PENGOLAHAN SINYAL DIGITAL

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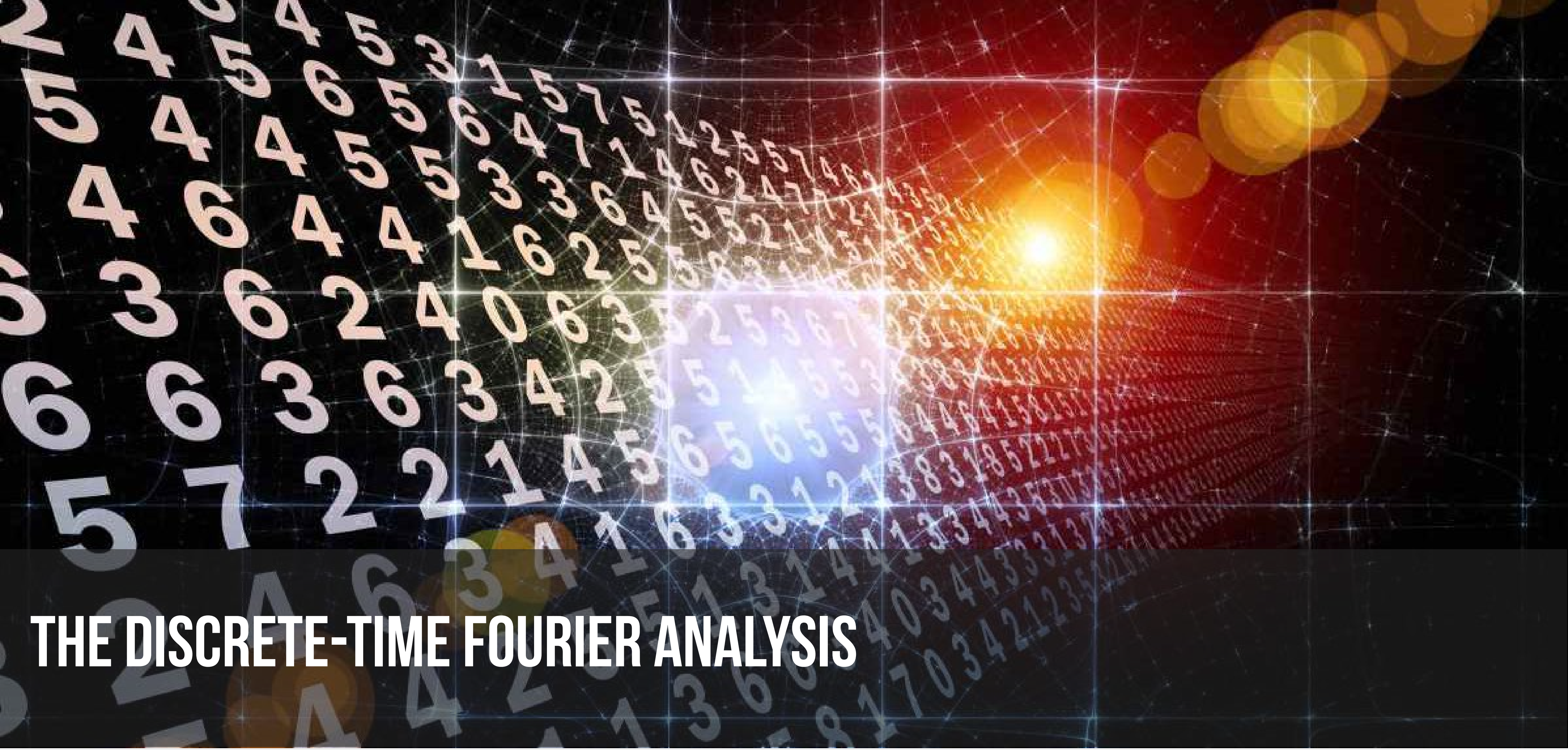

THE DISCRETE-TIME FOURIER ANALYSIS

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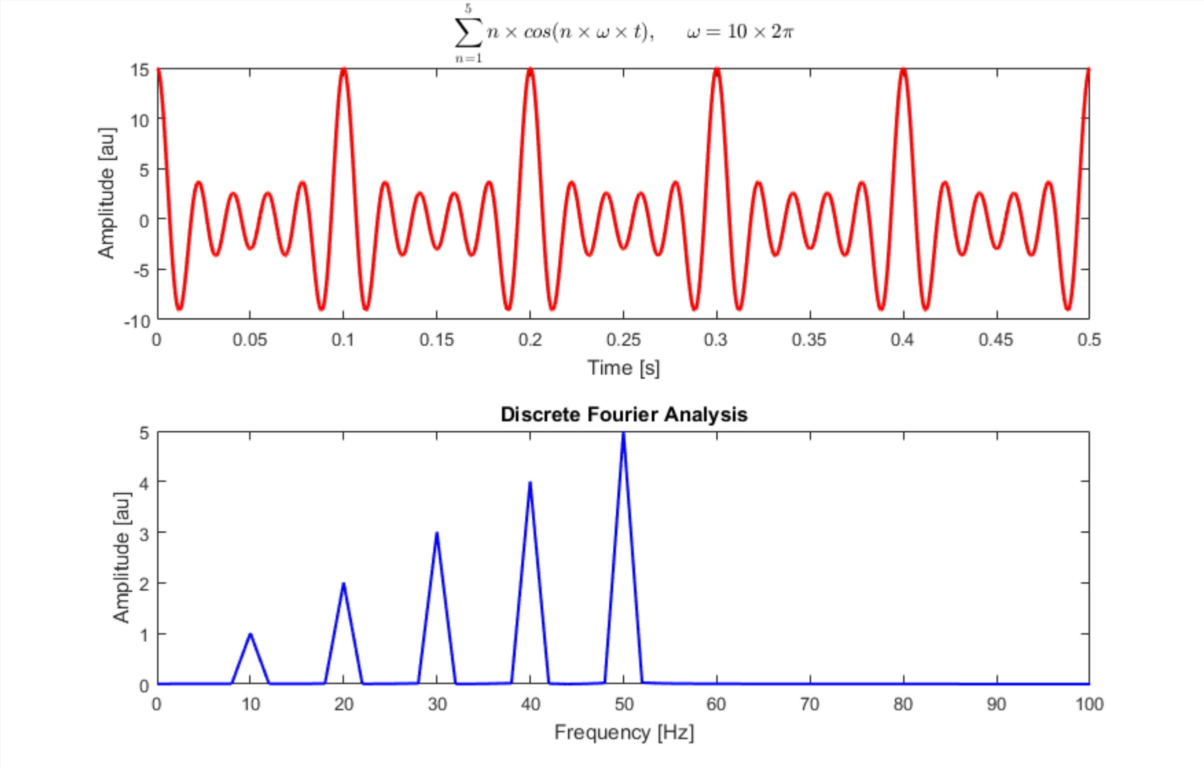
FOURIER TRANSFORM

