



## UNIT-V

### NUMERICAL METHODS

#### **Introduction:**

Numerical analysis plays a great role in engineering and in the quantitative parts of pure and applied science. Interpolation, the computing of values for a tabulated function at points not in the table, is historically a most important task. Many famous mathematicians have their names associated with procedures for interpolation: Gauss, Newton, Bessel, Stirling. The need to interpolate began with the early studies of astronomy when the motion of heavenly bodies was to be determined from periodic observations. Interpolation methods demonstrate some important theory about polynomials and the accuracy of numerical methods. Interpolating with polynomials serves as an excellent introduction to some techniques for drawing smooth curves. These methods are the basis of many other procedures. Among these procedures, we will focus on numerical differentiation and integration in this unit.

#### **Interpolation and Extrapolation:**

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

#### **Finite Differences:**

The finite difference deals with the changes in the value of the function (dependent variable) due to the changes in the values of independent variable. The values of the independent variable  $x$  are called Arguments and the corresponding values of dependent variables  $y$  are called Entries.

#### **Forward difference:**

If  $y_0, y_1, y_2, \dots, y_n$  denote the set of values of the function  $y=f(x)$ . Then  $\Delta y_0 = y_1 - y_0$ ,  $\Delta y_1 = y_2 - y_1, \dots, \Delta y_{n-1} = y_n - y_{n-1}$  are called the First Forward Differences of the function  $y=f(x)$  where  $\Delta$  is called the forward difference operator. The differences are called first forward differences denoted by  $\Delta y_0, \Delta y_1, \dots, \Delta y_n$ , second forward differences and are



denoted by  $\Delta^2 y_0, \Delta^2 y_1, \dots, \Delta^2 y_{n-1}$ . i.e.  $\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \Delta^2 y_1 = \Delta y_2 - \Delta y_1, \dots, \Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$ .

Similarly we can define third, fourth forward differences etc.

In general, the  $n^{\text{th}}$  forward differences is defined by the equation,  $\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$

### Forward Difference Table:

$x$	$y = f(x)$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_0$	$y_0$					
$x_1 = x_0 + h$	$y_1$	$\Delta y_0$				
$x_2 = x_0 + 2h$	$y_2$	$\Delta y_1$	$\Delta^2 y_0$			
$x_3 = x_0 + 3h$	$y_3$	$\Delta y_2$	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_0$	
$x_4 = x_0 + 4h$	$y_4$	$\Delta y_3$	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_0$
$x_5 = x_0 + 5h$	$y_5$	$\Delta y_4$	$\Delta^2 y_3$	$\Delta^3 y_2$		

Here  $y_0$  is called the First Entry and  $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots, \Delta^5 y_0$  are called the Leading Differences.

### Backward Difference:

Consider the function  $y = f(x)$ . If  $y_0, y_1, y_2, \dots, y_n$  denote the set of values of  $y$ . Then  $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$  are called the First Backward Differences and  $\nabla$  is called the backward difference operator.

The second backward difference of the function is given by,

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \dots, \nabla^n y_n = \nabla y_n - \nabla y_{n-1}$$

Similarly we can define higher order differences.

In general the  $n^{\text{th}}$  backward difference is given by  $\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}$

### Backward Difference Table:

$x$	$y = f(x)$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
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$x_0$	$y_0$					
$x_1 = x_0 + h$	$y_1$	$\nabla y_1$	$\nabla^2 y_2$	$\nabla^2 y_3$		
$x_2 = x_0 + 2h$	$y_2$	$\nabla y_2$	$\nabla^2 y_3$	$\nabla^3 y_4$	$\nabla^4 y_5$	
$x_3 = x_0 + 3h$	$y_3$	$\nabla y_3$	$\nabla^2 y_4$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$
$x_4 = x_0 + 4h$	$y_4$	$\nabla y_4$	$\nabla^2 y_5$			
$x_5 = x_0 + 5h$	$y_5$	$\nabla y_5$				

