Grover's

$N = 2^n$

M solutions.

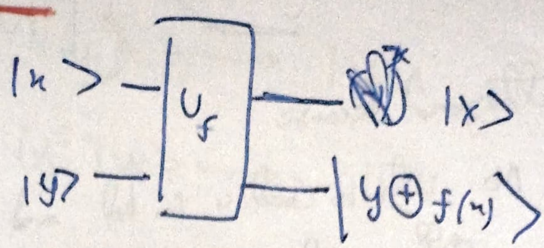
Given black box U_f for f.

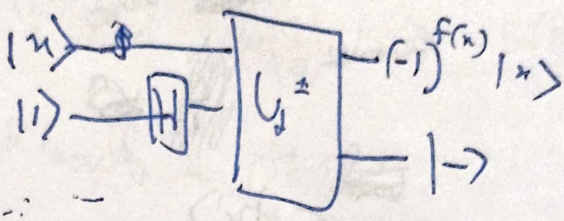
How many queries to find x^* s.t. $f(x^*) = 1$

Classical Complexity: Worst N. $\boxed{M=1}$

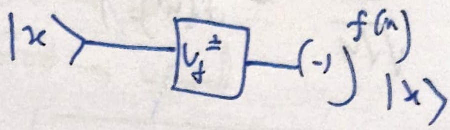
Avg $\frac{N}{2}$

$U_f^{\pm}$

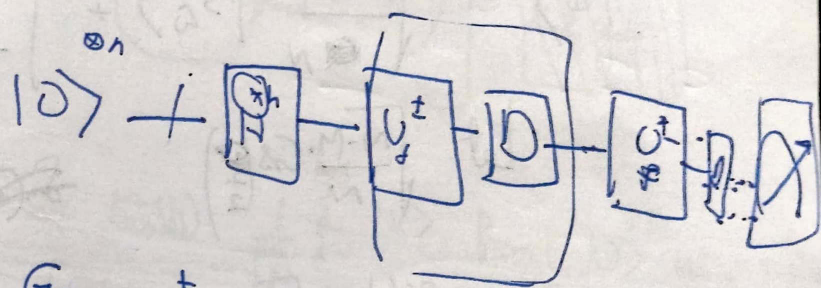




$U_f^{\pm}$:



Quantum Circuit



$$G = U_f^{\pm} D = O\left(\sqrt{\frac{N}{M}}\right) \text{ times}$$

$$I^{\otimes n} \otimes |0\rangle^{\otimes n} = \frac{1}{\sqrt{N}} \sum_{x=0}^{n-1} |x\rangle = |s\rangle$$

$$|s\rangle = \frac{1}{\sqrt{N}}\left(\sum_{x':f(x')=1} |x'\rangle + \sum_{x'':f(x'')=0} |x''\rangle\right)$$

$$= \frac{1}{\sqrt{N}}\left(\sqrt{M}\frac{1}{\sqrt{M}} \sum_{x':f(x')=1} |x'\rangle + \sum_{x'':f(x'')=0} |x''\rangle\right)$$

$$= \sqrt{\frac{M}{N}}|\omega\rangle + \sqrt{\frac{N-M}{N}}|s_{\bar{\omega}}\rangle$$

$\sqrt{\frac{M}{N}}$ (Good) $\left(\sqrt{\frac{N-M}{N}}\right)$ (Bad)

orthogonal because

No vects in Good is in bad and vice versa

ortho normal ~~...~~ also

because

~~Good~~

$$\frac{1}{\sqrt{M}} \sum_{f(x')=1} |x'\rangle$$

$f(x') = 1$ has m solutions.

$$|S\rangle = \sqrt{\frac{N-M}{N}}\,|S_{\bar{\omega}}\rangle + \sqrt{\frac{M}{N}}\,|\omega\rangle$$

Let $\sqrt{\frac{N-M}{N}} \cdot \cos\left(\frac{\theta}{2}\right)$ ~~$\sin(\theta)$~~