- Fourier transform decompose signal into its frequency component



DISCRETE-TIME FOURIER TRANSFORM

- A linear and time-invariant system can be represented using its response to the unit sample sequence that

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n) * x(n)$$

- A linear and time-invariant system can be represented using the complex exponential signal set

$$\left\{ e^{j\omega n} \right\}$$

DISCRETE-TIME FOURIER TRANSFORM

- General form of DTFT

$$X\left(e^{j\omega}\right) \triangleq \mathcal{F}\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- The inverse discrete-time Fourier transform (IDTFT) of $X(e^{j\omega})$ is given by

$$x(n) \triangleq \mathcal{F}^{-1}\left[X\left(e^{j\omega}\right)\right] = \frac{1}{2\pi} \sum_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

DISCRETE-TIME FOURIER TRANSFORM

- The operator $\mathcal{F}[.]$ transforms a discrete signal $x(n)$ into a complex-valued continuous function $X(e^{j\omega})$ of real variable ω, called a digital frequency, which is measured in radians/sample

EXAMPLE

- Determine the discrete-time Fourier transform of $x(n) = (0.5)^n\, u(n)$.

- The sequence $x(n)$ is absolutely summable; therefore its discrete-time Fourier transform exists.

$$X\left(e^{j\omega}\right)=\sum_{-\infty}^{\infty} x\left(n\right)e^{-j\omega n}=\sum_{0}^{\infty}\left(0,5\right)^{n} e^{-j\omega n}$$

$$=\sum_{0}^{\infty}\left(0,5e^{-j\omega}\right)^{n}=\frac{1}{1-0,5e^{-j\omega}}=\frac{e^{j\omega}}{e^{j\omega}-0,5}$$

THE GEOMETRIC SERIES

- A one-sided exponential sequence of the form

$$\left\{\alpha^n, \quad n \geq 0\right\}$$

- The series converges for $|\alpha| < 1$, while the sum of its components converges to

$$\sum_{n=0}^{\infty} \alpha^n \rightarrow \frac{1}{1-\alpha}, \quad \text{for } |\alpha| < 1$$

DTFT PAIRS

Signal Type	*Sequence* $x(n)$	DTFT $X\left(e^{j\omega}\right)$, $-\pi \leq \omega \leq \pi$
Unit impulse	$\delta(n)$	1
Constant	1	$2\pi\delta(\omega)$
Unit step	$u(n)$	$\dfrac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$
Causal exponential	$\alpha^n u(n)$	$\dfrac{1}{1-\alpha e^{-j\omega}}$
Complex exponential	$e^{j\omega_0 n}$	$2\pi\delta(\omega - \omega_0)$
Cosine	$\cos(\omega_0 n)$	$\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$
Sine	$\sin(\omega_0 n)$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
Double exponential	$\alpha^{\lvert n\rvert} u(n)$	$\dfrac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}$

PROPERTIES OF THE DTFT

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PROPERTIES OF THE DTFT

- **Linearity**: The discrete-time Fourier transform is a linear transformation

$$\mathcal{F}\left[\alpha x_1(n)+\beta x_2(n)\right]=\alpha\mathcal{F}\left[x_1(n)\right]+\beta\mathcal{F}\left[x_2(n)\right]$$

  - for every α, β, $x_1(n)$, and $x_2(n)$.
- **Time shifting**: A shift in the time domain corresponds to the phase shifting.

$$\mathcal{F}\left[x(n-k)\right]=X\left(e^{j\omega}\right)e^{-j\omega k}$$

PROPERTIES OF THE DTFT

- **Frequency shifting**: Multiplication by a complex exponential corresponds to a shift in the frequency domain

$$\mathcal{F}\left[x(n)e^{j\omega_o n}\right] = X\left(e^{j(\omega-\omega_o)}\right)$$

- **Conjugation**: Conjugation in the time domain corresponds to the folding and conjugation in the frequency domain

$$\mathcal{F}\left[x^*(n)\right] = X^*\left(e^{-j\omega}\right)$$

PROPERTIES OF THE DTFT

- **Folding:** Folding in the time domain corresponds to the folding in the frequency domain

$$\mathcal{F}\left[x(-n)\right] = X\left(e^{-j\omega}\right)$$

- **Symmetries in real sequences:** We have already studied the conjugate symmetry of real sequences. These real sequences can be decomposed into their even and odd parts

$$x(n) = x_e(n) + x_o(n)$$

$$\mathcal{F}\left[x_e(n)\right] = \mathrm{Re}\left[X\left(e^{j\omega}\right)\right] \qquad \mathcal{F}\left[x_o(n)\right] = j\,\mathrm{Im}\left[X\left(e^{j\omega}\right)\right]$$

PROPERTIES OF THE DTFT

- **Convolution:** This is one of the most useful properties that makes system analysis convenient in the frequency domain

$$\mathcal{F}\left[x_1(n) * x_2(n)\right] = \mathcal{F}\left[x_1(n)\right]\mathcal{F}\left[x_2(n)\right] = X_1\left(e^{j\omega}\right)X_2\left(e^{j\omega}\right)$$

