Eigenvalues & Eigenvectors ①

Def^n:- Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector x such that

$$Ax = \lambda x.$$

Vector x is called an eigenvector of A corresponding to λ.

Intuition $\rightarrow$ In Linear Algebra, it is often important to know which vectors have their directions unchanged by a given L.T. An eigenvector or characteristic vector is such vector.

Thus an eigenvalue v of a linear transformation T is scaled by a constant factor λ when the linear transformation is applied to it i.e. $Tv = \lambda v$.

Geometrically, vectors are multi-dimensional quantities with magnitude & direction. A L.T. rotates, stretches or shears the vector upon which it acts. Its eigenvectors are those vectors that are only stretched & the corresponding eigenvalue is the factor by which an eigenvector is stretched.

Example Show that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and find its corresponding eigenvalue.

Solⁿ Compute $Ax = \lambda x$

$$Ax = \underset{2\times 2}{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4x$$

and hence x is an eigenvector of A and $\lambda = 4$ is the eigenvalue.

Characteristic Equation:-

The characteristic eq^n is the eq^n which is solved to find the matrix's eigenvalue.

For a general $K \times K$ matrix A, the characteristic equation in variable λ is defined by

$$|A - \lambda I| = 0$$

where I is the identity matrix of order K.

Ques:) Find the eigenvalues of

a.) $A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$ (b) $B = \begin{bmatrix} 5 & -2 \\ 4 & -4 \end{bmatrix}$

Solⁿ:) a.) $A - \lambda I = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$= \begin{bmatrix} 5-\lambda & 3 \\ 2 & 10-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 3 \\ 2 & 10-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(10-\lambda) - (3)(2) = 0$$

$$50-5\lambda-10\lambda+\lambda^2-6=0$$

$$\lambda^2-15\lambda+44=0$$

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\lambda^2-11\lambda-4\lambda+44=0$$

$$\lambda(\lambda-11)-4(\lambda-11)=0$$

$$(\lambda-4)(\lambda-11)=0$$

$\lambda=4, 11$ are the eigenvalues.

(b) $B-\lambda I=\begin{bmatrix}5 & -2\\ 4 & -4\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}5 & -2\\ 4 & -4\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$

$$=\begin{bmatrix}5-\lambda & -2\\ 4 & -4-\lambda\end{bmatrix}$$

$$|B-\lambda I|=(5-\lambda)(-4-\lambda)-(-8)=0$$

$$\Rightarrow \lambda^2-\lambda-12=0$$

$$\Rightarrow \lambda^2-4\lambda+3\lambda-12=0$$

$$\Rightarrow \lambda(\lambda-4)+3(\lambda-4)=0$$

$$\Rightarrow (\lambda+3)(\lambda-4)=0$$

$\Rightarrow \lambda=-3, 4$ are the eigenvalues.

$\Rightarrow$ **How to find the eigenvectors:-**

1. Find the eigenvalues of a given matrix
2. Substitute the values in eqⁿ $AX=\lambda X$ or $(A-\lambda_1 I)X=0$
3. Calculate the value of eigenvector x, which is associated with the eigenvalue.
4. Repeat the steps to find the eigenvector for all eigenvalues.

Imp: The set of all eigenvectors corresponding to an eigenvalue λ of an $n\times n$ matrix A is just the set of nonzero vectors in the null space of $A-\lambda I$.

Ques: Find the eigenvectors of the given matrix. 7

4

$$A = \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix}$$

Solution → $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ -4 & -7-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(-7-\lambda) - (-16) = 0$

$\Rightarrow (\lambda+3)^2 = 0$

$\Rightarrow \lambda = -3, -3$

Now for eigenvector (for $\lambda_1 = -3$)

$(A - \lambda_1 I)X = 0$

$\Rightarrow (A - (-3)I)X = 0$

$\Rightarrow (A + 3I)X = 0$

$\Rightarrow \left(\begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) X = 0$

$\Rightarrow \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} X = 0$

Now compute its null space i.e

$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ -4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 + x_2 = 0$

~~say~~ $x_2 = -x_1$ $\boxed{x_1 = -x_2}$

$= -s$

So $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigenvector corresponding to eigenvalue -3.

corre — a./ $A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$ b./ $B = \begin{bmatrix} 5 & -2 \\ 4 & -2 \end{bmatrix}$ (5)

find eigenvectors.

