23MAT102
Class Notes

Adithya Nair

Contents

0.1 A Basic Order Of Importance

- **Axiom** - Statements taken as fact
- **Theorem** - Statements that are proven using axioms
- **Lemma** - Statements proven using theorems
- **Proposition** - Statements, regardless of whether it is true or false, is assumed to be true
- **Corollary** - A theorem that is proven using another theorem.*

Chapter 1

A Revision Of Sets And Functions

Sets are assumed to be sets on the basis of a theory known as **Naive Set Theory**. according to this theory, A set is defined as,

Definition 1.0.1: Sets

A set is a collection of objects

e.g. -

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

1.1 Notations

A,B...Z will denote sets a,b...z will denote elements a $\in$ A, a is an element of A a $\notin$ A, a is not an element of A

1.2 Roster Notation

$$\mathbb{N} = \{1, 2...\}$$

$$A = \{2, 4, 6, 8$$

$$B = \{x \in Z + |x < 10\}$$

B is written in set builder form

1.3 Basic Concepts Of Sets

Definition 1.3.1: Subsets

A and B are two sets. A is a subset of B, and we write A $\subset$ B, if every element of A is also an element of B

Theorem 1.3.1. Two sets A and B are equal and we write A=B if and only if A $\subset$ B and B $\subset$ A

Definition 1.3.2: Unions

The union of two sets A and B, denoted by A $\cup$ B, is

$$A \cup B = \{x|\ x \in A\ and\ x \in B\}$$

Definition 1.3.3: Intersections

The intersection of two sets denoted by A $\cap$ B, is

$$A \cap B = \{x|\ x \in A\ or\ x \in B\}$$

Definition 1.3.4: Set Difference

The difference of two sets denoted by A$\setminus$ B is

$$A \setminus B = x|\ x \in A\ and\ x \notin B$$

Definition 1.3.5: Set Complement

The complement of a set A, denoted by A ^{C} is,

$$A^C = \{x \in X|\ x \notin A\}$$

- $(B \cup C)^C = B^C \cap C^C$
- $(B \cap C)^C = B^C \cap C^C$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

1.4 Logical Notation

$\forall$ - for all $\exists$ - there exists $\exists!$ - there exists a unique

1.5 Functions

f: A $\rightarrow$ B f(a) = b, a $\in$ A, b $\in$ B A is the **domain** of the function, B is the **codomain** of the function and, {b$\in$ B |f(a) = b } - Range

1.6 Cartesian Product

$$A \times B = \{(a, b)|a \in A, b \in B\}$$

1.7 Composition Of Functions

(g$\circ$f)(x) = (g(f(x))

A function is the same as a mapping, which is the same as a transformation

1.8 Types Of Functions

1. f is injective(one-one) if,

$$f(a) = f(a')\ then\ a\ =\ a'$$

2. f is surjective(onto) if,

$$\forall b \in B, \exists\ a \in A,\ f(a) = b$$

3. f is bijective if f is injective and surjective

Reference

Knowles - Linear Vector Spaces and Cartesian Tensors
Halmos - Finite Dimensional Linear Spaces
Gelfand - Linear Algebra

Chapter 2

Linear Algebra

A vector space over a field F = $\mathbb{R}$ or $\mathbb{C}$ is a set V with two operations:

1. +:V× V → V i.e. "+" is closed under addition.
2. ·:F × V → V, i.e. " · " is closed under multiplication

having the following properties

1. **Associativity**
$$\forall\ v_1, v_2, v_3 \in V, (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$
2. **Existence of identity element**
$$\exists!\ 0 \in V, \forall v\ inV, such that 0 + v = v$$
3. **Existence of additive inverse**
$$\forall\ v \in V \exists(-v) \in V, v + (-v) = 0$$
4. **Commutativity**
$$\forall\ u, v \in V, u + v = v + u$$
Properties 1 to 4 constitute a group known as the "abelian group" or "commutative group"
5. **Existence of multiplactive identity**
$$\exists!\ 1 \in V,\ such\ that\ \forall\ v \in V, 1 \cdot v = v$$
6. **Associativity**
$$\mu, \lambda \in F, v \in V, \lambda(\mu \cdot v) = (\lambda\mu) \cdot v$$
7. **Distribution of + over ·**
$$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v, \forall\ \mu, \lambda \in F$$
8. **Distribution of · over +**
$$\lambda \cdot (u + v)\ = \lambda \cdot u + \lambda \cdot v, \forall \lambda \in\ F, u, v \in V$$

