~~Estimation~~ Quantum Amplitude Amplification

Generalization of Grover's method.

~~Like~~ In Grover's, we use $H^{\otimes n}|0\rangle_n$ to prepare $|S\rangle_n$

Assume there is some ~~~~ ~~~~ oracle that ~~prep~~ prepares some state we desire using an ancilla qubit such ~~that~~ the $n+1$ qubits state can be separated as good and bad states.

$$\Rightarrow \quad |0\rangle_{n+1} = |0\rangle_n |0\rangle$$

Assume our oracle is A [Only* real amplitudes]

$$\Rightarrow \quad A|0\rangle_n|0\rangle = \sqrt{p}\,|\omega\rangle_{n+1} + \sqrt{1-p}\,|S_{\bar{\omega}}\rangle$$

$$\boxed{\begin{array}{l}\text{On our promise of separation,} \\ |\omega\rangle \text{ and } |S_{\bar{\omega}}\rangle \text{ are orthogonal} \\ \Rightarrow \quad \langle \omega | S_{\bar{\omega}} \rangle = 0\end{array}}$$

We can rewrite it as

$$A|0\rangle_n|0\rangle = \sqrt{1-p}\,|\psi_0\rangle|0\rangle + \sqrt{p}\,|\psi_1\rangle_n|1\rangle$$

Last qubit is our ancilla

We do note that

$|\psi_0\rangle$ and $|\psi_1\rangle$ are normalized, but not necessarily orthogonal because:

~~Inner product~~ $\cancel{\langle\psi_0|\langle 0| \otimes |\psi_1\rangle|1\rangle}$

$\cancel{\langle\psi_0|\langle 0|\psi_1\rangle|1\rangle} = \cancel{\langle S_{\bar{\omega}}|\omega\rangle = 0}$

~~because~~

$$(\langle S_{\bar{\omega}}|) \cdot (|\omega\rangle) = 0$$

$$(\langle\psi_0| \otimes \langle 0|) \cdot (|\psi_1\rangle \otimes |1\rangle)$$

$$\Rightarrow \left[(\langle u| \otimes \langle v|) \cdot (|w\rangle \otimes |y\rangle) = \langle u|w\rangle \otimes \langle v|y\rangle \right]$$

$$= \langle\psi_0|\psi_1\rangle \otimes \underbrace{\langle 0|1\rangle}_{0}$$

$$= 0 \boxed{\langle\psi_0|\psi_1\rangle}$$

$\downarrow$

Not necessarily orthogonal.

So $A|0\rangle_n|0\rangle = \sqrt{1-a}\,|\psi_0\rangle|0\rangle + \sqrt{a}\,|\psi_1\rangle|1\rangle$

We define $\boxed{Q = A S_0 A^{\dagger} S_{\psi_0}}$

Where $\boxed{S_{\psi_0} = \mathbb{1} - 2|\psi_0\rangle\langle\psi_0| \otimes |0\rangle\langle 0|}$

$\boxed{S_0 = \mathbb{1} - 2|0\rangle_{n+1}\langle 0|_{n+1}}$

When we look at S_0

$$S_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}_{n+1} - 2\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & \cdots & & & 0 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & \vdots \\ 0 & & 1 & & \\ 0 & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \underbrace{-\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & & & \vdots \\ 0 & & -1 & & \\ \vdots & & & -1 & \\ 0 & \cdots & 0 & \cdots & -1 \end{bmatrix}_{(n+1)}}$$

This is the same in ~~as~~ analogy as the OR operation for V_0 in Grover's.

Diffuser $D = H V_0 H$ $\quad (\because H = H^{\dagger})$

$D = H V_0 H^{\dagger}$

Rewrite Q as

$$Q = A S_0' A^\dagger S_{\psi_0}' \qquad \boxed{S' = -S}$$

S_0' is just I_0

What about S_{ψ_0}' ?

$$\Rightarrow S_{\psi_0}' = 2|\psi_0\rangle\langle\psi_0|_n \otimes |0\rangle\langle 0|$$

What does it do to bad state? $\left[|\psi_0\rangle|0\rangle\right]$

$\Rightarrow$

$$S_{\psi_0}'\left(|\psi_0\rangle|0\rangle\right) = \left[2\left[|\psi_0\rangle\langle\psi_0| \otimes |0\rangle\langle 0|\right] - \mathbb{1}\right]|\psi_0\rangle|0\rangle$$

$$= 2\left[|\psi_0\rangle\langle\psi_0| \otimes |0\rangle\langle 0|\right]\left[|\psi_0\rangle|0\rangle\right] - |\psi_0\rangle|0\rangle$$