Solu^n: a./ $A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$ eigenvalues 4, 11.

eigenvector corresponding to eigenvalue $\lambda_1 = 4$ is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda_2 = 11$ is $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) corresponding to $\lambda_1 = 4$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ is $\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ Ans.

Eigenspace :→ Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the eigenspace of λ and denoted by E_λ.

Question:- Show that $\lambda = 6$ is an eigenvalue of $A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$ and find a basis for its eigenspace.

Solu^n: $\lambda = 6$ is an eigenvalue of A iff $|A - 6I| = 0$

$$A - 6I = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix}$$

$|A - 6I| = 0$ Hence $\lambda = 6$ is an eigenvalue of A.

Now eigenspace $E_6 \Rightarrow$

find the null space of $A-6I$.

i.e. $A-6I = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -3 & -3 & 6 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \quad x_1 + x_2 - 2x_3 = 0$ or $x_1 = -x_2 + 2x_3$

so $E_6 = \left\{ \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$ Ans.

Ques. Show that 5 is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and determine the basis for its eigenspace.

Soln $E_5 = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ Ans.

Ques. → $A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}$ $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. S.

then $Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \neq \lambda v$

Hence, v is not an eigenvector of A.

1.) Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

a.) over $\mathbb{R}$ b.) over $\mathbb{C}$.

Solnⁿ

$|A - \lambda I| = 0 \Rightarrow$ characteristic equation.

$$A - \lambda I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| \Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 + 1 = 0$$

a.) over $\mathbb{R}$, there are no solutions, so A has no real eigenvalues.

b.) over $\mathbb{C}$, the solutions are $\lambda = i$ and $\lambda = -i$..

$\Rightarrow$ Another way of finding the eigenvalues ~~without involving determinan~~ with help of matrices :-

· a.) for 2x2 matrix

say $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

characteristic eqⁿ: $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0$

i.e. $\lambda^2 - (\text{trace } A)\lambda + (\text{determinant } A) = 0$

$$\boxed{\text{trace of a matrix} = \text{sum of the diagonal entries}}$$

for eg: $A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$

characteristic eqⁿ: $\lambda^2 - 15\lambda + (50 - 6) = 0$

$\lambda^2 - 15\lambda + 44 = 0$

(b) for 3×3 matrix,

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

characteristic eq^n: $\lambda^3 - (\text{trace } A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det A = 0$

where A_{11}, A_{22}, A_{33} are the cofactors of a_{11}, a_{22} & a_{33}.

for eg: $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$

trace $A = 1+3+9 = 13$ $\quad$ $\det = 27 - 6 + 2 - 0 + (-6)$
$= 17$

Now $A_{11} = \begin{vmatrix} 3 & 2 \\ 3 & 9 \end{vmatrix} = 27 - 6 = 21$

$A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 9 \end{vmatrix} = 9 - 2 = 7$

$A_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$

Hence C.E. $\Rightarrow \lambda^3 - 13\lambda^2 + (31)\lambda - 17 = 0$

Cayley - Hamilton Theorem :-

Every matrix A is a root of its characteristic polynomial.

Imp: The eigenvalues of a triangular matrix are the
(1) entries on its main diagonal.

eg: Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$

eigenvalues of A are 3, 0, 2 eigenvalues of B are 4, 1, 4.

(2) The sum of n eigenvalues equals the sum of n diagonal entries.

