Inner Product Space

→ In the vector space, we generalized the linear structure (addition & scalar multiplication) of $\mathbb{R}^2$ and $\mathbb{R}^3$. We ignored the features, such as length and angles. These ideas are embedded in the concept we now investigate, inner products.

→ **The inner product**

If u and v are vectors in $\mathbb{R}^n$, then we say u and v as $n \times 1$ matrices. The no. $u^T v$ is called the inner product of u and v, and often it is written as $u \cdot v$ (also referred as dot product) and it is denoted by $\langle u, v \rangle = u^T v$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then, inner product of u and v is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example:- Compute $u \cdot v$ and $v \cdot u$ when $u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

Solⁿ $u \cdot v = u^T v = \begin{bmatrix} 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = -1$

$v \cdot u = v^T u = \begin{bmatrix} 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = -1$

Note:- 1) u.v = v.u holds here.
2) Commutativity of the inner product holds in general.

Dot Product - For $x, y \in \mathbb{R}^n$, the dot product of x and y, denoted by $x \cdot y$ is defined by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

where $x = (x_1, x_2, \dots, x_n)$ & $y = (y_1, y_2, \dots y_n)$

Note - ① Dot product of 2 vectors in $\mathbb{R}^n$ is a no / scalar not vector.

Properties ② Inner product is a generalization of dot product for real vector spaces.

of Inner product →

An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a no. $\langle u, v \rangle \in F$ and has the following properties:-

i) positivity: $\langle v, v \rangle \geq 0 \quad \forall \; v \in V$

ii) definiteness: $\langle v, v \rangle = 0$ iff $v = 0$

v) Symmetricity: $\langle u, v \rangle = \langle v, u \rangle$

iii) additivity in first slot:

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \forall \; u, v, w \in V$$

iv) homogeneity in first slot:

$$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \quad \forall \; \lambda \in F \; \& \; \forall \; u, v \in V$$

A vector space with inner product is an inner product space.

Example 1 Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$

Show that $\langle u,v \rangle = 2u_1v_1 + 3u_2v_2$

defines an inner product.

Solⁿ:- Check for all 4 properties:

i) $\langle v,v \rangle = 2v_1v_1 + 3v_2v_2 = 2v_1^2 + 3v_2^2 \geq 0$

ii) $\langle u,v \rangle = 2u_1v_1 + 3u_2v_2 = 2v_1u_1 + 3v_2u_2 = \langle v,u \rangle$

iii) $\langle u,v+w \rangle = 2u_1(v_1+w_1) + 3u_2(v_2+w_2)$

$= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2$

$= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2)$

$= \langle u,w \rangle + \langle u,v \rangle$

iv) $\langle cu,v \rangle = 2(cu_1)v_1 + 3(cu_2)v_2$

$= c(2u_1v_1 + 3u_2v_2)$

$= c\langle u,v \rangle$

Hence, $\langle u,v \rangle$ is an inner product.

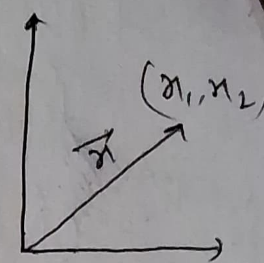
Example 2 Let f and g be in $C[a,b]$, the vector space of all cts functions on the closed interval $[a,b]$. Show that

$$\langle f,g \rangle = \int_a^b f(x)\,g(x)\,dx$$

defines an inner product on $C[a,b]$.

Norm of a vector:-

The length of a vector x in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called the norm of x, denoted by $\|x\|$.



The length of this x vector is $\sqrt{x_1^2+x_2^2}$

Thus for $x=(x_1, x_2) \in \mathbb{R}^2$, we have

$$\|x\| = \sqrt{x_1^2+x_2^2}$$

and in generalized form in $\mathbb{R}^n$

$$x=(x_1, x_2 \ldots x_n) \in \mathbb{R}^n$$

$$\|x\| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$

Note: i) The norm is not linear in $\mathbb{R}^n$.

ii) $\|v\| = \sqrt{v.v} = \sqrt{\langle v.v \rangle}$ in $\mathbb{R}^n$

iii) $\|v\|^2 = \langle v.v \rangle$ where $v \in \mathbb{R}^n$.

iv) If $\|v\|=1$ i.e. $\langle v.v \rangle = 1$, then v is called a unit vector and is said to be normalised.

$$\hat{v} = \frac{v}{\|v\|} \quad \text{--- (1)}$$

process of obtaining $\hat{v}$ is called normalization

Question: Consider $\mathbb{R}^3$

$u=(1,3,-4)$ $v=(4,2,2)$, $w=(5,1,-2)$

then compute $\langle 3u-2v, w \rangle$.

