Journal Society Notes

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Contents

Part I

Real Analysis

Chapter 1

Natural Numbers

Numbers were built to count. A system for counting was made, and that system is the number system.

Definition 1.0.1

A natural number is an element of the set $\mathbb{N}$ of the set

$$\mathbb{N} = \{0, 1, 2, 3 \cdots\}$$

is obtained from 0 and counting forward indefinitely.

1.1 Peano Axioms

We start with axioms to help clarify this.

- Axiom 1 : $0 \in \mathbb{N}$
- Axiom 2: If $n \in \mathbb{N}$,, then $n++ \in \mathbb{N}$
- Axiom 3: 0 is not an increment of any other natural number $n \in \mathbb{N}$
- Axiom 4: If $n \neq m$, $n++ \neq m++$
- Axiom 5: (Principle Of Mathematical Induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number.

We then make an assumption: That the set $\mathbb{N}$ which satisfies these five axioms is called the set of natural numbers. With these 5 axioms, we can construct sequences

1.2 Recursive Definitions

Proposition 1.2.1 (Recursive Definitions). Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.

1.3 Addition

Definition 1.3.1: Addition Of Natural Numbers

Let n be a natural number. ($n \in N$). To add zero to m, we define $0 + m := m$ Now suppose inductively that we have defined how to add n to m. Then we can add n++ to m by defining(n++) + m := (n+m)++

Lemma 1.3.1. For any natural number $n + 0 = n$

Proof. We use induction,

The base case, n = 0,

$$n = 0, 0 + 0 = 0$$
$$n + 0 = n$$
$$(n++) + 0 = (n + 0) + + = (n++)$$

Suppose inductively, that $n + 0 = n$,

For $n = n++$,

$$(n++) + 0 = (n + 0) + +$$
$$\text{We know that } n + 0 = n$$
$$(n++) + 0 = (n++)$$

□

Lemma 1.3.2. For any natural numbers n and m,

$$n + (m++) = (n + m) + +$$

Proof. Inducting on n while keeping m fixed,

$$n = 0,$$
$$0 + (m++) = (0 + m) + +$$
$$0 + (m++) = (m++)$$

This we know is true from the definition of addition $(0 + m := m)$

Suppose inductively, that $n + (m++) = (n + m) + +$ is true. For $n = (n++)$,

$$\begin{aligned}(n++) + (m++) &= ((n++) + m) + + && \text{From the definition of addition}\\ &= (n + (m++)) + +\\ &= ((n + m) + +)) + +\end{aligned}$$

□

Putting m = 0, we get $n + 1 = n++$

Proposition 1.3.1 (Addition is commutative). For any natural numbers n and m, $n + m = m + n$

Proof. We induct over n, For the base case, $n = 0$,

We must show that $m + 0 = 0 + m$ From the definition of addition, we have

$$0 + m = m$$

As shown earlier, we have

$$m + 0 = m$$

This is clearly true for $n = 0$.
Now suppose inductively that $m + n = n + m$
For $n = n++$, we must show that $m + (n++) = (n++) + m$
We know from the definition of addition that,

$$(n++) + m := (m + n)++$$

And we proved earlier that,

$$m + (n++) = (m + n)++$$

Therefore,

$$m + (n++) = (n++) + m$$

□

Proposition 1.3.2 (Addition is associative). For any natural numbers, a, b and c, we have $(a+b)+c = a + (b + c)$

Proof. We take $(a + b) + n = a + (b + n)$

Inducting over n,
For $n = 0$,
We have in the LHS,

$$\begin{aligned} &= (a + b) + 0 && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

On the RHS,

$$\begin{aligned} &= a + (b + 0) && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

Suppose inductively that $(a + b) + n = a + (b + n)$,
For $n = n++$, We have to show that $(a + b) + (n++) = a + (b + (n++))$
On the LHS we have,

$$
\begin{aligned}
&= (a+b)+(n++) \\
&= (a+b+n)++ && \text{(From the lemma } m+(n++)=(m+n)++\text{)}
\end{aligned}
$$

On the RHS we have,

$$
\begin{aligned}
&= a+(b+(n++)) \\
&= a+(b+n)++ && \text{(From the lemma } m+(n++)=(m+n)++\text{)} \\
&= (a+b+n)++
\end{aligned}
$$

LHS = RHS □

Proposition 1.3.3 (Cancellation Law). Let a, b, c be natural numbers such that $a+b=a+c$. Then we have $b=c$.

