UNIT-IV

INTEGRAL CALCULUS

Topic Learning Objectives:

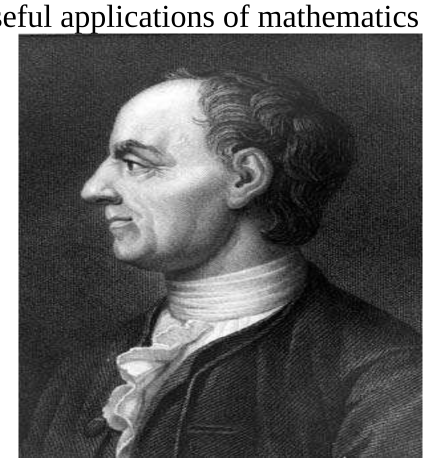
Upon Completion of this unit, student will be able to:

- Understand the existence of improper integrals- Beta and Gamma functions, multiple integrals and the mathematical expressions.
- Apply methods of evaluating multiple integrals by recognising the region of integration.
- Evaluate integrals by change of order of integration and by changing to polar coordinate system.
- Apply the concepts for computing areas and volume etc., associated with engineering problems.

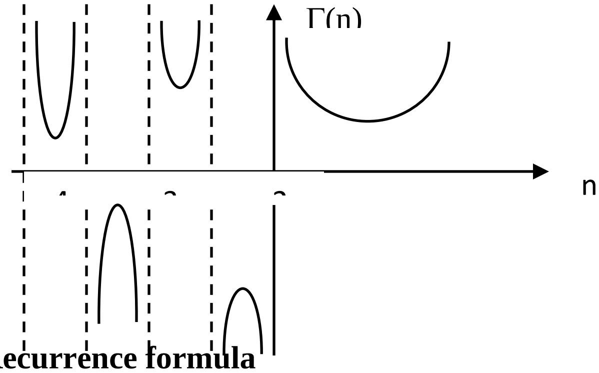
A definite integral $\int_a^b f(x)dx$ is said to be proper (regular) integral if the limits of integration are finite and the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$. If at least one of these conditions violated then the integral is known as an improper integral.
Certain mathematical functions which occur in various contexts have been designated as special functions. Beta and Gamma functions are two such functions which are defined in the form of integrals, these are more closely related to improper integrals.

Gamma and Beta functions

Gamma Function:
Introduced by the Swiss mathematician and physicist Leonhard Euler in the 18th century, one of the founders of pure mathematics. His contributions not only to the subjects of geometry, calculus, mechanics, and number theory but also developed methods for solving problems in observational astronomy and demonstrated useful applications of mathematics in technology.



Definition: The improper integral $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ is defined as the gamma function. Here n is a real number called the parameter of the function. $\Gamma(n)$ exists for all real values of n except 0, -1, -2., ………..the graph of which is shown below :



Recurrence formula

$$\Gamma(n)=\int_0^{\infty} e^{-x}x^{n-1}dx,\quad \mathrm{n}>0$$

$$=\left\{\frac{x^n}{n}e^{-x}\right\}_0^{\infty}+\int_0^{\infty}\frac{x^n}{n}e^{-x}dx, \text{ integrating by parts.}$$

Applying the definition of Gamma function and using $\frac{x^n}{e^x}\to 0$ as $x\to\infty$

$$\Gamma(n)=\frac{\Gamma(n+1)}{n}\text{.......................}(1)$$

$$or\quad \Gamma(n+1)=n\Gamma(n)\text{................}(2)$$

Note:

(1) $\Gamma(n)$ is not convergent when n = 0, -1, -2, …….

(2) If $\Gamma(n)$ is known for $0 < n < 1$, then its value for $1 < n < 2$ can be found using equation (2) and its values for $-1 < n < 0$ can be found using equation (1)

(3) If n is a positive integer, using the recurrence relation and $\Gamma(1) = 1$, we get

$\Gamma(n) = (n - 1)\,\Gamma(n - 1)$

$= (n - 1)(n - 2)\,\Gamma(n - 2)$ and so on

$= (n - 1)(n - 2)(n - 3)\,\dots\dots\, 1$

$= (n - 1)\,!$

Examples:

1. Compute the following:

(i) $\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)=\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$ **.**

(ii) $\Gamma\left(-\frac{7}{2}\right)=\dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{7}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}=\dfrac{8\sqrt{\pi}}{105}$.

(iii) $\Gamma(8)=7!$

2. Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

Solution: By definition, $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$

$$=\int_0^\infty e^{-t^2}t^{2n-1}dt \quad using \quad x=t^2$$

$$\therefore \Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx$$

$$=2\int_0^\infty e^{-y^2}y^{2n-1}dy$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-x^2}dx \quad =\int_0^\infty e^{-y^2}dy$$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2=4\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy$$

Transforming to polar coordinates with $x=r\cos\theta,\ y=r\sin\theta$, we get

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=2\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}e^{-r^2}(2r)drd\theta$$

$$=\int_0^{\pi/2}\left[e^{-r^2}\right]_0^\infty d\theta, \qquad \because\ -\int 2re^{-r^2}dr=e^{-r^2}$$

$$=2\int_0^{\pi/2}d\theta, \qquad \because e^{-r^2}\to 0 \text{ as } r\to\infty$$

$$=\pi$$

Taking square roots on both sides, we get $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

3. Show that $\int_0^\infty e^{-x^4}dx=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$.

Solution: Let $x^4=y\Rightarrow 4x^3dx=dy\Rightarrow dx=\frac{1}{4}y^{-3/4}dy$

$$\therefore \int_0^\infty e^{-x^4}\,dx = \frac{1}{4}\int_0^\infty e^{-y}y^{-3/4}\,dy \quad = \frac{1}{4}\Gamma\left(\frac{1}{4}\right), \text{ using the definition.}$$

4. Find $\int_0^1 (x\ln x)^5 dx$.

Solution: using $\ln x = -y$, we get $x = e^{-y}$, $dx = -e^{-y}\,dy$ and y ranges from ∞ to 0 when $x = 0$ to 1

$$\therefore \int_0^\infty (x\ln x)^5 dx = -\int_0^\infty y^5 e^{-6y}dy, \quad \text{interchanging the lower and the upper limits}$$

$$= \int_0^\infty e^{-t}\frac{t^5 dt}{6^6}, \quad \text{choosing} \quad 6y = t$$

$$= -\frac{1}{6^6}\Gamma(6) = -\frac{5!}{6^6}$$

5. Prove that $\displaystyle\int_0^a \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}} = a\sqrt{\pi}$.

Solution: Using $\ln\left(\frac{a}{x}\right) = t$, we get $\frac{a}{x} = e^t$

$dx = -ae^{-t}\,dt$ and $t = \infty$ to 0 when $x = 0$ to a

$$\therefore \int_0^a \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}} = \int_0^\infty t^{-1/2}a.e^{-t}dt = a\Gamma\left(\frac{1}{2}\right) = a\sqrt{\pi}$$

6. Show that $\displaystyle\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$.

Solution: Hence show that $\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}dx = -4$ when 'n' is a positive integer and m > -1.