Soln: With the help of additivity in 1st slot

$$\langle 3u-2v, w \rangle = \langle 3u, w \rangle - \langle 2v, w \rangle$$

$$= 3\langle u, w \rangle - 2\langle v, w \rangle \quad \text{--- (ii)}$$

Now $\langle 3u-2v, w\rangle = \langle (3(1,3,-4) - 2(4,2,2)), (5,1,-2)\rangle$ 5

$= \langle ((3,9,-12) - (8,4,4)), (5,1,-2)\rangle$

$= \langle (-5,5,-16), (5,1,-2)\rangle$

$= -25 + 5 + 32 = 12$

$3\langle u,w\rangle - 2\langle v,w\rangle = 3[(1,3,-4)(5,1,-2)] - 2[(4,2,2)(5,1,-2)]$

$= 3[16] - 2[18] = 12$ Ans.

ii) Normalize u and v,

$\|u\| = \sqrt{1+9+16} = \sqrt{26}$

$\hat{u} = \dfrac{u}{\|u\|} = \left(\dfrac{1}{\sqrt{26}}, \dfrac{3}{\sqrt{26}}, \dfrac{-4}{\sqrt{26}}\right)$

$\|v\| = \sqrt{v \cdot v} = \sqrt{16+4+4} = \sqrt{24}$

$\hat{v} = \dfrac{v}{\|v\|} = \left(\dfrac{4}{\sqrt{24}}, \dfrac{2}{\sqrt{24}}, \dfrac{2}{\sqrt{24}}\right)$

Orthogonal :- Two vectors $u, v \in V$ are called orthogonal if $\langle u, v\rangle = 0$.
Geometrically, the vectors are mutually perpend.

Note:- Orthogonality and 0 →

a) 0 is orthogonal to every vector in V.

b) 0 is the only vector in V that is orthogonal to itself.

Orthogonal Set of vectors →

A set of vectors $\{v_1, v_2 - - v_k\}$ in $\mathbb{R}^n$ is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal i.e., if

$$v_i \cdot v_j = 0 \quad \text{whenever } i \neq j$$
$$\text{for } i, j = 1, 2, \dots, k.$$

Note: If the orthogonal set also forms a basis for $\mathbb{R}^n$, it is called an orthogonal basis for the vector space $\mathbb{R}^n$.

Theorem:- If $\{v_1, v_2, \dots, v_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then

a.) $v_i \cdot v_j = 0$ for $i \neq j$

b.) $v_1, v_2, \dots, v_k$ are linearly independent.

c.) Any $w \in \mathbb{R}^n$, $w = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ where c_i's are unique scalars and they are (c_i's) given by

$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i} \quad \text{for } i = 1, 2, \dots, k.$$

Problems:- ① Show that $\{v_1, v_2, v_3\}$ is an orthogonal set in $\mathbb{R}^3$ if

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

→ We must show every pair of vectors from this set is orthogonal. i.e

$v_1 \cdot v_2 = 2(0) + 1.1 + -1.1 = 0$

$v_2 \cdot v_3 = 0(1) + 1(-1) + (1)(1) = 0$

$v_1 \cdot v_3 = 2(1) + 1(-1) + (-1)(1) = 0$

Hence v_1, v_2 and v_3 are mutually perpendicular

Note:- You can clearly check v_1, v_2 and v_3 are linearly independent too.

(ii) Determine which of the following sets are orthogonal

a) $u = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ $v = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

b) $u = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ $v = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

(iii) Show that $v_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an orthogonal basis in $\mathbb{R}^2$. For a given $w = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ find the coordinate vector $[w]_B$ of w w.r.t. basis.

