CS 420, Spring 2019
Homework 11 Solutions

1. **Exercise 5.4**

   No. For example, let $A = \{0^n1^n | n \geq 0\}$ and $B = \{0^n1^m | n, m \geq 0\}$. We will show that $A \leq_m B$ even though A is not regular and B is regular. A mapping reduction f from A to B is computed by the Turing machine F given by

   $F =$ "On input $w \in \{0, 1\}^*$,

   1. Determine if w belongs to A. (Since A is decidable, F can do this with no further information.)
   2. If w belongs to A, then output 01. If w does not belong to A, then output 10."

   (In fact, the same argument shows that if A is any decidable language and B is any language other than $\emptyset$ and Σ^*, then $A \leq_m B$.)

2. **Problem 5.22**

   A is Turing-recognizable if and only if $A \leq_m A_{TM}$.

   Proof: First suppose that A is Turing-recognizable and let M be a Turing machine that recognizes A. The function f defined by $f(w) = \langle M, w\rangle$ is a reduction from A to A_{TM} because it is obviously computable and we have

   $$w \in A \text{ iff } M \text{ accepts } w \text{ iff } \langle M, w\rangle \in A_{TM} \text{ iff } f(w) \in A_{TM}.$$

   Now suppose that $A \leq_m A_{TM}$. We know that A_{TM} is Turing-recognizable, so by Theorem 5.28, A is Turing-recognizable.

3. **Problem 5.23**

   A is decidable if and only if $A \leq_m 0^*1^*$.
   Proof: First suppose that A is decidable. Define f by $f(x) = 01$ if $x \in A$ and $f(x) = 10$ if $x \in \overline{A}$. Since A is decidable, f is computable and $x \in A$ if and only if $f(x) \in 0^*1^*$, so $A \leq_m 0^*1^*$.

   Conversely, suppose that $A \leq_m 0^*1^*$. Since 0^*1^* is decidable, A is decidable by Theorem 5.22.

4. **Problem 5.24**

   The set $\overline{A_{TM}}$ is mapping reduced to J by the function $f(y) = 1y$. Thus, J is not Turing-recognizable. The set A_{TM} is mapping reduced to J by the function $g(x) = 0x$. This shows that $\overline{A_{TM}}$ is mapping reducible to $\overline{J}$ and hence that $\overline{J}$ is not Turing-recognizable.

5. **Problem 5.25**

Consider the set J of Problem 5.24. According to that problem, J is not Turing-recognizable, so J is not decidable. We will show that $J \leq_m \overline{J}$, so $B = J$ is a solution to the problem.

First note that $\overline{J} = \{w | w = 0x \text{ for some } x \in \overline{A_{TM}} \text{ or } w = 1y \text{ for some } y \in A_{TM} \text{ or } w = \varepsilon \text{ or } w \text{ begins with a symbol other than 0 or 1}\}$. Let z_0 be some fixed string in J, for example, z_0 could be $0x_0$ for some particular x_0 in A_{TM}. Define $f : \Sigma^* \to \Sigma^*$ by

$$f(w) = \begin{cases} 1x & \text{if } w = 0x \\ 0y & \text{if } w = 1y \\ z_0 & \text{if } w = \varepsilon \text{ or } w \text{ starts with a symbol other than 0 or 1.} \end{cases}$$

Then, it is clear that f is computable. To see that f mapping reduces J to $\overline{J}$, suppose first that $w \in J$. We must show that $f(w) \in \overline{J}$. If $w \in J$, there are two possibilities. If $w = 0x$ with $x \in A_{TM}$, then $f(w) = 1x$ with $x \in A_{TM}$, so $f(w) \in \overline{J}$. If $w = 1y$ with $y \in \overline{A_{TM}}$, then $f(w) = 0y$ with $y \in \overline{A_{TM}}$, so $f(w) \in \overline{J}$. Thus, if $w \in J$, then $f(w) \in \overline{J}$.

Now suppose that $w \notin J$. We must show that $f(w) \notin \overline{J}$. There are four possibilities to consider. If $w = 0x$ with $x \in \overline{A_{TM}}$, then $f(w) = 1x$, so $f(w) \notin \overline{J}$. If $w = 1y$ with $y \in A_{TM}$, then $f(w) = 0y$, so $f(w) \notin \overline{J}$. If $w = \varepsilon$ or w starts with a symbol other than 0 or 1, then $f(w) = z_0$, so $f(w) \notin \overline{J}$. Thus, if $w \notin J$, then $f(w) \notin \overline{J}$.

This shows that f is a mapping reduction of J to $\overline{J}$

6. It is not possible to m-reduce E_{LBA} to A_{LBA}.
   Proof: Suppose that $E_{LBA} \leq_m A_{LBA}$. By Theorem 5.9, A_{LBA} is decidable, so by Theorem 5.22, E_{LBA} is decidable. This contradicts Theorem 5.10, so the m-reduction is not possible.

7. Is $\overline{A_{LBA}}$ m-reducible to 0^*1^*? Explain your answer.

   Solution: $\overline{A_{LBA}}$ is m-reducible to 0^*1^*. To prove this, first note that by Theorem 5.9, A_{LBA} is decidable, so by Problem 3.15d, $\overline{A_{LBA}}$ is decidable, so by Problem 5.23, $\overline{A_{LBA}}$ is m-reducible to 0^*1^*.

8. Is A_{TM} m-reducible to $\overline{REJECT_{TM}}$? Explain your answer.

   Solution: A_{TM} is not m-reducible to $\overline{REJECT_{TM}}$. To prove this, suppose that $A_{TM} \leq_m \overline{REJECT_{TM}}$. Then $\overline{A_{TM}} \leq_m REJECT_{TM}$. By Corollary 4.23, $\overline{A_{TM}}$ is not Turing recognizable, so by Corollary 5.29, $REJECT_{TM}$ is not recognizable. This contradicts Problem 1a on Homework 10. Thus, the reduction is not possible.