Orthogonal Complement:-

Let W be a subspace of $\mathbb{R}^n$. A vector v in $\mathbb{R}^n$ is orthogonal to W if v is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called orthogonal complement of W, denoted $W^\perp$. That is

$$W^\perp = \{v \text{ in } \mathbb{R}^n : v.w = 0 \ \forall \ w \text{ in } W\}$$

Note:→ i) A vector x is in $W^\perp$ iff x is orthogonal to every vector in a set that spans W.

ii) $W^\perp$ is a subspace of $\mathbb{R}^n$.

iii) $(W^\perp)^\perp = W$

iv) $W \cap W^\perp = \{0\}$

v) If $\{w_1, w_2, -- w_k\}$ spans W then z is in $W^\perp$

$\Rightarrow w_1 \cdot z = w_2 \cdot z = --- = w_k \cdot z = 0$

Since any $w \in W$, if we have to find z s.t.

$w_1 \cdot z = w_2 \cdot z = --- = w_k \cdot z = 0$ mean we have to find the null space of W.

Hence, if A is an $(m \times n)$ matrix, then the orthogonal complement of the row space of A is the null space of A. i.e.

$$\boxed{(\text{row } A)^\perp = \text{null } A}$$

as $\text{row } A^T = \text{col } A$

$$\therefore \boxed{(\text{col } A)^\perp = \text{Null } A^T}$$

Orthogonal complement of column space of A is the null space of A^T.

Reason:- Every vector x that satisfies $Ax=0$ is in the nullspace of A, and is perpendicular to every row of A. Thus x is also $\perp$ar to linear combination of rows of A and is thus $\perp$ar to every vector in the row space.

Problems:- ① Let W be the subspaces spanned by the vectors $\{w_1, w_2\}$. Find a basis for $W^\perp$ where $w_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $w_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

Solu^n:- $A = \begin{pmatrix} 2 & 4 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}$ Now $A^T = \begin{pmatrix} 2 & 1 & -2 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_2}$

$$\begin{pmatrix} 2 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & -5/2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -5/2 \end{pmatrix}$$

so: $y_1 = -\frac{1}{4} y_3$ basis $=$ span $\begin{pmatrix} -1/4 \\ 5/2 \\ 1 \end{pmatrix}$

$y_2 = \frac{5}{2} y_3$

$$\text{null}(A^T) = \left\{ \begin{pmatrix} -1 \\ 10 \\ 4 \end{pmatrix} \right\} = (\text{col } \cancel{\text{row}}\ A)^\perp$$

② Let W be a subspace of $\mathbb{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of the $W^\perp$ of W.

Solu^n

Let $w = (x, y, z, s, t)$

$w \cdot u = x + 2y + 3z - s + 2t = 0$

$w \cdot v = 2x + 4y + 7z + 2s - t = 0$

Now $A = \begin{pmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{pmatrix}$

then $x + 2y + 3z - s + 2t = 0$

$z + 4s - 5t = 0$

Let $t = a$ $\quad s = b$

$z = 5a - 4b$

$y = c$

$x = -2c - 3(5a - 4b) + b - 2a$

$= -2c - 15a + 12b + b - 2a$

$= -2c - 17a + 13b$

$$\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = a \begin{pmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Basis $=$ Span $\left\{ \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$= \text{null}(A) = (\text{row } A)^{\perp}$

Answer

Ques.) (ii) Find the orthogonal complement of $w^{\perp}$ of w and give a basis for $w^{\perp}$

Solu^n $\quad w = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 2x - y = 0 \right\}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} x$

Now $v = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad v \cdot w = 0$

$\Rightarrow a + 2b = 0$

$\Rightarrow a = -2b$

then $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2b \\ b \end{pmatrix} = +b \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

so basis for $w^\perp = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ Answer

(iv) $W = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x - y + 3z = 0\right\}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x + 3z \\ z \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{w_1} + z \underbrace{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}_{w_2}$$

say $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in W^\perp \Rightarrow w_1 \cdot v = 0 \Rightarrow a + 2b = 0$

$\Rightarrow w_2 \cdot v = 0 \Rightarrow 3b + c = 0$

Now let $c = t$, $b = -\frac{t}{3}$ $a = \frac{2}{3} t$

basis for $W^\perp = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}$ Ans

(v) $W = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}, x = t, y = -t, z = 3t\right\}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \underbrace{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}_{w_1}$$

Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in W^{\perp}$

So $w_1 \cdot v = 0 \Rightarrow a - b + 3c = 0$

$c = t$, $b = r$ $a = r - 3t$

basis for $W^{\perp} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ (Answer)

Problems:- i) Find bases for the row space and null space of A. Verify that every vector in row (A) is orthogonal to every vector in null (A).