Let $\log(x) = -t \Rightarrow dx = -e^{-t}\,dt$ and $t = \infty$ to 0 when $x = 0$ to 1.

$$\therefore \int_0^1 x^m(\log x)^n dx = \int_0^\infty e^{-mt}(-t)^n e^{-t}dt = (-1)^n\int_0^\infty t^n e^{-(m+1)t}dt$$

using $(m+1)t = y$, $\displaystyle\int_0^1 x^m(\log x)^n dx = (-1)^n\int_0^\infty \frac{y^n}{(m+1)^n}e^{-y}\frac{dy}{m+1}, \quad \because m > -1$

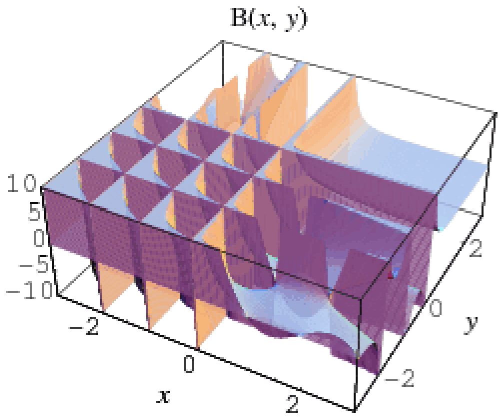
$$\therefore \int_0^1 x^m(\log x)^n dx = \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-y}y^n dy$$

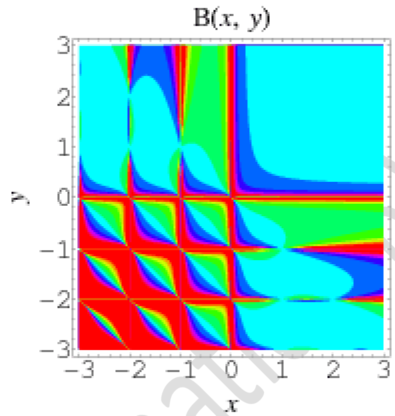
$$= \frac{(-1)^n}{(m+1)^{n+1}}n!, \quad \because \Gamma(n+1) = n!$$

choosing $n=1$ and $m=-\frac{1}{2}$, we get the required result.

Beta Function:

The beta function B(m,n) is the name used by Legendre and Whittaker and Watson (1990) for the beta integral (also called the Eulerian integral of the first kind).





Definition: Beta function, denoted by B(m,n) is defined by $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$,

where m and n are positive real numbers.

Note: **B(m,n) = B(n,m)** i,e $\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^1 (1-x)^{m-1}x^{n-1}dx$

Alternate Expressions for B(m,n)

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx\text{.........................}(1)$$

(i) Let $x=\frac{1}{1+t} \Rightarrow dx=-\frac{1}{(1+t)^2}dt$, $1-x=\frac{t}{1+t}$

and $t=\infty$ to 0 when $x=0$ to 1.

$\therefore (1)$ reduces to $B(m,n) = \int_0^{\infty}\left(\frac{1}{1+t}\right)^{m-1}\left(\frac{1}{1+t}\right)^{n-1}\left(\frac{1}{(1+t)^2}\right)dt$

$B(m,n)$ is also $=\int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m+n}}dt$ using $B(m,n)=B(n,m)$

(ii) Let $x = \sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta\, d\theta$ and $\theta=0$ to $\frac{\pi}{2}$ when $x=0$ to 1

Then (1) reduces to $2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\, d\theta$

$$\therefore B(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta.$$

Relation between Gamma and Beta functions:

Prove that $$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof: We know that $\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx$ *and* $\Gamma(m)=2\int_0^\infty e^{-y^2}x^{2m-1}dy$

Then $\Gamma(m)\,\Gamma(n)=4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}dxdy$

Transforming to polar coordinates with $x=r\cos\theta,\ y=r\sin\theta$, we get

$$=4\int_0^\infty e^{-r^2}r^{2(m+n)-1}dr\int_{\theta=0}^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

$= \Gamma(m+n)\ B(m,n)$ by definitions.

Note:

Using $\int_0^\infty \frac{x^{n-1}}{1+x}dx=\frac{\pi}{\sin n\pi},\quad 0<n<1$, we get

$\Gamma(n)\ \Gamma(1-n) = \Gamma(1)\ B(1-n, n)$ using the above relation

But $B(1-n,n)=\int_0^\infty\frac{x^{n-1}}{1+x}dx$ using definition of Beta function.

Therefore $\Gamma(n)\ \Gamma(1-n) = \pi/\sin n\pi,\quad 0 < n < 1$

Examples:

1. Show that $\int_0^1\frac{5x}{\sqrt{1-x^5}}dx=B\left(\frac{2}{5},\frac{1}{2}\right)$.

Solution: Let $x^5=t$, we get $x=t^{1/5}, dx=\frac{1}{5}t^{-4/5}dt$ and $t = 0$ to 1 when $x = 0$ to 1

$$\therefore\int_0^1\frac{5x}{\sqrt{1-x^5}}dx=\int_0^1 5t^{1/5}(1-t)^{-1/2}\frac{1}{5}t^{-4/5}dt$$

$$=\int_0^1 t^{-3/5}(1-t)^{-1/2}dt$$

$$=B\left(\frac{2}{5},\frac{1}{2}\right)\ \text{by definition}.$$

2. Evaluate $I=\int_0^\infty\frac{x^{10}(1-x^8)}{(1+x)^{30}}dx-\int_0^\infty\frac{x^{18}}{(1+x)^{30}}dx$.

Solution: $I = B(11, 19) - B(19,11) = 0$, using $B(m,n) = B(n,m)$.

3. Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta d\theta}$.