Trace of $A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}$

for eg: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ then its trace $= 3+3 = \underline{6}$

& its eigenvalues:

$$\lambda^2 - 6\lambda + 8 = 0$$
$$\lambda^2 - 4\lambda - 2\lambda + 8 = 0$$
$$(\lambda-4)(\lambda-2) = 0$$

sum of eigenvalues $= 2+4 = \underline{6}$

(3) The product of n eigenvalues equals the determinant of A.

for eg: from previous example

determinant $= 3\times3 - 1\times1 = 9-1 = \underline{8}$

product of eigenvalues $= 4\times2 = \underline{8}$

(4) If all the entries of a $n\times n$ matrix are same (say = c) then one eigenvalue is nc and rest all eigenvalues are zero.

for eg: $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}_{3\times3}$

then its one eigenvalue is $nC = 3\times5 = 15$
and other 2 eigenvalues are 0, 0.

⑤ The eigenvalues of A are equal to the eigenvalues of A^T.

for eg: $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

$$\lambda^2 - 5\lambda + 4 = 0$$
$$(\lambda-1)(\lambda-4) = 0$$
$$\lambda = 1, 4$$

$A^T = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

$$\lambda^2 - 5\lambda + 4 = 0$$
$$\lambda = 1, 4$$

$\boxed{\text{Reason:- because } |A-\lambda I| = |A^T-\lambda I|}$

⑥ If the sum of each row/column of a matrix is equal to 'c', then c is an eigenvalue of A.

for eg: $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ sum of each row = 4
$\Rightarrow$ 4 is an eigenvalue of A.

$$\lambda^2 - 3\lambda + (2-6) = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\lambda = 4, -1$$

(Ans.)

⑦ A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

Algebraic multiplicity & Geometric multiplicity :- (11)

The A.M. of an eigenvalue λ of A is the no. of times 'λ' appears as a root of characteristic ~~polynomial~~ equation.

→ The G.M. of an eigenvalue λ of A is the dimension of eigenspace 'E_λ'.

Ques.) Find the A.M. & G.M. of the eigenvalues of the given matrix $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

characteristic eqn →

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & 0-\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -1-\lambda \begin{vmatrix} 0-\lambda & -3 \\ 0 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & 0-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda) \begin{vmatrix} -\lambda & -3 \\ 0 & -1-\lambda \end{vmatrix} + 1|(0-(1)(-\lambda)| = 0$$

$$\Rightarrow (-1-\lambda)\,[(-\lambda)(-1-\lambda)-0] + 1(\lambda) = 0$$

$$\Rightarrow (-1-\lambda)\,[\lambda+\lambda^2] + \lambda = 0$$

$$\Rightarrow -\cancel{\lambda}-\lambda^2-\lambda^2-\lambda^3+\cancel{\lambda} = 0$$

$$\Rightarrow -2\lambda^2-\lambda^3 = 0$$

$$\Rightarrow \lambda^3+2\lambda^2 = 0 ; \Rightarrow \lambda^2(\lambda+2) = 0$$

$$\lambda = 0, 0, -2$$

Thus A.M. of $\lambda_1 = -2$ is 1.

A.M. of $\lambda_2 = 0$ is 2.

eigenvector for $\lambda_1 = \lambda_2 = 0$

$$[A - 0.I \mid 0] = [A \mid 0] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

thus $x_1 - x_3 = 0$

$\Rightarrow x_1 = x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so: $E_0 = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

thus G.M. for $\lambda = 0$ is 2.

eigenvector for $\lambda_3 = -2$

$$[A - (-2)I \mid 0] = [A + 2I \mid 0] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 + x_3 = 0$ & $x_2 - 3x_3 = 0$

$x_1 = -x_3$ and $x_2 = 3x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

so $E_{-2} = \left\{ x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$

thus G.M. for $\lambda = -2$ is 1.

E.V. cor

E.V.

Statements:- Let A be a square matrix with eigenvalue λ and corresponding eigenvector x.

i) for any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.

ii) If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector x.

iii) If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x

Ques. (i) Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Solⁿ → Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. find $A^{10}x$.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \; \lambda_2 = 2.$$

eigenvector corresponding to $\lambda_1 = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_1$

" " to $\lambda_1 = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $= v_2$

Now $A^{10}x = A^{10}(3v_1 + 2v_2)$ then $\boxed{x = 3v_1 + 2v_2}$

$$= 3(A^{10}v_1) + 2(A^{10}v_2)$$

$$= 3(\lambda_1^{10}v_1) + 2(\lambda_2^{10}v_2)$$

$$= 3(-1)^{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2(2)^{10}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \end{bmatrix} + 2^{11}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{11} \end{bmatrix}$$ Ans

eigenvector is ...