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 7 & -14 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

row (A) = span $\{u_1, u_2\}$

$u_1 = (1, 0, 1)$ & $u_2 = (0, 1, -2)$ — (i)

null(A) = $x + z = 0$

$y - 2z = 0$

$z = t$, $y = 2t$

∴ null (A) = span $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ and $v = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

— (ii)

$v \cdot u_1 = 1 - 2 + 3 = 0$

$v \cdot u_2 = 0 + 2 - 2 = 0$

To show $(\text{row } A)^{\perp} = \text{null } A$, it is enough to show that every u of Row(A) is orthogonal to v.

$\rightarrow$ $\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

$$u_1 = \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{Null space of } A^T = \left\{ \begin{pmatrix} 5 \\ -1 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 7 \end{pmatrix} \right\}$$

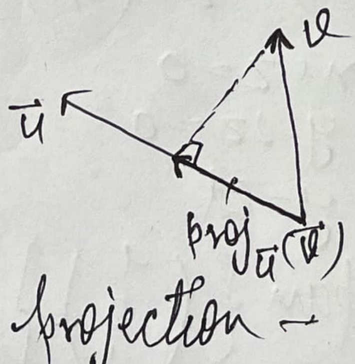
clearly $(\text{col} A)^{\perp} = \text{null } A^T$ **Proved**

Orthogonal Projection:-

In $\mathbb{R}^2$, the projection of $\vec{v}$ onto $\vec{u}$ is defined as

$$\text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \right) \vec{u}$$

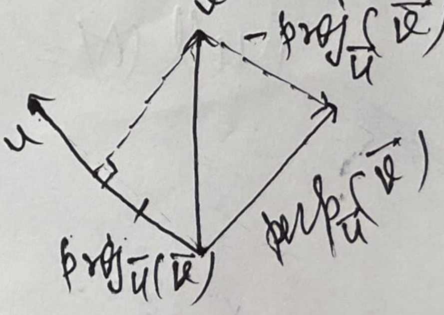
Geometrically:



Now orthogonal projection -

$\Rightarrow$ $\text{perp}_{\vec{u}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{u}}(\vec{v})$

$\vec{v} = \text{proj}_{\vec{u}}(\vec{v}) + \text{perp}_{\vec{u}}(\vec{v})$



Definition:- Let W be a subspace of $\mathbb{R}^n$ and let $\{u_1, u_2, \ldots, u_k\}$ be orthogonal basis for W. For any vector v in $\mathbb{R}^n$, the orthogonal projection of v onto W is defined as

$$\text{proj}_W(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)u_1 + \cdots + \left(\frac{u_k \cdot v}{u_k \cdot u_k}\right)u_k \quad \text{①}$$

The component of v orthogonal to W is the vector

$$\text{perp}_W(v) = v - \text{proj}_W(v)$$

eq^n ① can be rewritten as

$$\text{proj}_W(v) = \text{proj}_{u_1}(v) + \cdots + \text{proj}_{u_k}(v)$$

Question Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$ and let $v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of v onto W and the components of v orthogonal to W.

Solu^n- Orthogonal basis for $W = \{u_1, u_2\}$ where

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Now orthogonal projection of v onto W is

$$\text{proj}_W(v) = \text{proj}_{u_1}(v) + \text{proj}_{u_2}(v)$$

$$= \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)u_1 + \left(\frac{u_2 \cdot v}{u_2 \cdot u_2}\right)u_2$$

$$= \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{(-3-1+2)}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}5/3\\1/3\\-2/3\end{bmatrix}$$

and component of v orthogonal to W

$$perp_W(v) = v - proj_W(v)$$

$$= \begin{bmatrix}3\\-1\\2\end{bmatrix} - \begin{bmatrix}5/3\\1/3\\-2/3\end{bmatrix} = \begin{bmatrix}4/3\\-4/3\\8/3\end{bmatrix}$$ Answer

Note:- 1) It is easy to see that $proj_W(v)$ is in W, since it satisfies the eqn of plane.

2) It is equally easy to see that $perp_W(v)$ is orthogonal to W, since it is a scalar multiple of the normal vector $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ to W.