**NOTE:** Only the notations are changed not the differences.

- (i) Relation between forward and backward operators:  $\Delta^n y_r = \nabla^n y_{n+r}$
- (ii)  $f(x) = f(x + h) - f(x)$
- (iii)  $\nabla f(x) = f(x) - f(x - h)$

### Problems:

1. Evaluate  $\Delta \tan^{-1} x$

**Solution:**  $\Delta \tan^{-1} x = \tan^{-1} (x + h) - \tan^{-1} x$

$$= \tan^{-1} \left\{ \frac{x+h-x}{1+(x+h)x} \right\} = \tan^{-1} \left\{ \frac{h}{1+hx+x^2} \right\}$$

2. Evaluate  $\Delta^2 \cos 2x$

$$\begin{aligned}
 \text{Solution: } \Delta^2 \cos 2x &= \Delta \{ \cos 2(x + h) - \cos 2x \} \\
 &= \Delta \cos 2(x + h) - \Delta \cos 2x \\
 &= [\cos 2(x + 2h) - \cos 2(x + h)] - [\cos 2(x + h) - \cos 2x] \\
 &= -2 \sin (2x + 3h) \sin h + 2 \sin (2x + h) \sin h \\
 &= -2 \sin h [\sin (2x + 3h) - \sin (2x + h)] \\
 &= -2 \sin h [2 \cos (2x + 2h) \sin h] \\
 &= -4 \sin^2 h \cos (2x + 2h)
 \end{aligned}$$

### Differences of a Polynomial:

The  $n^{\text{th}}$  differences of a polynomial of the  $n^{\text{th}}$  degree are constant and all higher order differences are zero.

Let  $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ , then

$$\Delta^n f(x) = a_0 n(n-1)(n-2) \dots 1 \cdot h^n = a_0 n! h^n \dots \dots \dots (1)$$

$$\text{and then for higher orders } \Delta^{n+1} f(x) = \Delta^{n+2} f(x) = \dots = 0. \dots \dots \dots (2)$$

### Problems:

1. Evaluate  $\Delta^{10} [(1 - ax)(1 - bx^2)(1 - cx^3)(1 - dx^4)]$  with  $h=1, h=2$ .

**Solution:** Given,

$$\Delta^{10} [(1 - ax)(1 - bx^2)(1 - cx^3)(1 - dx^4)] = \Delta^{10} [abcd x^{10} + () x^9 + () x^8 + \dots + 1]$$



$$= abcd \Delta^{10} (x^{10}) \quad [\Delta^{10} (x^n) = 0 \text{ or } n <$$

10]

$$= abcd (10!) \quad (\text{for } h = 1)$$

Similarly for  $h = 2$ ,

$$= abcd (10!) (2)^{10}$$

2. Construct the forward difference table, given that

$x$	5	10	15	20	25	30
$y$	9962	9848	9659	9397	9063	8660

and find the values of  $\Delta^2 y_{10}$ ,  $\Delta^3 y_5$ ,  $\Delta^4 y_5$ .

**Solution:**

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962	-114			
10	9848	-189	-75	2	
15	9659	-262	-73	1	-1
20	9397	-334	-72	3	2
25	9063	-403	-69		
30	8660				

Now  $\Delta^2 y_{10} = -73$  which is second element for the column  $\Delta^2 y$ .

$\Delta^3 y_5 = 2$  which is the first element of the column  $\Delta^3 y$ . Similarly  $\Delta^4 y_5 = -1$ .

3. Construct the finite difference table for the function  $f(x) = x^3 + x + 1$  where  $x$  takes the values 0,1,2,3,4,5,6. Identify the leading forward and backward differences. Hence find  $\Delta^2 y_1$ ,  $\nabla^3 y_5$ .

**Solution:**

$x$	$y$	First difference	Second difference	Third difference	Fourth difference
0	1	2			
1	3	8	6		
2	11	20	12	6	0
3	31	38	18	6	0
4	69	62	24	6	0
5	131	92	30		
6	223				

The leading forward differences are 2, 6, 6 and leading backward differences are 92, 30, 6.  $\Delta^2 y_1 = 12$ ,  $\nabla^3 y_5 = 6$ .