Statement →

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $v_1, v_2, \ldots v_m$. Then $v_1, v_2, \ldots v_m$ are linearly indep.

Ques. from previous example $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ eigenvalues -1 & 2.

eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

clearly $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are L.I.

Determinants :-

Each n-square matrix $A=[a_{ij}]$ is assigned a special scalar called the determinant of A, denoted by $\det(A)$ or $|A|$.

Determinant of a $n\times n$ matrix →

Let $A=[a_{ij}]_{n\times n}$, where $n \geq 2$. Then the determinant of A is the scalar

$$\det A = |A| = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n} a_{1n}\det A_{1n}.$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}$$

where C_{ij} is the cofactor.

Ques.) Compute the determinant of $A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$

Solⁿ- First, notice that column 3 has only one non-zero entry, so we should expand along this column.

So $\det A = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$

$$= 0\cdot C_{13} + 2C_{23} + 0\cdot C_{33} + 0\cdot C_{43}$$

$$= -2 \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \\ -2 & 1 & 0 \end{vmatrix}$$

→ expanding by 3rd row

$$= -2\left(-2\begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}\right)$$

$$= -2\left(-2(-8) - 5\right) = -22$$ Ans.

Properties of a determinant:-

Let $A = [a_{ij}]$ be a square matrix

i) The determinant of a matrix and its transpose are equal i.e. $|A| = |A^T|$

ii) If A has a row (Column) of zeros, then $|A| = 0$

iii) If A has two identical rows (columns), then $|A| = 0$

iv) If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

v) A square matrix is invertible iff $\det A \neq 0$.

vi) $\det(KA) = K^n \det(A)$

vii) $\det(AB) = \det(A) \cdot \det(B)$.

viii) If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Cramer's Rule:-

Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \ldots n$$

Ques.) Use cramer's rule to solve the system

$$x_1 + 2x_2 = 2$$
$$-x_1 + 4x_2 = 1$$

Solu^n $\det A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6$

So $\det(A_1(b)) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 6$ & $\det(A_2(b)) = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$

By Cramer's Rule

$$x_1 = \frac{\det(A_1(b))}{\det A} = \frac{6}{6} = 1 \quad \& \quad x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{3}{6} = \frac{1}{2}$$

Ques.) Solve the system using Cramer's Rule

$$x + y + z = 5$$
$$x - 2y - 3z = -1$$
$$2x + y - z = 3$$

Solu^n $A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = 5$

Now $\det(A_1(b)) = \begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix} = 20$

$\det(A_2(b)) = \begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix} = -10$

$\det(A_3(b)) = \begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 15$

thus $x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{20}{5} = 4$

$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{-10}{5} = -2$

$x_3 = \frac{\det(A_3(b))}{\det(A)} = \frac{15}{5} = 3$

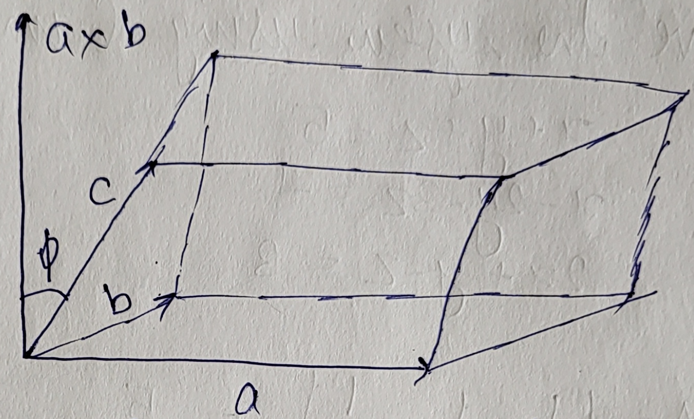
so $u = (x, y, z)$
$= (4, -2, 3)$

Ans.