- **Multiplication:** This is a dual of the convolution property

$$\mathcal{F}\left[x_1(n) \cdot x_2(n)\right] = \mathcal{F}\left[x_1(n)\right] \circledast \mathcal{F}\left[x_2(n)\right]$$

$$\triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\left(e^{j\theta}\right)X_2\left(e^{j\omega-\theta}\right)d\theta$$

- This convolution-like operation is called a *periodic convolution* and hence denoted by $\circledast$

PROPERTIES OF THE DTFT

- **Energy:** The energy of the sequence $x(n)$ can be written as

$$\varepsilon_x = \sum_{-\infty}^{\infty} \left|x(n)\right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

$$= \int_{0}^{\pi} \frac{\left|X\left(e^{j\omega}\right)\right|^2}{\pi} d\omega$$

EXAMPLE 1

- Verify the linearity property of DTFT using real-valued finite duration sequences

$$\mathcal{F}\left[\alpha x_1(n)+\beta x_2(n)\right]=\alpha\mathcal{F}\left[x_1(n)\right]+\beta\mathcal{F}\left[x_2(n)\right]$$

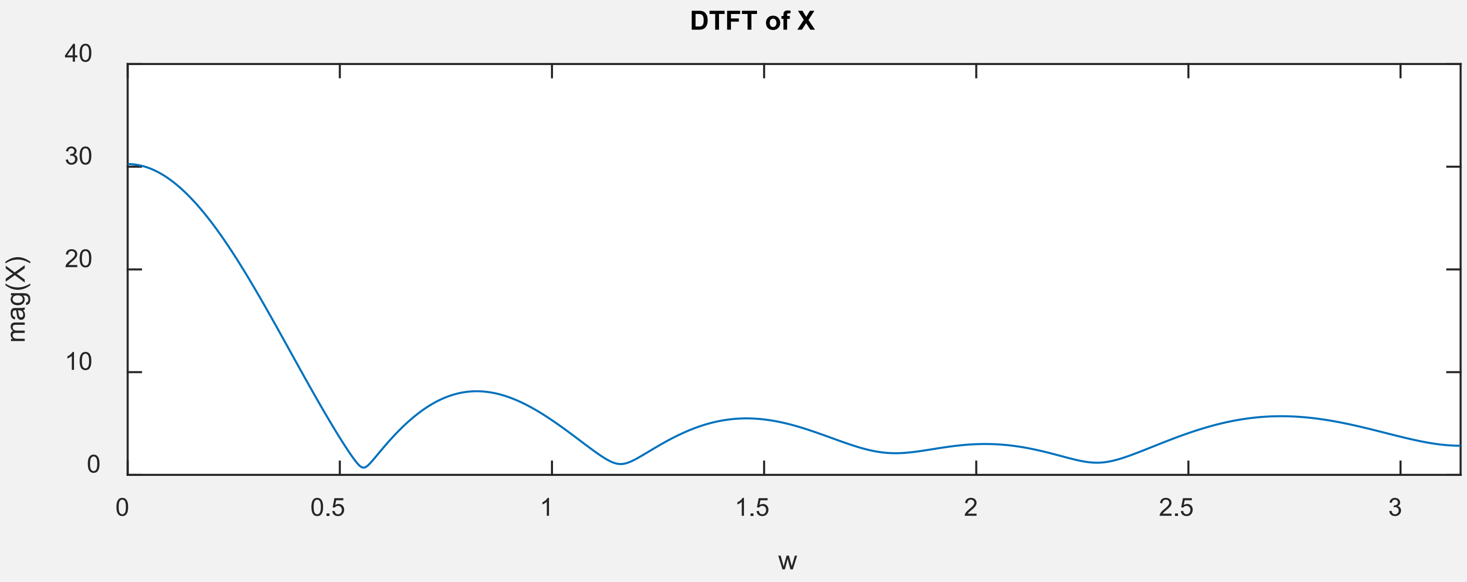
- Which $x_1(n)$ and $x_2(n)$ are two random sequences uniformly distributed between [0, 1] over $0 \le n \le 10$.

