Unit 2 : Span and Subspaces

1) Def^n: Vector Space

A non empty set V, equipped with two binary operations
a) vector addition and,
b) scalar multiplication
is said to be vector space over a field F, provided V is an abelian group (prop a-e) wrt vector addition and satisfies prop 1-5 wrt scalar multiplication.

a) closure : for $u, v \in V$, $u+v \in V$
b) Associative : for $u, v, w \in V$
$(u+v)+w = u+(v+w)$
c) Existence of additive identity — zero element
$\exists$ $e \in V$ such that for any $u \in V$
$u+e = e+u = u$
ie, e is the zero element.
d) Existence of additive inverse :
for any $u \in V$, $\exists$ $v \in V$ such that
$u+v = v+u = e$
e) commutative :
for any $u, v \in V$ $\exists$ $u+v = v+u$.

Abelian grp wrt vector addition

and,
1) for any $c \in F$, $u \in V$ ~~$\exists$ u+u~~
$\exists$ $c * u \in V$
2) for $c_1, c_2 \in F$ and $u \in V \Rightarrow$
$(c_1+c_2)u = (c_1 * u) + (c_2 * u)$

3) for $1 \in F$ and $u \in V$
$\Rightarrow 1 * u = u * 1 = u$

4) for $c \in F$, $u, v \in F$
$\Rightarrow c * (u + v) = (c * u) + (c * v)$

5) for $c_1, c_2 \in F$, $u \in V$
$\Rightarrow (c_1 c_2) * u = c_1 * (c_2 * u)$

} wrt scalar multiplication

eg. The set of all 2×3 matrices is a vector space wrt matrix addition and matrix scalar multiplication.

eg. $\mathbb{R}^n$, $\mathbb{C}^n$, $P_n(\mathbb{R})$, $M_{m \times n}$

2 Defn. Subspace

A non empty subset W of a vector space V over a field is called a subspace of V, if W is itself a vector space over F, under the same operations of addition and scalar multiplication as defined in V.

Note: A non empty subset W of a vector space V over a field F is a subspace of V, iff (if and only if)

i) $\forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$
ii) $c \in F$, $\forall \alpha \in W \Rightarrow c \cdot \alpha \in W$

OR

$\forall \alpha, \beta \in W$ and $c_1, c_2 \in F$
$c_1 \alpha_1 + c_2 \alpha_2 \in W$.

eg. The set W of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

eg. Matrix of the form $\begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix}$ is not a subspace of $M_{2\times 2}$.

3) Defn Spanning set

If $S = \{v_1, v_2, \dots v_k\}$ is a set of vectors in a vector space V; then set of all the linear combinations of $v_1, v_2 \dots v_k$ is called span.

$\Rightarrow u = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$

4) Defn Linearly independent

The set of vectors $\{u_1, u_2, \dots, u_n\}$ are linearly independent if $c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0$ has the ~~ste~~ only solution as $c_1 = c_2 = \dots = c_n = 0$.

$\Rightarrow$ No vector in a set can be written as a linear combination of other vectors.

eg. a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ linearly independent.

b) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ linearly dependent.

5) **Defⁿ** Basis: A subset B of a vector space $V(F)$ is called a basis if
a) B is a linearly independent set
b) Span $B = V$.

OR

The set of minimum no. of vectors that span the vector space.

6) **Defⁿ** The dimension of a finite dimensional vector space V over F is the no. of elements in any basis of V and is denoted by $d[V]$.

Note: Let $A = \{u_1, u_2, \dots u_n\}$ be a basis for a vector space V.
i) Any set more than n vectors in V must be linearly dependent.
(ii) Any set containing fewer than n vectors in V cannot span V.

7) **Matrix Spaces**

For $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

→ the subspace of R^n spanned by the row vectors of A is denoted by row(A) and is called the row space of A.

→ the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is denoted by col(A) and is called as column space of A

→ The solution space of the homogeneous system of equations $AX = 0$, which is a subspace of $\mathbb{R}^n$ is denoted by null(A) and is called the null space of A.

* <u>Relation b/w four subspaces</u>

For $A_{m\times n}$ matrix,

row space ; row(A) - subspace of $(n\times1)$ matrices $\mathbb{R}^n$
nullspace ; null(A) - subspace of $(n\times1)$ matrices $\mathbb{R}^n$
Column space; col(A) - subspace of $(m\times1)$ matrices $\mathbb{R}^m$
left null space : Null(A^T) - subspace of $(m\times1)$ matrices $\mathbb{R}^m$

Nullspace $AX = 0$
Nullspace is the set of all vectors that are orthogonal to the row space of A.

* dim(Null(A)) = No. of non pivot columns of row reduced echelon form.
dim(Row(A)) = No. of pivot columns.

→ Sum of these two subspaces → all $(n\times1)$ matrices
→ no. of columns.

⇒ Null(A) and Row(A) are orthogonal complements of each other.

* $\dim(\text{Col}(A))$ = no. of pivot columns.
= $\dim(\text{Row}(A))$
= $\text{Rank}(A)$.

⇒ No. of linearly independent rows
= No. of linearly independent columns.