Geometric Applications of Determinants :→

i) ~~Cross product~~ Area & Volume →

Volume of parallelepiped :→ (a polyhedron with 6 faces & a 3-D shape whose faces are all parallelograms)



The mixed product of 3 vectors is called triple product. Hence $a = (a_1, a_2, a_3)^T$, $b = (b_1, b_2, b_3)^T$ and $c = (c_1, c_2, c_3)^T$, the volume of parallelepiped =

$$V = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = |(a \times b) \cdot c|$$

Ques.) Find the volume of parallelepiped with one vertex at origin and adjacent vertices at $(1, 0, -2)$, $(1, 2, 4)$, $(7, 1, 10)$

Soln – $$V = \begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 10 \end{vmatrix} \Rightarrow 1(20-4) - 1(0-(-2)) + 7(0+4)$$

$$= 16 - 2 + 28$$

$$= 42$$ Ans.

Area of a parallelogram →

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then area of parallelogram determined by u and v is given by

$$= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

Ques.) a) $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ & $v = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

and then area of parallelogram $= \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8+3$
$= 11$

b.) $u = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

calculate the area of parallelogram.

ii) Lines and Planes →

two distinct points (x_1, y_1) and (x_2, y_2). Then there is a unique line passing through these points. and its equation is of the form

$$ax + by + c = 0 \quad \text{—(i)}$$

Since these 2 points are on the line, their coordi-nates satisfy this equation. Thus

$$ax_1 + by_1 + c = 0 \quad \text{—(ii)}$$
$$ax_2 + by_2 + c = 0 \quad \text{—(iii)}$$

→ Thus eq^n of line through the points (x_1, y_1) & (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Ques:) a.) (2,3) and (-1,0) to: find the eq^n of line through given points.

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0 \Rightarrow x(3-0) - y(2+1) + 1(+3) = 0$$

$$\Rightarrow 3x - 3y + 3 = 0$$

$$\Rightarrow x - y + 1 = 0$$

b.) (1,2) & (4,3) → do the same.

→ Three points are collinear iff

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

→ Four points are coplanar iff

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

iii) Curve fitting →

Suppose a_1, a_2, a_3 are distinct real no's. For any real no, b_1, b_2 & b_3, there is a unique quadratic with equation of the form $y = a + bx + cx^2$ passing through the points (a_1, b_1) (a_2, b_2) & (a_3, b_3) & it can be found by solving the system of equations.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & a_1 \\ \hline & 1 & a_2 \\ \hline & 1 & a_3 \\ \hline \end{array}$$

Ques.) a.) $A(1,-1), B(2,4), C(3,3)$

then $-1 = a + b\cdot 1 + c\cdot 1^2 \Rightarrow -1 = a+b+c$

$4 = a + 2\cdot b + c\cdot 2^2 \Rightarrow 4 = a+2b+4c$

$3 = a + 3\cdot b + c\cdot 3^2 \Rightarrow 3 = a+3b+9c$

then. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ then $|A| = 1(18-12) - 1(9-4) + 1(3-2)$
$= 6 - 1\times 5 + 1 = 2$

Now: $\det(A_1(b)) = \begin{vmatrix} -1 & 1 & 1 \\ 4 & 2 & 4 \\ 3 & 3 & 9 \end{vmatrix} = -24$

$\det(A_2(b)) = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 4 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 28$

$\det(A_3(b)) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 4 \\ 1 & 3 & 3 \end{vmatrix} = -6$

then $a = \dfrac{\det(A_1(b))}{\det A} = \dfrac{-24}{2} = -12$

$b = \dfrac{28}{2} = 14$ and $c = \dfrac{-6}{2} = -3$

thus curve is: $y = -12 + 14x - 3x^2 = -12 + 14x - 3x^2$

Ans.

Similarity :-

Similar Matrices → Let A and B are $n \times n$ matrices. Then we say A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$; or $\boxed{AP = BP}$

Note: If A is similar to B, we write $A \sim B$.

Ques. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$, then $A \sim B$.