Solun clearly $\langle v_1, v_2 \rangle = (4-4) = 0$

$\Rightarrow$ and v_1 & v_2 are l.i also

$\Rightarrow$ v_1, v_2 are orthogonal basis in $\mathbb{R}^2$.

for $w = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ in $\mathbb{R}^2$

$\Rightarrow$ $w = c_1 v_1 + c_2 v_2$

$\begin{pmatrix} 1 \\ -3 \end{pmatrix} = c_1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

for $c_i = \dfrac{w \cdot v_i}{v_i \cdot v_i}$

$\Rightarrow c_1 = \dfrac{(1,-3)\cdot(4,-2)}{(4,-2)\cdot(4,-2)} = \dfrac{4+6}{16+4} = \dfrac{10}{20} = \dfrac{1}{2}$

$c_2 = \dfrac{(1,-3)\cdot(1,2)}{(1,2)\cdot(1,2)} = \dfrac{1-6}{1+4} = \dfrac{-5}{5} = -1$

Hence $w = \frac{1}{2} v_1 - v_2$

thus $[w]_B = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$

iv) $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ & $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$, then show that $B = \{v_1, v_2, v_3\}$ forms an orthogonal basis in $\mathbb{R}^3$ and for given $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find the coordinate vector $\{w\}_B$ w.r.t. basis B.

Solⁿ – Its easy to check that B form an orthogonal basis for $\mathbb{R}^3$.

for $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$c_i = \dfrac{w \cdot v_i}{v_i \cdot v_i}$

$c_1 = 0, \quad c_2 = \dfrac{2}{3}, \quad c_3 = \dfrac{1}{3}$

then $w = 0 \cdot v_1 + \frac{2}{3} v_2 + \frac{1}{3} v_3$

then $(w)_B = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix}$

<u>Ques. VI</u> Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$

<u>Soln</u> — clearly, W is a plane going through origin.
From the equation of plane, we have $x = y - 2z$ — (I)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -2z \\ 0 \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

so, say $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are a basis of W, but they are not orthogonal.

So, find another non-zero vector w which is orthogonal to either u or v:

Let $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \langle u, w \rangle = 0$
$\Rightarrow x + y = 0$ — (II)

and from (I) & (II)

$\Rightarrow x = -z$ and $y = z$

thus $w = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

so $\{u, w\}$ is an orthogonal set in W and hence, an orthogonal basis for W, since $\dim W = 2$

~~Note:- a vector perpendicular to both will also work. Jery for cross product of basis vectors if they are not orthogonal.~~

Orthonormal -

i) A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all other vectors in the list.

In other vectors, a list $v_1, v_2, \ldots v_k$ of vectors in V is orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Example ① show that $S = \{q_1, q_2\}$ is an orthonormal set in $\mathbb{R}^3$.

$$q_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } q_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\langle q_1, q_2 \rangle = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$$

$$\langle q_1, q_1 \rangle = \frac{1}{(\sqrt{3})^2} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\langle q_2, q_2 \rangle = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$$

(ii) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is an orthonormal list in $\mathbb{R}^3$.

(iii) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$ is an orthonormal list in $\mathbb{R}^3$.

- If $\{q_1, q_2, \dots q_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and w be any vector in W, then

$$w = \langle w, q_1 \rangle q_1 + \langle w, q_2 \rangle q_2 + \dots + \langle w, q_k \rangle q_k$$

and this representation is unique.

Problems:- Construct an orthonormal basis for $\mathbb{R}^3$

(i) $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(ii) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Solⁿ:- $q_1 = \dfrac{v_1}{\|v_1\|}$, $q_2 = \dfrac{v_2}{\|v_2\|}$, $q_3 = \dfrac{v_3}{\|v_3\|}$

thus $\{q_1, q_2, q_3\}$ forms an orthonormal basis for $\mathbb{R}^3$.

non-square: The columns of an $m \times n$ matrix Q form an orthonormal set iff $Q^TQ = I_n$

Orthogonal Matrix → An $n \times n$ matrix Q whose columns form an orthonormal set is called an orthogonal matrix. $\boxed{A^TA = I}$

for square matrix, row & col norm = 1
non-square , col norm = 1

Example!
i) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

as it can be seen columns are orthonormal.

ii) $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ → columns are orthonormal

Note:- If Q is orthogonal matrix, then its rows also form an orthonormal set (only for square matrices)

Note - i) A square matrix Q is orthogonal iff
$Q^{-1} = Q^T$ (or $Q^TQ = I_n = Q \cdot Q^T$)

for eg: i) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

as $A^T = A^{-1}$

Hence orthogonal.

ii) $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$ orthogonal

as $B^{-1} = B^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Theorem:- Let Q be an orthogonal matrix, ~~then~~

~~i) its rows form an orthonormal set~~

ii) Q^T, Q^{-1} is orthogonal

iii) $\det Q = \pm 1$

iv) If λ is an eigenvalue of Q then $|\lambda| = 1$

v) If Q_1, Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$.

