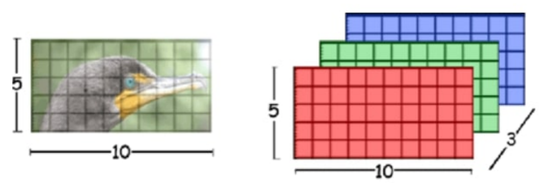
Linear Algebra and Digital Image Processing

Image processing can be defined as the manipulation of images using mathematical functions and operations. Linear algebra is widely made use of in image processing.

A digital image is made up of very small controllable square units called pixels. A pixel can be said as the atom or building block of an image. Each pixel is associated with a colour which is generated upon the primary (base) colours, red, green and blue (RGB). Each of these three colours has 256 gradations, and they could be represented as a number ranging from $0 - 255$ in order to represent a colour numerically (0 indicates black and 255 indicates white), so we can generate a total of $255 \times 255 \times 255$ in RGB system.

Thus an image can be represented as a 3D matrix, say having dimension $m \times n \times 3$, where $m \times n$ represents the dimension of the 2D image and 3 represents each of the three base colours, red, green and blue.



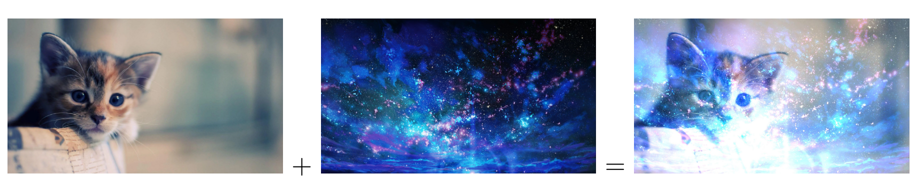
Real colour image (Left) and its three dimensional RGB colour matrix (Right)

Image Processing through Matrix Operations

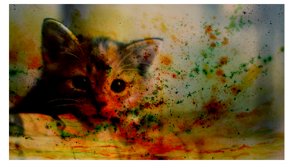
Images as matrices can undergo matrix operations and generate new images. Matrix operations play a vital role in digital image processing.

1. Matrix Addition/Subtraction

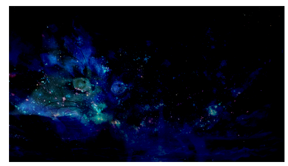
A resultant image can be generated from two images by adding two matrices associated to the images thereby obtaining the associated matrix for the resultant image. Each element in first matrix is added to the corresponding element in the second matrix to obtain the resultant. If the sum becomes greater than 255, that sum remains as 255 and gives white colour to the corresponding pixel. An example is given below.



Similarly a resultant image can be generated by subtracting one matrix from another, both of which are associated to the images, thereby obtaining the associated matrix for the resultant image. Each element in the second matrix is subtracted from the corresponding element in the first matrix to obtain the resultant. If the difference becomes less than 0, the difference remains as 0 and gives black colour to the corresponding pixel. An example is given below.

 $-$  $=$ 

The following result is obtained if first image is subtracted from the second.

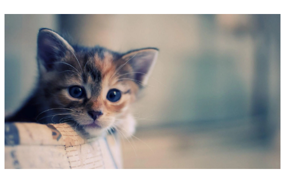
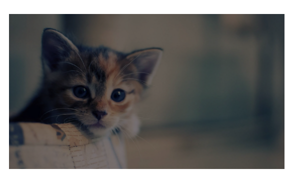
 $-$  $=$ 

2. Scalar Multiplication of Matrix

Each element in the matrix associated to an image can be multiplied by a positive scalar to obtain a new image, having intensity more or less depending upon the scalar. The intensity of the new image is more than the original if the scalar is greater than 1 and less if the scalar is less than 1. If the value becomes greater than 255, then it remains as 255 and gives white colour to the corresponding pixel. The image becomes completely black if we multiply by 0.

 $\times \quad 1.5 \quad =$ 

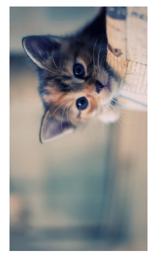
The intensity here is more than that of the original since each element is scaled by a factor greater than 1, so the value increases thereby become brighter. Below example shows an image multiplied by 0.5, where the intensity becomes less than that of original since the image is scaled by a factor less than 1.

 $\times \quad 0.5 \quad =$ 

3. Elementary Operations of Matrix

Elementary operations of matrices can be performed over images for rotating and flipping the images. Matrix transpose can also be applied over an image like the following transformation.

$$\xrightarrow{\text{Transpose}}$$



Let us discuss about the effect of elementary operations of matrices over digital images.