Proof. We have,

$$n+b=n+c$$

Inducting over n, For the base case, $n=0$

$$
\begin{aligned}
0+b &= 0+c \\
b &= c
\end{aligned}
$$

Suppose inductively that $n+b=n+c$ For $n=n++$,

$$(n++)+b=(n++)+c$$

On the LHS

$$
\begin{aligned}
&= (n++)+b \\
&= (n+b)++
\end{aligned}
$$

On the RHS

$$
\begin{aligned}
&= (n++)+c \\
&= (n+c)++
\end{aligned}
$$

We know from the inductive hypothesis that,

$$\text{If} n+b=n+c, \text{then} b=c$$

Thus we have,

$$b++=c++$$

□

Definition 1.3.2: Positive natural number

All numbers where,

$$n \neq 0, n \in \mathbb{N}$$

Proposition 1.3.4. If a is a positive natural number and b is a natural number, then $a+b$ is positive.

Proof. Inducting over b,

For $b = 0$,

$$a + 0 = a$$

This proves the base case, since we know a is positive.

Now, suppose inductively, that $(a + b)$ is positive.

For $(a + (n++))$,

$$a + (n++) = (a + n)++$$

We know from Axiom 3 that $n++ \neq 0$. Thus we close the inductive loop. □

Lemma 1.3.3. For every a, there exists a unique b such that $b++ = a$

Proof. Proof by contradiction, Suppose that there are two different increments, $m++$, $n++$ that equal to a,

We have,

$$m++ = a$$
$$n++ = a$$

Then we can say,

$$m++ = n++$$
$$m + 1 = n + 1$$
$$m = n \qquad \text{(By Cancellation Law)}$$

But we said that m and n are different numbers which increment to a.

Therefore, we can conclude that there is only one number b which increments to a □

1.4 Order

Definition 1.4.1: Order

Let n and m be natural numbers we say that n is greater than or equal to m, and write $n \geq m$ iff we have $n = m + a$ for some natural number a. We say that $n > m$ when $n \geq m$ and $n \neq m$

Proposition 1.4.1 (Basic properties of order for natural numbers). Let a, b, c be natural numbers then

1. (Order is reflexive) $a \geq a$
2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$
3. (Order is antisymmetric) If $a \geq b$ and $b \geq a$ then $a = b$
4. (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$
5. $a < b$ if and only if $a{+}{+} \leq b$
6. $a < b$ if and only if $b = a + d$ for some positive number d.

Proof. 1. Proving order is reflexive, $a \geq a$

   We know that,

   $a = a + 0$

   From the definition of order, We can write that $a \geq b$ when $a = b + d$ where $d \in \mathbb{N}$

   Thus $a \geq a$.

2. Proving order is transitive, $a \geq b$ and $b \geq c$ then $a \geq c$

   We write,

$$\begin{aligned} a &= b + d \\ b &= c + e \\ a &= c + e + d \end{aligned}$$

   We can say that since $(e + d) \in \mathbb{N}$

   We define $f := (e + d)$ Where $f \in \mathbb{N}$

$$a = c + (f)$$

   Thus we can say,

   If $a \geq b, b \geq c$ then $a \geq c$

3. Proving order is antisymmetric, If $a \geq b$ and $b \geq a$ then $a = b$ We can say,

$$\begin{aligned} a &= b + d \\ b &= a + e \end{aligned}$$

   Where $d, e \in \mathbb{N}$

$$\begin{aligned} a &= (a + e) + d \\ b &= (b + d) + e \end{aligned}$$

Then we can write,

$$\begin{aligned} a &= a + (e + d) \\ b &= b + (d + e) \end{aligned}$$

Then we can say that $(e + d)$ and $(d + e)$ are 0.