**Note:** Third differences are constants and higher order differences are zero as  $f(x)$  is a polynomial of third degree.

### Interpolation with equal intervals:

#### Newton's Forward Interpolation Formula:

Let the function  $y = f(x)$  takes the values  $y_0, y_1, \dots, y_n$  at the points  $x_0, x_1, \dots, x_n$  where  $x_i = x_0 + ih$ . Then Newton's Forward interpolation polynomial is given by,

$$y_p = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)\dots(p-n+1)}{n!}\Delta^n y_0,$$

where  $x = x_0 + ph$

#### Newton's Backward Interpolation Formula:

Let the function  $y = f(x)$  takes the values  $y_0, y_1, \dots, y_n$  at the points  $x_0, x_1, \dots, x_n$ , where  $x_i = x_0 + ih$ . The Newton's Backward Interpolation polynomial is given by,

$$y_p = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots + \frac{p(p+1)\dots(p+n-1)}{n!}\nabla^n y_n,$$

where  $x = x_n + ph$

#### Remark:

Newton's Forward Interpolation Formula is used to interpolate the values of  $y$  near the beginning of the set of tabulated values or for extrapolating values of  $y$  to the left of the beginning. Newton's Backward Interpolation Formula is used to interpolate the values of  $y$  near the end of the set of tabulated values or for extrapolating values of  $y$  to the right of the last tabulated value  $y$ .

#### Examples:

1. Find a cubic polynomial which takes the following data

$x$	0	1	2	3
$f(x)$	1	2	1	10

**Solution:** The forward difference table is given by,

$x$	$y = f(x)$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$
0	1			
1	2	1		
2	1	-1	-2	
3	10	9	10	12

Here,  $x_0 = 0$ ,  $h = 1$ ,  $p = \frac{x-x_0}{h} = \frac{x-0}{1} = x$ ,  $\Delta y_0 = 1$ ,  $\Delta^2 y_0 = -2$ ,  $\Delta^3 y_0 = 12$ ,

By Newton-Gregory forward interpolation formula we have

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$



$$f(x) = 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$f(x) = 2x^3 - 7x^2 + 6x + 1$  is the required polynomial.

2. The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

<b>x = height:</b>	150	200	250	300	350	400
<b>y = distance:</b>	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of  $y$  when  $x = 160ft$  and  $x = 410ft$ .

**Solution:**

The difference table is

$x$	$y$	First difference	Second difference	Third difference	Fourth difference	Fifth difference
150	<b>13.03</b>	<b>2.01</b>				
200	15.04	1.77	<b>-0.24</b>	<b>0.08</b>		
250	16.81	1.61	-0.16	0.03	<b>-0.05</b>	
300	18.42	1.48	-0.13	<b>0.02</b>	<b>-0.01</b>	<b>0.04</b>
350	19.90	<b>1.37</b>	<b>-0.11</b>			
400	<b>21.27</b>					

(i)  $x_0=160, y_0=13.03, \Delta y_0=2.01, \Delta^2 y_0=-0.24, \Delta^3 y_0=0.08, \Delta^4 y_0=-0.05, h=50$

$$p = \frac{x-x_0}{h} = \frac{10}{50} = 0.2$$

Using Newton's forward interpolation formula, we get

$$f(160) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$f(160) = 13.03 + 0.402 + 0.0192 + 0.00384 + 0.00168 + 0.001$$

$$f(160) = 13.46 \text{ nautical miles.}$$

(ii)  $x=410, x_n=400, y_n=21.27, \nabla y_n=1.37, \nabla^2 y_n=-0.11, \nabla^3 y_n=0.02, h=50$

$$p = \frac{x-x_n}{h} = \frac{10}{50} = 0.2$$

Using Newton's backward interpolation formula we get,

$$f(410) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$



$$f(410) = 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \dots$$

$$f(410) = 21.27 + 0.274 + 0.0132 + 0.0018 + 0 + 0.0024$$

$$f(410) = 21.53 \text{ nautical miles.}$$

3. From the following table, estimate the number of students who obtained marks between 40 and 45:

Marks (x):	30-40	40-50	50-60	60-70	70-80
No. of students(y):	31	42	51	35	31

**Solution:** we prepare cumulative frequency table as follows:

Marks less than(x):	40	50	60	70	80
No. of students(y):	31	73	124	159	190

Now the difference table is:

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42			
50	73	51	9		
60	124	35	-16	-25	
70	159	31	-4	12	37
80	190				

To find  $y(45)$  i.e. number of students with marks less than 45.

$$\text{Taking } x_0 = 40, x = 45, h = 10, p = \frac{x-x_0}{h} = \frac{5}{10} = 0.5$$

Using Newton's Forward Interpolation formula we get,

$$\begin{aligned} y(45) &= y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{6}\Delta^3 y_{40} + \dots \\ &= 31 + 0.5(42) + \frac{0.5(-0.5)}{2}(9) + \frac{0.5(0.5)(-1.5)}{6}(-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(37) \\ &= 47.87 \end{aligned}$$

The number of students with marks less than 45 is  $47.87 \cong 48$ .

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45  $= 48 - 31 = 17$ .

**Exercise:**

1. Fit a cubic polynomial to the following data using suitable interpolation formula.

$x$	0	1	2	3
$f(x)$	-2	2	12	34

2. Using Newton-Gregory Interpolation formula, estimate  $f(0.12)$  from the following data.

$x$	0.10	0.15	0.20	0.25	0.30
$f(x)$	0.1003	0.1511	0.2027	0.2553	0.3093

3. Apply Newton's backward difference interpolation formula to find  $f(7.5)$  from the following table:

$x$	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

4. Using Newton -Gregory Interpolation formulae, find  $f(17)$  from the following data.

$x$	0	4	8	12	16	20	24
$f(x)$	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4452

**Interpolation with unequal intervals****Lagrange's formula for unequal intervals:**

Let  $y = f(x)$  be a function whose values are  $y_0, y_1, y_2, \dots, y_n$  corresponding to  $x_0, x_1, x_2, \dots, x_n$  not necessarily equally spaced.

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}y_1 \\ + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}y_n$$

This formula is known as Lagrange's Interpolation formula.

**Inverse interpolation:**

The process of estimating the value of  $x$  for a given value of  $y$  is called Inverse interpolation. So far given a table of values of  $x$  and  $y$ , using one of the interpolation formulae we find the value of  $y$  corresponding to some value of  $x$  which is not in the table. On the other hand the process of





estimating the value of  $x$  for some value of  $y$  which is not in the table is called inverse interpolation.

This method is used when the values of  $x$  are not necessarily equally spaced. Lagrange's interpolation formula can be simply viewed as a relation between two variables and any one of the variable can be taken as an independent variable. Therefore, inverse interpolation formula can be obtained by interchanging the variables  $x$  and  $y$  in Lagrange's formula, we get.

$$x = \frac{(y - y_1)(y - y_2) \dots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \dots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \dots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \dots (y_1 - y_n)} x_1 + \dots + \frac{(y - y_0)(y - y_1) \dots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \dots (y_n - y_{n-1})} x_n$$

### Problems:

1. By using the Lagrange's interpolation formula to fit a polynomial to the data given

$x$	0	1	3	4
$y$	-12	0	6	12

Hence, find  $y$  when  $x = 2$ .

### Solution:

$$x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4$$

$$y_0 = -12 \quad y_1 = 0 \quad y_2 = 6 \quad y_3 = 12$$

By Lagrange's formula

$$y = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} (-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} (0) +$$

$$\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} (6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} (12)$$

$$y = (x-1)(x-3)(x-4) - (x)(x-1)(x-4) + x(x-1)(x-3)$$

$$= (x-1)[x^2 - 7x + 12 - x^2 + 4x + x^2 - 3x]$$

$$= (x-1)(x^2 - 6x + 12)$$

$$y = x^3 - 7x^2 + 18x - 12$$

for  $x = 2$ , we get  $y = 4$ .