Solution: $\int_0^{\pi/2} \sqrt{\tan\theta\, d\theta} = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta d\theta$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right), \quad \text{by definition}$$

4. Show that $\int_0^n \left(1-\frac{x}{n}\right)^n x^{t-1}dx = n^t B(t, n+1), \quad t>0, n>1$

Solution: Put $\frac{x}{n} = y \Rightarrow dx = ndy$ and $y = 0$ to 1 when $x = 0$ to n

$$\therefore \int_0^n \left(1-\frac{x}{n}\right)^n x^{t-1}dx = \int_0^1 (1-y)^n n^{t-1} y^{t-1} n dy$$

$$= n^t \int_0^1 (1-y)^n y^{t-1} dy = n^t B(t, n+1) \quad \text{by definition}.$$

5. Prove that $B(n,n) = \frac{1}{2^{2n-1}} \mathbf{B}\left(n, \frac{1}{2}\right)$.

Solution: We know that $B(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1} 2\theta}{2^{2n-1}} d\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \int_0^{\pi} \frac{\sin^{2n-1}\phi}{2^{2n-1}} d\phi, \quad \text{putting } 2\theta = \phi$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\phi}{2^{2n-1}} d\phi, \quad \sin\phi \text{ is even in } (0,\pi) \ .$$

Using definition of Beta function, we get

$$B(n,n) = \frac{1}{2^{2n-1}} \mathbf{B}\left(n, \frac{1}{2}\right).$$

6. Prove that $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Solution: By one of the definitions of Beta function, we have

$$B(m,n)=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$=\int_0^{1}\frac{x^{n-1}}{(1+x)^{m+n}}dx+\int_1^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$= I_1 + I_2$ (say)

Substituting $x=\frac{1}{y}$ in I_2, we get

$$I_2=\int_1^{0}\frac{y^{m+n}}{y^{n-1}(1+y)^{m+n}}\left(-\frac{1}{y^2}\right)dy=\int_0^{1}\frac{y^{m-1}}{(1+y)^{m+n}}dy$$

Hence $B(m,n)=\int_0^{1}\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Exercise:

Express the following integrals in terms of gamma function.

1. $\int_0^{\infty} a^{-bx^2}\,dx$ 2. $\int_0^{\pi/2}\sqrt{tan\theta}\;d\theta$

Answers:

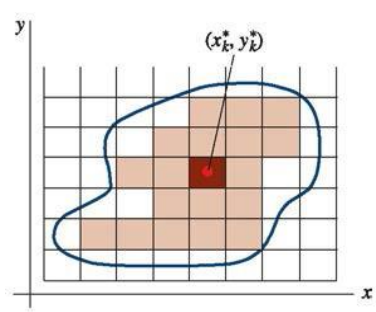
1. $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

2. $\frac{\sqrt{\pi}}{2\sqrt{b\log a}}$

MULTIPLE INTEGRALS

Introduction:

• Let $z = f(x,y)$ be defined in a closed and bounded region R of two dimensional space.
• By means of a grid of vertical and horizontal lines parallel to the coordinate axes, form a partition P of R into n rectangular subregions R_k of areas A_k that lie entirely in R.



• Choose a sample point (x^*,y^*) in each subregion R_k.
• Form the sum $\sum_{k=1}^{n} f\left(x^*,y^*\right)\Delta A_k$.

Let f be a function of two variables defined on a closed region R of two dimensional space. Then the double integral over R is given by

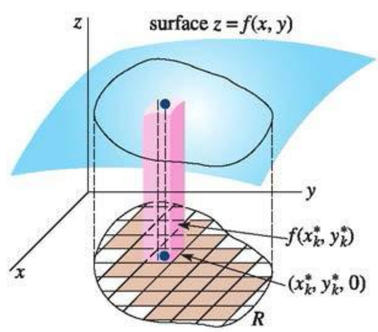
$$\iint_R f(x,y)dA=\lim_{\|P\to 0\|}\sum_{k=1}^{n} f(x_k^*,y_k^*)\Delta A_k$$

If the above limit exists then f is integrable over R and that R is the region of integration. When f is continuous on R, then f is necessarily integrable over R.

Area: When $f(x,y)=1$ on R, then $\lim_{\|P\to 0\|}\sum_{k=1}^{n}\Delta A_k$ simply gives the area A of the region, that is,

$$A=\iint_R dA$$

Note: $dA=dxdy$ is called as area element.



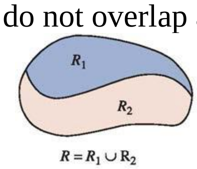
Volume: If $f(x,y)\geq 0$ on R, then as shown in figure, the product $f(x^*,y^*)\Delta A_k$ can be interpreted as the volume of a rectangular prism of height $f(x^*,y^*)$ and base of area ΔA_k. The summation of volumes $\sum_{k=1}^{n} f(x^*,y^*)\Delta A_k$ is an approximation to the volume V of the solid above the region R and below the surface $z=f(x,y)$. The limit of this sum as $\|P\|\to 0$, if it exists, gives the exact volume of this solid; that is, if f is nonnegative on R, then

$$V=\iint_R f(x,y)\,dA$$

Properties of Double Integrals:

Let f and g be functions of two variables that are integrable over a region R. Then

(i) $\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA$, where k is a constant
(ii) $\iint_R [f(x,y)\pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
(iii) $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$,
where R_1 and R_2 are subregions of R that do not overlap and $R=R_1\cup R_2$



Evaluation of double Integrals:

Regions of Type I and II

$R: a \le x \le b,\ \ g_1(x) \le y \le g_2(x)$, where the boundary functions g_1 and g_2 are continuous, is called a region of Type I.

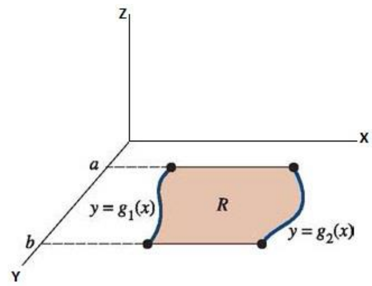


Figure 1: Type I

Since the partial integral $\int_{g_1(x)}^{g_2(x)} f(x,y)dy$ is a function of x alone, integrate the resulting function with respect to x. If f is continuous on a region of Type I, integral of f over the region is

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx.$$

The partial integral with respect to y gives a function of x, which is then integrated in the usual manner from $x=a$ to $x=b$. The end result of both integrations will be a real number.

$R: c \le y \le d,\ \ \ h_1(y) \le x \le h_2(y)$, where the boundary functions h_1 and h_2 are continuous, is called a region of Type II.

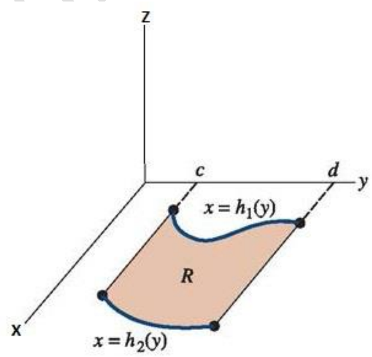


Figure 2: Type II

Since the partial integral $\int_{h_1(y)}^{h_2(y)} f(x,y)dx$ is a function of y alone, integrate the resulting function with respect to y. If f is continuous on a region of Type II, we define an iterated integral off over the region by

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right] dy.$$

The partial integral with respect to x gives a function of y, which is then integrated in the usual manner from $y=c$ to $y=d$. The end result of both integrations will be a real number.

