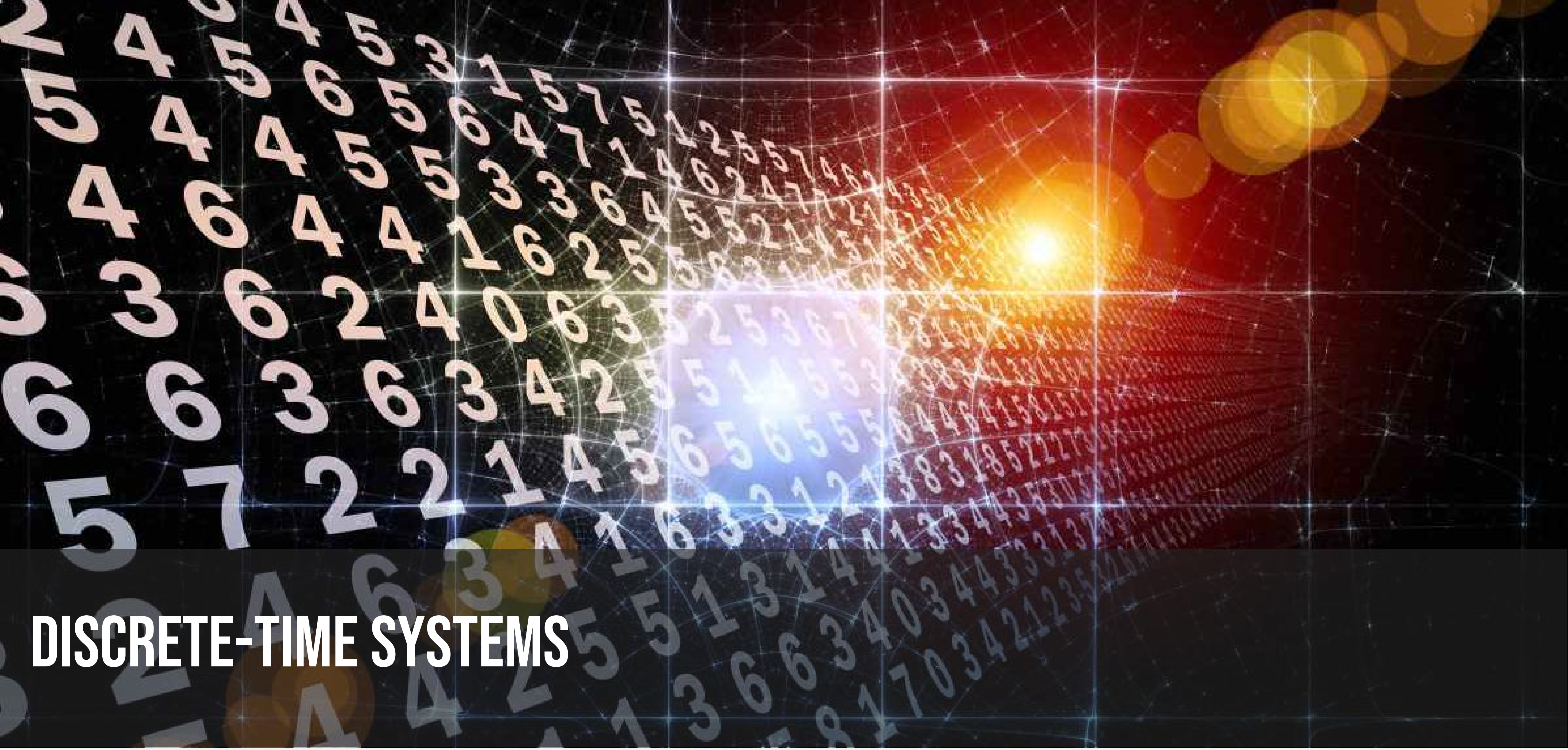

DISCRETE-TIME SYSTEMS

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DISCRETE-TIME SYSTEMS

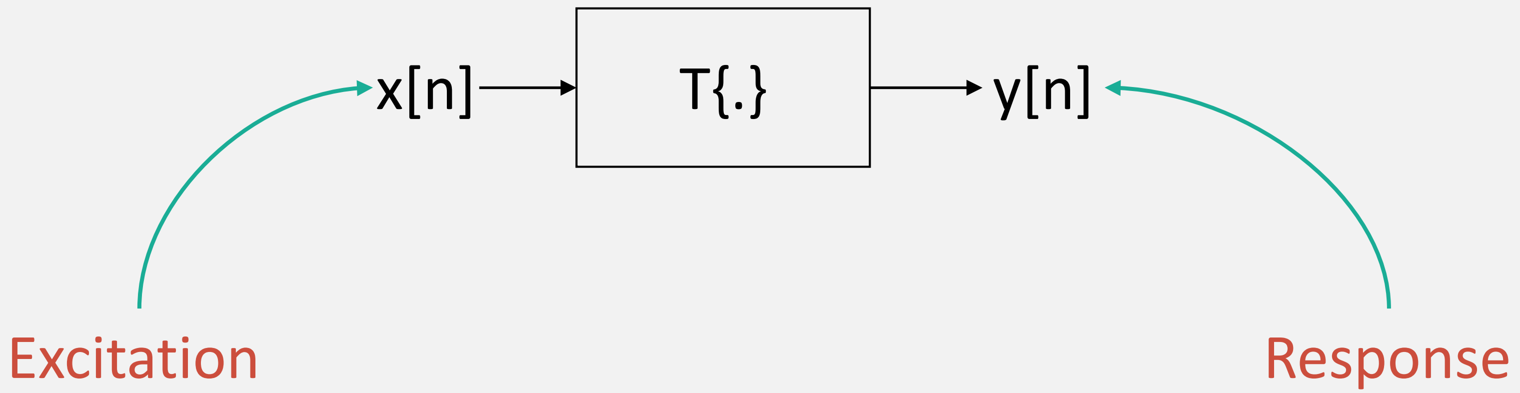
- A discrete time system operates on a discrete time signal input and produces a discrete time signal output

- Examples of useful discrete time systems: digital filters for images or sound

DISCRETE-TIME SYSTEMS

- Discrete-Time Sequence (described as an operator $T[\cdot]$) is a mathematical operation that maps a given input sequence x[n] into an output sequence y[n]

$$y[n] = T\{x[n]\}$$



DISCRETE-TIME SYSTEMS

- Example Discrete-Time Systems
  - Moving (Running) Average

