

DEPARTMENT OF PHYSICS

SECOND SEMESTER BE PROGRAMS (CS, CD, CY, IS, AIML, & BT)
ACADEMIC YEAR 2022-2023

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|-------------|-------------------------------|----------------------------|--------|
| Dated | 10th July 2023 | Maximum Marks | 50 |
| Course Code | 22PHY22C | Duration | 90 min |
| Course | QUANTUM PHYSICS FOR ENGINEERS | CIE-I (Test) - Long Scheme | |

1(a) An excited state of an atom with lifetime of 10^{-8} s shows more broadening in its emitted spectral line than a state with lifetime of 10^{-3} s. Justify this statement by deriving a relation for spectral line broadening. ④

The energy of the emitted photon is given by,

$$E = h\nu = \frac{hc}{\lambda} \quad (1)$$

Where h is Planck's constant, ν is the frequency, c is the velocity and λ is the wavelength of the emitted radiation. Differentiating this equation with respect to λ , we get

$$\Delta E = -hc \frac{\Delta \lambda}{\lambda^2} \quad (1)$$

Considering only the magnitude of the difference,

$$|\Delta E| = hc \frac{\Delta \lambda}{\lambda^2} \quad (2)$$

According to Heisenberg's uncertainty principle, the finite lifetime Δt of the excited state means there will be an uncertainty of ΔE in the energy of the state. Hence the emitted photon energy will also have an uncertainty of ΔE in its energy and is related by,

$$\Delta E \Delta t \geq \hbar/2 \quad \text{Or,} \quad \Delta E \geq \frac{h}{4\pi\Delta t} \quad (1)$$

Substituting for ΔE from (2) we get,

$$hc \frac{\Delta \lambda}{\lambda^2} \geq \frac{h}{4\pi\Delta t} \quad \text{Or} \quad \boxed{\Delta \lambda \geq \frac{1}{4\pi c} \frac{\lambda^2}{\Delta t}} \quad (1)$$

This shows that for a finite lifetime of the excited state, the measured value of the emitted photon wavelength will have a spread of wavelengths around the mean value λ . This demands that for a very narrow spread, the lifetime of the excited state must be very high. Justifying that if Δt is less $\Delta \lambda$ will be more. ①

1(b) Show that the de Broglie wavelength of an electron accelerated through a potential difference V is inversely proportional to \sqrt{V} . ④

Kinetic energy of a particle, with mass m , moving with non-relativistic velocity v is given by,

$$E = \frac{1}{2}mv^2$$

The equation can be rearranged as,

$$m^2v^2 = 2mE$$

where $mv = p$ is the momentum of the particle. Hence it can be rewritten as,

$$p = \sqrt{2mE} \quad (1)$$

Therefore de Broglie wavelength of the particle can be expressed in terms of kinetic energy as,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \quad (1)$$

Consider an electron of mass m and charge $-e$ accelerated from rest through an electric potential V . The work done on the electron, which is equal to eV , is then equal to the kinetic energy E gained by the electron.

$$E = eV \quad (1)$$

de Broglie wavelength of this electron in terms of the accelerating potential is,

$$\lambda = \frac{h}{\sqrt{2meV}} \quad (1)$$

1(c) What is i) Born interpretation of wave function and ii) Normalization of wave function. (2)

i) Probability of finding the particle in the position range $[a, b]$ is $P_{ab} = \int_a^b |\Psi(x, t)|^2 dx$ (1)

ii) Total probability of finding the particle anywhere is 1, $\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx = 1$ (1)

2(a) Making use of the concept of matter waves setup one dimensional time independent Schrodinger wave equation. (6)

Let $\Psi(x, t)$ be the wave function for the matter wave associated with a moving particle. The corresponding wave equation is as follows,

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v_p^2} \frac{\partial^2 \Psi}{\partial t^2} \quad (1)$$

where v_p is the phase velocity of the associated matter waves. The solution of eq. (1) is,

$$\Psi(x, t) = \psi(x)e^{-i\omega t} \quad (2)$$

where ω is the angular frequency.