→ If we let $W = span(u)$, then $w = proj_u(v)$ is in W and $w^\perp = perp_u(v)$ is in $W^\perp$. We therefore have a way of decomposing v into the sum of two vectors, one from W and other orthogonal to W, namely

$$v = w + w^\perp$$

formally;

The Orthogonal Decomposition Theorem :→

Let W be a subspace of $\mathbb{R}^n$ and let v be a vector in $\mathbb{R}^n$. Then there are unique vectors w in W and $w^\perp$ in $W^\perp$ such that

$$\boxed{v = w + w^\perp}$$

Eg: from previous example $\boxed{w + w^\perp = v}$

(Answer)

QR Factorization

If A is an $m\times n$ matrix with linearly independent columns ($m \geq n$), then applying the Gram-Schmidt process to these columns yields a very useful factorization of A into the product of a matrix Q with orthonormal columns and an upper triangular matrix R. This is QR factorization, and it has applications in solving the equations and finding the eigenvalues.

Theorem:- If A is an $m\times n$ matrix with linearly independent columns, then A can be formed and factored as $A=QR$, where Q is an $m\times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n\times n$ upper triangular invertible matrix with positive entries on its diagonal.

Example:- Find a QR factorization of $A=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\\1&1&1\end{bmatrix}$

Solⁿ:- An orthogonal basis for col $A = \{v_1, v_2, v_3\}$ is

$$v_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad v_2=\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \quad v_3=\begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix}$$

Now normalize v_1, v_2 and v_3

$$Q=\begin{bmatrix}\frac{1}{2} & -3/\sqrt{12} & 0\\ \frac{1}{2} & 1/\sqrt{12} & -2/\sqrt{6}\\ \frac{1}{2} & 1/\sqrt{12} & 1/\sqrt{6}\\ \frac{1}{2} & 1/\sqrt{12} & 1/\sqrt{6}\end{bmatrix} = [q_1\ q_2\ q_3]$$

as $A = QR$ for some upper triangular matrix.
To find R, we use the fact that Q has orthonormal columns and, hence $Q^T Q = I$, thus

$$A = QR$$
$$Q^T A = Q^T Q R$$
$$Q^T A = I R$$
$$= R$$

thus, $R = Q^T A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Question:- Find QR factorization of $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Orthogonal Diagonalization of Symmetric Matrices:-

As we know, not all square matrices are diagonalizable. The situation changes if we restrict our attention to real symmetric matrices. All the eigenvalues of a real symmetric matrix are real, and such matrix is always diagonalizable.

Eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

E. Values: $-3, 2$
$\Rightarrow$ diagonalizable

$B = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

E. Values: $8, 6, 3$
$\Rightarrow$ diagonalizable

$C.E. = -\lambda^3 + 17\lambda^2 - 90\lambda + 144$
$= -(\lambda - 8)(\lambda - 6)(\lambda - 3)$

f^n :- A square matrix is orthogonally diagonalizable if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

C.E. $= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$

E.V. for $\lambda = -3$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = v_1$

$\lambda = 2$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$

thus $P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Since v_1 & v_2 are orthogonal, thus

$u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$ $u_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

and then $Q = [u_1\ u_2] = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -2/\sqrt{5} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Since Q is an orthogonal matrix, then $Q^{-1} = Q^T$

then $Q^TAQ = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} - \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$= \begin{bmatrix} -\frac{3}{\sqrt{5}} & \frac{6}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = D$

Hence, A is orthogonally diagonalizable.

Theorem:- ~~① An n×n matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.~~

① If A is symmetric matrix, then any 2 eigenvector corresponding to distinct eigenvalues of A are orthogonal.

Example:- $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

C.E.: $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda-4)(\lambda-1)^2$

thus eigenvalues are 1, 1, 4.

Now $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ and $E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$

clearly $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0$

$\Rightarrow$ E_4 is orthogonal to every vector in E_1.

Note: ① $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1$. Thus, eigenvectors corresponding to the same eigenvalue need not to be orthogonal.

Ⅲ

The Spectral Theorem :→

The set of eigenvalues of a matrix is called spectrum and the spectral theorem is given below as:

Theorem: An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

Example:- $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \;\&\; E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

Now orthonormal vectors

$$Q = [q_1 \; q_2 \; q_3] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

Now Q is orthogonal matrix

$$\Rightarrow Q^{-1} = Q^T$$

Now $Q^T A Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Answer

$\Rightarrow$ A is orthogonally diagonalizable.