2.1 Examples Of Vector Spaces

1. V = 0
2. $\mathbb{R}$
3. All polynomials of order **at most** n

Reference

- Donald Knuth
- Marvin Mirsky, MIT
- Web Of Stories, Youtube Channel
- Axler, Chapter 1
- Olver, Shakiban, Chapter 2
- Terrence Tao Notes - AMS Open Math

2.2 Some Theorems And Proofs Regarding Vector Spaces

Theorem 2.2.1. Additive identity is unique

Proof. Suppose $\exists$ *additive identities* 0_1, 0_2 such that

$$\forall u \in V, 0_1 + u \ \& \ 0_2 + u = u$$

$$0_1 + 0_2 = 0_2$$

$$0_2 + 0_1 = 0_1$$

$$\therefore 0_1 = 0_2$$

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Theorem 2.2.2. Additive inverse is unique

Proof. Suppose additive inverses of u are v_1, v_2

$$u + v_1 = 0, u + v_2 = 0$$

$$v_2 + (u + v_1) = v_2 + 0$$

$$(v_2 + u) + v_1 = v_2$$

$$0 + v_1 = v_2$$

$$v_1 = v_2$$

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Theorem 2.2.3. $0 \cdot u = 0$

Proof. Let 0 . u = 0 Consider,

$$v + v = 0.u + 0.u = (0 + 0).u$$

$$= 0.u = v$$

$$\Rightarrow v + v = v$$

$$v + (v + (-v)) = v + -v$$

$$\Rightarrow v = 0$$

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Theorem 2.2.4 (Scalars And Inverses).

$$(-\lambda)u = -(\lambda.u) = \lambda.(-u)$$

Proof. Let

$$v = (-\lambda).u$$

Consider,

$$v + \lambda.u = (-\lambda).u + \lambda.u$$

$$= (\lambda + \lambda).u$$

$$= 0.u = 0$$

$$\therefore (\lambda.u) + -(\lambda.u) = 0$$

$$= (-\lambda.u) + (\lambda.u) + -(\lambda.u) = (-\lambda.u)$$

$$= (-\lambda).u + 0 = (-\lambda.u)$$

$$(-\lambda).u = -(\lambda.u)$$

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2.3 Fields

A) To every pair α and β of scalars, there corresponds a scalar $\alpha + \beta$ called the sum of α and β, in such a way that

1. addition is commutative, $\alpha + \beta = \beta + \alpha$

2. addition is associative, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

3. There exists a unique scalar 0, called zero, such that $\alpha + 0 = \alpha$ for every scalar α, and

4. to every scalar α, there corresponds a unique scalar $(-\alpha)$ such that $\alpha + (-\alpha) = 0$

B) To every pair α and β of scalars there corresponds a scalar $\alpha\beta$, called the product of α and β in such a way that

1. multiplication is commutative, $\alpha\beta = \beta\alpha$

2. multiplication is associative, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

3. there exists a unique non-zero scalar 1(called one) such that $\alpha 1 = \alpha$ for every scalar α and

4. to every non-zero scalar α, there corresponds a unique scalar $\alpha^{-1} = \frac{1}{\alpha}$ such that $\alpha\alpha^{-1} = 1$

C) Multiplication is distributive with respect to addition,

$$\alpha(\beta + \gamma) = (\alpha\beta + \alpha\gamma)$$

If addition and multiplication are defined within same set of objects(scalars) so that the conditions A,B and C are satisfied, then that set is called a field.

Note:-

The main difference between vector spaces and fields:
All fields are vector spaces over themselves. The main difference arises in the operation. Vector spaces have operations:

$$+ : V \times V \to V$$

$$\cdot : F \times V \to V$$

While fields have both operations:

$$+, \cdot : F \times F \to F$$

Another key difference is that fields have an axiom regarding a unique scalar known as the multiplicative inverse (α^{-1}) which is not an axiom for vector spaces.

2.4 Examples Of Vector Spaces And Fields

Examples of fields include: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

- For complex numbers

$$\mathbb{C}(\text{Complex Numbers}) = \{(a,b) : a,b \in R\}$$

$$(a,b) + (c,d) = (a+c, b+d)$$

$$(a,b)(c,d) = (ac-bd, ad+bc)$$

  This is a field.

- $\mathbb{R}^n$, n-tuple where $x = (x_1, x_2, x_2 \ldots x_n)$ This is a vector space over $\mathbb{R}$

- $P_n(\mathbb{R})$, the set of all polynomials with degree n is a vector space over $\mathbb{R}$.