Solⁿ → We are supposed to find invertible P such that
$$AP = PB.$$

Now $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$

$$\Rightarrow AP = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -c & -d \end{bmatrix}$$

and $$PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} a-2b & -b \\ c-2d & -d \end{bmatrix}$$

Now, as $AP = PB \Rightarrow \begin{bmatrix} a+2c & b+2d \\ -c & -d \end{bmatrix} = \begin{bmatrix} a-2b & -b \\ c-2d & -d \end{bmatrix}$

$\Rightarrow a+2c = a-2b$ | $b+2d = -b$ | $-c = c-2d$

$2c+2b = 0$ | $2b+2d = 0$ | $2c-2d = 0$

$\boxed{c+b=0}$ | $\boxed{b+d=0}$ | $\boxed{c-d=0}$

$c = -b$ | $\boxed{b=-d}$ | $\boxed{c=d}$

$= -(-d)$

$\boxed{c=d}$

thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -d \\ d & d \end{bmatrix} = d\begin{bmatrix} a/d & -1 \\ 1 & 1 \end{bmatrix}$ choose $a=1$ & $d=1$

we get $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ **Ans.**

⇒ How to find Similar matrices :–

If A is given then

i) characteristic polyⁿ of A to find eigenvalues

ii) find corresponding eigenvectors

iii) P is obtained by combining all eigenvectors into a matrix.

Ques: find similar matrix to $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

Solⁿ – Eigenvalues of A $\Rightarrow \lambda^2 - 0\lambda + (-4+3) = 0$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

E.V. corresponding to $\lambda = 1$ is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

E.V. " " $\lambda = -1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

so $P = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$ & $P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

Now $B = P^{-1}AP$

$$= \frac{-1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{-1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{-1}{2} \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Answer

Theorem:- Similarity is an equivalence relation

a.) $A \sim A$ (reflexive)

Proof → $I^{-1}AI = A$ choose $P = I$.

b.) $A \sim B$ then $B \sim A$ (symmetric)

Proof → As $A \sim B$ then $B = P^{-1}AP$ for $|P| \neq 0$

$\Rightarrow PB = P(P^{-1}AP)$ (premultiply by P)

$\Rightarrow PB = AP$

$\Rightarrow PBP^{-1} = APP^{-1}$ (postmultiply by P^{-1})

$\Rightarrow PBP^{-1} = A$

$\Rightarrow B \sim A$.

c.) If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitivity)

Proof - As $A \sim B \Rightarrow B = PAP^{-1}$ $(|P| \neq 0)$ — (I)

$B \sim C \Rightarrow C = QBQ^{-1}$ $(|Q| \neq 0)$

$\Rightarrow B = Q^{-1}CQ$ — (II)

$\Rightarrow PAP^{-1} = Q^{-1}CQ$

$\Rightarrow QPAP^{-1} = QQ^{-1}CQ$

$\Rightarrow (QP)AP^{-1} = CQ$

$\Rightarrow (QP)AP^{-1} \cdot Q^{-1} = CQQ^{-1}$

$\Rightarrow (QP)A(P^{-1}Q^{-1}) = CQQ^{-1}$

$\Rightarrow (QP)A(QP)^{-1} = C$

$\Rightarrow$ $\underline{A \sim C}$ $\underline{\text{Proved}}$

Theorem:- Let A and B be $n \times n$ matrices with $A \sim B$. Then

a.) $|A| = |B|$ b.) A and B have same rank

c.) A and B have the same characteristic polynomial.

d.) A and B have same eigenvalues.

e.) A is invertible iff B is invertible.

Note:- Converse need not to be true.

Ques./ $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$|A| = -3$ $|B| = 3$

hence A and B are not similar.

Ques./ $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

C.E. $\lambda^2 - 3\lambda - 4 = 0$ & $\lambda^2 - 4 = 0$

hence they are not similar.

Imp./- Example of converse -

say $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

- A and B have same determinant
- A and B have same rank
- A and B have same C.E.
- A and B have same eigenvalue

but $A \nsim B$ as $P^{-1}AP = P^{-1}IP = I \neq B$.

