UNIT-2

DISCRETE FOURIER SERIES & TRANSFORM

DISCRETE FOURIER SERIES :- < PERIODIC SIGNALS >

Consider a periodic sequence $x_p(n)$ with a period of 'N' samples so that,

$$x_p(n) = x_p(n+lN) \longrightarrow (1)$$

Since $x_p(n)$ is periodic, it can be represented as a weighted sum of complex exponentials whose frequencies are integer multiples of the fundamental frequency $2\pi/N$. These periodic complex exponentials are of the form,

$$e^{j2\pi kn/N} = e^{j2\pi k(n+lN)/N} \longrightarrow (2)$$

where 'L' is an integer.

Thus, the periodic sequence $x_p(n)$ can be expressed as,

$$\boxed{x_p(n) = \sum_{K=0}^{N-1} X_p(K) e^{j2\pi Kn/N}} , \quad n = 0, 1, 2, \ldots N-1 \longrightarrow (3)$$

where, $X_p(K)$, $K = 0, \pm 1, \pm 2, \ldots$ are called 'Discrete Fourier Series Coefficients'.

To obtain the DFS coefficients, $X_p(K)$ multiply both sides of eq (3) by $e^{-j2\pi mn/N}$ and summing the product from $n=0$ to $N-1$,

$$\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi mn/N} = \sum_{n=0}^{N-1} \sum_{K=0}^{N-1} X_p(K)\, e^{j2\pi (K-m)n/N}$$

$$= \sum_{K=0}^{N-1} X_p(K) \sum_{n=0}^{N-1} e^{j2\pi (K-m)n/N} \longrightarrow (4)$$

Using the formula,

$$\sum_{n=0}^{N-1} e^{j2\pi (K-m)n/N} = N \quad \text{for } K-m = 0, \pm N, \pm 2N, \ldots$$

$N\,\delta(K-m)$

$$= 0 \quad \text{otherwise} \quad \left[\because \sum_{n=0}^{N-1} a^n = N, \; a=1 \right.$$

$$\left. = \frac{1-a^N}{1-a}, \; a \neq 1 \right]$$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi mn/N} = N X_p(m)$$

$= \sum_{K=0}^{N-1} X_p(K)\, N\,\delta(K-m) = N \sum_{K} X_p(K)\,\delta(K-m) = N X_p(m)$

Changing the index from 'm' to 'K' the Fourier Series coefficients $X_p(K)$ are obtained from $x_p(n)$ by the relation,

$$\boxed{X_p(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi Kn/N}} \longrightarrow (5)$$

Eq's (3) and (5) are referred to as Discrete Fourier Series and Inverse Discrete Fourier Series respectively.

Frequency Analysis of Discrete-Time Periodic Signals:-

Synthesis Equation: $x_p(n) = \sum_{K=0}^{N-1} X_p(K)\, e^{j2\pi Kn/N}$

Analysis Equation: $X_p(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi Kn/N}$

Both the sequences $x_p(n)$ and $X_p(K)$ are periodic with period 'N' samples.

$$\boxed{DFS[x_p(n)] = X_p(K)}$$

Properties of Discrete Fourier Series :-

1. **Periodicity:** Discrete Fourier Series, $X_p(K)$ is periodic with a period of 'N' ie

$$X_p(K+N) = X_p(K)$$

Proof:

$$X_p(K+N) = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi(K+N)n/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}\, e^{-j2\pi Nn/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi Kn/N}\, e^{-j2\pi n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi Kn/N} \quad \left[\because e^{-j2\pi n} = 1\right]$$

$$= X_p(K)$$

2. **Linearity:** Consider two periodic sequences $x_{1p}(n)$, $x_{2p}(n)$ both with period 'N', such that

$$DFS[x_{1p}(n)] = X_{1p}(K) \text{ and}$$

$$DFS[x_{2p}(n)] = X_{2p}(K) \text{ then,}$$

$$DFS[a\,x_{1p}(n) + b\,x_{2p}(n)] = a\,X_{1p}(K) + b\,X_{2p}(K)$$

Proof: $DFS[a\,x_{1p}(n) + b\,x_{2p}(n)] = \frac{1}{N}\sum_{n=0}^{N-1}[a\,x_{1p}(n) + b\,x_{2p}(n)]\,e^{-j2\pi kn/N}$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[a\,x_{1p}(n)\,e^{-j2\pi kn/N} + b\,x_{2p}(n)\,e^{-j2\pi kn/N}\right]$$

$$= \frac{a}{N}\sum_{n=0}^{N-1} x_{1p}(n)e^{-j2\pi kn/N} + \frac{b}{N}\sum_{n=0}^{N-1} x_{2p}(n)\,e^{-j2\pi kn/N}$$

$$= a\,X_{1p}(K) + b\,X_{2p}(K)$$

3. Time Shifting: If $x_p(n)$ is periodic with period 'N' samples

and, $DFS[x_p(n)] = X_p(K)$, then

$$DFS[x_p(n-m)] = e^{-j2\pi mK/N}\,X_p(K)$$

where, $x_p(n-m)$ is a shifted version of $x_p(n)$

Proof: $DFS[x_p(n-m)] = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n-m)\,e^{-j2\pi kn/N}$

Let $q = n-m \Rightarrow n = q+m$

$$\Rightarrow DFS[x_p(n-m)] = \frac{1}{N}\sum_{q=-m}^{N-1-m} x_p(q)\,e^{-j2\pi K(q+m)/N}$$

$$= \frac{1}{N}\sum_{q=0}^{N-1} x_p(q)\,e^{-j2\pi Kq/N}\,e^{-j2\pi Km/N}$$

Note: Since x_p is periodic, the limits of summation can be changed.

$$= e^{-j2\pi Km/N}\,\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\,e^{-j2\pi nK/N}$$ [Replacing dummy variable 'q' with 'n']

$$= e^{-j2\pi mK/N} \, X_p(K)$$

4. Frequency Shifting :- If $x_p(n)$ is periodic with a period of 'N' samples and $DFS[x_p(n)] = X_p(K)$, then

$$DFS\left[e^{j2\pi Ln/N} x_p(n)\right] = X_p(K-L)$$

where $X_p(K-L)$ is a shifted version of $X_p(K)$

Proof:
$$DFS\left[e^{j2\pi Ln/N} x_p(n)\right] = \frac{1}{N}\sum_{n=0}^{N-1}\left\{e^{j2\pi Ln/N} x_p(n)\right\} e^{-j2\pi Kn/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi(K-L)n/N}$$

$$= X_p(K-L)$$

5. Periodic Convolution: Let $x_{1p}(n)$ and $x_{2p}(n)$ be two periodic sequences with a period of 'N' samples, such that

$$DFS[x_{1p}(n)] = X_{1p}(K) \text{ and}$$

$$DFS[x_{2p}(n)] = X_{2p}(K) \text{ then}$$

$$DFS[x_{1p}(n) \circledast x_{2p}(n)] = N X_{1p}(K)\, X_{2p}(K)$$

where, $x_{1p}(n) \circledast x_{2p}(n) = \sum_{m=0}^{N-1} x_{1p}(m)\, x_{2p}(n-m)$

Proof:
$$DFS[x_{1p}(n) \circledast x_{2p}(n)] = \frac{1}{N}\sum_{n=0}^{N-1}[x_{1p}(n) \circledast x_{2p}(n)]\, e^{-j2\pi Kn/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x_{1p}(m)\, x_{2p}(n-m)\, e^{-j2\pi Kn/N}$$

Interchanging the Summations,

$$= \frac{N}{N} \sum_{m=0}^{N-1} x_{1p}(m) \frac{1}{N} \sum_{n=0}^{N-1} x_{2p}(n-m)\, e^{-j2\pi kn/N}$$

$$= N \frac{1}{N} \sum_{m=0}^{N-1} x_{1p}(m)\, X_{2p}(K)\, e^{-j2\pi Km/N}$$ [From Time shifting Property]

$$= N X_{2p}(K) \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} x_{1p}(n)\, e^{-j2\pi Kn/N}}$$ [Replacing dummy variable 'm' with 'n']

$$= N X_{2p}(K)\, X_{1p}(K)$$

$$= N X_{1p}(K)\, X_{2p}(K)$$

6 Multiplication In Time Domain :

Let $x_{1p}(n)$ and $x_{2p}(n)$ be two periodic sequences with a period of 'N' samples, such that

$$DFS[x_{1p}(n)] = X_{1p}(K) \text{ and}$$

$$DFS[x_{2p}(n)] = X_{2p}(K) \text{ then}$$

$$DFS[x_{1p}(n)\, x_{2p}(n)] = \sum_{L=0}^{N-1} X_{1p}(L)\, X_{2p}(K-L)$$

$$= X_{1p}(K) \circledast X_{2p}(K)$$

Proof: $$IDFS[X_{1p}(K) \circledast X_{2p}(K)] = \sum_{K=0}^{N-1} [X_{1p}(K) \circledast X_{2p}(K)]\, e^{j2\pi Kn/N}$$

$$= \sum_{K=0}^{N-1} \left[\sum_{L=0}^{N-1} X_{1p}(L)\, X_{2p}(K-L) \right] e^{j2\pi Kn/N}$$

Interchanging the Summations

$$= \sum_{L=0}^{N-1} X_{1p}(L) \cdot \sum_{K=0}^{N-1} X_{2p}(K-L)\, e^{j2\pi Kn/N}$$

$$\cancel{X = \sum_{L=0}^{N-1} X_{1p}(L)\, e^{j2\pi KL/N} \cdot \sum_{K=0}^{N-1} X_{2p}(K-L)\, e^{j2\pi (K-L)n/N} \cdot X}$$

Let $K - L = m \Rightarrow K = L + m$

$$= x_{1p}(n) \cdot \sum_{m=-L}^{N-1-L} X_{2p}(m)\, e^{j2\pi mn/N}$$

$$= x_{1p}(n) \cdot \sum_{m=0}^{N-1} X_{2p}(m)\, e^{j2\pi mn/N}$$

$$= \cancel{x_{1p}(n) \cdot x_{2p}(n)}$$

$$= \sum_{l=0}^{N-1} X_p(l) \sum_{m=-l}^{N-1-l} X_{2p}(m)\, e^{\frac{j2\pi (l+m)n}{N}}$$

$$= \sum_{l=0}^{N-1} X_p(l)\, e^{\frac{j2\pi ln}{N}} \sum_{m=0}^{N-1} X_{2p}(m)\, e^{\frac{j2\pi mn}{N}}$$

$$= x_{1p}(n)\, x_{2p}(n)$$

7 **Symmetry Properties:**

(i) $DFS[x_p^*(n)] = X_p^*(-K)$

(ii) $DFS[x_p^*(-n)] = X_p^*(K)$

(iii) $DFS[Re(x_p(n))] = X_{pe}(K)$

(iv) $DFS[j\, Im(x_p(n))] = X_{po}(K)$

(v) $DFS[x_{pe}(n)] = Re\{X_p(K)\}$

(vi) $DFS[x_{po}(n)] = j\, Im\{X_p(K)\}$

Proof: i, $\text{DFS}[x_p^*(n)] = \frac{1}{N}\sum_{n=0}^{N-1} x_p^*(n)\, e^{-j2\pi kn/N}$

$$= \frac{1}{N}\left[\sum_{n=0}^{N-1} x_p(n)\, e^{j2\pi kn/N}\right]^* \quad (\because (a+jb)^*(c+jd)^* = [(a+jb)(c+jd)]^*)$$

$$\Rightarrow \text{DFS}[x_p^*(n)] = \left[\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi(-k)n/N}\right]^*$$

$$\Rightarrow \text{DFS}[x_p^*(n)] = X_p^*(-k)$$

ii, $\text{DFS}[x_p^*(-n)] = \frac{1}{N}\sum_{n=0}^{N-1} x_p^*(-n)\, e^{-j2\pi kn/N}$

$$= \left[\frac{1}{N}\sum_{n=0}^{N-1} x_p(-n)\, e^{j2\pi kn/N}\right]^*$$

Let $m = -n$

$$= \left[\frac{1}{N}\sum_{m=0}^{-(N-1)} x_p(m)\, e^{-j2\pi km/N}\right]^*$$

$$= \left[\frac{1}{N}\sum_{m=0}^{N-1} x_p(m)\, e^{-j2\pi km/N}\right]^*$$

$$= X_p^*(k)$$

iii, Note : Real and Imaginary parts of a complex sequence are defined as,

$$\text{Re}\{x(n)\} = \frac{x(n) + x^*(n)}{2}$$

$$\text{Im}\{x(n)\} = \frac{x(n) - x^*(n)}{2j}$$

Even and odd parts of the complex sequence are defined as,

$$x_e(n) = \frac{x(n) + x^*(-n)}{2}$$

$$x_o(n) = \frac{x(n) - x^*(-n)}{2}$$

$$DFS\{Re[x_p(n)]\} = \frac{1}{N}\sum_{n=0}^{N-1} Re[x_p(n)]\, e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \left[\frac{x_p(n) + x_p^*(n)}{2}\right] e^{-j2\pi kn/N}$$

$$= \frac{1}{2}\left[\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi nk/N} + \frac{1}{N}\sum_{n=0}^{N-1} x_p^*(n)\, e^{-j2\pi kn/N}\right]$$

$$= \frac{1}{2}\left[X_p(K) + X_p^*(-K)\right]$$

$$= X_{pe}(K)$$

IV, $$DFS\{j\,Im[x_p(n)]\} = \frac{1}{N}\sum_{n=0}^{N-1} j\,Im[x_p(n)]\, e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \left[\frac{x_p(n) - x_p^*(n)}{2}\right] e^{-j2\pi kn/N}$$

$$= \frac{1}{2}\left[\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N} - \frac{1}{N}\sum_{n=0}^{N-1} x_p^*(n)\, e^{-j2\pi kn/N}\right]$$

$$= \frac{1}{2}\left[X_p(K) - X_p^*(-K)\right]$$

$$= X_{po}(K)$$

v. $DFS\{x_{pe}(n)\} = \frac{1}{N}\sum_{n=0}^{N-1} x_{pe}(n)\, e^{-j2\pi kn/N}$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[\frac{x_p(n)+x_p^*(-n)}{2}\right] e^{-j2\pi kn/N}$$

$$= \frac{1}{2}\left[\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N} + \frac{1}{N}\sum_{n=0}^{N-1} x_p^*(-n)\, e^{-j2\pi kn/N}\right]$$

$$= \frac{1}{2}\left[X_p(K)+X_p^*(K)\right]$$

$$= Re\{X_p(K)\}$$

vi. $DFS\{x_{po}(n)\} = \frac{1}{N}\sum_{n=0}^{N-1} x_{po}(n)\, e^{-j2\pi kn/N}$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[\frac{x_p(n)-x_p^*(-n)}{2}\right] e^{-j2\pi kn/N}$$

$$= \frac{1}{2}\left[\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N} - \frac{1}{N}\sum_{n=0}^{N-1} x_p^*(-n)\, e^{-j2\pi kn/N}\right]$$

$$= \frac{1}{2}\left[X_p(K)-X_p^*(K)\right]$$

$$= j\, Im\{X_p(K)\}$$

Power Density Spectrum of Periodic signals:

The average power of a discrete time periodic signal with period 'N' is defined as,

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 \longrightarrow (6)$$

DSP We shall derive the expression for P_x interms of Fourier coefficients $X_p(K)$.

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, x^*(n)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left[\sum_{K=0}^{N-1} X_p^*(K)\, e^{-j2\pi Kn/N}\right]$$

Interchanging the order of summations, we get

$$P_x = \sum_{K=0}^{N-1} X_p^*(K)\left[\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi Kn/N}\right]$$

$$= \sum_{K=0}^{N-1} X_p^*(K)\, X_p(K)$$

$$= \sum_{K=0}^{N-1} |X_p(K)|^2 \longrightarrow (7)$$

Comparing eq (6) and eq (7), we get

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{K=0}^{N-1} |X_p(K)|^2 \longrightarrow (8)$$

Eq (8) can be viewed as parseval's relation for discrete time periodic signals.

∴ Average power in the signal is the sum of the powers of individual frequency components.