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

  - Maximum

$$y[n] = \max\{x[n], x[n-1], x[n-2]\}$$

  - Ideal Delay System

$$y[n] = x[n-n_o]$$

MEMORYLESS SYSTEM

- Memoryless System
  - A system is memoryless if the output y[n] at every value of n depends only on the input x[n] at the same value of n
- Example Memoryless Systems
  - Square $$y[n] = \left(x[n]\right)^2$$
  - Sign $$y[n] = sign\left\{x[n]\right\}$$

MEMORYLESS SYSTEM

- Counter Example
  - Ideal Delay System

$$y[n] = x[n - n_o]$$

LINEAR SYSTEMS

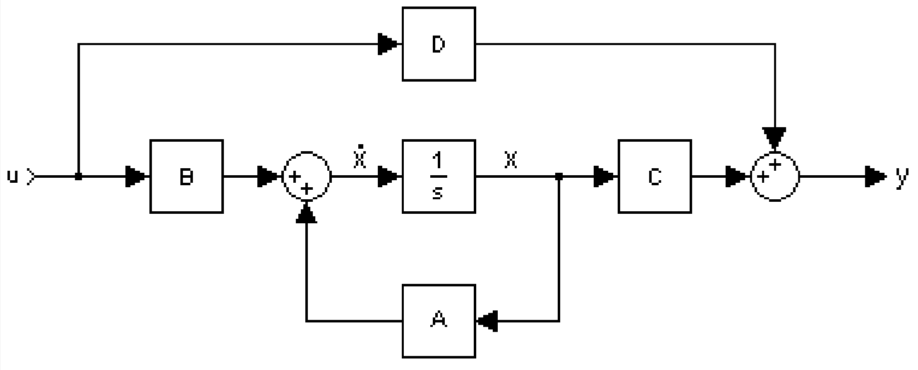
- Many times signal filtering is done with linear filters, which are an important instance of linear systems.