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{M}{N}}$$

$$|S\rangle = \cos\left(\frac{\theta}{2}\right)|S_\omega\rangle + \sin\left(\frac{\theta}{2}\right)|\omega\rangle$$

$$\boxed{\text{As } M \ll N \quad \sin\left(\frac{\theta}{2}\right) \approx \left(\frac{\theta}{2}\right) \approx \sqrt{\frac{M}{N}}}$$

Application of Oracle

$$U_f^{\pm}\, |s_{\bar{\omega}}\rangle \longrightarrow |s_{\bar{\omega}}\rangle$$

$$U_f^{\pm}\, |\omega\rangle \longrightarrow -|\omega\rangle$$

$$\Rightarrow \quad U_f^{\pm}\, |s\rangle = \cos\left(\tfrac{\theta}{2}\right) |s_{\bar{\omega}}\rangle - \sin\left(\tfrac{\theta}{2}\right) |\omega\rangle$$

So what exactly is $U_f^{\pm}$?

Bad $\to$ Bad

Good $\to$ $-$Good

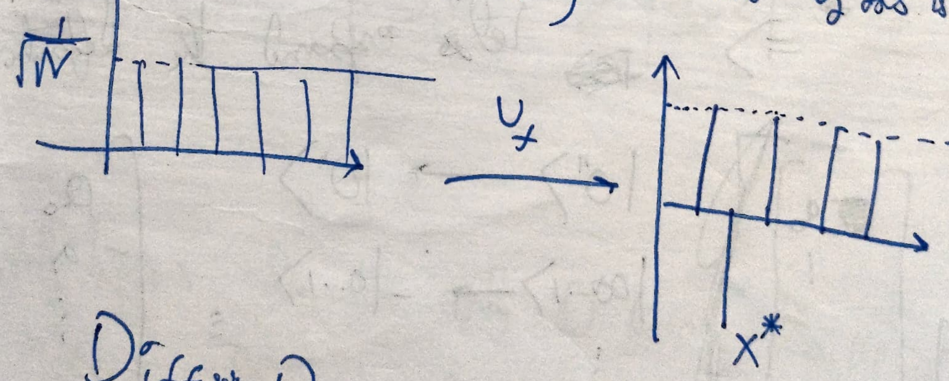
$$\Rightarrow \quad \boxed{U_f^{\pm} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{2D subspace}}$$

$\boxed{\text{Pauli - Z}}$

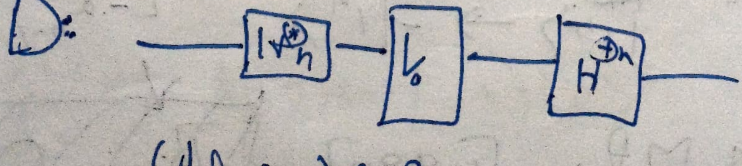
~~C-Z gate~~

But first we created $\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$ from $|0\rangle^{\otimes n}$ using $H^{\otimes n}$ (uniform superposition). So, what U_f does is

Amplitude



Diffuser D



What is V_0?

$$V_0 : |x\rangle \longrightarrow \begin{cases} |x\rangle, & \text{if } x = 0^n \\ -|x\rangle, & \text{if } x \neq 0^n \end{cases}$$

Performs a conditional phase shift on every computational basis other than 0^n.

$$V_0 = 2|0^n\rangle\langle 0^n| - \mathbb{1}$$

$$V_0 |0^n\rangle = |0^n\rangle$$

$$V_0 \begin{cases} |0^n\rangle = |0^n\rangle \\ |j\rangle = -|j\rangle \end{cases}$$

$\boxed{j = 1 \to N-1}$

ex $00, 01, 10, 11 \Rightarrow 00, -01, -10, -11$

V_0 How?