2. Apply Lagrange's formula to find  $f(5)$  and  $f(6)$  given that  $f(1)=2, f(2)=4, f(3)=8, f(7)=128$  and explain why the results differ from those obtained by  $f(x) = 2^x$ .

$$\text{Solution: } x_0 = 1 \quad x_1 = 2 \quad x_2 = 3 \quad x_3 = 7$$



By Lagrange's formula, we have

$$\begin{aligned}
 f(x) &= \frac{(x-2)(x-3)(x-7)}{(1-2)(1-3)(1-7)} f(x_0) + \frac{(x-1)(x-3)(x-7)}{(2-1)(2-3)(2-7)} f(x_1) + \\
 &\quad \frac{(x-1)(x-3)(x-17)}{(3-1)(3-2)(1-7)} f(x_2) + \frac{(x-1)(x-2)(x-3)}{(7-1)(7-2)(7-3)} f(x_3) \\
 f(x) &= \frac{(x-2)(x-3)(x-7)}{-12} (2) + \frac{(x-1)(x-3)(x-7)}{5} (4) + \\
 &\quad \frac{(x-1)(x-2)(x-17)}{-8} (8) + \frac{(x-1)(x-2)(x-3)}{120} (128) \\
 f(5) &= \frac{(3)(2)(-2)}{-12} (2) + \frac{(4)(2)(-2)}{5} (4) + \frac{(4)(3)(-2)}{-8} (8) + \frac{(4)(3)(2)}{120} (128) \\
 f(5) &= 2 - 12.8 + 24 + 25.6 = 38.8 \\
 f(6) &= \frac{(4)(3)(-1)}{-12} (2) + \frac{(5)(3)(-1)}{5} (4) + \frac{(5)(4)(-1)}{-8} (8) + \frac{(5)(4)(3)}{120} (128) \\
 f(6) &= 2 - 12 + 20 + 64 = 74
 \end{aligned}$$

But actual values of  $f(5)$  and  $f(6)$  are  $f(5) = 2^5 = 32$  and  $f(6) = 2^6 = 64$ .

The difference in values of  $f(5)$  and  $f(6)$  are due to the assumption of  $f(x)$  as a polynomial, when it is an exponential function of the form  $2^x$ .

3. Find  $x$  for  $y = 7$

$x$	1	3	4
$y$	4	12	19

**Solution:**

$$x_0=1 \quad x_1=3 \quad x_2=4$$

$$y_0=4 \quad y_1=12 \quad y_2=19$$

$$\begin{aligned}
 x &= \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2 \\
 &= \frac{(y-12)(y-19)}{(4-12)(4-19)} (1) + \frac{(y-4)(y-19)}{(12-4)(12-19)} (3) + \frac{(y-4)(y-12)}{(19-4)(19-12)} (4)
 \end{aligned}$$

at  $y = 7$

$$x = \frac{60}{120} + \frac{108}{56} - \frac{60}{105} = 1.85714$$

**Numerical Differentiation:**

Let the function  $y = f(x)$  is given by a table of values  $(x, y)$  then the process of computing the derivatives  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  etc. for some particular value of  $x$  is called Numerical Differentiation.

**Derivatives using Newton's forward interpolation formula:**

By Newton's Interpolation Formula, we have

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!} \Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!} \Delta^3 y_0 + \dots \quad (1)$$

$$\text{where, } x = x_0 + Ph \text{ (or) } P = \frac{x - x_0}{h} \quad \dots \quad (2)$$

Differentiating (1) w.r.t. P, we get

$$\frac{dy}{dP} = \Delta y_0 + \frac{(2P-1)}{2!} \Delta^2 y_0 + \frac{3P^2+6P+2}{3!} \Delta^3 y_0 + \frac{(4P^3-18P^2+22P-6)}{4!} \Delta^4 y_0 + \dots \quad (3)$$