DISCRETE TIME FOURIER TRANSFORM: <APERIODIC SIGNALS>

Consider a periodic sequence $x_p(n)$ with a period of 'N' samples.

As $N \to \infty$, the periodic sequence $x_p(n)$ becomes an aperiodic sequence $x(n)$ ie,

$$x(n) = \lim_{N \to \infty} x_p(n) \longrightarrow (6)$$

The Discrete Fourier Series of $x_p(n)$ can be expressed as,

$$x_p(n) = \sum_{K=0}^{N-1} X_p(K) e^{j2\pi nK/N}, \quad n = 0, \pm 1, \pm 2, \ldots \longrightarrow (7)$$

where $X_p(K)$ are the DFS coefficients given by

$$X_p(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nK/N} \longrightarrow (8)$$

Now, Let $\omega_K = \frac{2\pi}{N}(K)$. Then $X_p(K)$ can be rewritten as,

$$N X_p(K) = \sum_{n=0}^{N-1} x_p(n) e^{-j\omega_K n} \longrightarrow (9)$$

Since $x_p(n)$ is periodic the summation in the above equation can also be taken with in the range $n = -(\frac{N}{2}-1)$ to $N/2$

$$\Rightarrow N X_p(K) = \sum_{n=-(N/2-1)}^{N/2} x_p(n) e^{-j\omega_K n} \longrightarrow (10)$$

As $N \to \infty$, the periodic sequence $x_p(n)$ becomes aperiodic sequence $x(n)$ and $\frac{2\pi}{N} \to 0$ which implies that $\omega_K = \frac{2\pi K}{N}$ becomes a continuous variable 'ω' and limits on summation becomes $-\infty$ to ∞.

Thus as $N \to \infty$

$$N X_p(K) = \lim_{N\to\infty} \sum_{n=-(N/2-1)}^{N/2} x_p(n)\, e^{-j\omega_K n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\lim_{N\to\infty} x_p(n) \right) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= X(\omega)$$

$$\therefore \boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}} \quad \to (11)$$

Here $X(\omega)$ is called Fourier Transform of $x(n)$.

From eq (7), $x_p(n) = \sum_{K=0}^{N-1} X_p(K)\, e^{j2\pi nK/N}$

$$= \sum_{K=0}^{N-1} \frac{X(\omega)}{N}\, e^{j2\pi nK/N} \qquad \left[\because X_p(K) = \frac{N X_p(K)}{N} = \frac{X(\omega)}{N} \right]$$

Multiplying and dividing by a factor of 2π on RHS,

$$x_p(n) = \frac{1}{2\pi} \left(\frac{2\pi}{N} \right) \sum_{K=0}^{N-1} X(\omega)\, e^{j2\pi nK/N}$$

$$\Rightarrow x_p(n) = \frac{1}{2\pi} \sum_{K=0}^{N-1} X(\omega)\, e^{j2\pi nK/N} \left(\frac{2\pi}{N} \right) \qquad \left(\because \omega_K = \frac{2\pi K}{N} \right)$$

W.K.T $x(n) = \lim_{N\to\infty} x_p(n) = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{K=0}^{N-1} X(\omega)\, e^{j\omega_K n} \left(\frac{2\pi}{N} \right)$

$$= \frac{1}{2\pi} \lim_{N\to\infty} \left[\sum_{K=0}^{N-1} X(\omega)\, e^{j\omega_K n} \left(\frac{2\pi}{N} \right) \right]$$

As $N \to \infty$, $\frac{2\pi}{N} \to 0$ and $\omega_K = \frac{2\pi K}{N}$ becomes a continuous variable 'ω'. And $\frac{2\pi}{N}$ can be written as $d\omega$. Further summation becomes integral with limits 0 & 2π.

$$\Rightarrow x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

Limits can also be taken as $-\pi$ to π because $X(\omega)$ is periodic with a period of 2π.

1. Find the DFS coefficients of the signal shown in fig-1.

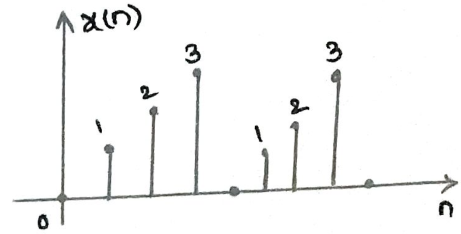


Fig-1

Sol:- The DFS coefficients are given by

$$X_p(K) = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi nK/N}$$

From fig N=4. Hence above eq'n becomes

$$X_p(K) = \frac{1}{4}\sum_{n=0}^{3} x_p(n)\, e^{-j2\pi nK/4} \quad ; K=0,1,2,3$$

$$X_p(0) = \frac{1}{4}\sum_{n=0}^{3} x_p(n) = \frac{1}{4}\left[x_p(0)+x_p(1)+x_p(2)+x_p(3)\right]$$

$$= \frac{1}{4}(0+1+2+3) = 6/4 = 3/2$$

$$X_p(1) = \frac{1}{4}\sum_{n=0}^{3} x_p(n)\, e^{-j2\pi n/4} = \frac{1}{4}\sum_{n=0}^{3} x_p(n)\, e^{-j\pi n/2}$$

$$= \frac{1}{4}\left[x_p(0) + x_p(1)e^{-j\pi/2} + x_p(2)e^{-j\pi} + x_p(3)e^{-j3\pi/2}\right]$$

$$= \frac{2j-2}{4}$$

$$X_p(2) = \frac{1}{4}\sum_{n=0}^{3} x_p(n)\, e^{-j\pi n} = \frac{1}{4}\left[x_p(0) + x_p(1)e^{-j\pi} + x_p(2)e^{-j2\pi} + x_p(3)e^{-j3\pi}\right]$$

$$= 1/4\left[0 + 1(-1) + 2(1) + 3(-1)\right] = -2/4$$

$$X_p(3) = \frac{1}{4}\sum_{n=0}^{3} x_p(n)\, e^{-j3\pi/2\, n}$$

$$= \frac{1}{4}\left[x_p(0) + x_p(1)\, e^{-j3\pi/2} + x_p(2)\, e^{-j3\pi} + x_p(3)\, e^{-j9\pi/2}\right]$$

$$= \frac{1}{4}\left[0 + 1(+j) + 2(-1) + 3(-j)\right]$$

$$= \frac{1}{4}\left[-2j-2\right] = \frac{-2-2j}{4}$$

Thus the DFS coefficients are $X_p(K) = \{3/2, -\frac{1}{2}+j1/2, -1/2, -\frac{1}{2}-j1/2\}$

2. Determine the spectra of the signals

(a) $x(n) = \cos\sqrt{2}\pi n$

(b) $x(n) = \cos \pi n/3$

(c) $x(n)$ is periodic with period $N=4$ and $x(n) = \{\underset{\uparrow}{1},1,0,0\}$

Sol:-

(a) For $\omega_0 = \sqrt{2}\pi$, we have $f_0 = 1/\sqrt{2}$. Since f_0 is not a rational number, the signal is not periodic. Consequently this signal can not be expanded in Fourier series. Nevertheless, the signal does not possess a spectrum. Its spectral content consists of the single frequency component at $\omega = \omega_0 = \sqrt{2}\pi$.

(b). In this case, $\omega_0 = \pi/3$, $f_0 = 1/6$ and hence $x(n)$ is periodic with fundamental period $N=6$.

$$X_p(K) = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi nK/N}$$

$$= \frac{1}{6}\sum_{n=0}^{5} x_p(n)\, e^{-j2\pi nK/6}$$

However, $x(n)$ can be expressed as,

$$x(n) = \cos\frac{2\pi n}{6} = \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6} \rightarrow (1)$$

which is already in the form of Exponential Fourier series.

ie $$x_p(n) = \sum_{K=0}^{N-1} X_p(K)\, e^{j2\pi nK/N} \rightarrow (2)$$

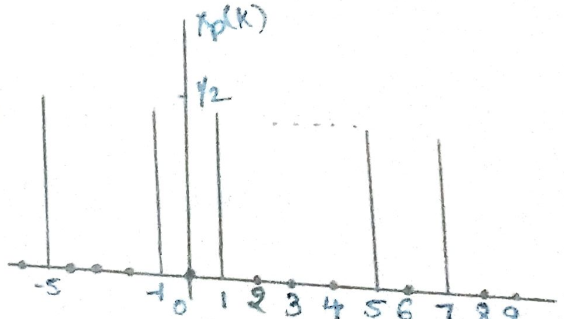
Comparing two exponentials in (1) & (2), it is apparent that $X_p(1) = 1/2$. The second exponential in $x(n)$ corresponds to the term $K=-1$ in eqn (2). However, this term can also be written as $e^{-j2\pi n/6} = e^{j2\pi(5-6)n/6} = e^{j2\pi(5n)/6}$

Which means that $X_p(-1) = X_p(5)$. As we know that, Fourier series coefficients form a periodic sequence of period N.

∴ We conclude that,

$X_p(0) = X_p(2) = X_p(3) = X_p(4) = 0$

$X_p(1) = 1/2 \quad X_p(5) = 1/2$



(c) W.K.T $$X_p(K) = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi nK/N}$$

$$= \frac{1}{4}\sum_{n=0}^{3} x_p(n)\, e^{-j2\pi nK/4} \quad ; K = 0, 1, 2, 3$$

$$\therefore X_p(0) = \frac{1}{4}\left[x_p(0) + x_p(1) + x_p(2) + x_p(3)\right] = 2/4 = 1/2$$

$$X_p(1) = \frac{1}{4}\left[x_p(0) + x_p(1)e^{-j\pi/2} + x_p(2)e^{-j\pi} + x_p(3)e^{-j3\pi/2}\right]$$

$\Rightarrow X_p(1) = \frac{1}{4}(1-j)$

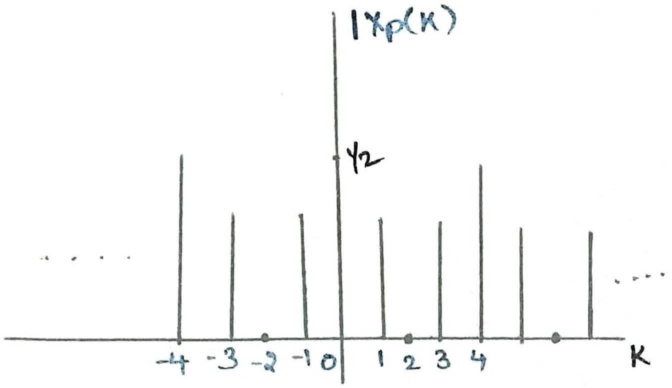
$X_p(2) = 0$

$X_p(3) = \frac{1}{4}(1+j)$

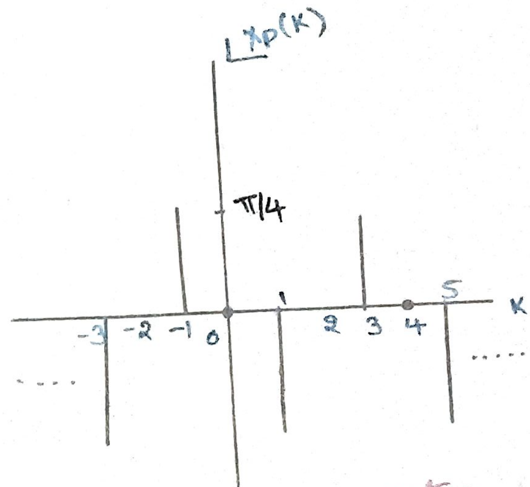
The magnitude and phase spectra are :

$|X_p(0)| = 1/2$; $|X_p(1)| = \sqrt{2}/4$; $|X_p(2)| = 0$, $|X_p(3)| = \sqrt{2}/4$

$\angle X_p(0) = 0$; $\angle X_p(1) = -\pi/4$, $\angle X_p(2)$ = undefined , $\angle X_p(3) = \pi/4$



(a) Magnitude spectra



(b) Phase spectra

3. Determine the Fourier series coefficients and the power density spectrum of the periodic signal shown in fig. below:

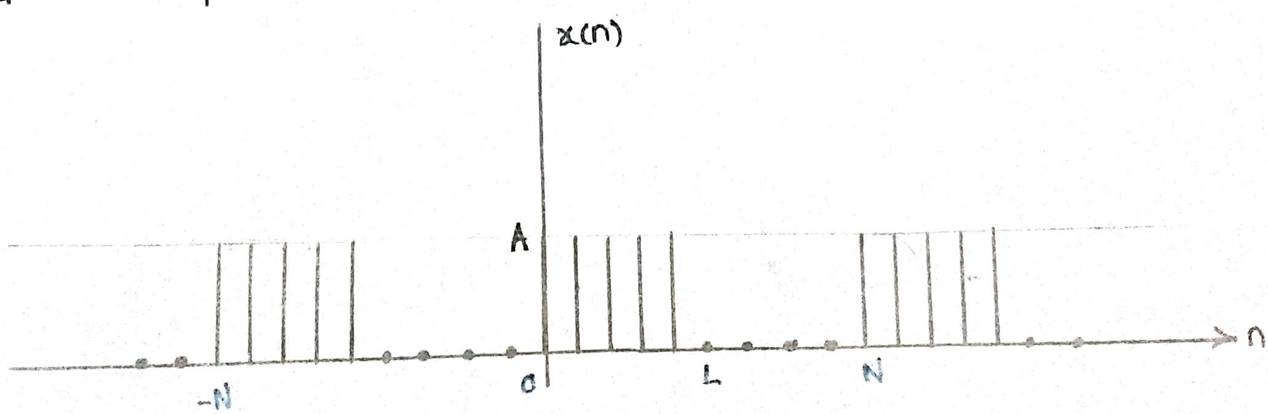


Fig: Discrete time periodic square wave signal.

Sol:- By applying the analyses equation to the signal shown in fig above, 2.18

we obtain,

$$X_p(K) = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi nK/N}$$

$$= \frac{1}{N}\sum_{n=0}^{L-1} A\, e^{-j2\pi nK/N} \quad ; K = 0, 1, 2, \dots N-1$$

$$= \frac{A}{N}\sum_{n=0}^{L-1} \left(e^{-j2\pi K/N}\right)^n$$

Note:

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a=1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

$$= \begin{cases} \dfrac{AL}{N} & ; K = 0 \\[2ex] \dfrac{A}{N}\,\dfrac{1-e^{-j2\pi KL/N}}{1-e^{-j2\pi K/N}} & ; K = 1, 2, \dots N-1 \end{cases}$$

The last expression can further be simplified, if we note that

$$\frac{1-e^{-j2\pi KL/N}}{1-e^{-j2\pi K/N}} = \frac{e^{-j\pi KL/N}}{e^{-j\pi K/N}}\;\frac{e^{j\pi KL/N} - e^{-j\pi KL/N}}{e^{j\pi K/N} - e^{-j\pi K/N}}$$

$$= e^{-j\pi K(L-1)/N}\;\frac{\sin(\pi KL/N)}{\sin(\pi K/N)}$$

Therefore,

$$X_p(K) = \begin{cases} \dfrac{AL}{N} & , k = 0, \pm N, \pm 2N, \dots \\[2ex] \dfrac{A}{N}\, e^{-j\pi K(L-1)/N}\;\dfrac{\sin(\pi KL/N)}{\sin(\pi K/N)}, & \text{otherwise} \end{cases}$$

The power density spectrum of this periodic signal is

$$|X_p(k)|^2 = \begin{cases} (AL/N)^2, & k = 0, \pm N, \pm 2N, \dots \\ (A/N)^2 \left(\dfrac{\sin \pi k L/N}{\sin \pi k/N}\right)^2, & \text{otherwise} \end{cases} \rightarrow (1)$$

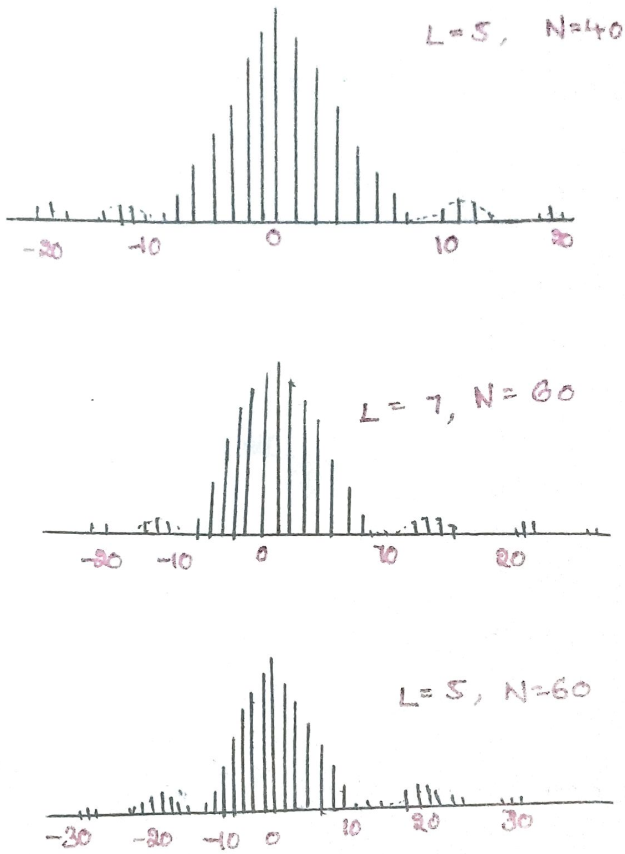


Fig: Plot of the PSD given by eq(1).