$\Rightarrow$ Let's expand the definition.

$$M \begin{bmatrix} a_0 \\ \vdots \\ \\ a_{n-1} \end{bmatrix} \quad \begin{matrix} |0^n\rangle \longrightarrow |0^n\rangle \\ |00\ldots1\rangle \longrightarrow -|0\ldots1\rangle \\ \vdots \\ |1^n\rangle \longrightarrow -|1^n\rangle \end{matrix} \quad \begin{bmatrix} a_0 \\ -a_1 \\ \vdots \\ \\ -a_{n-1} \end{bmatrix}$$

$$M \text{ Matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & & 0 \\ 0 & & -1 & \vdots \\ \vdots & & & \ddots \\ & & & -1 \end{bmatrix}$$

~~This is OR Gate~~

$\rightarrow$ This is OR gate

We are doing an n bit OR function.

$$M_{OR} = V_0$$

<u>What is D?</u>

$$D = H^{\otimes n} V_0 H^{\otimes n}$$

$$= H^{\otimes n}\left(2|0^n\rangle\langle 0^n| - \mathbb{1}\right) H^{\otimes n}$$

$$= 2\left[H^{\otimes n}|0^n\rangle\langle 0^n|H^{\otimes n}\right] - H^{\otimes n}H^{\otimes n}$$

S is superposition of all states.

$$\boxed{D = 2|s\rangle\langle s| - \mathbb{1}}$$

What happens when we apply D to any state $|\psi\rangle$?

$$|\psi\rangle = \sum_{k=0}^{N-1} \psi_k |k\rangle \ ?$$

$$D|\psi\rangle = \left(2|s\rangle\langle s| - \mathbb{1}\right)|\psi\rangle$$

$$= 2|s\rangle\langle s|\psi\rangle - |\psi\rangle$$

$$\boxed{|s\rangle\langle s| = |s\rangle \times \langle s|}$$

$$D|\psi\rangle = 2\left(|s\rangle \times \langle s|\right)|\psi\rangle - |\psi\rangle$$

$$D|\psi\rangle = 2|s\rangle\left[\langle s|\psi\rangle\right] - |\psi\rangle$$

$$D|\psi\rangle = 2\langle s|\psi\rangle\left[|s\rangle\right] - |\psi\rangle$$

$\cancel{D|\psi\rangle = 2}$.

$$\langle s|\psi\rangle = \langle s| \sum_k \alpha_k |k\rangle$$

$$= \sum_k \alpha_k \langle s|k\rangle$$

$$= \sum_{k,l} \frac{\alpha_k}{\sqrt{N}} \langle l|k\rangle$$

$$= \sum_{k,l} \frac{\alpha_k}{\sqrt{N}} \delta_{l,k}$$

only gives 1 when $l = k$

Computationally basis states

Mutually orthogonal

$$= \boxed{\sum_k \frac{\alpha_k}{\sqrt{N}}}$$

Another way of showing

$$|s\rangle\langle s| = |s\rangle \times \langle s|$$

$$= \frac{1}{\sqrt{N}}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\frac{1}{\sqrt{N}}[1\,1\,1\,1 \ldots 1]$$

$$= \frac{1}{N}\begin{bmatrix}1 & 1 & 1 & 1 & \cdots & 1\\ \vdots & \ddots & & & & \vdots\\ 1 & \cdots & & & \cdots & 1\end{bmatrix}$$

$$|s\rangle\langle s|\,|\psi\rangle = \frac{1}{N}\begin{bmatrix}1 & 1 & \cdots & 1\\ \vdots & \ddots & & \vdots\\ 1 & \cdots & \cdots & 1\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ \vdots\\ a_{n-1}\end{bmatrix}\frac{1}{\sqrt{N}}$$

$$= \frac{1}{N\sqrt{N}}\quad \sum_k a_k + \sum_k a_k \ldots\ldots \sum_k a_k$$

$$= \frac{\cancel{N}\sum_k a_k}{\cancel{N}\sqrt{N}} = \boxed{\frac{\sum_k a_k}{\sqrt{N}}}$$