Differentiating eq. (2) partially twice with respect to t , we get,

$$\frac{\partial \Psi}{\partial t} = (-i\omega)\psi(x)e^{-i\omega t} \quad \text{and} \quad \frac{\partial^2 \Psi}{\partial t^2} = (-i\omega)^2\psi(x)e^{-i\omega t} = -\omega^2\Psi \quad (3)$$

Substituting eq. (3) in eq. 1

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{\omega^2}{v_p^2} \Psi$$

We have, for matter waves, phase velocity $v_p = \lambda\nu$ and angular frequency $\omega = 2\pi\nu$, where ν is the frequency.

$$\frac{\omega^2}{v_p^2} = \left(\frac{2\pi f}{\lambda f}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 = \frac{4\pi^2}{\lambda^2}$$

Substituting this and in the above equation we get,

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} \Psi \quad (2)$$

Substituting Ψ from eq. (2) into it we get,

$$\begin{aligned} \frac{\partial^2 [\psi(x)e^{-i\omega t}]}{\partial x^2} &= -\frac{4\pi^2}{\lambda^2} [\psi(x)e^{-i\omega t}] \\ e^{-i\omega t} \frac{d^2 \psi(x)}{dx^2} &= -\frac{4\pi^2}{\lambda^2} \psi(x)e^{-i\omega t} \end{aligned}$$

$$\frac{d^2\psi(x)}{dx^2} + \frac{4\pi^2}{\lambda^2}\psi(x) = 0$$

In the above equations you can note that partial differential is changed into ordinary differential because ψ is a function of x alone. Substituting de Broglie's wavelength $\lambda = h/(mv)$ for the matter waves of a particle of mass m moving with velocity v into it, we get,

$$\frac{d^2\psi(x)}{dx^2} + 4\pi^2 \frac{m^2 v^2}{h^2} \psi(x) = 0 \quad (4)$$

If E is the total energy and V is the potential energy of the particle respectively, then the kinetic energy of the particle is,

$$E_k = \frac{1}{2}mv^2 = E - V, \quad \therefore \quad m^2 v^2 = 2m(E - V) \quad (2)$$

Substituting this into eqn. (4), we get,

$$\boxed{\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi(x) = 0} \quad (5)$$

Here ψ is a function of x alone and is independent of time. We use this equation for the cases where $V \equiv V(x)$ is a function of x alone so that the whole equation is independent of time. This equation is called one dimensional time-independent Schrödinger equation.

2(b) At time $t = 0$ a particle is represented by the wave function

$$\Psi(x, 0) = \begin{cases} \sqrt{(3/b)}(x/a), & 0 \leq x \leq a, \\ \sqrt{(3/b)}(b-x)/(b-a), & a \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

where a and b are (positive) real constants representing two locations on the x -axis. What is the probability of finding the particle i) to the left of a and ii) to the right of a ? (4)

$$P_{\text{left of } a} = \int_0^a \left[\sqrt{(3/b)}(x/a) \right] \left[\sqrt{(3/b)}(x/a) \right] dx = \int_0^a (3/(ba^2))x^2 dx = a/b \quad (2)$$

$$P_{\text{right of } a} = 1 - P_{\text{left of } a} = 1 - a/b = (b-a)/b \quad (2)$$