Examples:

1. Evaluate $\int_1^4 \int_3^5 x^2y\,dy\,dx$

Solution: Here it is given that the order of integration is first w.r.t. to y and then w.r.t. x. Therefore the limits for x are $x: 1 \to 4$ and that of y are $y: 3 \to 5$. As the limits for both x and y are constants, they repressent straight lines parallel to cordinate axes and the region bounded by them is a rectangle.

$$\therefore \int_1^4 \int_3^5 x^2y\,dy\,dx = \int_{x=1}^4 x^2 \left(\int_{y=3}^5 y\,dy \right) dx$$
$$= \int_{x=1}^4 x^2 \left(\frac{y^2}{2}\right)\Big|_{y=3}^5 dx$$
$$= \int_{x=1}^4 8x^2\,dx = 8\frac{x^3}{3}\Big|_1^4 = 168.$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$

Solution: Here it is given that one of the limits are variables i.e., $x \to \sqrt{x}$ therefore the order integration is first w.r.t. to y and then w.r.t. x. Limit of integration are given by $y: x \to \sqrt{x}$ and $x: 0 \to 1$

$$\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx = \int_{x=0}^1 \left(\int_{y=x}^{\sqrt{x}} y\,dy \right) dx = \int_{x=0}^1 x \left(\frac{y^2}{2}\right)_x^{\sqrt{x}} dx = \frac{1}{2}\int_{x=0}^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = \frac{1}{24}.$$

3. $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3y\,dx\,dy$

Solution: Here the limit of integration are given by $y: 0 \to 1$ and $x: 0 \to \sqrt{1-y^2}$

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3y\,dx\,dy = \int_0^1 y \left(\int_0^{\sqrt{1-y^2}} x^3\,dx \right) dy = \int_0^1 y \left(\frac{x^4}{4}\right)_0^{\sqrt{1-y^2}} dy = \frac{1}{4}\int_0^1 (y + y^5 - 2y^3)\,dy$$

$$= \frac{1}{4}\left(\frac{y^2}{2} + \frac{y^6}{6} - 2\frac{y^4}{4}\right)_0^1 = \frac{1}{24}.$$

4. Evaluate $\int_0^{\pi} \int_0^{\cos\theta} r\sin\theta\,dr\,d\theta$

Solution: Here the limit of integration are given by $r: 0 \to \cos\theta$ and $r: 0 \to \pi$

$$\int_0^{\pi} \int_0^{\cos\theta} r\sin\theta\,dr\,d\theta = \int_0^{\pi} \sin\theta \left(\int_0^{\cos\theta} r\,dr \right) d\theta = \int_0^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{\cos\theta} d\theta$$
$$= \frac{1}{2}\int_0^{\pi} \sin\theta\cos^2\theta\,d\theta$$

using the substitution $\cos\theta = t \to -\sin\theta\,d\theta = dt$, we get

$$= \frac{1}{2}\int_1^{-1} t^2\,dt = -\frac{1}{2}\frac{t^3}{3}\,1\text{-}1 = \frac{1}{3}.$$

5. Show that $\int_0^{\infty} \int_y^{\infty} x\,e^{\frac{-x^2}{y}}\,dx\,dy = \frac{1}{2}$

Solution: Here the limit of integration are given by $y: 0 \to \infty$ and $x: y \to \infty$

$$\int_0^{\infty}\int_y^{\infty} xe^{\frac{-x^2}{y}}\,dx\,dy=\int_0^{\infty}\left[\int_y^{\infty}\left(-\frac{y}{2}e^{\frac{-x^2}{y}}\left(-\frac{2x}{y}\right)\right)dx\right]dy=\int_0^{\infty}\left(-\frac{y}{2}e^{\frac{-x^2}{y}}\right)dy$$

$$=\int_0^{\infty}\frac{y}{2}e^{-y}\,dy=\left[\frac{1}{2}\frac{ye^{-y}}{-1}\right]_0^{\infty}-\int_0^{\infty}-\frac{e^{-y}}{-1}\,dy=\left[\frac{1}{2}\frac{e^{-y}}{-1}\right]_0^{\infty}=\frac{1}{2}.$$

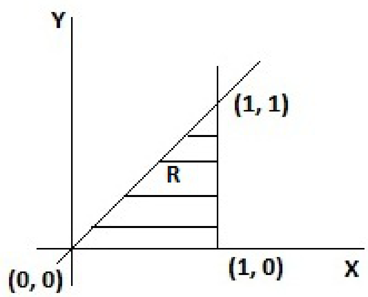
6. Evaluate $\iint_R \frac{dx\,dy}{x+y+1}$ over the space $R{:}0\le x\le 1,\ 0\le y\le 1,\ 0\le x,\ y\le 1$.

Solution: Here the limit of integration are given by $y{:}0\to 1$ and $x{:}0\to 1$

$$\iint_R \frac{dx\,dy}{x+y+1}=\int_0^1\left(\int_0^1\frac{dx}{x+y+1}\right)dy=\int_0^1 \log(x+y+1)_0^1 dy$$
$$=\int_0^1 (\log(y+2)-\log(y+1))\,dy$$
$$=[(y+2)\log(y+2)-(y+2)-(y+1)\log(y+1)+(y+1)]_0^1$$
$$=\log\left(\frac{27}{16}\right).$$

7. Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the triangle bounded by the lines $y=0$, $y=x$ and $x=1$.

Solution:

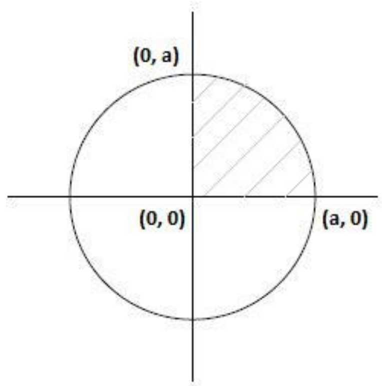


Here the minimum value of x is 0 and maximum value is 1. Therefore x varies from 0 to 1, and y varies from the line 0 to the line $y=x$ as shown in the figure. i.e. $x{:}0\to 1$ and $y{:}0\to x$

$$\iint_R (x^2+y^2)\,dx\,dy=\int_{x=0}^{1}\int_{y=0}^{x}(x^2+y^2)\,dy\,dx$$
$$=\int_0^1\left(\int_0^x (x^2+y^2)dy\right)dx=\int_0^1\left(x^2y+\frac{y^3}{3}\right)_0^x dx$$
$$=\int_0^1\left(x^3+\frac{x^3}{3}\right)dx=\int_0^1\frac{4x^3}{3}\,dx=\frac{4}{3}\left[\frac{x^4}{4}\right]_0^1=\frac{1}{3}.$$

8. Evaluate $\iint_A xy\,dx\,dy$ where A is area bounded by the circle $x^2+y^2=a^2$ in the first quadrant.

Solution:



Here the minimum value of x is 0 and maximum value is a. Therefore x varies from 0 to a, and y varies from the line 0 to the circle $x^2+y^2=a^2 \rightarrow y=\sqrt{a^2-x^2}$ as shown in the figure. i.e. $x:0\rightarrow a$ and $y:0\rightarrow\sqrt{a^2-x^2}$.