Spectral Decomposition :→

The spectral theorem allows us to write a real symmetric matrix A in the form $A = QDQ^T$, where Q is orthogonal and D is diagonal.

The diagonal entries of D are just the eigenvalues of A, and if columns of Q are orthonormal vectors $q_1, q_2, \ldots, q_n$, then we have

$$A = QDQ^T = [q_1 \ldots q_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= [\lambda_1 q_1 \ \cdots \ \lambda_n q_n] \cdot \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

This is called the spectral decomposition of A. ①

<u>Note</u> i) Each of the terms $\lambda_i q_i q_i^T$ is a rank 1 matrix

ii) $q_i q_i^T$ is actually the matrix of the projection onto the subspace spanned by q_i.

iii) eqⁿ ① is sometimes referred to as the projection form of the spectral theorem.

<u>Example</u>:- Find the spectral decomposition of the given matrix $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

<u>Solⁿ</u>:- C.E.: $\lambda^2 - 11\lambda + 24 = 0 \Rightarrow \lambda = 8, 3$

eigenvector corresponding to $\lambda = 8$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_1$

$\lambda = 3$ is $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 2 \end{bmatrix} = v_2$

as v_1 & v_2 are orthogonal.

$$\Rightarrow Q = [q_1 \ q_2] = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \rightarrow \text{orthogonal matrix.}$$

Now $q_1 q_1^T = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & 2/5 \\ \frac{2}{5} & 1/5 \end{bmatrix}$

$q_2 q_2^T = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

So: $\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T = 8\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + 3\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

$= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A$

Note:- Spectral decomposition expresses a symmetric matrix A explicitly in terms of eigenvalues & eigenvectors. This gives us a way of constructing a matrix with given eigenvalues and orthonormal eigenvectors.

Example:- Find a 2×2 matrix with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$ and its corresponding eigenvectors $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Solⁿ:- Now $q_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ $q_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

with the help of spectral decomposition:

$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T =$

$= 3\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} + (-2)\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$

$= 3\begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix} - 2\begin{bmatrix} \frac{16}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{9}{25} \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{6}{5} \end{bmatrix}$

Answer

Quadratic forms :-

Until now, our attention in the course has focused only on linear equations, except for the sums of squares when computing $x^T x$. Such sums and more general expressions, called quadratic forms, occurs frequently in applications of linear Algebra to engineering.

An expression of the form

$$ax^2 + by^2 + cxy \quad \text{—— (i)}$$

is called a quadratic form in x and y. Similarly

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz \quad \text{—— (ii)}$$

is a quadratic form in x, y and z.

Note - i) A quadratic form is the sum of terms, each of which has total degree two in the variables.

ii) $5x^2 - 3y^2 + 2xy$ is a quadratic form, but $x^2 + y^2 + x$ is not.

Definition :- A quadratic form in n variables is a funⁿ $f: \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = x^T A x$$

where A is symmetric $n \times n$ matrix and x in $\mathbb{R}^n$.

We refer to A as the matrix associated with f.

Examples :- ① Write the quadratic form associated with the matrices

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$$

Solⁿ: for matrix A

$$x^TAx = [x_1 \; x_2]\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 5x_2^2 - 6x_1x_2$$

for matrix B

$$x^TAx = [x_1 \; x_2]\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2$$

for matrix C

$$x^TAx = [x_1 \; x_2]\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 - 4x_1x_2 + 7x_2^2$$

Answer

Example ②:- Find the matrix associated with the quadratic form

i) $f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2 - 3x_1x_3$

ii) $f(x_1, x_2, x_3) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

Solⁿ → The coefficients of x_i^2 go on the diagonal of A. To make A symmetric, the coefficients of x_ix_j for $i \neq j$ must be split b/w (i, j) and (j, i) entries in A.

thus i) $A = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & -1 & 0 \\ -\frac{3}{2} & 0 & 5 \end{bmatrix}$

ii) $B = \begin{bmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$

Example ③ Let $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$. Compute the value of $Q(x)$ for $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Solⁿ → $Q(-3, +1) = (-3)^2 - 8(-3)(+1) - 5(1)^2 = 28$

$Q(2, -2) = 16 = (2)^2 - 8(2)(-2) - 5(-2)^2$

$Q(1, -3) = 1^2 - 8(1)(-3) - 5(-3)^2 = -20$

Answer

In some cases, quadratic forms are easier to use when they have no cross-product terms, i.e. when the matrix of the quadratic form is a diagonal matrix. Fortunately, the cross-product term can be eliminated by making a suitable change of variables.