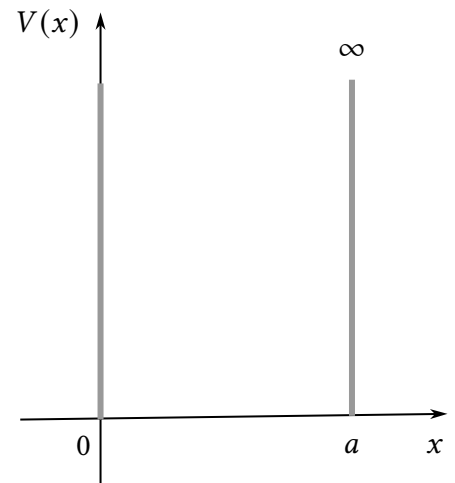
$$\begin{aligned} \text{Or, } P_{\text{right of } a} &= \int_a^b \left[\sqrt{(3/b)}(b-x)/(b-a) \right] \left[\sqrt{(3/b)}(b-x)/(b-a) \right] dx = \int_a^b \frac{3}{b(b-a)^2} (b-x)^2 dx \\ &= \left[\frac{3}{b(b-a)^2} \frac{(b-x)^3}{(-1)^3} \right]_a^b = \frac{(b-a)^3}{b(b-a)^2} = (b-a)/b = 1 - a/b \end{aligned}$$

3(a) For a particle in an one dimensional potential well of infinite depth, solve time independent Schrodinger wave equation and obtain normalized wave functions for first three allowed states. (6)

Suppose the potential is,

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ \infty, & \text{otherwise} \end{cases}$$

A particle in this potential is completely free, except at the two ends ($x = 0$ and $x = a$), where an infinite force prevents it from escaping. A classical model would be a cart on a frictionless horizontal air track, with perfectly elastic bumpers—it just keeps bouncing back and forth forever. (This potential is artificial, of course, but I urge you to treat it with respect. Despite its simplicity—or rather, precisely because of its simplicity—it serves as a wonderfully accessible test case for all the fancy machinery that comes later.)



We begin by time independent Schrödinger equation which reads,

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2m}{h^2}(E - V)\psi(x) = 0 \quad (1)$$

Outside the potential well ($0 \leq x \leq a$) it reads

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2m}{h^2}(E - \infty)\psi(x) = 0$$

This equation holds good only if $\psi(x) = 0$ for all points outside the well. The probability of finding the particle there is zero.

Inside the potential well, where $V = 0$, the time-independent Schrödinger equation (1) reads,

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2m}{h^2}E\psi(x) = 0$$

Or

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0, \quad \text{where } k^2 = \frac{8\pi^2m}{h^2}E \quad (2)$$

Solution: Equation (2) is the classical *simple harmonic oscillator* equation; the general solution is,

$$\psi(x) = A \sin(kx) + B \cos(kx) \quad (3)$$

where A and B are arbitrary constants. Typically, these constants are fixed by the *boundary conditions* of the problem.

Boundary conditions: What are the appropriate boundary conditions for $\psi(x)$? Ordinarily, both ψ and $d\psi/dx$ are continuous, but where the potential goes to infinity only the first of these applies. Continuity of $\psi(x)$ requires that

$$\psi(0) = \psi(a) = 0$$

so as to join onto the solution outside the well. What does this tell us about A and B ?

Condition 1: At $x = 0$, $\psi = 0$

Substituting it in equation (3), we get

$$\psi(0) = A \sin 0 + B \cos 0 = B$$

so $B = 0$ as $\psi(0) = 0$, and hence equation (3) will become,

$$\psi(x) = A \sin(kx) \quad (4)$$

Condition 2: At $x = a$, $\psi = 0$

Substituting it in equation 4, we get

$$\psi(a) = A \sin(ka)$$

Since $\psi(a) = 0$, $A \sin(ka) = 0$, so either $A = 0$ (in which case we're left with the trivial—non-normalizable—solution $\psi(x) = 0$), or else $\sin(ka) = 0$ which means that

$$ka = 0, \pm\pi, \pm2\pi, \pm3\pi, \dots$$

But $k = 0$ is no good (again, that would imply $\psi(x) = 0$, which means the particle is not inside the well), and the negative solutions give nothing new, since $\sin(-\theta) = -\sin(\theta)$ and we can absorb the minus sign into A . So the *distinct* solutions are

$$k = \frac{n\pi}{a}, \quad \text{with } n = 1, 2, 3, \dots \quad (5)$$

Normalization of wave function: Actually, it's the wave function $\Psi(x, t)$ that must be normalized, but in view of $\Psi(x, t) = \psi(x)e^{-i\omega t}$ we will get $\Psi^*\Psi = \psi^*\psi$. This entails the normalization of $\psi(x)$.