Differentiating (2) w.r.t. x, we get

$$\frac{dP}{dx} = \frac{1}{h} \dots \quad (4) \quad \text{But, } \frac{dy}{dx} = \frac{dy}{dP} \cdot \frac{dP}{dx}$$

Using (3) and (4) the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_0+Ph} = \frac{1}{h} \left[ \Delta y_0 + \frac{(2P-1)}{2!} \Delta^2 y_0 + \frac{(3P^2-6P+2)}{3!} \Delta^3 y_0 + \frac{(4P^3-18P^2+22P-6)}{4!} \Delta^4 y_0 + \dots \right] \dots \quad (5)$$

At  $x = x_0$ ,  $P = 0$ , the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right] \dots \quad (6)$$

Formula (5) is used to compute  $y'$  at any point  $x = x_0 + Ph$ , whereas formula (6) is used to compute  $y'$  at any of the value of  $x$  when  $y$  is specified.

Similarly,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0+Ph} = \frac{1}{h^2} \left[ \Delta y_0 + (P-1) \Delta^3 y_0 + \frac{(6P^2-18P+11)}{12} \Delta^4 y_0 + \dots \right] \dots \quad (7)$$

$$\text{and } \left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \dots \right] \dots \quad (8)$$

The formula (7) is used to compute  $y''$  at any point  $x = x_0 + Ph$  whereas formula (8) is used to compute  $y''$  at any value of  $x$  where  $y$  is specified.

**Derivatives using Newton's backward interpolation formula:**

By Newton's Backward Interpolation Formula, we have



$$y = y_n + \frac{P}{1!} \nabla y_n + \frac{P(P+1)}{2!} \nabla^2 y_n + \frac{P(P+1)(P+2)}{3!} \nabla^3 y_n + \dots \quad \dots\dots (1)$$

$$\text{Where, } x = x_n + Ph \quad (\text{or}) \quad P = \frac{x - x_n}{h} \quad \dots\dots (2)$$

Differentiating (1) w.r.t. P, we get

$$\frac{dy}{dP} = \nabla y_n + \frac{(2P+1)}{2!} \nabla^2 y_n + \frac{(3P^2+6P+2)}{3!} \nabla^3 y_n + \dots \quad \dots\dots (3)$$

Differentiating (2) w.r.t. x, we get

$$\frac{dP}{dx} = \frac{1}{h} \dots (4) \quad \text{But, } \frac{dy}{dx} = \frac{dy}{dP} \cdot \frac{dP}{dx}$$

Using (3) and (4) the above equation becomes,

$$\left( \frac{dy}{dx} \right)_{x=x_n+Ph} = \frac{1}{h} \left[ \nabla y_n + \frac{(2P+1)}{2!} \nabla^2 y_n + \frac{(3P^2+6P+2)}{3!} \nabla^3 y_n + \dots \right] \quad \dots (5)$$

At  $x = x_n$ ,  $P = 0$ , the above equation becomes,

$$\left( \frac{dy}{dx} \right)_{x=x_n} = \frac{1}{h} \left[ \nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \dots \right] \quad \dots (6)$$

Formula (5) is used to compute  $y'$  at any point  $x = x_n + Ph$  whereas formula (6) is used to compute  $y'$  at any of the values of  $x$  when  $y$  is specified.

Similarly

$$\left( \frac{d^2y}{dx^2} \right)_{x=x_n+Ph} = \frac{1}{h^2} \left[ \nabla^2 y_n + (P+1) \nabla^3 y_n + \frac{(6P^2+18P+11)}{12} \nabla^4 y_n + \dots \right] \quad \dots (7)$$

$$\left( \frac{d^2y}{dx^2} \right)_{x=x_n} = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n + \frac{137}{180} \nabla^6 y_n + \dots \right] \quad \dots (8)$$

The formula (7) is used to compute  $y''$  at any point  $x = x_n + Ph$  whereas formula (8) is used to compute  $y''$  at any of the values of  $x$  when  $y$  is specified.