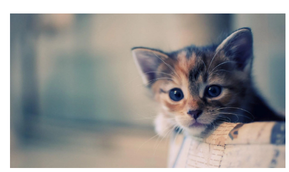
3.1. Horizontal Flip

Flipping an image horizontally can be done by applying elementary column operations to the matrix associated to the image. If the matrix associated is an $m \times n \times 3$ matrix, the j^{th} column of the matrix can be interchanged with $(n+1-j)^{th}$ column $\forall j \in \mathbb{N},\ 1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$ in order to flip the image horizontally. Below example shows the horizontal flip of a 3×4 matrix due to the column operations. Note that the operations applied for such a 2D matrix behave same for matrix with each of the three base colours red, green and blue, so there's no need to consider all the three layers of the 3D matrix amd it's enough to consider one such 2D matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{C_j \leftrightarrow C_{5-j}\ \ \forall j=1,\ 2} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \\ 12 & 11 & 10 & 9 \end{bmatrix}$$

Similarly, an image can be flipped horizontally like,

$$\xrightarrow{C_j \leftrightarrow C_{n+1-j}\ \ \forall j \in \mathbb{N},\ 1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor}$$



Since this is an elementary column operation series, it is equivalent to post - multiplication of the matrix associated to the image, with a matrix obtained by applying the same elementary column operations to $n \times n$ identity matrix in the same order. The operations applied to such an identity matrix results in,

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{n\times n} \xrightarrow{C_j \leftrightarrow C_{n+1-j}\ \forall j\in\mathbb{N},\ 1\le j\le \lfloor \frac{n+1}{2} \rfloor} \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n}$$

Post - multiplying with this matrix results in horizontal flipping. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \\ 12 & 11 & 10 & 9 \end{bmatrix}$$

Therefore,

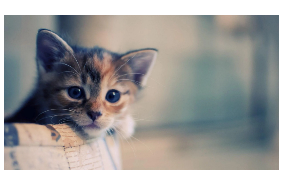
$$\times \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n} =$$

3.2. Vertical Flip

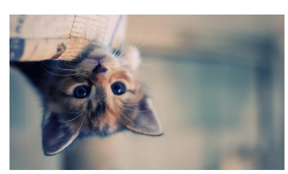
Flipping an image vertically can be done by applying elementary row operations to the matrix associated to the image. Here i^{th} row of an $m \times n \times 3$ matrix can be interchanged with $(m+1-i)^{th}$ row $\forall i \in \mathbb{N},\ 1 \le i \le \lfloor \frac{m+1}{2} \rfloor$. Below example shows the vertical flip of a 5×3 matrix due to row operations.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{bmatrix} \xrightarrow{R_i \leftrightarrow R_{6-i}\ \forall i=1,\ 2,\ 3} \begin{bmatrix} 13 & 14 & 15 \\ 10 & 11 & 12 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Similarly, an image can be flipped vertically like,



$$\xrightarrow{R_i \leftrightarrow R_{m+1-i}\ \forall i\in\mathbb{N},\ 1\le i\le \lfloor \frac{m+1}{2} \rfloor}$$



Since this is an elementary row operation series, it is equivalent to pre - multiplication of the matrix associated to the image, with a matrix obtained by applying the same elementary row operations to $m \times m$ identity matrix in the same order. The operations applied to such an identity matrix results in,

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{m\times m} \xrightarrow{R_i \leftrightarrow R_{m+1-i} \;\; \forall i \in \mathbb{N},\; 1 \le i \le \lfloor \frac{m+1}{2} \rfloor} \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{m\times m}$$

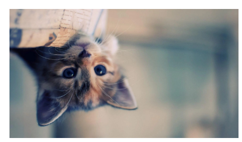
The matrix is the same as what we found for horizontal flipping.
Pre - multiplying with this matrix results in vertical flipping. For example,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{bmatrix} = \begin{bmatrix} 13 & 14 & 15 \\ 10 & 11 & 12 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{m\times m} \times$$

$=$



3.3. Rotation

Rotation can be done by combination of transposing and flipping.

A matrix can be rotated 90° clockwise by first transposing then flipping horizontally.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow{\text{Horizontal Flip}} \begin{bmatrix} 9 & 5 & 1 \\ 10 & 6 & 2 \\ 11 & 7 & 3 \\ 12 & 8 & 4 \end{bmatrix}$$

Therefore,

A matrix can be rotated 90° anticlockwise by first transposing then flipping vertically.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow{\text{Vertical Flip}} \begin{bmatrix} 4 & 8 & 12 \\ 3 & 7 & 11 \\ 2 & 6 & 10 \\ 1 & 5 & 9 \end{bmatrix}$$

Therefore,

A matrix can be rotated 180° by flipping horizontal and vertically.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{\text{Horizontal Flip}} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \\ 12 & 11 & 10 & 9 \end{bmatrix} \xrightarrow{\text{Vertical Flip}} \begin{bmatrix} 12 & 11 & 10 & 9 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Therefore,

We can add gradient also by using elementary operations. In the below example, i^{th} row undergoes the elementary row operation $R_i \to \frac{i}{m} R_i$ for each $i = 1,\ 2,\ 3,\ \dots\ ,\ m$ to obtain the gradient.

$$\xrightarrow{R_i \to \frac{i}{m} R_i\ \ \forall i=1,\ 2,\ 3,\dots,\ m}$$

Thus linear algebra is extensively made use of in digital image processing. Today, we as a social media user are able to alter and edit images easily using many digital image editing apps. Many social media apps like Facebook, Instagram, Snapchat etc. also allow us to take pictures and make whatever changes we would like. There are also other features in such apps to add/remove different filter effects to the image. Such effects read in those images, map each colour to their proper codes which allow us for special editing. Besides such social media apps, there are image processing programs like Photoshop that would allow us to make different edits to an image using mathematical concepts.

Image processing is a tool for some other areas like aerial photography, facial recognition, surveillance system etc. which widely make use of linear algebra, like facial recognition is carried out by generating a matrix for a particular face, with which the software is able to make up a pattern in the face which allows for recognising a person's face. Such a manipulation allows us to understand its working based on linear algebra, one of the applications of mathematics.

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