We know that if $a + b = 0$ then $a, b = 0$

Thus d and e are 0.

$$\begin{aligned} a &= b + d \\ a &= b \end{aligned}$$

4. Proving $a < b$ if and only if $b = a + d$ for some positive number d If $b = a + d$ where d is a positive natural number, $d \neq 0$

   Which means that $b \neq a + 0$ or $b \neq a$

   This means that b is strictly greater than a

   If $a < b$ then $a \geq b$ and $a \neq b$

   So if $a \geq b$ Then,

   $$a = b + d$$

   But,

   $$\begin{aligned} a &\neq b \\ a &\neq b + 0 \\ d &\neq 0 \end{aligned}$$

   Thus d cannot be 0. d can only be a positive natural number.

5. Proving addition preserves order, $a \geq b$ if and only if $a+c \geq b+c$ Proving $a \geq b$ if $a+c \geq b+C$

   Where $d \in \mathbb{N}$

   $$\begin{aligned} a + c &= b + c + d && \text{By definition} \\ a + c &= (b + d) + c \\ a &= (b + d) && \text{By cancellation law} \\ a &\geq b \end{aligned}$$

   Proving $a + c \geq b + c$ if $a \geq b$

   We know,

   $$a = b + d$$

   Where $d \in \mathbb{N}$

We write a+c using what we know from above,

$$\begin{aligned} a + c &= b + d + c \\ a + c &= b + c + d \\ (a + c) &= (b + c) + d \\ a + c &\geq b + c \end{aligned}$$

6. Proving $a < b$ if and only if $a++ \leq b$ Proving $a < b$ if $a++ \leq b$

   We can write,

$$\begin{aligned} a++ &= b + d && \text{Where } d \in \mathbb{N} \\ a + + + d &= b \\ a + (d++) &= b \end{aligned}$$

   Since from Axiom 3, we know that 0 is not an increment of any natural number, $(d++ \neq 0)$ Therefore,

$$a < b$$

□

Proposition 1.4.2 (Trichotomy of order for natural numbers). Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = bor a > b$

Proof. First we show that no more than one of the statements is true. If $a < b$ then $a \neq b$ by definition. If $a > b$ then $a \neq b$ by definition. If $a > b$ and $a < b$ then $a = b$, which we proved earlier.

Now to show that exactly one of these statements are true. We induct on a,

When a = 0, We know that,

$$\begin{aligned} b &= 0 + b && (\forall b \in \mathbb{N}) \\ b &\geq 0 \end{aligned}$$

Suppose inductively that exactly one of the above statements are true for a and b. For a++, We take each statement. First for $a > b$

$$\begin{aligned} a &> b \\ a &= b + d \\ (a++) &= (b + d)++ && \text{Incrementing both sides} \\ (a++) &= b + d++ && \text{From Lemma 1.3.2} \\ (a++) &> b && \text{If } d \in \mathbb{N} \text{ then } d++ \in \mathbb{N} \end{aligned}$$

For $a = b$

$$\begin{aligned} a &= b \\ (a++) &= (b)++ \\ (a++) &= b+1 \\ a &> b \end{aligned}$$

For $a < b$

$$\begin{aligned} a &< b \\ a+d &= b \\ (a+d)++ &= b++ \\ (a++)+d &= b++ \\ (a++)+d &= b+1 \end{aligned}$$

We have two cases, If $d = 1$, Then by cancellation law

$$a++ = b$$

If $d \neq 1$ Then

$$a++ < b$$

But never both, which concludes the inductive loop. □

1.5 Special Forms of Induction

Proposition 1.5.1 (Strong Principle Of Induction). Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.(In particular this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. For a property $Q(n)$, which is the property that $P(m')$ is true for $m_0 \leq m < n$, then $P(n)$ is true... Then it is true $\forall\ m \geq m_0$

For $Q(0)$, we can say that the statement is vacuous since the conditions are not satisfied for both when $m_0 = 0$ and when $m_0 < 0$

Suppose inductively that $Q(n)$ is true.

