FLAC Assignment 4

Exercise 1. Consider a class of automata in which the set of states is $S = \{0, \ldots, 7\}$, the alphabet is $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, the starting state is $s_0 = 0$, and the transition relation δ is given by:

δ	a	b
0	5	4
1	4	4
2	1	0
3	0	1
4	7	7
5	6	7
6	3	2
7	3	3

a. Assign a set of final states F so that $M = (S, \Sigma, s_0, \delta, F)$ is already a reduced automaton.

b. Assign a set of final states F, where $\emptyset \neq F \neq S$, so that M reduces to the smallest possible automaton.

Exercise 2. We say that an equivalence relation E on Σ^* is *invariant* iff for all $u, v \in \Sigma^*$, $w \, E \, w' \;\Rightarrow\; (uwv) \, E \, (uw'v)$. Prove that $L \subseteq \Sigma^*$ is a regular language iff there is an invariant equivalence relation E of finite index[1] such that $w \in L \;\Rightarrow\; \{w' \mid w \, E \, w'\} \subseteq L$.

Exercise 3. Prove that the following language is regular: The set of sequences of 1000×1000 binary matrices that, when multiplied together modulo 2, yield the 1000×1000 identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$. (I.e., each symbol of the alphabet is a 1000×1000 binary matrix. In a binary matrix, each entry is either 0 or 1.) (Hint: Nerode's Theorem.)

Exercise 4. If L is a language over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$, define L_{ERROR} so that a string is in L_{ERROR} iff it is the result of flipping a bit in a string in L; i.e.,

$$L_{\mathrm{ERROR}} = \{w \mid w = uxv, \text{ where } x \in \{\mathtt{0}, \mathtt{1}\} \text{ and } u\bar{x}v \in L\}$$

Prove that if L is regular, then L_{ERROR} must be also be regular.

[1] I.e., the number of equivalence classes must be finite.

Exercise 5. Suppose we wish to use an automaton to represent a machine that reads three binary numbers in parallel (starting with the least-significant bit). To do this, we will let each symbol of our alphabet be a vector of three components, each of which is in $\{\texttt{0}, \texttt{1}, \texttt{\#}\}$, where the # indicates we are past the end of the number.

Part A (binary addition). Is the following language regular? (Prove your answer.)

$$A = \left\{ \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} \begin{bmatrix} \texttt{\#} \\ \texttt{\#} \\ z_{s+1} \end{bmatrix} \begin{bmatrix} \texttt{\#} \\ \texttt{\#} \\ \texttt{\#} \end{bmatrix}^* \middle| \; x + y = z, \right.$$
$$\left. \text{where} \quad x = \sum_{i=0}^{s} x_i \cdot 2^i, \quad y = \sum_{i=0}^{s} y_i \cdot 2^i, \quad z = \sum_{i=0}^{s+1} z_i \cdot 2^i \right\}$$

Part B (binary multiplication) (extra credit). Is the following language regular? (Prove your answer.)

$$B = \left\{ \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} \begin{bmatrix} \texttt{\#} \\ \texttt{\#} \\ z_{s+1} \end{bmatrix} \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \begin{bmatrix} \texttt{\#} \\ \texttt{\#} \\ z_t \end{bmatrix} \begin{bmatrix} \texttt{\#} \\ \texttt{\#} \\ \texttt{\#} \end{bmatrix}^* \middle| \; x \cdot y = z, \right.$$
$$\left. \text{where} \quad x = \sum_{i=0}^{s} x_i \cdot 2^i, \quad y = \sum_{i=0}^{s} y_i \cdot 2^i, \quad z = \sum_{i=0}^{t} z_i \cdot 2^i \right\}$$