$$\iint_A xy\, dx\, dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a x\left(\int_0^{\sqrt{a^2-x^2}} y\, dy\right) dx = \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$=\frac{1}{2}\int_0^a (a^2x-x^3)\, dx = \frac{1}{2}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{a^4}{8}.$$

Exercise:

Evaluate the following integral

1. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\, dx\, dy$
2. $\int_1^2\int_0^x \frac{dy\, dx}{x^2+y^2}$
3. $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

Answers:

1. $8\ln 8 - 16 + e$
2. $\frac{\pi}{4}\log 2$
3. $\frac{\pi}{4}\log(1+\sqrt{2})$

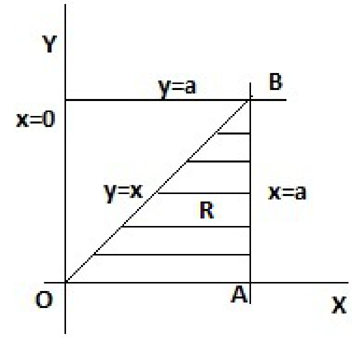
Change of order of integration:

Reversing the Order of Integration: A problem may become easier when the order of integration is changed or reversed. Also, some integrals that may be impossible to evaluate using one order of integration can perhaps be evaluated using the reverse order of integration.

Examples:

1. Change the order of integration in $\int_0^a\int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and hence evaluate.

Solution:



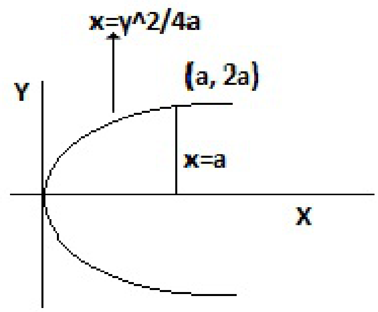
From the limits of integration , it is clear that the region of integration is bounded by x=y, x=a y=0 and y=a. Thus the region of integration is ΔOAB and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into verticle strips.
The new limits of integration becomes $y:0\to x$ and $x:0\to a$

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy \;=\int_0^a \int_o^x \frac{x}{x^2+y^2}\,dy\,dx=\int_0^a \left(x\frac{1}{x}\tan^{-1}\frac{y}{x}\right)_0^x dx=\int_0^a \frac{\pi}{4}\,dx$$

$$=\left[\frac{\pi}{4}x\right]_0^a=\frac{\pi a}{4}.$$

2. Change the order of integration in $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$ and hence evaluate.

Solution:



After changing the order of integration, limit of integration are given by

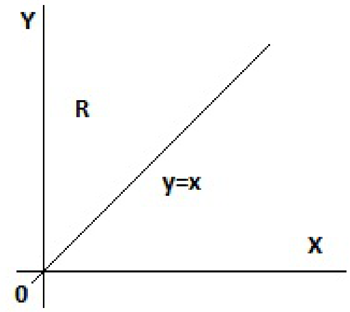
$y:0\to 2a$ and $x:\frac{y^2}{4a}\to a$

$$\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx \;=\int_0^{2a} \int_{\frac{y^2}{4a}}^{a} x^2\,dx\,dy=\int_0^{2a} \left[\frac{x^3}{3}\right]_{\frac{y^2}{4a}}^{a} dy=\int_0^{2a} \left[\frac{a^3}{3}-\frac{y^7}{192a^3}\right] dy$$

$$=\left[\frac{a^3}{3}y-\frac{y^6}{192a^3\times 7}\right]_0^{2a}=\frac{4}{7}a^4.$$

3. Change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ and hence evaluate.

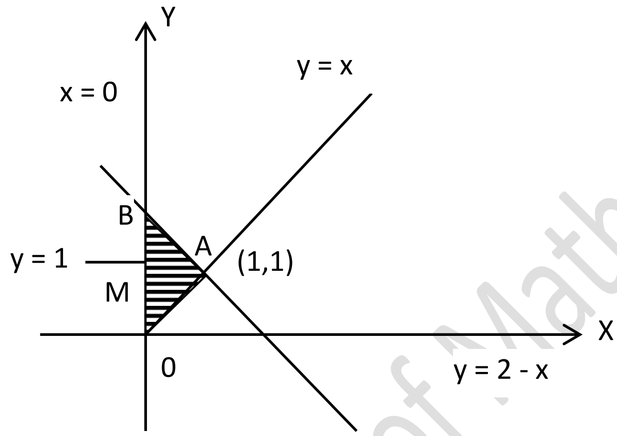
Solution: After changing the order of integration, limit of integration are given by $y:0\to\infty$ and $x:0\to y$



$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \frac{e^{-y}}{y}[x]_0^y dy = \int_0^\infty e^{-y}\, dy = [-e^{-y}]_0^\infty = 1.$$

4. Change the order of integration and hence evaluate the integral $\int_0^1 \int_x^{2-x} \frac{x}{y} dxdy$

Solution: The region of integration is the shaded portion shown in the figure below



To get the limits for x first, we need to divide the shaded area into two parts AMB and AMO where x values are 0 to 2-y and 0 to y respectively. The respective y values are 1 to 2 and 0 to 1.
The given integral is now

$$= \int_{y-0}^{1} \int_{x-0}^{y} \frac{x}{y} dxdy + \int_{y-1}^{2} \int_{x-0}^{2-y} \frac{x}{y} dxdy$$

$$= \int_{y-0}^{1} \frac{y}{2} dy + \int_{y-1}^{2} \left(\frac{2}{y} + \frac{y}{2} - 2 \right) dy$$

= 2 ln 2 -1.

Exercise:
Evaluate the following integrals by changing the order of integration.

1. $\int_0^\infty \int_0^x x e^{-x^2/y}\, dy\, dx$

2. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dy\, dx$

Answers:

1. ½

2. $\frac{a^2}{28}+\frac{a}{20}$

Change of variables:

In some instances a double integral $\iint_R f(x,y)dA$ that is difficult or even impossible to evaluate using rectangular coordinates may be readily evaluated when a change of variables is used.

An approximation choice of coordinates quite often facilitates the evaluation of a double integral. By changing the variables a given integral can be transformed into a simpler integral involving the new variables.

1. uv-plane: In a double integral, let the variables (x,y) be changed to the new variables (u,v) by the transformation. $x=\phi(u,v)$ and $y=\psi(u,v)$, where $\phi(u,v)$ and $\psi(u,v)$ are continuous and have continuous first order derivatives in some region R_{uv}' in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x,y)\, dx\, dy = \iint_{R_{uv}'} f(\phi(u,v),\psi(u,v))\, |J|\, du\, dv$$

where $J=\frac{\partial(x,y)}{\partial(u,v)}, (\neq 0)$ is the Jacobian of the transformation from (x,y) to (u,v) coordinates.

2. In Polar coordinates: If $x=r\cos\theta$ and $y=r\sin\theta$ with $r^2=x^2+y^2$ then

$$\iint_{R_{xy}} f(x,y)\, dx\, dy = \iint_{R_{r\theta}'} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where $J=\frac{\partial(x,y)}{\partial(r,\theta)}=r$.