- Linear systems are those systems that satisfy the properties of superposition and scaling

- A transfer function is a popular alternative for the representation of linear systems
  - Another way of representation is by using state-space equations, which can easily deal with multivariable systems

STATE-SPACE EQUATIONS

- A State-Space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations or difference equations



$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

$$y(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)$$

$$\dot{\mathbf{x}}(t) := \frac{d}{dt}x(t)$$

LINEAR SYSTEMS

- Linear System:
  - Satisfies the principle of superposition
  - A system is linear if and only if

$$T\{x_1[n]+x_2[n]\}=T\{x_1[n]\}+T\{x_2[n]\} \quad \text{(additivity)}$$

$$\text{and}$$

$$T\{ax[n]\}=aT\{x[n]\} \quad \text{(scaling)}$$

LINEAR SYSTEMS

- Examples
  - Ideal Delay System

$$y[n] = x[n - n_o]$$

$$\begin{aligned}
T\{x_1[n] + x_2[n]\} &= x_1[n - n_o] + x_2[n - n_o] \\
T\{x_2[n]\} + T\{x_1[n]\} &= x_1[n - n_o] + x_2[n - n_o] \\
T\{ax[n]\} &= ax_1[n - n_o] \\
aT\{x[n]\} &= ax_1[n - n_o]
\end{aligned}$$

TIME-INVARIANT SYSTEMS

TIME-INVARIANT SYSTEM

- A System has a time-dependent system function that is not a direct function of time.
- The function depends *only* indirectly on the time-domain (via the input function)

TIME-DEPENDENT SYSTEM

- The time-dependent system function is a function of the time-dependent input function.
- Any direct dependence on the time-domain of the system function could be considered as a "time-varying system"

TIME-INVARIANT SYSTEMS

- Time-Invariant (shift-invariant) Systems
  - A time shift at the input causes corresponding time-shift at output

$$y[n] = T\{x[n]\} \Rightarrow y[n-n_o] = T\{x[n-n_o]\}$$

- Example
  - Square

$$y[n] = \left(x[n]\right)^2$$

Delay the input the output is $\mathrm{y}_1\left[n\right] = \left(x[n-n_o]\right)^2$

Delay the output gives $\mathrm{y}\left[\mathrm{n}\text{-}\mathrm{n}_\mathrm{o}\right] = \left(x[n-n_o]\right)^2$

TIME-INVARIANT SYSTEMS

- Time-Invariant (shift-invariant) Systems
  - A time shift at the input causes corresponding time-shift at output

$$y[n] = T\{x[n]\} \Rightarrow y[n-n_o] = T\left\{x[n-n_o]\right\}$$

- Counter Example
  - Compressor System

$$y[n] = x[Mn]$$

Delay the input the output is $\mathrm{y}_1\left[n\right] = x[Mn - n_o]$

Delay the output gives $\mathrm{y}\left[\mathrm{n}\text{-}\mathrm{n}_\mathrm{o}\right] = x\left[M\left(n-n_o\right)\right]$

CAUSAL SYSTEM

- Causality
  - A system is causal it's output is a function of only the current and previous samples