Discrete Time Fourier Transform Pair of Aperiodic Signals:-

Synthesis Equation: $x(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$

Analysis Equation: $X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

The two basic differences between Fourier Transform of discrete time finite energy signals and Fourier transform of finite energy analog signals are :

i, The Fourier Transform and hence the spectrum of continuous time signals have a frequency range of $-\infty$ to $+\infty$.

The frequency range of a discrete time signal is unique over the frequency range $(-\pi, \pi)$ or $(0, 2\pi)$.

Indeed $X(\omega)$ is periodic with period 2π.

ie $$X(\omega+2\pi K) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+2\pi K)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi K n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = X(\omega)$$

Proof for Periodicity Property

Thus the frequency range of discrete time signal is limited to $(-\pi, \pi)$ or $(0, 2\pi)$ and any frequency outside this interval is equivalent to a frequency within this interval.

ii, Since the signal is discrete in time, the Fourier transform of the signal involves a summation of terms instead of an integral.

Existence of Discrete Time Fourier Transform:

The Discrete time Fourier Transform does not exist for every aperiodic sequence. A Sufficient condition for the existence of DTFT for an aperiodic sequence $x(n)$ is,

$$\boxed{\sum_{n=-\infty}^{\infty} |x(n)| < \infty}$$

ie if a sequence $x(n)$ is absolutely summable then DTFT exists for that sequence.

Prob's on DFT (later topic)

1. Compute the DFT of each of the following finite length sequences considered to be of length N:

(a) $x(n) = \delta(n)$ (b) $x(n) = \delta(n-n_0)$ where $0 < n_0 < N$

(c) $x(n) = a^n, \; 0 \le n \le N-1$ (d) $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

Sol:-

(a) Given $x(n) = \delta(n)$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nK/N}$$

$$= \sum_{n=0}^{N-1} \delta(n)\, e^{-j2\pi nK/N} = 1.$$

ie $X(K) = 1$ for $0 \le K \le N-1$.

(b) Given $x(n) = \delta(n-n_0)$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nK/N} = \sum_{n=0}^{N-1} \delta(n-n_0)\, e^{-j2\pi nK/N}$$

$$= e^{-j2\pi K n_0/N} \quad \text{for } 0 \le K \le N-1$$

DSP

(c) Given $x(n) = a^n$

$$X(K) = \sum_{n=0}^{N-1} a^n e^{-j(2\pi/N)nK} = \sum_{n=0}^{N-1} \left[a e^{-j2\pi K/N}\right]^n \text{ for } 0 \le K \le N-1$$

$$= \frac{1 - a^N e^{-j2\pi K}}{1 - a e^{-j2\pi K/N}}$$

(d) Given $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nK}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(2\pi/N)2nK} + \sum_{n=0}^{(N/2)-1} x(2n+1) e^{-j(2\pi/N)(2n+1)K}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j4\pi nK/N} = \sum_{n=0}^{(N/2)-1} e^{-j4\pi nK/N}$$

$$= \frac{1 - e^{-j2\pi K}}{1 - e^{-j4\pi K/N}}$$

2. Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ and IDFT of $X(K) = \{4, 2, 0, 4\}$ directly.

Sol:- Given sequence is $x(n) = \{1, -1, 2, -2\}$. Here the DFT $X(K)$ to

(a) be found is $N = 4$-point and length of the sequence $L = 4$. So no padding of zeros is required. We know that the DFT $\{x(n)\}$ is given by

$$X(K) = \sum_{n=0} x(n) W_N^{nK} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nK/N}, \quad K = 0,1,2,3$$

$$\therefore X(0) = \sum_{n=0}^{3} x(n) e^{0} = x(0) + x(1) + x(2) + x(3)$$

$$= 1-1+2-2 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j2\pi n/4} = \sum_{n=0}^{3} x(n)\, e^{-jn\pi/2}$$

$$= x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 + (-1)(0-j) + 2(-1-j0) - 2\,(0+j)$$

$$= -1-j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n} = x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1-1(-1-j0) + 2(1-j0) - 2(-1-j0) = 6$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j(3\pi/2)n}$$

$$= x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-j(9\pi/2)}$$

$$= 1-1(0+j) + 2(-1-j0) - 2(0-j) = -1+j$$

$$\therefore X(K) = \{0, -1-j, 6, -1+j\}$$

(b) Given DFT is $X(K) = \{4,2,0,4\}$. The IDFT of $X(K)$ ie $x(n)$ is given by

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-nK} = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j(2\pi/N)nK}$$

ie $$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K)\, e^{j\pi/2\, nK}$$

$$\therefore \; x(0) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{0}$$

$$= \frac{1}{4} \left[X(0) + X(1) + X(2) + X(3) \right] = \frac{1}{4} \left[4 + 2 + 0 + 4 \right] = 2.5$$

$$x(1) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{j\pi/2\, K}$$

$$= \frac{1}{4} \left[X(0) + X(1) e^{j\pi/2} + X(2) e^{j\pi} + X(3) e^{j3\pi/2} \right]$$

$$= \frac{1}{4} \left[4 + 2(0 + j) + 0 + 4(0 - j) \right] = 1 - j0.5$$

$$x(2) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{j\pi/2\, K}$$

$$= \frac{1}{4} \left[X(0) + X(1) e^{j\pi} + X(2) e^{j2\pi} + X(3) e^{j3\pi} \right]$$

$$= \frac{1}{4} \left[4 + 2(-1 + j0) + 0 + 4(-1 + j0) \right] = -0.5$$

$$x(3) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{j(3\pi/2) K}$$

$$= \frac{1}{4} \left[X(0) + X(1) e^{j(3\pi/2)} + X(2) e^{j3\pi} + X(3) e^{j(9\pi/2)} \right]$$

$$= \frac{1}{4} \left[4 + 2(0 - j) + 0 + 4(0 + j) \right] = 1 + j0.5$$

$$\therefore \; x_3(n) = \{ 2.5,\ 1,\ -j0.5,\ -0.5,\ 1 + j0.5 \}$$

3. Find the DTFT of the following sequences :

(a) $u(n-m)$ (b) $\delta(n+3)-\delta(n-3)$ (c) $-a^n u(-n-1)$

(d) $u(n+3)-u(n-3)$ (e) $(0.5)^n u(n)+2^n u(-n-1)$ (f) $a^{|n|}$

(g) $(1/4)^n u(n+1)$

Sol:- (a) $x(n) = u(n-m)$

$$u(n-m) = \begin{cases} 1 & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases}$$

$$X(\omega) = F[u(n-m)] = \sum_{n=-\infty}^{\infty} u(n-m) e^{-j\omega n} = \sum_{n=m}^{\infty} (1) e^{-j\omega n}$$

$$= e^{-j\omega m} + e^{-j\omega(m+1)} + e^{-j\omega(m+2)} + \cdots$$

$$= e^{-j\omega m}(1+e^{-j\omega}+e^{-j2\omega}+\cdots)$$

$$= \frac{e^{-j\omega m}}{1-e^{-j\omega}} \qquad \left[\because 1+x+x^2+x^3+\cdots = \frac{1}{1-x}\right]$$

$$\therefore \boxed{F[u(n-m)] = \frac{e^{-j\omega m}}{1-e^{-j\omega}}}$$

(b) $x(n) = \delta(n+3)-\delta(n-3)$

$$X(\omega) = F\{\delta(n+3)-\delta(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} [\delta(n+3)-\delta(n-3)] e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=-3} - e^{-j\omega n}\Big|_{n=3} = e^{j3\omega} - e^{-j3\omega} = 2j\sin 3\omega$$

$$\therefore \boxed{F[\delta(n+3)-\delta(n-3)] = 2j\sin 3\omega}$$

(c) $x(n) = -a^n u(-n-1)$

$$X(\omega) = F[-a^n u(-n-1)]$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} -a^n e^{-j\omega n} = -\sum_{n=1}^{\infty} a^{-n} e^{j\omega n} = -\sum_{n=1}^{\infty} (a^{-1} e^{j\omega})^n$$

$$= -\left[a^{-1} e^{j\omega} + (a^{-1} e^{j\omega})^2 + (a^{-1} e^{j\omega})^3 + \cdots\right]$$

$$= -a^{-1} e^{j\omega}\left[1 + a^{-1} e^{j\omega} + (a^{-1} e^{j\omega})^2 + \cdots\right]$$

$$= \frac{-a^{-1} e^{j\omega}}{1 - a^{-1} e^{j\omega}} = \frac{1}{1 - a e^{-j\omega}}$$

$$\therefore \quad -a^n u(-n-1) \longleftrightarrow \frac{1}{1 - a e^{-j\omega}}$$

(d) $x(n) = u(n+3) - u(n-3)$

Sol:- $X(\omega) = F[u(n+3) - u(n-3)]$

$$= \sum_{n=-\infty}^{\infty} [u(n+3) - u(n-3)] e^{-j\omega n} = \sum_{n=-3}^{\infty} (1) e^{-j\omega n} - \sum_{n=3}^{\infty} (1) e^{-j\omega n}$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \cdots - e^{-j3\omega} - e^{-j4\omega} - \cdots$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

(e) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

Sol:- $X(\omega) = F[x(n)] = \sum_{n=-\infty}^{\infty} [(0.5)^n u(n) + 2^n u(-n-1)]\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} [(0.5)^n u(n)] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} [2^n u(-n-1)] e^{-j\omega n}$$

$$= \sum_{n=0} (0.5)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$$ 2.27

$$= \sum_{n=0}^{\infty} (0.5 e^{-j\omega})^n + \sum_{n=-\infty}^{-1} (2^{-1} e^{j\omega})^n$$

$$= \frac{1}{1-0.5e^{-j\omega}} + \frac{2^{-1}e^{j\omega}}{1-2^{-1}e^{j\omega}}$$

$$= \frac{1}{1-0.5e^{-j\omega}} - \frac{1}{1-2e^{-j\omega}}$$

f $\quad x(n) = a^{|n|}$

$$X(\omega) = F[a^{|n|}] = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} (ae^{j\omega})^n + \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega+a^2}$$

g $\quad x(n) = (1/4)^n u(n+1)$

$$X(\omega) = F\left[(1/4)^n u(n+1)\right] = \sum_{n=-\infty}^{\infty} (1/4)^n u(n+1) e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} (1/4)^n e^{-j\omega n} = \sum_{n=-1}^{\infty} \left(\tfrac{1}{4}e^{-j\omega}\right)^n$$

$$= \left(\tfrac{1}{4}e^{-j\omega}\right)^{-1} + \sum_{n=0}^{\infty} \tfrac{1}{4} e^{-j\omega n}$$

$$= 4e^{j\omega} \left[\frac{1}{1 - \frac{1}{4} e^{-j\omega}} \right]$$

$$= \frac{4e^{j\omega}}{1 - \frac{1}{4} e^{-j\omega}}$$

4. Find the DTFT of the following sequences

(a) $\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)$ (b) $\left(\frac{1}{2}\right)^{n-2} u(n-2)$

Soln $x(n) = \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)$

$$X(\omega) = F[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)\right] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{ \frac{e^{jn\pi/4} - e^{-jn\pi/4}}{2j} \right\} e^{-j\omega n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{j[\pi/4 - \omega]n} - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j[\pi/4 + \omega]n} \right]$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} \left(\frac{1}{2} e^{j[\pi/4 - \omega]}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j[\pi/4 + \omega]}\right)^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - \frac{1}{2} e^{j(\pi/4 - \omega)}} - \frac{1}{1 - \frac{1}{2} e^{-j(\pi/4 + \omega)}} \right]$$

$$= \frac{\frac{1}{2} e^{-j\omega} \sin \pi/4}{1 + \frac{1}{4} e^{-j2\omega} - e^{-j\omega} \cos \pi/4} = \frac{(1/2\sqrt{2})\, e^{-j\omega}}{1 - (1/\sqrt{2})\, e^{-j\omega} + \frac{1}{4} e^{-j2\omega}}$$

b)

Sol:- $x(n) = (1/2)^{n-2} u(n-2)$

$$X(\omega) = F[x(n)] = \sum_{n=-\infty}^{\infty} \left[(1/2)^{n-2} u(n-2) \right] e^{-j\omega n}$$

$$= \sum_{n=2}^{\infty} (1/2)^{n-2} e^{-j\omega n}$$

$$= e^{-j2\omega} + \frac{1}{2} e^{-j3\omega} + (1/2)^2 e^{-j4\omega} + \cdots$$

$$= e^{-j2\omega} \left[1 + \frac{1}{2} e^{-j\omega} + (1/2)^2 e^{-j2\omega} + \cdots \right]$$

$$= \frac{e^{-j2\omega}}{1 - 1/2 \, e^{-j\omega}}$$

5. Find the inverse Fourier transform of $X(\omega) = 2 + e^{-j\omega} + 3e^{-j3\omega} + 4e^{j4\omega}$

Sol:- Given $X(\omega) = 2 + e^{-j\omega} + 3e^{-j3\omega} + 4e^{-j4\omega}$

W.K.T $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$= \cdots + x(-2) e^{j2\omega} + x(-1) e^{j\omega} + x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + \cdots$$

On comparing the above two values of $X(\omega)$, we get

$x(0) = 2$, $x(1) = 1$, $x(2) = 0$, $x(3) = 3$, $x(4) = 4$

$\therefore x(n) = \{2, 1, 0, 3, 4\}$

6. Find the IFT of the following:

$$X(\omega) = \begin{cases} 1, & \pi/3 \le |\omega| \le 2\pi/3 \\ 0, & 2\pi/3 \le |\omega| \le \pi \end{cases}$$

$\ldots \quad -j2\omega \quad -j3\omega$

Sol:-

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-2\pi/3}^{-\pi/3} 1\cdot e^{j\omega n}\,d\omega + \int_{\pi/3}^{2\pi/3} 1\cdot e^{j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-2\pi/3}^{-\pi/3} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi/3}^{2\pi/3}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{-jn(\pi/3)} - e^{-jn(2\pi/3)} + e^{jn(2\pi/3)} - e^{jn(\pi/3)}}{2j}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn(2\pi/3)} - e^{-jn(2\pi/3)}}{2j} - \frac{e^{jn(\pi/3)} - e^{-jn(\pi/3)}}{2j}\right]$$

$$= \frac{1}{n\pi}\left[\sin n\frac{2\pi}{3} - \sin n\frac{\pi}{3}\right]$$

7. The impulse response of a LTI system is $h(n) = \{1, 2, 1, -2\}$. Find the response of the system for the input $x(n) = \{1, 3, 2, 1\}$

Sol:- The response of the system $y(n)$ for an input $x(n)$ and impulse response $h(n)$ is given by

$$y(n) = x(n) * h(n)$$

using convolution property of Fourier Transform, we get

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$\therefore\ y(n) = F^{-1}\{X(\omega)\,H(\omega)\}$$

Given $x(n) = \{1, 3, 2, 1\}$

$\therefore X(\omega) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$

Given $h(n) = \{1, 2, 1, -2\}$

$\therefore H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega}$

$Y(\omega) = X(\omega) H(\omega)$

$= [1 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}][1 + 2e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega}]$

$= 1 + 5e^{-j\omega} + 9e^{-j2\omega} + 6e^{-j3\omega} - 2e^{-j4\omega} - 3e^{-j5\omega} - 2e^{-j6\omega}$

Taking Inverse Fourier Transform on both sides, we get

$y(n) = 1 + 5\delta(n-1) + 9\delta(n-2) + 6\delta(n-3) - 2\delta(n-4) - 3\delta(n-5) - 2\delta(n-6)$

& $y(n) = \{1, 5, 9, 6, -2, -3, -2\}$

Relation Between DTFT and Z-Transform:

The z-transform of a discrete sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable.