Which means that,

P(m) is true for $m_0 \leq m \leq n$

Then for Q(n++),

We know that the property $P(m)$ is true for $m_0 \leq m \leq n$ Take the upper limit of the range,

$$m_0 \leq n$$

We can rewrite this as,

$$m_0 < n++$$

Since $a < b$ if and only if $a++ \leq b$

So we have the property is true for $m_0 \leq m < n++$, the property is true for $P(n++)$, since that is how we chose $Q(n)$

Thus $Q(n++)$ is true, closing the inductive loop.

Since we know that $P(m)$ is true for any n larger than m, we can then say that $m \geq m_0$ □

Proposition 1.5.2 (Induction starting from the base case n). Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m++)$ is true. Show that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$. (This principle is sometimes referred to as the principle of induction starting from the base case n.)

Proof. We can cast this into a standard inductive proof. Consider a property $Q(m)$ defined as $P(n+m)$. Inducting over m:
When $m = 0$,

$$\begin{aligned} Q(0) &= P(n+0) \\ &= P(n) \end{aligned}$$

which we have taken to be true. Suppose inductively that $Q(m) = P(n+m)$ is true. Then, from the definition of $P(m)$, we know that $P((n+m)++)$ is true.

$$\begin{aligned} P((n+m)++) &= P(n+m++) \\ &= Q(m++) \\ &\Rightarrow Q(m++) \text{is true.} \end{aligned}$$

□

1.6 Multiplication

Definition 1.6.1: Multiplication

Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n++$ to m by defining $(n++) \times m := (n \times m) + m$

We can say $0 \times m = 0$, $1 \times m = 0 + m$, $2 \times m = 0 + m + m$ and so on.

Lemma 1.6.1. Prove that multiplication is commutative

Proof. We use the way we proved that addition is commutative as a blueprint. There are two things we need to prove first.

1. For any natural number, n, $n \times 0 = 0$
2. For any natural numbers, n and m, $n \times (m++) = (n \times m) + m$

First we prove,

1. For any natural number, n, $n \times 0 = n$ We induct over n, For $n = 0$,

$$0 \times 0 = 0$$

Which is true from the definition

Now suppose inductively, that $n \times 0 = 0$, For $(n++) \times 0$, From the definition we can write this as,

$$(n++) \times 0 = (n \times 0) + 0$$
$$\text{We know that } n \times 0 = 0 (n++) \times 0 = 0 + 0$$
$$(n++) \times 0 = 0$$

Therefore,

$$n \times 0 = n$$

2. For any natural numbers, n and m, $n \times (m++) = (n \times m) + m$ We induct over n, (keeping m fixed)

For $n = 0$, We know from the definition for multiplication with zero that,

$$0 \times (m++) = 0$$
$$\text{We also know that}$$
$$(m++) \times 0 = (m \times 0) + 0$$
$$(m++) \times 0 = 0$$
$$(m++) \times 0 = 0 \times (m++) = (0 \times m) + m$$

Suppose inductively that $n \times (m++) = (n \times m) + m$ For $n = (n++)$ To prove $(n++) \times (m++) = ((n++) \times m) + m$,

$$(n++) \times (m++) = (n \times (m++)) + m++$$
$$\text{We can rewrite RHS using the inductive hypothesis}$$
$$(n++) \times (m++) = ((n \times m) + m) + m++$$

Taking the LHS, we write

$$\begin{aligned}
(n++) \times (m++) &= (n \times (m++)) + m++ && \text{From the definition} \\
&= (n \times m) + m) + m++ && \text{From the inductive hypothesis} \\
&= (n \times m) + m + (m++) && \text{Associativity of addition} \\
&= (n \times m) + (m++) + m && \text{Commutativity of addition}
\end{aligned}$$

We can say that LHS = RHS, closing the inductive loop

Now we can finally get on with proving that multiplication is commutative, armed with the lemmas we've proved here we can prove this similarly to how we proved addition is commutative We are proving, $m \times n = n \times m$ We fix m and induct over n

