23MAT102
Class Notes

Adithya Nair

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0.1 A Basic Order Of Importance

- **Axiom -** Statements taken as fact
- **Theorem -** Statements that are proven using axioms
- **Lemma -** Statements proven using theorems
- **Proposition -** Statements, regardless of whether it is true or false, is assumed to be true
- **Corollary -** A theorem that is proven using another theorem.*

Chapter 1

A Revision Of Sets And Functions

Sets are assumed to be sets on the basis of a theory known as **Naive Set Theory**. according to this theory, A set is defined as,

Definition 1.0.1: Sets

A set is a collection of objects

e.g. -

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

1.1 Notations

A,B...Z will denote sets
a,b...z will denote elements
a $\in$ A, a is an element of A
a $\notin$ A, a is not an element of A

1.2 Roster Notation

$$\mathbb{N} = \{1, 2...\}$$

$$A = \{2, 4, 6, 8$$

$$B = \{x \in Z + |x < 10\}$$

B is written in set builder form

1.3 Basic Concepts Of Sets

Definition 1.3.1: Subsets

A and B are two sets. A is a subset of B, and we write A $\subset$ B, if every element of A is also an element of B

Theorem 1.3.1. Two sets A and B are equal and we write A=B if and only if A $\subset$ B and B $\subset$ A

Definition 1.3.2: Unions

The union of two sets A and B, denoted by A $\cup$ B, is

$$A \cup B = \{x |\ x \in A \ and\ x \in B\}$$

Definition 1.3.3: Intersections

The intersection of two sets denoted by A $\cap$ B, is

$$A \cap B = \{x |\ x \in A \ or\ x \in B\}$$

Definition 1.3.4: Set Difference

The difference of two sets denoted by A$\setminus$ B is

$$A \setminus B = x |\ x \in A \ and\ x \notin B$$

Definition 1.3.5: Set Complement

The complement of a set A, denoted by A ^{C} is,

$$A^C = \{x \in X |\ x \notin A\}$$

- $(B \cup C)^C = B^C \cap C^C$
- $(B \cap C)^C = B^C \cap C^C$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

1.4 Logical Notation

$\forall$ - for all
$\exists$ - there exists
$\exists!$ - there exists a unique

1.5 Functions

f: A $\rightarrow$ B
f(a) = b, a $\in$ A, b $\in$ B
A is the **domain** of the function, B is the **codomain** of the function and,
$\{$b$\in$ B $|$f(a) = b $\}$ - Range

1.6 Cartesian Product

$$A \times B = \{(a,b)|a \in A, b \in B\}$$

1.7 Composition Of Functions

(g$\circ$f)(x) = (g(f(x))

A function is the same as a mapping, which is the same as a transformation

1.8 Types Of Functions

1. f is injective(one-one) if,

$$f(a) = f(a') \ then \ a \ = \ a'$$

2. f is surjective(onto) if,

$$\forall b \in B, \exists \ a \in A, \ f(a) = b$$

3. f is bijective if f is injective and surjective

Reference

Knowles - Linear Vector Spaces and Cartesian Tensors
Halmos - Finite Dimensional Linear Spaces
Gelfand - Linear Algebra

Chapter 2

Vector Spaces

A vector space over a field F = $\mathbb{R}$ or $\mathbb{C}$ is a set V with two operations:

1. +:V$\times$ V $\to$ V i.e. "+" is closed under addition.
2. $\cdot$:F $\times$ V $\to$ V, i.e. " $\cdot$ " is closed under multiplication

having the following properties

1. **Associativity**

$$\forall\ v_1, v_2, v_3 \in V, (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

2. **Existence of identity element**

$$\exists!\ 0 \in V, \forall v\ inV, such that 0 + v = v$$

3. **Existence of additive inverse**

$$\forall\ v \in V \exists (-v) \in V, v + (-v) = 0$$

4. **Commutativity**

$$\forall\ u, v \in V, u + v = v + u$$

Properties 1 to 4 constitute a group known as the "abelian group" or "commutative group"

5. **Existence of multiplactive identity**

$$\exists!\ 1 \in V,\ such\ that\ \forall\ v \in V, 1 \cdot v = v$$

6. **Associativity**

$$\mu, \lambda \in F, v \in V, \lambda(\mu \cdot v) = (\lambda\mu) \cdot v$$

7. **Distribution of + over $\cdot$**

$$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v, \forall\ \mu, \lambda \in F$$

8. **Distribution of $\cdot$ over +**

$$\lambda \cdot (u + v)\ = \lambda \cdot u + \lambda \cdot v, \forall \lambda \in\ F, u, v \in V$$

2.1 Examples Of Vector Spaces

1. V = 0
2. $\mathbb{R}$
3. All polynomials of order **at most** n

Reference

- Donald Knuth
- Marvin Mirsky, MIT
- Web Of Stories, Youtube Channel
- Axler, Chapter 1
- Olver, Shakiban, Chapter 2
- Terrence Tao Notes - AMS Open Math

2.2 Some Theorems And Proofs Regarding Vector Spaces

Theorem 2.2.1. Additive identity is unique

Proof. Suppose $\exists$ *additive identities* 0_1, 0_2 such that

$$\forall u \in V, 0_1 + u \ \&\ 0_2 + u = u$$

$$0_1 + 0_2 = 0_2$$

$$0_2 + 0_1 = 0_1$$

$$\therefore 0_1 = 0_2$$

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Theorem 2.2.2. Additive inverse is unique

Proof. Suppose additive inverses of u are v_1, v_2

$$u + v_1 = 0, u + v_2 = 0$$

$$v_2 + (u + v_1) = v_2 + 0$$

$$(v_2 + u) + v_1 = v_2$$

$$0 + v_1 = v_2$$

$$v_1 = v_2$$

😄

Theorem 2.2.3. $0 \cdot u = 0$

Proof. Let 0 . u = 0 Consider,

$$v + v = 0.u + 0.u = (0 + 0).u$$

$$= 0.u = v$$

$$\Rightarrow v + v = v$$

$$v + (v + (-v)) = v + -v$$

$$\Rightarrow v = 0$$

😄

Theorem 2.2.4 (Scalars And Inverses)**.**

$$(-\lambda)u = -(\lambda.u) = \lambda.(-u)$$

Proof. Let $v = (-\lambda).u$ Consider,

$$v + \lambda.u = (-\lambda).u + \lambda u$$

😄

2.3 Fields

A) To every pair α and β of scalars, there corresponds a scalar $\alpha + \beta$ called the sum of α and β, in such a way that

1. addition is commutative, $\alpha + \beta = \beta + \alpha$
2. addition is associative, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

3. There exists a unique scalar 0, called zero, such that $\alpha + 0 = \alpha$ for every scalar α, and

4. to every scalar α, there corresponds a unique scalar $(-\alpha)$ such that $\alpha + (-\alpha) = 0$