The Fourier transform of a discrete time sequence $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The $X(z)$ can be viewed as a unique representation of the sequence $x(n)$ in the complex z-plane

Let $z = re^{j\omega}$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (r^{-n} x(n)) e^{-j\omega n}$$

When $r = 1$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(\omega)$$

So we can conclude that the Fourier transform of $x(n)$ is same as the z-transform of $x(n)$ evaluated along the unit circle centered at the origin of the z-plane.

$$\therefore X(\omega) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

For $X(\omega)$ to exist, the ROC must include the unit circle. Since ROC cannot contain any poles of $X(z)$ all the poles must lie inside the unit circle. Therefore, we can conclude that Fourier transform can be obtained for any sequence $x(n)$, from its Z-transform $X(z)$ if the poles of $X(z)$ are inside the unit circle.

Problems on DTFS

1. Find the DFS of $x(n) = \cos(\pi/3\, n)$ and plot its mag. spectrum

Sol:- $x_p(n) = \cos\left(\frac{\pi}{3}n\right)$

Here $\Omega_0 = \frac{\pi}{3} = \frac{2\pi}{N} \times m = \frac{2\pi}{6} \times 1$

$\therefore N = 6, \quad m = 1$

WKT $x_p(n) = \sum_{K=\langle N \rangle} X_p(K)\, e^{j2\pi nK/N} \rightarrow (1)$

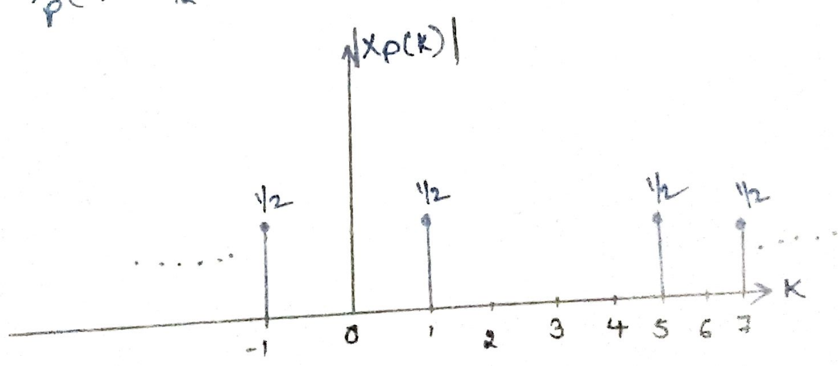
Represent $\cos(\pi/3\, n)$ interms of exponentials & compare with eq (1)

$$\therefore x_p(n) = \frac{e^{j\pi/3\, n} + e^{-j\pi/3\, n}}{2}$$

$$= \frac{1}{2} e^{j\frac{\pi}{3}n} + \frac{1}{2} e^{-j\pi/3\, n}$$

$$= \frac{1}{2} e^{j(1)\pi/3\, n} + \frac{1}{2} e^{j(-1)\frac{\pi}{3}n}$$

$\therefore X_p(1) = 1/2 \quad ; \quad X_p(-1) = 1/2$



Mag spectrum of $x_p(n)$

2. $x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1$

Sol:- $\Omega_0 = \gcd\left(\frac{4\pi}{21}, \frac{10\pi}{21}\right)$

(or)

$\Omega_1 = \frac{4\pi}{21} \quad \Omega_2 = \frac{10\pi}{21}$

$\therefore \frac{\Omega_2}{\Omega_1} = \frac{10\pi}{21} \times \frac{21}{4\pi} = 5/2$

$\therefore$ ~~Fundamental period~~ $N = 2\Omega_2$

$N_1 = 2\pi \cdot \frac{m}{\Omega_1} = 2\pi \cdot \frac{m}{4\pi/21}$

$= \frac{21}{2} \cdot 2 \quad (\because m = 2)$

$\Rightarrow N_1 = 21$

$N_2 = 2\pi \cdot \frac{m}{\Omega_2}$

$= 2\pi \cdot \frac{m}{\frac{10\pi}{21}} = \frac{21}{5} \cdot m$

$= \frac{21}{5} \cdot 5$

$= 21$

$\therefore \frac{N_1}{N_2} = \frac{21}{21} = 1$

$\therefore N = N_1 = N_2 = 21$

$$x(n) = \frac{e^{j\frac{4\pi}{21}n} - e^{-j\frac{4\pi}{21}n}}{2j} + \frac{e^{j\frac{10\pi}{21}n} + e^{-j\frac{10\pi}{21}n}}{2} + 1$$

$$\Rightarrow x(n) = \frac{1}{2j} e^{j2 \cdot \frac{2\pi n}{21}} - \frac{1}{2j} e^{j(-2) \cdot \frac{2\pi n}{21}} + \frac{1}{2} e^{j5 \cdot \frac{2\pi n}{21}} + \frac{1}{2} e^{j(-5) \cdot \frac{2\pi n}{21}} + 1 \quad \text{—— (1)}$$

compare eq (1) with $x_p(n) = \sum_{k=\langle N \rangle} X_p(k) e^{jk\Omega_0 n}$

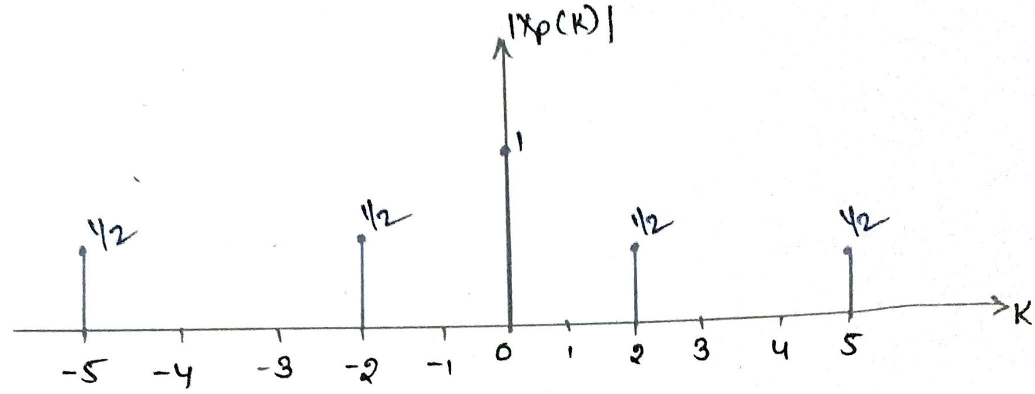
$\therefore X_p(2) = 1/2j = -0.5j \Rightarrow |X_p(2)| = \sqrt{0^2 + 0.5^2} = 0.5$

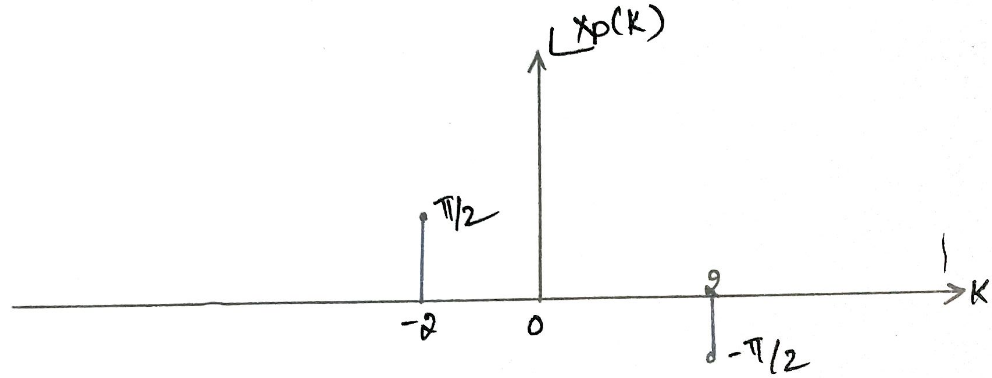
$X_p(-2) = -1/2j = 0.5j \Rightarrow |X_p(-2)| = \sqrt{0.5^2} = 0.5$

$X_p(5) = 1/2 \Rightarrow |X_p(5)| = 1/2$

$X_p(-5) = 1/2 \Rightarrow |X_p(-5)| = 1/2$

$X_p(0) = 1 \Rightarrow |X_p(0)| = 1$





$\angle X_p(-2) = \tan^{-1}(0.5/0) = \tan^{-1}(\infty) = \pi/2$

$\angle X_p(2) = \tan^{-1}(-0.5/0) = \tan^{-1}(-\infty) = -\pi/2$

$\angle X_p(5) = \tan^{-1}(0/1/2) = \tan^{-1}0 = 0$

$\angle X_p(-5) = \tan^{-1}(0/1/2) = \tan^{-1}0 = 0$

$\angle X_p(0) = \tan^{-1}(0/1) = \tan^{-1}0 = 0$

3. $x(n) = \cos\left(\frac{6\pi}{13} n + \pi/6\right)$

Sol:

$$\Omega_0 = \frac{6\pi}{13}$$

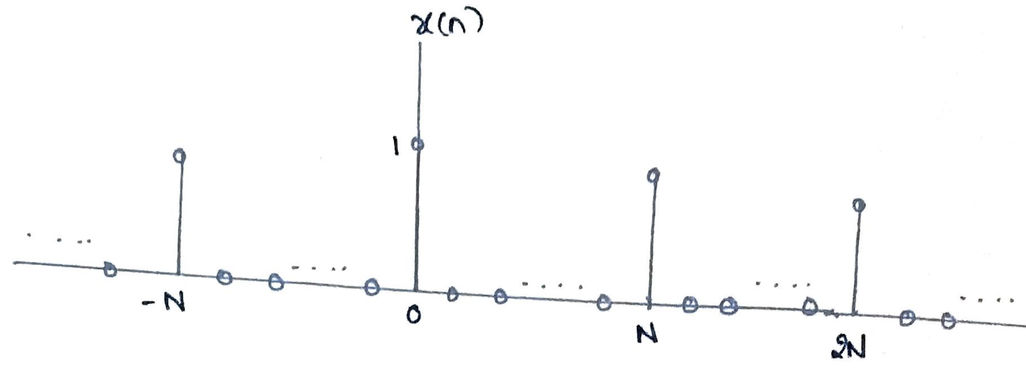
$$= \frac{2\pi}{13} \times 3$$

$$= \frac{2\pi}{N} \times m$$

$\therefore$ $N = 13$ Samples

$$x(n) = \sum_{k=\langle N \rangle}$$

4 Find the DTFS coefficients of the N-periodic impulse train $x(n) = \sum_{l=-\infty}^{\infty} \delta(n - lN)$, as shown in fig below.

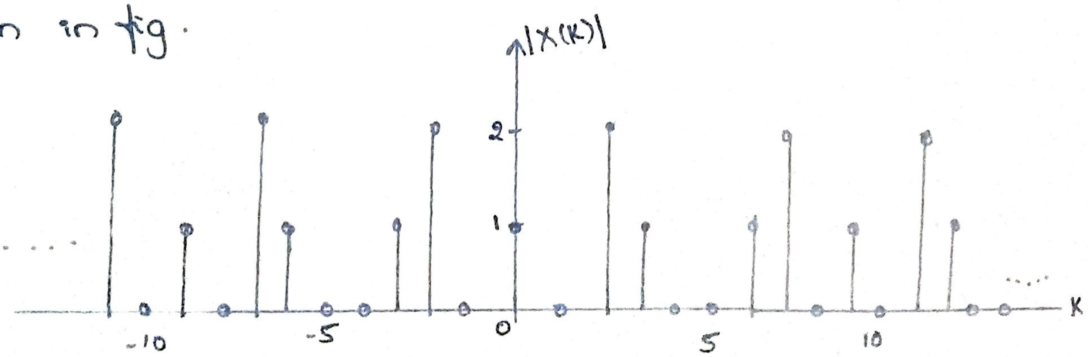


Sol: Since there is only one Non-Zero value in $x(n)$ per period, it is convenient to evaluate $X_p(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-jK\Omega_0 n}$ over the interval $n=0$ to $n=N-1$ to obtain

$$X_p(K) = \frac{1}{N} \sum_{n=0}^{N-1} \delta(n) e^{-jK2\pi n/N}$$

$$= \frac{1}{N}$$

5 Determine the time-domain signal $x_p(n)$ from DTFS coefficients shown in fig.



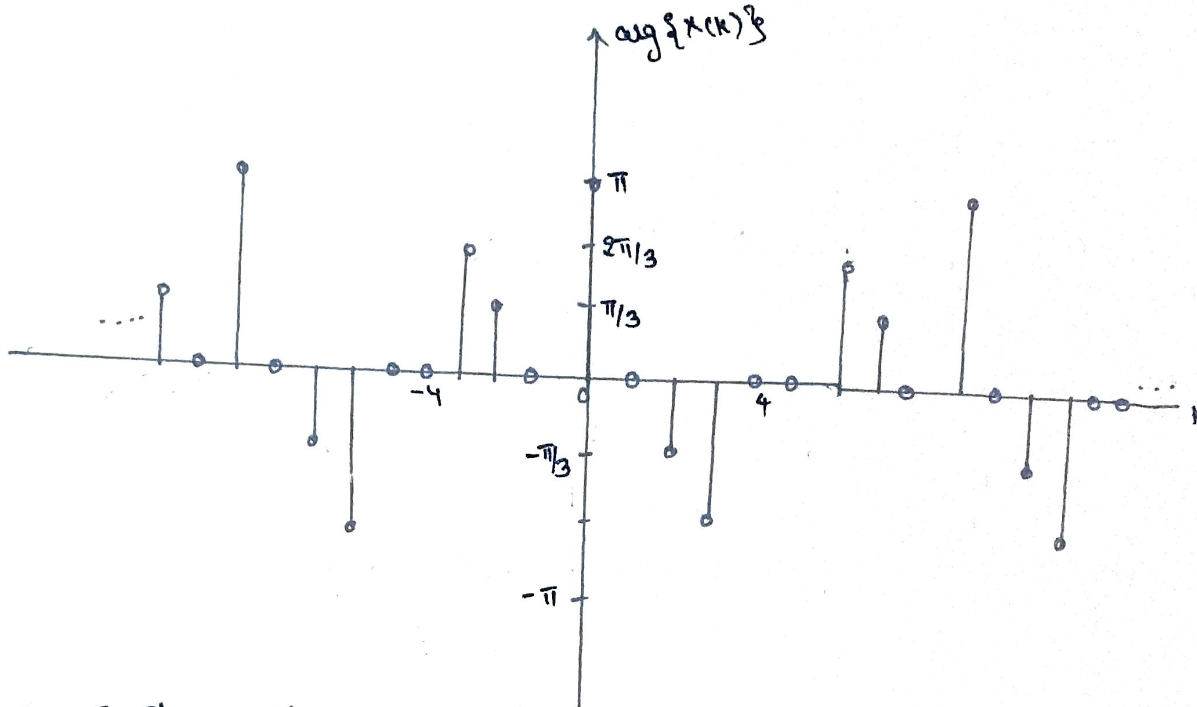


Fig: Mag & Phase of DTFS coefficients

Sol: The DTFS coefficients have period 9, so $\Omega_0 = 2\pi/9$.