$$D|\psi\rangle = \frac{2\sum_k a_k}{\sqrt{N}}|s\rangle - |\psi\rangle$$

$$D|\psi\rangle = 2\frac{\sum_k a_k}{N}\sum_l |l\rangle - |\psi\rangle$$

$$= 2\langle a\rangle \sum_l |l\rangle - \sum_l a_l |l\rangle$$

=

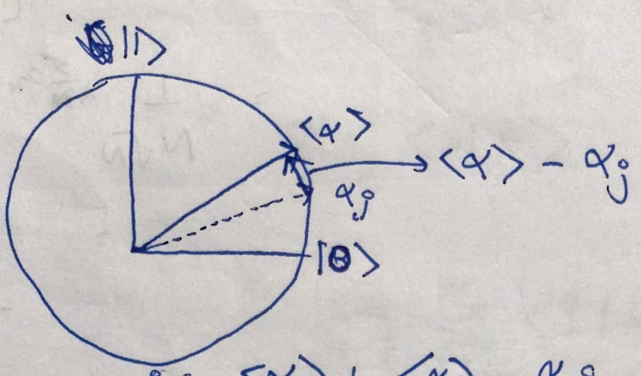
$$D|\psi\rangle = \sum_{l} \left[2\langle\alpha\rangle - \alpha_l \right] |l\rangle$$

$\downarrow$

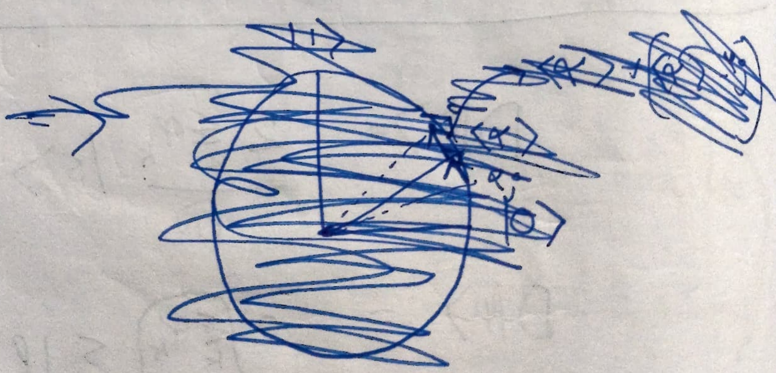
Inverting around the average

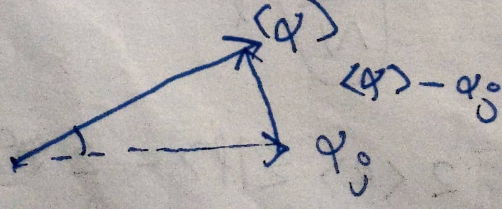
$\boxed{\text{D is also known as "inversion around mean" operator.}}$

Geometric visualization

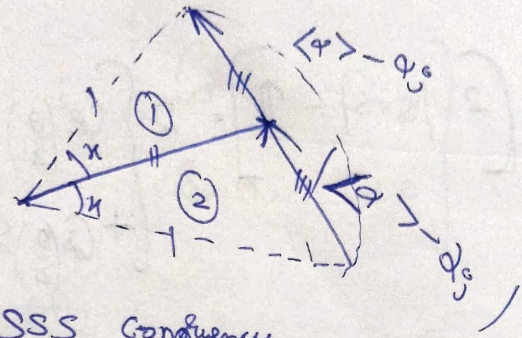


if $\langle\alpha\rangle + \langle\alpha\rangle - \alpha_j$?