- Examples
  - Backward Difference $y[n] = x[n] - x[n-1]$

- Counter Example
  - Forward Difference $y[n] = x[n+1] + x[n]$

STABLE SYSTEM

- Stability (in the sense of bounded-input bounded-output BIBO)
  - A system is stable if and only if every bounded input produces a bounded output

$$|x[n]| \le B_x < \infty \Rightarrow \ |y[n]| \le B_y < \infty$$

- Example
  - Square

$$y[n] = \left(x[n]\right)^2$$

$$\text{if input is bounded by } |x[n]| \le B_x < \infty$$

$$\text{output is bounded by } |y[n]| \le B_x^2 < \infty$$

STABLE SYSTEM

- Counter Example
  - Log

$$y[n] = \log_{10}\left(\left|x[n]\right|\right)$$

$$\text{even if input is bounded by } \left|x[n]\right| \le B_x < \infty$$

$$\text{output not bounded for } \mathrm{x}\left[\mathrm{n}\right] = 0 \;\Rightarrow \mathrm{y}\left[0\right] = \log_{10}\left(\left|x\left[n\right]\right|\right) = -\infty$$

EXAMPLE 1

- Determine whether the following systems are linear:
  1. $y(n) = T[x(n)] = 3x^2(n)$
  2. $y(n) = 2x(n - 2) + 5$
  3. $y(n) = x(n + 1) - x(n - 1)$

EXAMPLE – TIPS

- Let $y_1[n] = T[x_1[n]]$ and $y_2[n] = T[x_2[n]]$
  - Determine the response of each system to the linear combination $a_1x_1[n]+a_2x_2[n]$
  - Check whether it is equal to the linear combination $a_1x_1[n] + a_2x_2[n]$
  - a_1 and a_2 are arbitrary constants.

EXAMPLE – SOLUTION

1. $y(n) = T[x(n)] = 3x^2(n)$:

   $T[a_1x_1(n) + a_2x_2(n)] = 3[a_1x_1(n) + a_2x_2(n)]^2$

   $= 3a_1{}^2x_1{}^2(n) + 3a_2{}^2x_2{}^2(n) + 6a_1a_2x_1(n)x_2(n)$

   which is not equal to

   $a_1y_1(n) + a_2y_2(n) = 3a_1{}^2x_1{}^2(n) + 3a_2{}^2x_2{}^2(n)$

   Hence the given system is nonlinear.

EXAMPLE – SOLUTION

2. $y(n) = 2x(n - 2) + 5$

$$T[a_1x_1(n) + a_2x_2(n)] = 2[a_1x_1(n-2) + a_2x_2(n-2)] + 5$$
$$= a_1y_1(n) + a_2y_2(n) - 5$$