$\left[\because \langle a|a\rangle = 1\right]$

$$= 2\left[|\psi_0\rangle\langle\psi_0|\psi_0\rangle\right] \otimes \left[|0\rangle\langle 0|0\rangle\right] - |\psi_0\rangle|0\rangle$$

$$= 2|\psi_0\rangle \otimes |0\rangle - |\psi_0\rangle|0\rangle$$

$$= |\psi_0\rangle|0\rangle$$

$\boxed{\text{It did nothing to bad states!}}$

What does it do to good states? $[|\psi_1\rangle|1\rangle]$

$$S'_{\psi_0}\left[|\psi_1\rangle|1\rangle\right] = \left[2\left[|\psi_0\rangle\langle\psi_0|\right]\otimes|0\rangle\langle 0| - I\right]\left[|\psi_1\rangle|1\rangle\right]$$

$$= 2\left[|\psi_0\rangle\langle\psi_0|\right]\otimes|\psi_1\rangle|1\rangle - |\psi_1\rangle|1\rangle$$

$$= 2\left[|\psi_0\rangle\langle\psi_0| \otimes |0\rangle\langle 0|\right] |\psi_0\rangle|1\rangle - |\psi_1\rangle|1\rangle$$

$[\because \langle 0|1\rangle = 0]$

$$= 2\left[|\psi_0\rangle\langle\psi_0|\psi_1\rangle \otimes \langle 0|\langle 0|1\rangle\right] - |\psi_1\rangle|1\rangle$$

$$= \boxed{-|\psi_1\rangle|1\rangle}$$

Phase kickback!

S'_{ψ_0} is our $U_f^{\perp}$ oracle in Grover's but just generalized. It works for any general ~~...~~ state kickback

Why extra Ancilla? [This is where we mark our good and bad states]

★ [We can just use Z gate to flip the Ancilla Phase]

We are done with showing generalized estimation if we can find out what $AS_0' A^\dagger$ is doing.

What does $AS_0' A^\dagger$ do to any general state $|\phi\rangle$?

We ~~know~~ have $|\psi\rangle = A|0\rangle$ [Our State Preparation]

Any state $|\phi\rangle$ can be decomposed as components parallel and perpendicular to $|\psi\rangle$

$$\therefore \; |\phi\rangle = \alpha|\psi\rangle + \beta|\psi^\perp\rangle$$

Now apply $AS_0' A^\dagger$

$$\cancel{AS_0' A^\dagger|\phi\rangle = \alpha AS_0' A^\dagger|\psi\rangle + \beta|\psi^\perp\rangle}$$

$$AS_0' A^\dagger|\phi\rangle = \alpha AS_0' A^\dagger|\psi\rangle + \beta AS_0' A^\dagger|\psi^\perp\rangle$$

$[\because A^\dagger = A^{-1}]$

$$= \alpha AS_0'|0\rangle + \beta AS_0'|0^\perp\rangle$$

As A is unitary, orthogonal part remains orthogonal $\Rightarrow$ let $A^{-1}|\psi^\perp\rangle = |0^\perp\rangle$

[We saw S_0' is like OR gate that flips everything except 0]

$$= \alpha A|0\rangle + \beta A[-|0^\perp\rangle]$$

$\boxed{D = AS_0' A^\dagger}$

$$= \alpha A|0\rangle - \beta A[|0^\perp\rangle]$$

$$D[|\phi\rangle] = \alpha|\psi\rangle - \beta[\psi^\perp]$$

$$|\phi\rangle = \alpha|\psi\rangle + \beta\left[|\psi^\perp\rangle\right]$$

$$\boxed{\beta|\psi^\perp\rangle = |\phi\rangle - \alpha|\psi\rangle}$$

$$D|\phi\rangle = \alpha|\psi\rangle - \left[|\phi\rangle - \alpha|\psi\rangle\right]$$

$$\boxed{D|\phi\rangle = 2\alpha|\psi\rangle - |\phi\rangle}$$

As $\alpha = \langle\psi|\phi\rangle$ [Projection]

$$\boxed{D|\phi\rangle = 2\langle\psi|\phi\rangle|\psi\rangle - |\phi\rangle}$$

Remember from Grover's for any general $|\phi\rangle$

$$D|\phi\rangle = 2\langle s|\phi\rangle|s\rangle - |\phi\rangle$$

This is exactly the same! Here are $|s\rangle \approx |\psi\rangle$

As 'A' operator has just H gate

$A|0\rangle = |\psi\rangle$, $H|0\rangle = |s\rangle$

$\therefore$ D acts as inversion around mean.