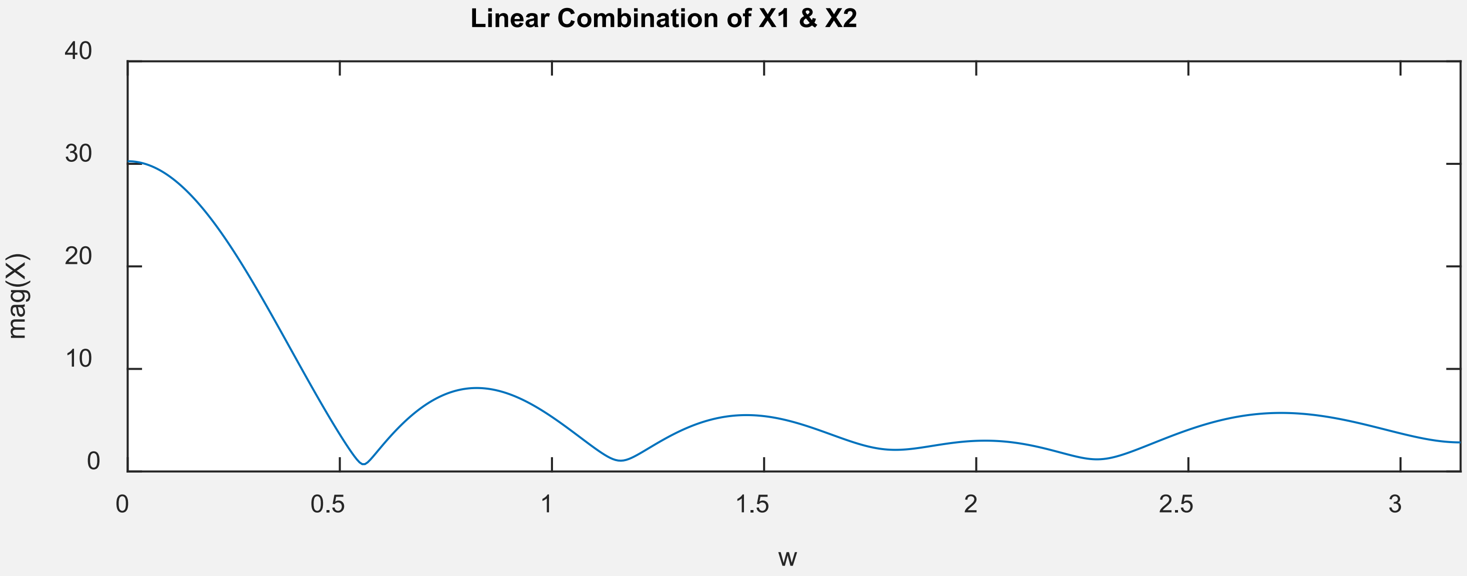
EXAMPLE 1

```
x1 = rand(1,11); x2 = rand(1,11); n = 0:10;
alpha = 2; beta = 3; k = 0:500; w = (pi/500)*k;
X1 = x1 * (exp(-j*pi/500)).^(n'*k); % DTFT of x1
X2 = x2 * (exp(-j*pi/500)).^(n'*k); % DTFT of x2
x = alpha*x1 + beta*x2; % Linear combination of x1 & x2
X = x * (exp(-j*pi/500)).^(n'*k); % DTFT of x

% verification
X_check = alpha*X1 + beta*X2; % Linear Combination of X1 & X2
error = max(abs(X-X_check)) % Difference
```

EXAMPLE 1


DTFT of X
40
30
20
10
0
mag(X)
0
0.5
1
1.5
2
2.5
3
w


Linear Combination of X1 & X2
40
30
20
10
0
mag(X)
0
0.5
1
1.5
2
2.5
3
w

EXAMPLE 2

- Verify the sample shift property

$$\mathcal{F}\left[x\left(n-k\right)\right]=X\left(e^{j\omega}\right)e^{-j\omega k}$$

- which $x(n)$ is a random sequence uniformly distributed between [0, 1] over $0 \leq n \leq 10$ and let $y(n) = x(n-2)$.

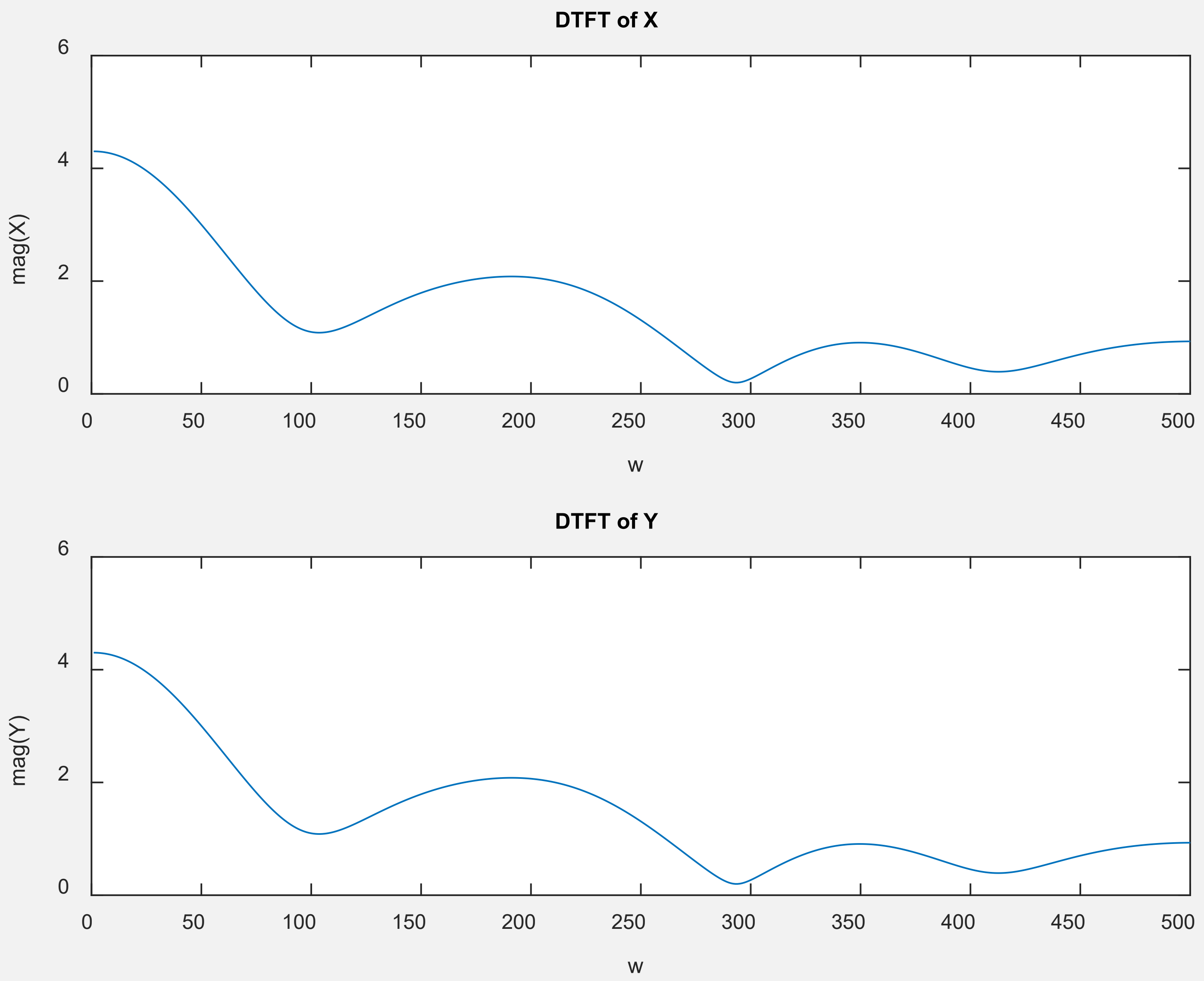
EXAMPLE 2

```
x = rand(1,11); n = 0:10;
k = 0:500; w = (pi/500)*k;
X = x * (exp(-j*pi/500)).^(n'*k); % DTFT of x

% signal shifted by two samples
y = x; m = n+2;
Y = y * (exp(-j*pi/500)).^(m'*k); % DTFT of y

% verification
Y_check = (exp(-j*2).^w).*X; % multiplication by exp(-j2w)
error = max(abs(Y-Y_check)) % Difference
```