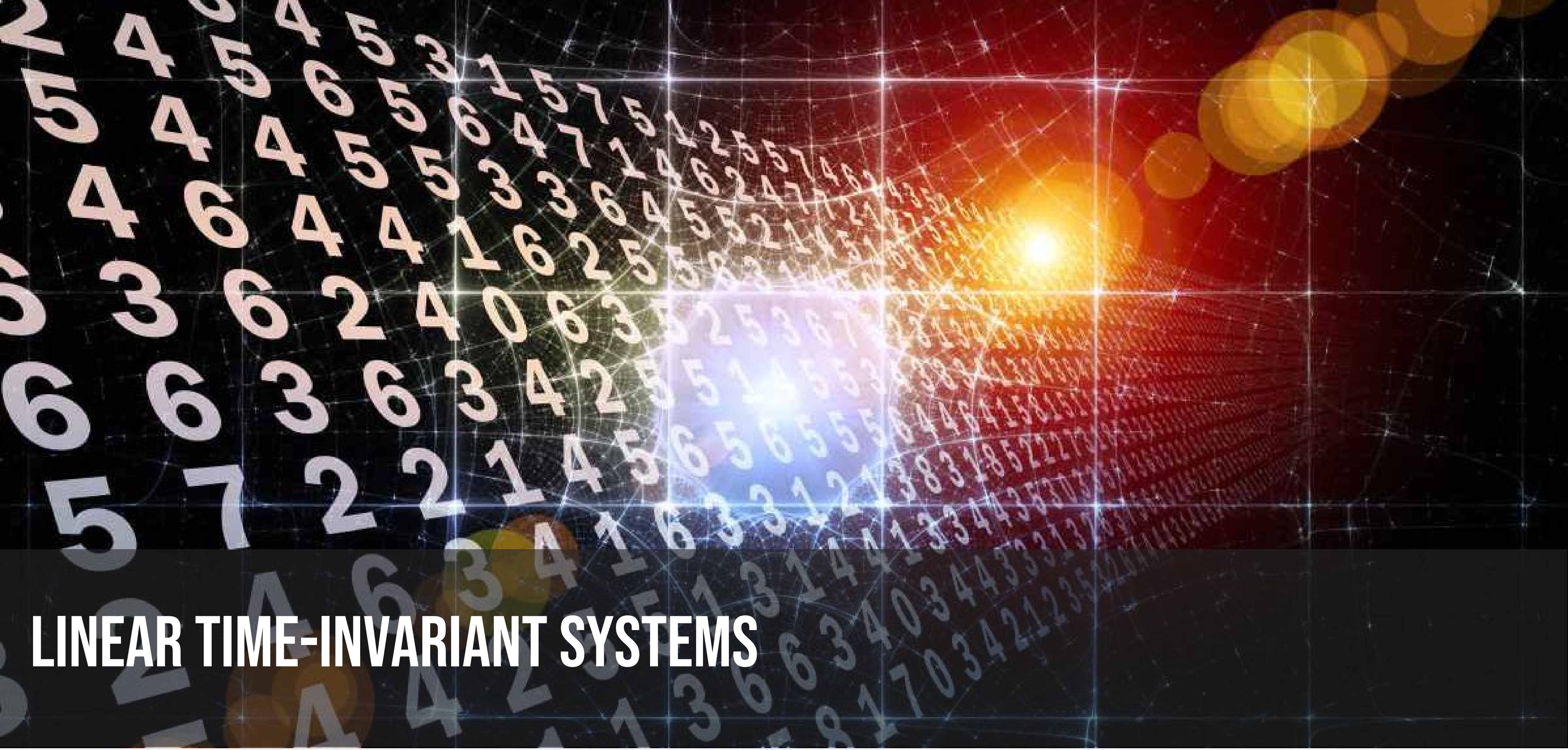
Clearly, the given system is nonlinear even though the input-output relation is a straight-line function.

EXAMPLE – SOLUTION

3. $y(n) = x(n + 1) - x(1 - n)$

$$\begin{aligned} T[a_1x_1(n) + a_2x_2(n)] &= a_1x_1(n + 1) + a_2x_2(n + 1) + a_1x_1(1 - n) + a_2x_2(1 - n) \\ &= a_1[x_1(n + 1) - x_1(1 - n)] + a_2[x_2(n + 1) - x_2(1 - n)] \\ &= a_1y_1(n) + a_2y_2(n) \end{aligned}$$

Hence the given system is linear



LINEAR TIME-INVARIANT SYSTEMS

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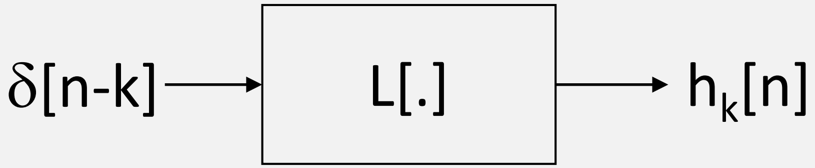
LINEAR-TIME INVARIANT SYSTEM

- A linear system in which an input-output pair, *x*(*n*) and *y*(*n*), is invariant to a shift *k* in time
- Special importance for their mathematical tractability
- Most signal processing applications involve LTI systems
- LTI system can be completely characterized by their impulse response



