DEPARTMENT OF PHYSICS

SECOND SEMESTER BE PROGRAMS (CS, CD, CY, IS, AIML, & BT)

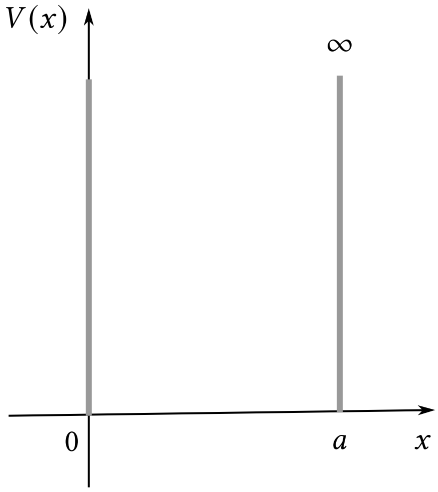
ACADEMIC YEAR 2022-2023

Dated	05th September 2023	Maximum Marks	50
Course Code	22PHY22C	Duration	90 min
Course	Quantum Physics For Engineers	**CIE-III (Test) - Long Scheme**	

1(a) *Solve the problem of a particle in an infinite well to arrive at the un-normalized eigen functions and eigen values.* ⑦

Suppose the potential is,

$$V(x) = \begin{cases} 0, & 0 \le x \le a, \\ \infty, & \text{otherwise} \end{cases}$$



A particle in this potential is completely free, except at the two ends ($x = 0$ and $x = a$), where an infinite force prevents it from escaping. A classical model would be a cart on a frictionless horizontal air track, with perfectly elastic bumpers—it just keeps bouncing back and forth forever. (This potential is artificial, of course, but I urge you to treat it with respect. Despite its simplicity—or rather, precisely because of its simplicity—it serves as a wonderfully accessible test case for all the fancy machinery that comes later.)

We begin by time independent Schrödinger equation which reads,

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi(x) = 0 \tag{1}$$

Outside the potential well ($0 \le x \le a$) it reads

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2 m}{h^2}(E - \infty)\psi(x) = 0$$

This equation holds good only if $\psi(x) = 0$ for all points outside the well. The probability of finding the particle there is zero.

Inside the potential well, where $V = 0$, the time-independent Schrödinger equation (1) reads,

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi(x) = 0$$

Or

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0, \qquad \text{where } k^2 = \frac{8\pi^2 m}{h^2}E \tag{2}$$

Solution: Equation (2) is the classical *simple harmonic oscillator* equation; the general solution is,

$$\psi(x) = A\sin(kx) + B\cos(kx) \tag{3}$$

where A and B are arbitrary constants. Typically, these constants are fixed by the *boundary conditions* of the problem.

Boundary conditions: What are the appropriate boundary conditions for $\psi(x)$? Ordinarily, both ψ and $d\psi/dx$ are continuous, but where the potential goes to infinity only the first of these applies.

Continuity of $\psi(x)$ requires that

$$\psi(0) = \psi(a) = 0$$

so as to join onto the solution outside the well. What does this tell us about A and B?

Condition 1: At $x = 0$, $\psi = 0$

Substituting it in equation (3), we get

$$\psi(0) = A \sin 0 + B \cos 0 = B$$

so $B = 0$ as $\psi(0) = 0$, and hence equation (3) will become,

$$\boxed{\psi(x) = A \sin(kx)} \tag{4}$$

Condition 2: At $x = a$, $\psi = 0$

Substituting it in equation 4, we get

$$\psi(a) = A \sin(ka)$$

Since $\psi(a) = 0$,

$$A \sin(ka) = 0$$

so either $A = 0$ (in which case we're left with the trivial—non-normalizable—solution $\psi(x) = 0$), or else $\sin(ka) = 0$ which means that

$$ka = 0,\ \pm\pi,\ \pm 2\pi\ \pm 3\pi,\ \ldots$$

But $k = 0$ is no good (again, that would imply $\psi(x) = 0$, which means the particle is not inside the well), and the negative solutions give nothing new, since $\sin(-\theta) = -\sin(\theta)$ and we can absorb the minus sign into A. So the *distinct* solutions are

$$k = \frac{n\pi}{a}, \qquad \text{with } n = 1, 2, 3, \ldots \tag{5}$$

Curiously, the boundary condition at $x = a$ does not determine the constant A, but rather the constant k, and hence the possible values of E by substituting k^2 into equation (2). We get,

$$\frac{n^2\pi^2}{a^2} = \frac{8\pi^2 m}{h^2} E$$

or

$$\boxed{E = \frac{n^2 h^2}{8ma^2}} \qquad \text{with } n = 1, 2, 3, \ldots \tag{6}$$

In radical contrast to the classical case, a quantum particle in the infinite square well cannot have just any old energy—it has to be one of these special ("allowed") values.

