6.045 Pset 2

Assigned: Friday, February 11, 2011
Due: Thursday, February 24, 2011

To facilitate grading, remember to solve each problem on a separate sheet of paper!

1. Show that the following languages are context-free.

   (a) $L = \{a^n b^{2n} : n \geq 0\}$

   (b) The language $L \subset \{(,)\}^*$ that consists of all strings of *balanced parentheses*: for example, $(()())()(())$ is in L, but $())(()$ is not in L.

   (c) $L = \{x \in \{a, b\}^* \mid x \text{ contains an equal number of } a\text{'s and } b\text{'s}\}$

   (d) $L = \{x \in \{a, b\}^* \mid x \text{ contains more } a\text{'s than } b\text{'s}\}$

   (e) *[Extra credit]* $L = \{x\#y \mid x, y \in \{a, b\}^* \text{ and } x \neq y\}$

2. Show that context-free languages are *closed under union*: that is, if A and B are both CFLs, then $A \cup B$ is a CFL also.

3. Show that every regular language is also a CFL. [*Hint:* Explain how to convert any regular expression into a CFG that generates the same language.]

4. Let $L_1 = \{a^n b^n c^m | n, m \geq 0\}$ and $L_2 = \{a^n b^m c^m | n, m \geq 0\}$.

   (a) Show that L_1 and L_2 are both CFLs. [*Note:* You only need to give a CFG generating L_1; for L_2 you can appeal to the symmetry with L_1.]

   (b) Recall from pset1 that regular languages are *closed under intersection*: that is, if A and B are both regular, then so is $A \cap B$. Using problem 4a together with a result from class, show that CFLs are *not* similarly closed under intersection.

   (c) Show that CFLs are not *closed under complement*: that is, even if L is a CFL, the complementary language $\overline{L} = \{x \mid x \notin L\}$ need not be a CFL. [*Hints:* problem 2, L_1 and L_2, De Morgan's Law.]

easy to prove using pumping lemma

5. Let L be language consisting of 1, 101, 101001, 1010010001, etc. Show that L is not context-free.

the complement is actually context-free

6. Let $L = \{1^n \mid n \text{ is prime}\}$.

   (a) Show that L is not regular. (You can use the fact that there are infinitely many prime numbers.)

   (b) Show that the regular languages are closed under complement. Conclude that $\overline{L} = \{1^n \mid n \text{ is composite}\}$ is not regular either.

   (c) Show that L is not context-free.

   (d) *[Extra credit]* Show that $\overline{L} = \{1^n \mid n \text{ is composite}\}$ is not context-free. (You can use Dirichlet's Theorem, which says that if $\text{GCD}(n, k) = 1$, then the sequence $n,\ n+k,\ n+2k\ldots$ contains infinitely many primes.)

suppose the pumping lemma gives k.
we will select the first prime after k, there will always exist one.
let the square of this number be p.
then p and k are coprimes.

7. Let $L = \big\{ \#x\# \mid x \in \{0,1\}^* \text{ is a palindrome} \big\}$. Design a Turing machine, over the alphabet $\{0, 1, \#\}$, that recognizes L. Give the complete state transition diagram.

8. Let $L = \big\{ \#x\# \mid x \in \{0,1\}^* \text{ contains an equal number of 0's and 1's} \big\}$. Verbally describe a Turing machine, over the alphabet $\{0, 1, \#\}$, that recognizes L. (*Note:* You can't just write "count the number of 0's and 1's"—explain how the counting is done!)