Examples:

1. Transform to polar coordinates and hence evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.

Solution: In polar form, we have $x=r\cos\theta$, $y=r\sin\theta$ $x^2+y^2=r^2$ and $dxdy=|J|drd\theta$ with $|J|=r$.
Since x, y varies from 0 to ∞, r also varies from 0 to ∞. In the first quadrant θ varies form 0 to $\pi/2$.

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

put $r^2=t \Rightarrow rdr=\frac{dt}{2}$

$$\therefore \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} \frac{e^{-e^{-t}}}{2}\, dt\, d\theta = \int_0^{\pi/2} [-e^{-t}]_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \left[\frac{1}{2}\theta\right]_0^{\pi/2} = \frac{\pi}{4}.$$

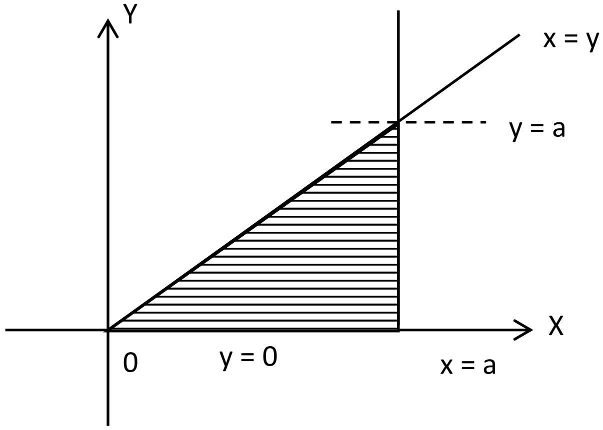
2. Evaluate $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ by changing to polar coordinates.

Solution: In polar form, $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$
and $dxdy=|J|drd\theta$ with $|J|=r$.

$$\therefore \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx \quad = \int_{\theta=-\pi}^{\pi} \int_{r=0}^{a} r^2 dr\, d\theta = \int_{\theta=-\pi}^{\pi} d\theta \times \int_{r=0}^{a} r^2 dr = [\theta]_{-\pi}^{\pi} \text{-}\pi\pi \times \left[\frac{r^3}{3}\right]_0^a$$
$$= \frac{2\pi a^3}{3}.$$

3. Change to polar coordinates and hence evaluate $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}} dxdy$

Solution: The region of integration is as shown below



We see that θ ranges from 0 to π/4 in the shaded region. R ranges from 0 to x = a i.e., a sec θ. The given integral

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} r\cos^2\theta\, dr\, d\theta$$

$$= \int_0^{\pi/4} \frac{a^2}{2} \sec\theta\, d\theta = \frac{a^2}{2}\left[\ln\left(\sec\theta + \tan\theta\right)\right]_0^{\pi/4}$$

$$= \frac{a^2}{2} \ln\left(\sqrt{2}+1\right).$$

Exercise:

Evaluate the following integrals by changing to polar coordinates.

1. $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx\, dy$.

2. $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy$.

Answers:

1. $\frac{a^3}{3} \log(1+\sqrt{2})$

2. $8a^2(\frac{\pi}{2}-\frac{5}{3})$

Area enclosed by plane curves:

Evaluation of area:

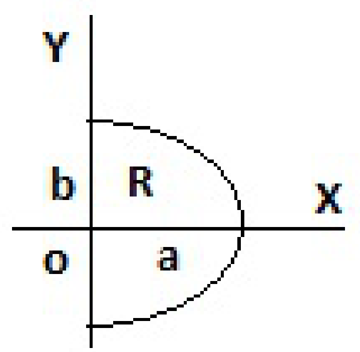
Note: 1. In the Cartesian system, area of the region $R=\iint_R dxdy$.

2. In the polar coordinate system, area of the region $R=\iint_R rdrd\theta$.

Examples:

1. Find the area bounded by one quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:



In the given region x varies from 0 to a and for each x, y varies from 0 to a point on the ellipse. i.e., $y=b\left(1-\frac{x^2}{a^2}\right)^{1/2}$.

$$\therefore A \;=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x^2}{a^2}\right)^{1/2}} dy\,dx=\int_0^a\left[b\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{\frac{a^2}{2}\sin^{-1}\frac{x}{a}}{2}\right]_0^a=\frac{b}{a}\left[\frac{a^2}{2}\frac{\pi}{2}\right]=\frac{\pi}{4}ab.$$

Note: It follows that the whole area bounded by the ellipse is $4\times\frac{\pi}{4}ab=\pi ab$.

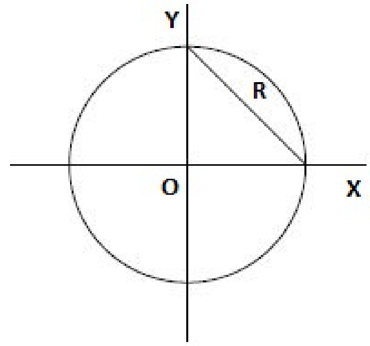
2. Find the area between the parabolas $y^2=4ax$ and $x^2=4ay$.

Solution:

$$A \;=\int_0^{4a}\int_{\frac{y^2}{4a}}^{\sqrt{4ay}} dx\,dy=\int_0^{4a} x_{\frac{y^2}{4a}}^{\sqrt{4ay}}dy=\int_0^{4a}\left[\sqrt{4ay}-\frac{y^2}{4a}\right]dy$$

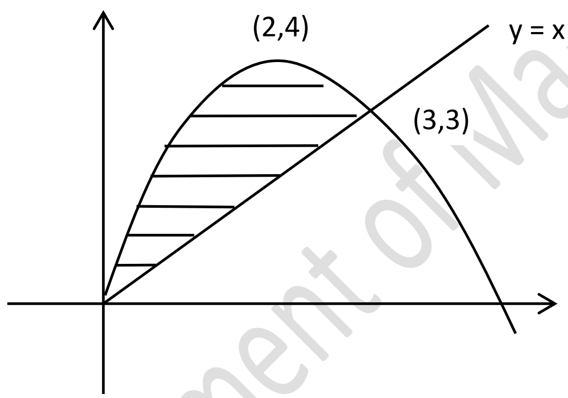
$$=\left[\frac{\sqrt{4a}y^{3/2}}{3/2}-\frac{y^3}{4a(3)}\right]_0^{4a} a=\frac{16}{3}a^2.$$

3. Find the area bounded by the circle $x^2 + y^2 = a^2$ and the line $x + y$ in the first quadrant.
Solution: In the given region x varies from 0 to a and y varies from $x + y$ to $x^2 + y^2 = a^2$.



$$A \quad = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a [x]_{a-x}^{\sqrt{a^2-x^2}} dx = \int_0^a \left[\sqrt{a^2-x^2}-(a-x)\right] dx$$
$$= \left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)-ax+\frac{x^2}{2}\right]_0^a = \frac{a^2}{4}(2\pi-1).$$