LINEAR-TIME INVARIANT SYSTEM

- Represent any input

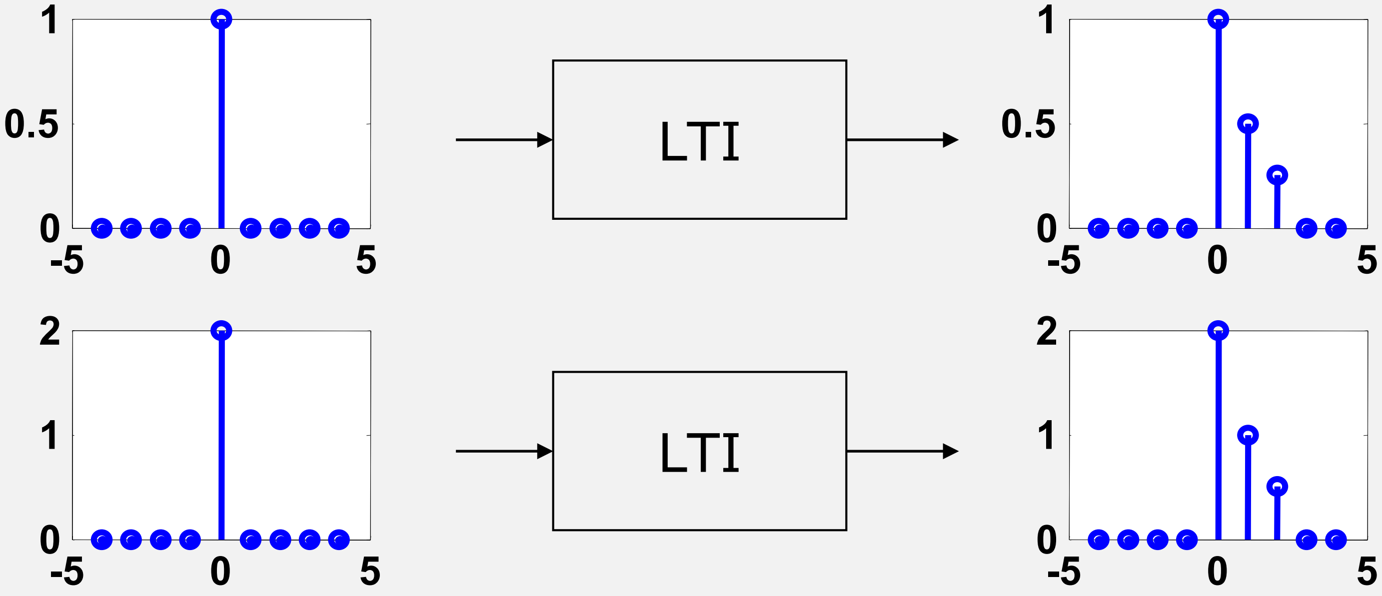
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- From time invariance we arrive at convolution

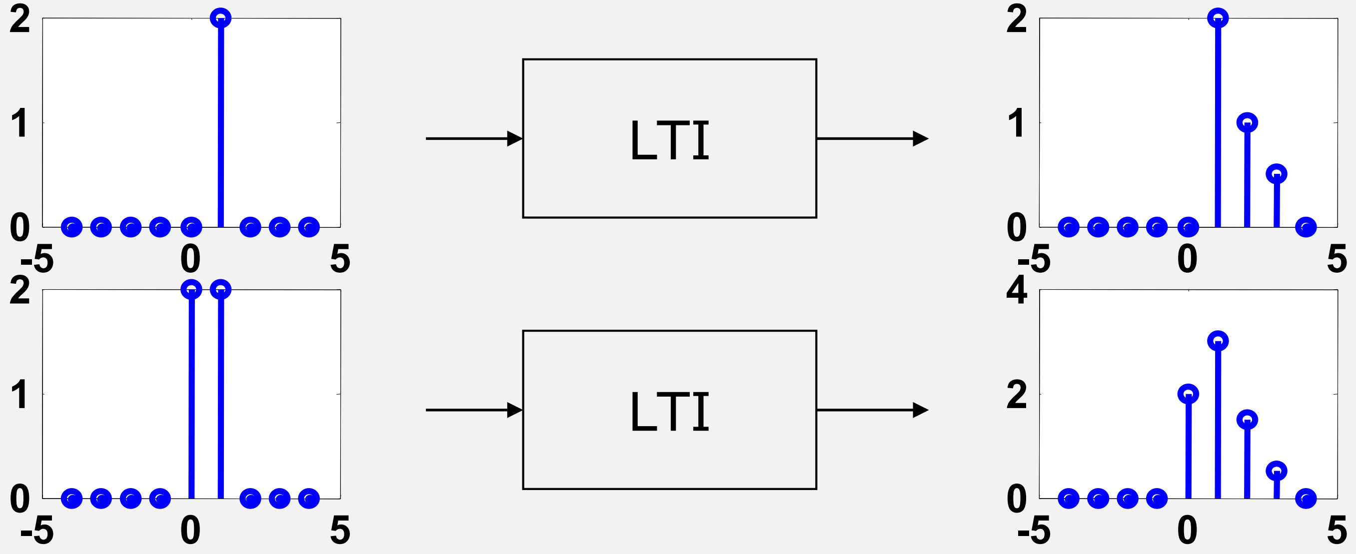
$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\ h[n-k] = x[k] * h[k]$$