1(b) *The 1st excited state wave function of a particle in an infinite well is given by* $\psi = B \sin(10^9 \pi x)$. *Calculate B and energy of the state.* ③

For first excited state $n = 2$ and

$$\psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a} x\right)$$

Comparing it with the given wave function,

$$\psi = B\sin\left((10^9\mathrm{m}^{-1})\pi x\right)$$

we identify,

$$(10^9\mathrm{m}^{-1})\pi x = \frac{2\pi}{a}x \qquad \Longrightarrow a = \frac{2}{10^9\mathrm{m}^{-1}} = 2\times 10^{-9}\mathrm{m}$$

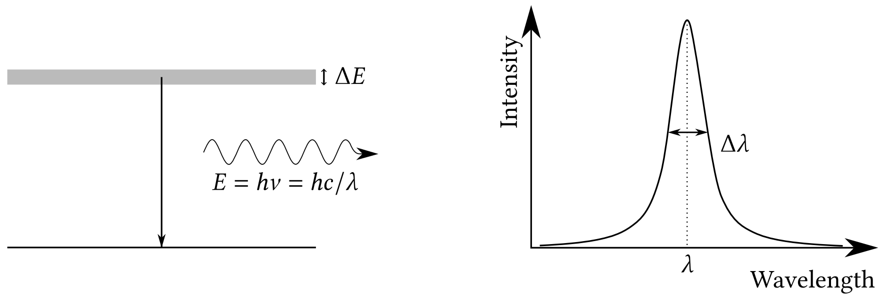
and using this a,

$$B = \sqrt{\frac{2}{a}} = \sqrt{\frac{2}{2\times 10^{-9}\mathrm{m}}} = 10^{9/2}\mathrm{m}^{-1/2}$$

Assuming the particle as an electron with $m = 9.109\times 10^{-31}$kg the energy of the state is given by,

$$E = \frac{n^2h^2}{8ma^2} = \frac{2^2\times(6.626\times 10^{-34}\,\mathrm{J\,s})^2}{8\times 9.109\times 10^{-31}\mathrm{kg}\times(2\times 10^{-9}\mathrm{m})^2} = 6.025\times 10^{-20}\mathrm{J} = 0.376\mathrm{eV}$$

2(a) *Using Heisenberg's uncertainty principle, explain the broadening of atomic spectral lines. Hence derive an expression for the minimum line broadening.* ⑦



The energy of the emitted photon is given by,

$$E = h\nu = \frac{hc}{\lambda} \tag{1}$$

Where h is Planck's constant, ν is the frequency, c is the velocity and λ is the wavelength of the emitted radiation. Differentiating this equation with respect to λ, we get,

$$\Delta E = -hc\frac{\Delta\lambda}{\lambda^2}$$

Considering only the magnitude of the difference,

$$|\Delta E| = hc\frac{\Delta\lambda}{\lambda^2} \tag{2}$$

According to Heisenberg's uncertainty principle, the finite lifetime Δt of the excited state means there will be an uncertainty of ΔE in the energy of the state. Hence the emitted photon energy will also have an uncertainty of ΔE in its energy and is related by,

$$\Delta E\,\Delta t \geq \hbar/2$$

Or,

$$\Delta E \geq \frac{h}{4\pi\Delta t}$$

Substituting for ΔE from (2) we get,

$$hc\frac{\Delta\lambda}{\lambda^2} \geq \frac{h}{4\pi\Delta t}$$

Or

$$\boxed{\Delta\lambda \geq \frac{1}{4\pi c}\frac{\lambda^2}{\Delta t}}$$

This shows that for a finite lifetime of the excited state, the measured value of the emitted photon wavelength will have a spread of wavelengths around the mean value λ. This demands that for a very narrow spread, the lifetime of the excited state must be very high. Such excited levels are called metastable states whose lifetimes will be of the order of 10^{-3}s. This concept is used in the production of laser light, which will be highly monochromatic due to the involvement of a metastable state.

2(b) *Calculate the difference in energy levels, given that the broadening of the emission line spectrum between them is* 100 Å *and lifetime of the higher energy level is* 100 µs. ③

Broadening of the spectral line is given by,

$$\Delta\lambda \geq \frac{1}{4\pi c}\frac{\lambda^2}{\Delta t}$$

We need to find out λ and hence $E = h\nu = hc/\lambda$ from it. It is given that $\Delta\lambda = 100$Å and $\Delta t = 100\,\mu$s.

$$\therefore \quad \lambda = \sqrt{4\pi c\,\Delta\lambda\,\Delta t} = \sqrt{4 \times 3.142 \times 3 \times 10^8\text{ms}^{-1} \times 100 \times 10^{-10}\text{m} \times 100 \times 10^{-6}\text{s}} = 0.061\,38\,\text{m}$$

$$\text{and} \quad E = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34}\,\text{J s} \times 3 \times 10^8\text{ms}^{-1}}{0.061\,38\,\text{m}} = 3.2364 \times 10^{-24}\text{J} = 20.2 \times 10^{-6}\text{eV}$$

3(a) *State de-Broglie's hypothesis. Use the expression relating momentum of a particle to the wavelength of its equivalent wave to arrive at the expression for the energy of a particle of mass m in the ground state of an infinite well of width a.* ⑦

According to the de Broglie's hypothesis, any material particle of mass m moving with velocity v and momentum p also shows wave particle duality similar to photons. The wavelength of such matter waves is given by

$$\boxed{\lambda = \frac{h}{mv} = \frac{h}{p}}$$

This is called the de Broglie wavelength.