EXAMPLE 2


DTFT of X
mag(X)
6
4
2
0
0
50
100
150
200
250
300
350
400
450
500
w
DTFT of Y
mag(Y)
6
4
2
0
0
50
100
150
200
250
300
350
400
450
500
w

EXAMPLE 3

- Verify the frequency shift property

$$\mathcal{F}\left[x(n)e^{j\omega_o n}\right] = X\left(e^{j(\omega-\omega_o)}\right)$$

- Which

$$x(n) = \cos(\pi n/2), \quad 0 \le n \le 100$$
$$y(n) = e^{j\pi n/4}x(n)$$

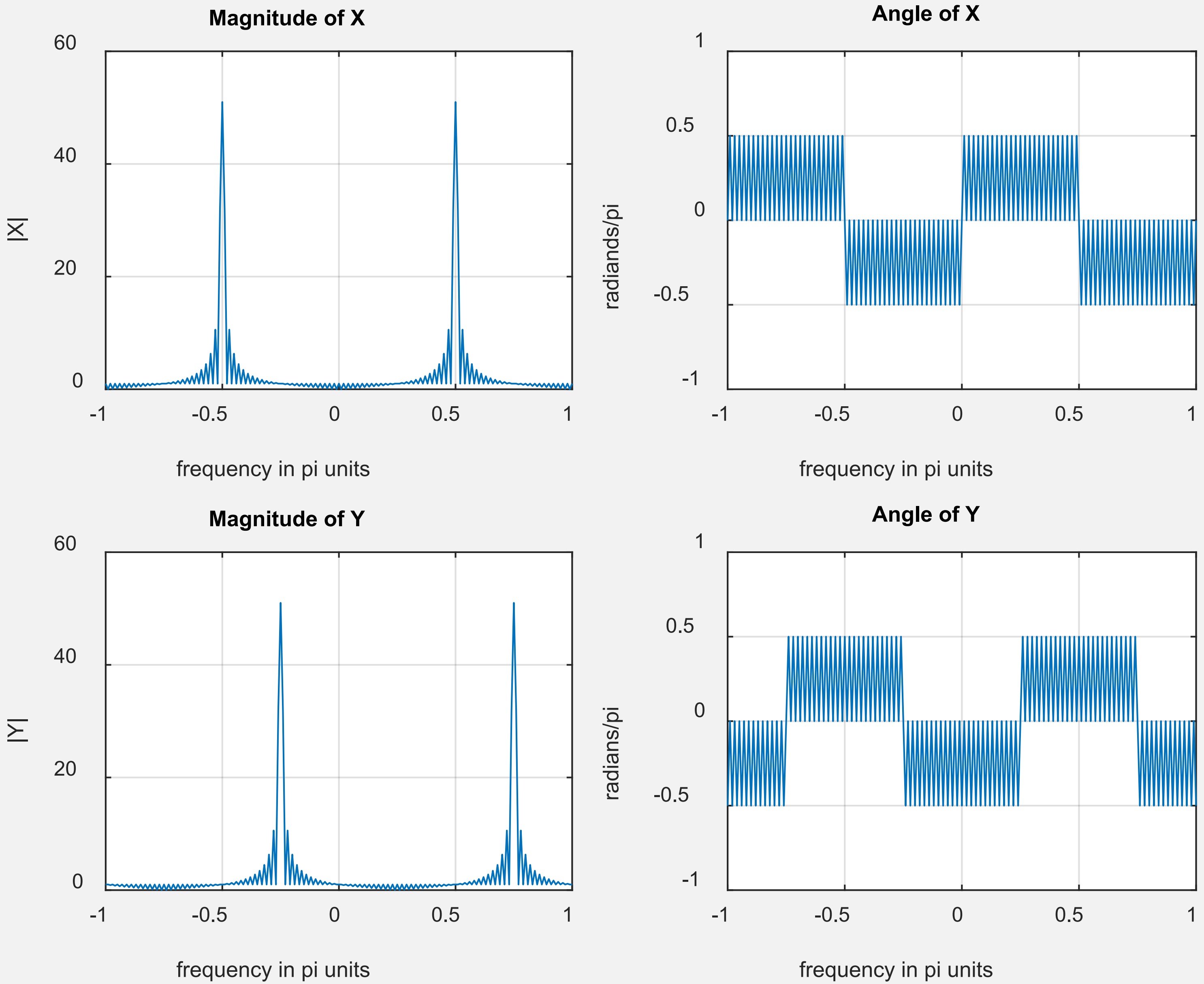
EXAMPLE 3

```
n = 0:100; x = cos(pi*n/2);
k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x
y = exp(j*pi*n/4).*x; % signal multiplied by exp(j*pi*n/4)
Y = y * (exp(-j*pi/100)).^(n'*k); % DTFT of y
```