### Problems:

#### 1. Given

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y$	2.72	3.32	4.06	4.96	6.05	7.39

Find  $y'$  and  $y''$  at  $x = 1.2$ .

**Solution:** Here, the step-length is  $h = 0.2$ . We first form the following difference table.

$x$	$y$	first differences	Second differences	Third differences	Fourth differences
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1.0	2.27				
1.2	3.32	0.60			
1.4	4.06	0.74	0.14	0.02	0.01
1.6	4.96	0.90	0.16	0.03	
1.8	6.05	1.09	0.19	0.06	0.03
2.0	7.39	1.34	0.25		

We have to compute  $y'$  and  $y''$  at  $x = 1.2$ , which is a specified value of  $x$ , for this purpose, we take  $x_0 = 1.2$ . Then we find from the table that,

$$\Delta y_0 = 0.74, \Delta^2 y_0 = 0.16, \Delta^3 y_0 = 0.03, \Delta^4 y_0 = 0.03$$

Using the formula we have,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right]$$

$$\text{Then, } \left(\frac{dy}{dx}\right)_{(1.2)} = \frac{1}{(0.2)} \left[ (0.74) - \frac{1}{2} (0.16) + \frac{1}{3} (0.03) - \frac{1}{4} (0.03) \right] = 3.3125$$

Using the formula we have,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \dots \right]$$

$$\text{And } \left(\frac{d^2y}{dx^2}\right)_{1.2} = \frac{1}{(0.2)^2} \left[ (0.16) - (0.03) + \frac{11}{12} (0.03) \right] = 3.39375$$

2. Using appropriate interpolation formulas, find the values of  $y'$  and  $y''$  when  $x = 4$  using the following table.

$x$	1	2	3	4
$y$	4	12	20	36

**Solution:** Here  $x = 4$  is a specified value of  $x$  which is at the end of the given table.

For this purpose we take  $x_n = 4$ .

The difference table is given by,

$x$	$y$	First differences	Second differences	Third differences
1	4			
2	12	8		
3	20	8	0	
4	36	16	8	8

Here  $x_n = 4$ ,  $\nabla y_n = 16$ ,  $\nabla^2 y_n = 8$ ,  $\nabla^3 y_n = 8$  and  $h=1$



$$\text{Then, } y'(4) = \frac{1}{h} \left[ (\nabla y_n) + \frac{1}{2} (\nabla^2 y_n) + \frac{1}{3} (\nabla^3 y_n) \right] = \frac{1}{1} \left[ 16 + \frac{1}{2} (8) + \frac{1}{3} (8) \right]$$

$$y'(4) = 16 + 4 + \frac{8}{3} = 22.667$$

$$\text{And } y''(4) = \frac{1}{h^2} \left[ \frac{1}{2} (\nabla^2 y_n) + (\nabla^3 y_n) \right] = \frac{1}{1^2} [8 + 8] = 16$$

**3.** A rod is rotating in a plane, the following table gives the angle  $\theta$  (in radians) through which the rod is turned for various values of time  $t$  (in seconds): Find the angular velocity and angular acceleration at  $t = 0.4$  sec.

**Solution:** Given

$t$	0	0.2	0.4	0.6	0.8	1.0	1.2
$\theta$	0	0.12	0.49	1.12	2.02	3.20	4.67

To compute velocity  $= \frac{d\theta}{dt}$  and Acceleration  $= \frac{d^2\theta}{dt^2}$  at  $t = 0.4 \text{ sec}$

The finite difference table is given by

$x$	$\theta$	First differences	second differences	Third differences	Fourth differences
0	0				
0.2	0.12	0.12			
<b>0.4</b>	<b>0.49</b>	0.37	0.25	0.01	
0.6	1.12	<b>0.63</b>	0.26	0.01	0.00
0.8	2.02	0.90	<b>0.27</b>	<b>0.01</b>	0.00
1.0	3.20	1.18	0.28	0.01	<b>0.00</b>
1.2	4.67	1.47	0.29		