LTI SYSTEM EXAMPLE



LTI SYSTEM EXAMPLE



PROPERTIES OF LTI SYSTEMS

- Convolution is commutative

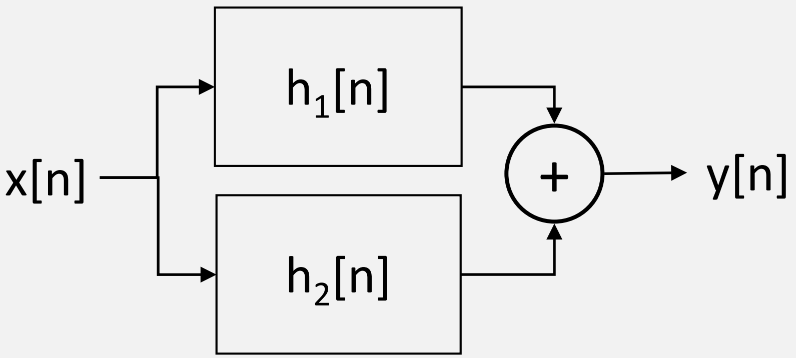
$$x[k]*h[k]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=h[k]*x[k]$$

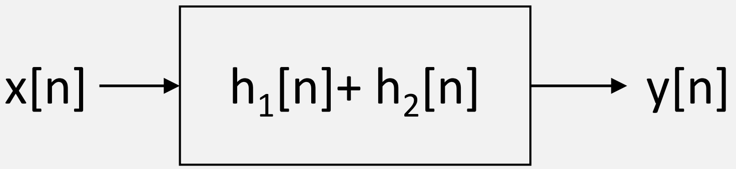
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad h[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]$$

PROPERTIES OF LTI SYSTEMS

- Convolution is distributive

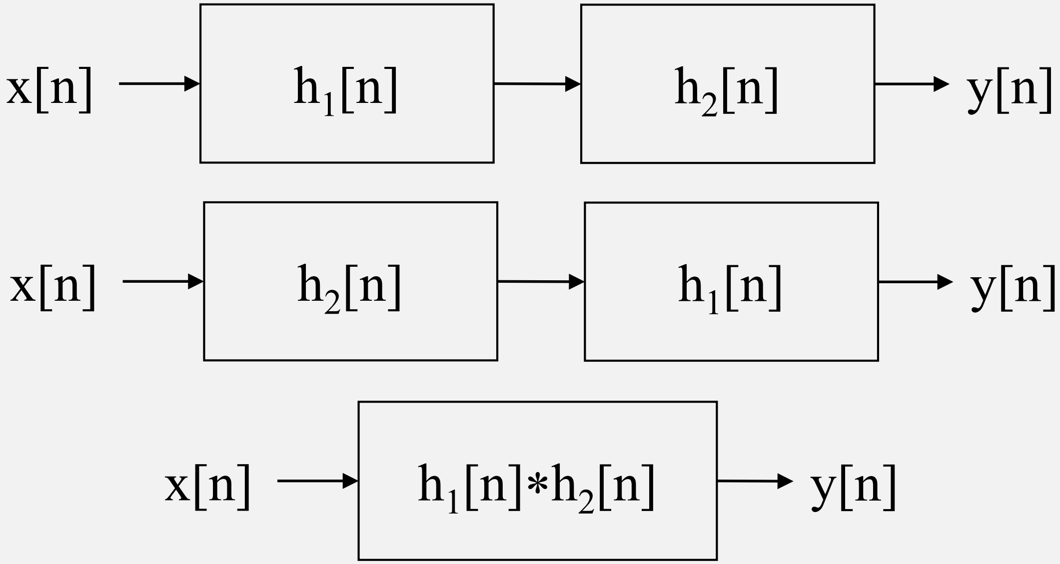
$$x[k]*\left(h_1[k]+h_2[k]\right)=x[k]*h_1[k]+x[k]*h_2[k]$$