$$\therefore \quad x(n) = \sum_{K=-4}^{4} X_p[K]\, e^{j2\pi nk/9}$$

$$= 0 + e^{j2\pi/3} e^{-j6\pi n/9} + 2e^{j\pi/3} e^{-j4\pi n/9} - 1 + 2e^{-j\pi/3} e^{j4\pi n/9} + e^{-j2\pi/3} e^{j6\pi n/9}$$

$$= e^{-j(6\pi n/9 - 2\pi/3)} + 2e^{-j(4\pi n/9 - \pi/3)} - 1 + 2e^{j\left(\frac{4\pi n}{9} - \pi/3\right)} + e^{j(6\pi n/9 - 2\pi/3)}$$

$$= 2\cos(6\pi n/9 - 2\pi/3) + 4\cos(4\pi n/9 - \pi/3) - 1$$

6 Use the method of inspection to determine the DTFS Coeffi-cients for the following signals.

a) $x(n) = 1 + \sin(n\pi/12 + 3\pi/8)$

b) $x(n) = \cos(n\pi/30) + 2\sin(n\pi/90)$

Sol:

a) $x(n) = 1 + \sin\left(\frac{n\pi}{12} + \frac{3\pi}{8}\right)$

$\Omega_0 = \frac{\pi}{12}$

$= \frac{2\pi}{24} \times 1 = \frac{2\pi}{N} m$

$\therefore N = 24$

$\therefore x(n) = 1 + \dfrac{e^{j(n\pi/12 + 3\pi/8)} + e^{-j(n\pi/12 + \frac{3\pi}{8})}}{2j}$

$= 1 + \frac{1}{2j} e^{jn\pi/12} e^{j3\pi/8} + \frac{1}{2j} e^{-jn\pi/12} e^{-j3\pi/8}$

$= 1 + \frac{1}{2j} e^{j3\pi/8} e^{\frac{j2\pi n(1)}{24}} + \frac{1}{2j} e^{-j3\pi/8} e^{\frac{j2\pi n(-1)}{24}}$ — (1)

$\therefore$ Compare eq (1) with $x(n) = \sum_{K=\langle N \rangle} X_p(K) e^{j2\pi nK/N}$

$\therefore X_p(0) = 1$

$X_p(1) = \frac{e^{j3\pi/8}}{2j}$

$X_p(-1) = \frac{e^{-j3\pi/8}}{2j}$

$\therefore X_p(K) = 0$; others

b) $x(n) = \cos(n\pi/30) + 2\sin(n\pi/90)$

$\Omega_1 = \frac{\pi}{30} = \frac{2\pi}{60}(1)$ $\quad$ $\Omega_2 = \frac{\pi}{90} = \frac{2\pi}{180}(1)$ $\quad$ period

$\therefore N_1 = 60$ $\quad$ $\therefore N_2 = 180$

$\Rightarrow N = \frac{N_1}{N_2} = \frac{\cancel{60}}{\cancel{180}_{3}} = \frac{1}{3}$

$\Rightarrow N = 3N_1 = N_2 = 180$

$$x(n) = \frac{e^{jn\pi/30} + e^{-jn\pi/30}}{2} + 2\,\frac{e^{jn\pi/90} - e^{-jn\pi/90}}{2j}$$

$$= \frac{1}{2}e^{\frac{j2\pi n}{60}(1)} + \frac{1}{2}e^{\frac{j2\pi n(-1)}{60}} + \frac{1}{j}e^{\frac{j2\pi n(1)}{180}} - \frac{1}{j}e^{j2\pi n(-1)/180}$$

$$= \frac{1}{2}e^{\frac{j2\pi n(3)}{180}} + \frac{1}{2}e^{\frac{j2\pi n(-3)}{180}} + \frac{1}{j}e^{\frac{j2\pi n(1)}{180}} - \frac{1}{j}e^{\frac{j2\pi n(-1)}{180}}$$

$\therefore X_p(1) = 1/j$; $X_p(-1) = -1/j$

$X_p(3) = 1/2$; $X_p(-3) = 1/2$

$X_p(K) = 0$; others

7. Determine the DTFS coefficients of $x(n) = \cos(\pi n/3 + \phi)$, using the method of inspection.

Sol:- The period of $x(n)$ is $N = 6$. We expand the cosine by using Euler's formula and move any phase shifts infront of the complex sinusoids.

$$\therefore \quad x(n) = \frac{e^{j(\pi n/3 + \phi)} + e^{-j(\pi n/3 + \phi)}}{2}$$

$$= \frac{1}{2} e^{j\phi} e^{j\pi n/3} + \frac{1}{2} e^{-j\phi} e^{j\pi n(-1)/3}$$

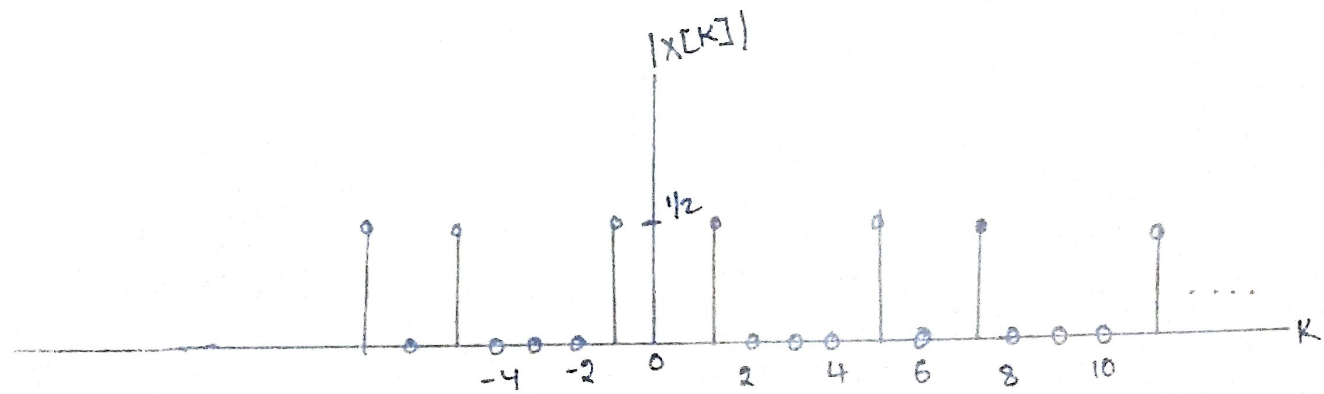
Compare the above eq with

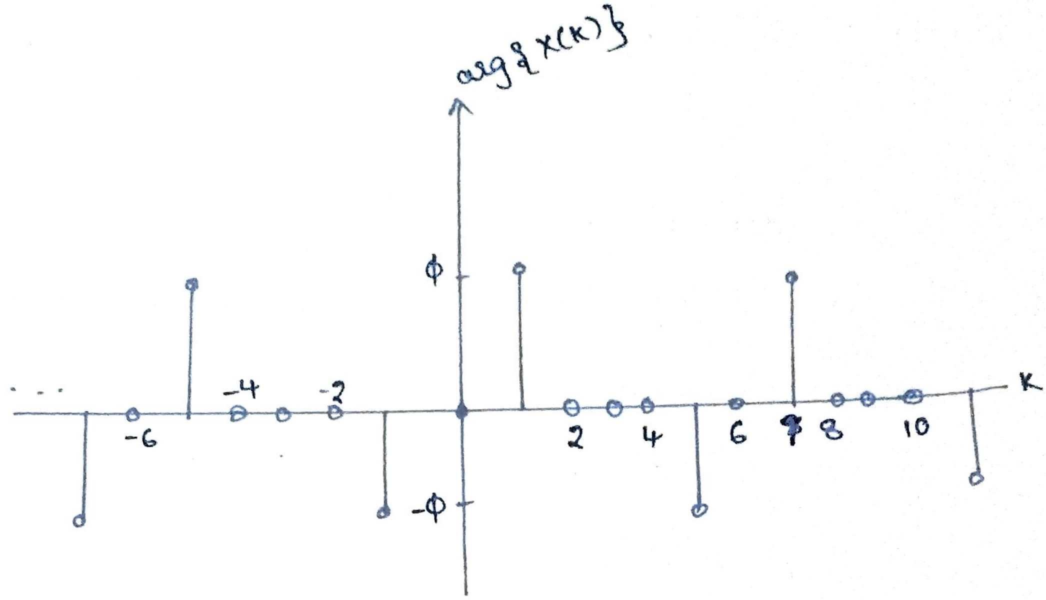
$$x(n) = \sum_{K=-2}^{3} X[K] e^{j2\pi nK/3}$$

$$\therefore \quad X[1] = \frac{e^{j\phi}}{2}$$

$$X[-1] = \frac{e^{-j\phi}}{2}$$

$$X[K] = 0 \; ; \text{ others}$$





arg{X(k)}
φ
-φ
k
-6
-4
-2
2
4
6
8
10

Properties Of Discrete Time Fourier Transform :

1. **Periodicity** :- The DTFT $X(\omega)$ is periodic in 'ω' with a period of 2π. $X(\omega) = X(\omega + 2\pi K)$ for any integer value of 'K'.

Proof:
$$X(\omega + 2\pi K) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega + 2\pi K)n}$$
$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi K n}$$
$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = X(\omega)$$

2. **Linearity** : If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$, then
$$F[a\,x_1(n) + b\,x_2(n)] = a\,X_1(\omega) + b\,X_2(\omega)$$

Proof :
$$F[a\,x_1(n) + b\,x_2(n)] = \sum_{n=-\infty}^{\infty} [a\,x_1(n) + b\,x_2(n)]\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} [a\,x_1(n)\, e^{-j\omega n} + b\,x_2(n)\, e^{-j\omega n}]$$
$$= a \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j\omega n}$$
$$= a\,X_1(\omega) + b\,X_2(\omega)$$

3. **Time Shifting** :- If $F[x(n)] = X(\omega)$ then $F[x(n-K)] = e^{-j\omega K} X(\omega)$ where 'K' is an integer.

Proof :
$$F[x(n-K)] = \sum_{n=-\infty}^{\infty} x(n-K)\, e^{-j\omega n}$$

Let $n - K = P \Rightarrow n = P + K$

$$= \sum_{P=-\infty}^{\infty} x(P)\, e^{-j\omega(P+K)} = \sum_{P=-\infty}^{\infty} x(P)\, e^{-j\omega P}\, e^{-j\omega K}$$
$$= e^{-j\omega K} \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \text{(Replacing } P \text{ with } n\text{)}$$

$$\Rightarrow F[x(n-k)] = e^{-j\omega k} X(\omega)$$

The above result shows that time shifting of signal by 'k' units does not change its amplitude spectrum but phase spectrum is changed by $-\omega k$.

4. **Frequency Shifting :-** If $F[x(n)] = X(\omega)$ then

$$F[x(n)\, e^{j\omega_0 n}] = X[e^{j(\omega-\omega_0)}] = X(\omega-\omega_0)$$

Proof :- $F[x(n)\, e^{j\omega_0 n}] = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n} e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$$

$$= X(\omega-\omega_0)$$

5. **Time Reversal :-** If $F[x(n)] = X(\omega)$ then $F[x(-n)] = X(-\omega)$

Proof :- $F[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$

Let $m = -n$

$$= \sum_{m=\infty}^{-\infty} x(m)\, e^{j\omega m} = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\omega)m}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}$$

$$= X(-\omega)$$

Folding in Time domain corresponds to folding in frequency domain.

6. **Differentiation In Frequency :-** If $F[x(n)] = X(\omega)$ then

$$F[n\,x(n)] = j\frac{d}{d\omega} X(\omega)$$

Proof: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Differentiating on both sides w.r.t 'ω'

$$\frac{d}{d\omega} X(\omega) = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}\right] = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\omega} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-jn) e^{-j\omega n} = -j\sum_{n=-\infty}^{\infty} n\, x(n) e^{-j\omega n}$$

$$\Rightarrow j\frac{d}{d\omega} X(\omega) = \sum_{n=-\infty}^{\infty} n\,x(n) e^{-j\omega n}$$

$$\therefore F[n\,x(n)] = j\frac{d}{d\omega} X(\omega)$$

7. **Time Convolution :-** If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$ then

$F[x_1(n) * x_2(n)] = X_1(\omega)\, X_2(\omega)$ where $x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K) x_2(n-K)$

Proof:- $F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)\right] e^{-j\omega n}$$

Interchanging the summations,

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{n=-\infty}^{\infty} x_2(n-K) e^{-j\omega n}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) e^{-j\omega K} X_2(\omega) \quad (\because \text{Time shifting Property})$$

$$= X_2(\omega) \sum_{K=-\infty}^{\infty} x_1(K) e^{-j\omega K} = X_2(\omega) \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n}$$

$$= X_2(\omega) X_1(\omega)$$

$$= X_1(\omega) X_2(\omega)$$

The convolution involved in time domain is linear convolution

8. Frequency Convolution :- If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$ then $F[x_1(n) x_2(n)] = X_1(\omega) \circledast X_2(\omega)$

where $X_1(\omega) \circledast X_2(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega-\lambda) \, d\lambda$

Proof:- $F[x_1(n) x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n) x_2(n) e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} x_2(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) e^{j\lambda n} d\lambda\right] e^{-j\omega n}$$

Interchanging the order of summation and integration,

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \left[\sum_{n=-\infty}^{\infty} x_2(n) e^{-j(\omega-\lambda)n}\right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega-\lambda) \, d\lambda$$

$$= X_1(\omega) \circledast X_2(\omega)$$

The convolution involved in frequency domain is circular convolution.

9. Correlation Theorem: If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$

then $F[\gamma_{x_1x_2}(l)] = S_{x_1x_2}(\omega) = X_1(\omega) X_2(-\omega)$ where, $\gamma_{x_1x_2}(l)$ is cross correlation function and $S_{x_1x_2}(\omega)$ is the cross energy density spectrum of the signals $x_1(n)$ & $x_2(n)$.

Proof:- $\gamma_{x_1 x_2}(l) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(K-L)$

$$F[\gamma_{x_1 x_2}(L)] = \sum_{L=-\infty}^{\infty} \gamma_{x_1 x_2}(L) e^{-j\omega L} = \sum_{L=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x_1(K)\, x_2(K-L) \right] e^{-j\omega L}$$

Interchanging the Summations

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{L=-\infty}^{\infty} x_2(K-L) e^{-j\omega l}$$

Let $K-L = m \Rightarrow l = K-m$ (in 2^{nd} Summation)

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega K} e^{j\omega m}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) e^{-j\omega K} \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j(-\omega)m}$$

$$= X_1(\omega)\, X_2(-\omega)$$

$$= S_{x_1 x_2}(\omega)$$

10, **Modulation Theorem:** If $F[x(n)] = X(\omega)$ then

$$F[x(n)\cos\omega_0 n] = \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

Proof: $F[x(n)\cos\omega_0 n] = \sum_{n=-\infty}^{\infty} x(n)\cos\omega_0 n\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right] e^{-j\omega n}$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{j\omega_0 n} e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega_0 n} e^{-j\omega n} \right]$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+\omega_0)n} \right]$$

Proof:- $\gamma_{x_1x_2}(l) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(K-L)$

$$F[\gamma_{x_1x_2}(l)] = \sum_{l=-\infty}^{\infty} \gamma_{x_1x_2}(l)\, e^{-j\omega l} = \sum_{l=-\infty}^{\infty}\left[\sum_{K=-\infty}^{\infty} x_1(K)\, x_2(K-L)\right] e^{-j\omega l}$$

Interchanging the Summations

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{l=-\infty}^{\infty} x_2(K-L)\, e^{-j\omega l}$$

let $K-L = m \Rightarrow l = K-m$ (in 2^{nd} Summation)

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega K}\, e^{j\omega m}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K)\, e^{-j\omega K} \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j(-\omega)m}$$

$$= X_1(\omega)\, X_2(-\omega)$$

$$= S_{x_1x_2}(\omega)$$

10, **Modulation Theorem:** If $F[x(n)] = X(\omega)$ then

$$F[x(n)\cos\omega_0 n] = \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

Proof: $F[x(n)\cos\omega_0 n] = \sum_{n=-\infty}^{\infty} x(n)\cos\omega_0 n\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} x(n)\left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] e^{-j\omega n}$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n}\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega_0 n}\, e^{-j\omega n}\right]$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+\omega_0)n}\right]$$

$$= \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

11. **Parseval's Theorem :-** If $F[x(n)] = X(\omega)$ then

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Proof :- $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n)$

$$= \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\right]^* d\omega$$

Interchanging Summation and integration,

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\, X(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

12. **Symmetry Properties :** The Fourier Transform $X(\omega)$ is a complex function of 'ω' and can be expressed as,

$$X(\omega) = X_R(\omega) + j\, X_I(\omega)$$

where, $X_R(\omega)$ is real part and $X_I(\omega)$ is imaginary part of $X(\omega)$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)\left[\cos\omega n - j\sin\omega n\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cos\omega n + j\left[-\sum_{n=-\infty}^{\infty} x(n)\sin\omega n\right]$$

$$= X_R(\omega) + j\, X_I(\omega)$$

$$\therefore\ X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n \qquad X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

$$X_R(-\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n = X_R(\omega)$$

ie., $X_R(-\omega) = X_R(\omega)$

$\therefore$ Real part of $X(\omega)$ exhibits Even Symmetry w.r.t 'ω'

$$X_I(-\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\sin\omega n = -X_I(\omega)$$

ie., $X_I(-\omega) = -X_I(\omega)$

$\therefore$ Imaginary part of $X(\omega)$ exhibits odd Symmetry. w.r.t 'ω'.