For $n = 0$,

$$\begin{aligned} 0 \times m &= 0 && \text{From the definition} \\ m \times 0 &= 0 && \text{Proved earlier} \\ m \times 0 &= 0 \times m \end{aligned}$$

Suppose inductively that $m \times n = n \times m$ is true.
For proving $m \times (n++) = (n++) \times m$, Take the LHS,

$$m \times (n++) = (n \times m) + m \qquad \text{Proved earlier}$$

Take the RHS,

$$(n++) \times m = (n \times m) + m \qquad \text{From the definition}$$

We can see that LHS = RHS, closing the inductive loop
Thus multiplication is commutative ☐

Lemma 1.6.2 (Positive natural numbers have no zero divisors). Let n, m be natural numbers, then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if both n, m are both positive, then nm is positive

Proof. First we prove if both n, m are both positive, then nm is positive We have, n and m such that $n, m > 0$
Let $a, b \in N$ such that $a++ = m, b++ = n$

$$\begin{aligned} n \times m &= (a++)(b++) \\ &= (a++)b + (a++) && m \times n++ = m \times n + m \\ &= ab + b + (a++) && m++ \times n = m \times n + n \\ &= ab + b + (m) && m++ \times n = m \times n + n \end{aligned}$$

Since we know that $m > 0$, then $m + d$ where $d \in \mathbb{N}$ is also greater than 0. Thus it's proved.
Now onto proving the first statement, Let n, m be natural numbers, then $n \times m = 0$ if and only if at least one of n, m is equal to zero.
We have to prove it both ways, if $n \times m = 0$ then at least one of n, m is equal to zero.
Proving by contradiction when $nm = 0$ let's assume $n, m \neq 0$ meaning n, m are positive natural numbers.
☐

Proposition 1.6.1 (Distributive Law). For any natural numbers, a, b, c We have, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

Proof. We only need to show the first identity, then the other would be implied by commutativity We keep a and b fixed, and induct over c.

For $c = 0$, We have on the LHS,

$$a(b+0) = ab \qquad (b+0=b)$$

We have on the RHS,

$$\begin{aligned} ab + ac &= ab + a0 \\ &= ab + 0 && \text{Since } m \times 0 = 0 \\ &= ab \end{aligned}$$

LHS=RHS
This is true for the base case. Now suppose inductively that $a(b+c) = ab + ac$
For $c = (n++)$, We have on the LHS,

$$\begin{aligned} a(b+(n++)) &= a((b+n)++) && \text{Proved earlier} \\ &= a(b+n) \times a && m \times (n++) = (m \times n) + m \end{aligned}$$

We have on the RHS,

$$ab + ac = a(b+c) \qquad \text{Inductive hypothesis}$$

Thus, LHS = RHS
Closing the inductive loop. □

Proposition 1.6.2 (Multiplication is associative). Prove,

$$(a \times b) \times c = a \times (b \times c)$$

Proof. We fix a, b and induct over c
For $c = 0$, On the LHS,

$$(ab)0 = 0 \qquad m \times 0 = 0$$

On the RHS,

$$\begin{aligned} a(b \times 0) &= a(0) && m \times 0 = 0 \\ &= 0 \end{aligned}$$

This covers the base case, suppose inductively that $(a \times b) \times c = a \times (b \times c)$
For $c = c++$,

We have on the LHS,

$$\begin{aligned}(a \times b) \times (c++) &= ((ab)c) + ab && m \times 0 = m \\ &= (ab)c + ab\end{aligned}$$

We have on the RHS,

$$\begin{aligned}a \times (b \times (c++)) &= a((bc) + b) && m \times n++ = m \times n + m \\ &= a(bc) + ab && \text{Distributive Law}\end{aligned}$$

From the inductive hypothesis, $(ab)c = a(bc)$, the LHS and the RHS statements are equivalent
Thus we have LHS = RHS
Closing the inductive hypothesis □

Proposition 1.6.3 (Multiplication preserves order). If a, b are natural numbers such that $a > b$ and c is positive, then $ac > bc$