To compute  $\frac{d\theta}{dt}$  and  $\frac{d^2\theta}{dt^2}$  at  $t = 0.4 \text{ sec}$ , which is specified value of  $t$ . We take  $t_0 = 0.4$

$$\Delta\theta_0 = 0.63, \Delta^2\theta_0 = 0.21, \Delta^3\theta_0 = 0.01, \Delta^4\theta_0 = 0.00$$

Using the formula,

$$\left( \frac{d\theta}{dt} \right)_{x=x_0} = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right]$$

$$\left( \frac{d\theta}{dt} \right)_{(0.4)} = \frac{1}{0.2} \left[ (0.63) - \frac{1}{2} (0.27) + \frac{1}{3} (0.01) \right] = 2.49$$

And using the formula,

$$\left( \frac{d^2\theta}{dt^2} \right)_{x=x_0} = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \dots \right]$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{(0.4)} = \frac{1}{(0.2)^2} [(0.27) - 0.01] = 6.5$$

4. The following table gives the temperature  $\theta$  (in degree Celsius) of a cooling body at different instants of time  $t$  (in seconds)

$T$	1	3	5	7	9
$\theta$	85.3	74.5	67.0	60.5	54.3

Find approximately the rate of cooling at  $t = 8$  seconds.

**Solution:** Rate of cooling  $= \frac{d\theta}{dt}$

The backward difference table is given by,

$T$	$\theta$	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
1	85.3				
3	74.5	-10.8			
5	67.0	-7.5	3.3		
7	60.5	-6.5	1.0	-2.3	
9	54.3	-6.2	0.3	-0.7	1.6

Here, to compute  $\frac{d\theta}{dt}$  at  $t = 8 \text{ sec.}$ , which is not the specified value of  $t$

We have,  $t_n = t_4 = 9$ ,  $\theta_n = \theta_4 = 54.3$

$$\nabla\theta_4 = -6.2, \nabla^2\theta_4 = 0.3, \nabla^3\theta_4 = -0.7, \nabla^4\theta_4 = 1.6$$

$$P = \frac{t-t_4}{h} = -0.5$$

Using the formula, we have

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x=x_n} &= \frac{1}{h} \left[ \nabla y_n + \frac{2p+1}{2!} \nabla^2 y_n + \frac{3p^2+6p+2}{3!} \nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{4!} \nabla^4 y_n + \dots \right] \\ \left(\frac{d\theta}{dt}\right)_{(t=8)} &= \frac{1}{2} \left[ -6.2 + \frac{2(-.05) + 1}{2} (0.3) + \frac{3(-0.5)^2 + 6(.5) + 2}{6} (-0.7) \right. \\ &\quad \left. + \frac{4(-0.5)^3 + 18(-0.5)^2 + 22(-.05) + 6}{24} (1.6) \right] \\ &= \frac{1}{2} [-6.2 + 0 + 0.029166 - 0.066666] = -3.11875 \end{aligned}$$

Thus, the body cools at the rate of 3.11875 degree/second.

**Exercise:**

1. Given that,

$x$	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$y$	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  at  $x = 1.1$  and  $x = 1.6$ .2. A slider in a machine moves along a fixed straight rod. Its distance  $x$  cm along the rod is given below for various values of the time ' $t$ ' seconds. Find the velocity of the slider and its acceleration when  $t = 0.1$  second.

$t$	0	1.0	1.2		1.4	1.6	1.8	2.0
$x$	30.13	31.62	32.87		33.64	33.95	33.81	33.24

3. Find  $\frac{dx}{dy}$  and  $\frac{d^2y}{dx^2}$  at  $x = 1.1$  of the function tabulated below:

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.128	-0.544	1.296	2.432	4.00

4. Compute  $y'(0)$  and  $y''(0)$  from the following table:

$x$	0	1	2	3	4	5
$y$	4	8	15	7	6	2

5. Find  $f'(1.5)$  using the differentiation formula based on Newton's interpolation for the following data:

$x$	1	2	3	4	5	6
$f(x)$	4	9	16	25	36	49