⇒ Let $f(x) = x^T A x$ be a quadratic form in n variables, with A a symmetric $n \times n$ matrix. By the spectral theorem, there is a diagonal matrix Q that diagonalizes A; i.e. $Q^T A Q = D$, where D is a diagonal matrix displaying the eigenvalues of A. Now set

$x = Qy$ or, equivalently $y = Q^{-1}x = Q^T x$

substitute this into the quadratic form yields

$$
\begin{aligned}
x^T A x &= (Qy)^T A (Qy) \\
&= y^T Q^T A Q y \\
&= y^T D y \quad (\text{as } Q^T A Q = D)
\end{aligned}
$$

which is a quadratic form without cross-product term. Thus

$$y^T D y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

This process is called diagonalizing a quadratic form.

The Principal Axes Theorem :-

Every quadratic form can be diagonalized; i.e. Let A be an $n \times n$ symmetric matrix, then there is an orthogonal change of variables, $x = Qy$, that transforms the quadratic form $x^T A x$ into a quadratic form $y^T D y$ with no cross-product term.

Note:- (i) The columns of Q in the theorem are called the principal axes of the quadratic form x^TAx. (11)
(ii) The vector y is the coordinate vector of x relative to the orthonormal basis of $\mathbb{R}^n$ given by these principal axes.

Example:- Find a change of variables that transforms the quadratic form
$$f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$$
into one with no cross-product terms.

Solu^n:- The matrix of f is $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$

C.E.: $\lambda^2 - 7\lambda + 6 = 0$
$\Rightarrow \lambda_1 = 6$ & $\lambda_2 = 1$

E.V. corresponding to $\lambda = 6$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_1$
$\lambda = 1$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = v_2$

clearly v_1 & v_2 are orthogonal.

Now $Q = [q_1 \; q_2] = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$

then $Q^TAQ = D$. The change of variables $x = Qy$ where
$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

converts f into $f(y) = f(y_1, y_2) = [y_1 \; y_2] \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
$= 6y_1^2 + y_2^2$

Note:- The original quadratic form x^TAx and the new one y^TDy are equal in the following sense, suppose we want to evaluate,

$f(x) = x^T A x$ at $x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. We have

$$f(-1,3) = 5(-1)^2 + 4(-1)(3) + 2(3)^2 = 11$$

$$f(y_1, y_2) = \cancel{f(-1,3)} = 6y_1^2 + y_2^2 = 6\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{-7}{\sqrt{5}}\right)^2$$

$$= \frac{55}{5} = 11$$

where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y = Q^T x = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -7/\sqrt{5} \end{bmatrix}$

Answer

Exercise :- $f(x_1, x_2) = x_1^2 + 5x_2^2 - 8x_1x_2$

Ans :- $f(y_1, y_2) = 3y_1^2 - 7y_2^2$

Classification of Quadratic forms :-

A quadratic form $f(x) = x^T A x$ is classified as one of the following :

i) positive definite if $f(x) > 0 \ \forall \ x \neq 0$.

ii) positive semidefinite if $f(x) \geq 0 \ \forall \ x$.

iii) negative definite if $f(x) < 0 \ \forall \ x \neq 0$.

iv) negative semidefinite if $f(x) \leq 0 \ \forall \ x$.

v) indefinite if $f(x)$ takes on both positive & negative values.

Note :→ A symmetric matrix A is called +ve definite, positive semidefinite, negative definite, negative semidefinite or indefinite if the associated quadratic form $f(x) = x^T A x$ has corresponding property.

Quadratic form and eigenvalues :- (13)

Let A be an $n\times n$ symmetric matrix. The quadratic form $f(x) = x^T A x$ is

a.) +ve definite iff all eigenvalues of A are +ve.

b.) +ve semi-definite iff all the eigenvalues of A are nonnegative.

c.) -ve definite iff all of the eigenvalues of A are negative.

d.) -ve semidefinite iff all of the eigenvalues of A are nonpositive.

e.) indefinite iff A has both positive and negative eigenvalues.

Example:- classify the following quadratic forms

i) $f(x,y,z) = 3x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz$

ii) $f(x,y,z) = 3x^2 + 2y^2 + z^2 + 4xy + 4yz$

Solⁿ. i) $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$

it has eigenvalues 1, 4, 4

$\Rightarrow$ positive definite quadratic form.

ii) $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

it has eigenvalues 5, 2, -1.