$X(\omega)$ can be expressed in polar form as,

$$X(\omega) = |X(\omega)|\, e^{j\theta(\omega)}$$

where, $|X(\omega)|$ is magnitude and $\theta(\omega)$ is phase of $X(\omega)$

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

$$|X(\omega)| = \sqrt{[X_R(\omega)]^2 + [X_I(\omega)]^2}$$

and $\theta(\omega) = Tan^{-1}\left[\dfrac{X_I(\omega)}{X_R(\omega)}\right]$

$$|X(-\omega)| = \sqrt{[X_R(-\omega)]^2 + [X_I(-\omega)]^2}$$

$$= \sqrt{[X_R(\omega)]^2 + [-X_I(\omega)]^2} = \sqrt{[X_R(\omega)]^2 + [X_I(\omega)]^2}$$

$\Rightarrow |X(-\omega)| = |X(\omega)|$

$\therefore |X(\omega)|$ is an even function of 'ω'

$$\theta(-\omega) = Tan^{-1}\left[\frac{X_I(-\omega)}{X_R(-\omega)}\right] = Tan^{-1}\left[\frac{-X_I(\omega)}{X_R(\omega)}\right] = -Tan^{-1}\left[\frac{X_I(\omega)}{X_R(\omega)}\right]$$

$$\theta(-\omega) = -\theta(\omega)$$

$\therefore \angle X(\omega) = \theta(\omega)$ is an odd function of 'ω'.

* Due to these Properties, to plot $X(\omega)$, only half-Period of $X(\omega)$ can be considered. Generally, the period chosen is $[0, \pi]$.

- A sequence $x(n)$ can be expressed as,

$$x(n) = x_e(n) + x_o(n)$$

where, $x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$

and $x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$

Similarly, Fourier Transform $X(\omega)$ can be decomposed as,

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

where, $X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$

and $X_o(\omega) = 1/2\, [X(\omega) - X^*(-\omega)]$

Implication: If $F[x(n)] = X(\omega)$, then,

(i) $F[x^*(n)] = X^*(-\omega)$

(ii) $F[x^*(-n)] = X^*(\omega)$

(iii) $F[x_e(n)] = Re[X(\omega)]$

(iv) $F[x_o(n)] = j\,Im[X(\omega)]$

(v) $F[Re(x(n))] = X_e(\omega)$

(vi) $F[j\,Im(x(n))] = X_o(\omega)$

Proof: 1) $F[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\omega n} = \left[\sum_{n=-\infty}^{\infty} x(n) e^{j\omega n}\right]^*$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j(-\omega)n}\right]^* = [X(-\omega)]^* = X^*(-\omega)$$

DSP

(ii) $F[x^*(-n)] = \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n} = \left[\sum_{n=-\infty}^{\infty} x(-n) e^{j\omega n}\right]^*$

let $m=-n$

$= \left[\sum_{m=+\infty}^{-\infty} x(m) e^{-j\omega m}\right]^* = \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}\right]^* = [X(\omega)]^*$

$= X^*(\omega)$

(iii) $F[x_e(n)] = \sum_{n=-\infty}^{\infty} x_e(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \frac{1}{2}[x(n) + x^*(-n)] e^{-j\omega n}$

$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n}\right]$

$= \frac{1}{2}[X(\omega) + X^*(\omega)] = Re[X(\omega)]$

(iv) $F[x_o(n)] = \sum_{n=-\infty}^{\infty} x_o(n) e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} \frac{1}{2}[x(n) - x^*(-n)] e^{-j\omega n}$

$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} - \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n}\right]$

$= \frac{1}{2}[X(\omega) - X^*(\omega)] = j\,Im[X(\omega)]$

(v) $F[Re(x(n))] = \sum_{n=-\infty}^{\infty} Re[x(n)] e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} \frac{1}{2}[x(n) + x^*(-n)] e^{-j\omega n}$

$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n}\right]$

$= \frac{1}{2}[X(\omega) + X^*(-\omega)] = X_e(\omega)$

(vi) $F[j\,Im(x(n))] = \sum_{n=-\infty}^{\infty} j\,Im[x(n)]\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}[x(n) - x^*(-n)]\, e^{-j\omega n}$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} - \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n}\right]$$

$$= \frac{1}{2}[X(\omega) - X^*(\omega)]$$

$$= X_o(\omega)$$

Energy Density Spectrum of Aperiodic Signals :-

Energy of a discrete-time signal $x(n)$ is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 \rightarrow (1)$$

Let us now express the energy E_x interms of the Spectral characteristic $X(\omega)$.

$$E_x = \sum_{n=-\infty}^{\infty} x(n) x^*(n) = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} d\omega\right]$$

Interchange the order of integration and summation, we have

$$E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Therefore, the energy relation between $x(n)$ and $X(\omega)$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \rightarrow (2)$$

Eq(2) represents parseval's relation for discrete-time aperiodic signals with finite energy.

The spectrum $X(\omega)$ is a complex-valued function of frequency. It may be expressed as,

$$X(\omega) = |X(\omega)| e^{j\theta(\omega)}$$

where $\theta(\omega)$ - phase spectrum $|X(\omega)|$ - magnitude spectrum

Let $x(n)$ is real, then it follows that,

$$X^*(\omega) = X(-\omega)$$

or $\quad |X(-\omega)| = |X(\omega)|$ (Even Symmetry)

and $\quad \theta(-\omega) = -\theta(\omega)$

From these symmetry properties, we can conclude that the frequency range of real discrete time signals can be limited to the range $0 \leq \omega \leq \pi$. Indeed if we know $X(\omega)$ in the range $0 \leq \omega \leq \pi$, we can determine it for the range $-\pi \leq \omega < 0$ using the symmetry properties given above.

As in the

$$S_{XX}(\omega) = |X(\omega)|^2$$

Problems on DTFT

1. Find the DTFT of the sequence $x(n) = \alpha^n u(n)$

Sol:- WKT DTFT of $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

This sum diverges for $|\alpha| \geq 1$. For $|\alpha| < 1$, we have the convergent geometric series.

$$X(\omega) = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n \quad \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} ; \ |a| < 1\right]$$

$$= \frac{1}{1-\alpha e^{-j\omega}}, \ |\alpha| < 1. \longrightarrow (1)$$

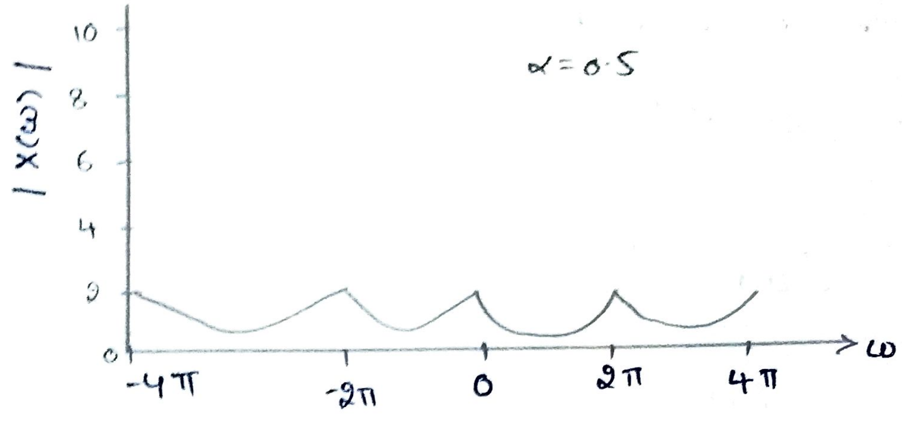
If α is real valued, we may expand the denominator of eq (1) using Euler's formula to obtain

$$X(\omega) = \frac{1}{1-\alpha\cos\omega + j\alpha\sin\omega}$$

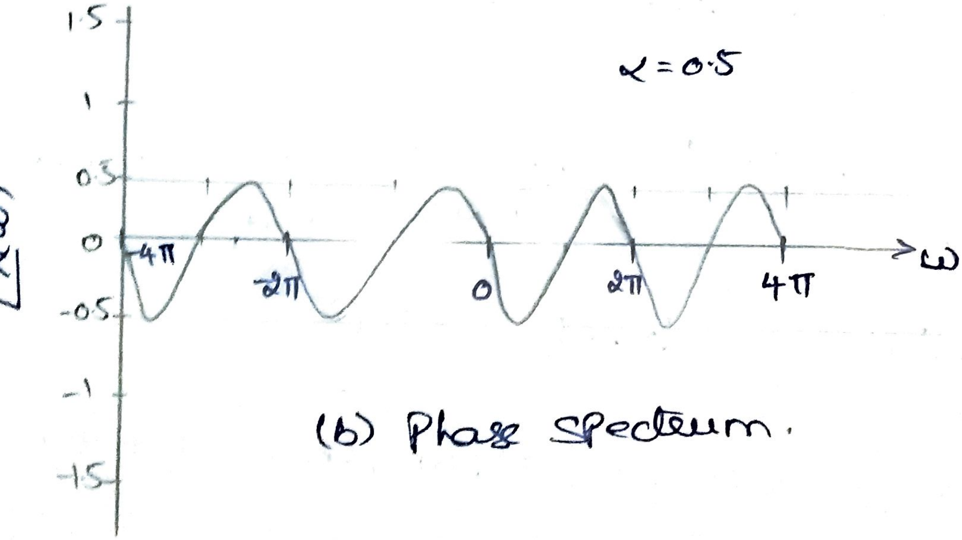
From this form, we see that the magnitude and phase spectra are given by

$$|X(\omega)| = \frac{1}{[(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega]^{1/2}} = \frac{1}{(\alpha^2+1-2\alpha\cos\omega)^{1/2}}$$

& $\angle X(\omega) = -\tan^{-1}\left(\dfrac{\alpha \sin\omega}{1-\alpha\cos\omega}\right)$



(a) Mag. Spectrum



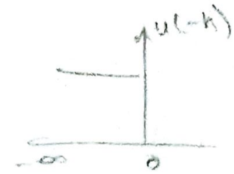
(b) Phase Spectrum.

Fig: DTFT of an exponential signal $x(n) = (\alpha)^n u(n)$

The magnitude is even and the phase is odd.

Both are 2π periodic.

d the DTFT of $x(n) = 2(3)^n u(-n)$



WKT $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} 2(3)^n u(-n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} 2(3)^n e^{-j\omega n}$$

$$= 2 \sum_{n=-\infty}^{0} 3^n e^{-j\omega n}$$

$$= 2 \sum_{n=0}^{\infty} 3^{-n} e^{j\omega n}$$

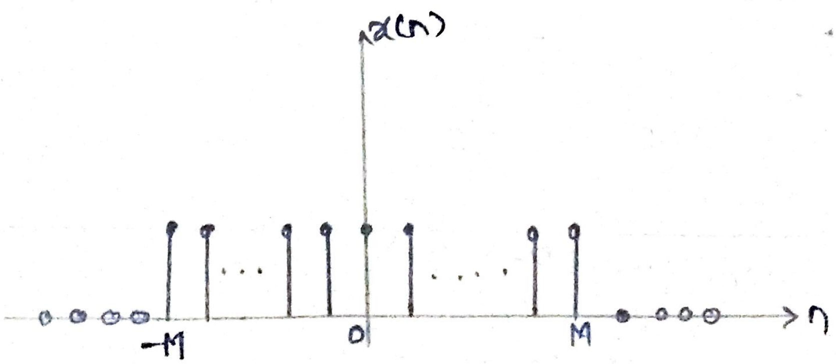
$$= 2 \sum_{n=0}^{\infty} \left(\frac{e^{j\omega}}{3}\right)^n$$

$$= \frac{2}{1 - e^{j\omega}/3}$$

3. DTFT of a Rectangular pulse.

$$x(n) = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases}$$

as shown in below fig. Find the DTFT of $x(n)$.



Sol:-

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-M}^{M} 1 \cdot e^{-j\omega n}$$

Now we perform the change of variable $m = n+M$, obtaining

$$X(\omega) = \sum_{m=0}^{2M} e^{-j\omega(m-M)}$$

$$= e^{j\omega M} \sum_{m=0}^{2M} e^{-j\omega m}$$

$$= e^{j\omega M} \left[\frac{1 - e^{-j\omega(2M+1)}}{1 - e^{-j\omega}} \right] ; \omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

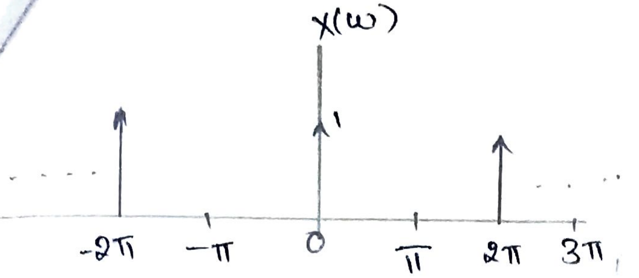
$$= 2M+1 \quad , \quad \omega = 0, \pm 2\pi, \pm 4\pi, \ldots$$

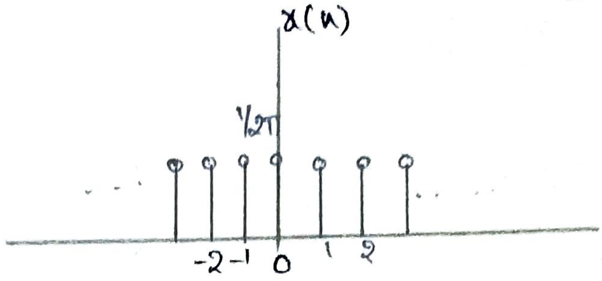
The expression for $X(\omega)$ when $\omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots$ may be simplified by symmetrizing the powers of the exponential in the Nr & Dr as follows:

$$X(\omega) = e^{j\omega M} \left[\frac{e^{-j\omega(2M+1)/2} \left(e^{j\omega(2M+1)/2} - e^{-j\omega(2M+1)/2} \right)}{e^{-j\omega/2} \left(e^{j\omega/2} - e^{-j\omega/2} \right)} \right]$$

$$= \frac{e^{j\omega(2M+1)/2} - e^{-j\omega(2M+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)}$$





6. Determine and sketch the energy density spectrum $S_{xx}(\omega)$ of the signal $x(n) = a^n u(n)$; $-1 < a < 1$

sol: Since $|a| < 1$, the sequence $x(n)$ is absolutely summable.