PROPERTIES OF LTI SYSTEMS

- Cascade connection of LTI systems



STABLE LTI SYSTEMS

- An LTI system is (BIBO) stable if and only if
  - Impulse response is absolute summable $\sum_{k=-\infty}^{\infty} \left|h[k]\right| < \infty$
  - Let's write the output of the system as $\left|y[n]\right| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} \left|h[k]\right|\left|x[n-k]\right|$
  - If the input is bounded $\left|x[n]\right| \leq B_x$
  - Then the output is bounded by $\left|y[n]\right| \leq B_x \sum_{k=-\infty}^{\infty} \left|h[k]\right|$
  - The output is bounded if the absolute sum is finite

CAUSAL LTI SYSTEMS

- An LTI system is causal if and only if

$$h[k]=0 \quad \text{for k} < 0$$

EXAMPLE

- Determine whether the following linear systems are time-invariant

1. . $y(n)=L\left[x(n)\right]=10\sin(0.1\pi n)x(n)$
2. . $y(n)=L\left[x(n)\right]=x(n+1)-x(1-n)$
3. . $y(n)=L\left[x(n)\right]=\frac{1}{4}x(n)+\frac{1}{2}x(n-1)+\frac{1}{4}x(n-2)$

EXAMPLE

1. Compute the response $y_k(n) = L[x(n - k)]$ to the shifted input sequence
    - Subtracting k from the arguments of every input sequence term on the right-hand side of the linear transformation

2. Compare it to the shifted output sequence $y(n - k)$
    - Replacing every n by $(n - k)$ on the right-hand side of the linear transformation

EXAMPLE

1. $y(n) = L[x(n)] = 10 \sin(0.1\pi n)x(n)$:

   The response due to shifted input is

   $$y_k(n) = L[x(n-k)] = 10 \sin(0.1\pi n)x(n-k)$$

   while the shifted output is

   $$y(n-k) = 10 \sin[0.1\pi(n-k)]x(n-k) \neq y_k(n)$$

   Hence the given system is not time-invariant

EXAMPLE

2. $y(n) = L[x(n)] = x(n+1) - x(1-n)$

   The response due to shifted input is

$$y_k(n) = L[x(n-k)] = x(n+1-k) - x(1-n-k)$$

   while the shifted output is

$$y(n-k) = x(n-k) - x(1-[n-k]) = x(n+1-k) - x(1-n+k) \neq y_k(n).$$

   Hence the given system is not time-invariant.

EXAMPLE

3. .

$$y(n)=L\left[x(n)\right]=\frac{1}{4}x(n)+\frac{1}{2}x(n-1)+\frac{1}{4}x(n-2)$$

The response due to shifted input is

$$y_k(n)=L\left[x(n-k)\right]=\frac{1}{4}x(n-k)+\frac{1}{2}x(n-k-1)+\frac{1}{4}x(n-k-2)$$

while the shifted output is

$$y(n-k)=\frac{1}{4}x(n-k)+\frac{1}{2}x(n-k-1)+\frac{1}{4}x(n-k-2)=y_k(n)$$

Hence the given system is time-invariant.

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