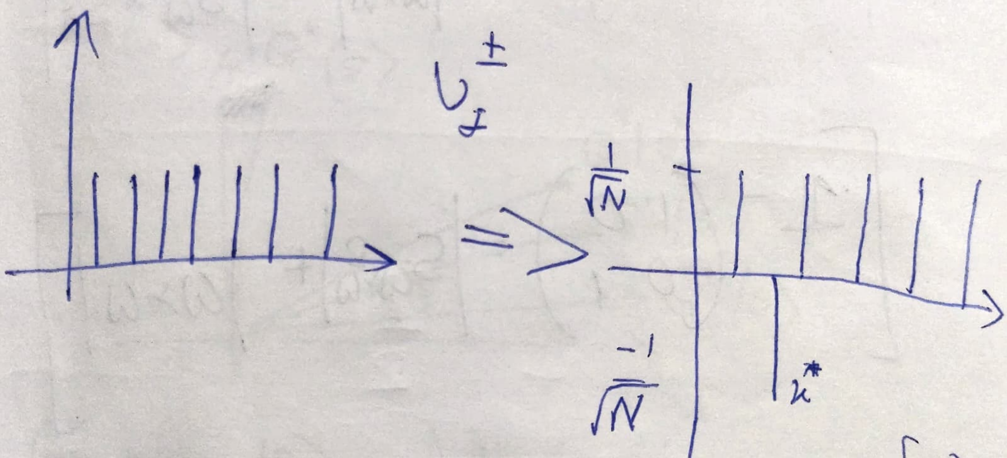


$$|\varphi_j| = |2\langle\alpha\rangle - \varphi_j| = |x|0\rangle + y|1\rangle| = \sqrt{x^2+y^2} = 1$$



(1) & (2) Congruent triangles

SSS Congruency



$$\langle\alpha\rangle = \frac{1}{N}\left[(N-1)\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N}}\right]$$

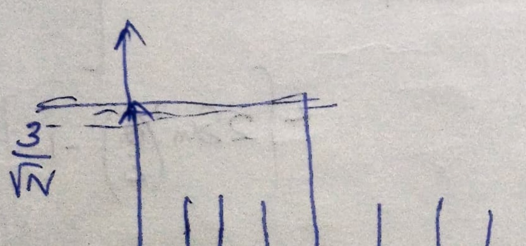
$$\approx \frac{1}{\sqrt{N}}$$

$$\varphi_j \xrightarrow{D} 2\langle\alpha\rangle - \varphi_j$$

$$\frac{1}{\sqrt{N}} \xrightarrow{D} \frac{1}{\sqrt{N}}$$

$$\frac{-1}{\sqrt{N}} \xrightarrow{D} \frac{3}{\sqrt{N}}$$

D



Amplification of amplitude of the solution.

Grover iterate : $\boxed{G = DU_f^{\pm}}$ In $\left\{ |s_{\bar{\omega}}\rangle, |\omega\rangle \right\}$ subspace

$$|s\rangle = \cos\left(\frac{\theta}{2}\right)|s_{\bar{\omega}}\rangle + \sin\left(\frac{\theta}{2}\right)|\omega\rangle$$

$$D = \left[2|s\rangle\langle s| - \mathbb{1}\right] = 2\left\{ \cos^2\left(\frac{\theta}{2}\right)|s_{\bar{\omega}}\rangle\langle s_{\bar{\omega}}| + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)|\omega\rangle\langle s_{\bar{\omega}}| + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)|s_{\bar{\omega}}\rangle\langle\omega| + \sin^2\left(\frac{\theta}{2}\right)|\omega\rangle\langle\omega| \right\}$$

$$- |\omega\rangle\langle\omega| - |s_{\bar{\omega}}\rangle\langle s_{\bar{\omega}}|$$

$$\left[\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |s_{\bar{\omega}}\rangle\langle s_{\bar{\omega}}| + |\omega\rangle\langle\omega| \right]$$