Proof. If $a > b$, we say that $a = b + d$ where d is a positive natural number.
Multiplying by c on both sides,

$$\begin{aligned}ac &= (b + d)c \\ ac &= bc + cd && \text{Distributive Law}\end{aligned}$$

We know that cd is a positive natural number, so this is expressed in the form $m = n + d$ which is the same as writing $m > c$
Thus we can write,

$$ac > bc$$

□

Corollary 1.6.1 (Cancellation Law). Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero. Then $a = b$

Proof. Let's examine the three cases possible for a, b
When $a < b$, then $ac < bc$ from the previous proposition which is a contradiction of the assumption.
When $a > b$, then $ac > bc$ from the previous proposition which is another contradiction of the assumption.
Thus, the only possible case is $a = b$ □

Proposition 1.6.4 (Euclid's division lemma). Let n be a natural number and let q be a postive number. Then there exists natural numbers m, r such that $0 \leq r \leq q$ and $n = mq + r$

Proof. Inducting over n,
For $n = 0$,

$$0 = mq + r$$

When $a, b \in \mathbb{N}$, $a + b = 0$ if and only if $a = b = 0$ We know that,

$$m + 0 = m$$
$$0 + 0 = 0$$

The converse is easily true, since $a + b = 0$, Thus the base case is true.
Suppose inductively that there exists $m, r \in \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$
For $n++$,
We can write,

$$\begin{aligned} n++ &= (mq + r)++ \\ &= mq + r++ && \text{Proved earlier} \end{aligned}$$

Now we have two cases to look at here, We know that $0 \leq r < q$ from the inductive hypothesis. But for $r++$, that is not necessarily the case, The range of (r++) becomes $0++ \leq r++ < q++$ or $1 \leq r++ \leq q$ since $a < b$ if and only if $a++ \leq b$

When $1 \leq r++ < q$, we have closed the inductive loop but there is one more case to look at.
When $r++ = q$ We have,

$$\begin{aligned} n++ &= mq + q \\ &= (m++)q \end{aligned}$$

Thus we have for this case, $r = 0$, which satisfies the condition $0 \leq r < q$
Closing the inductive loop. □

Definition 1.6.2: Exponentiation For Natural Numbers

Let m be a natural number. To raise m to the power m^0, we define $m^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n then we define $m^{n++} = m^n \times m$

Thus we can write, $m^0 = 1, m^1 = m^0 + m = m, m^2 = m^1 + m = 2m$ and so on

Proposition 1.6.5. Prove,

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Proof. Take the RHS, We have,

$$
\begin{aligned}
(a+b)^2 &= (a+b)(a+b) && \text{From the definition} \\
&= (a+b)a + (a+b)b && \text{Distributive Law} \\
&= aa + ba + ab + bb \\
&= a^2 + ba + ab + b^2 && m^2 = m \times m \\
&= a^2 + ab + ab + b^2 && \text{Commutativity of multiplication} \\
&= a^2 + 2ab + b^2 && \text{Definition of multiplication} \\
&= a^2 + b^2 + 2ab && \text{Commutativity}
\end{aligned}
$$

Thus we have LHS $=$ RHS □

Chapter 2

Set Theory

2.1 Fundamentals

We define first what a set is:

Definition 2.1.1: Sets

We define set A to be any unordered collection of objects. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection. Otherwise $x \in A$

We start with some axioms:

1. (Sets are objects) If A is a set, then A is also an object. A side track about "Pure Set Theory" - This theory states that everything in the mathematical universe is a set. We can write 0 as or an empty set, 1 can be written as 0 and 2 as 0,1 and so on. Terence Tao argues that they are the 'cardinalities of the set.'

2. (Equality of sets) Two sets A and B are equal, A = B, iff every element of A is an element of B. A = B, if and only if every element of x of A also belongs to B, and every element y of B belongs to A.