$\Rightarrow$ indefinite quadratic form.

Answer

Singular Value Decomposition :→

We know that every symmetric matrix A can be factored as $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix consisting of eigenvalues of A. If A is not symmetric, such a factorization is not possible. But again we know to factor a square matrix A as $A = PDP^{-1}$, here P is simply an invertible matrix.

Not every matrix is diagonalizable, but with the help of SVD (Singular value decomposition), we will show that every matrix has a factorization of the form $A = PDQ^T$, where P and Q are orthogonal matrices and D is a diagonal matrix.

Singular Values of a Matrix :→

If A is an $m \times n$ matrix, the singular values of A are the square ~~values of A~~ roots of the eigenvalues of A^TA and are denoted by $\sigma_1, \sigma_2, --, \sigma_n$. It is conventional to arrange the singular values so that $\sigma_1 \geq \sigma_2 \geq -- \geq \sigma_n$.

Eg: Find the singular values of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

C.E.: $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = 1$

thus $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1} = 1$ are the singular values of A.

Note → Since $[A]_{m\times n}$ and A^TA → symmetric and orthogonally diagonalizable.

$$\|Av_i\|^2 = (Av_i)^TAv_i = v_i^TA^TAv_i$$
$$= v_i^T\lambda_i v_i$$
$$= \lambda_i \quad (v_i \text{ is a unit vector})$$

So, the singular values of A are the lengths of the vectors $Av_1, Av_2, \dots Av_n$.

SVD :→

The decomposition of A involves an $m\times n$ diagonal matrix Σ of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \leftarrow m-r \text{ rows}$$

$\hookrightarrow$ $n-r$ columns

where D is an $r\times r$ diagonal matrix for some r not exceeding the smaller of m and n. If r equals m or n or both, some or all of the zero matrices do not appear.

Theorem:→ Let A be an $m\times n$ matrix with rank r. Then there exists an $m\times n$ matrix Σ for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, and there exists an $m\times m$ orthogonal matrix U and an $n\times n$ orthogonal matrix V such that

$$\boxed{A = U\Sigma V^T}$$

Note:- i) A factorization of $A = U\Sigma V^T$ is called SVD of A.

ii) The columns of U are called the left singular vectors of A, and the columns of V are called right singular vectors of A.

iii) The matrices U and V are not uniquely determined by A, but Σ must contain the singular values of A.

Example:- ① Find the singular Value decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Soln :- It is divided into three steps:

Step ①: Find an orthogonal diagonalization of A^TA.

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

eigenvalues of A^TA are: $\lambda_1 = 360$, $\lambda_2 = 90$, $\lambda_3 = 0$

& its corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

as v_1, v_2 and v_3 are already unit vectors we need to normalize them.

Step ②: Set up V and Σ.

Arrange the eigenvalues of A^TA in decreasing order. & find the singular values

$$\sigma_1 = 6\sqrt{10}, \quad \sigma_2 = 3\sqrt{10}, \quad \sigma_3 = 0.$$

The non zero singular values are the diagonal entries of D. The matrix Σ is the same size as A with D in its upper-left corner and 0's elsewhere

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \text{ and } \Sigma = [D\ 0] = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$V = [v_1\ v_2\ v_3]$

Step ③: Construct U:

When A has rank r, the first r columns of U are the normalized vectors obtained from $Av_1, \dots Av_r$

$$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

Now singular value decomposition of A is

$$A = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}}_{V^T}$$

(11) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ → Find SVD

Solⁿ Step ①:- $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

eigenvalues $\lambda_1 = 3$ & $\lambda_2 = 1$ & corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Since eigenvectors are orthogonal, so we will normalize it

$$V = [q_1\ q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Step ②:- $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$

so; $D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} D \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

and $V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Step ③:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

~~Now singular value decomposition of A is~~

~~A=~~ Now U should be a 3×3 matrix, but we are getting only 2 vectors, so we will extend $\{u_1, u_2\}$ to an orthonormal basis for $\mathbb{R}^3$.

say $\{u_1, u_2, e_3\}$ is linearly independent.

Apply the Gram-Schmidt Process:

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

so: $U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

so; $A = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (Ans.)

11) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find SVD.

Ans.) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} =$

$= U \Sigma V^T$