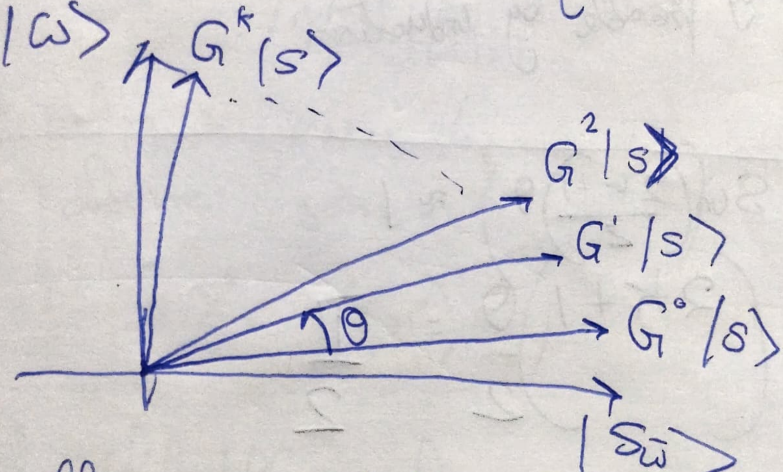
$$= \left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right)|s_{\bar{\omega}}\rangle\langle s_{\bar{\omega}}| + \sin\theta\left[|\omega\rangle\langle s_{\bar{\omega}}| + |s_{\bar{\omega}}\rangle\langle\omega|\right] + \left[2\sin^2\left(\frac{\theta}{2}\right) - 1\right]\left[|\omega\rangle\langle\omega|\right]$$

$$= \cos\theta\,|s_{\bar{\omega}}\rangle\langle s_{\bar{\omega}}| + \sin\theta\left[|\omega\rangle\langle s_{\bar{\omega}}| + |s_{\bar{\omega}}\rangle\langle\omega|\right] - \cos\theta\,|\omega\rangle\langle\omega|$$

$$G = D U_f^{-f} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

G induces a rotation by an ~~ag~~ angle θ in ~~the~~ the 2D subspace spanned by $\{ |s_{\bar{\omega}}\rangle, |\omega\rangle \}$



G rotates $|s\rangle$ gradually towards $|\omega\rangle$.

$$\boxed{G = -R_s R_\omega}$$

$$R_\omega = 2|\omega\rangle\langle\omega| - \mathbb{1}$$

$$R_s = 2|s\rangle\langle s| - \mathbb{1}$$

Find k such that $|\langle \omega | G^k | s \rangle|^2 \approx 1$

$$G = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } |s\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

We will show

$$G^k |s\rangle = \cos\left(\frac{(2k+1)\theta}{2}\right) |s_{\bar{\omega}}\rangle + \sin\left(\frac{(2k+1)\theta}{2}\right) |\omega\rangle$$

easily provable by induction

$$\sin\left(\frac{(2k+1)}{2}\theta\right) \approx 1$$

$$\Rightarrow (2k+1)\frac{\theta}{2} = \frac{\pi}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{2k+1}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{M}{N}}$$

$$\theta = 2\sin^{-1}\left(\sqrt{\frac{M}{N}}\right)$$

$$2\sin^{-1}\sqrt{\frac{M}{N}} = \frac{\pi}{2k+1}$$

$M \ll N$ $\quad$ $\sin x \approx x$

$$\Rightarrow 2\sqrt{\frac{M}{N}} = \frac{\pi}{2k+1} \Rightarrow 2k+1 \cdot \frac{\pi}{2}\sqrt{\frac{N}{M}}$$

$$\boxed{K_{opt} = \left\lfloor \frac{\pi}{4}\sqrt{\frac{N}{M}} - \frac{1}{2} \right\rfloor}$$