For particle in ground state of an infinite square well, $\lambda = 2a$ (wave function is a half sine wave) where a is the width of the well. Using de Broglie's wavelength, its momentum is given by,

$$p = \frac{h}{\lambda}$$

Therefore, energy of this particle is,

$$E = \frac{p^2}{2m} = \frac{1}{2m}\frac{h^2}{\lambda^2} = \frac{h^2}{2m(2a)^2} = \frac{h^2}{8ma^2}$$

3(b) *An infinite well between* 0 *to* a *is shifted to the new position* $-0.5a$ *to* $0.5a$. *Will the eigenvalues and eigenfunctions change for this new configuration? Explain why.* ③

Energy of the particle in the potential well is inversely dependent on square of the well width.

$$E = \frac{n^2h^2}{8ma^2}$$

As the well width $= 0.5a - (-0.5a) = a$ does n't change due to shifting the energy eigenvalues remain same.

However the wave functions will change to account for the new symmetry of the well.

4(a) *State the condition of unitarity of a matrix. Show that the Pauli Matrices* σ_x, σ_y, σ_z *are unitary matrices.* ⑦

A matrix **U** is said to be unitary if it satisfies,

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$$

To show that the Pauli's matrices are unitary matrices:

$$\sigma_x\sigma_x^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_x^\dagger\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_y^\dagger\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_z\sigma_z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_z^\dagger\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4(b) *Prove that the matrix given below is a unitary matrix. All intermediate steps need to be shown explicitly.* ③

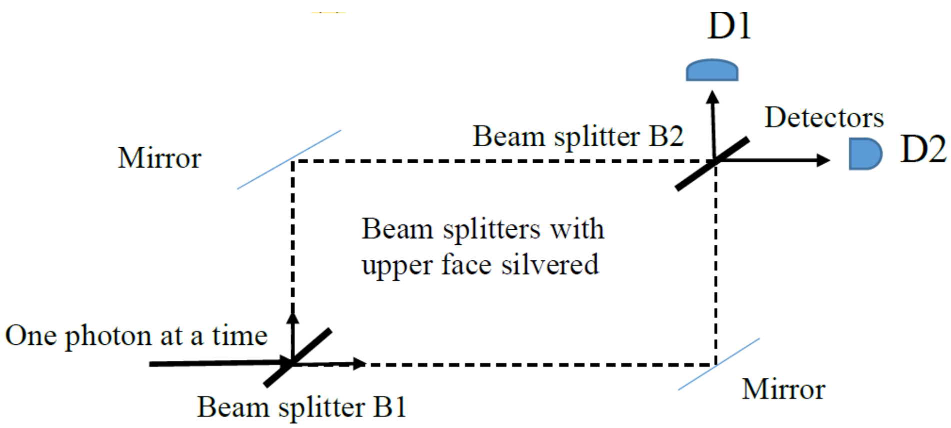
$$\mathbf{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{and} \quad \mathbf{U}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\therefore \quad \mathbf{U}\mathbf{U}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+1 & -i+i \\ i-i & -i^2-i^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\therefore \quad \mathbf{U}^\dagger\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i^2 & 1+i^2 \\ 1+i^2 & 1-i^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

5(a) *With a neat labeled diagram involving a single photon source, two beam splitters, two mirrors and two detectors, prove that a single photon simultaneously travels through both the paths at the same time and does not choose any one of the paths in a random fashion.* ⑦



5(b) Calculate the inner product between two vectors $|\psi\rangle$ and $|\chi\rangle$, given that for a particular basis, the two kets can be written as : $|\psi\rangle = 0.707|\phi_1\rangle + 0.707|\phi_2\rangle$ and $|\chi\rangle = 0.5|\phi_1\rangle + 0.866|\phi_2\rangle$. ③

In matrix form $|\psi\rangle$ and $|\chi\rangle$ can be written as,

$$|\psi\rangle = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \text{and} \qquad |\chi\rangle = \begin{pmatrix} 0.5 \\ 0.866 \end{pmatrix}$$

Therefore their inner product is,

$$\langle\psi|\chi\rangle = \begin{pmatrix} 0.707 & 0.707 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.866 \end{pmatrix} = 0.9658$$