To find A , we normalize ψ from equation 4 with normalization condition:

$$\int_{-\infty}^{+\infty} \psi^* \psi dx = \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

Since $\psi(x) = 0$ outside the well, it is sufficient to integrate between 0 to a . With $\psi(x) = A \sin(kx)$ inside the well we have,

$$\int_0^a |\psi(x)|^2 dx = \int_0^a |A|^2 \sin^2(kx) dx = 1$$

With the identity, $\sin^2 \theta = (1 - \cos(2\theta))/2$ the above integral can be written as,

$$\frac{|A|^2}{2} \int_0^a [1 - \cos(2kx)] dx = 1 \quad \text{or,} \quad \frac{|A|^2}{2} \left[x - \frac{1}{2k} \sin(2kx) \right]_0^a = 1$$

By substituting limits and k from equation (5) we get,

$$\frac{|A|^2}{2} \left[a - \frac{a}{2n\pi} \sin(2n\pi) \right] = 1 \quad \therefore |A|^2 = \frac{2}{a}$$

This only determines the magnitude of A , but it is simplest to pick the positive real root and the phase of A carries no physical significance anyway. Hence,

$$A = \sqrt{\frac{2}{a}}$$

Inside the well, then, the normalized solutions (also called eigenfunctions) are,

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

The eigenfunctions for $n = 1, 2, 3$ are as follows. $\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$, $\psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a}x\right)$, $\psi_3 = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi}{a}x\right)$ ②

3(b) A particle in an one dimensional potential well of infinite depth makes two consecutive transitions from energy levels $E_{n+1} \rightarrow E_n \rightarrow E_{n-1}$ releasing energy of 3.384 eV and 2.632 eV respectively. Determine n . ④

$$E_{n+1} - E_n = \frac{(n+1)^2 h^2}{8ma^2} - \frac{n^2 h^2}{8ma^2} = (2n+1) \frac{h^2}{8ma^2} \quad ①$$

$$E_n - E_{n-1} = \frac{n^2 h^2}{8ma^2} - \frac{(n-1)^2 h^2}{8ma^2} = (2n-1) \frac{h^2}{8ma^2} \quad ①$$

$$\frac{E_{n+1} - E_n}{E_n - E_{n-1}} = \frac{2n+1}{2n-1} = \frac{3.384}{2.632} = 1.286 = f \text{ (say)} \quad ①$$

$$2n+1 = 2fn - f \text{ and } 2fn - 2n = f + 1$$

$$n = \frac{f+1}{2f-2} = \frac{1.286+1}{2 \times 1.286-2} = 3.9965 \approx 4 \quad ①$$

4(a) Obtain an expression for energy density of photons in terms of Einstein's coefficients. ⑦

Consider an atomic system interacting with radiation field of energy density U_ν . Let E_1 and E_2 , be the two energy states of atomic system ($E_2 > E_1$). Let us consider atoms are to be in thermal equilibrium with radiation field, which means that the energy density U_ν is constant in spite of the interaction that is taking place between itself and the incident radiation. This is possible only if the number of photons absorbed by the system per second is equal to the number of photons it emits per second by both the stimulated and spontaneous emission processes. We know that

- The rate of induced absorption = $B_{12}U_\nu N_1$
- The rate of spontaneous emission = $A_{21}N_2$
- The rate of stimulated emission = $B_{21}U_\nu N_2$