4. Find the area between the parabola $y = 4x - x^2$ and the line $y = x$, using double integration.
Solution: The required area is $\iint_R dxdy$



$$= \int_{x-0}^{3} \int_{y-x}^{4x-x^2} dxdy = \int_{x-0}^{3} \left(3x - x^2\right) dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}.$$

5. Find the area which is inside the cardioid $r = 2(1 + cos\theta)$ and outside the circle r = 2.

Solution: Since r = 2 is a circle centred at the origin and of radius 2.

$$A = \int_{-\pi/2}^{\pi/2} \int_2^{1+cos\theta} r\, dr\, d\theta$$
$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_2^{1+cos\theta} d\theta$$

$= 4\int_0^{\pi/2}(2\cos\theta + \cos^2\theta)\, d\theta$

$= 4\left[2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$

$= \pi + 8$

Exercise:

1. Find the area enclosed by $y^2 = 2x$ and the line $y = x$, using double integration.
2. Find the area included by the Lemniscate $r^2 = a^2 \cos 2\theta$ by double integral.
3. Find the area enclosed by the curve $r = a(1 + \cos\theta)$ and lying above the initial line

Answers:
1. 2/3

2. a^2
3. $\frac{3\pi a^2}{4}$

Evaluation of triple integrals:
For the purpose of evaluating the $\iiint_R f(x,y,z)dv$ over the region R can be expressed as an iterated integral or repeated integral in the form

$$\iiint_R f(x,y,z)dx\,dy\,dz = \int_a^b \left[\int_{g(x)}^{h(x)} \left\{\int_{\psi x,y}^{\phi x,y} f(x,y,z)dz\right\} dy\right] dx$$

The above integral indicates three successive integrations to be performed in the following order first w.r.t z keeping x and y as constants, then w.r.t y keeping x as constant and finally w.r.t to x.

Note:

1. When an integration is performed w.r.t. a variable, that variable is eliminated completely from the remaining integral.
2. If the limits are not constants then integration should be in the order in which dx, dy, dz is given in the integral.
3. Evaluation of the integral may be performed in any order if all the limits are constants.
4. If $f(x,y,z)=1$ then the triple integral gives the volume of the region.

One of the silmple application of triple integral is to find the volume of the solid. The volume element is dV = dx dy dz (length× breadth× height), summation of all volume elements gives the volume of a solid.

i,e $V = \iiint_R dx\,dy\,dz$

Examples:

1. Evaluate $\int_0^1 \int_0^1 \int_0^1 (x+y+z)dx\,dy\,dz$.
Solution:

$$\int_0^1\int_0^1\int_0^1 (x+y+z)dx\,dy\,dz = \int_0^1\int_0^1\left[\int_0^1 (x+y+z)dx\right]dy\,dz$$
$$= \int_0^1\int_0^1 [x^2/2+yx+zx]_{z=0}^1\,dy\,dz = \int_0^1\int_0^1 [1/2+y+z]\,dy\,dz$$
$$= \int_0^1 [1/2y+y^2/2+zy]_{y=0}^1\,dz = \int_0^1 [1+z]\,dz$$
$$=[z+z^2/2]_0^1 = 3/2$$

2. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\,dy\,dx.$

Solution:

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\right]dy\,dx$$
$$= \int_0^1\int_0^1\left[\frac{1}{-2(x+y+z+1)^2}\right]_{z=0}^{1-x-y}dy\,dx$$
$$= \frac{1}{-2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-\frac{1}{(x+y+1)^2}\right]dy\,dx$$
$$= \frac{1}{-2}\int_0^1\left[\frac{y}{4}+\frac{1}{x+y+1}\right]_{y=0}^{1-x}dx$$
$$= \frac{1}{-2}\int_0^1\left[\frac{1-x}{4}+\frac{1}{2}-\frac{1}{1+x}\right]dx$$
$$= \frac{1}{-2}\left[-\frac{(1-x)^2}{8}+\frac{x}{2}-\log(1+x)\right]_{x=0}^1$$
$$= \frac{1}{-2}\left[\frac{5}{8}-\log 2\right] = \frac{\log 2}{2}-\frac{5}{16}.$$

3. Evaluate $\int_{r=0}^a\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin\theta\,d\phi\,d\theta\,dr$

Solution:

$$\int_{r=0}^a\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin\theta\,d\phi\,d\theta\,dr = \int_{r=0}^a\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin\theta\,d\phi\,d\theta\,dr$$
$$= \int_{r=0}^a r^2dr\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\int_{\phi=0}^{\pi/2}d\phi \because \text{all the limits of integration are constant}$$
$$= \left[\frac{r^3}{3}\right]_0^a\times[-\cos\theta]_0^{\pi/2}\times[\phi]_0^{\pi/2} = \frac{\pi a^3}{6}.$$

Exercise:

Evaluate the following integral

1. $\int_0^a\int_0^{a-x}\int_0^{a-x-y}(x^2+y^2+z^2)dz\,dy\,dx.$
2. $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$
3. $\int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dz\,dy\,dx$

Answers:

1. $\frac{a^5}{20}$
2. $\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a-\frac{3}{8}$
3. $\frac{1}{2}(e^2-8e+13)$

Volume of solids
Volume using triple integrals:
The volume V of the region R is given by
$V=\iiint_R dx\,dy\,dz$ Cartesian form

Examples:

1. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
Solution: Required volume $V=\iiint_R dx\,dy\,dz$
Since the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is symmetrical about each of the co-ordinate planes, required volume $V = 8V_1$ where V_1 is the volume bounded by the ellipsoid in all positive octant.

$$V=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}[z]_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dy\,dx$$

$$=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$=c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{b}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\,dx$$

By using formula $\left[\int(\sqrt{k^2-y^2}\,dy=\frac{y}{2}\sqrt{k^2-y^2}+\frac{k^2}{2}\sin^{-1}\left(\frac{y}{k}\right)\right]$

$$=\frac{c}{b}\int_0^a\left\{\left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}+\frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\sin^{-1}\left(\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}}\right\}dx$$

$$=\frac{c}{b}\int_0^a\frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}\,dx$$

$$=\frac{\pi bc}{4}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$=\frac{\pi bc}{4}\int_0^a\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$=\frac{\pi abc}{6}$$

$$\therefore V=8\left(\frac{\pi abc}{6}\right)=\frac{4\,\pi abc}{3}$$

2. Find the volume of the tetrahedron $x\geq 0, y\geq 0, z\geq 0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\leq 1$
Solution: Required volume $V=\iiint_R dx\,dy\,dz$

$$V=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [z]_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \, dy\, dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$= c\int_0^a \left\{\left[\left(1-\frac{x}{a}\right) y - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)}\right\} dx$$

$$= c\int_0^a \left\{b\left(1-\frac{x}{a}\right)^2 - \frac{b\left(1-\frac{x}{a}\right)^2}{2}\right\} dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right) dx$$

$$= \frac{bc}{2}\left[x-\frac{x^2}{a}+\frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{abc}{6}$$

Exercise:

1. Using triple integration, find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

2. Find the volume bounded by cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Answers:

1. $\frac{4\pi a^3}{3}$

2. 16π

Center of gravity of a plane region

The total weight of the object concentrated in a single point called the object's centre of gravity or it is a point on which object is in balance.