3. (Empty set) There exists a set $\emptyset$ known as the empty set, which contains no elements. $x \notin \emptyset$

Proposition 2.1.1 (Partial Order). If $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

Proof. If $x \in A$, then $x \in B$, If $x \in B$, then $x \in C$, Then $x \in A$, then $x \in C$

Thus, $A \subseteq C$ $\square$

Lemma 2.1.1 (Single choice). Let A be a non-empty set. Then there exists an object x such that $x \in A$

Proof. Proving by contradiction, Suppose there is no object x that belongs to A. For all x, we have $x \notin A$. We know from Axiom 3, that $x \notin \emptyset$

For the statement,

$$x \in A \Leftrightarrow x \in \emptyset$$

Is false both ways, which gives us the result true, which is a contradiction.

Thus we also prove that $\emptyset$ is unique. $\square$

4. (Singleton sets and pair sets) If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e. for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore if a and b are objects, then there exists a set $\{a, b\}$ if and only if $y = a$ or $y = b$, we refer to this set as the *pair set* formed by a and b.

5. (Single choice) a is an object, $\{a\}$. $y \in \{a\}$, y = a

6. (Pairwise Union) $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Lemma 2.1.2. $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. Taking the left hand side, We have $x \in A$ or $x \in (B \cup C)$. If we look to the right hand side, we have $x \in (A \cup B)$ or $x \in C$ If we break the statement down further. We have $x \in A$ or $x \in B$ or $x \in C$, and on the right $x \in A$ or $x \in B$ or $x \in C$

The two statements are equivalent. □

7. (Axiom Of Specification) A, $x \in A$, let P(x) be a property pertaining to x. Then there exists a set called $\{x \in A, P(x) \text{is true}\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.

8. (Replacement) Let A be a set, for any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$

9. (Infinity) There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n++$ assigned to every natural number $n \in \mathbb{N}$ such that the Peano axioms hold.

10. **Russel's Paradox** (Axiom Of Universal Specification) Suppose for every x we have a property $P(x)$ pertaining to x, Then there exists a set $\{x : P(x) \text{ is true}\}$ such that for every object y:

$$y \in \{x : P(x) \text{is true}\} \Leftrightarrow P(y) \text{is true.}$$

This is quite a handy axiom, we can even find all the axioms through this axiom, but there's an issue.

Let's say we defined the property $P(x) \Leftrightarrow$ x is a set, and $x \notin x$ And we defined a set Ω for which this property is true. A paradox is created, where we don't know whether Ω belongs in this set or not, it's simultaneously true and untrue. If Ω is not in the set, then it is a set that is not in its own set, thereby being contained in the set. If Ω is in the set, then it's a set that is in itself, therefore it cannot be in the set. To resolve this paradox we have, the next axiom.

Note:-

There is a similar paradox known as the "Grellig-Nelson paradox" with the words heterological and autological, the word autological means that the word is an example of itself, for example, the word 'word' is a word or 'pronounceable' is pronounceable or a 'noun' is a noun. The word heterological means that the word is not an example of itself. The word 'triangle' is clearly not a $\triangle$ and so on. The question arises, is heterological autological or heterological?

Proposition 2.1.2. The Axiom Of Universal Specification implies the third axiom onwards.

Proof. Proving for each axiom:

(a) For the axiom of null sets,
We can construct null sets in many ways. We can define the set such that:

$$\{x : x \neq x\}$$

A property that is not true for any given object. This allows us to then say that, for $P(y)$ is true, where $y \in \phi$ then that means $y \neq y$ which is not true. The third axiom is true.

(b) For the axiom of single choice,
We define the property $P(x)$ such that $x = a$,
Then for $y \in \{x : P(x) \text{is true}\}$, then $y \in A$

(c) For the axiom of pair sets,
We define the property similarly $P(x)$ such that $x = a$ or $x = b$
For the axiom of pairwise unions,
We take $P(x)$ to be $x \in A$ or $x \in B$ For the axiom of specification,
We define a property $Q(x) : x \in A, P(x)$ is true.
Then we have,

$$y \in \{x : x \in A, Q(x) \text{is true}\}$$

This means that when $Q(y)$ is true, $y \in A$ $P(y)$ is true

(d) For the axiom of replacement, We define $Q(y) : x \in A, P(x, y) istrue$
$z \in \{y : Q(y) \text{ is true}\} \Leftrightarrow Q(z)$is true or $P(x, z)$is true

(e) For the axiom of infinity, We take the property $P(x)$ that x is a natural number.