Ans.

Diagonalization :-

An $n\times n$ matrix A is diagonalizable if there is a diagonal matrix D such that A is similar to D – i.e if there is an invertible $n\times n$ matrix P such that

$$P^{-1}AP = D.$$

Ex:- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is diagonalizable since

$P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, then

$$P^{-1}AP = D \;.\; \text{or} \;\; AP = DP.$$

Theorem:- Let A be an $n\times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Ques:) If possible, find a matrix 'P' that diagonalizes

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Solⁿ:- C.E : $-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$

(since sum of each row is 1, so '1' is an eigenvalue)

$\lambda - 1)\; -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \;(\, -\lambda^2 + 3\lambda - 2$

$\cancel{-\lambda^3} + \lambda^2$

$+ \quad -$

$3\cancel{\lambda^2} - 5\lambda$

$\cancel{3\lambda^2} - 3\lambda$

$- \quad +$

$-2\lambda + 2$

$\cancel{-2\lambda + 2}$

so: $-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (\lambda - 1)(-\lambda^2 + 3\lambda - 2)$

And hence eigenvalues are 1, 1, 2.

Now $E_1 = \boxed{(A-I)X=0}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_1} \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now $-x_1 + x_2 = 0$ and $-x_2 + x_3 = 0$

$x_1 = x_2$ $\qquad$ $x_2 = x_3$

$$\Rightarrow x_1 = x_2 = x_3$$

E.V. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 1$

IIIly E_2 has E.V. $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

So, for $\lambda_1 = 1 = \lambda_2$, E_1 has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda_3 = 2$, E_2 has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Since A has only 2 l.i. eigenvectors

$\Rightarrow$ A is not diagonalizable. Ans.

Ques.) If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Soln:- C.E. : $-\lambda^2(\lambda+2)=0$

E.V. $0, 0, -2$

$$E_0 = \left\{ \overset{p_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{p_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\} \quad \& \quad E_{-2} = \left\{ \overset{p_3}{\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}} \right\}$$

clearly vectors in E_0 & E_{-2} are l.i. i.e.

$$P = [p_1 \ p_2 \ p_3] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

is invertible. Furthermore,

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D.$$

Hence A is diagonalizable. Ans.

Theorem:- If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Ques.) $A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

C.E. : $\lambda^3 - (6)\lambda^2 + (-5 + (-2) + (10))\lambda + 10 = 0$

$\lambda^3 - 6\lambda^2 + 3\lambda + 10 = 0$

E.V. $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = -1$

Since all its eigenvalues are distinct, so given A is diagonalizable.

Note:- G.M. of each eigenvalue $\leq$ A.M. of each eigen-value.

heorem:- The Diagonalization Theorem-

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2 \cdots, \lambda_k$. Then following statements are equivalent:

a.) A is diagonalizable.

b.) The A.M. of each eigenvalue equals its G.M.

Ques.)

a.) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$

E.V. 1, 1, 2

$E_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow$ A.M. of $\lambda_1 = 2$ (for $\lambda_1 = 1$)
G.M. of $\lambda_1 = 1$ (for $\lambda_1 = 1$)

as A.M. $\neq$ G.M. for $\lambda_1 = 1$

$\Rightarrow$ not diagonalizable.

b.) $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

E.V. 0, 0, -2

$E_0 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ A.M. of $\lambda = 0$ is 2
G.M. of $\lambda = 0$ is 2

$E_{-2} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$ A.M. of $\lambda = -2$ is 1
G.M. of $\lambda = -2$ is 1

Since A.M. = G.M.

$\Rightarrow$ diagonalizable <u>Answer</u>

Ques.) Compute A^{10} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

Soln.- E.Val. of A are $-1, 2$

and eigenvector corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

" " " $\lambda = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

thus $P = [v_1 \; v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

thus $P^{-1}AP = D \Rightarrow \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = D$$

Now,

$$A' = PDP^{-1}$$

$$A^{10} = PD^{10}P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^{10} & 0 \\ 0 & (2)^{10} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$$

Answer