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

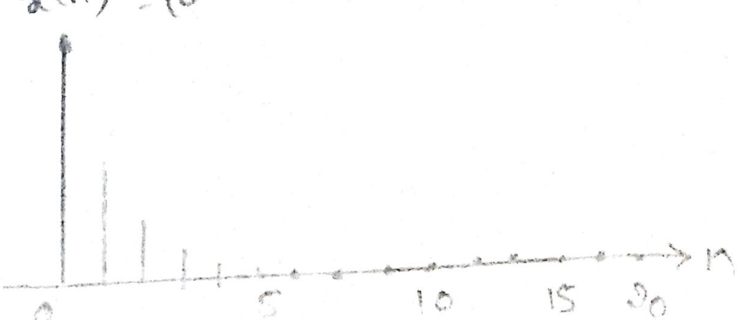
$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

The energy density spectrum is given by

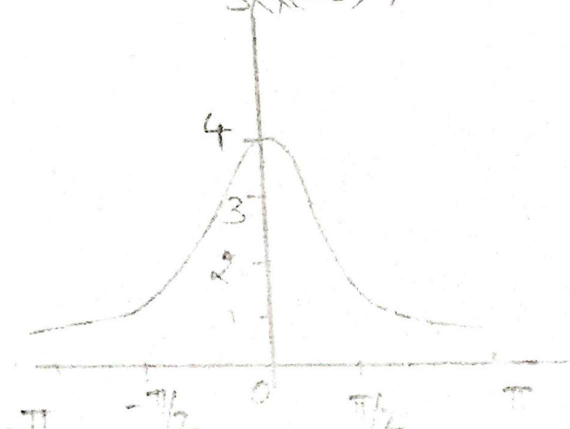
$$S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)\, X^*(\omega) = \frac{1}{(1 - a e^{-j\omega})(1 - a e^{j\omega})}$$

& $$S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}$$

$x(n) = (0.5)^n u(n)$

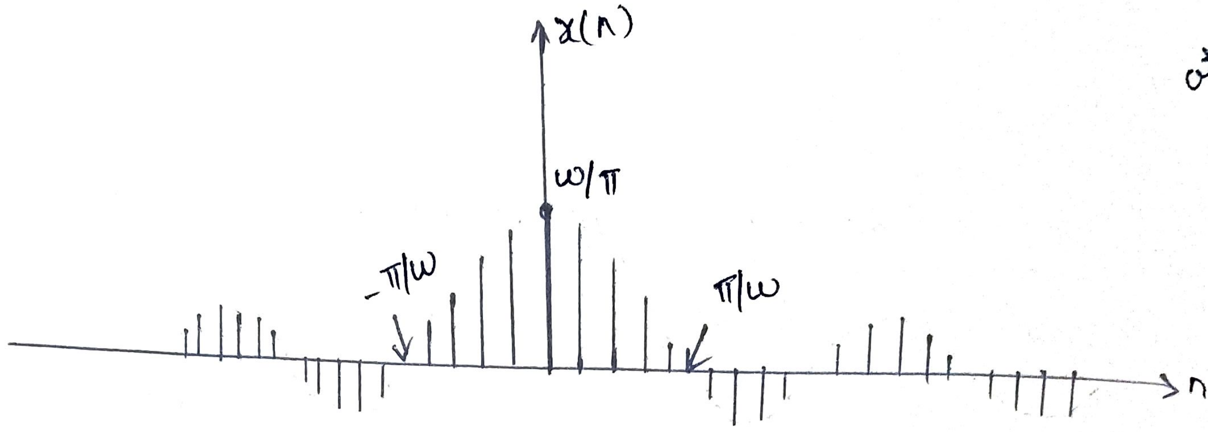


$S_{xx}(\omega)$; $a = 0.5$

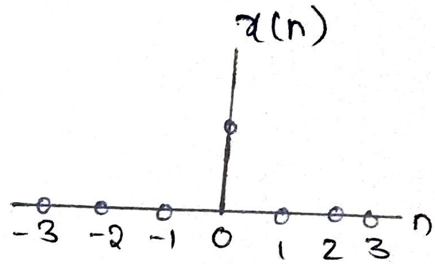


we may also write

$$x(n) = \frac{\omega}{\pi} \operatorname{Sinc}(\omega n/\pi)$$



5. Find the DTFT of $x(n) = \delta(n)$

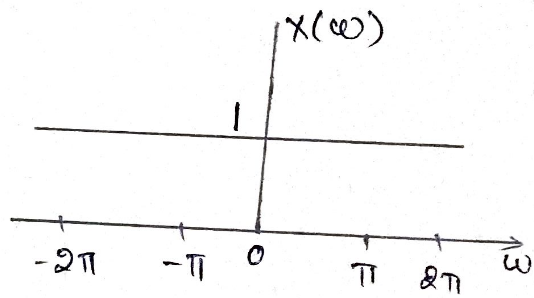


Sol:- For $x(n) = \delta(n)$ we have

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}$$

$$= 1$$

$$\delta(n) \xleftrightarrow{DTFT} 1$$



6. Find the inverse DTFT of $X(\omega) = \delta(\omega), -\pi < \Omega \leq \pi$

Sol: $$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega$$

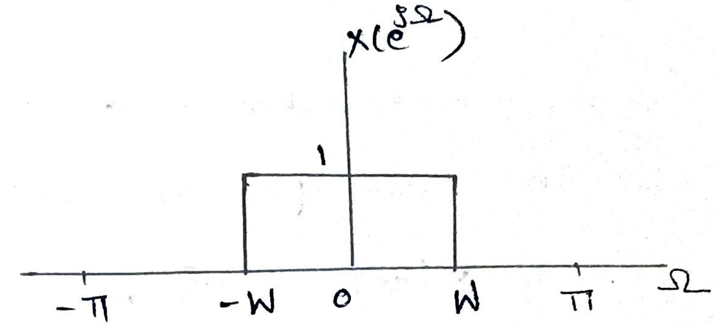
$$= \frac{1}{2\pi}$$

$$\frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega), \quad -\pi < \Omega \leq \pi$$

the inverse DTFT of

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$

which is shown in fig below.



Sol:-
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} 1\cdot e^{j\Omega n}\, d\Omega \quad , n \neq 0$$

$$= \frac{1}{2\pi n j}\, e^{j\Omega n}\Big|_{-W}^{W} \quad , n \neq 0$$

$$= \frac{\sin(Wn)}{\pi n} \quad , n \neq 0$$

Using L-Hopital's rule, we show that

$$\lim_{n\to 0} \frac{\sin(Wn)}{\pi n} = \lim_{n\to 0} \frac{W\cos(Wn)}{\pi}$$

$$= \frac{W}{\pi}$$

we usually write $x(n) = \frac{1}{\pi n}\sin(Wn)$

L' Hopital's rule gives

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$$\lim_{\omega \to 0, \pm 2\pi, \pm 4\pi, \ldots} \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)} = 2M+1$$

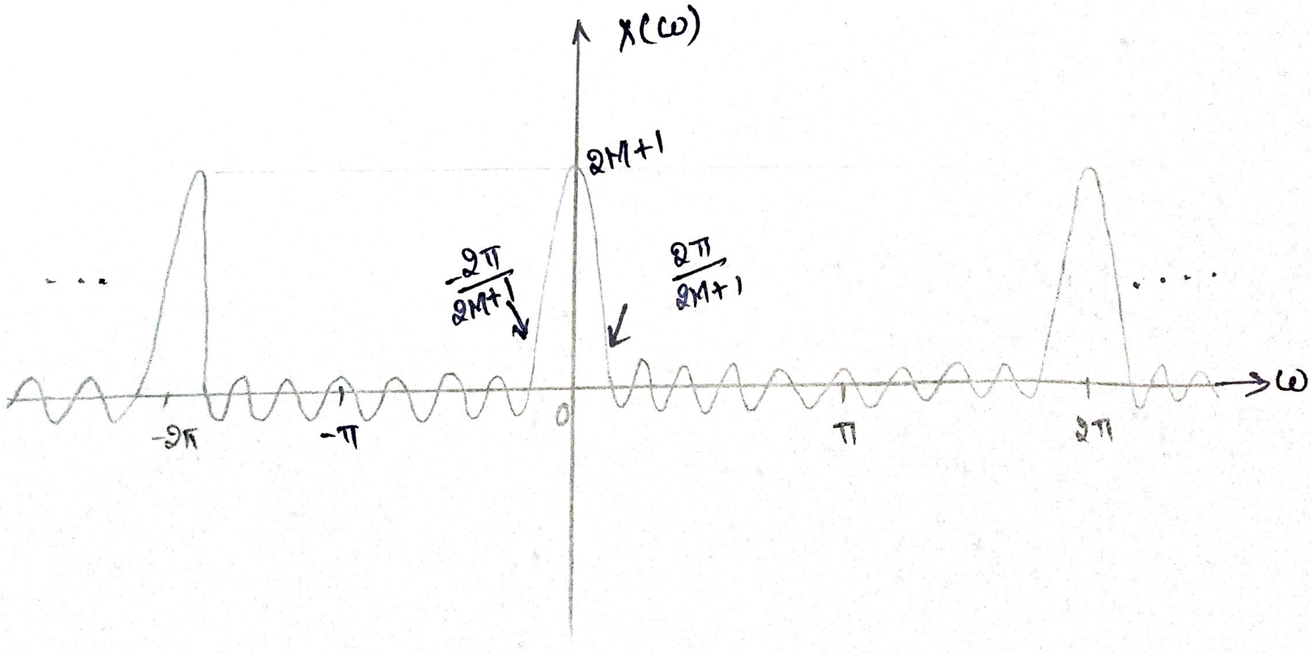
∴ hence rather than write $X(\omega)$ as two forms dependent on the value of ω, we simply write

$$X(\omega) = \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)}$$

with the understanding that $X(\omega)$ for $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ is obtained as limit.

In this $X(\omega)$ is purely real → Mag. Spectrum.

As M increases, the time extent of $x(n)$ ↑, while the energy in $X(\omega)$ becomes more concentrated near $\omega = 0$



Representation of an Arbitrary Sequence :

An arbitrary sequence $x(n)$ can be represented interms of delayed and scaled impulse sequence $\delta(n)$.

* Let $x(n)$ is an infinite sequence as shown in fig-1.
* The sample $x(0)$ can be obtained by multiplying $x(0)$, the magnitude, with unit impulse $\delta(n)$ as shown in fig-1(c).

i.e $$x(0)\,\delta(n) = \begin{cases} x(0) ; & \text{for } n=0 \\ 0 ; & \text{for } n\neq 0 \end{cases}$$

* Similarly, the sample $x(-1)$ can be obtained by multiplying $x(-1)$ the magnitude, with one sample advanced unit impulse $\delta(n+1)$ as shown in fig-1(d).

ie $$x(-1)\,\delta(n+1) = \begin{cases} x(-1); & \text{for } n=-1 \\ 0 ; & \text{for } n\neq -1 \end{cases}$$

* In the same way

$$x(-2)\,\delta(n+2) = \begin{cases} x(-2) ; & \text{for } n=-2 \\ 0 ; & \text{for } n\neq -2 \end{cases}$$

$$x(1)\,\delta(n-1) = \begin{cases} x(1) ; & \text{for } n=1 \\ 0 ; & \text{for } n\neq 1 \end{cases}$$

$$x(2)\,\delta(n-2) = \begin{cases} x(2); & \text{for } n=2 \\ 0 ; & \text{for } n\neq 2 \end{cases}$$

* The sum of the five sequences in the fig-1(a)

$x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\ldots$

equals $x(n)$ for $-2 \le n \le 2$.

* In general we can write $x(n)$ for $-\infty \le n \le \infty$ as

$$x(n) = \ldots\ldots x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n)$$
$$+ x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + \ldots\ldots$$

$$\Rightarrow \boxed{x(n) = \sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)} \rightarrow (1)$$

where $\delta(n-K)$ is unity for $n=K$ and zero for all other terms.

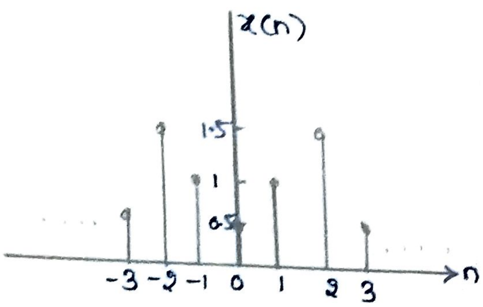
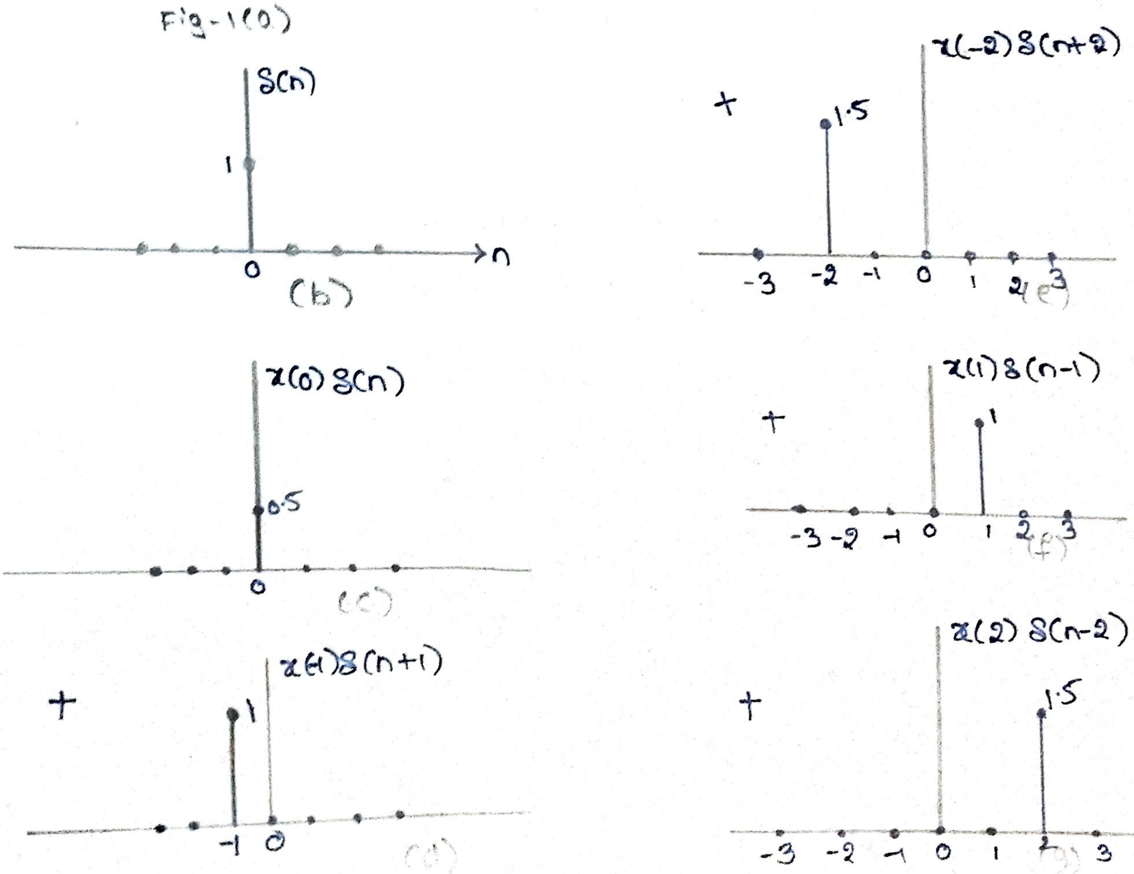


Fig-1(a)

Fig-1: Representation of a sequence as a sum of delayed impulses



Represent the sequence $x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$ as sum of shifted unit impulses.

Sol:- Given $x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$

$n = -3 \ -2 \ -1 \ 0 \ 1 \ 2 \ 3 \ 4$

$$\therefore x(n) = x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n)$$
$$+ x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3) + x(4)\,\delta(n-4)$$

$$= 4\,\delta(n+3) + 2\,\delta(n+2) - \delta(n+1) + \delta(n) + 3\,\delta(n-1) + 2\,\delta(n-2)$$
$$+ \delta(n-3) + 5\,\delta(n-4)$$

Impulse Response & Convolution Sum:

* **Convolution:** Convolution is a mathematical operation which is used to express the input-output relationship of an LTI system.

* A discrete-time s/m performs an operation on an input signal based on a predefined criteria to produce a modified output signal.