$$\boxed{K = O\left(\sqrt{\frac{N}{M}}\right)}$$

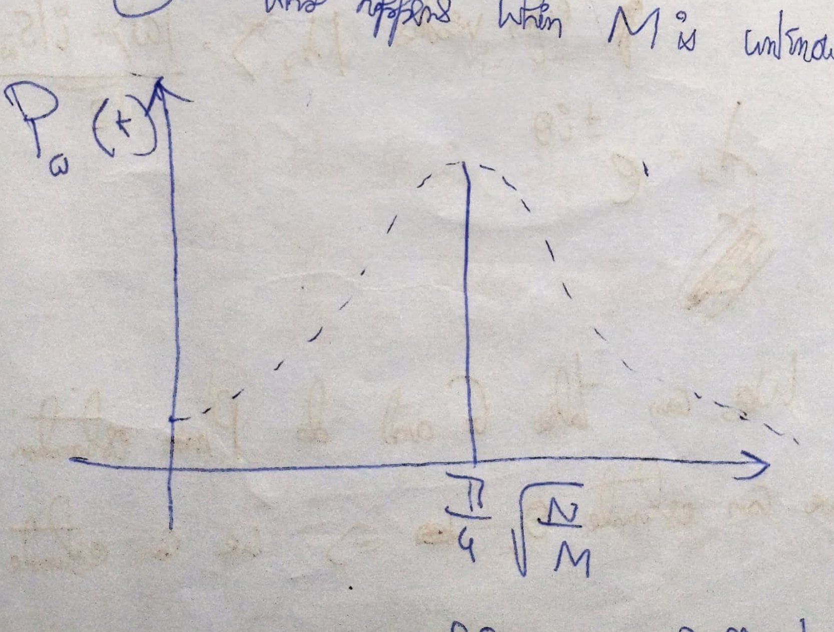
After these many iterations, we end up with a state close to $|\omega\rangle$.

$$|\omega\rangle = \frac{1}{\sqrt{M}} \sum_{x : f(x) = 1} |x\rangle$$

By making a measurement in the computational basis, we observe some

$$x^* : f(x^*) = 1$$

But what happens when M is unknown?



Overshooting is a problem!

Solution →

~~We guess $M = 1$~~ There are 2 ways:-

~~If~~

1. Randomized Quantum Search:

$\Theta(\sqrt{N})$ iterations, you can obtain x^* with $\Theta(1)$ probability.

2. Quantum Counting ⇒ USE QPE on G to estimate M to some accuracy.

$$G = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

⇒ G eigen values are $e^{\pm i\theta}$

eigen vectors $|\lambda_{\pm}\rangle = \dfrac{|\omega\rangle + i|S_{\bar{\omega}}\rangle}{\sqrt{2}}$

$\lambda_{\pm} = e^{\pm i\theta}$

We can take G and do Phase-estimation.

If we can estimate θ, ⇒ we can estimate M

Randomized Quantum Search Method:

Let M' be the actual number of solutions to the search problem (M' is unknown).

We will ~~choose~~ ~~some~~ guess some M to be the ~~solutio~~ solutions.

Idea → If we keep repeating the Grover's algorithm with different guesses of M, then there will be a value of M that is close ~~eno~~ enough to M', so ~~that~~ the solution is obtained with a constant probability.

⇒ ① Set $k=0$

② Set $M = 2^k$

③ Run Grover's algo for $\frac{\pi}{4}\sqrt{\frac{N}{M}}$ iterations

④ If a solution is found, EXIT. Otherwise go to step 5.

⑤ $k = k+1$

⑥ If $k > \log_2 N$, then EXIT. Otherwise

→ Run Grover's algorithm using $M=1$, if it fails, try again with $M = 2, 4, 8, \ldots\ldots 2^{\log_2 N}$

→ Total No of queries in the worst case

$$= \sum_{l=0}^{\log_2 N} \frac{\pi}{4}\sqrt{\frac{N}{2^l}} = \frac{\pi}{4}\sqrt{N}\sum_{l=0}^{\log_2 N}\left(\frac{1}{\sqrt{2}}\right)^l$$

$$= O(\sqrt{N})$$

⇒ There will be a guess such ~~that~~

$$\frac{M'}{2} \le M \le 2M'$$

$$\boxed{\begin{array}{c}\text{Box: } \exists\, j \text{ s.t } P/2 \le 2^j \le 2P \text{ for } P \in \mathbb{N} \\ \equiv \\ \exists \text{ a positive integer between} \\ \log_2 P - 1 \;\&\; \log_2 P + 1\end{array}}$$

⇒ What is success probability ?

- For M s.t $\frac{M'}{2} \le M \le 2M'$, the Grover's algorithm run for $T = \frac{\pi}{4}\sqrt{\frac{N}{M}}$ iterations. This means that $\boxed{T}$ is within a factor of $\sqrt{2}$ of $T = \frac{\pi}{4}\sqrt{\frac{N}{M}}$, the optimal no. of iterations.