EXAMPLE 3

```
% Graphical verification
subplot(2,2,1); plot(w/pi,abs(X)); grid; axis([-1,1,0,60])
xlabel('frequency in pi units'); ylabel('|X|')
title('Magnitude of X')
subplot(2,2,2); plot(w/pi,angle(X)/pi); grid; axis([-1,1,-1,1])
xlabel('frequency in pi units'); ylabel('radiands/pi')
title('Angle of X')
subplot(2,2,3); plot(w/pi,abs(Y)); grid; axis([-1,1,0,60])
xlabel('frequency in pi units'); ylabel('|Y|')
title('Magnitude of Y')
subplot(2,2,4); plot(w/pi,angle(Y)/pi); grid; axis([-1,1,-1,1])
xlabel('frequency in pi units'); ylabel('radians/pi')
title('Angle of Y')
```

EXAMPLE 3


Magnitude of X
60
40
20
0
|X|
-1
-0.5
0
0.5
1
frequency in pi units
Angle of X
1
0.5
0
-0.5
-1
radiands/pi
-1
-0.5
0
0.5
1
frequency in pi units
Magnitude of Y
60
40
20
0
|Y|
-1
-0.5
0
0.5
1
frequency in pi units
Angle of Y
1
0.5
0
-0.5
-1
radians/pi
-1
-0.5
0
0.5
1
frequency in pi units

EXAMPLE 4

- Verify the conjugation property

$$\mathcal{F}\left[x^*(n)\right] = X^*\left(e^{-j\omega}\right)$$

- Which $x(n)$ is a complex-valued random sequence over $-5 \le n \le 10$ with real and imaginary parts uniformly distributed between [0, 1]

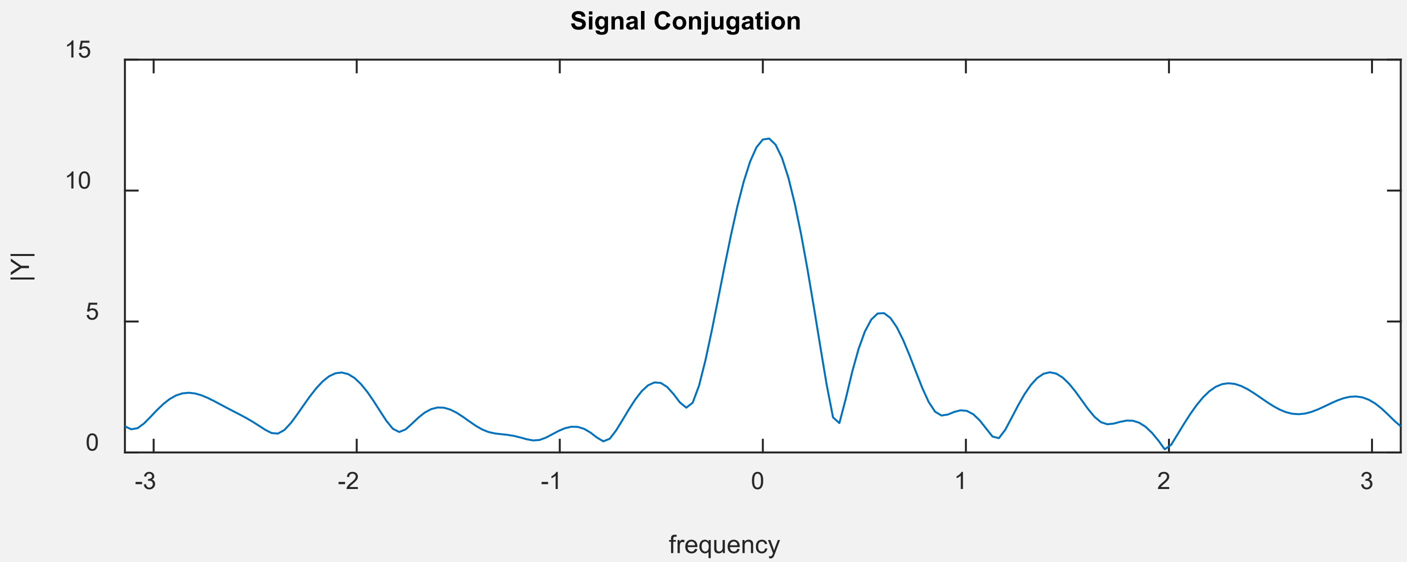
EXAMPLE 4

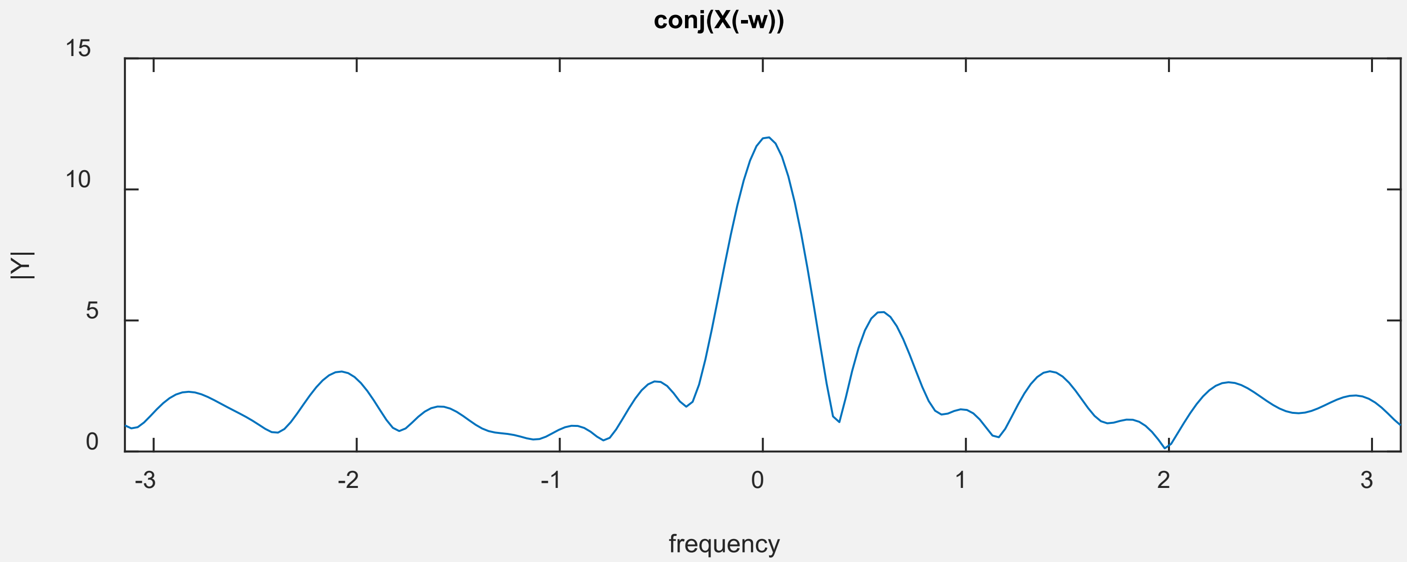
```
n = -5:10; x = rand(1,length(n)) + j*rand(1,length(n));
k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x

% conjugation property
y = conj(x); % signal conjugation
Y = y * (exp(-j*pi/100)).^(n'*k); % DTFT of y

% verification
Y_check = conj(fliplr(X)); % conj(X(-w))
error = max(abs(Y-Y_check)) % Difference
```