Example $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}$ say $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

then all above holds. (orthogonal)

then $A \cdot A^T = I_n =$ (orthogonal)

Note- i) An orthogonal matrix of 2×2 must have the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector.

ii) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\det A = 1$

but A is not orthogonal matrix.

iii) Converse of above theorem need not to be true.

Theorem: Let Q be an $m\times n$ matrix. The following statements are equivalent:

a) Q is orthogonal.

b) $\|Qx\| = \|x\|$ for every $x \in \mathbb{R}^n$

c) $Qx \cdot Qy = x \cdot y$ for every x and y in $\mathbb{R}^n$

Ques: Let $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$ and $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$

verify $\|Ux\| = \|x\|$

Soln- as $U^TU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence U has orthonormal columns

$$Ux = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\|Ux\| = \sqrt{9+1+1} = \sqrt{11}$

$\|x\| = \sqrt{2+9} = \sqrt{11}$

Note → 1) The above example depicts that $\|Ux\| = \|x\|$ preserves length.

Gram-Schmidt Process :-

It is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$.

Process :-

Let $\{x_1, x_2, \ldots x_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following :

$$v_1 = x_1$$

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 \longrightarrow \text{(projection of } x_2 \text{ onto } v_1)$$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$\vdots$$

$$v_p = x_p - \left(\frac{x_p \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_p \cdot v_2}{v_2 \cdot v_2}\right) v_2 - \cdots - \left(\frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}}\right) v_{p-1}$$

Then $\{v_1, v_2, \ldots v_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{v_1, v_2, \ldots v_k\} = \text{Span}\{x_1, x_2, \ldots x_k\}$$

for $1 \le k \le p$

mple 1 Apply the Gram-Schmidt process to construct an orthonormal basis for the subspace $W = \text{Span}\{x_1, x_2, x_3\}$ of $\mathbb{R}^4$, where

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution – Since $\{x_1, x_2, x_3\}$ is l.i. set, so it forms a basis for W.

$$v_1 = x_1 = [1 \ -1 \ -1 \ 1]^T$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{(3+3+\frac{1}{2}+1)}{\frac{9}{4}+\frac{9}{4}+\frac{1}{4}+\frac{1}{4}} \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{\frac{15}{2}}{5} \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{15}{10} \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Now we have orthogonal basis $\{v_1, v_2, v_3\}$ for W.

Now $q_1 = \frac{1}{\|v_1\|} \cdot v_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$q_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{3\sqrt{5}}{10} \\ \frac{3\sqrt{5}}{10} \\ \frac{\sqrt{5}}{10} \\ \frac{\sqrt{5}}{10} \end{bmatrix}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ 0 \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$ Ans.

Then $\{q_1, q_2, q_3\}$ is an orthonormal basis for W.

(ii) $x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(iii) $x_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 8 \\ 1 \\ 6 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Try (ii) & (iii)rd

Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solⁿ:- Find a basis for $\mathbb{R}^3$ containing v_1. If we take

$x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

then $\{v_1, x_2, x_3\}$ is clearly a basis for $\mathbb{R}^3$.

Now $v_2 = x_2 - \left(\dfrac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ \frac{5}{7} \\ -\frac{3}{7} \end{bmatrix}$$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2$$

$$= \begin{bmatrix} -\frac{3}{10} \\ 0 \\ \frac{1}{10} \end{bmatrix}$$

Then $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$ that contains v_1.

Practice Problems:-

Determine whether the following matrices are orthogonal and hence find its inverse.

ⓐ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ⓑ $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ ⓒ $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ -\frac{1}{3} & 0 & \frac{2}{5} \end{pmatrix}$

(iii) Find the missing entries of Q to make Q an orthogonal matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & - \\ 0 & \frac{1}{\sqrt{3}} & - \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & - \end{pmatrix}$$

(e) Determine whether the given orthogonal matrix represents a rotation or a reflection. If it is a rotation, give the angle of rotation, if it is a reflection, give the line of reflection.

(i) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ → rotation by $\theta = 45^\circ$

(ii) $\begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ → rotation by $\theta = 120^\circ \times 2$
$= 240^\circ$

(iii) $\begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ → reflection by $x = \sqrt{3}y$

(iv) $\begin{pmatrix} \frac{-3}{5} & \frac{-4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix}$ → reflection by $x = 2y$.

* If the diagonal values are same, then it is rotation.
diff, reflection.

$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$ → reflection. $\boxed{m = \sqrt{3},\ y = mx}$