□

11. (Axiom Of Regularity) If A is a non-empty set, then there is at least one element x of A which is either not a set or is disjoint from A. How exactly does this mitigate the issues that come in from Russel's Paradox?

Proposition 2.1.3. Show that if A is a set then $A \notin A$

Proof. We take the set A and form a singleton set, $\{A\}$

A, if $x \in \{A\} \Rightarrow x = A$

$$A \cap \{A\} = \phi$$

$$\Rightarrow A \notin A$$

□

Proposition 2.1.4. If A and B are two sets then either $A \notin B$ or $B \notin A$

Proof. Let's say we have A, B such that $x \in \{A, B\} \Rightarrow x = A$ or $x = B$

x = A,

$$A \cap \{A, B\} = \phi$$

$$B \notin A \cup \{A, B\}$$

$$B \in \{A, B\}$$

So if $A \cap \{A, B\} = \phi$ Then we know that,

$$B \notin A$$

Because if it were then we'd have$A \cap \{A, B\} = B$ □

Definition 2.1.2: Intersection Of Sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Proposition 2.1.5. The axiom of universal specification is equivalent to an axiom postullating the existence of a universal set consisting of all objects(for all objects x we have $x \in \Omega$). Conversely if a universal set exists, then the axiom of universal specification is true

Proof. We can say from the axiom of universal specification,

$$\Omega = \{x : x \text{ is an object}\}$$

Conversely,
Assuming the universal set exists, we can write

$$\{x \in \Omega, P(x) \text{ is true }\}$$

Since every object belongs to the universal set, this gives us the axiom of universal specificiation □

Proposition 2.1.6. Show that the axiom of replacement implies the axiom of specification

Proof. We define the property $P(x, y)$ to be $y = x$ Thus, we get the axiom of specification. Since $y = x$ we are saying something about $P(x)$ □

Proposition 2.1.7. Define a proper subset of a set A to be a subset B of A with $B \neq A$. Let A be a non-empty set. Show that A does not have any non-empty proper subsets if and only if A is of the form $A = \{x\}$ for some object x.

Proof. Proving the converse statement is relatively straightforward,
Given that, $A = \{x\}$
We must prove that there no non-empty proper subsets of A.
A proper subset of A is defined as $A \subseteq B, A \neq B$,

□

When we take out the Axiom Of Universal Specification, we have a set of axioms known as "Zermelo-Fraenkel Set Theory".

We will discuss another axiom, known as the "Axiom of choice" in upcoming sessions, which allows us to talk about unions and intersections of sets that aren't countable.

2.2 Functions

Definition 2.2.1: Functions

Let A, B be sets and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$ such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true. Then we define the function $f : X \to Y$ defined by P on the domain X and the codomain to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$ defined to be the unique object $f(x) \in Y$ for which $P(x, f(x))$ is true. Thus for any $x \in X$ and $y \in Y$

$$y = f(x) \Leftrightarrow P(x, y)$$

Definition 2.2.2: One-To-One Functions

A function f is one-to-one(or injective) if different elements map to different elements:

$$x \neq x' \Rightarrow f(x) \neq f(x')$$

Definition 2.2.3: Onto functions

A function f is onto if every element in Y comes from applying f to some element in X:
For every $y \in Y$ there exists $x \in X$ such that $f(x) = y$

These functions are extremely important for modelling the real world. We represent bodies as a set of points. And the way we model the real world is by making these sets have a one-to-one mapping with the Euclidean space.

Definition 2.2.4: Bijective function

Functions $f : X \to Y$ which are both one-one and onto are called bijective.

Definition 2.2.5: Composition Of Functions

Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that the codomain fo f is the same set as the domain of g. We then define the composition $gof : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(gof)(x) := g(f(x))$$

If the codomain of f does not match the domain of g, we leave the composition gof undefined.