  > **Note:-**
  >
  > $P_n(\mathbb{R})$ has a direct correlation to $(\mathbb{R}^{n+1})$ which means they are **isomorphic** spaces

- C($\mathbb{R}$) - the space of all continuous functions is a vector space over field $\mathbb{R}$

- $M_{m\times n}$: set of all $m \times n$ matrices is a vector space

- **Linear maps/operations/transformations/functions** - Suppose U,V are two vector spaces over a field F, then $T : u \to v$ is linear for some scalars $\alpha, \beta \in F$, we have $T(\alpha u_1 + \beta u_2 = \alpha T(u_1)) + \beta T(u_2)$

2.5 Subspaces

(V,+, $\cdot$) - is a vector space
(W $\subset$ V, +, $\cdot$) - is a subspace of V

A subspace is a vector space, where the set is a subset of another vector space.

> **Note:-**
>
> All lines that pass through the origin in a 2-dimensional plane are subspaces of $\mathbb{R}^2$

Lemma 2.5.1. Let V be a vector space and let W be a subset of V . Then W is a subspace of V if and only if the following properties hold:

1. W is closed under addition,

$$\text{If } w_1, w_2 \in W, \text{ then}$$
$$w_1 + w_2 \in W$$

2. W is closed under scalar multiplication.

$$\text{If } \alpha \in F, w \in W, \text{ then } \alpha w \in W$$

Proof.

1. W is a subspace $\Rightarrow$ 1. and 2.
W is closed under addition and scalar multiplication, because this is implicit in the definition of subspaces.

2. 1. and 2. $\Rightarrow$ W is a subspace
$-u \in W$, because $-1.u = -u$
$0 \in W$ because $0.u = 0$

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2.6 Span

Definition 2.6.1: Spans

Let S be a finite collection of vectors in a vector space V. A linaer combination of S is defined to be any vector in V of the form

$$\alpha_1 v_1 + \alpha_2 v_2 \cdots + \alpha_n v_n$$

Theorem 2.6.1. Let S be a subset of a vector space V. Then span(S) is a subspace of V which contains S is a subspace of V which contains S. Moreoever, ny subspace of V which contains S as a subset must in fact contain all of span(S)

Proof. To prove, i) span(S) is a subspace of V ii) span(S) $\subseteq V$

Proving span(S) $\subseteq V$, Let $\alpha_1 v_1 + \alpha_2 v_2 \cdots + \alpha_n v_n$ be any element in span(S), where $\alpha_i \in F, v_i \in S$ and $i = \{1, \dots n\}$ Then

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in span(S)$$

Since, $\cdot$ and $+$ are closed under V, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V$
$\therefore span(S) \in V$ 😄

2.7 Linear dependence and independence

Definition 2.7.1: Linear dependence

Any collection S of vectors in a vector space V are linearly dependent, if we can find scalars

$$\alpha_1, \alpha_2 \ldots, \alpha_n \in \text{ F not all zero, such that}$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$$

2.8 Basis

Definition 2.8.1: Basis

A basis in a vector space V is a set S of linearly independent vectors such that every vector in V is a linear combination of elements in S, i.e. S is the set of linearly independent vectors and is the spanning set of V.

Definition 2.8.2: Dimension

The dimension of a finite dimensional vector space V is the number of elements in a basis of V.

Lemma 2.8.1. Let $\{v_1, v_2, \ldots v_n\}$ be a basis for a vector space V. Then every vector v can be written in the form,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \tag{2.1}$$

Proof. Let,

$$v = \beta_1 v_1 + \beta_2 v_2 + \ldots \beta_n v_n \tag{2.2}$$

be another representation of v in $v_1, v_2 \ldots v_n$

$$2.2 - 2.1$$

$$0 = (\beta_1 - \alpha_1)v_1 + (\beta_2 - \alpha_2)v_2 \cdots + (\beta_n - \alpha_n)v_n$$

$$\therefore \alpha_1 = \beta_1, \alpha_2 = \beta_2 \ldots \alpha_n = \beta_n$$

ii) W is any subspace of V, which contains S.

Take any term v, which is a linear combination of $\alpha_i v_i$

$$v = \alpha_1 v_1 + \alpha_2 v_2 \cdots + \alpha_n v_n,$$

$$\therefore v \in span(S) \in W$$

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Theorem 2.8.1. Let V be a vector space, and let S be a linearly independent subset of V. Let v be a vector which does not lie in S. a) If v lies in span(S) then the set $S \cup \{v\}$ is linearly dependent and span($S \cup \{v\} =$ span(S)