The same diffuser as in Grover's!

Q operator is ~~amplifying~~ amplifying amplitude of a general state!

So $|\psi\rangle = A|0\rangle$

$$|\psi\rangle = \sqrt{a}\,|\omega\rangle + \sqrt{1-a}\,|\psi_{\bar{\omega}}\rangle$$

$$|\psi\rangle = \cos\left(\frac{\theta_a}{2}\right)|\psi_{\bar{\omega}}\rangle + \sin\left(\frac{\theta_a}{2}\right)|\omega\rangle \quad \left[a = \sin^2\left(\frac{\theta_a}{2}\right)\right]$$

In the basis $\{|\psi_{\bar{\omega}}\rangle, |\omega\rangle\}$

$$|\psi_{\bar{\omega}}\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\omega\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$S'_{\psi_0}|\psi_{\bar{\omega}}\rangle = |\psi_{\bar{\omega}}\rangle$$

$$S'_{\psi_0}|\omega\rangle = -|\psi_{\bar{\omega}}\rangle$$

$$\boxed{S_{\psi'} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}$$

Diffuser $D = AS'_0A^\dagger$

$\cancel{D|\psi_{\bar{\omega}}\rangle}$

$$D|\phi\rangle = 2\langle\psi|\phi\rangle|\psi\rangle - |\phi\rangle$$

$$D|\psi_{\bar{\omega}}\rangle = 2\langle\psi|\psi_{\bar{\omega}}\rangle|\psi\rangle - |\psi_{\bar{\omega}}\rangle$$

$$\boxed{\therefore \langle\psi|\psi_{\bar{\omega}}\rangle = \cos\frac{\theta}{2}}$$

$$= 2\cos\left(\frac{\theta_a}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)|\psi_{\bar{\omega}}\rangle + \sin\left(\frac{\theta}{2}\right)|\omega\rangle\right] - |\psi_{\bar{\omega}}\rangle$$

$$= 2\cos^2\left(\frac{\theta_a}{2}\right)|\psi_{\bar{\omega}}\rangle + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)|\omega\rangle$$

$$D|\psi_{\bar{\omega}}\rangle = \cos\theta\,|\psi_{\bar{\omega}}\rangle + \sin\theta\,|\omega\rangle$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ first column}$$

$$D|\omega\rangle = 2\langle\psi|\omega\rangle|\psi\rangle - |\omega\rangle$$

$$= 2\sin\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)|\psi_{\bar{\omega}}\rangle + \sin\left(\frac{\theta}{2}\right)|\omega\rangle\right] - |\omega\rangle$$

$$= \sin\theta\,|\psi_{\bar{\omega}}\rangle - \cos\theta\,|\omega\rangle$$

$$= \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \text{ second column}$$

$$D = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$DS'_{\psi_0} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The same as G.

As $G^k|S\rangle = \cos\left(\frac{(2k+1)\theta}{2}\right)|S_{\bar{\omega}}\rangle + \sin\left(\frac{(2k+1)\theta}{2}\right)|\omega\rangle$

$$\Rightarrow \boxed{Q^k|\psi\rangle = \cos\left((2k+1)\left(\frac{\theta}{2}\right)\right)|\psi_{\bar{\omega}}\rangle + \sin\left((2k+1)\frac{\theta}{2}\right)|\omega\rangle}$$

$$\mathbb{P}\left[|1\rangle\right] = \sin^2\left[(2k+1)\left(\frac{\theta}{2}\right)\right].$$

$$(2k+1)\frac{\theta}{2} \approx 1$$

$$\theta = \frac{\pi}{2k+1}$$

$$\frac{\pi}{2k+1} = 2\sin^{-1}\sqrt{\frac{M}{(N+1)}} \quad (M <<< N)$$

$$\frac{\pi}{2k+1} = 2\sqrt{\frac{M}{N+1}}$$

$$k = \left\lfloor \frac{\pi}{4}\sqrt{\frac{N}{M}} - \frac{1}{2} \right\rfloor$$

$$= O\left[\sqrt{\frac{N}{M}}\right] \quad \boxed{\text{Quadratic speedup}}$$

Hence Proved

Quantum Amplitude Amplification