Let $f(x,y)$ be the density (ρ, mass per unit area) of a distribution of mass in the xy-plane

Then the total mass M in the region R is given by

$$M = \iint_R f(x,y)dxdy$$

The center of gravity of a mass in R has the coordinates $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{M}\iint_R xf(x,y)dxdy$$

and

$$\overline{y} = \frac{1}{M}\iint_R yf(x,y)dxdy$$

For a solid if the density at the point $P(x,y,z)$ be $f(x,y,z)$ then total mass of the solid is given by

$$M = \iiint_R f(x,y,z)dxdydz$$

The center of gravity of a mass in R has the coordinates $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{1}{M}\iiint_R xf(x,y,z)dxdydz,$$

$$\overline{y} = \frac{1}{M}\iiint_R yf(x,y,z)dxdydz$$

and

$$\overline{z} = \frac{1}{M}\iiint_R zf(x,y,z)dxdydz$$

Examples:

1. *Find the center of gravity $(\overline{x}, \overline{y})$ of a mass of density $f(x,y) = 1$ in the region R the semidisk $x^2 + y^2 \leq a^2,\ y \geq 0$.*

Solution: Given region a circle with centered at origin and radius equal to a. It is easy to evaluate if we make use of polar coordinates, using the transformation $x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2$ and $|J| = r$. Here $r: 0 \to a$ and $\theta: 0 \to \pi$

Mass is given by

$$M = \iint_R f(x,y)dxdy = \int_0^{\pi}\int_0^{a} rdrd\theta = \frac{1}{2}\int_0^{\pi} a^2 d\theta = \frac{a^2}{2}\pi$$

The center of gravity $(\overline{x}, \overline{y})$ is given by

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\iint_R xf(x,y)dxdy = \frac{1}{a^2\pi/2}\int_0^{\pi}\int_0^{a} r\cos\theta drd\theta \\
&= \frac{2}{a^2\pi}\int_0^{\pi}\cos\theta d\theta \times \int_0^{a} r^2 dr \\
&= \frac{2}{a^2\pi} \times [\sin\theta]_0^{\pi} \times \left[\frac{r^3}{3}\right]_0^{a} = 0
\end{aligned}$$

and

$$\begin{aligned}
\overline{y} &= \frac{1}{M}\iint_R yf(x,y)dxdy = \frac{1}{a^2\pi/2}\int_0^{\pi}\int_0^{a} r\sin\theta drd\theta \\
&= \frac{2}{a^2\pi}\int_0^{\pi}\sin\theta d\theta \times \int_0^{a} r^2 dr \\
&= \frac{2}{a^2\pi} \times [-\cos\theta]_0^{\pi} \times \left[\frac{r^3}{3}\right]_0^{a} = -\frac{2}{a^2\pi}(-1-1)\frac{a^3}{3} = \frac{4a}{3\pi}
\end{aligned}$$

$\therefore$ Center of gravity $(\overline{x}, \overline{y})$ is given by $\left(0, \frac{4a}{3\pi}\right)$

2. *Find the center of gravity in a volume of solid, which is in the form of positive octant in the sphere $x^2 + y^2 + z^2 = 1$, the density ρ at any point (x,y) is given by $\rho = \mu xyz$, where μ is a constant.*

Solution: This gives the region of the positive octant of the unit sphere $x^2 + y^2 + z^2 = 1$.

The mass is given by

$$\mathrm{M} = \iiint_V f(x,y,z)dxdydz$$

$$M = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \mu\, xyz\, dz\, dy\, dx$$

$$=\mu \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{1-x^2-y^2}} \text{dy dx}$$

$$=\frac{\mu}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2) \text{ dy dx}$$

$$=\frac{\mu}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3) \text{ dy dx}$$

$$= \frac{\mu}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_{z=0}^{\sqrt{1-x^2}} \text{dx}$$

$$= \frac{\mu}{2}\int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right] \text{dx}$$

$$= \frac{\mu}{2}\int_0^1 \left[\frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4}\right] \text{dx}$$

$$= \frac{\mu}{2}\left[\frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24}\right]_{z=0}^{1} = \frac{\mu}{48}.$$

Center of gravity is given by

$$\bar{x} = \frac{1}{M}\iiint_R xf(x,y,z)dxdydz = \frac{1}{M}\iiint_R x^2yz\, dzdydx = \frac{\mu}{\left(\frac{\mu}{48}\right)}\int_0^1 \int_0^{\sqrt{1-x^2}} x^2y \left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{1-x^2-y^2}} \text{dy dx}$$

$$= 24\int_0^1 \int_0^{\sqrt{1-x^2}} x^2y(1-x^2-y^2) \text{ dy dx}$$

$$= 24\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2y - x^4y - x^2y^3) \text{ dy dx}$$

$$= 24\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{x^2y^2}{2} - \frac{x^4y^2}{2} - \frac{x^2y^4}{4}\right]_{z=0}^{\sqrt{1-x^2}} \text{dx}$$

$$= 24\int_0^1 \left[\frac{x^2(1-x^2)}{2} - \frac{x^4(1-x^2)}{2} - \frac{x^2(1-x^2)^2}{4}\right] \text{dx}$$

$$= 24\int_0^1 \left[\frac{x^3}{12} - \frac{x^5}{10} + \frac{x^7}{28}\right] \text{dx}$$

$$= \frac{16}{35}.$$

$$\bar{y} = \frac{1}{M}\iiint_R yf(x,y,z)dxdydz = \frac{1}{M}\iiint_R xy^2z\, dzdydx \frac{16}{35}.$$

$$\bar{z} = \frac{1}{M}\iiint_R zf(x,y,z)dxdydz = \frac{1}{M}\iiint_R xyz^2\, dzdydx = \frac{16}{35}$$

$\therefore$ Center of gravity is given by $\left(\frac{16}{35}, \frac{16}{35}, \frac{16}{35}\right)$.

Exercise:

1. Find the Center of gravity of the area of the circle $x^2 + y^2 = a^2$ in the I quadrant.

2. Find the Centroid of the region in the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with density at any point given by *xyz*

Answers:

1. $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$

2. $\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$

Video Links:

1. Double integrals and Polar integrals
https://www.youtube.com/watch?v=GHBMiscPE-g

2. Change of order of integration
https://www.youtube.com/watch?v=rkvwEkM5RVo
https://www.youtube.com/watch?v=LcG5gvRwYdY