- We have $(2T'+1)\sin^{-1}\left(\sqrt{\frac{M'}{N}}\right) = \frac{\pi}{2}$

- Success prob: $\sin^2\left((2T+1)\sin^{-1}\left(\sqrt{\frac{M'}{N}}\right)\right)$

$$= \sin^2\left[\frac{2T+1}{(2T'+1)}\underbrace{(2T'+1)\sin^{-1}\left(\sqrt{\frac{M'}{N}}\right)}_{:=\frac{\pi}{2}}\right]$$

$$= \sin^2\left[\frac{2T+1}{2T'+1}\frac{\pi}{2}\right]$$

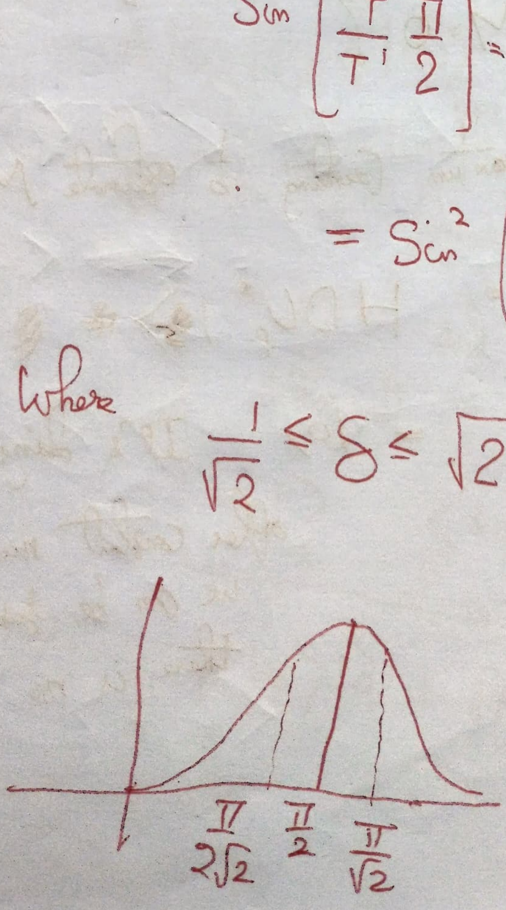
$$= \sin^2\left[\frac{T}{T'}\frac{\left(1+\frac{1}{2T}\right)}{\left(1+\frac{1}{2T'}\right)}\frac{\pi}{2}\right]$$

$$\approx \sin^2\left[\frac{T}{T'}\frac{\pi}{2}\right] = \sin^2\left(\sqrt{\frac{M'}{M}}\frac{\pi}{2}\right)$$

$$= \sin^2\left(\frac{\delta\pi}{2}\right)$$

Where

$$\frac{1}{\sqrt{2}} \leq \delta \leq \sqrt{2} = \Theta(1)$$



• Randomized Quantum Search finds x^* with constant probability in $O(\sqrt{N})$ queries even when M is unknown.

• What happens when M is very close to N? $\left(M > \frac{N}{2}\right)$

$$|s\rangle = \underbrace{\sqrt{\frac{M}{N}}}_{= \frac{1}{\sqrt{2}}} |\omega\rangle + \underbrace{\sqrt{\frac{N-M}{N}}}_{\frac{1}{\sqrt{2}}} |s_{\bar{\omega}}\rangle$$

Prob is already half. We don't even need to run Grover's. [Some basic statistics]

$\Rightarrow |\langle \omega | s \rangle|^2 \geq \frac{1}{2}$, so just preparing $|s\rangle$ followed by measurement would suffice

• What if $M = 0$?

$\Rightarrow$ Use Quantum Counting to estimate M.

Simpler method = do $HDU_f^{\pm} |s\rangle$

$= |0\rangle$ It's doing nothing after constant number of checks we can be sure there is no answer.