* The input signal $x(n)$ is the system excitation, and $y(n)$ is the system response. This transform operation is shown in fig-2.

$$x(n) \longrightarrow \boxed{H} \longrightarrow y(n) = H[x(n)]$$

Fig-2: A DTS representation

* The impulse response of a discrete time system is obtained by setting the input equal to the impulse $\delta(n)$

ie If $x(n) = \delta(n)$ then impulse response is denoted as $h(n)$. ($\because h(n) = H[\delta(n)]$)

* Given the impulse response, we determine the o/p due to an arbitrary input signal by expressing the i/p signal as a weighted superposition of the time shifted impulse responses.

* The weighted superposition is termed the "Convolution Sum" for discrete-time systems.

* An arbitrary signal is expressed as a weighted superposition of shifted impulses.

* Then the convolution sum is obtained by applying a signal represented in this manner to an LTI system

$$\therefore \quad x(n) = \sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)$$

weights of the signal at the corresponding time shifts.

* Let the operator H denote the system to which the i/p $x(n)$ is applied. The system response is given by

$$\therefore \quad y(n) = H[x(n)]$$

$$\Rightarrow y(n) = H\left[\sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)\right]$$

we use the linearity property to interchange the system operator H with the summation and obtain

$$y(n) = \sum_{K=-\infty}^{\infty} H[x(K)\,\delta(n-K)]$$

* Since 'n' is the time index, the quantity $x(K)$ is a constant w.r.t the system operator H.

ie $$y(n) = \sum_{K=-\infty}^{\infty} x(K)\,H[\delta(n-K)]$$

* Further assuming that the system is time invariant, then a time shift in the i/p results in a time shift in the o/p.

ie $$H[\delta(n-K)] = h(n-K)$$

where $h(n) = H[\delta(n)]$ is the impulse response of LTI s/m H.

$$\therefore \boxed{y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K)} \rightarrow (2)$$

Procedure:

1. choose an initial value of n, the starting time for evaluating the o/p sequence $y(n)$.

   If $x(n)$ starts at $n=n_1$ and $h(n)$ starts at $n=n_2$ then $n=n_1+n_2$ is a good choice.

2. Express both sequences interms of the index K.
3. Fold $h(K)$ about $K=0$ to obtain $h(-K)$ and shift by n to the right if 'n' is +ve and left if 'n' is -ve to obtain $h(n-K)$

4. Multiply the two sequences $x(K)$ and $h(n-K)$ element by element and sum the products to get $y(n)$.

5. Increment the index n, shift the sequence $h(n-K)$ to right by one sample and do step 4

6. Repeat step-5 until the sum of products is zero for all remaining values of n.

ie Total length of convolution $= N_1+N_2-1$

where $N_1 =$ No. of samples in $x(n)$ & length of seq $x(n)$

$N_2 =$ No. of samples in $h(n)$ & length of seq $h(n)$

Properties:

1. commutative Law : $x(n) * h(n) = h(n) * x(n)$

2. Associative Law : $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

3. Distributive Law : $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

1. Find the convolution of two finite duration sequences

$$h(n) = a^n u(n) \ \forall n$$

$$x(n) = b^n u(n) \ \forall n$$

i) when $a \neq b$

ii) When $a = b$

is causal

Using eq(2)

Sol

4.4

The impulse response $h(n)=0$ for $n<0$ so the given system is causal and $x(n)=0$ for $n<0$, hence the sequence is a causal sequence.

Using eqn ②, we have $y(n) = \sum_{K=0}^{n} x(K)\, h(n-K)$

$$\Rightarrow y(n) = \sum_{K=0}^{n} b^K a^{n-K}$$

$$= a^n \sum_{K=0}^{n} (b/a)^K \longrightarrow ③$$

$$= a^n \left[1 + (b/a) + (b/a)^2 + \cdots\cdots (n+1) \text{ terms}\right]$$

$$= a^n \left[\frac{1-(b/a)^{n+1}}{1-b/a}\right] \quad \left(\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}\right) \longrightarrow ④$$

When $a=b$ the eqn ④ reduces to indeterminate form. Therefore, applying L'Hopital's rule we get

$$y(n) = a^n \lim_{b\to a} \left[\frac{-(1/a)^{n+1} (n+1) b^n}{(-1/a)}\right]$$

$$= a^n \lim_{b\to a} (n+1)(b/a)^n \longrightarrow ⑤$$

(or)

When $a=b$ eqn ③ reduces to

$$y(n) = a^n \sum_{K=0}^{n} (1)^K = a^n (n+1) \quad \left(\because 1+1+1+\cdots n+1 \text{ terms} = n+1\right)$$

2. Find $y(n)$ if $x(n)=n+2$ for $0\leq n\leq 3$

$h(n)=a^n u(n)$ for all n.

Sol:- We have $y(n)=\sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$

Given $x(n)=n+2$; $0\leq n\leq 3$

$h(n) = a^n u(n)$; $\forall n$

ie $h(n) = 0$; $n<0$, so the system is causal.

$x(n)$ is a causal finite sequence whose value is zero for $n>3$.

$\therefore y(n) = \sum_{K=0}^{3} x(K)\, h(n-K)$

$= \sum_{K=0}^{3} (K+2)\, a^{n-k} u(n-k)$

$= 2a^n u(n) + 3a^{n-1} u(n-1) + 4a^{n-2} u(n-2) + 5a^{n-3} u(n-3)$

3. Determine the response of the relaxed system characterized by the impulse response $h(n) = (1/2)^n u(n)$ to the i/p signal $x(n) = 2^n u(n)$

Sol: Given $x(n) = 2^n u(n)$; $h(n) = (1/2)^n u(n)$

A causal signal applied to a causal S/m.

$\therefore y(n) = \sum_{K=0}^{n} x(K)\, h(n-k)$

$= \sum_{K=0}^{n} 2^K (1/2)^{n-K} = (1/2)^n \sum_{K=0}^{n} 2^{2K}$

$$= (1/2)^n \left[1 + 2^2 + 2^4 + 2^6 + \dots \text{(n+1 terms)}\right]$$

$$= (1/2)^n \left[\frac{1 - (2^2)^{n+1}}{1 - 2^2}\right]$$

$$= (1/2)^n \left[\frac{4.4^n - 1}{4 - 1}\right]$$

$$= (1/2)^n \left[\frac{4^{n+1} - 1}{3}\right]$$

4. Determine the convolution sum of two sequences $x(n) = \{3, 2, 1, 2\}$

$$h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$$

Sol:- $x(n)$ starts at $n_1 = 0$ & $h(n)$ starts at $n_2 = -1$

$\therefore$ The starting value of $n = n_1 + n_2 = -1$

Length of $x(n)$ $N_1 = 4$

Length of $h(n)$ $N_2 = 4$

$\therefore$ Total length of convolution sum $= N_1 + N_2 - 1 = 4 + 4 - 1 = 7$

For $n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

$$\Rightarrow y(-1) = 3 \cdot 1 = 3 \quad \text{(from fig)}$$

Illy for $n = 0$

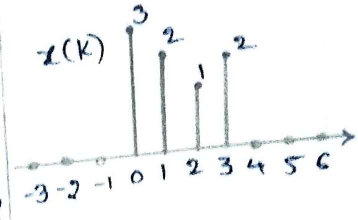
$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) = 3 \cdot 2 + 2 \cdot 1 = 8$$

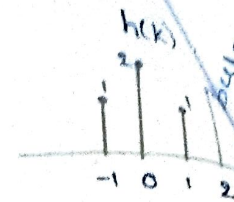
For $n = 1$, $y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) = 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 8$

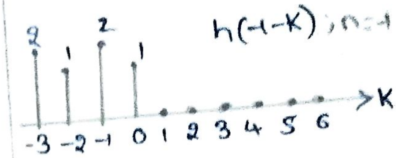
For $n=2$

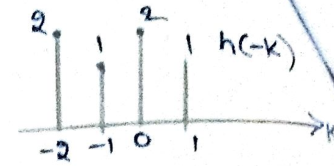
$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$

$$= 3\cdot2+2\cdot1+1\cdot2+2\cdot1 = 12$$







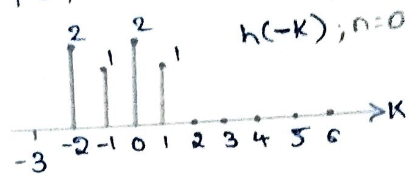


For $n=3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= 3\cdot0+2\cdot2+1\cdot1+2\cdot2+0\cdot1$$

$$= 9$$

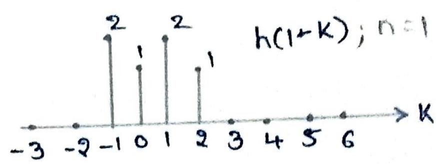


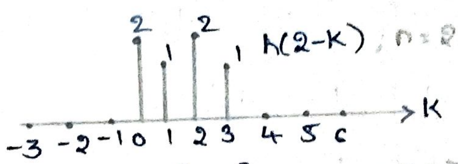
For $n=4$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= 1\cdot2+2\cdot1$$

$$= 4$$

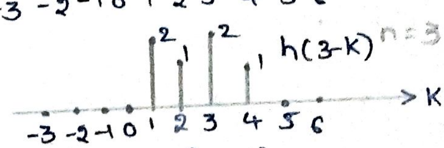




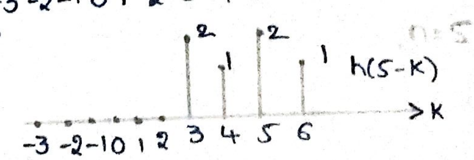
For $n=5$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k)$$

$$= 2\cdot2 = 4$$



$$\therefore y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$$



Check: In given prob, $\sum_n x(n) = 8$, $\sum_n h(n) = 6$, $\sum_n y(n) = 48$

$$\Rightarrow \sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n) \text{ proved.}$$

$\therefore$ The result is correct.

he starting value of $n=-1$

K		-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x(K)$						3	2	1	2				
$n=-1$	$h(-1-k)$		2	1	2	1							
$n=0$	$h(0-K)$			2	1	2	1						
$n=1$	$h(1-K)$				2	1	2	1					
$n=2$	$h(2-K)$					2	1	2	1				
$n=3$	$h(3-K)$						2	1	2	1			
$n=4$	$h(4-K)$							2	1	2	1		
$n=5$	$h(5-K)$								2	1	2	1	

$y(-1) = 3 \cdot 1 = 3$

$y(0) = 3 \cdot 2 + 2 \cdot 1 = 8$

$y(1) = 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 8$

$y(2) = 3 \cdot 2 + 2 \cdot 1 + 1 \cdot 2 + 2 \cdot 1 = 12$

$y(3) = 2 \cdot 2 + 1 \cdot 1 + 2 \cdot 2 = 9$

$y(4) = 1 \cdot 2 + 2 \cdot 1 = 4$

$y(5) = 2 \cdot 2 = 4$

$\therefore y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$

Method-3

S-1: Write down the sequence $x(n)$ & $h(n)$ as shown.

S-2: Multiply each & every sample in $h(n)$ with the samples of $x(n)$ and tabulate the values.

S-3: Divide the elements in the table by drawing diagonal lines as shown.

S-4: starting from the left sum all the elements in each strip & write down in the same order.

$x(n)$

$h(n)$ \ $x(n)$	3	2	1	2
1	3	2	1	2
2	6	4	2	4
1	3	2	1	2
2	6	4	2	4

$$y(n) = \{3,\ 6+2,\ 3+4+1,\ 6+2+2+2,\ 4+1+4,\ 2+2,\ 4\}$$

$$= \{3,\ 8,\ 8,\ 12,\ 9,\ 4,\ 4\}$$

S-5: starting value of $n = -1$, mark the symbol $\uparrow$ at time origin ($n=0$)

$$\therefore y(n) = \{3,\ \underset{\uparrow}{8},\ 8,\ 12,\ 9,\ 4,\ 4\}$$

4.7

the convolution of the signals $x(n) = 1 \;;\; n=-2,0,1$
$\qquad\qquad\qquad\qquad\qquad = 2 \;;\; n=-1$
$\qquad\qquad\qquad\qquad\qquad = 0 \;;\; \text{elsewhere}$

$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$

Given $x(n) = \{1, 2, \underset{\uparrow}{1}, 1\}$ $h(n) = \{\underset{\uparrow}{1}, -1, 1, -1\}$

Starting value of $n = n_1 + n_2 = -2 + 0 = -2$

Length of convolution $= N_1 + N_2 - 1 = 4 + 4 - 1 = 7$

For $n=-2$,

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)\, h(-2-k) = 1\cdot 1 = 1$$

For $n=-1$,

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k) = -1\cdot 1 + 1\cdot 2 = 1$$

For $n=0$,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) = 1\cdot 1 + (-1\cdot 2) + 1\cdot 1 = 0$$

For $n=1$,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) = -1\cdot 1 + 1\cdot 2 + (-1\cdot 1) + 1\cdot 1 = 1$$

For $n=2$,

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k) = -1\cdot 2 + 1\cdot 1 + (-1\cdot 1) = -2$$

For $n=3$

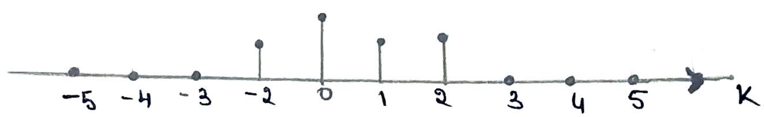
$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k) = -1\cdot 1 + 1\cdot 1 = 0$$

For $n=4$

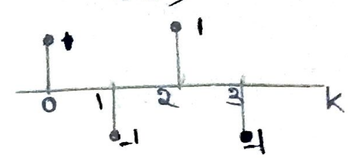
$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k) = -1\cdot 1 = -1$$

$$\therefore y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

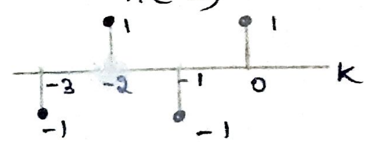
$x(k)$



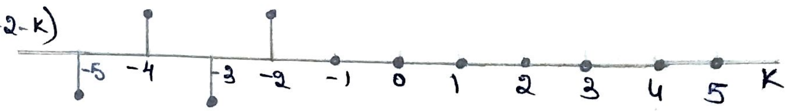
$h(k)$



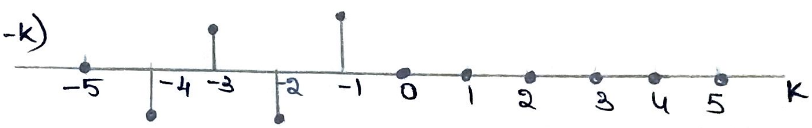
$h(-k)$



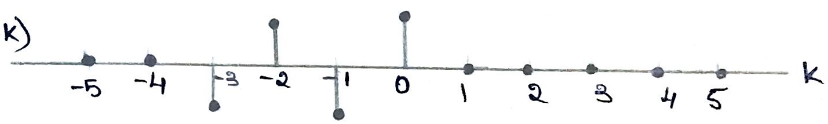
$n=-2$ $h(-2-k)$



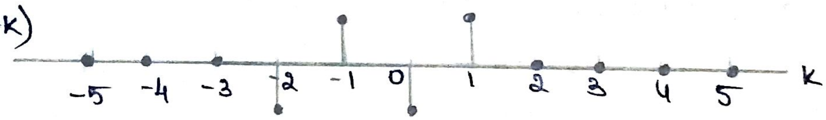
$n=-1$ $h(-1-k)$



$n=0$ $h(-k)$

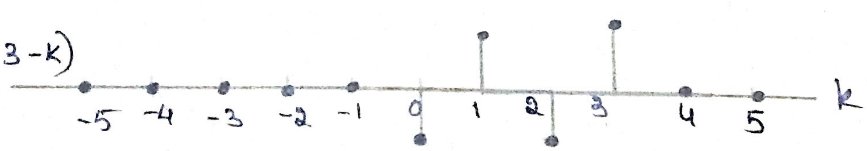


$n=1$ $h(1-k)$



$n=2$ $h(2-k)$

$n=3$ $h(3-k)$



$n=4$ $h(4-k)$