EXAMPLE 4


Signal Conjugation
15
10
5
0
|Y|
-3
-2
-1
0
1
2
3
frequency


conj(X(-w))
15
10
5
0
|Y|
-3
-2
-1
0
1
2
3
frequency

EXAMPLE 5

- Verify the folding property

$$\mathcal{F}\left[x(-n)\right] = X\left(e^{-j\omega}\right)$$

- Which $x(n)$ be a random sequence over $-5 \le n \le 10$ uniformly distributed between [0, 1].

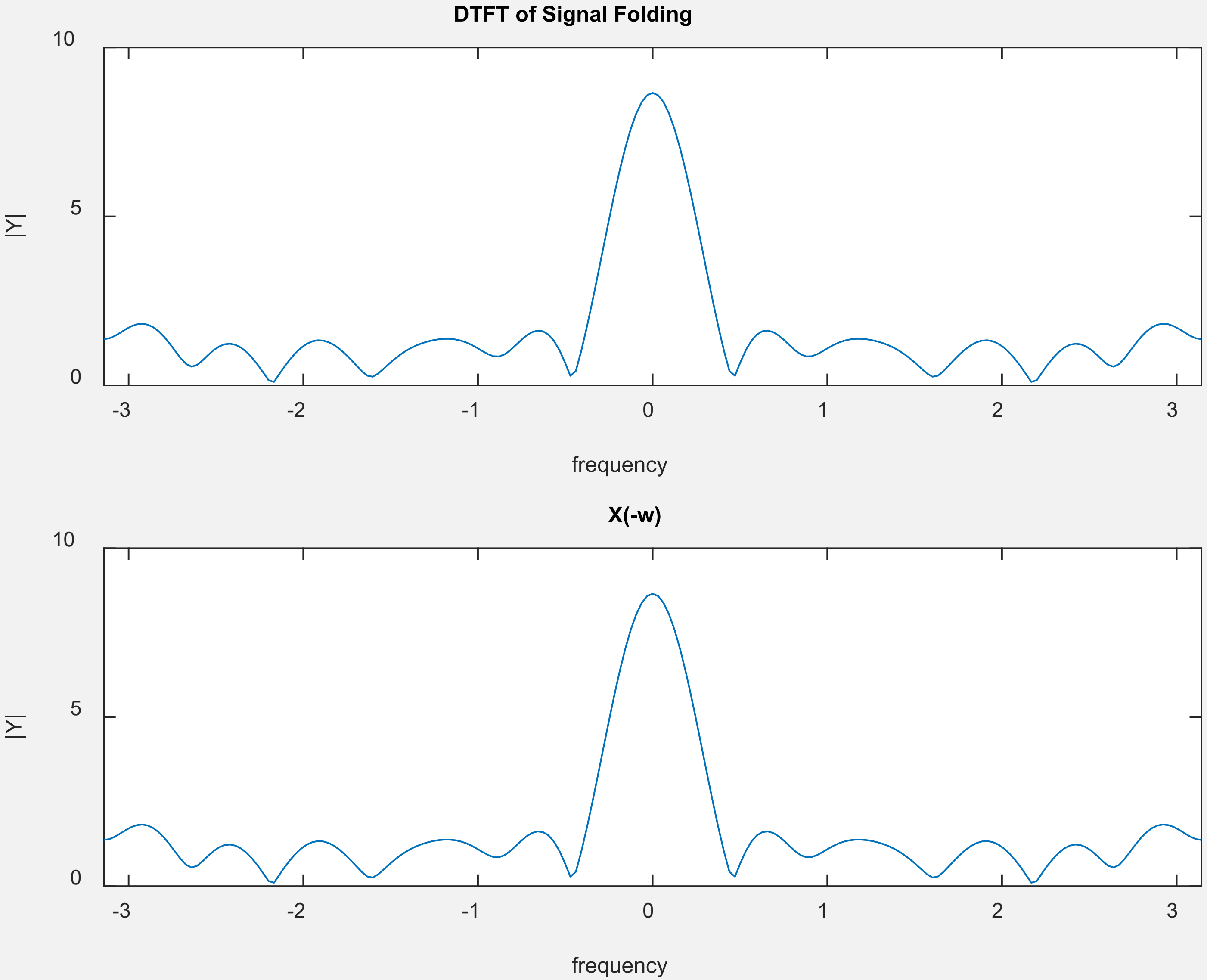
EXAMPLE 5

```
n = -5:10; x = rand(1,length(n));
k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x

% folding property
y = fliplr(x); m = -fliplr(n); % signal folding
Y = y * (exp(-j*pi/100)).^(m'*k); % DTFT of y

% verification
Y_check = fliplr(X); % X(-w)
error = max(abs(Y-Y_check)) % Difference
```