N_1 and N_2 are the number of atoms in the energy state E_1 and E_2 respectively. B_{12} , A_{21} and B_{21} are the Einstein coefficients for induced absorption, spontaneous emission and stimulated emission respectively. ①
At thermal equilibrium, *Rate of induced absorption = Rate of spontaneous emission + Rate of stimulated emission*

$$\therefore B_{12}N_1U_\nu = A_{21}N_2 + B_{21}N_2U_\nu \quad \text{or} \quad U_\nu(B_{12}N_1 - B_{21}N_2) = A_{21}N_2$$

$$\text{or,} \quad U_\nu = \frac{A_{21}}{B_{21}} \left[\frac{1}{\frac{B_{12}N_1}{B_{21}N_2} - 1} \right] \quad \text{②} \quad (1)$$

In a state of thermal equilibrium, the populations of energy levels E_2 and E_1 are fixed by the Boltzmann factor. The population ratio is given by,

$$\frac{N_2}{N_1} = e^{-(E_2-E_1)/(kT)} \quad \text{with } E_2 - E_1 = h\nu \text{ it can written as, } \frac{N_1}{N_2} = e^{h\nu/(kT)} \quad \text{①}$$

Substituting it into eqn. (1) we get,

$$U_\nu = \frac{A_{21}}{B_{21}} \left[\frac{1}{\frac{B_{12}}{B_{21}} e^{h\nu/(kT)} - 1} \right] \quad \text{①} \quad (2)$$

According to Planck's law of black body radiation, the equation for U_ν is,

$$U_\nu = \frac{8\pi h\nu^3}{c^3} \left[\frac{1}{e^{h\nu/(kT)} - 1} \right] \quad \text{③} \quad (3)$$

Now comparing the eqns. (2) and (3) term by term on the basis of positional identity we have

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3} \quad \text{and} \quad \frac{B_{12}}{B_{21}} = 1 \quad \text{or} \quad B_{12} = B_{21}$$

This implies that the probability of induced absorption is equal to the probability of stimulated emission. Due to this identity the subscripts could be dropped, and A_{21} and B_{21} can be simply represented as A and B and eqn. (2) can be rewritten. Therefore, at thermal equilibrium the equation for energy density is,

$$U_\nu = \frac{A}{B} \left[\frac{1}{e^{h\nu/(kT)} - 1} \right] \quad \text{②}$$

4(b) A laser operating at temperature of 300 K and wavelength of 680 nm is at thermal equilibrium. Determine the ratio of Einstein's coefficients A and B . ③

$$\begin{aligned} \frac{A}{B} &= \frac{8\pi h\nu^3}{c^3} = \frac{8\pi h}{\lambda^3} \\ \frac{A}{B} &= \frac{8 \times 3.142 \times 6.626 \times 10^{-34} \text{Js}}{(680 \times 10^{-9} \text{m})^3} \quad \text{①} \\ \text{Simplification} &\quad \text{①} \\ &= 5.296 \times 10^{-14} \text{Pa s} \quad \text{①} \end{aligned}$$

5(a) With energy band diagram explain the construction and working of semiconductor laser. ⑦

- Schematic diagram ①
- Construction explanation ①
- Energy band diagram ②
- Working explanation ③

5(b) Two levels of an atomic system at thermal equilibrium has energy difference of 1.8 eV. If the system is at temperature 300 K, determine the ratio of population of these two energy levels. ③

$$\begin{aligned} \frac{N_2}{N_1} &= e^{-h\nu/(kT)} = e^{-\Delta E/(kT)} = e^{-1.8 \text{eV}/(1.381 \times 10^{-23} \text{J K}^{-1} \times 300 \text{K})} \quad \text{①} \\ &= e^{-1.8 \times 1.602 \times 10^{-19} \text{J}/(1.381 \times 10^{-23} \text{J K}^{-1} \times 300 \text{K})} \quad \text{①} \\ &= 5.772 \times 10^{-31} \quad \text{Or} \quad \frac{N_1}{N_2} = 0.173 \times 10^{31} \quad \text{①} \end{aligned}$$