# PROBLEM 4.21

To find:-

$\mathbb{E}[MS_{PE}] = ?$

$\mathbb{E}[MS_{LOF}] = ?$

Working:-

$$\mathbb{E}[MS_{PE}] = \frac{1}{n-m}\,\mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{y}_i)^2\right] \text{ ~~where~~ } \text{———— } (1)$$

$$\mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{y}_i)^2\right] = E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}^2 - 2y_{ij}\bar{y}_i + \bar{y}_i^2)\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left\{E(y_{ij}^2) - 2E\left(y_{ij}\sum_{j'=1}^{n_i}\frac{y_{ij'}}{n_i}\right) + E(\bar{y}_i^2)\right\}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\sigma^2 - 2E\left[y_{ij}\sum_{j'=1}^{n_i}\frac{y_{ij'}}{n_i}\right] + \frac{\sigma^2}{n_i}\right\}$$

$$= n\sigma^2 + m\sigma^2 - 2\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{j'=1}^{n_i}\frac{y_{ij}y_{ij'}}{n_i}\right)$$

$$= n\sigma^2 + m\sigma^2 - 2\sum_{i=1}^{m}\frac{n_i\sigma^2}{n_i} = n\sigma^2 + m\sigma^2 - 2m\sigma^2$$

$$= (n-m)\sigma^2 \text{ ———— } (2)$$

Sub (2) in (1) :-

$$\mathbb{E}[MS_{PE}] = \frac{1}{n-m}\times(n-m)\sigma^2 = \boxed{\sigma^2} \Rightarrow \boxed{\mathbb{E}[MS_{PE}] = \sigma^2}$$

$$SS_{Res} = SS_{LOF} + SS_{PE} \Rightarrow \boxed{SS_{LOF} = SS_{Res} - SS_{PE}}$$

$$\Rightarrow \mathbb{E}[SS_{LOF}] = \mathbb{E}[SS_{Res}] - \mathbb{E}[SS_{PE}] = (n-2)\sigma^2 + \sum_{i=1}^{m}\left(\mathbb{E}(y_i) - \beta_0 - \beta_1 x_i\right)^2 - (n-m)\sigma^2$$

$$= (m-2)\sigma^2 + \sum_{i=1}^{m}\left[\mathbb{E}[y_i) - \beta_0 - \beta_1 x_i\right)^2$$

$$\Rightarrow \mathbb{E}[MS_{LOF}] = \frac{\mathbb{E}[SS_{LOF}]}{m-2} \Rightarrow \mathbb{E}[MS_{LOF}] = \sigma^2 + \frac{\sum_{i=1}^{m}\left[\mathbb{E}(y_i) - \beta_0 - \beta_1 x_i\right)^2}{m-2}$$