EXAMPLE 5


DTFT of Signal Folding
10
5
0
|Y|
-3
-2
-1
0
1
2
3
frequency
X(-w)
10
5
0
|Y|
-3
-2
-1
0
1
2
3
frequency

EXAMPLE 6

- Verify the symmetry property of real signals

$$x(n) = x_e(n) + x_o(n)$$

$$\mathcal{F}\left[x_e(n)\right] = \mathrm{Re}\left[X\left(e^{j\omega}\right)\right] \qquad \mathcal{F}\left[x_o(n)\right] = j\,\mathrm{Im}\left[X\left(e^{j\omega}\right)\right]$$

- Which $x(n) = \sin(\pi n/2),\ -5 \leq n \leq 10$

EXAMPLE 6

```
n = -5:10; x = sin(pi*n/2);
k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x
% signal decomposition
[xe,xo,m] = evenodd(x,n); % even and odd parts
XE = xe * (exp(-j*pi/100)).^(m'*k); % DTFT of xe
XO = xo * (exp(-j*pi/100)).^(m'*k); % DTFT of xo
% verification
XR = real(X); % real part of X
error1 = max(abs(XE-XR)) % Difference
XI = imag(X); % imag part of X
```

EXAMPLE 6

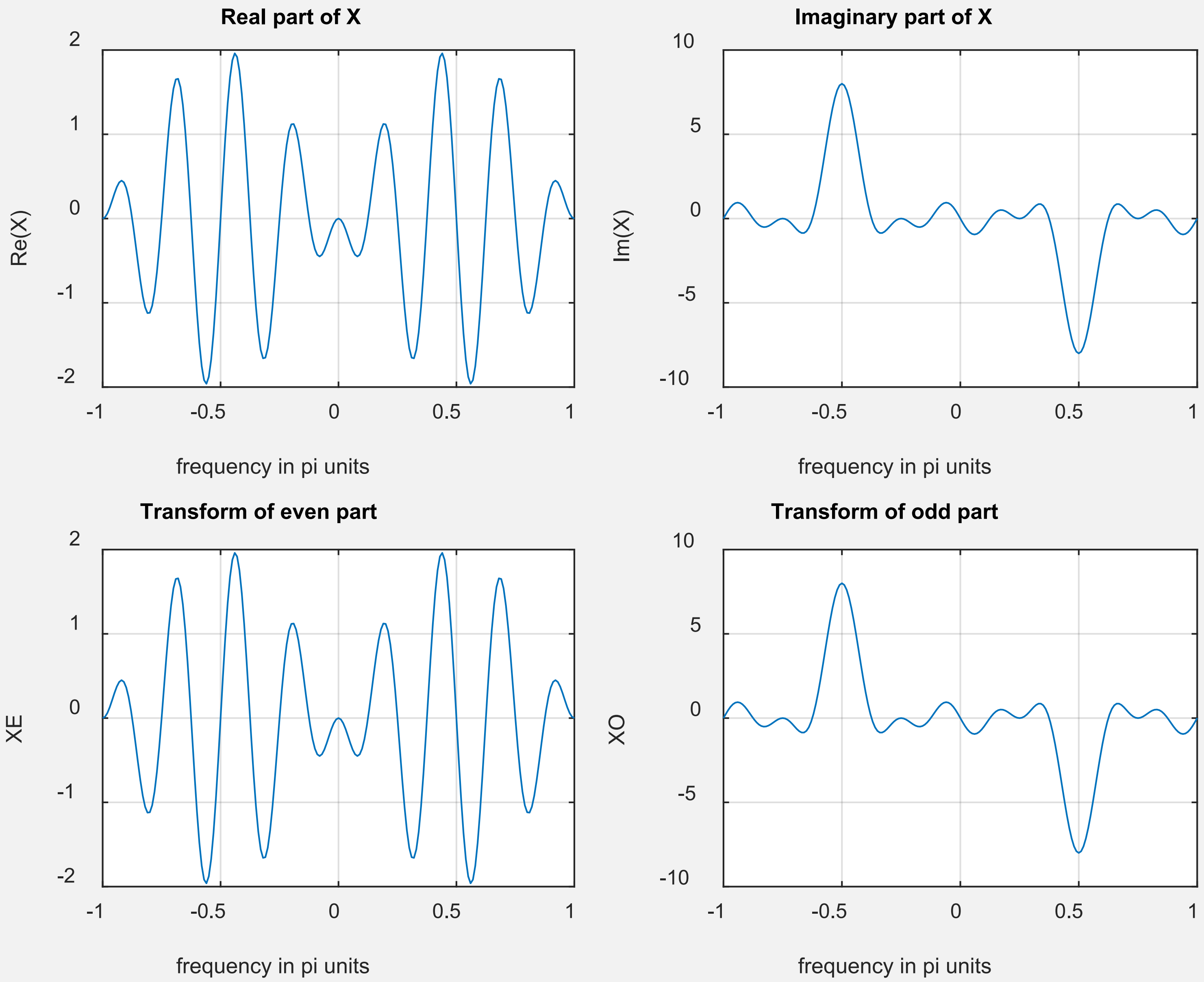
```
error2 = max(abs(XO-j*XI)) % Difference
error2 = 1.8033e-019

% graphical verification
subplot(2,2,1); plot(w/pi,XR); grid; axis([-1,1,-2,2])
xlabel('frequency in pi units'); ylabel('Re(X)');
title('Real part of X')
subplot(2,2,2); plot(w/pi,XI); grid; axis([-1,1,-10,10])
xlabel('frequency in pi units'); ylabel('Im(X)');
title('Imaginary part of X')
```

EXAMPLE 6

```
subplot(2,2,3); plot(w/pi,real(XE)); grid; axis([-1,1,-2,2])
xlabel('frequency in pi units'); ylabel('XE');
title('Transform of even part')
subplot(2,2,4); plot(w/pi,imag(XO)); grid; axis([-1,1,-10,10])
xlabel('frequency in pi units'); ylabel('XO');
title('Transform of odd part')
```

MATLAB SCRIPT



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